We have to count triangles that satisfy two rules (to have three white vertices and to share at least one edge with the initial polygon). So I think a useful point of view might be to try to find the answer in two steps. First, we determine the number of triangles that share at least one edge with a polygon that has only white vertices. Second, we find the number of triangles satisfying both rules in a polygon having two black vertices.

**Step 1.** The idea is to count the triangles constructed on a given edge of the polygon. We move around the polygon counterclockwise with vertices *P*1

, *P*2, ..., and consider each edge independently. Now, if we count the triangles for a fixed edge *e*, we obtain *N*−2, one for each vertex not on *e*. But there is a problem with the fact that some triangles are counted two times (those for which the determining vertex is next to one of the vertices of *e*

).

To get rid of this problem, and with an eye on the next step, we assume that an edge can see all the vertices ahead of it, but the last. Explicitly, if *e*=[*PiPi*+1]

, then *e* cannot see the vertex *Pi*−1

.

So, for the edge *e*

, we have *N*−3 triangles. If follows that the number of triangles that share at least one edge with the polygon is *N*(*N*−3)

.

**Step 2.** We keep using the previous point of view and introduce some vocabulary. We say that an edge is black if one of its vertices is black. Otherwise, it is white.

To answer the question, we have to determine how many triangles among the *N*(*N*−3)

triangles that share at least an edge with the polygon have a black edge. Let us call this number *b*2(*N*). Notice that the number of black edges is either 3 or 4

.

If the number of black edges is 3

(i.e. the black points are next to one another), suppose that the black vertices are *P*1 and *P*2. Then we have

*b*2(*N*)=3(*N*−3)+1+2(*N*−4)=5*N*−16

with

* 3(*N*−3)

 for the triangles constructed on the black edges,

 1 for the triangle [*P*3*P*4*P*1] (remember that [*P*3*P*4] cannot see *P*2

 , and

 2(*N*−4)

* for the other triangles constructed on white edges with the third vertex a black point.

If the number of black edges is 4

(i.e. the black points do not determine an edge of the polygon), assume that *P*1 is black and *PN* is white. Then there are again two cases: either *P*3 is black, or *Pk* is black with 3<*k*<*N*

. Geometrically, in the second case, there are white edges on both sides of a black vertex.

If the black points are *P*1

and *P*3, then

*b*2(*N*)=4(*N*−3)+1+2(*N*−5)=6(*N*−6)+15*N*−21

with a similar counting as before. In the other case, that is if the distance between the black points is bigger than 2, then

*b*2(*N*)=4(*N*−3)+2+2(*N*−6)=6*N*−22

with 4(*N*−3) for the triangles constructed on the black edges, 2 for the triangles [*P*2*P*3*P*4] and [*P*5*P*6*P*1] (I assume *k*=4 to simplify the discussion), and 2(*N*−6)

for the other triangles constructed on white edges with the third vertex a black point.

Summing up, the number of triangles is *N*(*N*−3)−*b*2(*N*)

which equals

* (*N*−2)(*N*−6)+4

in case the distance between the black points is 1

 ,

 (*N*−3)(*N*−6)+3 in case the distance between the black points is 2

 , and

 (*N*−3)(*N*−6)+4 in case the distance between the black points is bigger than 2.