Rank of The matrix

19.1 RANK OF A MATRIX

The rank of a matrix is said to be r if

(a) It has at least one non-zero minor of order r.

(b) Every minor of A of order higher than r is zero.

Note: (i) Non-zero row is that row in which all the elements are not zero.

(ii) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

19.2 NORMAL FORM (CANONICAL FORM)

By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A :

(i) I_r (ii) $[I_r \ \ 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A. Since both row and column transformations may be used here, the element 1 of the first row obtained can be moved in the first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

Example 1. *Find the rank of the following matrix by reducing it to normal form –*

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad C_2 \to C_2 - 2C_1,\ C_3 \to C_3 + C_1,\ C_4 \to C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad C_3 \to C_3 + \frac{6}{7} C_2, C_4 \to C_4 - \frac{11}{7} C_2,$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 + 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to -1/7\, R_2 \\ R_3 \to -1/2\, R_3 \end{matrix}$$

Rank of A = 3

Number of non zero row is 3

Rank of A = 3

Example 3. *Reduce the matrix to normal form and find its rank.*

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Solution. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -\frac{1}{2} & -1 & \frac{-3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & \frac{-7}{2} & -7 & \frac{-21}{2} \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{3}{2}R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - \frac{9}{2}R_1 \end{matrix}$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & \frac{-3}{2} \\ 0 & -1 & -2 & -3 \\ 0 & \frac{-7}{2} & -7 & \frac{-21}{2} \end{bmatrix} \begin{matrix} C_2 \to C_2 - \frac{3}{2}C_1 \\ C_3 \to C_3 - 2\,C_1 \\ C_4 \to C_4 - \frac{5}{2}C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_1 \to \frac{1}{2}R_1 \\ R_2 \to -2R_2 \\ R_3 \to -R_3 \\ R_4 \to \frac{-2}{7}R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3 - 2C_2 \\ C_4 \to C_4 - 3C_2 \end{matrix}$$

$$= \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence its rank = 2

Example 4. *Find the rank of the matrix.*

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix},$$ *by reducing it to normal form.*

Solution. We have, $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \begin{matrix} \\ C_2 \to C_2 - 3C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_3 \leftrightarrow R_4 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3 - \frac{5}{7} C_2 \\ C_4 \to C_4 - \frac{2}{7} C_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{7} R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \text{ which is normal form.}$$

Hence, Rank (A) = 3.

Example for Practice Purpose:

Find the rank of the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ **Ans.** 2

2. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ **Ans.** 3

3. $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **Ans.** 2

4. $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$ **Ans.** 3

5. $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ **Ans.** 4

6. $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ **Ans.** 2