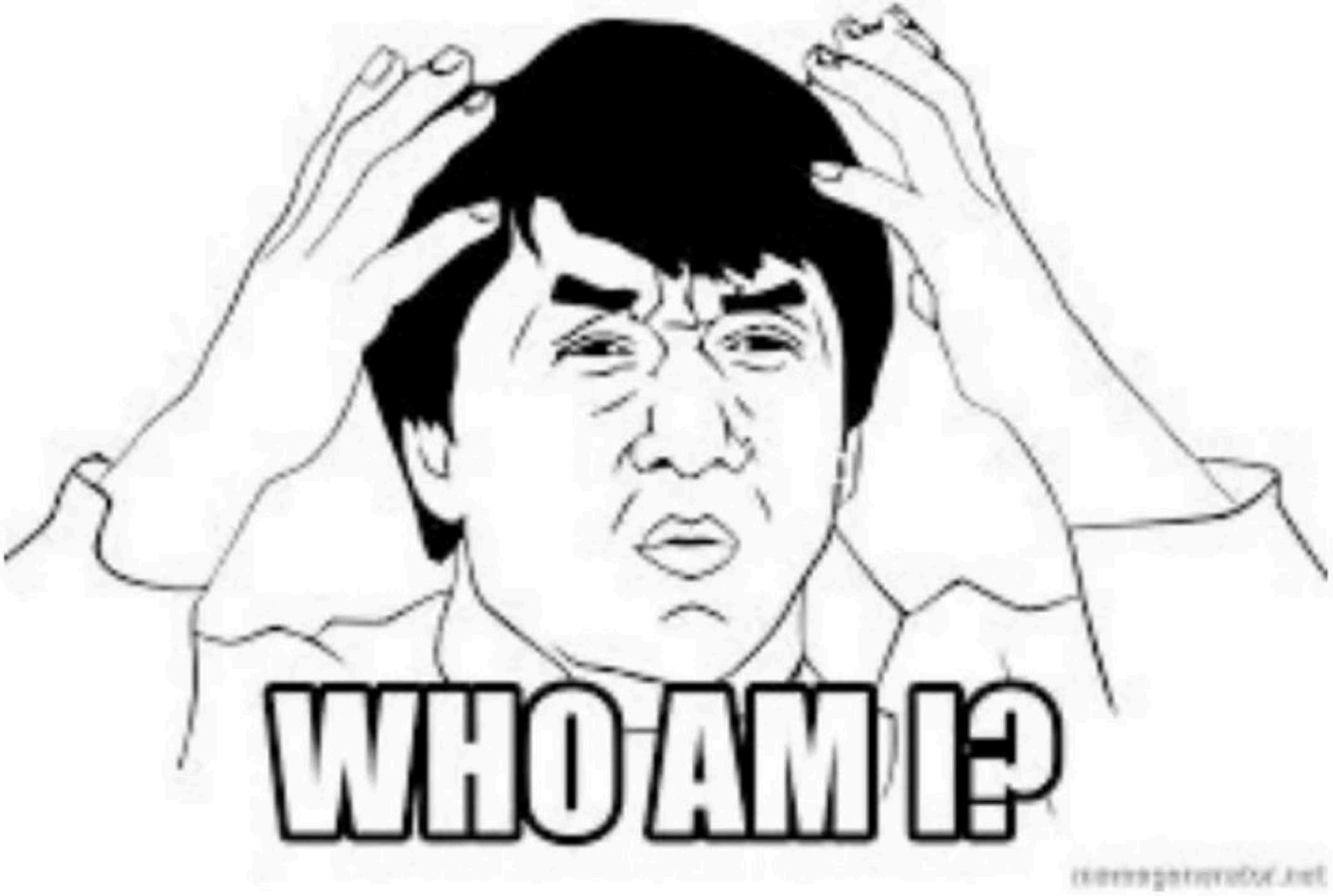
Introduction to Game Theory

Special class

Surya Kiran Adury • Nov 20, 2020

Introduction
to
Game Theory

-Surya Kiran Adury


WHO AM I?

ACM ICPC World Finalist (2014, 2015)

- Work Experience
    - @Google London (2015-2017)
    - @Google MTV (2017-2020)
    - Self Employed (2020-?)

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- Education
  - B.Tech in ECE from IIT Roorkee
- Teaching Experience
  - Weekly lectures to my juniors
  - Programming camps

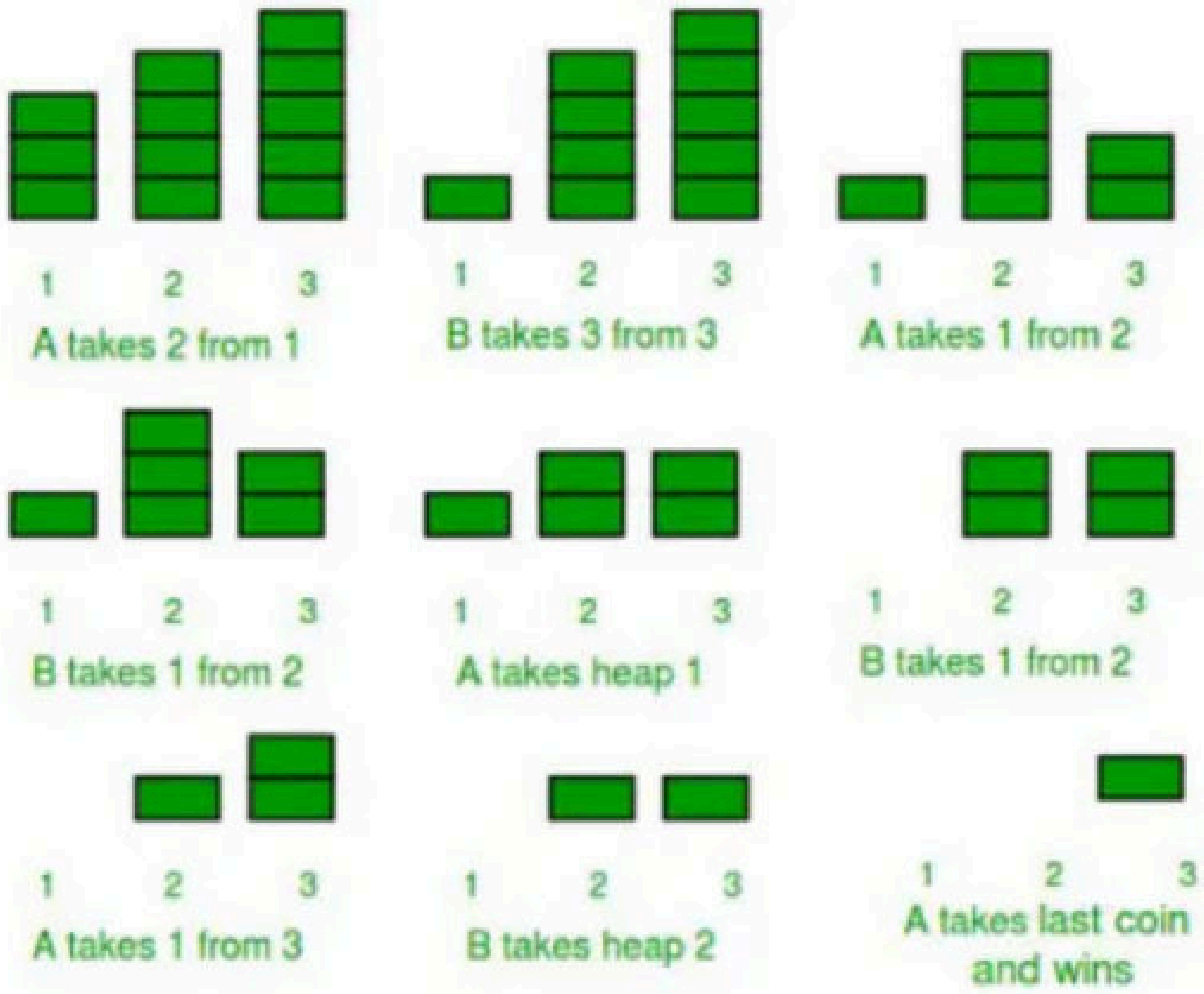

Question
What are we doing?

Objective

1. Basics of game theory, simple games
2. Nim game
3. Composite games - Grundy numbers (Nimbers)
4. Sprague grundy theorem

Games!!

Games!!



1 2 3
A takes 2 from 1
1 2 3
B takes 3 from 3
1 2 3
A takes 1 from 2
1 2 3
B takes 1 from 2
1 2 3
A takes heap 1
1 2 3
B takes 1 from 2
1 2 3
A takes 1 from 3
1 2 3
B takes heap 2
1 2 3
A takes last coin and wins

Games!!

- **Two Person**

Games!!

- Two Person
- **Perfect information**

Games!!

- Two Person
- Perfect information
- **No chance moves**

Games!!

- Two Person
- Perfect information
- No chance moves
- **Win or lose outcome**

Games!!

- Two Person
- Perfect information
- No chance moves
- Win or lose outcome

Question:

Does **Poker** fall under the above criteria? (Yes/No)

Games!!

- Two Person
- Perfect information
- No chance moves
- Win or lose outcome

Question:

Does **Chess** fall under the above criteria? (Yes/No)

Games!!

- Two Person
- Perfect information
- No chance moves
- Win or lose outcome

Question:

Does **Monopoly** fall under the above criteria? (Yes/No)

Problem, Simple Game

- At the beginning there are n coins.
- When it is a player's turn they can take away 1, 3 or 4 coins.
- The player who takes the last one away is declared the winner

n = 1 W

n = 2 L

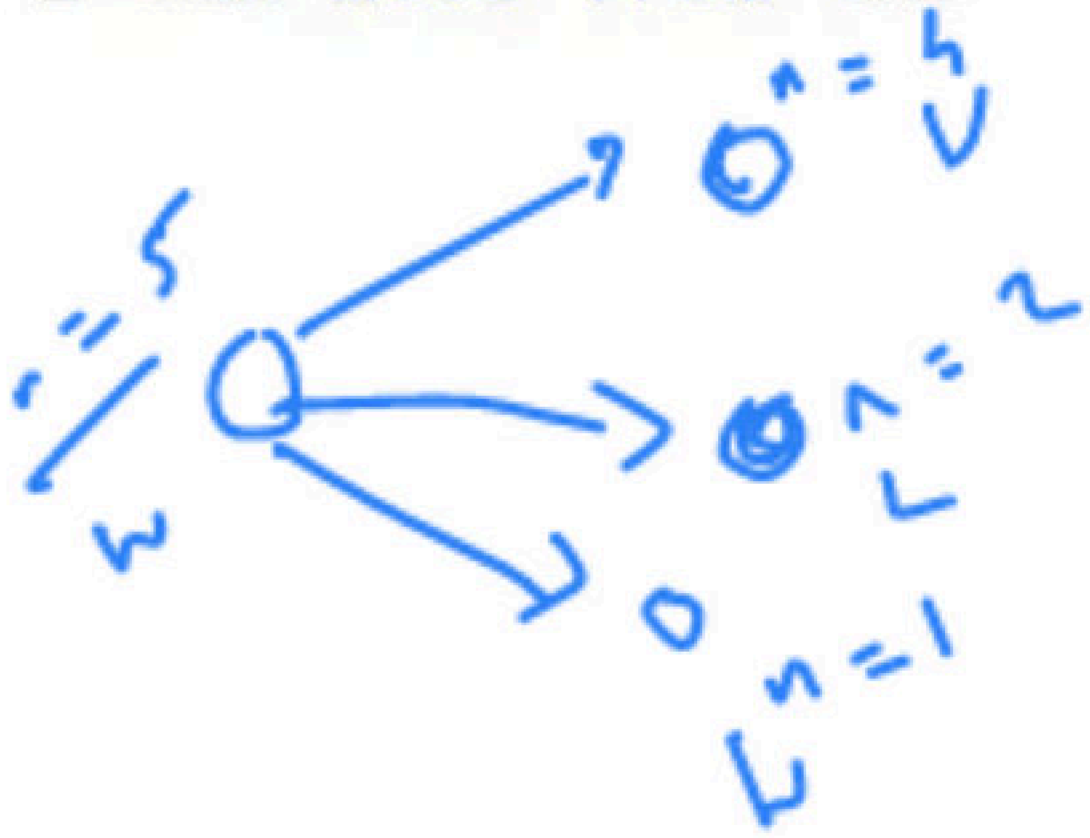
n : 3 W

n = 4 W

n : 5 W

```
boolean isWinning(position pos) {
  moves[] = possible positions to which I can move from the
  position pos;
  for (all x in moves)
    if (!isWinning(x)) return true;

  return false;
}
```



Question

n = 0 L
n = 1 W
n = 2 W
n = 3 L
n = 4 W
n = 5 W ..... n = 15 L

15

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$

- At the beginning there are 15 coins.
- When it is a player's turn they can take away 2^k coins (k can be any whole number).
- The player who takes the last one away is declared the winner.
- Does the First player win? (Yes/No)

Question

$2^{k} \% 3 > 0$

- At the beginning there are 10^7 coins.

n % 3 > 0

n % 3 1, 2

- When it is a player's turn they can take away 2^k coins (k can be any whole number).

1 2 4 8 16

- The player who takes the last one away is declared the winner.
- Does the First player win? (Yes/No)

Questions?

The Game of Nim

- Very famous.
- Lots of problems based on this game.
- Requires clever ideas and less code.

The Game of Nim - Statement

- There are **n** piles of coins.
- When it is a player's turn he chooses one pile and takes at least one coin from it.
- The one who removes the last coin is the winner

The Game of Nim - Solution

- Say **c1, c2, c3, ...cn** be the number of coins in each pile.

cn

The Game of Nim - Solution

- Say **c1, c2, c3, ...cn** be the number of coins in each pile.

- It is a losing position for the player whose turn it is if and only if

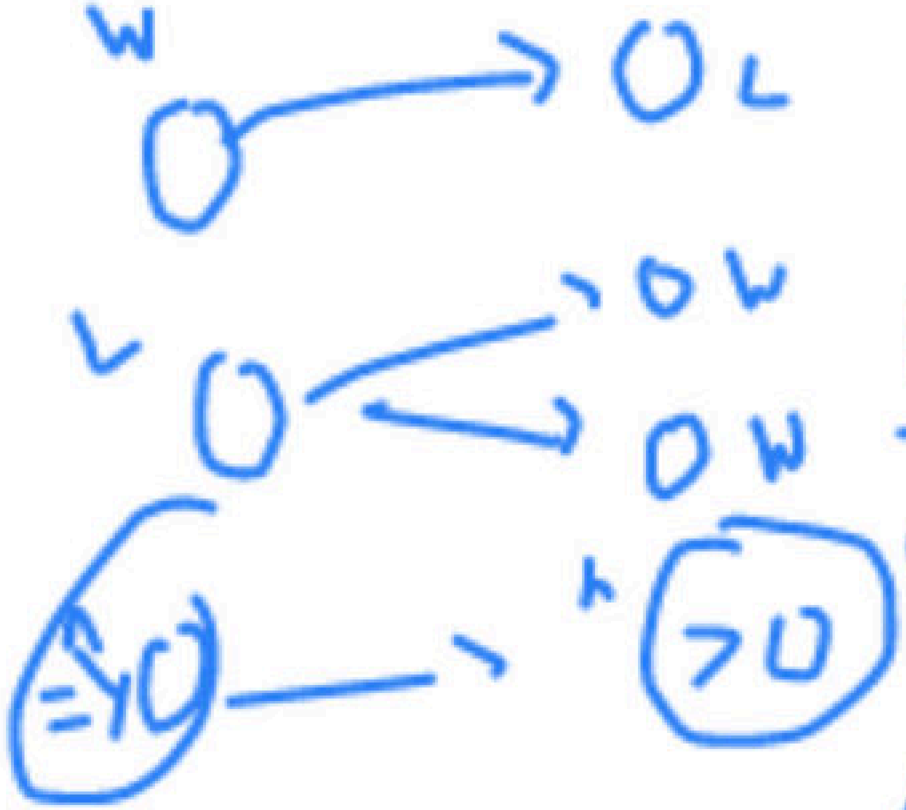
  c1 ^ c2 ^ c3 ^ .... ^ cn = 0

The Game of Nim

$0 \wedge 1 = 1$

$0 \wedge 0 = 0$

$1 \wedge 1 = 0$



$S \; \boxed{1}\,1\,1\,0\,0\,0$

$c_i \; \boxed{1}\,0\,1 \ldots$

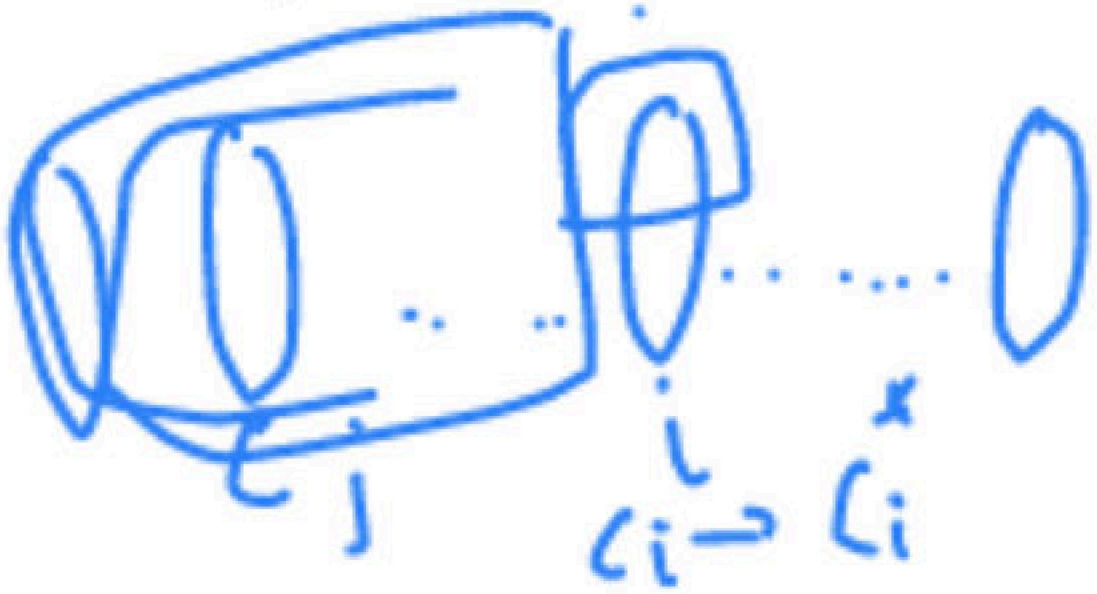
$S > 0$

$(c_i - x) = S \wedge c_i$

$x = c_i - \ldots$

Why does it work?

- From the losing positions we can move only to the winning ones.
- From the winning positions it is possible to move to at least one losing.



$S = 0 = c_1 \wedge c_2 \wedge c_3 \ldots c_n$

$= c_i \wedge k$

$k = c_i$

The Game of Nim

Why does it work?

- From the losing positions we can move only to the winning ones.

- From the winning positions it is possible to move to at least one losing.

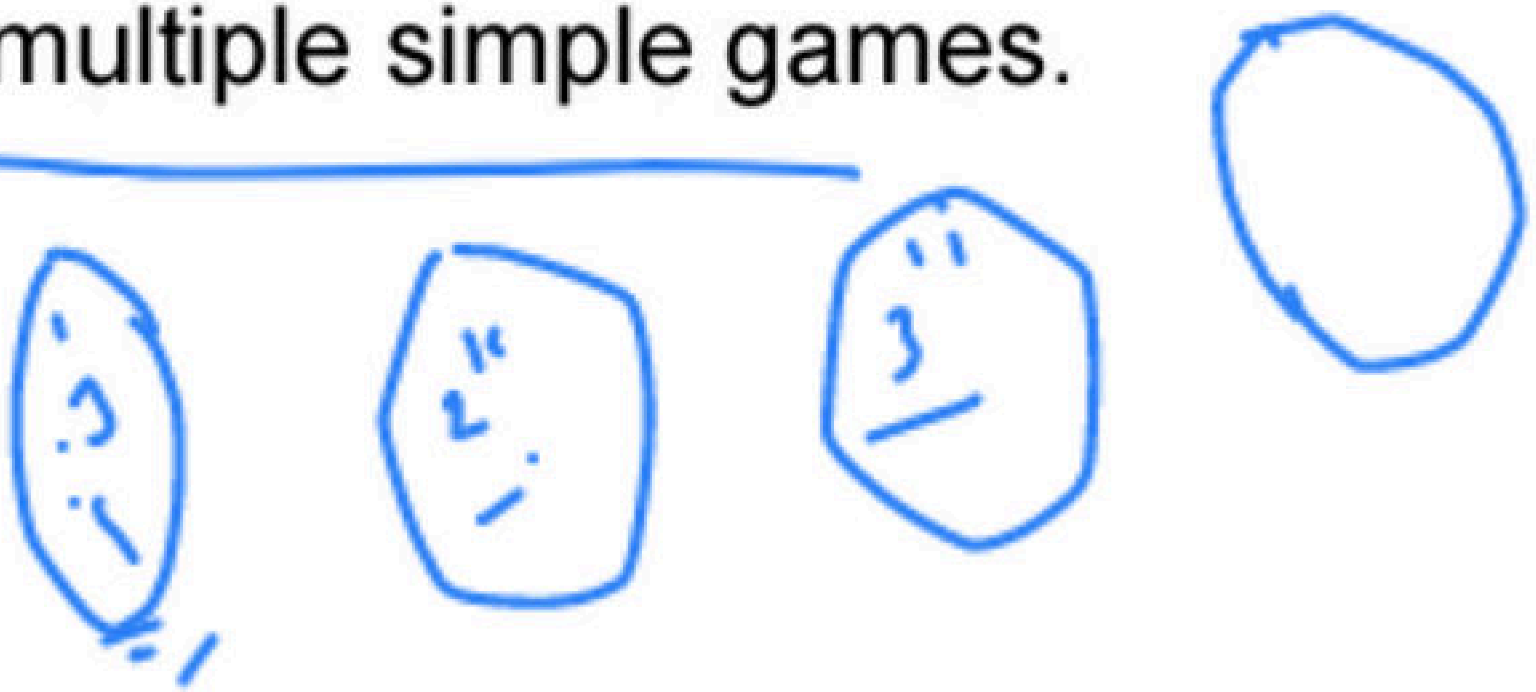
Questions?

$$c_j - n = c_i \wedge S$$

$$n = c_j - c_i \wedge S$$

Composite Games

- Composite games are combination of multiple simple games.
- Example:
  - Say there are n piles of coins.
  - When it is a player's turn they choose one pile and can take away 1, 3 or 4 coins.
  - The one who removes the last coin is the winner.



Grundy numbers

- Each position of a simple game can be identified by a unique integer.

Grundy numbers

- Each position of a simple game can be identified by a unique integer.
- **“MEX”** of all reachable integer set of all reachable positions.

Sprague Grundy Theorem

Sprague-Grundy Theorem says that if both A and B play optimally (i.e., they don't make any mistakes), then the player starting first is guaranteed to win if the XOR of the grundy numbers of position in each sub-games at the beginning of the game is non-zero.

Why does this work?

Questions?

Problem : **RRTREGAM**

Statement:

Ross and Rachel are playing a game. They have a tree rooted at **1**. Each vertex is having some number of stones in it. In one move a player can choose **2 stones** from some same node and move it to any of the ancestors of that node in the tree. The player not able to make a move loses. Help them find the winner of the game if they play optimally. **Rachel** starts first.

Constraints

- $1 \le n \le 10^5$
- $1 \le stones[i] \le 10^9$
- $1 \le x,y \le n$

Problem : CHGM

- The game is simple, there is a stack contains N numbers of disks initially.
- In each move, a player can remove **X** (>0) numbers of disks such that **X** divides **K** where **K** in the number of disks present at that time.
- The player who removes the last disk loses the game.

Resources

- http://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem
- http://www.topcoder.com/tc?module=Static&d1=tutorials&d2=algorithmGames
- http://www.ams.org/samplings/feature-column/fcarc-games1
- http://www.codechef.com/wiki/tutorial-game-theory

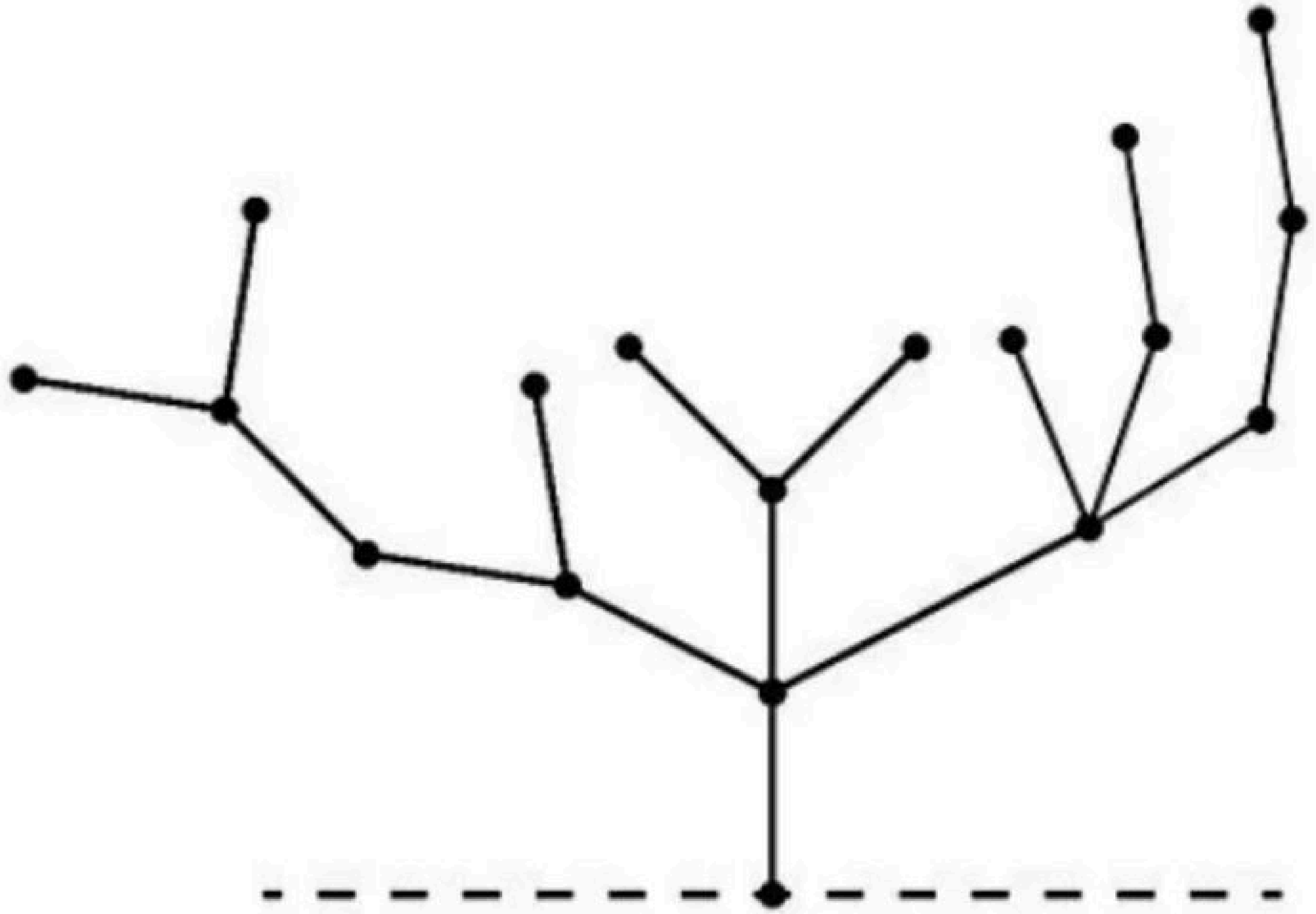
Questions?

Summary

1. Basics of game theory, simple games
2. Nim game
3. Composite games - Grundy numbers (Nimbers)
4. Sprague grundy theorem

One more thing

Hackenbush



Hackenbush - Resources

1. Suggested readings
    a. http://en.wikipedia.org/wiki/Hackenbush
    b. http://www.ams.org/samplings/feature-column/fcarc-partizan1
2. Suggested problems
    a. https://www.codechef.com/problems/GERALD08
    b. http://www.spoj.com/problems/PT07A/


THANK YOU !
meme-arsenal.ru