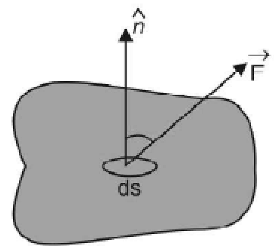
Surface Integral

Introduction

A surface $r = f(u, v)$ is called smooth if $f\ (u, v)$ posses continous first order partial derivative.

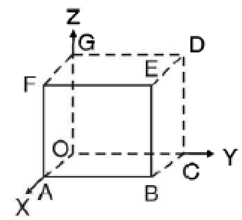
Let $\vec{F}$ be a vector function and S be the given surface.



Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal

$= \vec{F}\,.\,\hat{n}$, where n is the unit normal vector to an element ds and



$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|} \qquad ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

Surface integral of F over S

$$= \Sigma\,\vec{F}\cdot\hat{n} \qquad = \iint_S (\vec{F}\cdot\hat{n})\,ds$$

Note. (1) Flux $= \iint_S (\vec{F}\cdot\hat{n})\,ds$ where, $\overline{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F}\cdot\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a *solenoidal* vector point function.

Example 11. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\,ds$ *where* $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (*Nagpur University, Summer 2000*)

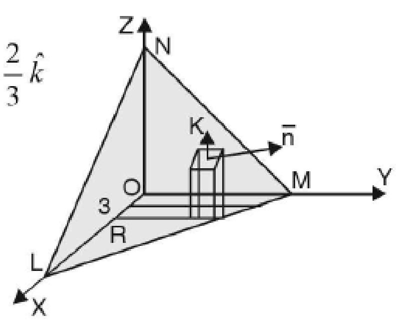
Solution. A vector normal to the surface "S" is given by

$$\nabla\,(2x + y + 2z) = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

And $\hat{n}$ = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}$$

$$\hat{k}\,.\,\hat{n} = \hat{k}\,.\left(\frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}\right) = \frac{2}{3}$$



$$\therefore \qquad \iint_S \overline{A}\,.\,\hat{n}\,ds = \iint_R \overline{A}\,.\,\hat{n}\,\frac{dx\,dy}{\hat{k}\,.\,\overline{n}}$$

Where R is the projection of S.

Now, $\vec{A}\cdot\hat{n} = [\,(x+y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}]\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

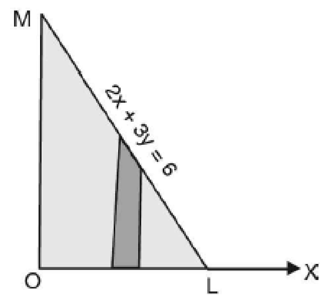
$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz \qquad \text{...(1)}$$

Putting the value of z in (1), we get

$$\vec{A}\cdot\hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right) \qquad \left(\begin{array}{l}\because \text{ on the plane } 2x+y+2z=6,\\ z = \dfrac{(6-2x-y)}{2}\end{array}\right)$$

$$\vec{A}\cdot\hat{n} = \frac{2}{3}y\,(y+6-2x-y) = \frac{4}{3}y\,(3-x) \qquad \text{...(2)}$$

Hence, $\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \overline{A}\cdot\overline{n}\,\frac{dx\,dy}{|\hat{k}\cdot\overline{n}|}$ ...(3)



Putting the value of $\vec{A}\cdot\hat{n}$ from (2) in (3), we get

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \frac{4}{3}y\,(3-x)\cdot\frac{3}{2}\,dx\,dy = \int_0^3\int_0^{6-2x} 2y\,(3-x)\,dy\,dx$$

$$= \int_0^3 2\,(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3-x)\,(6-2x)^2\,dx = 4\int_0^3 (3-x)^3\,dx$$

$$= 4.\left[\frac{(3-x)^4}{4\,(-1)}\right]_0^3 = -(0-81) = 81$$

Ans.

Example 12. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\,d\,S$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\,\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

Solution. Here, $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $f(x, y, z) = 2x + 3y + 6z - 12$

$$\text{Normal vector} = \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$

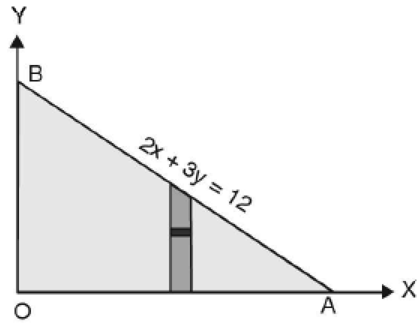
$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\cdot\hat{k}} = \frac{dxdy}{\frac{6}{7}} = \frac{7}{6}\,dx\,dy$$

$$\text{Now, } \iint \vec{A}\cdot\hat{n}\,dS = \iint (18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k})\cdot\frac{1}{7}(2\,\hat{i} + 3\,\hat{j} + 6\,\hat{k})\,\frac{7}{6}\,dx\,dy$$

$$= \iint (36z - 36 + 18y)\,\frac{dx\,dy}{6} \qquad = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get



$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\, dx\, dy$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\, dx\, dy$$

$$= \int_0^6 (6 - 2x)\, dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\, dx\, (y)_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6 - 2x) \frac{1}{3} (12 - 2x)\, dx = \frac{1}{3} \int_0^6 (4x^2 - 36x + 72)\, dx$$

$$= \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6 = \frac{1}{3}[4 \times 36 \times 2 - 18 \times 36 + 72 \times 6] = \frac{72}{3}[4 - 9 + 6] = 24 \textbf{ Ans.}$$

Example 13. *Evaluate* $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \vec{ds}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* (*U.P., I Semester, Dec. 2004*)

Solution. Here, $\phi = x^2 + y^2 + z^2 - a^2$

Vector normal to the surface $= \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad [\because x^2 + y^2 + z^2 = a^2]$$

Here, $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\vec{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

Now, $\iint_S F \cdot \hat{n}\, ds = \iint_S (\vec{F}\cdot\hat{n}) \dfrac{dx\, dy}{|\hat{k}\cdot\hat{n}|} = \displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{3xyz\, dx\, dy}{a\left(\frac{z}{a}\right)}$

$$= 3\int_0^a \int_0^{\sqrt{a^2 - x^2}} xy\, dy\, dx = 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2 - x^2}} dx$$

$$= \frac{3}{2}\int_0^a x(a^2 - x^2)\, dx = \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}. \qquad \textbf{Ans.}$$

Example 14. *Show that* $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$, *where* $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by the planes,*

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

S.No.	Surface	Outward normal	ds	
1	OABC	$-k$	$dx\,dy$	$z = 0$
2	DEFG	k	$dx\,dy$	$z = 1$
3	OAGF	$-j$	$dx\,dz$	$y = 0$
4	BCED	j	$dx\,dz$	$y = 1$
5	ABDG	i	$dy\,dz$	$x = 1$
6	OCEF	$-i$	$dy\,dz$	$x = 0$

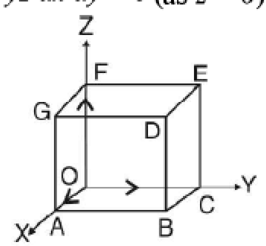
Solution. $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds$

$$+\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAGF} \vec{F}\cdot\hat{n}\,ds$$

$$+\iint_{BCED} \vec{F}\cdot\hat{n}\,ds + \iint_{ABDG} \vec{F}\cdot\hat{n}\,ds$$

$$+\iint_{OCEF} \vec{F}\cdot\hat{n}\,ds \qquad ...(1)$$

Now, $\iint_{OABC} \vec{F}\cdot n\,ds = \iint_{OABC} (4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\,(-k)\,dx\,dy = \int_0^1\int_0^1 - yz\,dx\,dy = 0$ (as $z = 0$)



$$\iint_{DEFG} (4\,xz\hat{i} - y^2\,\hat{j} + yz\,\hat{k})\cdot\hat{k}\,dx\,dy$$

$$= \iint_{DEFG} yz\,dx\,dy = \int_0^1\int_0^1 y\,(1)\,dx\,dy \quad (\text{as } z = 1)$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = [x]_0^1\,\frac{1}{2} = \frac{1}{2}$$

$$\iint_{OAGF} (4\,xz\,\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-j)\,dx\,dz = \iint_{OAGF} y^2\,dx\,dz = 0 \qquad (\text{as } y = 0)$$

$$\iint_{BCED} (4\,xz\,\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{j}\,dx\,dz = \iint_{BCED} (-y^2)\,dx\,dz$$

$$= -\int_0^1 dx\int_0^1 dz = -(x)_0^1\,(z)_0^1 = -1 \qquad (\text{as } y = 1)$$

$$\iint_{ABDG} (4\,xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}\,dy\,dz = \iint 4\,xz\,dy\,dz = \int_0^1\int_0^1 4\,(1)\,z\,dy\,dz \quad (\text{as } x = 1)$$

$$= 4\,(y)_0^1\left(\frac{z^2}{2}\right)_0^1 = 4\,(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCEF} (4\,xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\,(-\hat{i})\,dy\,dz = \int_0^1\int_0^1 - 4\,xz\,dy\,dz = 0 \qquad (\text{as } x = 0)$$

On putting these values in (1), we get

$$\iint_S \bar{F}\cdot\hat{n}\,ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$

Proved.

Example for Practice Purpose

1. Evaluate $\iint_S \vec{A}\,.\,\hat{n}\,ds$, where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. **Ans.** 90

2. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\,\hat{k}$ and $\vec{S} = 2t^2\hat{i} + 6t\hat{k}$, evaluate $\int_0^2 \vec{r}\cdot\vec{S}\,dt$. **Ans.** 12

3. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where, $F = 2yx\hat{i} - yz\hat{j} + x^2\hat{k}$ over the surface S of the cube bounded by the coordinate planes and planes $x = a$, $y = a$ and $z = a$. **Ans.** $\frac{1}{2}a^4$

4. If $\vec{F} = 2y\hat{i} - 3\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, and $z = 6$, then evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$. **Ans.** 132