Divergence

23.8 DIVERGENCE OF A VECTOR FUNCTION

The divergence of a vector point function $\vec{F}$ is denoted by *div F* and is defined as below.

Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

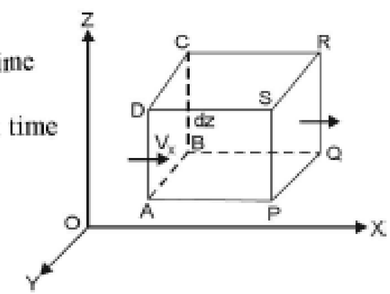
$$\operatorname{div} \vec{F} = \vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

23.9 PHYSICAL INTERPRETATION OF DIVERGENCE

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to x, y and z axes respectively.

Let $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ be the velocity of the fluid at $P(x, y, z)$.



$\therefore$ Mass of fluid flowing in through the face $ABCD$ in unit time

$= \text{Velocity} \times \text{Area of the face} = V_x\,(dy\,dz)$

Mass of fluid flowing out across the face $PQRS$ per unit time

$$= V_x(x + dx)\,(dy\,dz)$$

$$= \left(V_x + \frac{\partial V_x}{\partial x}dx\right)(dy\,dz)$$

Net decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time

$$= V_x\,dy\,dz - \left(V_x + \frac{\partial V_x}{\partial x}dx\right)dy\,dz$$

$$= -\frac{\partial V_x}{\partial x}dx\,dy\,dz \qquad \text{(Minus sign shows decrease)}$$

Similarly, the decrease in mass of fluid to the flow along y-axis $= \dfrac{\partial V_y}{\partial y}dx\,dy\,dz$

and the decrease in mass of fluid to the flow along z-axis $= \dfrac{\partial V_z}{\partial z}dx\,dy\,dz$

Total decrease of the amount of fluid per unit time $= \left(\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}\right)dx\,dy\,dz$

Thus the rate of loss of fluid per unit volume $= \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z) = \vec{\nabla}.\vec{V} = \operatorname{div}\vec{V}$$

If the fluid is compressible, there can be no gain or loss in the volume element. Hence

$$\operatorname{div}\vec{V} = 0 \qquad \text{...(1)}$$

and V is called a *Solenoidal* vector function.

Equation (1) is also called the *equation of continuity or conservation of mass.*

Example

Example 1 :- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Prove that

(i) div $\vec{r} = 3$ i.e, $\nabla . \vec{r} = 3$

Solution :- (i) div $\vec{r} = \nabla . \vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \qquad \because \hat{i}.\hat{i} = 1,\ \hat{i}.\hat{j} = 0 \text{ etc.}$$

$= 1+1+1$

$= 3$

Example 2 :- Prove that, for a constant vector $\vec{a}$

(ii) div $(\vec{a} \times \vec{r}) = 0$

Proof :- Let us suppose that

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore \vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= \hat{i}\ (a_3y - a_2z) - \hat{j}\,(a_3x - a_1z) + \hat{k}\ (a_2x - a_1y)$

$$\text{(ii) } \Delta.\left(\vec{a} \times \vec{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . \{\hat{i}\ (a_3y - a_2z) - \hat{j}\,(a_3x - a_1z) + \hat{k}\ (a_2x - a_1y)\}$$

$$= \frac{\partial}{\partial x}(a_3y - a_2z) - \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_2x - a_1y)$$

$= 0$

Example 3 :- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ show that

div grad $r^m = m(m+1)r^{m-2}$

(U.P.P.C.S 1996, U.P.T.U. 2002, 03, 04, 05)

Solution :-

$$\text{grad } r^m = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^m$$

$$= \hat{i}\, mr^{m-1}\frac{\partial r}{\partial x} + \hat{j}\ mr^{m-1}\frac{\partial r}{\partial y} + \hat{k}\ mr^{m-1}\frac{\partial r}{\partial z}$$

$$= mr^{m-1}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

$$= mr^{m-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right]$$

$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}$$

$= mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})$

$\therefore$ div grad $r^m = \nabla.\ [mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[mr^{m-2}(x\hat{i} + y\hat{j} + z\hat{k})]$$

$$= m\left[\frac{\partial}{\partial x}(xr^{m-2}) + \frac{\partial}{\partial y}(yr^{m-2}) + \frac{\partial}{\partial z}(zr^{m-2})\right]$$

$$= m\left[\left\{r^{m-2} + (m-2)\,xr^{m-3}\frac{\partial r}{\partial x}\right\} + \left\{r^{m-2} + (m-2)\,yr^{m-3}\frac{\partial r}{\partial y}\right\} + \left\{r^{m-2} + (m-2)zr^{m-3}\frac{\partial r}{\partial z}\right\}\right]$$

$$= 3m\,r^{m-2} + m(m-2)r^{m-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3m\,r^{m-2} + m(m-2)r^{m-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$= 3m\,r^{m-2} + m(m-2)r^{m-4}(x^2+y^2+z^2)$

$= 3m\,r^{m-2} + m(m-2)r^{m-4}(r^2)$

$\because x^2+y^2+z^2 = r^2$

$= [3m + m(m-2)]\,r^{m-2}$

$= m(m+1)\,r^{m-2}$

Hence proved.

Example 27. *If $u = x^2 + y^2 + z^2$, and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find div $(u\bar{r})$ in terms of u.*

(*A.M.I.E.T.E., Summer 2004*)

Solution. $\quad \text{div}\ (u\vec{r}) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right).[(x^2 + y^2 + z^2)(x\hat{i} + y\hat{j} + z\hat{k})]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)x\hat{i} + (x^2+y^2+z^2)y\hat{j} + (x^2+y^2+z^2)z\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^3 + xy^2 + xz^2) + \frac{\partial}{\partial y}(x^2 y + y^3 + yz^2) + \frac{\partial}{\partial z}(x^2 z + y^2 z + z^3)$$

$$= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2) = 5\,(x^2 + y^2 + z^2) = 5\,u$$ **Ans.**

Example 28. *Find the value of n for which the vector $r^n \vec{r}$ is solenoidal, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.*

Solution. Divergence $\vec{F} = \vec{\nabla}.\vec{F} = \vec{\nabla}.r^n\,\vec{r} = \nabla.(x^2 + y^2 + z^2)^{n/2}\,(x\hat{i} + y\hat{j} + z\hat{k})$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].[(x^2+y^2+z^2)^{n/2}x\hat{i} + (x^2+y^2+z^2)^{n/2}y\hat{j} + (x^2+y^2+z^2)^{n/2}z\hat{k}]$$

$$= \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2x^2) + (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2y^2)$$

$$+ (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2z^2) + (x^2 + y^2 + z^2)^{n/2}$$

$$= n(x^2 + y^2 + z^2)^{n/2-1}(x^2 + y^2 + z^2) + 3\,(x^2 + y^2 + z^2)^{n/2}$$

$$= n(x^2 + y^2 + z^2)^{n/2} + 3(x^2 + y^2 + z^2)^{n/2} = (n + 3)\,(x^2 + y^2 + z^2)^{n/2}$$

If $r^n\vec{r}$ is solenoidal, then $(n + 3)\,(x^2 + y^2 + z^2)^{n/2} = 0$ or $n + 3 = 0$ or $n = -3$. **Ans.**

Example 30. *Let* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ *and* $\vec{a}$ *is a constant vector. Find the value of*

$$div\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$$

Solution. Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{a}\times\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n} = \frac{(a_2 z - a_3 y)\hat{i} - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$

$$\text{div}\left(\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}\right) = \vec{\nabla}.\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\frac{(a_2 z - a_3 y)\hat{i} - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$

$$= \frac{\partial}{\partial x}\frac{a_2 z - a_3 y}{(x^2+y^2+z^2)^{n/2}} - \frac{\partial}{\partial y}\frac{a_1 z - a_3 x}{(x^2+y^2+z^2)^{n/2}} + \frac{\partial}{\partial z}\frac{(a_1 y - a_2 x)}{(x^2+y^2+z^2)^{n/2}}$$

$$= -\frac{n}{2}\frac{(a_2 z - a_3 y)\,2x}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} + \frac{n}{2}\frac{(a_1 z - a_3 x)\,2y}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} - \frac{n}{2}\frac{(a_1 y - a_2 x)\,2z}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}$$

$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}[(a_2 z - a_3 y)x - (a_1 z - a_3 x)y + (a_1 y - a_2 x)z]$$

$$= -\frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}[a_2 zx - a_3 xy - a_1 yz + a_3 xy + a_1 yz - a_2 zx] = 0$$ **Ans.**

Example 31. *Find the directional derivative of div* $(\vec{u})$ *at the point (1, 2, 2) in the direction of the outer normal of the sphere* $x^2 + y^2 + z^2 = 9$ *for* $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.

Solution. $\text{div}(\vec{u}) = \nabla.\vec{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}) = 4x^3 + 4y^3 + 4z^3$$

Outer normal of the sphere $= \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Outer normal of the sphere at (1, 2, 2) $= 2\hat{i} + 4\hat{j} + 4\hat{k}$ ...(1)

Directional derivative $= \vec{\nabla}(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^3 + 4y^3 + 4z^3) = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$$

Directional derivative at (1, 2, 2) $= 12\hat{i} + 48\hat{j} + 48\hat{k}$ ...(2)

Directional derivative along the outer normal $= (12\hat{i} + 48\hat{j} + 48\hat{k}) . \dfrac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}}$ [From (1), (2)]

$$= \frac{24 + 192 + 192}{6} = 68 \qquad \textbf{Ans.}$$

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Example 32. *Show that div (grad r^n) = $n\,(n+1)r^{n-2}$, where*

$$r = \sqrt{x^2 + y^2 + z^2}$$

Hence, show that $\nabla^2\left(\dfrac{1}{r}\right) = 0$. *(AMIETE, 2010, U.P. I Semester, Dec. 2004, Winter 2002)*

Solution. $\quad \text{grad}\,(r^n) = \hat{i}\dfrac{\partial}{\partial x}r^n + \hat{j}\dfrac{\partial}{\partial y}r^n + \hat{k}\dfrac{\partial}{\partial z}r^n$ by definition

$$= \hat{i}\, n\, r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n\, r^{n-1}\frac{\partial r}{\partial y} + \hat{k}. n\, r^{n-1}\frac{\partial r}{\partial z} = n\, r^{n-1}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

$$= n\, r^{n-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right] = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\bar{r}.$$

$$\left[\because r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}\right]$$

Thus, $\quad \text{grad}\,(r^n) = n\, r^{n-2}\, x\hat{i} + n\, r^{n-2}\, y\hat{j} + n\, r^{n-2}\, z\hat{k}$ ...(1)

$\therefore \quad \text{div grad } r^n = \text{div}\,[\, n\, r^{n-2}\, x\hat{i} + n\, r^{n-2}\, y\hat{j} + n\, r^{n-2}\, z\hat{k}\,]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(nr^{n-2}\, x\hat{i} + nr^{n-2}\, y\hat{j} + nr^{n-2}\, z\hat{k}) \qquad \text{[From (1)]}$$

$$= \frac{\partial}{\partial x}(n\, r^{n-2}x) + \frac{\partial}{\partial y}(n\, r^{n-2}y) + \frac{\partial}{\partial z}(n\, r^{n-2}z) \qquad \text{(By definition)}$$

$$= \left(n\, r^{n-2} + nx\,(n-2)\, r^{n-3}\frac{\partial r}{\partial x}\right) + \left(n\, r^{n-2} + ny\,(n-2)\, r^{n-3}\frac{\partial r}{\partial y}\right)$$

$$+ \left(n\, r^{n-2} + nz\,(n-2)\, r^{n-3}\frac{\partial r}{\partial z}\right)$$

$$= 3n\, r^{n-2} + n(n-2)r^{n-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3n\, r^{n-2} + n(n-2)r^{n-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$$\left[\because r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc.}\right]$$

$$= 3nr^{n-2} + n\,(n-2)r^{n-4}\,[x^2 + y^2 + z^2]$$

$$= 3nr^{n-2} + n\,(n-2)\, r^{n-4}.r^2 \qquad (\because\ r^2 = x^2 + y^2 + z^2)$$

$$= r^{n-2}\,[3n + n^2 - 2n] = r^{n-2}\,(n^2 + n) = n(n+1)\, r^{n-2}$$

If we put $n = -1$

$$\text{div grad}\,(r^{-1}) = -1\,(-1+1)\, r^{-1-2}$$

$$\Rightarrow \qquad \nabla^2\left(\frac{1}{r}\right) = 0$$

Ques. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, and $r = |r|$ find div $\left(\dfrac{\boldsymbol{r}}{r^2}\right)$. *(U.P. I Sem., Dec. 2006)* **Ans.** $\dfrac{1}{r^2}$

Example for Practice Purpose

EXERCISE 23.3

1. If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that (*i*) div $\left(\dfrac{\vec{r}}{|\bar{r}|^3}\right) = 0$,

   (*ii*) div $(r\,\phi) = 3\phi + r$ grad ϕ.

2. Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

   (*R.G.P.V., Bhopal, Dec. 2003*)

3. Show that $\nabla.(\phi\, A) = \nabla\phi A + \phi(\nabla A)$
4. If ρ, ϕ, z are cylindrical coordinates, show that grad $(\log \rho)$ and grad ϕ are solenoidal vectors.
5. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their corresponding expression in orthogonal curvilinear coordinates.

Prove the following:

6. (*a*) $\nabla.(\nabla\phi) = \nabla^2\phi$
7. $\vec{\nabla} \times \dfrac{(\vec{A} \times \vec{R})}{r^n} = \dfrac{(2-n)\vec{A}}{r^n} + \dfrac{n(\vec{A}.\vec{R})\vec{R}}{r^{n+2}}, r = |\vec{R}|$
8. div $(f\,\nabla g)$ – div $(g\,\nabla f) = f\,\nabla^2 g - g\,\nabla^2 f$