Laplacian Operator

Introduction

7.5 Laplacian Operator : ∇^2

7.5.1 $\nabla^2 = \nabla.\nabla = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right)\left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right)$

$= \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right)$ is called the Laplacian Operator

'∇^2' can be applied to both scalar and vector functions as shown below.

$\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

where 'ϕ' is scalar function
If $A = A_1 i + A_2 j + A_3 k$, is a vector function, then

$\nabla^2 A = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right)(A_1 i + A_2 j + A_3 k)$

$= \left(\nabla^2 A_1\right)i + \left(\nabla^2 A_2\right)j + \left(\nabla^2 A_3\right)k$

Examples:

Ex. 7.5.4 If $f = x^2 y^3 z^2$, find $\nabla^2 f$ at $(1, 2, 1)$

Sol: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

$= 2y^3 z^2 + 6x^2 y z^2 + 2x^2 y^3$

$\therefore$ at $(1, 2, 1)$, $\nabla^2 f = 16 + 12 + 16 = 44$.

Ex. 7.5.5 Show that, if $r = xi + yj + zk$, $r = |r|$, then $\nabla^2 r^n = n(n + 1) r^{n-2}$

Sol: $r^n = (x^2 + y^2 + z^2)^{n/2}$, $(\because r = \sqrt{x^2 + y^2 + z^2}\,)$

$\dfrac{\partial}{\partial x}(r^n) = \dfrac{n}{2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}-1}.2x = nx(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$

$\dfrac{\partial^2}{\partial x^2}(r^n) = n\left[x.\dfrac{n-2}{2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}-1}.2x + (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \right]$

$= n\left[(n-2)x^2.(x^2 + y^2 + z^2)^{\frac{n-4}{2}} + (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \right]$

$$= n(n-2)\ x^2.r^{n-4} + nr^{n-2} \qquad \qquad \text{...(1)}$$

Similarly, $\dfrac{\partial^2}{\partial y^2}(r^n) = n(n-2)y^2 r^{n-4} + nr^{n-2}$ \qquad ...(2)

and $\dfrac{\partial^2}{\partial z^2}(r^n) = n(n-2)z^2 r^{n-4} + nr^{n-2}$ \qquad ...(3)

$$\nabla^2 (r^n) = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) r^n$$

$$= 3nr^{n-2} + n(n-2)\ r^{n-4}\ (x^2 + y^2 + z^2) \qquad \text{adding (1), (2) \& (3)}$$

$$= 3nr^{n-2} + n(n-2)\ r^{n-2} \qquad (\because x^2 + y^2 + z^2 = r^2)$$

$$= n(n+1)\ r^{n-2}$$

Example 25. *Prove that* $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$.

$$\nabla f(r) = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right) f(r)$$

$$\left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r} \text{ and } \dfrac{\partial r}{\partial z} = \dfrac{z}{r} \right]$$

$$= i\ f'(r)\dfrac{\partial r}{\partial x} + j\ f'(r)\dfrac{\partial r}{\partial y} + k\ f'(r)\dfrac{\partial r}{\partial z} = f'(r)\left[i\dfrac{x}{r} + j\dfrac{y}{r} + k\dfrac{z}{r} \right]$$

$$= f'(r)\dfrac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla\ [\nabla f(r)] = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right)\left[f'(r)\dfrac{xi + yj + zk}{r} \right]$$

$$= \dfrac{\partial}{\partial x}\left[f'(r)\dfrac{x}{r} \right] + \dfrac{\partial}{\partial y}\left[f'(r)\dfrac{y}{r} \right] + \dfrac{\partial}{\partial z}\left[f'(r)\dfrac{z}{r} \right]$$

$$= \left(f''(r)\dfrac{\partial r}{\partial x} \right)\left(\dfrac{x}{r} \right) + f'(r)\dfrac{r1 - x\dfrac{\partial r}{\partial x}}{r^2} + \left(f''(r)\dfrac{\partial r}{\partial y} \right)\left(\dfrac{y}{r} \right) + f'(r)\dfrac{r.1 - y\dfrac{\partial r}{\partial y}}{r^2} +$$

$$\left(f''(r)\dfrac{\partial r}{\partial z} \right)\left(\dfrac{z}{r} \right) + f'(r)\dfrac{r.1 - z\dfrac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\dfrac{x}{r} \right)\left(\dfrac{x}{r} \right) + f'(r)\dfrac{r - \dfrac{x^2}{r}}{r^2} + \left(f''(r)\dfrac{y}{r} \right)\left(\dfrac{y}{r} \right) + f'(r)\dfrac{r - \dfrac{y^2}{r}}{r^2} + \left(f''(r)\dfrac{z}{r} \right)\left(\dfrac{z}{r} \right) + f'(r)\dfrac{r - \dfrac{z^2}{r}}{r^2}$$

$$= \left(f''(r)\dfrac{x}{r} \right)\left(\dfrac{x}{r} \right) + f'(r)\dfrac{r^2 - x^2}{r^3} + \left(f''(r)\dfrac{y}{r} \right) + f'(r)\dfrac{r^2 - y^2}{r^3} + \left(f''(r)\dfrac{z}{r} \right)\left(\dfrac{z}{r} \right) + f'(r)\dfrac{r^2 - z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2+z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2+z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2+y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2+z^2}{r^3} + \frac{z^2+x^2}{r^3} + \frac{x^2+y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2+y^2+x^2}{r^2} + f'(r)\frac{2(x^2+y^2+z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r} \qquad\qquad\qquad \textbf{Ans.}$$

Example for Practice Purpose

(1)   Show that $\nabla^2(\log r) = \dfrac{1}{r^2}$

(2)   Show that $(a)\ \nabla.(\nabla\phi) = \nabla^2\phi$

(3)   Prove that $\nabla^2(fg) = f(\nabla^2 g) + 2(\nabla f)(\nabla g) + g(\nabla^2 f)$