

Vector Differential Operator Del i.e. ∇

The vector differential operator Del is denoted by ∇ . It is defined as

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

GRADIENT OF A SCALAR FUNCTION

If $\phi(x, y, z)$ be a scalar function then $\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ .

And is denoted by $\text{grad } \phi$.

Thus,

$$\text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi(x, y, z)$$

$$\text{grad } \phi = \nabla \phi$$

(∇ is read del or nebula)

Example If $\phi = 3x^2y - y^3z^2$; find $\text{grad } \phi$ at the point $(1, -2, -1)$.

Solution.

$$\text{grad } \phi = \nabla \phi$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3x^2y - y^3z^2)$$

$$= \hat{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \hat{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2) + \hat{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2)$$

$$= \hat{i} (6xy) + \hat{j} (3x^2 - 3y^2z^2) + \hat{k} (-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = \hat{i} (6) (1) (-2) + \hat{j} [(3) (1) - 3(4) (1)] + \hat{k} (-2)(-8)(-1)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Ans.

Problem 13 : What is the greatest rate of increase of $\phi \equiv xyz^2$ at the point $(1, 0, 3)$.

Solution : $\text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$

$$= \hat{i} yz^2 + \hat{j} xz^2 + \hat{k} 2xyz$$

$$= \hat{i} 0 + \hat{j} 9 + \hat{k} 0 \text{ at } (1, 0, 3)$$

$$= 9 \hat{j}$$

Since we know the greatest rate of increase of $\phi = |\nabla \phi|$

$$= \sqrt{(9)^2}$$

$$= 9 \text{ Answer.}$$

Example 3 :- Show that $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$ where \vec{a} is a constant vector.

Proof : Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\text{Then } \vec{a} \cdot \vec{r} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= a_1 x + a_2 y + a_3 z$$

$$\text{Therefore } \nabla(\vec{a} \cdot \vec{r}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (a_1 x + a_2 y + a_3 z)$$

$$= \hat{i} a_1 + \hat{j} a_2 + \hat{k} a_3$$

$$= \vec{a}, \text{ hence proved.}$$

Example 24. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that :

(i) $\text{grad } r = \frac{\vec{r}}{r}$ (ii) $\text{grad } \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3}$. (Nagpur University, Summer 2002)

Solution. (i) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

$$\therefore 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r = \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z}$$

$$= \hat{i} \frac{x}{r} + \hat{j} \frac{y}{r} + \hat{k} \frac{z}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$

Proved.

(ii) $\text{grad } \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{r} \right) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(\frac{1}{r} \right) = \hat{i} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \hat{j} \frac{\partial}{\partial y} \left(\frac{1}{r} \right) + \hat{k} \frac{\partial}{\partial z} \left(\frac{1}{r} \right)$

$$= \hat{i} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) + \hat{j} \left(-\frac{1}{r^2} \frac{\partial r}{\partial y} \right) + \hat{k} \left(-\frac{1}{r^2} \frac{\partial r}{\partial z} \right)$$

$$= \hat{i} \left(-\frac{1}{r^2} \frac{x}{r} \right) + \hat{j} \left(-\frac{1}{r^2} \frac{y}{r} \right) + \hat{k} \left(-\frac{1}{r^2} \frac{z}{r} \right) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = -\frac{\vec{r}}{r^3} \quad \text{Proved.}$$

Example 8. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$ prove that $\text{grad } u$, $\text{grad } v$ and $\text{grad } w$ are coplanar vectors. [U.P., I Semester, 2001]

Solution. We have,

$$\text{grad } u = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (yz + zx + xy) = \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x)$$

[For vectors to be coplanar, their scalar triple product is 0]

$$\begin{aligned} \text{Now, } \text{grad } u \cdot (\text{grad } v \times \text{grad } w) &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \quad [\text{Applying } R_2 \rightarrow R_2 + R_3] \\ &= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0 \end{aligned}$$

Since the scalar product of $\text{grad } u$, $\text{grad } v$ and $\text{grad } w$ are zero, hence these vectors are coplanar vectors. **Proved.**

Example 25. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. (K. University, Dec. 2008)

Solution.

$$\begin{aligned} \nabla f(r) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) f(r) \\ &= \left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r} \right] \\ &= \hat{i} f'(r) \frac{\partial r}{\partial x} + \hat{j} f'(r) \frac{\partial r}{\partial y} + \hat{k} f'(r) \frac{\partial r}{\partial z} = f'(r) \left[\hat{i} \frac{x}{r} + \hat{j} \frac{y}{r} + \hat{k} \frac{z}{r} \right] \\ &= f'(r) \frac{xi + yj + zk}{r} \\ \nabla^2 f(r) &= \nabla \cdot [\nabla f(r)] = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left[f'(r) \frac{xi + yj + zk}{r} \right] \\ &= \frac{\partial}{\partial x} \left[f'(r) \frac{x}{r} \right] + \frac{\partial}{\partial y} \left[f'(r) \frac{y}{r} \right] + \frac{\partial}{\partial z} \left[f'(r) \frac{z}{r} \right] \\ &= \left(f''(r) \frac{\partial r}{\partial x} \right) \left(\frac{x}{r} \right) + f'(r) \frac{r \cdot 1 - x \frac{\partial r}{\partial x}}{r^2} + \left(f''(r) \frac{\partial r}{\partial y} \right) \left(\frac{y}{r} \right) + f'(r) \frac{r \cdot 1 - y \frac{\partial r}{\partial y}}{r^2} + \\ &\quad \left(f''(r) \frac{\partial r}{\partial z} \right) \left(\frac{z}{r} \right) + f'(r) \frac{r \cdot 1 - z \frac{\partial r}{\partial z}}{r^2} \end{aligned}$$

$$\begin{aligned}
&= \left(f''(r) \frac{x}{r} \right) \left(\frac{x}{r} \right) + f'(r) \frac{r-x^2}{r^2} + \left(f''(r) \frac{y}{r} \right) \left(\frac{y}{r} \right) + f'(r) \frac{r-y^2}{r^2} + \left(f''(r) \frac{z}{r} \right) \left(\frac{z}{r} \right) + f'(r) \frac{r-z^2}{r^2} \\
&= \left(f''(r) \frac{x}{r} \right) \left(\frac{x}{r} \right) + f'(r) \frac{r^2-x^2}{r^3} + \left(f''(r) \frac{y}{r} \right) \left(\frac{y}{r} \right) + f'(r) \frac{r^2-y^2}{r^3} + \left(f''(r) \frac{z}{r} \right) \left(\frac{z}{r} \right) + f'(r) \frac{r^2-z^2}{r^3} \\
&= f''(r) \frac{x^2}{r^2} + f'(r) \frac{y^2+z^2}{r^3} + f''(r) \frac{y^2}{r^2} + f'(r) \frac{x^2+z^2}{r^3} + f''(r) \frac{z^2}{r^2} + f'(r) \frac{x^2+y^2}{r^3} \\
&= f''(r) \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \right] + f'(r) \left[\frac{y^2+z^2}{r^3} + \frac{z^2+x^2}{r^3} + \frac{x^2+y^2}{r^3} \right] \\
&= f''(r) \frac{x^2+y^2+z^2}{r^2} + f'(r) \frac{2(x^2+y^2+z^2)}{r^3} = f''(r) \frac{r^2}{r^2} + f'(r) \frac{2r^2}{r^3} \\
&= f''(r) + f'(r) \frac{2}{r}
\end{aligned}$$

Ans.

GEOMETRICAL MEANING OF GRADIENT, NORMAL

If a surface $\phi(x, y, z) = c$ passes through a point P . The value of the function at each point on the surface is the same as at P . Then such a surface is called a *level surface* through P . For example, If $\phi(x, y, z)$ represents potential at the point P , then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ . Consider another level surface passing through Q , where the value of the function is $\phi + d\phi$.

Let \vec{r} and $\vec{r} + \delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\vec{r}$

$$\begin{aligned}
\nabla\phi \cdot d\vec{r} &= \left(\hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z} \right) \cdot (\hat{i} dx + \hat{j} dy + \hat{k} dz) \\
&= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi \quad \dots(1)
\end{aligned}$$

If Q lies on the level surface of P , then $d\phi = 0$

Equation (1) becomes $\nabla\phi \cdot d\vec{r} = 0$. Then $\nabla\phi$ is \perp to $d\vec{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

Let $\nabla\phi = |\nabla\phi| \hat{N}$, where \hat{N} is a unit normal vector. Let δn be the perpendicular distance between two level surfaces through P and R . Then the rate of change of ϕ in the direction of the

normal to the surface through P is $\frac{\partial \phi}{\partial n}$.

$$\begin{aligned}\frac{d\phi}{dn} &= \lim_{\delta n \rightarrow 0} \frac{\delta\phi}{\delta n} = \lim_{\delta n \rightarrow 0} \frac{\nabla\phi \cdot d\vec{r}}{\delta n} \\ &= \lim_{\delta n \rightarrow 0} \frac{|\nabla\phi| \hat{N} \cdot d\vec{r}}{\delta n} \quad \left\{ \begin{aligned} \hat{N} \cdot d\vec{r} &= |\hat{N}| |d\vec{r}| \cos \theta \\ &= |d\vec{r}| \cos \theta = \delta n \end{aligned} \right\} \\ &= \lim_{\delta n \rightarrow 0} \frac{|\nabla\phi| \delta n}{\delta n} = |\nabla\phi| \\ \therefore |\nabla\phi| &= \frac{\partial \phi}{\partial n}\end{aligned}$$

Hence, **gradient ϕ** is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

Example 12 :- Find a unit vector which is perpendicular to the surface of the paraboloid of revolution.

$z = x^2 + y^2$ at the point $(1, 2, 5)$

(B.P.S.C 1997)

Solution :- $\phi = x^2 + y^2 - z$

$$\therefore \text{grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$= \hat{i} \cdot 2x + \hat{j} \cdot 2y - \hat{k} \cdot 1$$

$$= 2\hat{i} + 4\hat{j} - \hat{k} \text{ at point } (1, 2, 5)$$

$$\text{Hence unit normal} = \frac{\text{grad } \phi}{|\text{grad } \phi|}$$

$$= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{4 + 16 + 1}}$$

$$= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}} \text{ Answer.}$$

NORMAL AND DIRECTIONAL DERIVATIVE

(i) **Normal.** If $\phi(x, y, z) = c$ represents a family of surfaces for different values of the constant c . On differentiating ϕ , we get $d\phi = 0$

$$\text{But} \quad d\phi = \nabla \phi \cdot d\vec{r} \quad \text{so} \quad \nabla \phi \cdot d\vec{r} = 0$$

The scalar product of two vectors $\nabla\phi$ and $d\vec{r}$ being zero, $\nabla\phi$ and $d\vec{r}$ are perpendicular to each other. $d\vec{r}$ is in the direction of tangent to the given surface.

Thus $\nabla\phi$ is a vector *normal* to the surface $\phi(x, y, z) = c$.

(ii) **Directional derivative.** The component of $\nabla\phi$ in the direction of a vector \vec{d} is equal to $\nabla\phi \cdot \hat{d}$ and is called the directional derivative of ϕ in the direction of \vec{d} .

$$\frac{\partial\phi}{\partial r} = \lim_{\delta r \rightarrow 0} \frac{\delta\phi}{\delta r} \quad \text{where, } \delta r = PQ$$

$\frac{\partial\phi}{\partial r}$ is called the *directional derivative* of ϕ at P in the direction of PQ .

Let a unit vector along PQ be \hat{N}' .

$$\frac{\delta n}{\delta r} = \cos \theta \Rightarrow \delta r = \frac{\delta n}{\cos \theta} = \frac{\delta n}{\hat{N} \cdot \hat{N}'} \quad \dots(1)$$

$$\begin{aligned} \text{Now} \quad \frac{\partial\phi}{\partial r} &= \lim_{\delta r \rightarrow 0} \left[\frac{\frac{\delta\phi}{\delta n}}{\frac{\delta n}{\hat{N} \cdot \hat{N}'}} \right] = \hat{N} \cdot \hat{N}' \frac{\partial\phi}{\partial n} \quad \left[\text{From (1), } \delta r = \frac{\delta n}{\hat{N} \cdot \hat{N}'} \right] \\ &= \hat{N}' \cdot \hat{N} |\nabla\phi| = \hat{N}' \cdot \nabla\phi \quad (\because \hat{N} |\nabla\phi| = \nabla\phi) \end{aligned}$$

Hence, $\frac{\partial\phi}{\partial r}$, directional derivative is the component of $\nabla\phi$ in the direction \hat{N}' .

$$\frac{\partial\phi}{\partial r} = \hat{N}' \cdot \nabla\phi = |\nabla\phi| \cos \theta \leq |\nabla\phi|$$

Hence, $\nabla\phi$ is the maximum rate of change of ϕ .

Example 10. Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$. (M.U. 2008)

Solution. Let $\phi(x, y, z) = xy^3z^2 - 4$

We know that $\nabla\phi$ is the vector normal to the surface $\phi(x, y, z) = c$.

$$\text{Normal vector} = \nabla\phi = \hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z}$$

$$\text{Now} \quad = \hat{i} \frac{\partial}{\partial x} (xy^3z^2) + \hat{j} \frac{\partial}{\partial y} (xy^3z^2) + \hat{k} \frac{\partial}{\partial z} (xy^3z^2)$$

$$\Rightarrow \quad \text{Normal vector} = y^3z^2 \hat{i} + 3xy^2z^2 \hat{j} + 2xy^3z \hat{k}$$

$$\text{Normal vector at } (-1, -1, 2) = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

Unit vector normal to the surface at $(-1, -1, 2)$.

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$$

Ans.

Example 9. Find the directional derivative of $x^2y^2z^2$ at the point $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.

(Nagpur University, Summer 2005)

Solution. Let $\phi = x^2 y^2 z^2$

Directional Derivative of ϕ

$$= \nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 y^2 z^2)$$

$$\nabla\phi = 2xy^2z^2 \hat{i} + 2yx^2z^2 \hat{j} + 2zx^2y^2 \hat{k}$$

Directional Derivative of ϕ at $(1, 1, -1)$

$$= 2(1)(1)^2(-1)^2 \hat{i} + 2(1)(1)^2(-1)^2 \hat{j} + 2(-1)(1)^2(1)^2 \hat{k}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k} \quad \dots(1)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = e^t \hat{i} + (\sin 2t + 1)\hat{j} + (1 - \cos t)\hat{k}$$

Tangent vector,

$$\vec{T} = \frac{d\vec{r}}{dt} = e^t \hat{i} + 2 \cos 2t \hat{j} + \sin t \hat{k}$$

$$\text{Tangent(at } t = 0) = e^0 \hat{i} + 2(\cos 0) \hat{j} + (\sin 0) \hat{k} = \hat{i} + 2\hat{j} \quad \dots(2)$$

$$\text{Required directional derivative along tangent} = (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j})}{\sqrt{1+4}}$$

[From (1), (2)]

$$= \frac{2+4+0}{\sqrt{5}} = \frac{6}{\sqrt{5}} \quad \text{Ans.}$$

Example 11. Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point $(1, 1, 1)$.
(Nagpur University, Summer 2001)

Solution. Rate of change of $\phi = \Delta \phi$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xyz) = \hat{i} yz + \hat{j} xz + \hat{k} xy$$

$$\text{Rate of change of } \phi \text{ at } (1, 1, 1) = (\hat{i} + \hat{j} + \hat{k})$$

Normal to the surface $\Psi = x^2y + y^2x + yz^2 - 3$ is given as -

$$\nabla\Psi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2xy + z^2) + \hat{k} 2yz$$

$$(\nabla\Psi)_{(1, 1, 1)} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\text{Unit normal} = \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{9+16+4}}$$

$$\text{Required rate of change of } \phi = (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}} \quad \text{Ans.}$$

Example 13. Find the values of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$, $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$.

(AMIEETE, II Sem., Dec. 2010, June 2009)

Solution. Here, we have

$$\lambda x^2 - \mu yz = (\lambda + 2)x \quad \dots(1)$$

$$4x^2y + z^3 = 4 \quad \dots(2)$$

$$\begin{aligned} \text{Normal to the surface (1), } &= \nabla [\lambda x^2 - \mu yz - (\lambda + 2)x] \\ &= \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] [\lambda x^2 - \mu yz - (\lambda + 2)x] \\ &= \hat{i} (2\lambda x - \lambda - 2) + \hat{j} (-\mu z) + \hat{k} (-\mu y) \end{aligned}$$

$$\begin{aligned} \text{Normal at } (1, -1, 2) &= \hat{i} (2\lambda - \lambda - 2) - \hat{j} (-2\mu) + \hat{k} \mu \\ &= \hat{i} (\lambda - 2) + \hat{j} (2\mu) + \hat{k} \mu \end{aligned} \quad \dots(3)$$

Normal at the surface (2)

$$\begin{aligned} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (4x^2y + z^3 - 4) \\ &= \hat{i} (8xy) + \hat{j} (4x^2) + \hat{k} (3z^2) \end{aligned}$$

$$\text{Normal at the point } (1, -1, 2) = -8\hat{i} + 4\hat{j} + 12\hat{k} \quad \dots(4)$$

Since (3) and (4) are orthogonal so

$$\begin{aligned} [\hat{i} (\lambda - 2) + \hat{j} (2\mu) + \hat{k} \mu] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] &= 0 \\ -8(\lambda - 2) + 4(2\mu) + 12\mu &= 0 \Rightarrow -8\lambda + 16 + 8\mu + 12\mu = 0 \\ -8\lambda + 20\mu + 16 &= 0 \Rightarrow 4(-2\lambda + 5\mu + 4) = 0 \\ -2\lambda + 5\mu + 4 &= 0 \Rightarrow 2\lambda - 5\mu = 4 \end{aligned} \quad \dots(5)$$

Point $(1, -1, 2)$ will satisfy (1)

$$\therefore \lambda(1)^2 - \mu(-1)(2) = (\lambda + 2)(1) \Rightarrow \lambda + 2\mu = \lambda + 2 \Rightarrow \mu = 1$$

Putting $\mu = 1$ in (5), we get

$$2\lambda - 5 = 4 \Rightarrow \lambda = \frac{9}{2}$$

$$\text{Hence } \lambda = \frac{9}{2} \text{ and } \mu = 1$$

Ans.

Example 14. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
(Nagpur University, Summer 2002)

Solution. Normal on the surface $(x^2 + y^2 + z^2 - 9 = 0)$

$$\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 9) = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\text{Normal at the point } (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \dots(1)$$

$$\begin{aligned} \text{Normal on the surface } (z = x^2 + y^2 - 3) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 - z - 3) \\ &= 2x\hat{i} + 2y\hat{j} - \hat{k} \end{aligned}$$

$$\text{Normal at the point } (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k} \quad \dots(2)$$

Let θ be the angle between normals (1) and (2).

$$(4\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{16 + 4 + 16} \sqrt{16 + 4 + 1} \cos \theta$$

$$16 + 4 - 4 = 6\sqrt{21} \cos \theta \quad \Rightarrow \quad 16 = 6\sqrt{21} \cos \theta$$

$$\Rightarrow \quad \cos \theta = \frac{8}{3\sqrt{21}} \quad \Rightarrow \quad \theta = \cos^{-1} \frac{8}{3\sqrt{21}} \quad \text{Ans.}$$

Example 15. Find the directional derivative of $\frac{1}{r}$ in the direction \vec{r} where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
(Nagpur University, Summer 2004, U.P., I Semester, Winter 2005, 2002)

$$\text{Solution. Here, } \phi(x, y, z) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\begin{aligned} \text{Now } \nabla \left(\frac{1}{r} \right) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{i} + \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{j} + \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{k} \\ &= \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} 2x \right\} \hat{i} + \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} 2y \right\} \hat{j} + \left\{ -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{3}{2}} 2z \right\} \hat{k} \end{aligned}$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} \quad \dots(1)$$

and \hat{r} = unit vector in the direction of $x\hat{i} + y\hat{j} + z\hat{k}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad \dots(2)$$

So, the required directional derivative

$$= \nabla\phi \cdot \hat{r} = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}} \cdot \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} \quad [\text{From (1), (2)}]$$

$$= \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2} \quad \text{Ans.}$$

Example 16. Find the direction in which the directional derivative of $\phi(x, y) = \frac{x^2 + y^2}{xy}$ at

$(1, 1)$ is zero and hence find out component of velocity of the vector $\vec{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$ in the same direction at $t = 1$.
(Nagpur University, Winter 2000)

Solution. Directional derivative $= \nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(\frac{x^2 + y^2}{xy} \right)$

$$= \hat{i} \left[\frac{xy \cdot 2x - (x^2 + y^2)y}{x^2 y^2} \right] + \hat{j} \left[\frac{xy \cdot 2y - x(y^2 + x^2)}{x^2 y^2} \right]$$

$$= \hat{i} \left[\frac{x^2 y - y^3}{x^2 y^2} \right] + \hat{j} \left[\frac{xy^2 - x^3}{x^2 y^2} \right]$$

Directional Derivative at $(1, 1) = \hat{i} 0 + \hat{j} 0 = 0$

Since $(\nabla\phi)_{(1,1)} = 0$, the directional derivative of ϕ at $(1, 1)$ is zero in any direction.

Again $\vec{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$

Velocity, $\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j}$

Velocity at $t = 1$ is $= 3\hat{i} + 2\hat{j}$

The component of velocity in the same direction of velocity

$$= (3\hat{i} + 2\hat{j}) \cdot \left(\frac{3\hat{i} + 2\hat{j}}{\sqrt{9 + 4}} \right) = \frac{9 + 4}{\sqrt{13}} = \sqrt{13}$$

Ans.

Example 17. Find the directional derivative of $\phi(x, y, z) = x^2 y z + 4 x z^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find the greatest rate of increase of ϕ .

(Uttarakhand, I Semester, Dec. 2006)

Solution. Here, $\phi(x, y, z) = x^2 y z + 4 x z^2$

Now, $\nabla\phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 y z + 4 x z^2)$

$$= (2xyz + 4z^2)\hat{i} + (x^2 z)\hat{j} + (x^2 y + 8xz)\hat{k}$$

$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\hat{i} + \{1 \times 1\}\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k}$$

$$= (-4 + 4)\hat{i} + \hat{j} + (-2 + 8)\hat{k} = \hat{j} + 6\hat{k}$$

Let $\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$

So, the required directional derivative at $(1, -2, 1)$

$$= \nabla\phi \cdot \hat{a} = (\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1 - 12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = |\hat{j} + 6\hat{k}| = \sqrt{1 + 36}$

$$= \sqrt{37}$$

Ans.

Example 18. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$.

(AMIEETE, Dec. 2010, Nagpur University, Summer 2008, U.P., I Sem., Winter 2000)

Solution. Directional derivative $= \nabla \phi$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 - y^2 + 2z^2) = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

$$\text{Directional Derivative at the point } P(1, 2, 3) = 2\hat{i} - 4\hat{j} + 12\hat{k} \quad \dots(1)$$

$$\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P} = (5, 0, 4) - (1, 2, 3) = (4, -2, 1) \quad \dots(2)$$

$$\text{Directional Derivative along } PQ = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16 + 4 + 1}} \quad [\text{From (1) and (2)}]$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

Ans.

Example 19. Find the directional derivative of $\phi = 4e^{2x-y+z}$ at the point $(1, 1, -1)$ in the direction towards the point $(-3, 5, 6)$. (Nagpur University, Winter 2003, Summer 2000)

Solution. Directional derivative $= \nabla \phi$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) 4e^{2x-y+z}$$

$$= 4[2\hat{i}e^{2x-y+z} - \hat{j}e^{2x-y+z} + \hat{k}e^{2x-y+z}] = 4[2\hat{i} - \hat{j} + \hat{k}]e^{2x-y+z}$$

Directional Derivative at $(1, 1, -1)$

$$= 4[2\hat{i} - \hat{j} + \hat{k}]e^{2-1-1} = 4[2\hat{i} - \hat{j} + \hat{k}] \quad \dots(1)$$

Direction of Directional Derivative

$$= (-3\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = -4\hat{i} + 4\hat{j} + 7\hat{k} \quad \dots(2)$$

Directional Derivative in the direction of $(-4\hat{i} + 4\hat{j} + 7\hat{k})$

$$= \left| (8\hat{i} - 4\hat{j} + 4\hat{k}) \cdot \frac{(-4\hat{i} + 4\hat{j} + 7\hat{k})}{\sqrt{16 + 16 + 49}} \right| \quad [\text{From (1) and (2)}]$$

$$= \left| \frac{1}{9} [-32 - 16 + 28] \right| = \left| -\frac{20}{9} \right| = \frac{20}{9}$$

Ans.

Example 20. For the function $\phi(x, y) = \frac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at $(0, 2)$.
(A.M.I.E.T.E., Winter 2002)

Solution. Directional derivative $= \vec{\nabla} \phi$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \frac{x}{x^2 + y^2} = \hat{i} \left(\frac{1}{x^2 + y^2} - \frac{x(2x)}{(x^2 + y^2)^2} \right) - \hat{j} \frac{x(2y)}{(x^2 + y^2)^2}$$

$$= \hat{i} \frac{y^2 - x^2}{(x^2 + y^2)^2} - \hat{j} \frac{2xy}{(x^2 + y^2)^2}$$

Directional derivative at the point $(0, 2)$

$$= \hat{i} \frac{4 - 0}{(0 + 4)^2} - \hat{j} \frac{2(0)(2)}{(0 + 4)^2} = \frac{\hat{i}}{4}$$

Directional derivative at the point $(0, 2)$ in the direction \vec{CA} i.e. $\left(\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \right)$

$$= \frac{\hat{i}}{4} \cdot \left(\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \right) \quad \left\{ \begin{array}{l} \vec{CA} = \vec{OB} + \vec{BA} = \hat{i} \cos 30^\circ + \hat{j} \sin 30^\circ \\ = \left(\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \right) \end{array} \right.$$

$$= \frac{\sqrt{3}}{8}$$

Ans.

Example 21. Find the directional derivative of \vec{V}^2 , where $\vec{V} = xy^2 \hat{i} + zy^2 \hat{j} + xz^2 \hat{k}$, at the point $(2, 0, 3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.
(A.M.I.E.T.E., Dec. 2007)

Solution. $V^2 = \vec{V} \cdot \vec{V}$

$$= (xy^2 \hat{i} + zy^2 \hat{j} + xz^2 \hat{k}) \cdot (xy^2 \hat{i} + zy^2 \hat{j} + xz^2 \hat{k}) = x^2 y^4 + z^2 y^4 + x^2 z^4$$

Directional derivative

$$= \vec{\nabla} V^2$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 y^4 + z^2 y^4 + x^2 z^4)$$

$$= (2xy^4 + 2xz^4) \hat{i} + (4x^2 y^3 + 4y^3 z^2) \hat{j} + (2y^4 z + 4x^2 z^3) \hat{k}$$

$$\text{Directional derivative at } (2, 0, 3) = (0 + 2 \times 2 \times 81) \hat{i} + (0 + 0) \hat{j} + (0 + 4 \times 4 \times 27) \hat{k}$$

$$= 324 \hat{i} + 432 \hat{k} = 108 (3 \hat{i} + 4 \hat{k}) \quad \dots(1)$$

Normal to $x^2 + y^2 + z^2 - 14 = \nabla \phi$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 14)$$

$$= (2x \hat{i} + 2y \hat{j} + 2z \hat{k})$$

$$\text{Normal vector at } (3, 2, 1) = 6 \hat{i} + 4 \hat{j} + 2 \hat{k} \quad \dots(2)$$

$$\text{Unit normal vector} = \frac{6 \hat{i} + 4 \hat{j} + 2 \hat{k}}{\sqrt{36 + 16 + 4}} = \frac{2(3 \hat{i} + 2 \hat{j} + \hat{k})}{2\sqrt{14}} = \frac{3 \hat{i} + 2 \hat{j} + \hat{k}}{\sqrt{14}} \quad [\text{From (1), (2)}]$$

$$\text{Directional derivative along the normal} = 108(3\hat{i} + 4\hat{k}) \cdot \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}.$$

$$= \frac{108 \times (9 + 4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$

Ans.

Example 22. Find the directional derivative of $\nabla(\nabla f)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$. (U.P., I Semester, Dec 2008)

Solution. Here, we have

$$f = 2x^3y^2z^4$$

$$\nabla f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (2x^3y^2z^4) = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\begin{aligned} \nabla(\nabla f) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}) \\ &= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \end{aligned}$$

Directional derivative of $\nabla(\nabla f)$

$$\begin{aligned} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2) \\ &= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j} \\ &\quad + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k} \end{aligned}$$

$$\begin{aligned} \text{Directional derivative at } (1, -2, 1) &= (48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k} \\ &= 348\hat{i} - 144\hat{j} + 400\hat{k} \end{aligned}$$

$$\text{Normal to } (xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$$

$$\begin{aligned} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xy^2z - 3x - z^2) \\ &= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k} \end{aligned}$$

$$\text{Normal at } (1, -2, 1) = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\text{Unit Normal Vector} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}} = \frac{1}{\sqrt{21}} (\hat{i} - 4\hat{j} + 2\hat{k})$$

Directional derivative in the direction of normal

$$\begin{aligned} &= (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot \frac{1}{\sqrt{21}} (\hat{i} - 4\hat{j} + 2\hat{k}) \\ &= \frac{1}{\sqrt{21}} (348 + 576 + 800) = \frac{1724}{\sqrt{21}} \end{aligned}$$

Ans.

Example 23. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a , b and c . (U.P. I semester, Winter 2001)

Solution. Given $\phi = ax^2y + by^2z + cz^2x$

$$\begin{aligned}\therefore \bar{\nabla}\phi &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (ax^2y + by^2z + cz^2x) \\ &= \hat{i}(2axy + cz^2) + \hat{j}(ax^2 + 2byz) + \hat{k}(by^2 + 2czx)\end{aligned}$$

$$\bar{\nabla}\phi \text{ at the point } (1, 1, 1) = \hat{i}(2a+c) + \hat{j}(a+2b) + \hat{k}(b+2c) \quad \dots(1)$$

We know that the maximum value of the directional derivative is in the direction of $\bar{\nabla}\phi$.

$$\text{i.e. } |\nabla\phi| = 15 \Rightarrow (2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$$

But, the directional derivative is given to be maximum parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \text{ i.e., parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}. \quad \dots(2)$$

On comparing the coefficients of (1) and (2)

$$\begin{aligned}\Rightarrow \frac{2a+c}{2} &= \frac{2b+a}{-2} = \frac{2c+b}{1} \\ \Rightarrow 2a+c &= -2b-a \Rightarrow 3a+2b+c=0 \quad \dots(3)\end{aligned}$$

$$\begin{aligned}\text{and } 2b+a &= -2(2c+b) \\ \Rightarrow 2b+a &= -4c-2b \Rightarrow a+4b+4c=0 \quad \dots(4)\end{aligned}$$

Rewriting (3) and (4), we have

$$\begin{aligned}\left. \begin{aligned} 3a+2b+c &= 0 \\ a+4b+4c &= 0 \end{aligned} \right\} &\Rightarrow \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \text{ (say)} \\ \Rightarrow a &= 4k, \quad b = -11k \quad \text{and} \quad c = 10k.\end{aligned}$$

Now, we have

$$\begin{aligned}(2a+c)^2 + (2b+a)^2 + (2c+b)^2 &= (15)^2 \\ \Rightarrow (8k+10k)^2 + (-22k+4k)^2 + (20k-11k)^2 &= (15)^2 \\ \Rightarrow k &= \pm \frac{5}{9} \\ \Rightarrow a &= \pm \frac{20}{9}, \quad b = \pm \frac{55}{9} \quad \text{and} \quad c = \pm \frac{50}{9}\end{aligned}$$

Ans.

Example 6 : Find the directional derivative of $\phi = xy + yz + zx$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$ at $(1, 2, 0)$

Solution :- Since we know

directional derivative = $\hat{a} \cdot \text{grad } \phi$

$$\text{Now grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$= (y + z) \hat{i} + (z + x) \hat{j} + (x + y) \hat{k}$$

$$= 2\hat{i} + \hat{j} + 3\hat{k} \text{ at } (1, 2, 0)$$

$$\text{Also } \hat{a} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$$\therefore \text{directional derivative} = \frac{1}{3} (\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (2\hat{i} + \hat{j} + 3\hat{k})$$

$$= \frac{1}{3} (2 + 2 + 6)$$

$$= \frac{10}{3} \text{ Answer.}$$

Problem 11 :- Find the directional derivative of a function $\phi = x^2 y^3 z^4$ at the point $(2, 3, -1)$ in the direction making equal angles with the positive x , y , & z axis.

Solution :- Given $\phi = x^2 y^3 z^4$

$$\text{Now grad } \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$= 2xy^3z^4 \hat{i} + 3x^2y^2z^4 \hat{j} + 4x^2y^3z^3 \hat{k}$$

If \hat{a} be the unit vector in the required direction and α be the angle which \hat{a} makes with the axes, then

$$\hat{a} = (\cos \alpha) \hat{i} + (\cos \alpha) \hat{j} + (\cos \alpha) \hat{k}$$

$$\text{where } \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\text{which gives } \cos \alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \hat{a} = \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k})$$

$$\therefore \text{directional derivative} = \hat{a} \cdot \text{grad } \phi$$

$$= \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) \cdot (2xy^3z^4 \hat{i} + 3x^2y^2z^4 \hat{j} + 4x^2y^3z^3 \hat{k})$$

$$= \frac{1}{\sqrt{3}} (2xy^3z^4 + 3x^2y^2z^4 + 4x^2y^3z^3)$$

$$= \frac{1}{\sqrt{3}} (108 + 108 - 432) \text{ at the point } (2, 3, -1)$$

$$= -\frac{216}{\sqrt{3}} \text{ Answer.}$$

Example for Practice Purpose

EXERCISE 23.1

1. Evaluate $\text{grad } \phi$ if $\phi = \log (x^2 + y^2 + z^2)$ **Ans.** $\frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$
2. Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 5$ at the point $(0, 1, 2)$. **Ans.** $\frac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$
(AMIE TE, June 2010)
3. Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point $(1, -1, 1)$ in the direction of $(3, 1, -1)$ (A.M.I.E.T.E. Winter 2009, 2000) **Ans.** $\frac{5}{\sqrt{11}}$
4. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at $(1, 1)$ is zero.
(Nagpur Winter 2000) **Ans.** $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$
5. Find the directional derivative of the scalar function of $(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point $(3, 1, 3)$. **Ans.** $\frac{27}{\sqrt{11}}$
6. The temperature of the points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move? **Ans.** $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$
7. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find $\text{grad } \phi$ at the point $(1, -2, -1)$ **Ans.** $-(16\hat{i} + 9\hat{j} + 4\hat{k})$
8. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$. **Ans.** $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$
9. What is the greatest rate of increase of the function $u = xyz^2$ at the point $(1, 0, 3)$? **Ans.** 9
10. If θ is the acute angle between the surfaces $xyz^2 = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$ show that $\cos \theta = 3/7\sqrt{6}$.