Intermediate Dynamic Programming

Special class

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Intermediate Dynamic Programming

IPC Div - 2, Day - 2

$O(2^n n)$

Speedrun of basic DP

- Given N integers a_1, a_2, ..., a_N, find a subset of elements with maximum sum, but which don't have any two consecutive elements.
- For eg. {a_1, a_3, a_5} is a valid subset but {a_1, a_3, a_4} isn't.
- DP is just a smart exhaustive search
- The technique is in choosing good subproblems and reusing already computed values.
- Prefixes are natural subproblems

5 1 4 10 3 → 15

$$DP[i] = \max\left\{ DP[i-1],\ A_i + DP[i-2] \right\}$$

$A_1 \quad A_2 \ \ldots\ A_i \ \ldots \ - \ \cancel{-\ -} \ -$

10 5

$A_1 \quad A_2$

$$DP[1] = A_1$$

$$DP[2] = \max\{A_1, A_2\}$$

$DP(i) = \max\{$ $DP(1), DP(2) \ldots DP(i-1)$

$A_1 \quad A_2 \quad A_3 \ldots \quad A_{i-1} \quad \boxed{A_i}$ $\quad \{a_1, a_4\}$

$\nexists A_i$ $\qquad \exists A_i$

$\{a_1, a_3\}$ $\{a_1, a_4\}$ $\{a_3\}$ $\qquad \{a_1, a_5\}$ $\{a_1, a_3, a_5\}$ $\{a_5\}$

$DP(i-1)$

$A_i + DP(i-2)$

$A_i +$ $\qquad DP(i-2)$

DP[i] = max { DP[i-1] , A_i + DP[i-2] }

Final Answer = DP[N]

O(N)

NP-Hard

Knapsack DP

- There is a shop which has bars of many metals
- Each bar of Gold weighs W_1 Kg and hae a value of Rs. V_1
- Each bar of Silver weighs W_2 Kg and hae a value of Rs. V_2
- There are N such metals
- You have a bag (knapsack) which can carry at most W Kgs
- Find the maximum value that you can steal
- A bar cannot be broken. So you either take it whole, or you don't take it at all
- ~~Infinite bars of each metal~~ → Exactly 1 bar of each metal

0-1 KS

5 Kg Rs.100

- Suppose N = 1?

22 ~~200~~ Kg $= W$

$22 = W$

W_1 V_1

W

$\left(\frac{20}{5}\right) \times 100 = 400$

$\boxed{\text{floor}\left(\frac{W}{W_1}\right) \times V_1} \rightarrow N = 1$

- Suppose N = 2
- W_1 = 1, V_1 = 100
- W_2 = 1, V_2 = 50
- W = 5

- Suppose N = 2
- W_1 = 5, V_1 = 100
- W_2 = 1, V_2 = 50
- W = 5

$\frac{3}{2} = 1.5$ 1

55 > 16

250 100

$w_1 = 2$ $v_1 = 100$

$w_1 = 1$ $v_1 = 40$

$W = 5$

$\boxed{240}$

$\underline{250} > 100$

W_1 V_1

W_2 V_2

W

Liquids instead of metals?

Fractional Knapsack

$$\left(\max_{n} \frac{v_i}{w_i}\right) \times W$$

$w_1 = 10$ kg $\qquad v_1 = 100$

$w_2 = 1$ kg $\qquad v_1 = 99$

$\boxed{99/\text{kg}}$

990

$W = 10$

$\frac{v_i}{w_i}$

$O(n)$

0-1 Knapsack

Exactly 1 bar of each metal.

$\frac{v_1}{w_1} = 33.3$

$\frac{v_2}{w_2} = 25$

$w_1 = 3$ $\quad v_1 = 100$

$w_2 = 2$ $\quad v_2 = 50$

$W = 5$

1 100

1 50

2

$w_1 = 2 \qquad v_1 = 10$

$w_2 = 3 \qquad v_2 = 20$

$W = \cancel{8}\ 7$

$\frac{v_1}{w_1} = \cancel{0.5}\ \frac{\cancel{4}}{\cancel{2.5}}\ 5$

$\frac{v_2}{W_2} = \cancel{\frac{1}{20}}\ 6 \ldots$

$w_1 \quad v_1$

$w_2 \quad v_2$

$\vdots$

$w_n \quad v_n$

$\underline{\underline{W}} = 1000$

$dp[i][w]$ = max value I can get using first i items, but sum of weight $\leq w$

$dp[20][1000]$

$DP[1][w] =$

$$\text{if}(w < w_1)$$

$$0$$

$$\text{else}$$

$$v_1$$

$O(NW)$

$O(NW) \rightarrow O(kW)$

$$DP[i][w] = \max \begin{cases} DP[i-1][w], \\ v_i + DP[i-1][w - w_i] \end{cases}$$

i: 1 to N

w: 0 to W

1 2 3 . . . . . . $i-1$ i

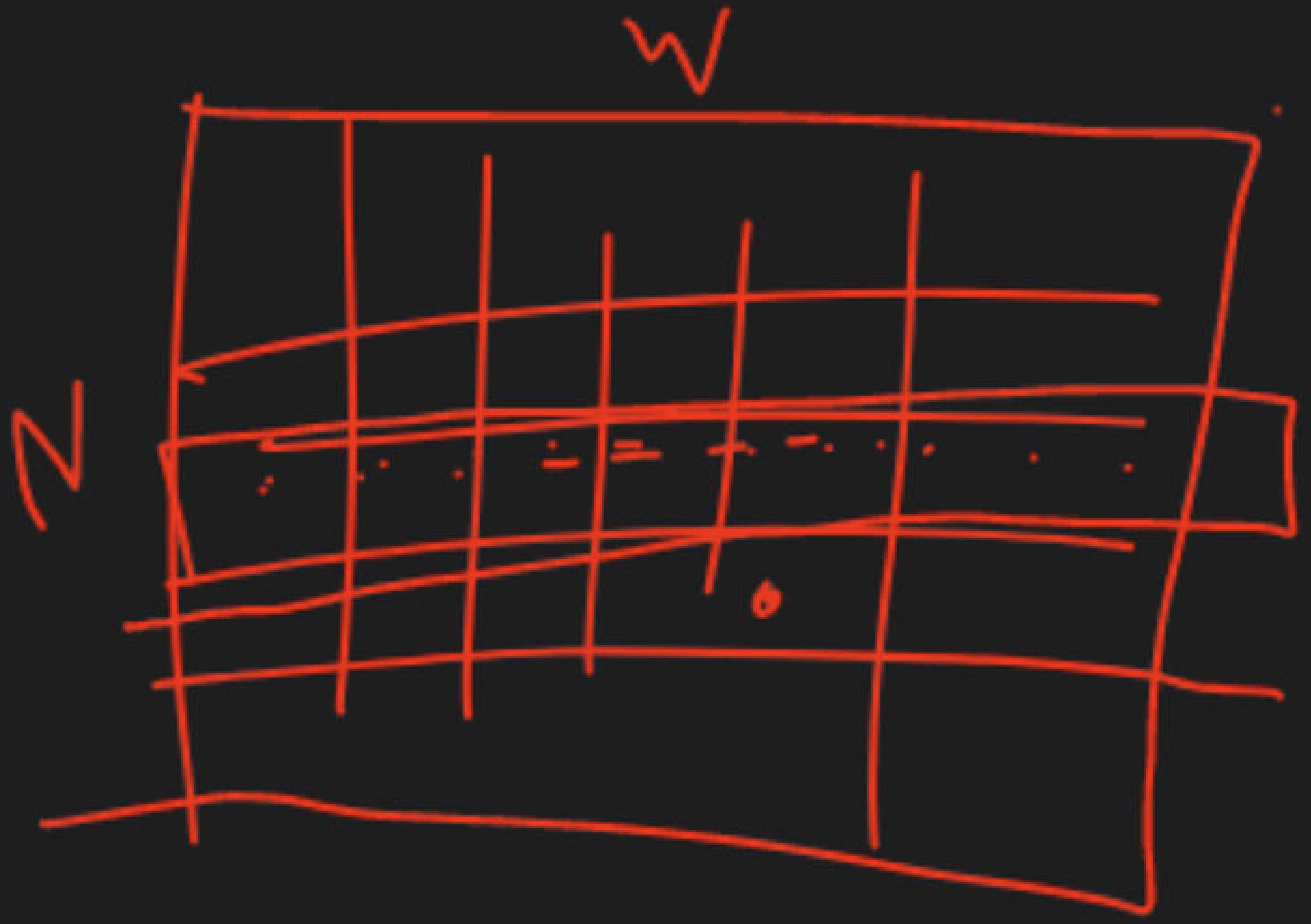
$DP[1][w]$

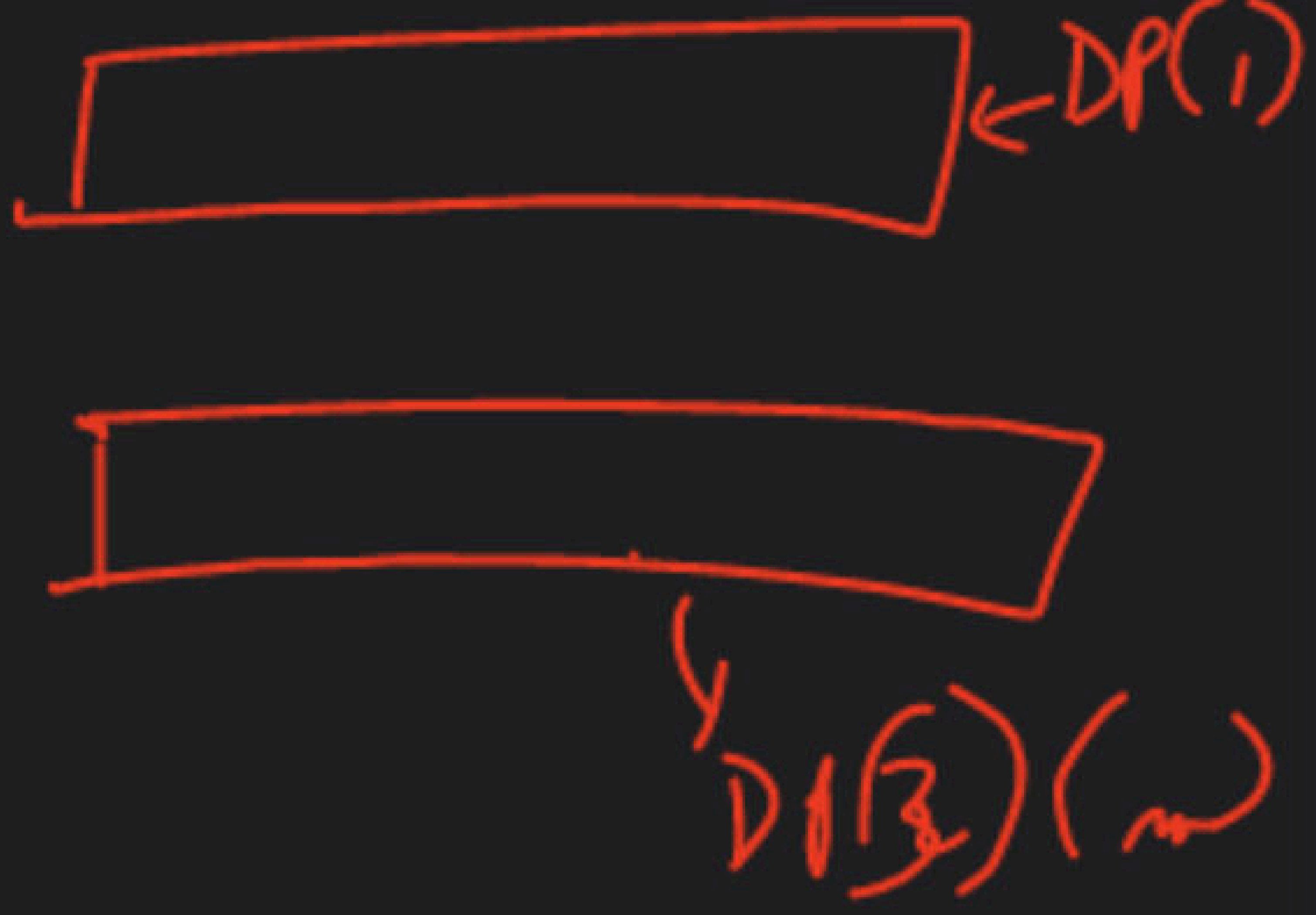
$DP[2][w]$

$w_i > w$

$w_i \le w$

$DP[N][W]$





$O(W)$

Unbounded Knapsack

$$DP(i)(w) =$$

$$DP(1)(w) = floor\left(\frac{w}{w_1}\right) \times v_1$$

$$DP(i)(w) = max\begin{cases} DP(i-1)(w), \\ v_i + DP(i-1)(w - w_i), \\ 2v_i + DP(i-1)(w - 2w_i), \end{cases}$$

$NW \times W$

$O(NW^2)$

$O(NW)$

$$B_i \times V_i + DP(i-1)(w - B_i w_i)$$

$$\vdots$$

$$\underline{\underline{x}}\, V_i + DP(i-1)(w - x\, w_i)$$

$$\text{floor}\left(\frac{W}{w_i}\right)$$

$O(NW \times \max(B_i))$

$O(N \times W^2)$

$w_5 = 10$

$$DP(i)(w) = \begin{cases} 0 \text{ bars?} \checkmark \\ \geq 1 \text{ bars?} \end{cases}$$

$DP[5][100]$

$\to v_5 + DP[5][100-10]$

$\to v_5 + DP(4)(80)$ — ; v_i, $w - w_i < w$

$$DP(i)(w) = \max \begin{cases} DP(i-1)(w), \\ v_i + DP[i][w - w_i] \end{cases}$$

$\underline{\underline{O(NW)}}$

$O(W)$

Unbounded Knap

$DP(0) = 0$

$DP[W] = \text{ans.}$

$$DP[w] = \max\{ V_1 + DP(w - w_1), \; V_2 + DP[w - w_2], \; \vdots \; V_n + DP(w - w_n) \}$$

w between 0 and W

$O(N\,W)$

Bounded Knapsack

B_i

$O(n)$

$O(n^2)$

$O(2^n)$

$O(n!)$

$O(N \times W^2)$

$O(NW \times max(B_i))$

$O(NW)$

NP-Hard

P vs NP

$\boxed{\text{Input Size.}}$

Input size.

N, W

$w_1, v_1,$

w_2, v_2

$\vdots$

w_n, v_n

$W = 2^{100}$ $W \le 1000$

$\log N + \log W \le 2N \log M \le cN$

$\log_2 W$

$O(N^3)$

$O(W^3)$

$O(NW)$

$W = 2$

for x

exp ✓

DP over subsets

Shortest Hamiltonian Path

Number of Hamiltoinan Paths

Shortest Hamiltonian Cycle