Wave Packets and
Phase and Group Velocity

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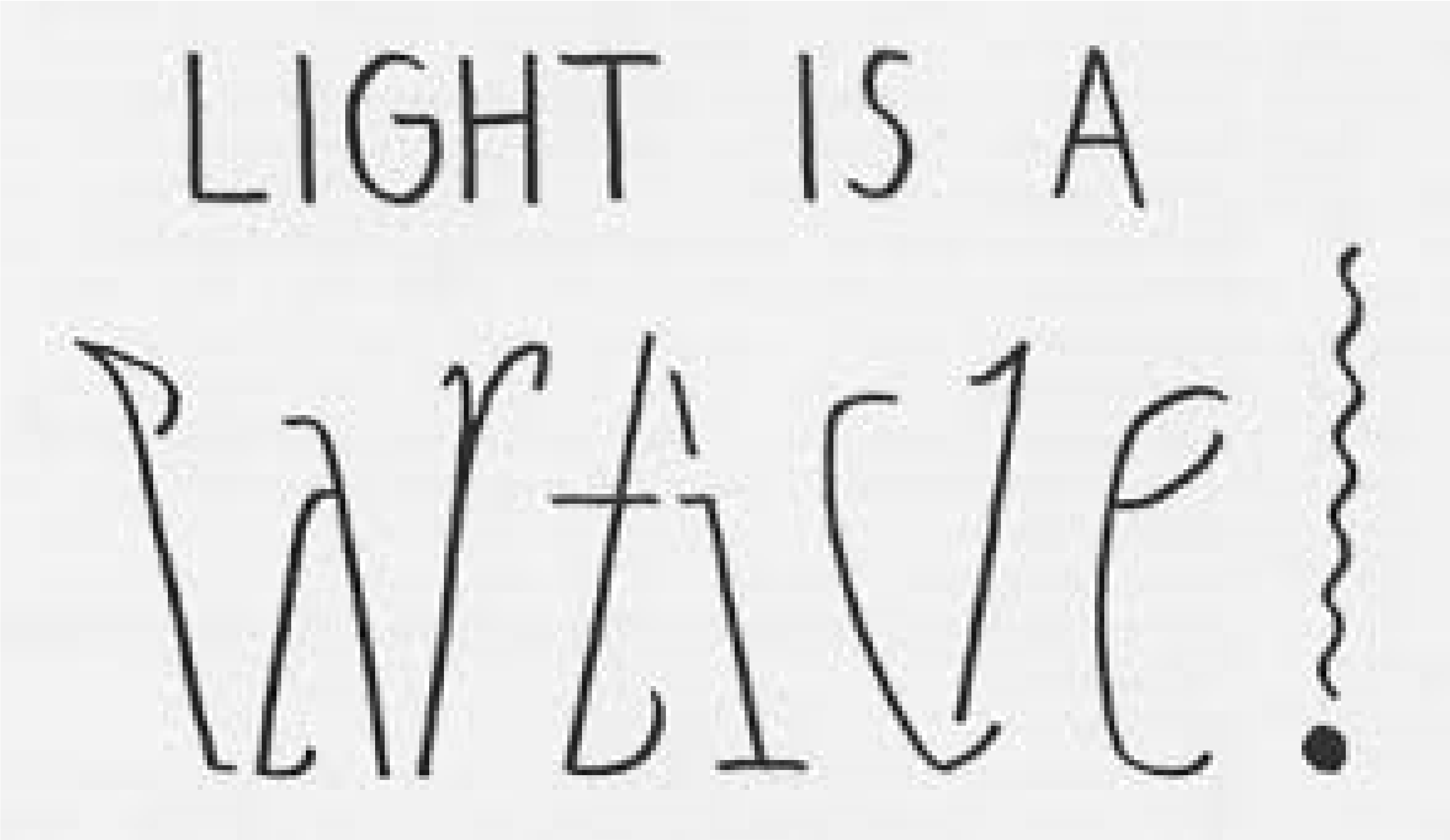
Recapitulate



• Everything (matter and radiation) has both wave and particle properties.

de Broglie Wavelength

For a photon, momentum $p = h\nu / c = h / \lambda$

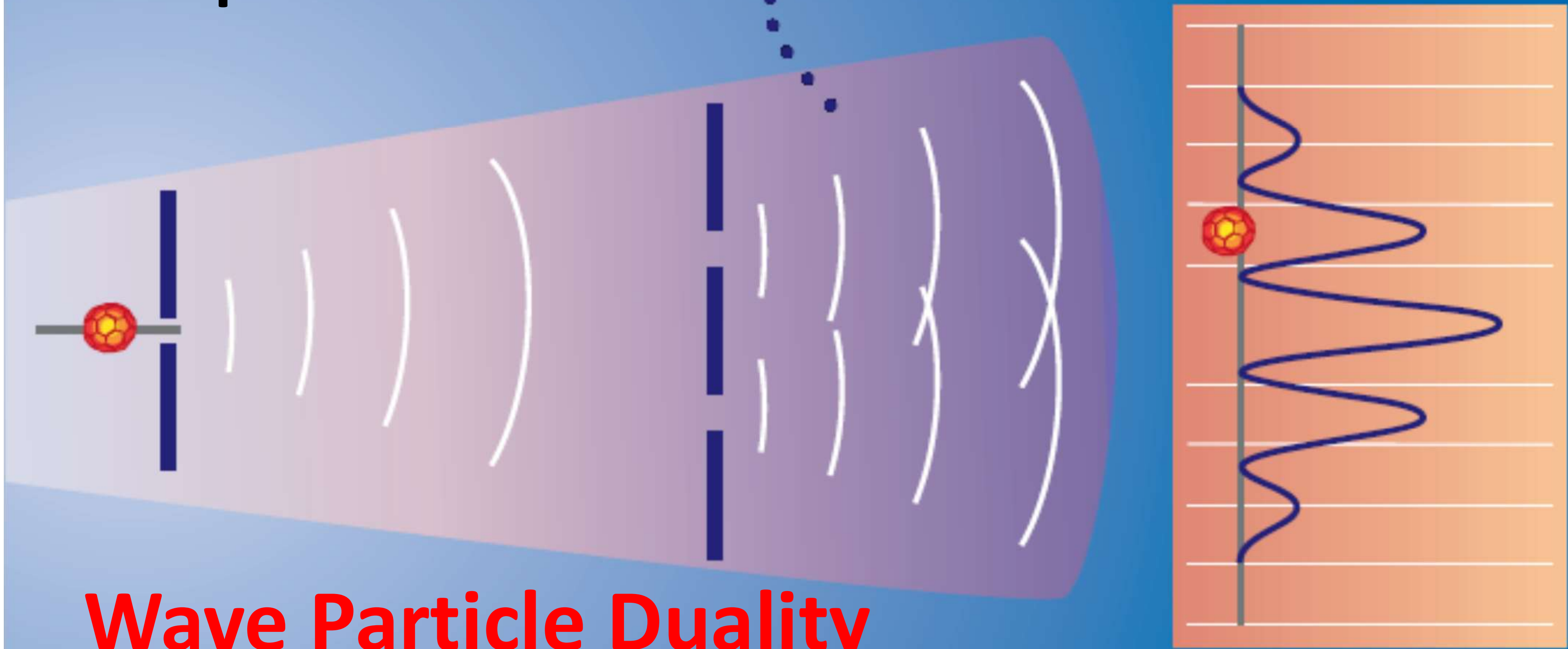
So for a particle of momentum *p*, the wavelength is

$$\lambda_{dB} = h / p = h / m\mathrm{v} = h / \gamma m_0 \mathrm{v}$$

λ_{dB} ***= de Broglie wavelength***

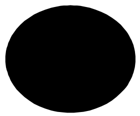
Interference is a result of wave property

When the object reaches the screen, it is detected as a particle.

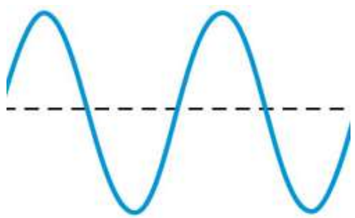


Wave Particle Duality

Wave Particle Duality



Particle: Localized, Definite position, momentum, confined in space



Wave: ‘Delocalized’, spread out in space and time

How to find a description of a particle which

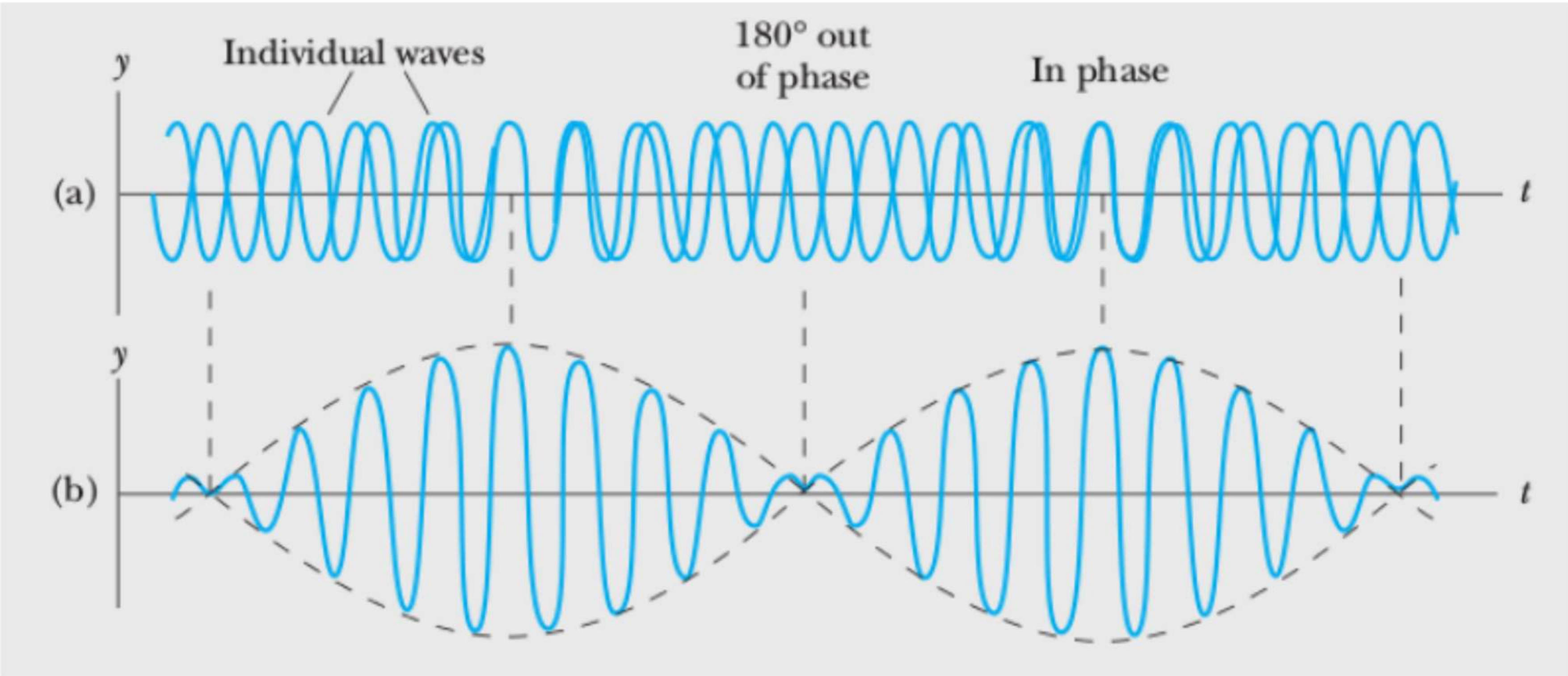
- *Fits the wave description*
- *And localized in space*

Wave Packet

Wave Packet

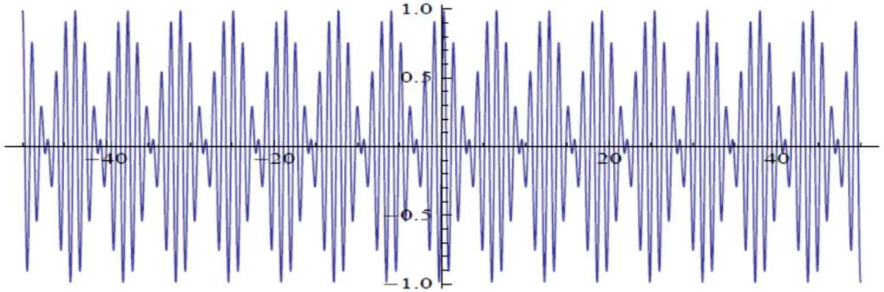
If several waves of different wavelengths and phases are superimposed together, what we get is a localized wave packet.

Example: Beat formation in superposition of two sinusoidal waves

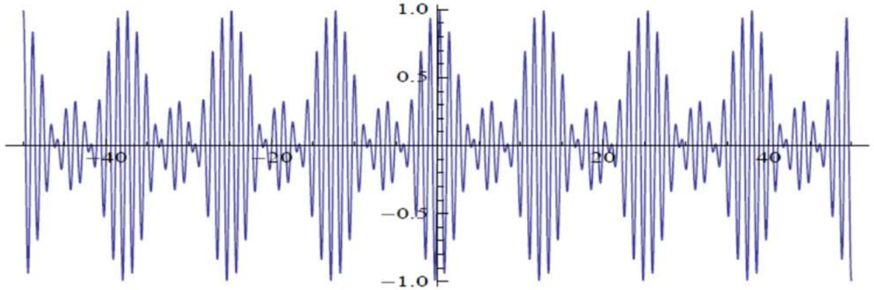


Spatial beats by superposition of sinusoidal waves of nearby wavelengths

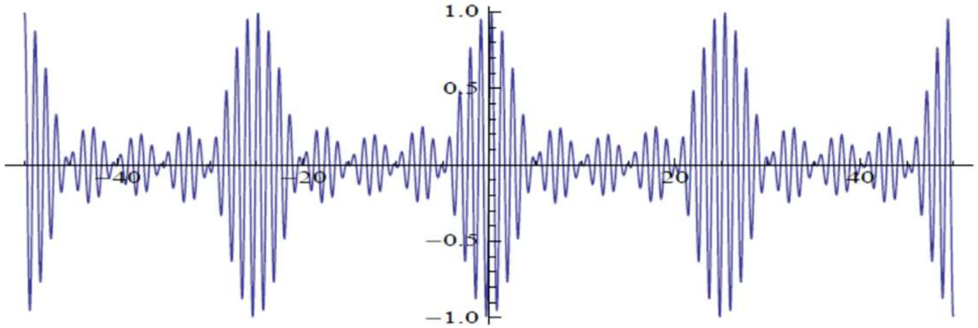
$$\psi = A\sin\left(\frac{2\pi}{\lambda}x\right) = A\sin kx$$



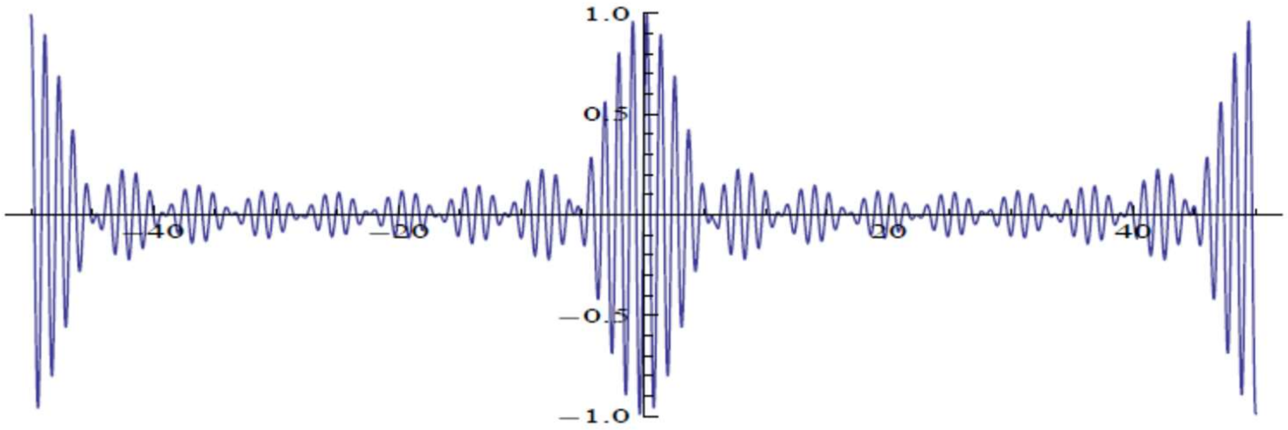
[sin(5x) + sin(6x)]/2



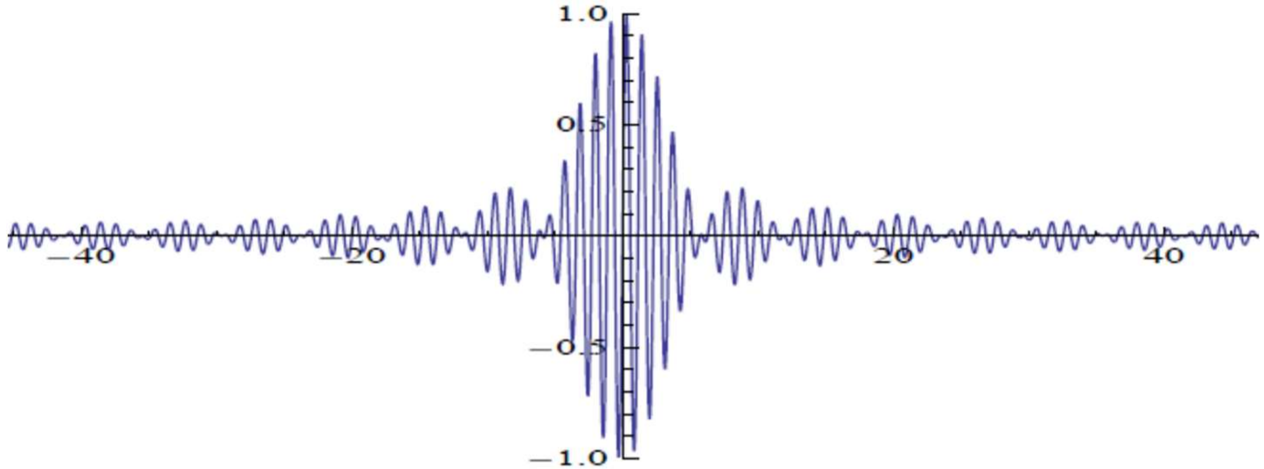
[sin(5x) + sin(5.5x) + sin(6x)]/3



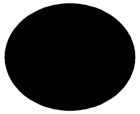
[sin(5x) + sin (5.25x) + sin(5.5x) + sin (5.75x) + sin(6x)]/5



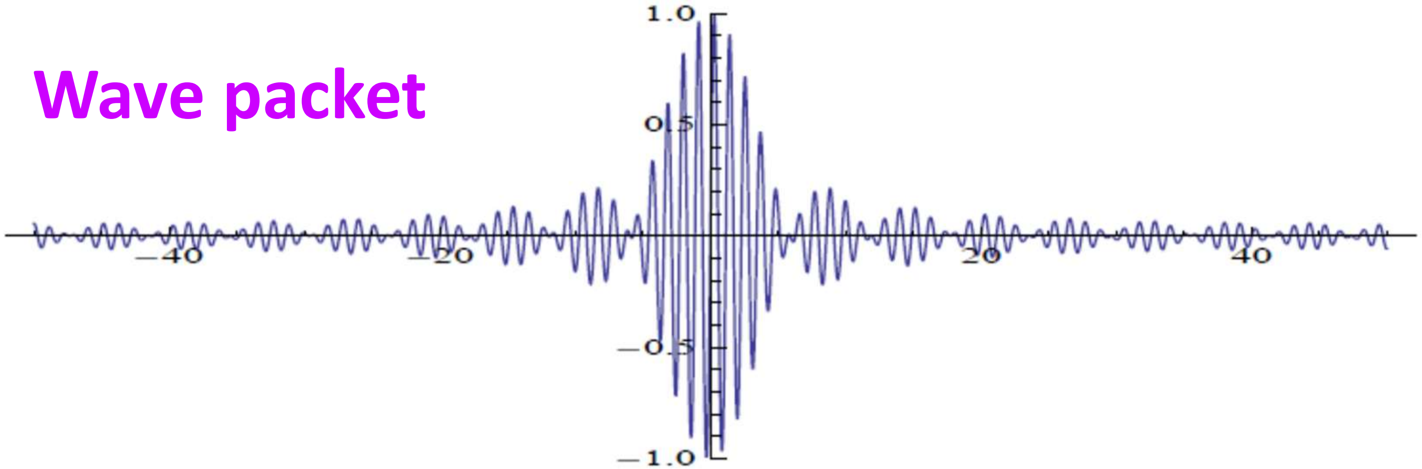
[sin(5x) + sin (5.125x) + sin (5.25x) + sin(5.375x) + sin(5.5x) + sin (5.625x) + sin (5.75x) + sin(5.875x) + sin(6x)]/9



[sin(5x) + sin(5.0625x) + sin (5.125x) + sin(5.1875x) + sin (5.25x) + sin(5.3125x) + sin(5.375x) + sin(5.4375x) + sin(5.5x) + sin(5.5625x) + sin (5.625x) + sin(5.6875x) + sin (5.75x) + sin(5.8125x) + sin(5.875x) + sin(5.9375x) + sin(6x)]/17

Particle 

Wave packet



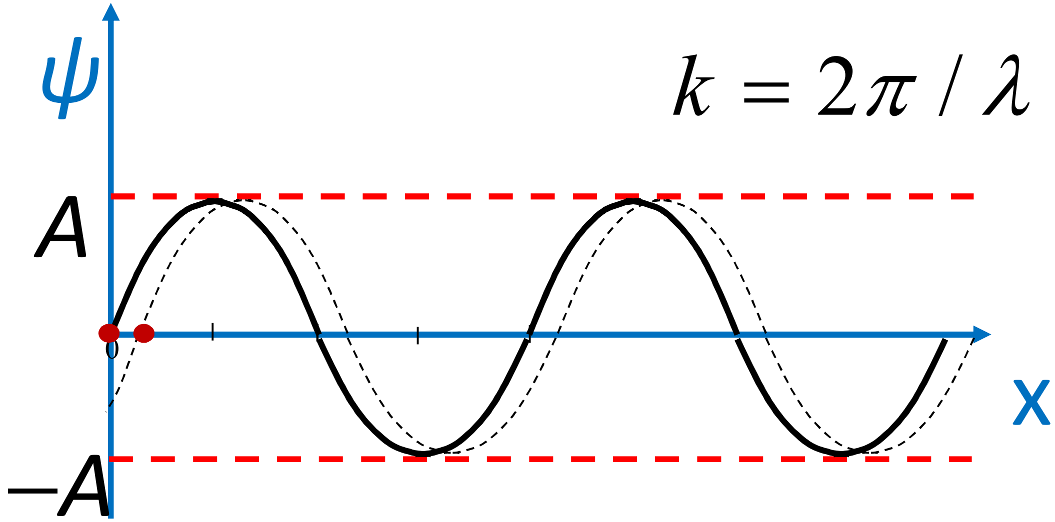
A wave packet is a **group of waves** with **slightly different wavelengths** interfering with one another in a way that the **amplitude of the group (envelope)** is **non-zero** in the neighbourhood of the particle.

A wave packet is localized; it is a good representation of a particle

Phase Velocity and Group Velocity

Consider an ideal wave $\psi = A\sin(kx - \omega t)$

$$k = 2\pi / \lambda \qquad \omega = 2\pi\nu$$



k measured in wavenumber

Take a point at t = 0 for which ψ = 0. Let time increase to Δt. What would be Δx to maintain ψ = 0.

$$k\Delta x - \omega\Delta t = 0$$

$$\mathrm{v}_p = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$$

Phase Velocity

Phase velocity is the velocity of a point of constant phase on the wave.

Now consider superposition of two waves

$$\psi_1 = A\sin(kx - \omega t)$$

$$\psi_1 = A\sin[(k + \Delta k)x - (\omega + \Delta\omega)t]$$

$$\psi_1 + \psi_2 = \psi = A\sin(kx - \omega t) + A\sin[(k + \Delta k)x - (\omega + \Delta\omega)t]$$

$$= 2A\sin\left[\frac{(2k + \Delta k)x}{2} - \frac{(2\omega + \Delta\omega)t}{2}\right]\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

Since Δk and $\Delta\omega$ are infinitesimally small quantities

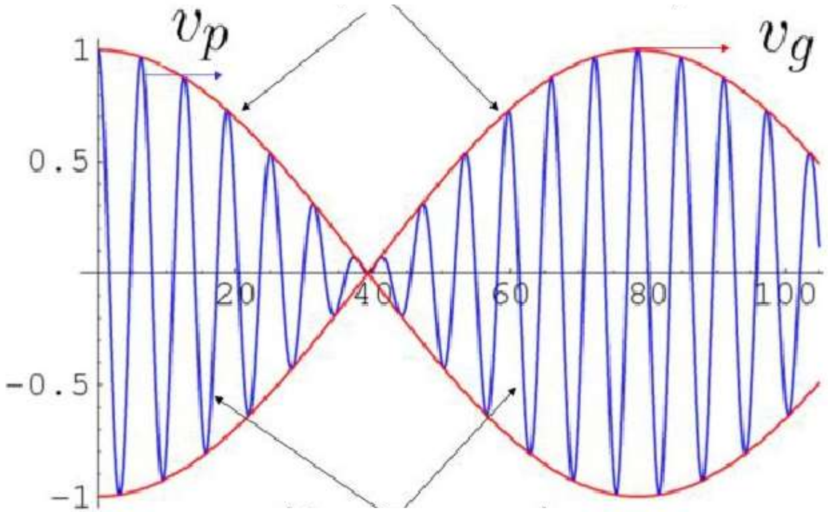
$$2k + \Delta k \approx 2k, \quad 2\omega + \Delta\omega \approx 2\omega$$

$$\boxed{\psi = 2A\sin(kx - \omega t)\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)}$$

$$\psi = 2A\sin(kx-\omega t)\cos\left(\frac{\Delta kx}{2}-\frac{\Delta \omega t}{2}\right)$$

Slowly varying envelope of frequency $\Delta\omega$ and propagation constant Δk

$$\cos\left(\frac{\Delta kx}{2}-\frac{\Delta \omega t}{2}\right)$$



$$\sin(kx-\omega t)$$

Group velocity is the velocity with which the envelope of the wave packet moves.

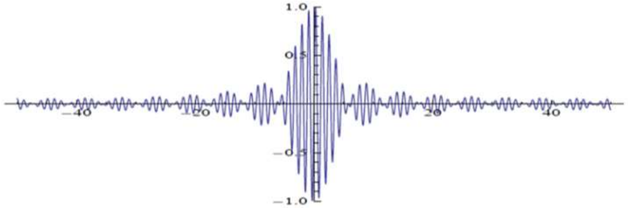
$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \qquad \text{as } k \rightarrow \infty$$

v_g is the velocity with which the wave packet moves.

Particle

V

Wave packet



$$\mathrm{v}_g = d\omega / dk$$

$$\mathrm{v}_p = \omega / k$$

Phase velocity

$$\mathrm{v}_p = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu$$

Relation between p and k

$$p = h/\lambda = 2\pi h / 2\pi\lambda = \hbar k$$

Wavelength

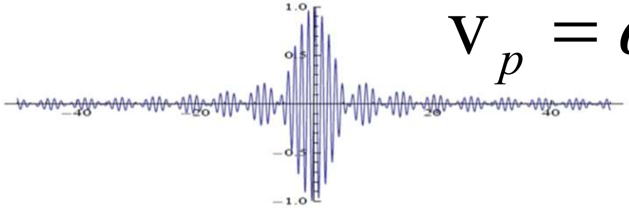
$$\lambda = h / p = h / m\mathrm{v}$$

Frequency

$$\nu = E/h = mc^2/h = \frac{m_0 c^2}{h\sqrt{1-\mathrm{v}^2/c^2}} = \frac{\sqrt{p^2c^2 + m_0^2 c^4}}{h}$$

Particle $\quad \mathrm{v}$

Wave packet



$$\mathrm{v}_p = \omega / k \qquad \mathrm{v}_\mathrm{g} = d\omega / dk$$

Suppose the velocity of the de Broglie wave associated with the moving particle is $\mathbf{v}_p$

$$\mathrm{v}_p = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu$$

$$\lambda = h/p = h/m\mathrm{v}$$

$$\nu = E/h = mc^2/h$$

$$\mathrm{v}_p = \frac{c^2}{\mathrm{v}}$$

$$\mathrm{v}_p > c \quad \text{since} \quad \mathrm{v} < \mathrm{c}$$

The de Broglie wave associated with the particle would leave the particle behind. This is against the wave concept of the particle.

$$\mathrm{v} \neq \mathrm{v}_\mathrm{p}$$

Is v = v$_g$?

$$E = h\nu = mc^2$$

$$\nu = mc^2 / h$$

$$\lambda = h / p = h / m\mathrm{v}$$

$$\omega = 2\pi\nu = 2\pi mc^2 / h$$

$$k = 2\pi / \lambda = 2\pi m\mathrm{v} / h$$

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{1 - \mathrm{v}^2 / c^2}}$$

$$k = \frac{2\pi m_0 \mathrm{v}}{h\sqrt{1 - \mathrm{v}^2 / c^2}}$$

$$\mathrm{v_g} = \frac{d\omega}{dk} = \frac{d\omega / d\mathrm{v}}{dk / d\mathrm{v}}$$

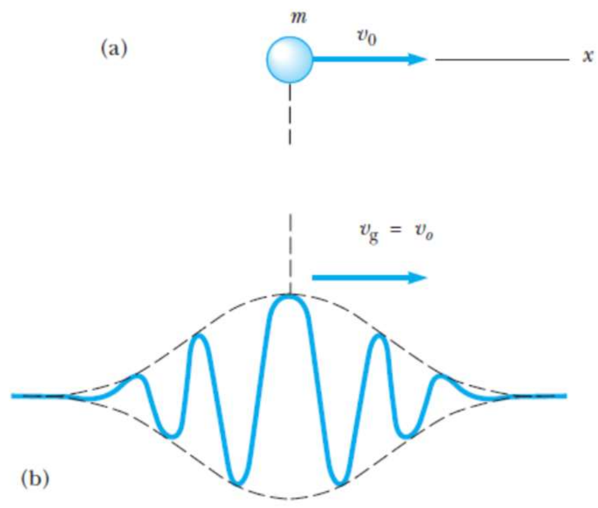
$$d\omega / d\mathrm{v} = \frac{2\pi m_0 \mathrm{v}}{h(1 - \mathrm{v}^2 / c^2)^{3/2}}$$

$$dk / d\mathrm{v} = \frac{2\pi m_0}{h(1 - \mathrm{v}^2 / c^2)^{3/2}}$$

$$\boxed{\mathrm{v_g} = \mathrm{v}}$$

de Broglie wave group associated with a moving body travels with the same velocity as the body!

de Broglie wave group associated with a moving body travels with the same velocity as the body!



In general, many waves having a continuous distribution of wavelengths must be added to form a packet that is finite over a limited range and really zero everywhere else. In this case,

$$\mathrm{v}_g = \frac{d\omega}{dk} \qquad \Longrightarrow \qquad \mathrm{v}_g = \left.\frac{d\omega}{dk}\right|_{k_0}$$

where the derivative is to be evaluated at the central k_0 .

Relationship between v_g and v_p

$$\mathrm{v}_p = \frac{\omega}{k}$$

$$\mathrm{v}_\mathrm{g} = \frac{d\omega}{dk}$$

$$\mathrm{v}_\mathrm{g} = \frac{d}{dk}(k\mathrm{v}_\mathrm{p}) = \left[\mathrm{v}_\mathrm{p} + k\frac{d\mathrm{v}_\mathrm{p}}{dk}\right]_{k_0}$$

Since $k = 2\pi / \lambda$

$$k\frac{d\mathrm{v}_\mathrm{p}}{\mathrm{dk}} = -\lambda\frac{d\mathrm{v}_\mathrm{p}}{d\lambda}$$

$$\mathrm{v}_\mathrm{g} = \left[\mathrm{v}_\mathrm{p} - \lambda\frac{d\mathrm{v}_\mathrm{p}}{d\lambda}\right]_{\lambda_0}$$

Dispersion Relations

Relation between ω and k is known as dispersion relation.

Plot of ω vs k is called the dispersion curve.

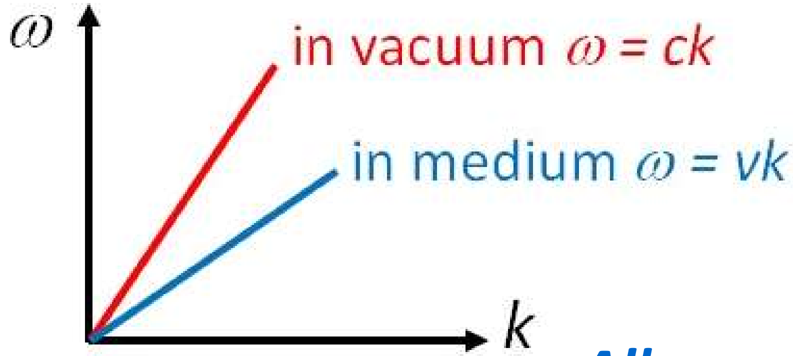
$$\mathrm{v_g} = \left[\mathrm{v_p} + k\frac{d\mathrm{v_p}}{dk} \right]_{k_0} = \left[\mathrm{v_p} - \lambda\frac{d\mathrm{v_p}}{d\lambda} \right]_{\lambda_0}$$

If $d\mathrm{v_p}/dk = 0$

$$\Downarrow$$

$$\mathrm{v_g} = \mathrm{v_p}$$

Since $\mathrm{v}_p = \omega/k$, $\quad d\mathrm{v_p}/dk = 0 \Rightarrow \omega = k\mathrm{v_g} \Rightarrow \omega = k\mathrm{v}$



Non-dispersive medium:

$$d\mathrm{v_p}/dk = 0 \qquad \mathrm{v_g} = \mathrm{v_p}$$

$$\mathrm{v_p} = c/n_r \qquad \textit{n_r = Refractive index}$$

All component waves have the same speed!

Dispersion Relations

$$\mathrm{v}_g = \left[\mathrm{v}_p + k\frac{d\mathrm{v}_p}{dk} \right]_{k_0} = \left[\mathrm{v}_p - \lambda\frac{d\mathrm{v}_p}{d\lambda} \right]_{\lambda_0}$$

Non-dispersive medium:

$$d\mathrm{v}_p / dk = 0 \qquad \mathrm{v}_g = \mathrm{v}_p$$

Dispersive medium: $d\mathrm{v}_p / dk \neq 0 \qquad \mathrm{v}_g \neq \mathrm{v}_p$

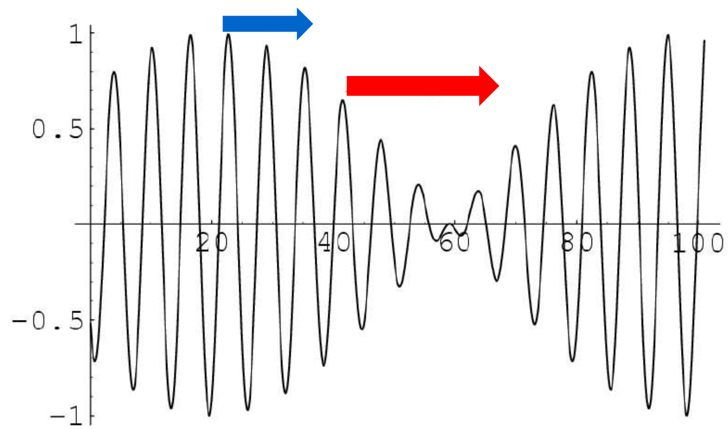
Dispersive occurs when phase velocity depends on k (or λ): $\mathrm{v}_p = c / n_r(\lambda)$

Normal dispersion

$$d\mathrm{v}_p / d\lambda > 0$$

$$n_r(red) < n_r(blue), \quad dn_r / d\lambda < 0,$$

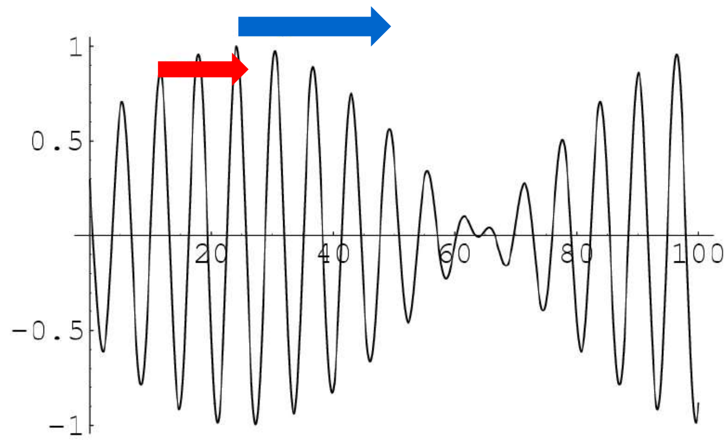
$$\mathrm{v}_g < \mathrm{v}_p$$



Anomalous dispersion

$$d\mathrm{v}_p / d\lambda < 0$$

$$\mathrm{v}_g > \mathrm{v}_p$$



Dispersion Relation for de Broglie Wave

Phase velocity $\quad \mathrm{v}_p = \lambda \nu$

$$\lambda = h / p \qquad p = \hbar k$$

$$\nu = E / h = \frac{\sqrt{p^2c^2 + m_0^2c^4}}{h}$$

$$\mathrm{v_p} = \frac{\sqrt{p^2c^2 + m_0^2c^4}}{p} = c\sqrt{1 + \left(\frac{m_0 c}{\hbar k}\right)^2}$$

$$d\mathrm{v_p} / dk \neq 0$$

All media are dispersive for de Broglie wave

Group velocity

$$\mathrm{v_g} = \left[\mathrm{v_p} + k\frac{d\mathrm{v_p}}{dk}\right]_{k_0}$$

$$\mathrm{v_g} = c\left[1 + \left(\frac{mc}{\hbar k_0}\right)^2\right]^{-1/2} = \frac{c^2}{\mathrm{v_p}|_{k_0}}$$

Since $\mathrm{v_p} = c^2 / \mathrm{v}$

$$\mathrm{v_g} = \mathrm{v}$$

v is the particle velocity

[This derivation is identical to the derivation on slide no. 15]

De Broglie Wave: Dispersion relation

ω Vs k relation

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\omega = \left(\frac{h}{2m}\right) k^2$$

$$E = h\nu = \hbar\omega$$