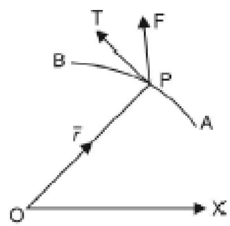
Line Integral

Introduction:

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.



Component of $\vec{F}$ along a tangent PT at P

= Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F} \cdot \frac{\vec{dr}}{ds} \left(\frac{\vec{dr}}{ds} \text{ is a unit vector along tangent PT} \right)$$

Line integral $= \sum \vec{F} \cdot \frac{\vec{dr}}{ds}$ from A to B along the curve

$$\therefore \text{ Line integral} = \int_c \left(\vec{F} \cdot \frac{\vec{dr}}{ds} \right) ds = \int_c \vec{F} \cdot \vec{dr}$$

Note (1) Work. If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F} \cdot \vec{dr}$

(2) Circulation. If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V} \cdot \vec{dr}$ is called the circulation of V round the closed curve c.

If the circulation of V round every closed curve is zero then V is said to be irrotational there.

(3) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

Example 1. *If a force $\vec{F} = 2x^2 y\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve $y = 4x^2$. Find the work done.*

Solution. Work done $= \int_c \vec{F} \cdot \vec{dr}$

$$\left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ \vec{dr} = dx\hat{i} + dy\hat{j} \end{array} \right]$$

$$= \int_c (2x^2 y\hat{i} + 3xy\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$

$$= \int_c (2x^2 y\, dx + 3xy\, dy)$$

Putting the values of y and dy, we get $\left(\begin{array}{l} y = 4x^2 \\ dy = 8x\, dx \end{array} \right)$

$$= \int_0^1 [2x^2 (4x^2)\, dx + 3x(4x^2)\, 8x\, dx]$$

$$= 104 \int_0^1 x^4\, dx = 104 \left(\frac{x^5}{5} \right)_0^1 = \frac{104}{5}$$ **Ans.**

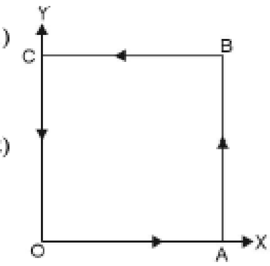
Example 2. *Evaluate $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

Solution. $\int_C \vec{F}.\vec{d}r = \int_{OA} \vec{F}.\vec{dr} + \int_{AB} \vec{F}.\vec{dr} + \int_{BC} \vec{F}.\vec{dr} + \int_{CO} \vec{F}.\vec{dr}$

Here $\quad \vec{r} = x\hat{i} + y\hat{j}, \quad \vec{d}r = dx\hat{i} + dy\hat{j}, \quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$

$$\vec{F}.\vec{dr} = x^2dx + xydy \qquad ...(1)$$



On $OA, y = 0$ $\qquad \therefore \vec{F}.\vec{dr} = x^2dx$

$$\int_{OA} \vec{F}.\vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ...(2)$$

On $AB, x = a$ $\qquad \therefore dx = 0$
(1) becomes

$$\therefore \vec{F}.\vec{dr} = aydy$$

$$\int_{Ab} \vec{F}.\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \qquad ...(3)$$

On $BC, y = a$ $\qquad \therefore dy = 0$

$\Rightarrow$ (1) becomes $\qquad \vec{F}.\vec{dr} = x^2dx$

$$\int_{BC} \vec{F}.\vec{dr} = \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \qquad ...(4)$$

On $CO, x = 0,$ $\qquad \therefore \vec{F}.\vec{dr} = 0$
(1) becomes

$$\int_{CO} \vec{F}.\vec{dr} = 0 \qquad ...(5)$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F}.\vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ **Ans.**

Example 3. *A vector field is given by*

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}.\vec{dr}$ *along the path c is* $x = 2t,$ $y = t, z = t^3$ *from* $t = 0$ *to* $t = 1$. *(Nagpur University, Winter 2003)*

Solution. $\int_C \vec{F}.\vec{dr} = \int_C (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$

$$\left[\begin{array}{lll} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore \frac{dx}{dt} = 2 & \frac{dy}{dt} = 1 & \frac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t+3)\,(2\,dt) + (2t)\,(t^3)\,dt + (t^4 - 2t)\,(3t^2\,dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**

Example 4. *If* $\bar{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, *evaluate* $\int_C \vec{F} \times \vec{dr}$ *along the curve*

$x = \cos t,\ y = \sin t,\ z = 2\cos t$ *from* $t = 0$ *to* $t = \frac{\pi}{2}$. *(Nagpur University, winter 2002)*

Solution. We have, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F} \times \vec{dr} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= (-zdz - xdy)\,\hat{i} - (2y\,dz - xdx)\,\hat{j} + (2ydy + zdx)\,\hat{k}$$

$$= [-2\cos t\,(-2\sin t)\,dt - \cos t\,(\cos t)\,dt]\,\hat{i}$$

$$- [\,2\sin t\,(-2\sin t)\,dt - \cos t\,(-\sin t)\,dt\,]\hat{j}$$

$$+ [\,2\sin t\,(\cos t)\,dt + 2\cos t\,(-\sin t)\,dt\,]\,\hat{k}$$

$$= [\,(4\cos t\sin t - \cos^2 t)\,\hat{i} + (4\sin^2 t - \cos t\sin t)\,\hat{j}\,]\,dt$$

$$\therefore \int_C \vec{F} \times \vec{dr} = \int_0^{\frac{\pi}{2}} [\,(4\cos t\sin t - \cos^2 t)\,\hat{i} + (4\sin^2 t - \cos t\sin t)\,\hat{j}\,]\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left[\left\{2\sin 2t - \frac{\cos 2t + 1}{2}\right\}\hat{i}\right] dt + \int_0^{\frac{\pi}{2}} \left[\left\{2\,(1 - \cos 2t) - \frac{1}{2}\sin 2t\right\}\hat{j}\right] dt$$

$$= \left[-\cos 2t - \frac{1}{4}\sin 2t - \frac{1}{2}t\right]_0^{\frac{\pi}{2}} \hat{i} + \left[2t - \sin 2t + \frac{1}{4}\cos 2t\right]_0^{\frac{\pi}{2}} \hat{j}$$

$$= \left[-\cos\pi - \frac{1}{4}\sin\pi - \frac{1}{2}\left(\frac{\pi}{2}\right) + \cos 0 + \frac{1}{4}\sin 0 + \frac{1}{2}(0)\right]\hat{i} +$$

$$\left[\pi - \sin\pi + \frac{1}{4}\cos\pi - 0 + \sin 0 - \frac{1}{4}\cos 0\right]\hat{j}$$

$$= \left[1 - 0 - \frac{\pi}{4} + 1 + 0\right]\hat{i} + \left[\pi - 0 - \frac{1}{4} + 0 - \frac{1}{4}\right]\hat{j} = \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}$$ **Ans.**

Example 5. *The acceleration of a particle at time t is given by*

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at* $t = 0$, *find* $\vec{v}$ *and* $\vec{r}$ *at any point t.*

Solution. Here, $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$

On integrating, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$$

$$\Rightarrow \quad \vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + \vec{c} \qquad ...(1)$$

At $\quad t = 0,\ \vec{v} = \vec{0}$

Putting $t = 0$ and $\vec{v} = 0$ in (1), we get

$$\vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$$

$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\,\hat{j} + 3t^2\hat{k}$$

Again integrating, we have

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$$

$$\Rightarrow \quad \vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\,\hat{k} + \vec{c_1} \qquad ...(2)$$

At, $\quad t = 0,\ \vec{r} = 0$

Putting $t = 0$ and $\vec{r} = 0$ in (2), we get

$$\therefore \quad \vec{0} = -2\hat{i} + \vec{C_1} \quad \Rightarrow \quad \vec{C_1} = 2\hat{i}$$

Hence, $\quad \vec{r} = 2(1 - \cos 3t)\,\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}$ **Ans.**

Example 6. *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\,d\vec{r}$ *from* (0, 0, 0) *to* (1, 1, 1) *along the curve C.*

$x = t,\ y = t^2,\ z = t^3.$ (*Uttarakhand, I Semester, Dec. 2006*)

Solution. We have,

$$\int_C \vec{A}.\,d\vec{r} = \int_C [(3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= \int_C [(3x^2 + 6y)\,dx - 14yzdy + 20xz^2dz]$$

If $x = t,\ y = t^2,\ z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

Now, $\displaystyle\int_C \vec{A}.\,d\vec{r} = \int_{t=0}^{t=1}[(3t^2 + 6t^2)\,d(t) - 14t^2\,t^3\,d(t^2) + 20t\,(t^3)^2\,d(t^3)]$

$$= \int_{t=0}^{t=1}[9t^2\,dt - 14t^5\,.\,2t\,dt + 20\,t^7\,.\,3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 = 3 - 4 + 6 = 5$$ **Ans.**

Example 7. *Compute* $\int_c \vec{F}\,.\,\vec{dr}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and c is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*

Solution. $\quad \vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z,\ d\,\overline{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\int_c \vec{F}\,.\,d\,\vec{r} = \int_c \frac{\hat{i}y - \hat{j}x}{x^2 + y^2}\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \frac{ydx - xdy}{x^2 + y^2} = \int_c (ydx - xdy) \qquad ...(1)\ [\because\ x^2 + y^2 = 1]$$

Parametric equation of the circle are $x = \cos\theta$, $y = \sin\theta$.
Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$ in (1), we get

$$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} \sin\theta\,(-\sin\theta\, d\theta) - \cos\theta\,(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta = -(\theta)_0^{2\pi} = -2\pi \qquad \textbf{Ans.}$$

Example 8. *Show that the vector field* $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ *is conservative. Find its scalar potential and the work done in moving a particle from* (–1, 2, 1) to (2, 3, 4).
(*A.M.I.E.T.E. June 2010, 2009*)

Solution. Here, we have

$$\vec{F} = 2x(y^2+z^3)\,\hat{i} + 2x^2y\,\hat{j} + 3x^2z^2\hat{k}$$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix} = (0-0)\hat{i} - (6xz^2 - 6xz^2)\hat{j} + (4xy - 4xy)\hat{k} = 0$$

Hence, vector field $\vec{F}$ is irrotational.
To find the scalar potential function ϕ

$$\vec{F} = \vec{\nabla}\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.\left(d\vec{r}\right) = \nabla\phi . d\vec{r} = \vec{F}.\,d\vec{r}$$

$$= \left[2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}\right](\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2x(y^2+z^3)dx + 2x^2y\,dy + 3x^2z^2dz$$

$$\phi = \int\left[2x(y^2+z^3)dx + 2x^2ydy + 3x^2z^2dz\right] + C$$

$$\int(2xy^2dx + 2x^2ydy) + (2xz^3dx + 3x^2z^2dz) + C = x^2y^2 + x^2z^3 + C$$

Hence, the scalar potential is $x^2y^2 + x^2z^3 + C$
Now, for conservative field

$$\text{Work done} = \int_{(-1,2,1)}^{(2,3,4)} \vec{F}.\,d\vec{r} = \int_{(-1,2,1)}^{(2,3,4)} d\phi = [\phi]_{(-1,2,1)}^{(2,3,4)} = \left[x^2y^2 + x^2z^3 + c\right]_{(-1,2,1)}^{(2,3,4)}$$

$$= (36 + 256) - (2 - 1) = 291 \qquad \textbf{Ans.}$$

Example 9. *A vector field is given by* $\vec{F} = (\sin y)\,\hat{i} + x\,(1+\cos y)\,\hat{j}$. *Evaluate the line integral over a circular path* $x^2 + y^2 = a^2$, $z = 0$. (*Nagpur University, Winter 2001*)

Solution. We have,

$$\text{Work done} = \int_C \vec{F}.\,\vec{dr}$$

$$= \int_C [(\sin y)\,\hat{i} + x(1+\cos y)\,\hat{j}]\,.\,[dx\hat{i} + dy\hat{j}] \quad (\because\ z = 0 \text{ hence } dz = 0)$$

$$\Rightarrow \qquad \int_C \vec{F}\,.\,\vec{d}\,r = \int_C \sin y\,dx + x(1+\cos y)\,dy = \int_C (\sin y\,dx + x\cos y\,dy + x\,dy)$$

$$= \int_C d\,(x \sin y) + \int_C x\,dy$$

(where d is differential operator).

The parametric equations of given path

$x^2 + y^2 = a^2$ are $x = a\cos\theta$, $y = a\sin\theta$,

Where θ varies form 0 to 2π

$$\therefore \qquad \int_C \vec{F}\,.\,\vec{d}\,r = \int_0^{2\pi} d\,[\,a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a\cos\theta\,.\,a\cos\theta\,d\,\theta$$

$$= \int_0^{2\pi} d\,[a\cos\theta\sin(a\sin\theta)\,] + \int_0^{2\pi} a^2\cos^2\theta\,.\,d\,\theta$$

$$= [a\cos\theta\sin(a\sin\theta)\,]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta\,d\,\theta$$

$$= 0 + a^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\,.\,2\pi = \pi a^2$$ **Ans.**

Example 10. *Determine whether the line integral* $\int (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$ *is independent of the path of integration ? If so, then evaluate it from (1, 0, 1) to* $\left(0, \frac{\pi}{2}, 1\right)$.

Solution. $\int_c (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$

$$= \int_c [(2xyz^2\hat{i}) + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c \vec{F}\cdot\vec{dr}$$

This integral is independent of path of integration if

$$\vec{F} = \nabla\phi \;\Rightarrow\; \nabla\times\vec{F} = 0$$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)\,\hat{i} - (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$
$= 0$

Hence, the line integral is independent of path.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{(Total differentiation)}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \nabla\phi\cdot dr = \vec{F}\cdot\vec{d}\,r$$

$$= [(2xyz^2)\,\hat{i} + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}].\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$

$$= [(2x\,dx)\,yz^2 + x^2(dy)\,z^2 + x^2y\,(2z\,dz)] + [(\cos yz\,dy)\,z + (\cos yz\,dz)\,y]$$

$$= d\,(x^2yz^2) + d\,(\sin yz)$$

$$\phi = \int d\,(x^2yz^2) + \int d\,(\sin yz) = x^2yz^2 + \sin yz$$

$$[\phi]_A^B = \phi\,(B) - \phi\,(A)$$

$$= [x^2yz^2 + \sin yz]_{(0,\,\frac{\pi}{2},\,1)} - [x^2yz^2 + \sin yz]_{(1,\,0,\,1)} = \left[0 + \sin\left(\frac{\pi}{2}\times 1\right)\right] - [0+0]$$

$$= 1$$

Ans.

Example for Practice Purpose

1. Find the work done by a force $y\hat{i} + x\hat{j}$ which displaces a particle from origin to a point $(\hat{i} + \hat{j})$. **Ans.** 1

2. Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$ moves a particle from origin to (1, 1) along a parabola $y^2 = x$. **Ans.** $\frac{2}{3}$

3. Show that $\vec{V} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find its scalar potential ϕ such that $\vec{V}$ = grad ϕ. Find the work done by the force $\vec{V}$ in moving a particle from (1, – 2, 1) to (3, 1, 4). **Ans.** $x^2y + xz^3$, 202

4. Show that the line integral $\int_c (2xy + 3)\,dx + (x^2 - 4z)\,dy - 4y\,dz$ where c is any path joining (0, 0, 0) to (1, – 1, 3) does not depend on the path c and evaluate the line integral. **Ans.** 14

6. If $\vec{\nabla}\phi = (y^2 - 2xyz^3)\,\hat{i} + (3 + 2xy - x^2z^3)\,\hat{j} + (z^3 - 3x^2yz^2)\,\hat{k}$, find ϕ. **Ans.** $3y + \frac{z^4}{4} + xy^2 - x^2yz^3$