Laplacian Operator

Introduction

7.5 Laplacian Operator : ∇^2

7.5.1 $\nabla^2 = \nabla.\nabla = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)$

$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$ is called the Laplacian Operator

'∇^2' can be applied to both scalar and vector functions as shown below.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

where 'ϕ' is scalar function

If $A = A_1 i + A_2 j + A_3 k$, is a vector function, then

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(A_1 i + A_2 j + A_3 k)$$

$$= (\nabla^2 A_1)i + (\nabla^2 A_2)j + (\nabla^2 A_3)k$$

Examples:

Ex. 7.5.4 If $f = x^2y^3z^2$, find $\nabla^2 f$ at (1, 2, 1)

Sol: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$

$= 2y^3z^2 + 6x^2yz^2 + 2x^2y^3$

$\therefore$ at (1, 2, 1), $\nabla^2 f = 16 + 12 + 16 = 44.$

Ex. 7.5.5 Show that, if $r = xi + yj + zk$, $r = |r|$, then $\nabla^2 r^n = n(n+1)r^{n-2}$

Sol: $r^n = (x^2+y^2+z^2)^{n/2}$, ($\because r = \sqrt{x^2+y^2+z^2}$)

$$\frac{\partial}{\partial x}(r^n) = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}-1}.2x = nx(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$\frac{\partial^2}{\partial x^2}(r^n) = n\left[x.\frac{n-2}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}-1}.2x + (x^2+y^2+z^2)^{\frac{n-2}{2}}\right]$$

$$= n\left[(n-2)x^2.(x^2+y^2+z^2)^{\frac{n-4}{2}} + (x^2+y^2+z^2)^{\frac{n-2}{2}}\right]$$

$$= n(n-2)\ x^2.r^{n-4} + nr^{n-2} \qquad \ldots (1)$$

Similarly, $\frac{\partial^2}{\partial y^2}(r^n) = n(n-2)y^2 r^{n-4} + nr^{n-2}$ ... (2)

and $\frac{\partial^2}{\partial z^2}(r^n) = n(n-2)z^2 r^{n-4} + nr^{n-2}$ ... (3)

$$\nabla^2(r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$= 3nr^{n-2} + n(n-2)\ r^{n-4}\ (x^2 + y^2 + z^2)$ adding (1), (2) & (3)

$= 3nr^{n-2} + n(n-2)\ r^{n-2}$ $(\because x^2 + y^2 + z^2 = r^2)$

$= n(n+1)\ r^{n-2}$

Example 25. *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$\left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= i\ f'(r)\frac{\partial r}{\partial x} + j\ f'(r)\frac{\partial r}{\partial y} + k\ f'(r)\frac{\partial r}{\partial z} = f'(r)\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= f'(r)\frac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla\,[\nabla f(r)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[f'(r)\frac{xi + yj + zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= \left(f''(r)\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r1 - x\frac{\partial r}{\partial x}}{r^2} + \left(f''(r)\frac{\partial r}{\partial y}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y\frac{\partial r}{\partial y}}{r^2} +$$
$$\left(f''(r)\frac{\partial r}{\partial z}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r.1 - z\frac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r - \frac{x^2}{r}}{r^2} + \left(f''(r)\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r - \frac{y^2}{r}}{r^2} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r - \frac{z^2}{r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r^2 - x^2}{r^3} + \left(f''(r)\frac{y}{r}\right) + f'(r)\frac{r^2 - y^2}{r^3} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r^2 - z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2+z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2+z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2+y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2+z^2}{r^3}+\frac{z^2+x^2}{r^3}+\frac{x^2+y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2+y^2+x^2}{r^2} + f'(r)\frac{2(x^2+y^2+z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r}$$

Ans.

Example for Practice Purpose

(1) Show that $\nabla^2(\log r) = \frac{1}{r^2}$

(2) Show that $(a)\ \nabla.(\nabla\phi) = \nabla^2\phi$

(3) Prove that $\nabla^2(fg) = f(\nabla^2 g) + 2(\nabla f).(\nabla g) + g(\nabla^2 f)$