Square Root Decomposition

Course: https://unacademy.com/a/i-p-c-intermediate-track

tanujkhattar@

Objective

- MO’s Algorithm
  - Introduction and Problem Discussion
- MO’s with Updates
  - Introduction and Problem Discussion
- Other MO’s Variants
  - MO’s with Hilbert Curves
  - MO’s On Trees
- Overview of other Types of Square Root Decomposition
  - Array Square Root Decomposition
  - Query Square Root Decomposition
  - Heavy Set / Light Set Based Square Root Decomposition
- Conclusion

* MO's Algorithm

A : [| | | | | | | |]

Q [L, R] → Sum of elements in the range L...R.

* Easily be solved using Prefix Sums.

- Find a way to quickly "add" and "remove" an element to a range.
    - Given some DS and an answer for the range [L, R], we should be able to quickly "add"/"remove" an element s.t. we have updated DS and updated answer for range [L, R+1] / [L, R - 1].

answer + DS [L, R]

int Sum

⟹ O(1) or O(logN)

[L, R+1] : "add"

sum += A[R+1];

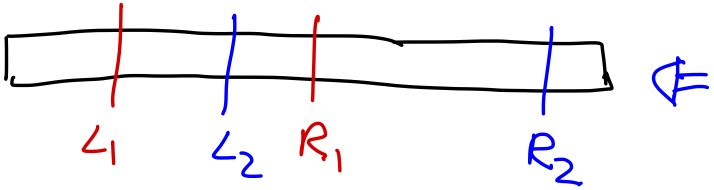
Still maintain correct ans. + correct DS.

OR

[L, R-1] : "remove"

sum -= A[R];

MO's Algorithm



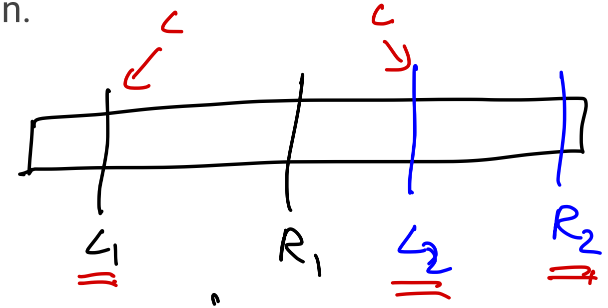
- Notice that it takes $|L_1 - L_2| + |R_1 - R_2|$ operations to go from $[L_1, R_1]$ to $[L_2, R_2]$.
    - Here an "operation" refers to the add or remove operation.

Q1) $L_1\ R_1$ ← add/remove.

Q2) $L_2\ R_2$

Apply your add/remove.

$\boxed{|L_1-L_2| + |R_1-R_2|}$. → # of add/remove operations.



$|R_2-R_1|$ * while $(R < R_2)$ add$(++R)$

Sum = Sum(L_1, R_2)

$|L_2-L_1|$ * while $(L < L_2)$ Remove$(L++)$

Sum = Sum(L_2, R_2)

MO's Algorithm

- Notice that it takes $|L_1 - L_2| + |R_1 - R_2|$ operations to go from $[L_1, R_1]$ to $[L_2, R_2]$.
    - Here an "operation" refers to the add or remove operation.

M

$Q_1 \quad L_1 \; R_1 \rightarrow [1, 1]$

$|L_2 - L_1| + |R_2 - R_1| \Rightarrow O(N)$

$Q_2 \quad L_2 \; R_2 \rightarrow [N, N]$

$|L_3 - L_2| + |R_3 - R_2| \Rightarrow O(N)$

$\rightarrow [1, 1]$

$\rightarrow [N, N]$

$Q_M \quad L_M \; R_M$

$O(N * M) \propto$ Bad.

MO's Algorithm

- Sort the queries offline such that $\sum (|L_i - L_{i+1}| + |R_i - R_{i+1}|)$ is minimized.
    - Reduces to TSP - NP Hard. Minimum.
    - Can sort the queries smartly such that this summation is $\boxed{O((N + Q) * Sqrt(Q))}$
    - bool cmp(Query a, Query b) {
        - return (a.lb < b.lb) || (a.lb == b.lb && a.r < b.r);
    - }

"Online" $Q_1 \rightarrow$ Answer of Q_1 Before you can read Q_2

Q_2

$\vdots$

M

"Offline"

Read M

Process

Print Answer.

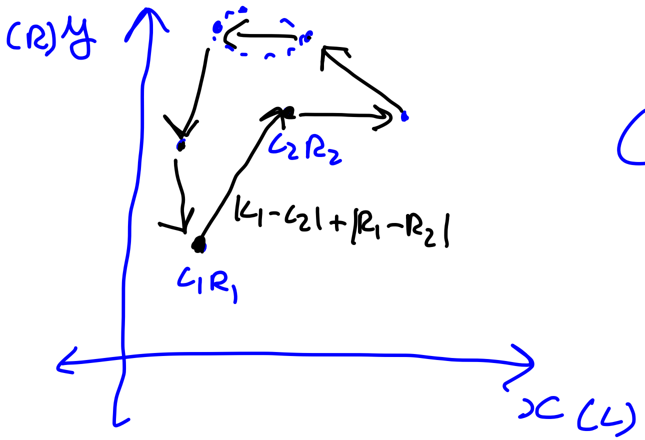
M

Q_1

Q_2

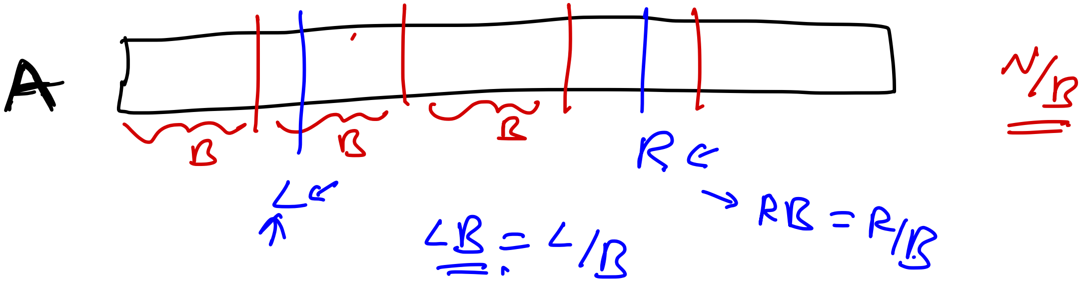
$\vdots$

Q_M



$$Cost(P_1 \rightarrow P_2) = |L_1 - L_2| + |R_1 - R_2|$$

$$O\left(\sum_i |L_i - L_{i+1}| + |R_i - R_{i+1}|\right)$$



MO's Algorithm

- Find a way to quickly "add" and "remove" an element to a range. + Offline
    - Given some DS and an answer for the range [L, R], we should be able to quickly "add"/"remove" an element s.t. we have updated DS and updated answer for range [L, R+1] / [L, R - 1].
- Notice that it takes $|L_1 - L_2| + |R_1 - R_2|$ operations to go from $[L_1, R_1]$ to $[L_2, R_2]$.
    - Here an "operation" refers to the add or remove operation.
- Sort the queries offline such that $\sum (|L_i - L_{i+1}| + |R_i - R_{i+1}|)$ is minimized.
    - Reduces to TSP - NP Hard.
    - Can sort the queries smartly such that this summation is O((N + Q) * Sqrt(Q))
    - bool cmp(Query a, Query b) {
        - return (a.lb < b.lb) || (a.lb == b.lb && a.r < b.r);
    - }

$Q_2 < Q_1 < Q_3$

MO's Algorithm - Sorting Approach 1

- Two queries with L in the same block are sorted as per increasing R. →
- Two queries with L in different blocks are sorted as per increasing LB (L → Block)

```
bool cmp(Query a, Query b) {
    return (a.lb < b.lb) ||
           (a.lb == b.lb && a.r < b.r);
}
```

If $a.lb < b.lb$

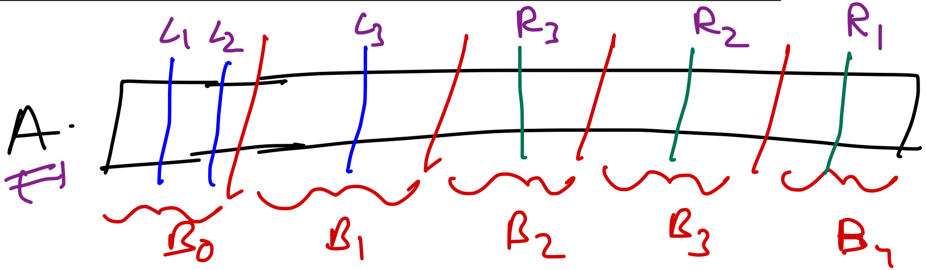
$\Rightarrow a.l < b.l.$

$Q_1 \; L_1 \; R_1$

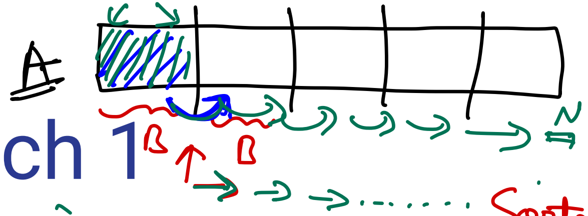
$Q_2 \; L_2 \; R_2$

$Q_3 \; L_3 \; R_3$

$\Rightarrow Q_2 \; Q_1 \; Q_3$



A Query [L,R] belongs to block ⌊L/B⌋.



MO's Algorithm - Sorting Approach 1

O(N) ← ↑sing.

Sorted Based on ↑sing R.

- We add/remove at most O(B) elements on the left side for every query -- O(B * Q).
- For every block, we add at-most O(N) elements on the right side -- O(N * N / B)
- For B = Sqrt(N), we get O((N + Q) * Sqrt(B)).

```
bool cmp(Query a, Query b) {
    return (a.lb < b.lb) ||
           (a.lb == b.lb && a.r < b.r);
}
```

B1.

LeftEnd

* $|L_1 - L_2| \le B$.
* $O(B * Q_1)$

*Right End.

* O(N) ← across all Q queries.

Left. $O(B * Q + N * N/B)$

$B \sim \sqrt{N} \Rightarrow O((N+Q)\sqrt{N})$

B2

$B * Q_2$

O(N)

⋮

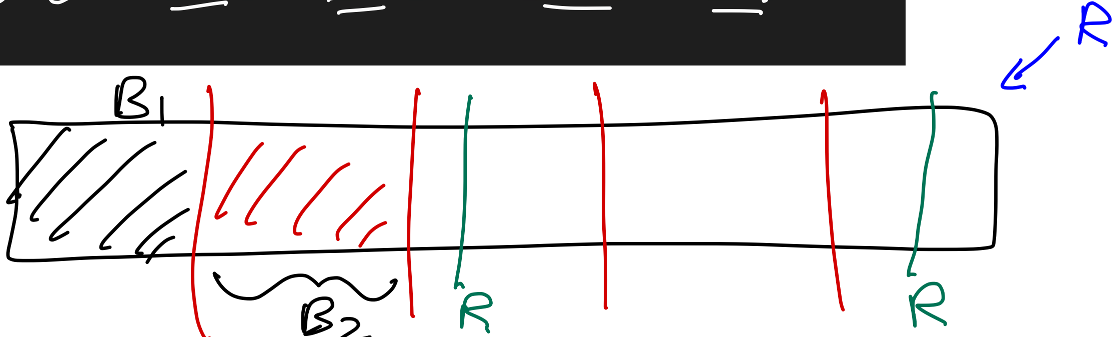
$\boxed{1/2}$

MO's Algorithm - Sorting Approach 2

- We can (slightly) optimize the previous approach by sorting the R in reverse order for even blocks.

Saves $O(N)$ / Block

```
bool cmp(Query a, Query b) {
    return (a.lb < b.lb) ||
           (a.lb == b.lb &&
           (a.lb & 1 ? a.r < b.r : a.r > b.r));
}
```



Practice Problem - 1 (spoj DQUERY)

- Given an array A of N integers, there are Q queries of the form:
  - Range Query: Given **L, R** - Find number of distinct elements in [L, R]

* Can we maintain some DS/Information s.t we can efficiently "add" / "remove".

"Freq Array" → O(1)

" Map / Unordered Map " → O(1) (Bad Constant)
or O(logN)

Practice Problem - 2 (spoj XXXXXXXX)

- Given an array A of N integers, there are Q queries of the form:
  - Range Query: Given **L, R - ** Find number of distinct elements in [L, R] → sum
  - Point Update: Given **i, x** - set **A[i] = x** ←

MO’s with Updates

- Given the DS maintained for MO’s without updates, find a way to quickly “apply” and “undo” the update on the DS.
  - Eg: For point updates, store the previous value so that the point update can be “undone”.

MO’s with Updates

- Let every query be represented as (L, R, T) where T = number of updates before this query.
  - Now to go from [L1, R1, T1] to [L2, R2, T2], we need |L1 - L2| + |R1 - R2| + |T1 - T2| operations.

MO's with Updates

- Sort the queries such that the $\sum (|L_i - L_{i+1}| + |R_i - R_{i+1}| + |T_i - T_{i+1}|)$ is minimized.
  - Process the queries whose Left and Right blocks are same together.
  - Sort such queries based on T s.t. Every pair of blocks, we spend O(Q) time iterating over increasing T.
  - Therefore, if $B = N^{(2/3)}$, $N / B = N^{(1/3)}$. So total complexity = $O(Q * (N / B)^2) = O(Q * N^{(2/3)})$

MO’s with Updates

- Given the DS maintained for MO’s without updates, find a way to quickly “apply” and “undo” the update on the DS.
  - Eg: For point updates, store the previous value so that the point update can be “undone”.
- Let every query be represented as (L, R, T) where T = number of updates before this query.
  - Now to go from [L1, R1, T1] to [L2, R2, T2], we need |L1 - L2| + |R1 - R2| + |T1 - T2| operations.
- Sort the queries such that the $\sum (|L_i - L_{i+1}| + |R_i - R_{i+1}| + |T_i - T_{i+1}|)$ is minimized.
  - Process the queries whose Left and Right blocks are same together.
  - Sort such queries based on T s.t. Every pair of blocks, we spend O(Q) time iterating over increasing T.
  - Therefore, if B = N^(⅔), N / B = N^(⅓). So total complexity = O(Q * (N / B)^2) = O(Q * N ^ (⅔))

MO’s with updates sorting approach.

```
bool cmp(Query a, Query b) {
  return (a.lb < b.lb) || (a.lb == b.lb && a.rb < b.rb) ||
         (a.lb == b.lb && a.rb == b.rb && a.t < b.t);
}
```

MO’s with updates iteration loop.

```
for (int i = 1, T = 0, L = 1, R = 0; i <= nq; i++) {
  while (T < q[i].t) reflect_update(++T, true);
  while (T > q[i].t) reflect_update(T--, false);
  while (R < q[i].r) add_element(++R);
  while (L > q[i].l) add_element(--L);
  while (R > q[i].r) remove_element(R--);
  while (L < q[i].l) remove_element(L++);
  ans[q[i].idx] = curr_ans;
}
```

Practice Problem - 2

- Given an array A of N integers, there are Q queries of the form:
    - Range Query: Given **L, R -** Find number of distinct elements in [L, R]
    - Point Update: Given **i, x** - set **A[i] = x**

MO’s on Trees

- https://codeforces.com/blog/entry/43230
- Linearize the tree using Euler Tour Traversal.
- A path in the tree reduces to a continuous range in an array
  - Elements on the path occurs once.
  - Elements not on the path occur twice.
- Use standard MOs on linearized array

MO’s with Hilbert Curves

- https://codeforces.com/blog/entry/61203
- Better sorting algorithm based on hilbert curves.

Other Types of Square Root Decomposition

- https://unacademy.com/course/course-on-data-structures-square-root-decomposition/XKRCFDJV
- Array Square Root Decomposition
  - Divide the array into blocks of Size B ~ Sqrt(N) & maintain some information for each block.
  - Divide a query/update into two parts
    - Individual elements in blocks of L & R
    - Complete Blocks which lie between L & R
  - Perform Range Updates Lazily
- Query Square Root Decomposition
  - Divide Q queries into blocks of size B ~Sqrt(Q)
  - Process all updates at the end of each block and maintain a “hard-to-update” DS for queries.
  - Reflect contribution of B updates on a query “quickly”.
- Heavy Set / Light Set based Query Decomposition
  - Identify a property whose sum(count) is bounded and can divided into heavy and light sets.
  - Process the heavy and light sets separately - Eg: O(N * #HeavySets) + O(SizeOf(LightSet)^2).

Conclusion

- Square Root Decomposition is a powerful tool with minimal prerequisites.
- It’s usually simple to code and is a great alternative to more complex data structures like Segment Trees etc.