The Central Force Problem

Dr. Yogesh Sonvane
Assistant Professor
Department of Applied Physics
Sardar Vallabhbhai National Institute of Technology, Surat

Central Forces

Introduction

- Interested in the **"2 body"** problem!

  Start out generally, but eventually restrict to motion of **2 bodies** interacting through a **central force**.

- ***Central Force*** ≡ Force between 2 bodies which is directed along the line between them.
- ***Important*** physical problem! Solvable ***exactly***!
  - Planetary motion & Kepler's Laws.
  - Nuclear forces
  - Atomic physics (H atom). Needs quantum version!

Reduction to Equivalent 1-Body Problem

- **General** 3d, 2 body problem. **2 masses** $\mathbf{m_1}$ & $\mathbf{m_2}$: Need **6 coordinates:** For example, components of 2 position vectors $\vec{\mathbf{r}}_1$ & $\vec{\mathbf{r}}_2$ (arbitrary origin).
- **Assume** only forces are due to an interaction potential **U**. At first, **U** = any function of the vector between 2 particles, $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, of their relative velocity $\dot{\vec{\mathbf{r}}} = \dot{\vec{\mathbf{r}}}_1 - \dot{\vec{\mathbf{r}}}_2$, & possibly of higher derivatives of $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$: $\mathbf{U} = \mathbf{U}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},..)$
  - Very soon, will restrict to central forces!

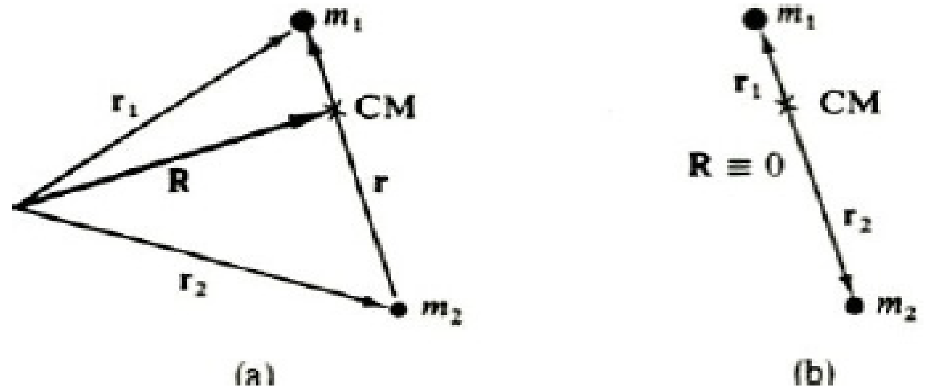
Lagrangian: $L = (½)m_1|\dot{\vec{r}}_1|^2 + (½)m_2|\dot{\vec{r}}_2|^2 - U(\vec{r},\dot{\vec{r}}, ..)$

- Instead of 6 components of 2 vectors $\mathbf{r}_1$ & $\mathbf{r}_2$, usually ***transform*** to (6 components of) Center of Mass (CM) & Relative Coordinates.
- ***Center of Mass Coordinate:*** $(M \equiv (m_1+m_2))$

$$\vec{R} \equiv (m_1\vec{r}_1 + m_2\vec{r}_2)/(M)$$

- ***Relative Coordinate:***

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$



- ***Define:*** **Reduced Mass:** $\mu \equiv (m_1 m_2)/(m_1+m_2)$

Useful relation: $\mu^{-1} \equiv (m_1)^{-1} + (m_2)^{-1}$

- Algebra $\Rightarrow$ Inverse coordinate relations:

$$\vec{r}_1 = \vec{R} + (\mu/m_1)\vec{r};\ \vec{r}_2 = \vec{R} - (\mu/m_2)\vec{r}$$

Lagrangian: $L = (\frac{1}{2})m_1|\dot{\vec{r}}_1|^2 + (\frac{1}{2})m_2|\dot{\vec{r}}_2|^2 - U(\vec{r},\dot{\vec{r}}, ..)$ (1)

- **Velocities related** by

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + (\mu /m_1)\dot{\vec{r}}; \quad \dot{\vec{r}}_2 = \dot{\vec{R}} - (\mu /m_2)\dot{\vec{r}} \quad (2)$$

- Combining **(1) & (2)** + algebra gives Lagrangian in terms of $\vec{R},\vec{r},\dot{\vec{r}}$: $\quad L = (\frac{1}{2})M|\dot{\vec{R}}|^2 + (\frac{1}{2})\mu|\dot{\vec{r}}|^2 - U$

Or: $\quad L = L_{CM} + L_{rel}$. Where: $\quad L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$

$$L_{rel} \equiv (\tfrac{1}{2})\ \mu\ |\dot{\vec{r}}|^2 - U$$

$\Rightarrow$ **Motion separates into 2 parts:**

1. **CM motion,** governed by $L_{CM} \equiv (\frac{1}{2})M|\vec{R}|^2$
2. **Relative motion,** governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

CM & Relative Motion

- Lagrangian for **2 body problem**: $L = L_{CM} + L_{rel}$

$$L_{CM} \equiv (½)M|\dot{\vec{R}}|^2 \; ; \; L_{rel} \equiv (½)\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

$\Rightarrow$ Motion separates into **2 parts:**

1. Lagrange's Eqtns for 3 components of CM coordinate vector $\vec{\mathbf{R}}$ clearly gives eqtns of motion independent of $\vec{\mathbf{r}}$.

2. Lagrange Eqtns for 3 components of relative coordinate vector $\vec{\mathbf{r}}$ clearly gives eqtns of motion independent of $\vec{\mathbf{R}}$.

- By transforming from $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ to $(\vec{\mathbf{R}},\vec{\mathbf{r}})$:

The 2 body problem has been separated into 2 one body problems!

- Lagrangian for **2 body problem**

$$L = L_{CM} + L_{rel}$$

$\Rightarrow$ Have transformed the 2 body problem

into ***2 one body problems!***

1. **Motion of the CM**, governed by

$$L_{CM} \equiv (½)M|\dot{\vec{R}}|^2$$

2. **Relative Motion**, governed by

$$L_{rel} \equiv (½)\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

- <u>Motion of CM</u> is governed by $L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$
  - Assuming no external forces.
- $\vec{R} = (X,Y,Z) \quad \Rightarrow$ 3 Lagrange Eqtns; each like:

$$(d/dt)(\partial[L_{CM}]/\partial\dot{X}) - (\partial[L_{CM}]/\partial X) = 0$$

$$(\partial[L_{CM}]/\partial X) = 0 \Rightarrow (d/dt)(\partial[L_{CM}]/\partial\dot{X}) = 0$$

$\Rightarrow \quad \ddot{X} = 0$, <u>*CM acts like a free particle!*</u>

- Solution: $\dot{X} = V_{x0} =$ constant
  - Determined by initial conditions!

$\Rightarrow \quad X(t) = X_0 + V_{x0}t$, <u>*exactly like a free particle!*</u>

- Same eqtns for **Y, Z**:

$\Rightarrow \quad \vec{R}(t) = \vec{R}_0 + \vec{V}_0 t$, <u>*exactly like a free particle!*</u>

<u>*CM Motion is identical to trivial motion of a free particle.*</u>

Uniform translation of CM. Trivial & uninteresting!

- **2 body** Lagrangian: $L = L_{CM} + L_{rel}$

$\Rightarrow$ 2 body problem is transformed to 2 one body problems!

1. Motion of the CM, governed by $L_{CM} \equiv (\tfrac{1}{2})M|\dot{\vec{R}}|^2$

Trivial free particle-like motion!

2. Relative Motion, governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

$\Rightarrow$ 2 body problem is transformed to 2 one body problems, *one of which is trivial!*

All interesting physics is in relative motion part!

$\Rightarrow$ ***Focus on it exclusively!***

Relative Motion

- **Relative Motion** is governed by

$$\mathcal{L}_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

  – Assuming no external forces.

  – Henceforth: $\mathcal{L}_{rel} \equiv \mathcal{L}$ (Drop subscript)

  – For convenience, take **origin of coordinates at CM:**

$$\Rightarrow \quad \vec{R} = 0$$

$$\vec{r}_1 = (\mu/m_1)\vec{r}; \qquad \vec{r}_2 = -(\mu/m_2)\vec{r}$$

$$\mu \equiv (m_1 m_2)/(m_1+m_2)$$

$$(\mu)^{-1} \equiv (m_1)^{-1} + (m_2)^{-1}$$

- **The 2 body, central force problem** has been formally reduced to an

<u>EQUIVALENT ONE BODY PROBLEM</u>

in which the motion of a "particle" of mass $\boldsymbol{\mu}$ in $\mathbf{U}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},\ldots)$ is what is to be determined!

  – Superimpose the uniform, free particle-like translation of CM onto the relative motion solution!

  – If desired, if get $\vec{\mathbf{r}}(\mathbf{t})$, can get $\vec{\mathbf{r}}_1(\mathbf{t})$ & $\vec{\mathbf{r}}_2(\mathbf{t})$ from above. **Usually, the relative motion (orbits) only is wanted & we stop at $\vec{\mathbf{r}}(\mathbf{t})$.**

Eqtns of Motion & 1st Integrals

- **System:** "Particle" of mass **μ** (**μ** → **m** in what follows) moving in a force field described by potential $\mathbf{U}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},..)$.
- Now, restrict to **conservative Central Forces:**

$$\mathbf{U} \rightarrow \mathbf{V} \qquad \text{where } \mathbf{V} = \mathbf{V}(\mathbf{r})$$

  – **Note:** **V(r)** depends only on $\mathbf{r} = |\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|$ = distance of particle from force center. No orientation dependence. ⇒ **System has spherical symmetry**

  ⇒ Rotation about any fixed axis can't affect eqtns of motion.

  ⇒ **Expect the angle representing such a rotation to be cyclic &** ***the corresponding generalized momentum (angular momentum) to be conserved*.**

Angular Momentum

- By the discussion in Ch. 2: **Spherical symmetry**

$\Rightarrow$ ***The Angular Momentum of the system is conserved:***

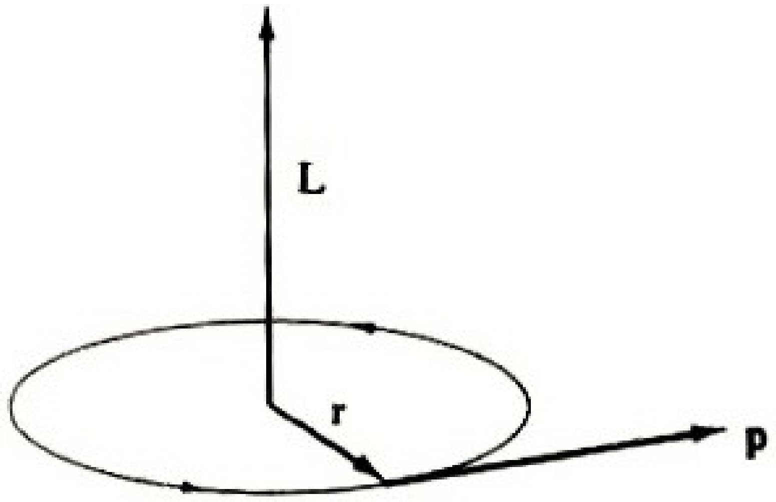
$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \text{constant (magnitude \& direction!)}$$

Angular momentum conservation!

$\Rightarrow$ $\vec{\mathbf{r}}$ & $\vec{\mathbf{p}}$ (& thus the particle motion!) always **lie in a plane** $\perp \vec{\mathbf{L}}$, which is fixed in space.

Figure:

(See text discussion for $\vec{\mathbf{L}} = \mathbf{0}$)



$\Rightarrow$ ***The problem is effectively reduced from 3d to 2d***

(particle motion in a plane)!

Motion in a Plane

- **Describe 3d motion in spherical coordinates** (Goldstein notation!): $(\mathbf{r},\theta,\psi)$. θ = angle in the plane (plane polar coordinates). ψ = azimuthal angle.
- $\vec{L}$ ***is fixed,*** as we saw. $\Rightarrow$ **The motion is in a plane.** Effectively reducing the 3d problem to a 2d one!
- Choose the polar (z) axis along $\vec{L}$.

$\Rightarrow$ $\psi = (\frac{1}{2})\pi$ ***& drops out of the problem.***

- **Conservation of angular momentum** $\vec{L}$

$\Rightarrow$ ***3 independent constants of the motion***

(1st integrals of the motion): Effectively we've used 2 of these to limit the motion to a plane. The third ($|\vec{L}|$ = constant) will be used to complete the solution to the problem.

Summary So Far

- Started with **6d, 2 body problem.** Reduced it to **2, 3d 1 body problems,** one (CM motion) of which is trivial. **Angular momentum conservation** reduces 2nd 3d problem (relative motion) **from 3d to 2d** (motion in a plane)!
- **Lagrangian** ($\mu \rightarrow$ **m** , conservative, central forces):

$$L = (½)m|\dot{\vec{r}}|^2 - V(r)$$

- **Motion in a plane**

  $\Rightarrow$ Choose plane polar coordinates to do the problem:

$$\Rightarrow \qquad L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- The Lagrangian is cyclic in θ

$\Rightarrow$ **The generalized momentum p_θ is conserved:**

$$p_\theta \equiv (\partial L/\partial\dot{\theta}) = mr^2\dot{\theta}$$

Lagrange's Eqtn: $(d/dt)[(\partial L/\partial\dot{\theta})] - (\partial L/\partial\theta) = 0$

$\Rightarrow \quad \dot{p}_\theta = 0, \qquad p_\theta = \text{constant} = mr^2\dot{\theta}$

- **Physics:** $p_\theta = mr^2\dot{\theta}$ = angular momentum about an axis $\perp$ the plane of motion. ***Conservation of angular momentum,*** as we already said!
- The problem symmetry has allowed us to integrate one eqtn of motion. $p_\theta \equiv$ a **"1st Integral"** of motion. Convenient to define: $\ell \equiv p_\theta \equiv mr^2\dot{\theta}$ = constant.

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- In terms of $\ell \equiv mr^2\dot{\theta}$ = constant, the Lagrangian is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Symmetry** & the resulting conservation of angular momentum has reduced the effective 2d problem (2 degrees of freedom) to an effective **1d problem!**

1 degree of freedom, one generalized coordinate r!

- Now: Set up & **solve the problem** using the above Lagrangian. Also, follow authors & do with **energy conservation.** However, first, **a side issue.**

Kepler's 2nd Law

- Const. angular momentum $\ell \equiv \mathbf{m r^2 \dot{\theta}}$
- Note that ℓ could be < 0 or > 0.
- **Geometric interpretation:** $\ell =$ const: See figure:

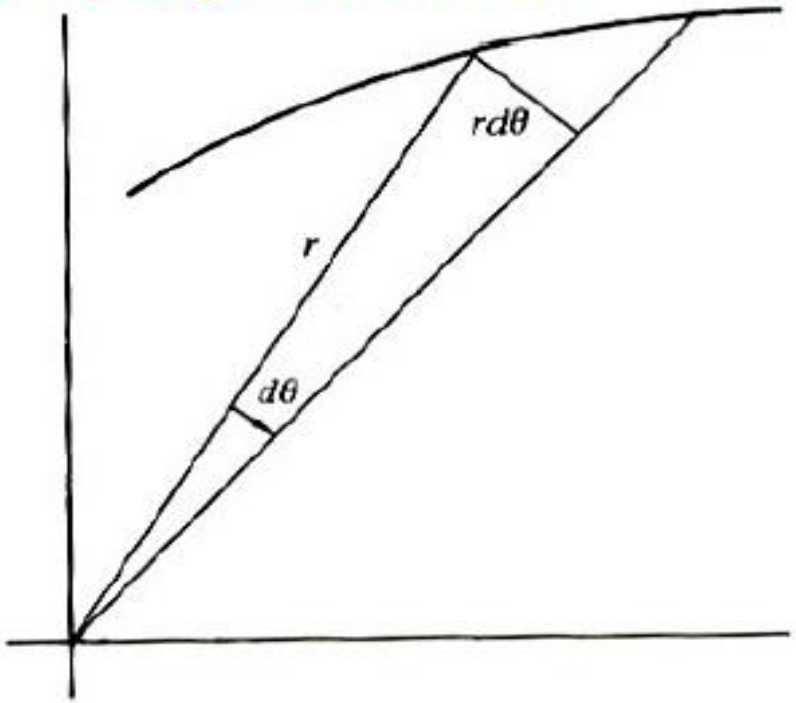


FIGURE 3.2 The area swept out by the radius vector in a time dt.

- In describing the path $\mathbf{r(t)}$, in time $\mathbf{dt}$, the radius vector sweeps out an area: $\mathbf{dA = (½)r^2 d\theta}$

- In **dt**, radius vector sweeps out area $\mathbf{dA = (½)r^2d\theta}$
  - Define ***Areal Velocity*** $\equiv$ **(dA/dt)**

$\Rightarrow$ $$\mathbf{(dA/dt) = (½)r^2(d\theta/dt) = dA = (½)r^2\dot{\theta}} \qquad (1)$$

But $\ell \equiv \mathbf{mr^2\,\dot{\theta}}$ = constant

$\Rightarrow$ $$\dot{\theta} = (\ell/mr^2) \qquad (2)$$

- Combine (1) & (2):

$\Rightarrow$ $$\mathbf{(dA/dt) = (½)(\ell/m)} = \text{constant!}$$

$\Rightarrow$ Areal velocity is constant in time!

$\equiv$ ***the Radius vector from the origin sweeps out equal areas in equal times*** $\equiv$ **Kepler's 2nd Law**

- First derived by empirically by Kepler for planetary motion.

General result for central forces!

Not limited to the gravitational force law ($\mathbf{r^{-2}}$).

Lagrange's Eqtn for r

- In terms of $\ell \equiv mr^2\dot{\theta}$ = const, the **Lagrangian** is:

$$L = (\tfrac{1}{2})m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Lagrange's Eqtn** for **r**:

$$(d/dt)[(\partial L/\partial \dot{r})] - (\partial L/\partial r) = 0$$

$$\Rightarrow \quad m\ddot{r} - [\ell^2/(mr^3)] = -(\partial V/\partial r) \equiv f(r)$$

$(f(r) \equiv$ force along **r**$)$

Rather than solve this directly, its easier to use **Energy Conservation.** Come back to this later.

Energy

- **Note:** Linear momentum is conserved also:
  - Linear momentum of CM.

  $\Rightarrow$ Uninteresting free particle motion
- **Total mechanical energy is also conserved** since the central force is conservative:

$$\mathbf{E = T + V = constant}$$

$$\mathbf{E = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)}$$

- Recall, **angular momentum** is:

$$\ell \equiv \mathbf{mr^2\dot{\theta}} = \text{const}$$

$$\Rightarrow \quad \dot{\theta} = [\ell/(mr^2)]$$

$$\Rightarrow \quad \mathbf{E = (½)m\dot{r}^2 + (½)[\ell^2/(mr^2)] + V(r)} = \text{const}$$

Another **"1st integral"** of the motion

$r(t)$ & $\theta(t)$

$$E = (\tfrac{1}{2})m\dot{r}^2 + [\ell^2/(2mr^2)] + V(r) = \text{const}$$

- **Energy Conservation** allows us to get solutions to the eqtns of motion in terms of $\mathbf{r(t)}$ & $\boldsymbol{\theta}\mathbf{(t)}$ and $\mathbf{r}(\boldsymbol{\theta})$ or $\boldsymbol{\theta}\mathbf{(r)}$ $\equiv$ **The orbit of the particle!**
  - Eqtn of motion to get $\mathbf{r(t)}$: One degree of freedom

  $\Rightarrow$ Very similar to a 1 d problem!
- Solve for $\dot{r} = (dr/dt)$:

$$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{\frac{1}{2}}$$

  - **Note:** This gives $\dot{\mathbf{r}}\mathbf{(r)}$, the phase diagram for the relative coordinate & velocity. Can qualitatively analyze (**r** part of) motion using it, just as in 1d.
- Solve for **dt** & formally integrate to get **t(r).** In principle, invert to get **r(t).**

$$\dot{r} = \pm\left(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)]\right)^{1/2}$$

- Solve for **dt** & **formally integrate** to get **t(r):**

$$t(r) = \pm\int dr\left(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)]\right)^{-1/2}$$

  – Limits $r_0 \rightarrow r$, r_0 determined by initial condition
  – Note the square root in denominator!

- Get $\theta(t)$ in terms of $r(t)$ using **conservation of angular momentum** again: $\ell \equiv mr^2\dot{\theta} = \text{const}$

$$\Rightarrow \qquad (d\theta/dt) = [\ell/(mr^2)]$$

$$\Rightarrow \qquad \theta(t) = (\ell/m)\int(dt[r(t)]^{-2}) + \theta_0$$

  – Limits $0 \rightarrow t$

  θ_0 determined by initial condition

- *Formally,* **the 2 body Central Force problem has been reduced to the evaluation of 2 integrals:**

(Given $\mathbf{V(r)}$ can do them, in principle.)

$$\mathbf{t(r) = \pm \int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}}$$

  - Limits $\mathbf{r_0 \to r}$, $\mathbf{r_0}$ determined by initial condition

$$\boldsymbol{\theta}\mathbf{(t)} = \mathbf{(\ell/m)\int(dt[r(t)]^{-2}) + \boldsymbol{\theta}_0}$$

  - Limits $\mathbf{0 \to t}$, $\boldsymbol{\theta}_0$ determined by initial condition

- To solve the problem, need 4 integration constants:

$$\mathbf{E, \ell, r_0, \boldsymbol{\theta}_0}$$

Orbits

- Often, we are much more interested in the path in the $\mathbf{r}, \boldsymbol{\theta}$ plane: $\mathbf{r}(\boldsymbol{\theta})$ or $\boldsymbol{\theta}(\mathbf{r}) \equiv$ ***The orbit.***
- Note that (chain rule):

$$(d\theta/dr) = (d\theta/dt)(dt/dr) = (d\theta/dt)/(dr/dt)$$

Or: $$(d\theta/dr) = (\dot{\theta}/\dot{r})$$

Also, $\ell \equiv mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} = [\ell/(mr^2)]$

Use $$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

$$\Rightarrow (d\theta/dr) = \pm [\ell/(mr^2)](\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}$$

Or:

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

- **Integrating this will give $\theta(r)$.**

- ***Formally:***

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}$$

- Integrating this gives a formal eqtn for the orbit:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}\, dr$$

- Once the central force is specified, we know **V(r)** & can, in principle, do the integral & get the orbit **θ(r),** or, (if this can be inverted!) **r(θ)**.

$\Rightarrow$ This is quite remarkable! Assuming only a central force law & nothing else:

We have reduced the original 6 d problem of 2 particles to a 2 d problem with only 1 degree of freedom. The solution for the orbit can be obtained simply by doing the above (1d) integral!

Equivalent "1d" Problem

- ***Formally,*** the 2 body Central Force problem has been **reduced to evaluation of 2 integrals**, which will give $\mathbf{r(t)}$ & $\boldsymbol{\theta}\mathbf{(t)}$: (Given $\mathbf{V(r)}$ can do them, in principle.)

$$\mathbf{t(r) = \pm\int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}} \quad \mathbf{(1)}$$

  - Limits $\mathbf{r_0 \rightarrow r}$, $\mathbf{r_0}$ determined by initial conditions
  - Invert this to get $\mathbf{r(t)}$ & use that in $\boldsymbol{\theta}\mathbf{(t)}$ (below)

$$\boldsymbol{\theta}\mathbf{(t)} = \mathbf{(\ell/m)\int(dt/[r^2(t)]) + \boldsymbol{\theta}_0} \qquad \mathbf{(2)}$$

  - Limits $\mathbf{0 \rightarrow t}$, $\boldsymbol{\theta}_0$ determined by initial condition
- Need **4 integration constants:** $\mathbf{E, \ell, r_0, \boldsymbol{\theta}_0}$
- Most cases: **(1), (2)** can't be done except numerically
- Before looking at cases where they can be done: Discuss the ***PHYSICS*** of motion obtained from conservation theorems.

- Assume the system has **known energy** **E** & **angular momentum** ℓ ($\equiv mr^2\dot{\theta}$).
  - Find the magnitude & direction of velocity **v** in terms of **r**:
- ***Conservation of Mechanical Energy:***

$\Rightarrow$ $$E = (\tfrac{1}{2})mv^2 + V(r) = \text{const} \qquad (1)$$

Or: $$E = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{const} \qquad (2)$$

v^2 = square of total (2d) velocity: $v^2 \equiv \dot{r}^2 + r^2\dot{\theta}^2 \qquad (3)$

(1) $\Rightarrow$ **Magnitude** of **v**:

$$v = \pm\left(\{2/m\}[E - V(r)]\right)^{\frac{1}{2}} \qquad (4)$$

(2) $\Rightarrow$ $$\dot{r} = \pm\left(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)]\right)^{\frac{1}{2}} \qquad (5)$$

Combining **(3), (4), (5)** gives the **direction** of **v**

- Alternatively, $\ell = mr^2\dot{\theta}$ = const, gives $\dot{\theta}$. Combined with (5) gives both magnitude & direction of **v**.

- **Lagrangian :** $L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$
- In terms of $\ell \equiv mr^2\theta$ = const, this is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- Lagrange Eqtn for **r**: $(d/dt)[(\partial L/\partial \dot{r})] - (\partial L/\partial r) = 0$

$\Rightarrow$ $$m\ddot{r} - [\ell^2/(mr^3)] = -(\partial V/\partial r) \equiv f(r)$$

(**f(r)** ≡ force along **r**)

Or: $$m\ddot{r} = f(r) + [\ell^2/(mr^3)] \qquad (1)$$

- **(1)** involves only **r** & **r**. ⇒ **Same Eqtn** of motion (Newton's 2nd Law) as for a **fictitious** (or effective) 1d **(r)** problem of mass **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

Centrifugal "Force" & Potential

- Effective 1d **(r)** problem: **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

- ***PHYSICS:*** Using $\ell \equiv mr^2\dot{\theta}$:

$$[\ell^2/(mr^3)] \equiv mr\dot{\theta}^2 \equiv m(v_\theta)^2/r \equiv \text{"Centrifugal Force"}$$

  – Return to this in a minute.

- Equivalently, energy:

$$E = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r) = \text{const}$$

- **Same energy Eqtn as** for a **fictitious** (or effective) 1d **(r)** problem of mass **m** subject to a potential:

$$V'(r) = V(r) + (\tfrac{1}{2})[\ell^2/(mr^2)]$$

  – Easy to show that $f'(r) = -(\partial V'/\partial r)$
  – Can clearly write $E = (\tfrac{1}{2})m\dot{r}^2 + V'(r) = \text{const}$

Comments on Centrifugal "Force" & Potential:

- Consider: $\mathbf{E} = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r)$
- *Physics* of $[\ell^2/(2mr^2)]$. Conservation of angular momentum: $\ell = mr^2\dot{\theta} \;\Rightarrow\; [\ell^2/(2mr^2)] \equiv (\tfrac{1}{2})mr^2\dot{\theta}^2$

  $\equiv$ *Angular part of kinetic energy* of mass **m**.
- Because of the form $[\ell^2/(2mr^2)]$, this contribution to the energy depends only on **r**: ***When analyzing the* r *part of the motion***, can treat this as **an additional part of the *potential energy.***

$\Rightarrow$ It's often convenient to call it another potential energy term $\equiv$ ***"Centrifugal" Potential Energy***

- $[\ell^2/(2mr^2)] \equiv$ *"Centrifugal" PE* $\equiv V_c(r)$
  - As just discussed, this is really the angular part of the ***Kinetic Energy***!

$\Rightarrow$ "Force" associated with $V_c(r)$:

$$f_c(r) \equiv -(\partial V_c/\partial r) = [\ell^2/(mr^3)]$$

Or, using $\ell = mr^2\dot{\theta}$:

$$f_c(r) = [\ell^2/(mr^3)] = mr\theta^2 \equiv m(v_\theta)^2/r$$

$$\equiv$$ *"Centrifugal Force"*

- $\mathbf{f_c(r) = [\ell^2/(mr^3)]}$ ≡ ***"Centrifugal Force"***

- $\mathbf{f_c(r)}$ = ***Fictitious*** "force" arising due to fact that the reference frame of the relative coordinate **r** (of "particle" of mass **m**) is ***not* an inertial frame!**

  – ***NOT (!!)*** a force in the Newtonian sense! A part of the "**ma**" of Newton's 2^{nd} Law, rewritten to appear on the "**F**" side.

Direction of $\mathbf{f_c}$: **Outward from the force center!**

- Particle moving in a circular arc: Force ***in an Inertial Frame*** is directed **INWARD TOWARDS THE CIRCLE CENTER**

  ≡ *Centripetal Force*

Planetary Motion

- General result for *Orbit* $\theta(r)$ was:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-½}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-½}\, dr + \theta'$$

  - θ' = integration constant

- Put $V(r) = -(k/r)$ into this:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-½}[E + (k/r) - \{\ell^2/(2mr^2)\}]^{-½}\, dr + \theta'$$

- Integrate by first changing variables: Let $u \equiv (1/r)$:

$$\theta(u) = \ell(2m)^{-½}\int du\, [E + k\,u - \{\ell^2/(2m)\}u^2]^{-½} + \theta'$$

- Tabulated. Result is: $(r = 1/u)$

$$\theta(r) = \cos^{-1}[G(r)] + \theta'$$

$$G(r) \equiv [(\alpha/r) - 1]/e\,;\ \alpha \equiv [\ell^2/(mk)]$$

$$e \equiv [\,1 + \{2E\ell^2/(mk^2)\}]^{½}$$

- **Orbit for inverse square law force:**

$$\cos(\theta - \theta') = [(\alpha/r) - 1]/e \qquad (1)$$

$$\alpha \equiv [\ell^2/(mk)]; \quad e \equiv [\ 1 + \{2E\ell^2/(mk^2)\}]^{1/2}$$

- Rewrite **(1)** as:

$$(\alpha/r) = 1 + e\cos(\theta - \theta') \qquad (2)$$

- (2) ≡ <u>*CONIC SECTION*</u> (analytic geometry!)
- Orbit properties:

$$e \equiv \textit{Eccentricity}$$

$$2\alpha \equiv \textit{Latus Rectum}$$

Conic Sections

$\Rightarrow$ A very important result!

***All** orbits for inverse r-squared forces*

(attractive or repulsive) *are conic sections*

$$(\alpha/r) = 1 + e\cos(\theta - \theta^{\prime})$$

with

$$\text{Eccentricity} \equiv e = [\,1 + \{2E\ell^2/(mk^2)\}]^{1/2}$$

and

$$\text{Latus Rectum} \equiv 2\alpha = [2\ell^2/(mk)]$$

Conic Sections

- **Conic sections:** Curves formed by the intersection of a plane and a cone.
- **A conic section:** A curve formed by the loci of points (in a plane) where the ratio of the distance from a fixed point (*the focus*) to a fixed line (*the directorix*) is a constant.
- **Conic Section**

$$(\alpha/r) = 1 + e\cos(\theta - \theta')$$

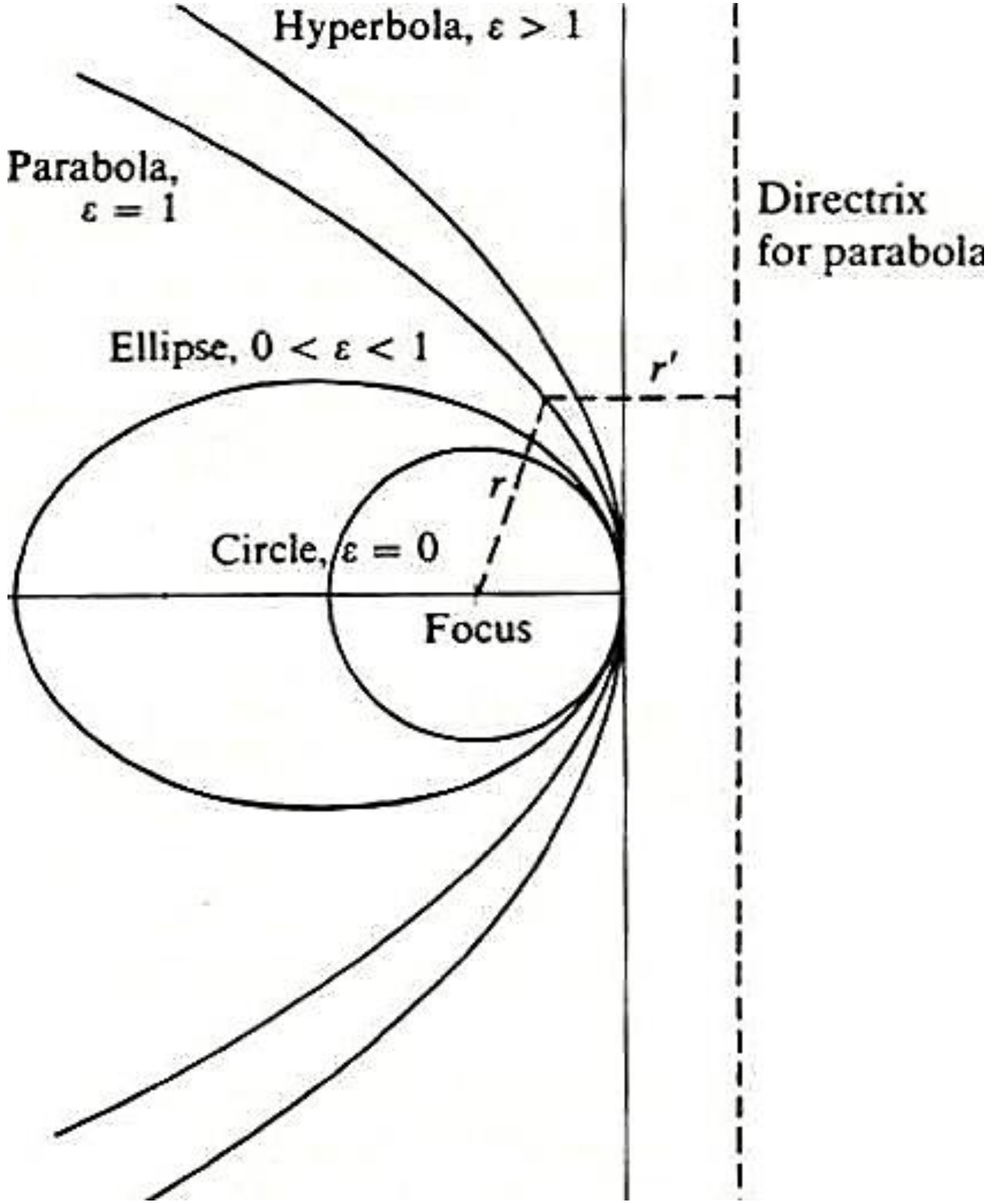
- The specific type of curve depends on eccentricity **e.** For objects in orbit, this, in turn, depends on the energy **E** and the angular momentum ℓ.

› Conic Section

$(\alpha/r) = 1 + e\cos(\theta - \theta')$

- Type of curve depends on eccentricity **e.**

  In Figure, **ε ≡ e**



Conic Section Orbits

› In the following discussion, we need 2 properties of the effective (1d, r-dependent) potential, which (as we've seen) **governs the orbit behavior** for a fixed energy E & angular momentum ℓ. For $V(r) = -(k/r)$ this is:

$$V'(r) = -(k/r) + [\ell^2/\{2m(r)^2\}]$$

1. It is easily shown that **the $r = r_0$ where $V'(r)$ has a minimum is:** $r_0 = [\ell^2/(2mk)]$. (We've seen in our general discussion that this is the radius of a circular orbit.)

2. Its also easily shown that **the value of V' at r_0 is:**
$$V'(r_0) = -(mk^2)/(2\ell^2) \equiv (V')_{min} \equiv E_{circular}$$

- We've shown that all orbits for inverse **r**-squared forces (attractive or repulsive) are **conic sections**

$$(\alpha/r) = 1 + e\cos(\theta - \theta')$$

  - As we just saw, ***the shape*** of curve (orbit) depends on the eccentricity $\mathbf{e} \equiv [\ 1 + \{2E\ell^2/(mk^2)\}]^{1/2}$
  - Clearly this depends on energy E, & angular momentum ℓ!
  - Note: $(V')_{min} \equiv -(mk^2)/(2\ell^2)$

$e > 1 \Rightarrow E > 0 \quad \Rightarrow$ ***Hyperbola***

$e = 1 \Rightarrow E = 0 \quad \Rightarrow$ ***Parabola***

$0 < e < 1 \Rightarrow (V')_{min} < E < 0 \quad \Rightarrow$ ***Ellipse***

$e = 0 \Rightarrow E = (V')_{min} \quad \Rightarrow$ ***Circle***

$e =$ *imaginary* $\Rightarrow E < (V')_{min} \quad \Rightarrow$ ***Not Allowed!***

- **Terminology for conic section orbits:**

Integration const $\Rightarrow$ $r = r_{min}$ when $\theta = \theta'$

$r_{min} \equiv$ *Pericenter*; $r_{max} \equiv$ *Apocenter*

Any radial turning point $\equiv$ *Apside*

Orbit about sun: $r_{min} \equiv$ *Perihelion*

$r_{max} \equiv$ *Aphelion*

Orbit about earth: $r_{min} \equiv$ *Perigee*

$r_{max} \equiv$ *Apogee*

- Conic Section: $(\alpha/r) = 1 + e\cos(\theta - \theta')$

$$e \equiv [\,1 + \{2E\ell^2/(mk^2)\}]^{1/2} \qquad \alpha \equiv [\ell^2/(mk)]$$

- $e > 1 \quad \Rightarrow \quad E > 0 \quad \Rightarrow$ *Hyperbola*

Occurs for the *repulsive* Coulomb force: See scattering discussion,

$0 < e < 1 \quad \Rightarrow \quad V_{min} < E < 0 \quad \Rightarrow$ *Ellipse*

($V_{min} \equiv -(mk^2)/(2\ell^2)$) Occurs for the *attractive* Coulomb force & the Gravitational force:

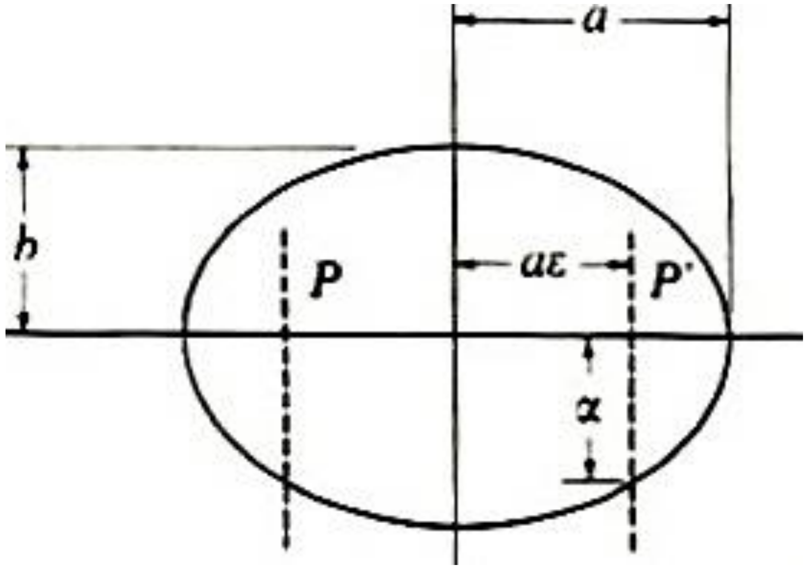
The Orbits of all of the planets (& several other solar system objects) are ellipses with the Sun at one focus. (Again, see table).

Most planets, $e \ll 1$ (see table) $\Rightarrow$ Their orbit is almost circular!

Planetary Orbits

- Planetary orbits in terms of ellipse geometry.

  In the figure, $\varepsilon \equiv e$



- Compute **major & minor axes** (2a & 2b) as in text.

  Get (recall k = GmM):

  $a \equiv (\alpha)/[1 - e^2] = (k)/(2|E|)$

  (depends only on energy E)

  $b \equiv (\alpha)/[1 - e^2]^{½} = (\ell)/(2m|E|)^{½} \equiv a[1 - e^2]^{½} \equiv (\alpha a)^{½}$

  (Depends on both energy E & angular momentum ℓ)

- **Apsidal distances** r_{min} & r_{max} (or r_1 & r_2):

  $r_{min} = a(1- e) = (\alpha)/(1 + e)$, $r_{max} = a(1+ e) = (\alpha)/(1 - e)$

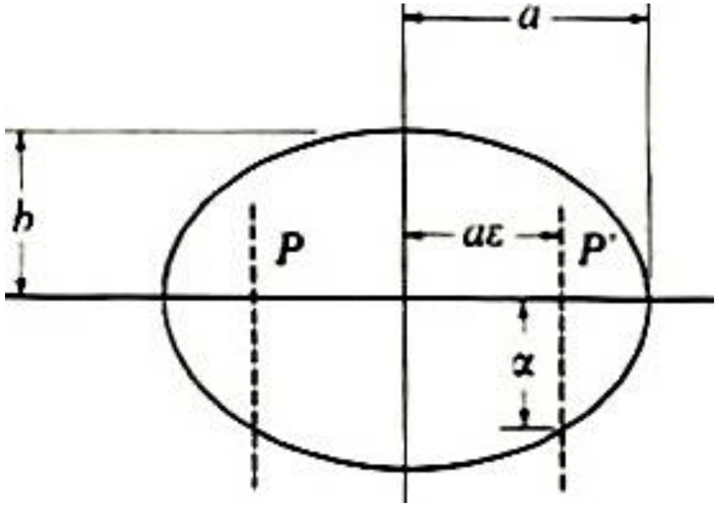
$\Rightarrow$ **Orbit eqtn** is: $r = a(1- e^2)/[1 + e \cos(\theta - \theta^{\prime})]$

- **Planetary orbits = ellipses, sun at one focus:** Fig:
- For a general central force,

  we had **Kepler's 2nd Law:**

  (Constant areal velocity!):

  $(dA/dt) = (\ell)/(2m) = \text{const}$



Use to compute **orbit period:**

$\Rightarrow \; dt = (2m)/(\ell)\, dA$

Period = time to sweep out ellipse area:

$\tau = \int dt = [(2m)/(\ell)] \int dA = [(2m)/(\ell)]A$

- **Period of elliptical orbit:**

$$\tau = [(2m)/(\ell)]\ A \qquad (A = \text{ellipse area}) \qquad (1)$$

- Analytic geometry: Area of ellipse:

$$A \equiv \pi ab \qquad (2)$$

- In terms of k, E & ℓ, we just had:

$$a = (k)/(2|E|);\ b = (\ell)/(2mE)^{½} \qquad (3)$$

$$(1), (2), (3) \Rightarrow \tau = \pi k(m/2)^{½}|E|^{-(3/2)}$$

- Alternatively: $b = (\alpha a)^{½}\ ;\ \alpha \equiv [\ell^2/(mk)]$

$$\Rightarrow \qquad \tau^2 = [(4\pi^2 m)/(k)]\ a^3$$

The square of the period is proportional to cube of semimajor axis of the elliptic orbit

$\equiv$ ***Kepler's Third Law***

› *Kepler's Third Law*

$\Rightarrow \quad \tau^2 = [(4\pi^2 m)/(k)]\ a^3$

The square of period is proportional to the cube of the semimajor axis of the elliptic orbit

› *Note:* Actually, $m \to \mu$. The reduced mass μ actually enters! As derived empirically by Kepler: Kepler's 3rd Law states that this is true with the same proportionality constant for all planets. This ignores the difference between the reduced mass μ & the mass m of the planet: $\mu = (m)[1 + mM^{-1}]^{-1}$

$$\mu \cong m[1 - (m/M) + (m/M)^2 - \dots\ ]$$

Note: $k = GmM$; $\mu \cong m \quad (m \ll M)$

$\Rightarrow \quad (\mu/k) \cong 1/(GM)$

$\Rightarrow \quad \tau^2 = [(4\pi^2)/(GM)]a^3 \quad (m \ll M)$

So Kepler was only approximately correct!

Kepler's Laws

- *Kepler's First Law:*

The planets move in elliptic orbits with the Sun at one focus.

  - Kepler proved empirically. Newton proved this from Universal Law of Gravitation & calculus.

- *Kepler's Second Law:*

The area per unit time swept out by a radius vector from sun to a planet is constant. (Constant areal velocity).

$$(dA/dt) = (\ell)/(2m) = \text{constant}$$

  - Kepler proved empirically. We've proven in general for any central force.

- *Kepler's Third Law:* $\tau^2 = [(4\pi^2 m)/(k)]\ a^3$

The square of a planet's period is proportional to cube of semimajor axis of the planet's elliptic orbit.

Example

- Halley's Comet, which passed around the sun early in 1986, moves in a highly elliptical orbit: Eccentricity **$e = 0.967$**; period **$\tau = 76$ years.** Calculate its minimum and maximum distances from the sun.

- Use the formulas just derived & find:

$$r_{min} = 8.8 \times 10^{10}\ \mathrm{m}$$

(Inside Venus's orbit & almost to Mercury's orbit)

$$r_{max} = 5.27 \times 10^{12}\ \mathrm{m}$$

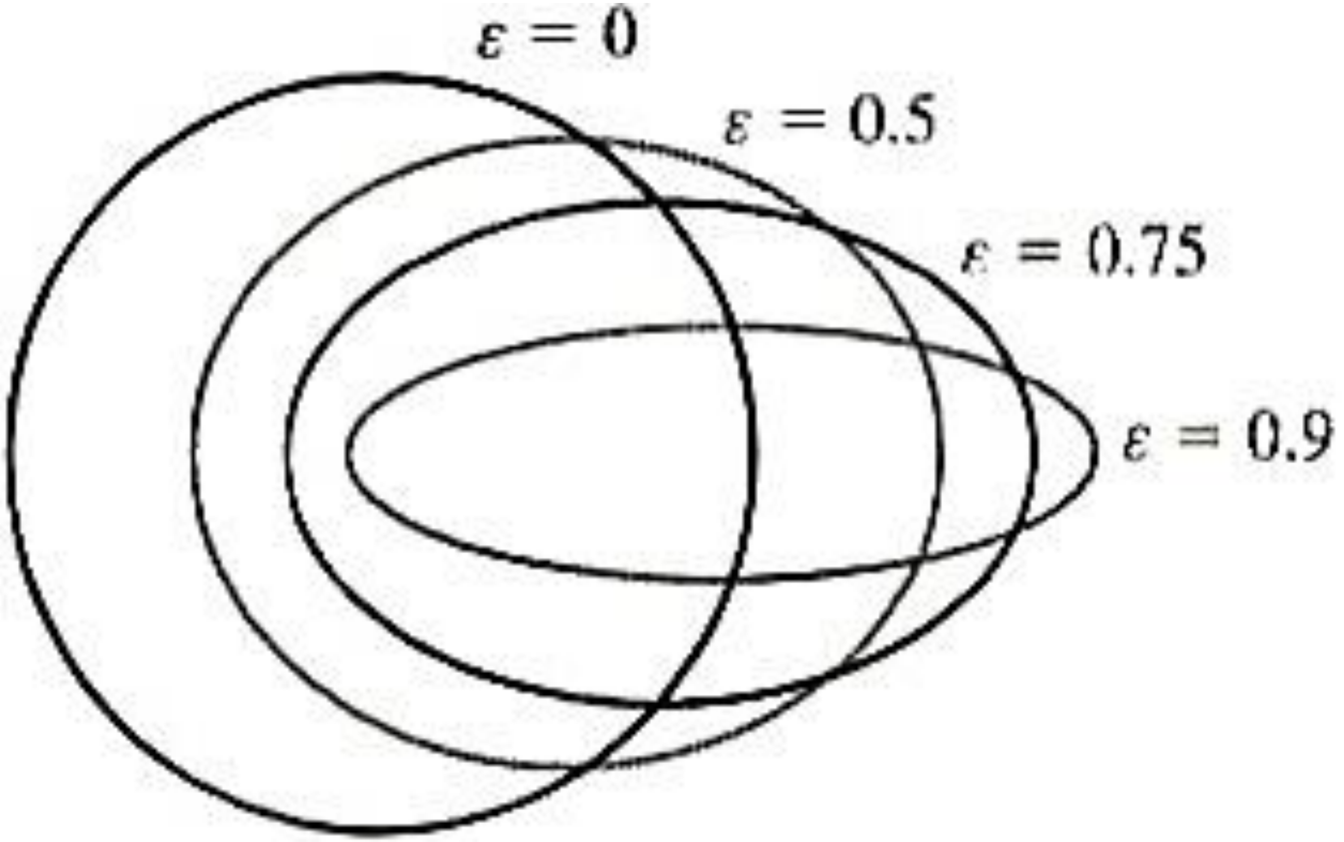
(Outside Neptune's orbit & near to Pluto's orbit)

› **Elliptical orbits:** Same semimajor axis $\mathbf{a} = (k)/(2|E|)$

$\Rightarrow$ Same energy **E** & mass **m**, different eccentricities

$$e = [1+\{2E\ell^2/(mk^2)\}]^{1/2} \quad \text{(\& semiminor axes)}$$

$$b = (\ell)/(2m|E|)^{1/2} \quad \Rightarrow \text{ Different angular momenta } \ell$$



- **Orbit properties:** r_1, r_2 ≡ apsidal distances, p_r, p_θ ≡ angular momenta, θ_1, θ_2 ≡ angular velocities at the apsidal distances, with respect to circular orbit, radius **a**. In Table, $\varepsilon \equiv e$

TABLE 3.1 Ellipse Properties

Ellipticity	$\frac{p_\theta}{l_0}$	$\frac{p_r a}{l_0}$ for $r = a$	$\frac{r_1}{a}$	$\frac{r_1}{a}$	$\frac{\dot\theta_1}{\dot\theta_0}$	$\frac{\dot\theta_2}{\dot\theta_0}$	$\frac{v_{\theta_1}}{v_0}$	$\frac{v_{\theta_2}}{v_0}$
ε	$\sqrt{1-\varepsilon^2}$	ε	$1-\varepsilon$	$1+\varepsilon$	$\sqrt{\frac{1-\varepsilon}{(1+\varepsilon)^3}}$	$\sqrt{\frac{1+\varepsilon}{(1-\varepsilon)^3}}$	$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{3/2}$	$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{3/2}$
0	1	0	1	1	1	1	1	1
0.1	0.995	0.1	0.9	1.1	0.822	1.23	0.740	1.35
0.25	0.968	0.25	0.75	1.25	0.620	1.72	0.465	2.15
0.5	0.867	0.5	0.5	1.5	0.384	3.46	0.192	5.20
0.75	0.661	0.75	0.25	1.75	0.216	10.58	0.054	18.5
0.9	0.435	0.9	0.1	1.9	0.121	43.6	0.012	82.8

- Velocity along particle path ≡ $\vec{v} = v_r\hat{r} + v_\theta\hat{\theta}$

$$v_r \equiv (p_r/m) = \dot{r},\ v_\theta \equiv r\dot{\theta} = [p_\theta/(mr)]$$

› Orbit phase space properties:

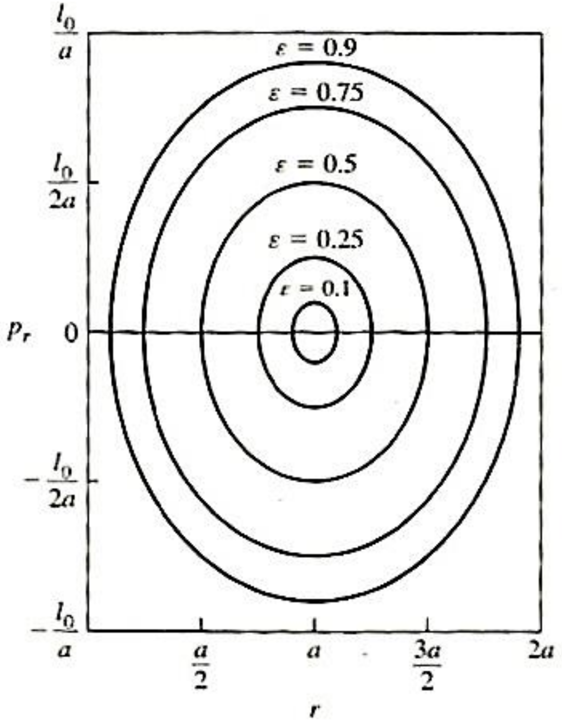


FIGURE 3.15 Phase-space plot for three ellipses in rp_r space.

p_r *vs.* r

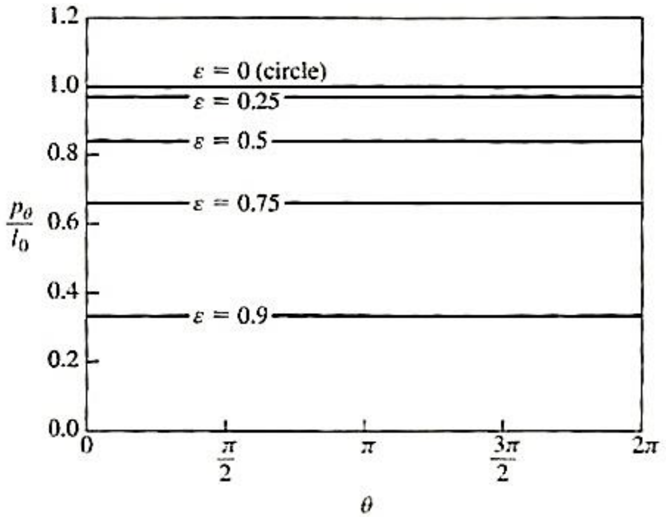


FIGURE 3.16 Phase-space plot for three ellipses in θp_θ space.

p_θ *vs.* θ

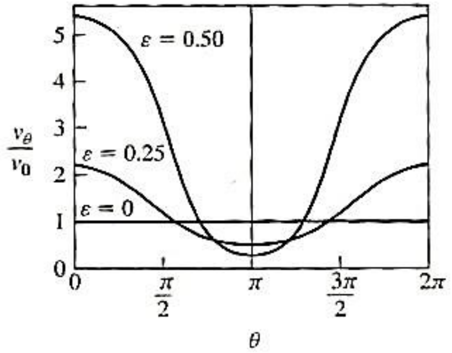


FIGURE 3.17 Velocity versus angle plot for three ellipses.

v_θ *vs.* θ

The Virial Theorem

- Skim discussion. Read details on your own!
- Many particle system. Positions $\vec{\mathbf{r}}_i$, momenta $\vec{\mathbf{p}}_i$. Bounded. ***Define*** $\mathbf{G} \equiv \sum_i \vec{\mathbf{r}}_i \bullet \vec{\mathbf{p}}_i$
- **Time derivative** of **G**:

$$(dG/dt) = \sum_i(\vec{\mathbf{r}}_i \bullet \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_i \bullet \vec{\mathbf{p}}_i) \qquad (1)$$

- **Time average: (dG/dt)** in interval τ:

$$\langle(dG/dt)\rangle \equiv \tau^{-1}\int(dG/dt)dt \qquad (2)$$

(limits $0 < t < \tau$)

$$\langle(dG/dt)\rangle = [G(\tau) - G(0)]/\tau \qquad (3)$$

- **Periodic motion** $\Rightarrow G(\tau) = G(0)$:

$$(3) \Rightarrow \langle(dG/dt)\rangle = 0$$

$$\langle (dG/dt) \rangle = [G(\tau) - G(0)]/\tau \qquad (3)$$

- If motion ***isn't periodic,*** still make $\langle (dG/dt) \rangle = \langle \dot{G} \rangle$ as small as we want if τ is very large. $\Rightarrow$ For a periodic system or for a non-periodic system with large τ can (in principle) **make** $\langle \dot{G} \rangle = 0$
- When $\langle \dot{G} \rangle = 0$, (long time average) **(1) & (2)** combine:

$$\langle \textstyle\sum_i (\dot{\vec{p}}_i \bullet \vec{r}_i) \rangle = - \langle \textstyle\sum_i (\vec{p}_i \bullet \dot{\vec{r}}_i) \rangle \qquad (4)$$

- Left side of **(4)**: $\vec{p}_i \bullet \dot{\vec{r}}_i = 2T_i$

or $\langle \sum_i (\vec{p}_i \bullet \dot{\vec{r}}_i) \rangle = \langle 2\sum_i T_i \rangle = 2\langle T \rangle$ **(5)**

T_i = KE of particle **i; T** = total KE of system

- **Newton's 2nd Law:** $\Rightarrow \dot{\vec{p}}_i = \vec{F}_i$ = force on particle **i**

$\Rightarrow$ Right side of **(4)** : $\langle \sum_i (\dot{\vec{p}}_i \bullet \vec{r}_i) \rangle = \langle \sum_i (\vec{F}_i \bullet \vec{r}_i) \rangle$ **(6)**

Combine **(5) & (6):**

$$\Rightarrow \quad \langle T\rangle = -(\tfrac{1}{2})\langle \textstyle\sum_i(\vec{F}_i \bullet \vec{r}_i)\rangle \qquad (7)$$

$$-(\tfrac{1}{2})\langle \textstyle\sum_i(\vec{F}_i \bullet \vec{r}_i)\rangle \equiv$$ ***The Virial** (of Clausius)*

$\equiv$ ***The Virial Theorem:***

The time average kinetic energy of a system is equal to its virial.

- Application to Stat Mech (ideal gas):

$\langle T\rangle = -(\tfrac{1}{2})\langle\sum_i(\vec{F}_i \bullet \vec{r}_i)\rangle \equiv$ ***The Virial Theorem:***

- **Application to classical dynamics:**
- For a conservative system in which a PE can be defined: $F_i \equiv -\nabla V_i \;\Rightarrow\; \langle T\rangle = (\tfrac{1}{2})\langle\sum_i(\nabla V_i \bullet r_i)\rangle$
- Special case: ***Central Force***, which (for each particle **i**):

$|\mathbf{F}| \propto \mathbf{r^n}$, **n** *any power* (**r** = distance between particles) $\Rightarrow$

$$\mathbf{V = kr^{n+1}}$$

$$\Rightarrow \qquad \nabla \mathbf{V \bullet r = (dV/dr)r = k(n+1)r^{n+1}}$$

or:

$$\nabla \mathbf{V \bullet r = (n+1)V}$$

$\Rightarrow$ **Virial Theorem gives:**

$$\langle \mathbf{T}\rangle = (\tfrac{1}{2})(\mathbf{n+1})\langle \mathbf{V}\rangle \qquad \mathbf{(8)}$$

(Central forces ONLY!)

- ***Virial Theorem, Central Forces:***

$$(F(r) = kr^n \;,\; V(r) = kr^{n+1})$$

$$\langle T\rangle = (\tfrac{1}{2})(n+1)\langle V\rangle \qquad (8)$$

- **Case 1:** Gravitational (or electrostatic!) Potential:

$$n = -2 \quad \Rightarrow \quad \langle T\rangle = -(\tfrac{1}{2})\langle V\rangle$$

- **Case 2:** Isotropic Simple Harmonic Oscillator Potential:

$$n = +1 \quad \Rightarrow \quad \langle T\rangle = \langle V\rangle$$

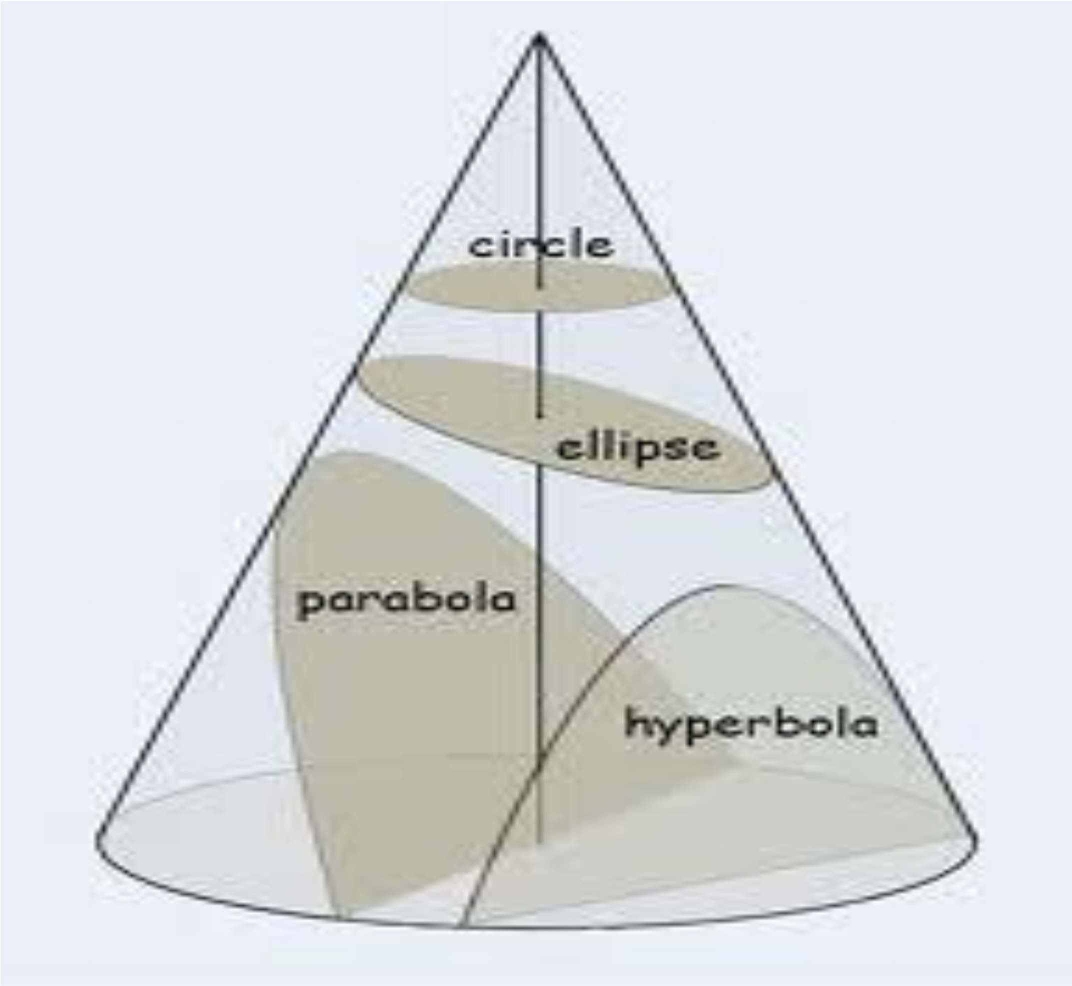
- **Case 3:** $\quad n = -1 \quad \Rightarrow \quad \langle T\rangle = 0$
- **Case 4:** **n** $\neq$ integer (real power **x**):

$$n = x \quad \Rightarrow \quad \langle T\rangle = (\tfrac{1}{2})(x+1)\langle V\rangle$$

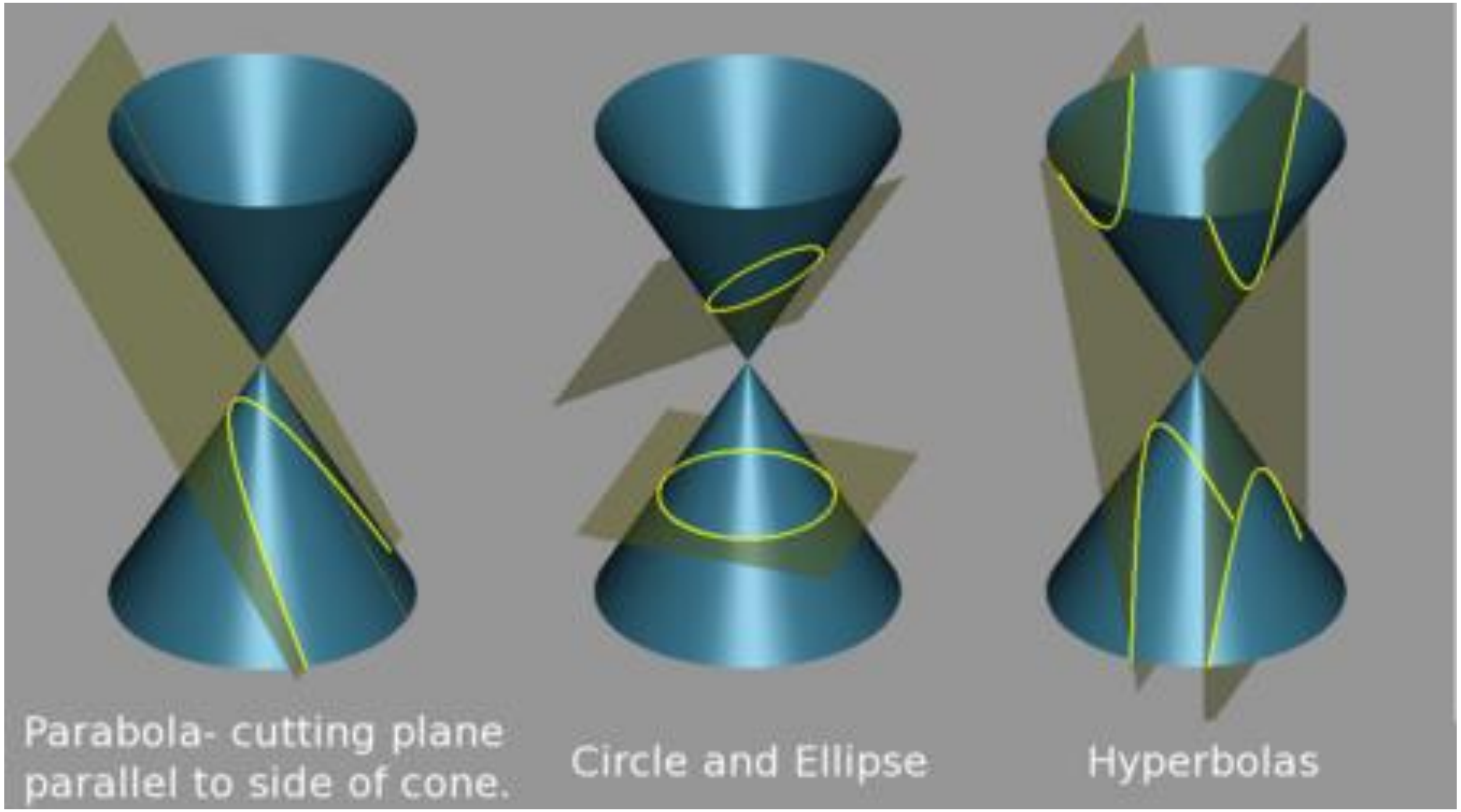
Thank You

CONIC SECTIONS

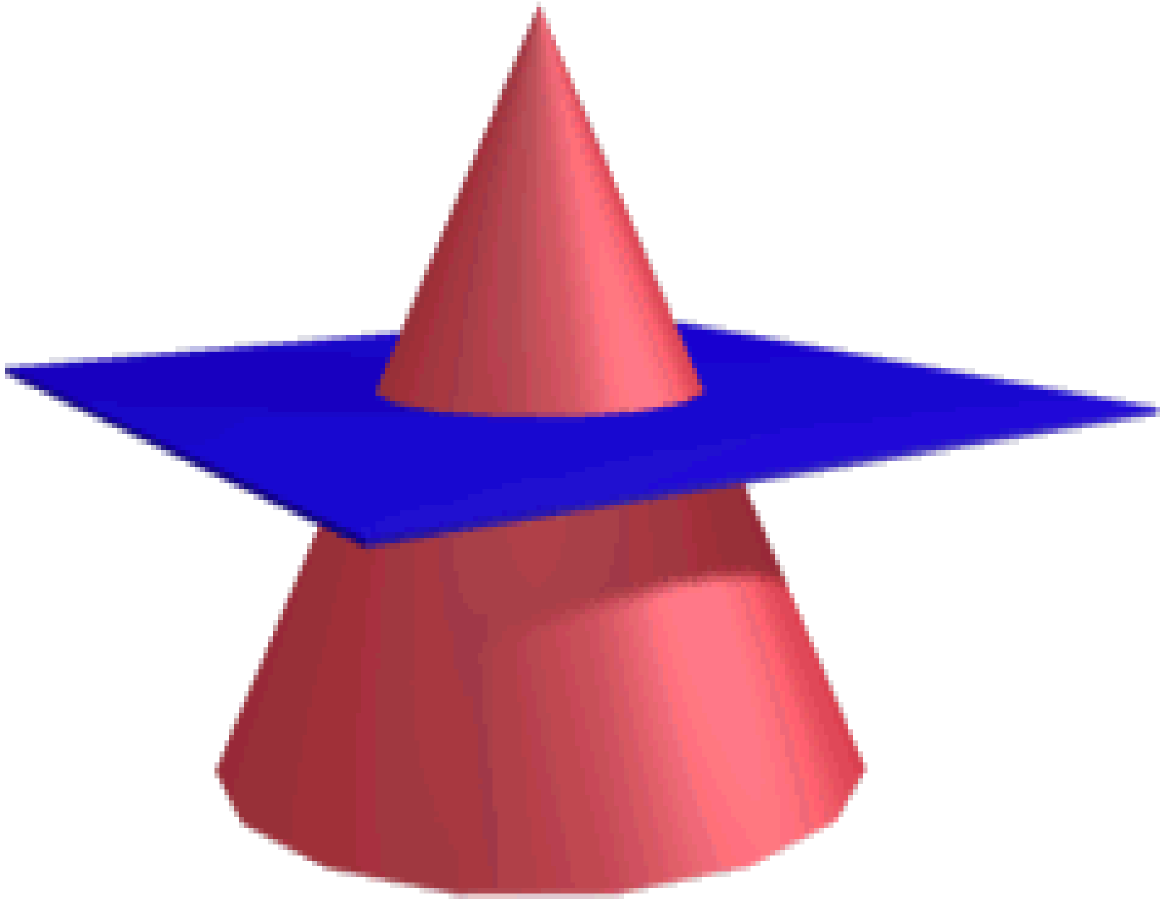
Conic Section: Any figure that can be formed by slicing a double cone with a plane



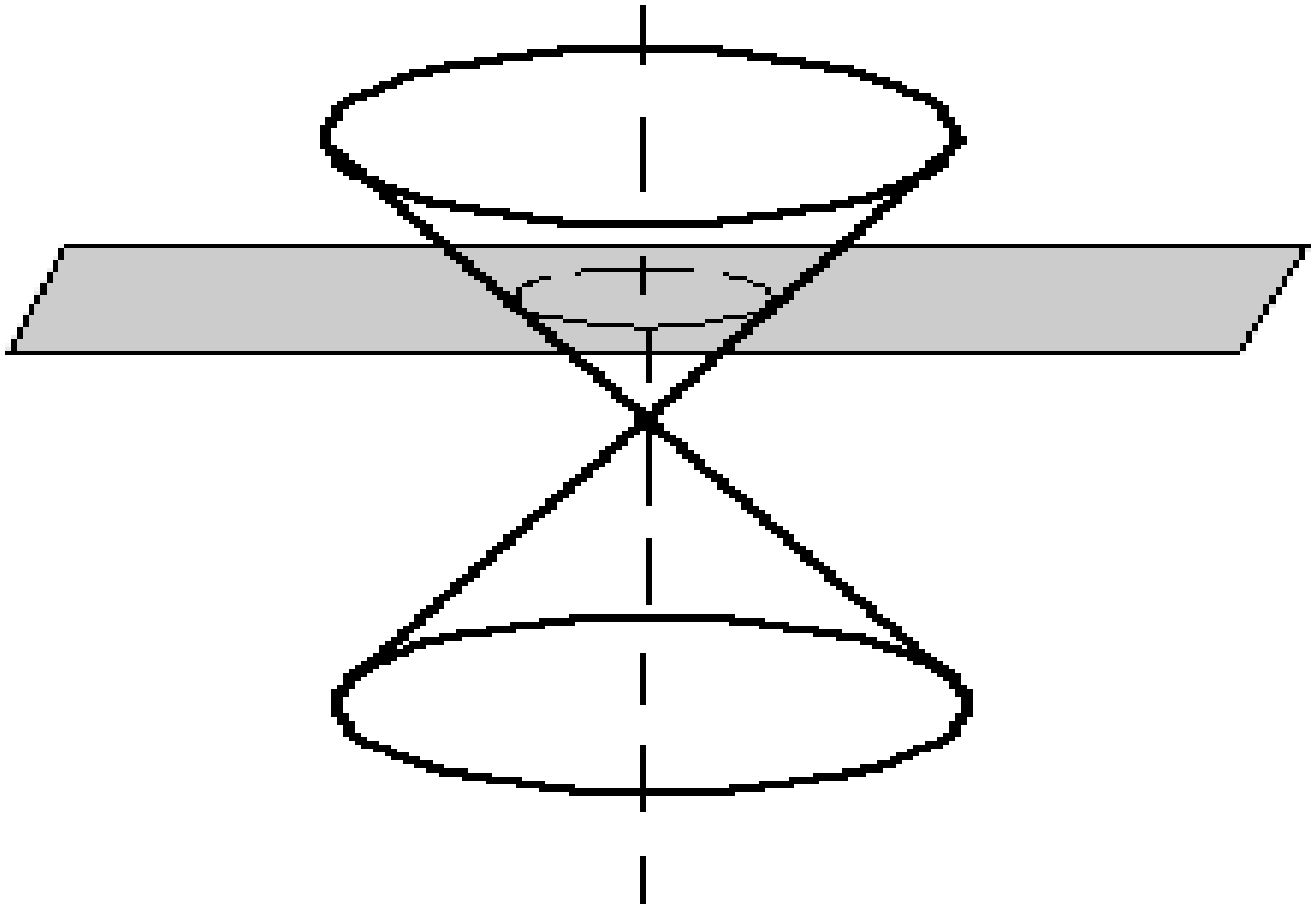
OTHER VIEW OF CONIC SECTIONS



THE CIRCLE



CONIC SECTION – THE CIRCLE



Equation for a Circle

- Standard Form: $x^2 + y^2 = r^2$
- You can determine the equation for a circle by using the distance formula then applying the standard form equation.
- Or you can use the standard form.
- Most of the time we will assume the center is (0,0). If it is otherwise, it will be stated.
- It might look like: $(x-h)^2 + (y - k)^2 = r^2$

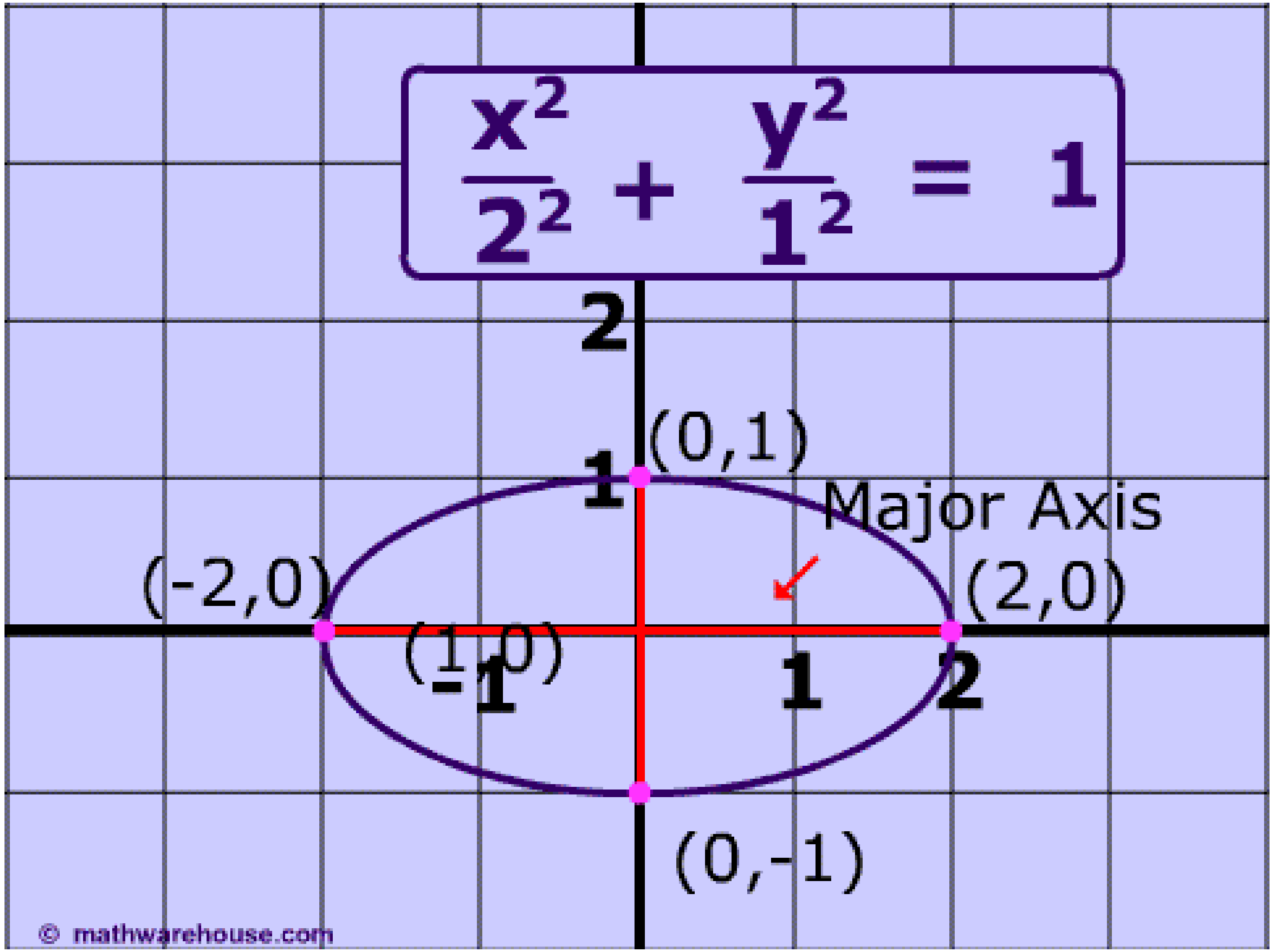
ELLIPSES

- Ellipse – A set of points in A plane such that the sum of the distance from two foci to any point on the ellipse is constant
- focus (foci - plural) – one of two fixed points within in an ellipse such that the sum of the distances from the points to any other point on the ellipse is constant

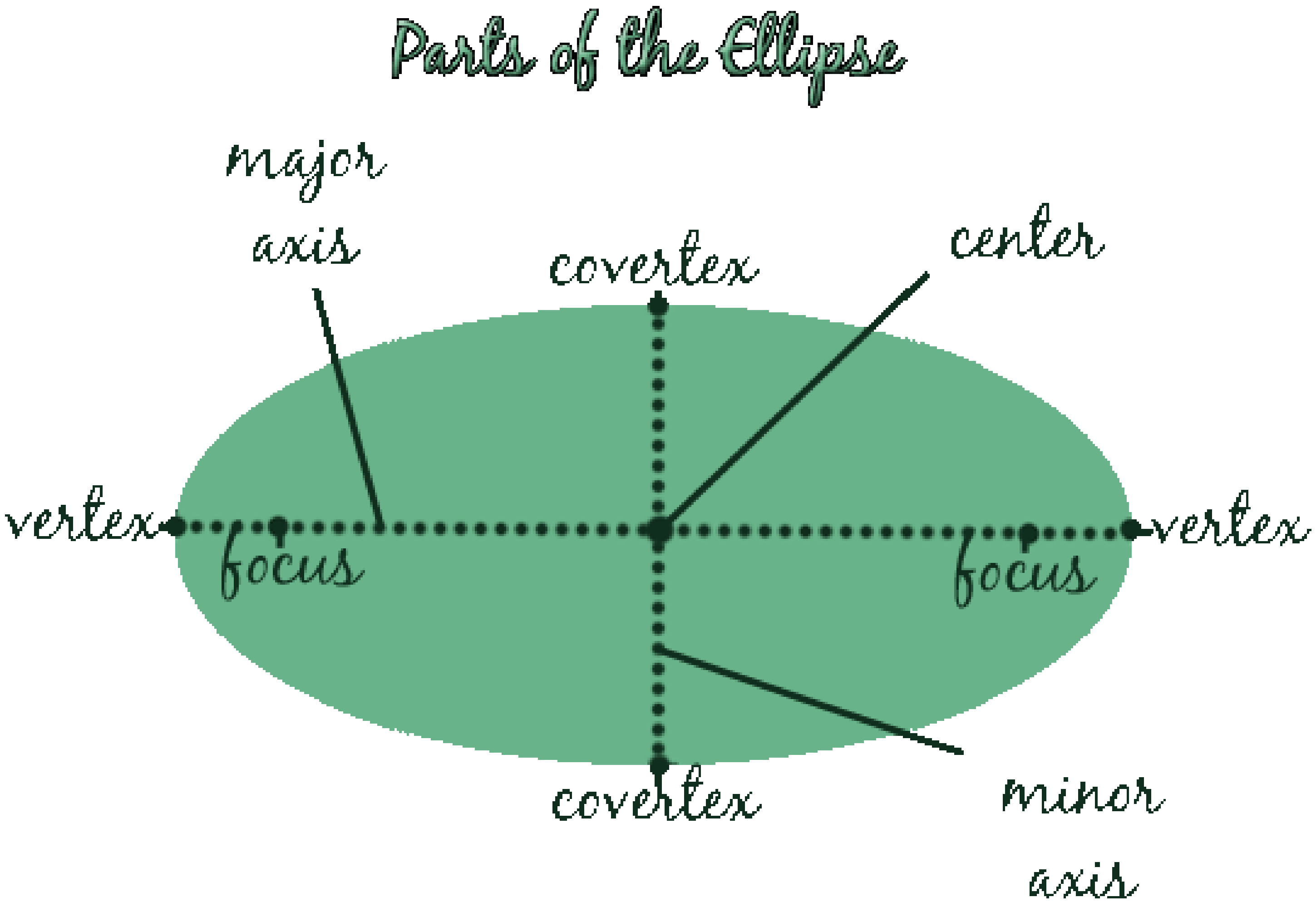
Vocabulary for Ellipses

- Vertices – for an ellipse, the y and x intercepts are the vertices
- Major axis – for an ellipse, the longer axis of symmetry, the axis that contains the foci
- Minor axis – for an ellipse, the shorter axis of symmetry
- Center – for an ellipse, the intersection of the major and minor arcs

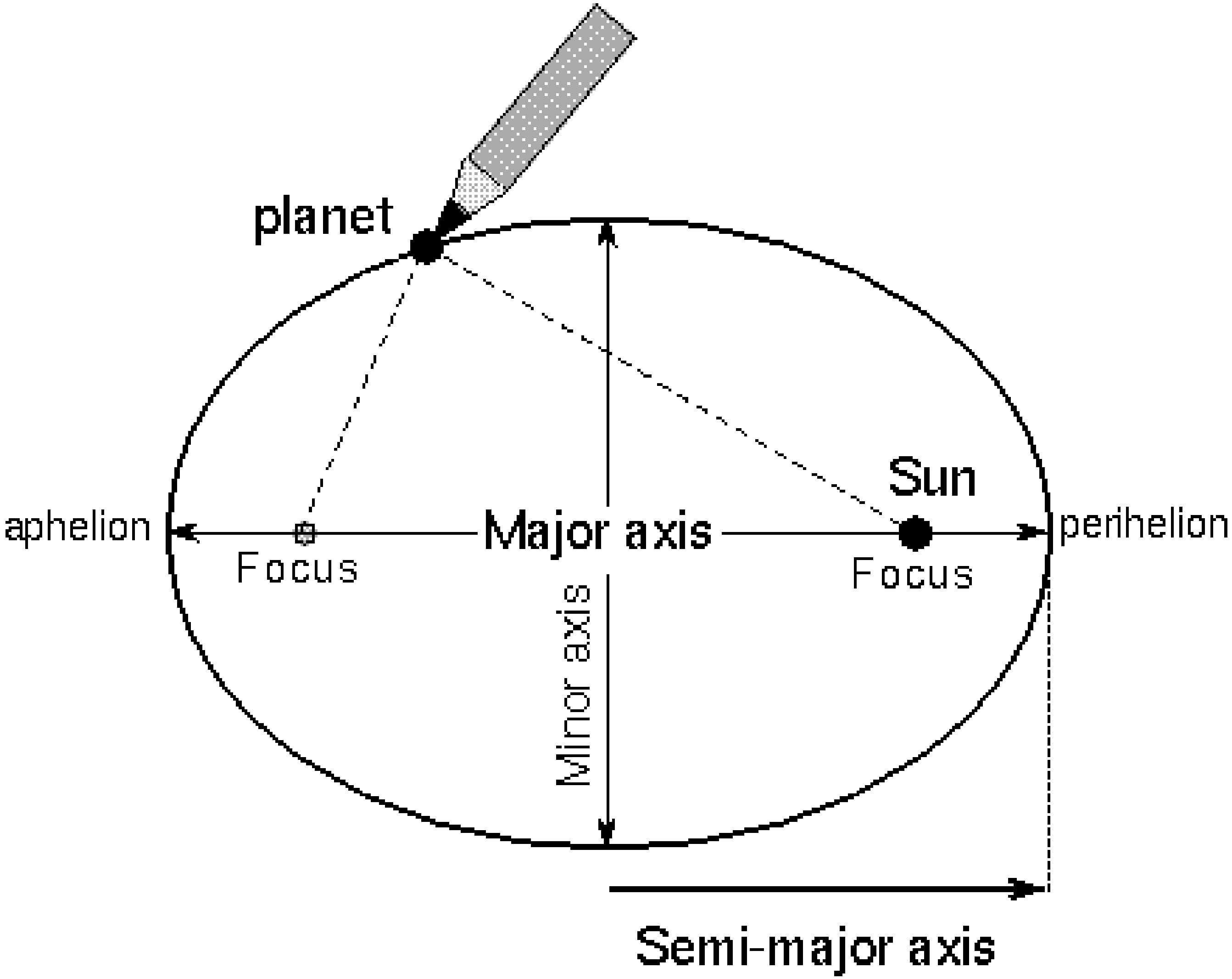
Equation for an Ellipse



Parts of an Ellipse



EXAMPLES



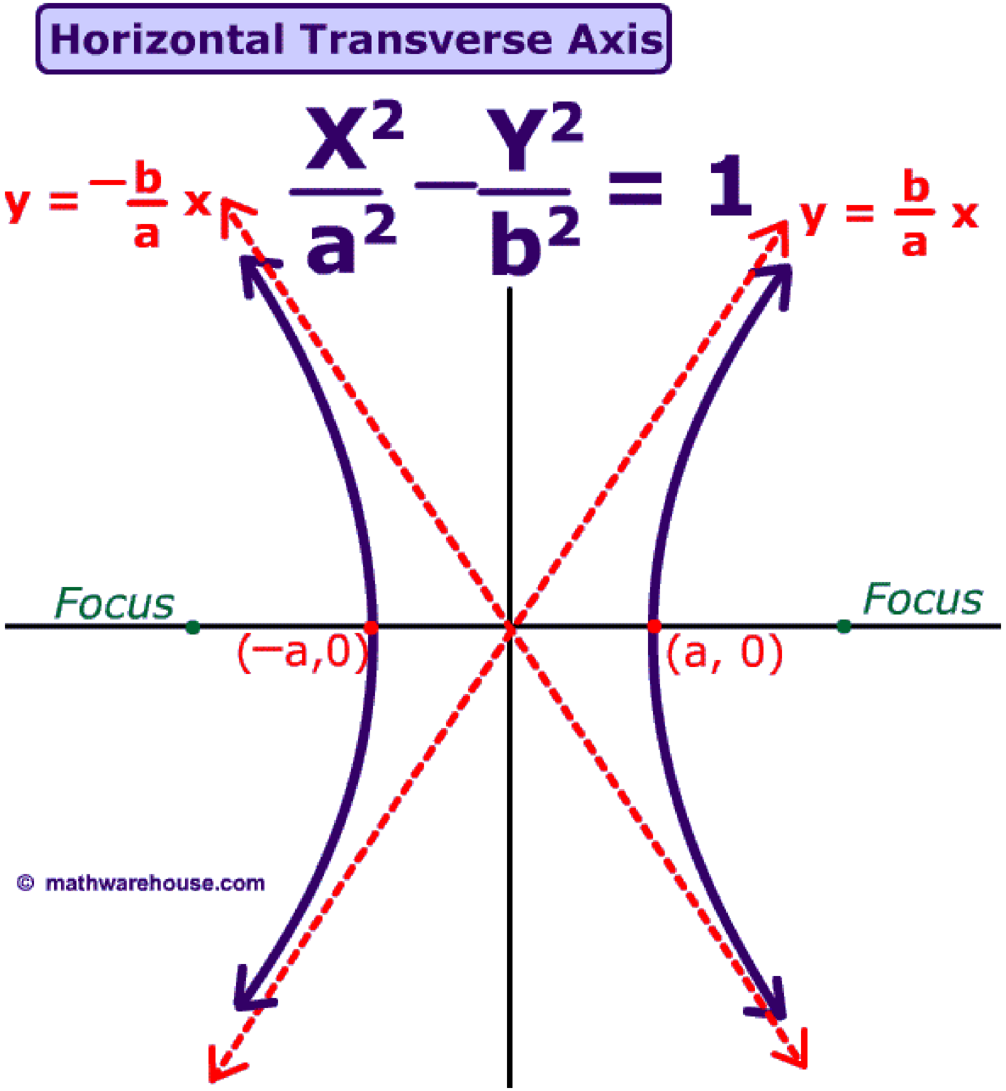
Perihelion – point *closest* to the sun in such an orbit

Aphelion – point *farthest* from the sun in such an orbit

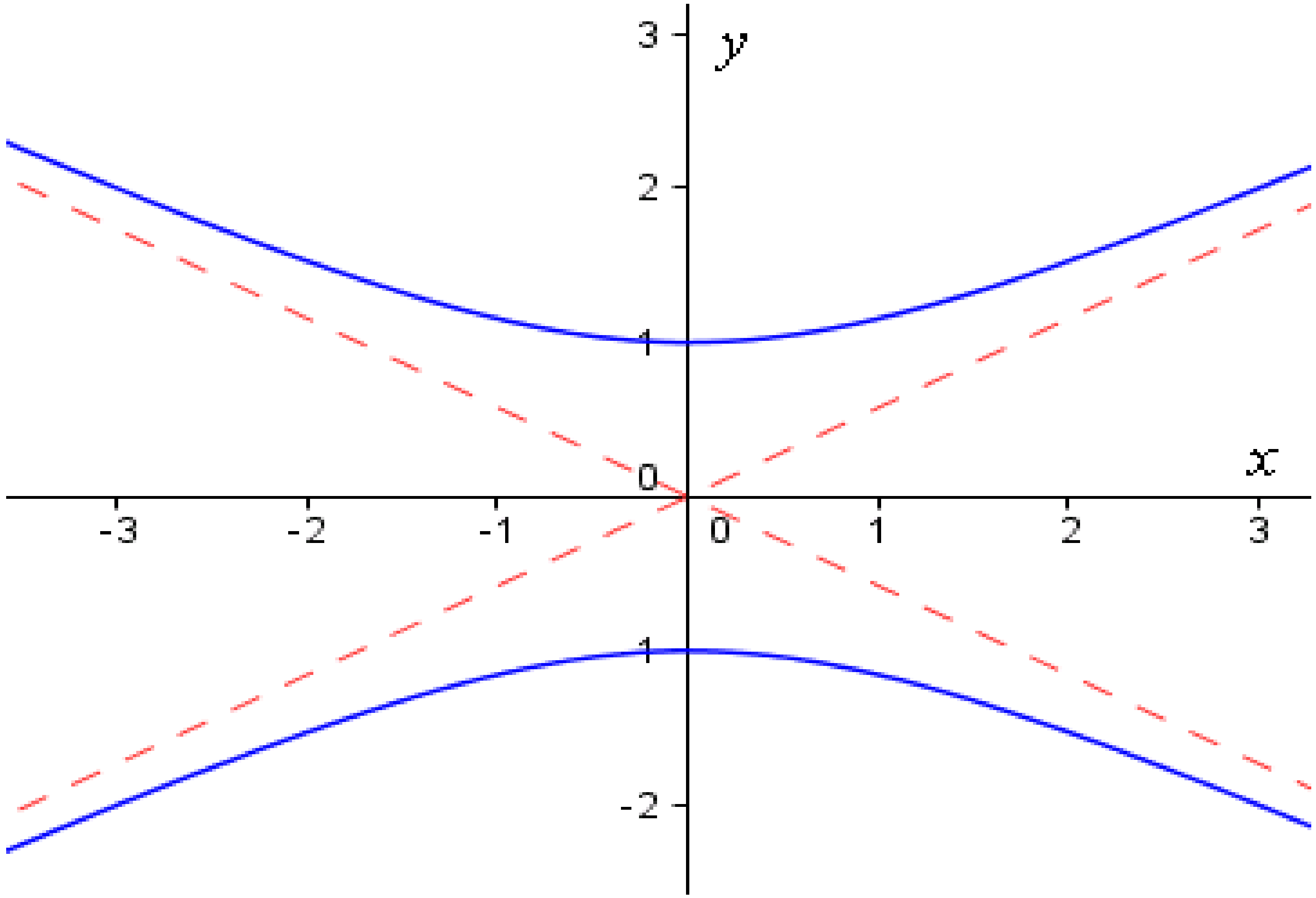
HYPERBOLAS

- Hypoberla – a set of points such that the difference of the distances from two fixed points to any point on the hyperbola is constant
- Vertices – x or y intercepts of a hyperbola
- Asymptote – a straight line that a curve approaches but never reaches

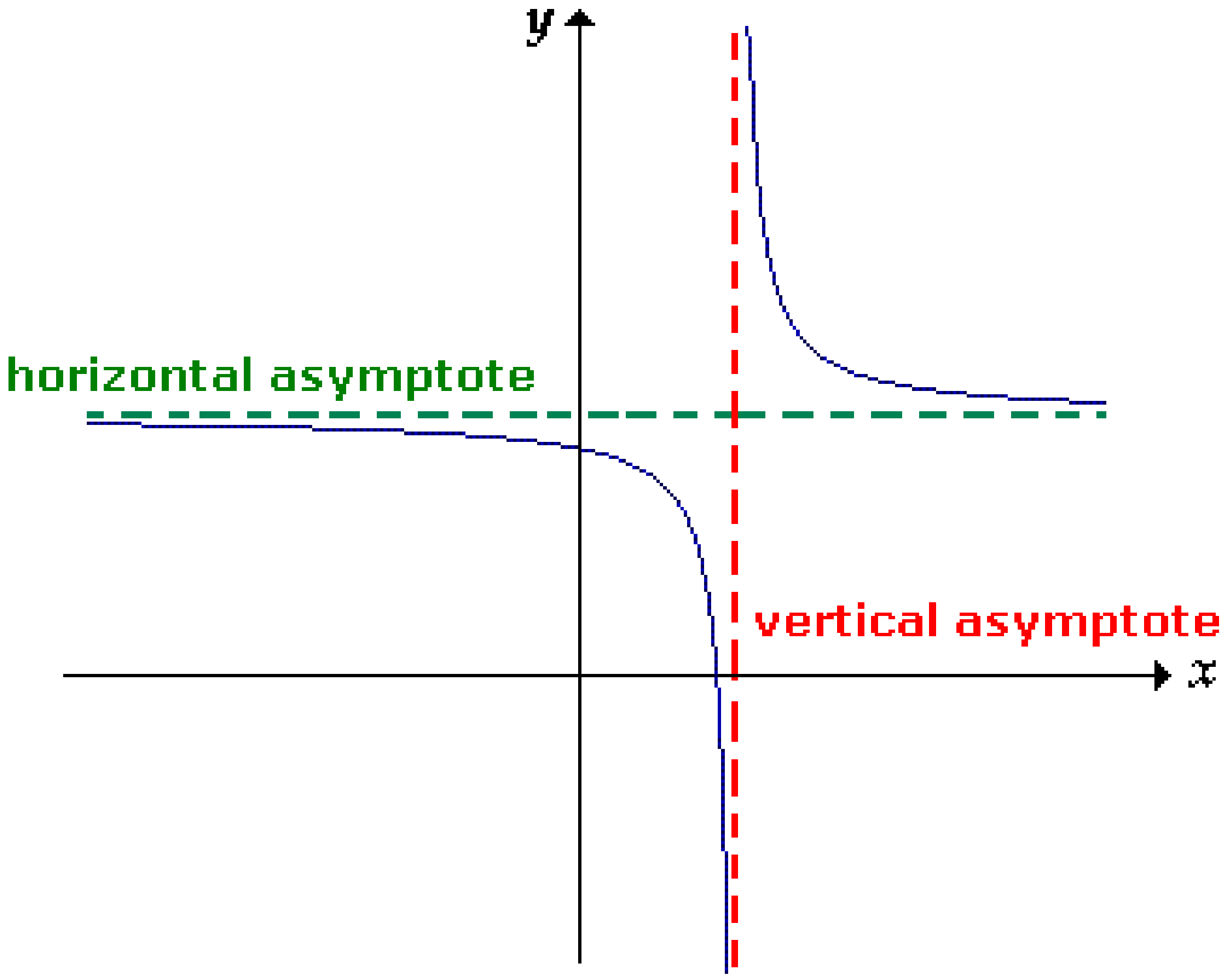
WHAT DOES IT LOOK LIKE?AND WHAT IT ITS FORMULA?



ASYMPTOTES (IN RED)



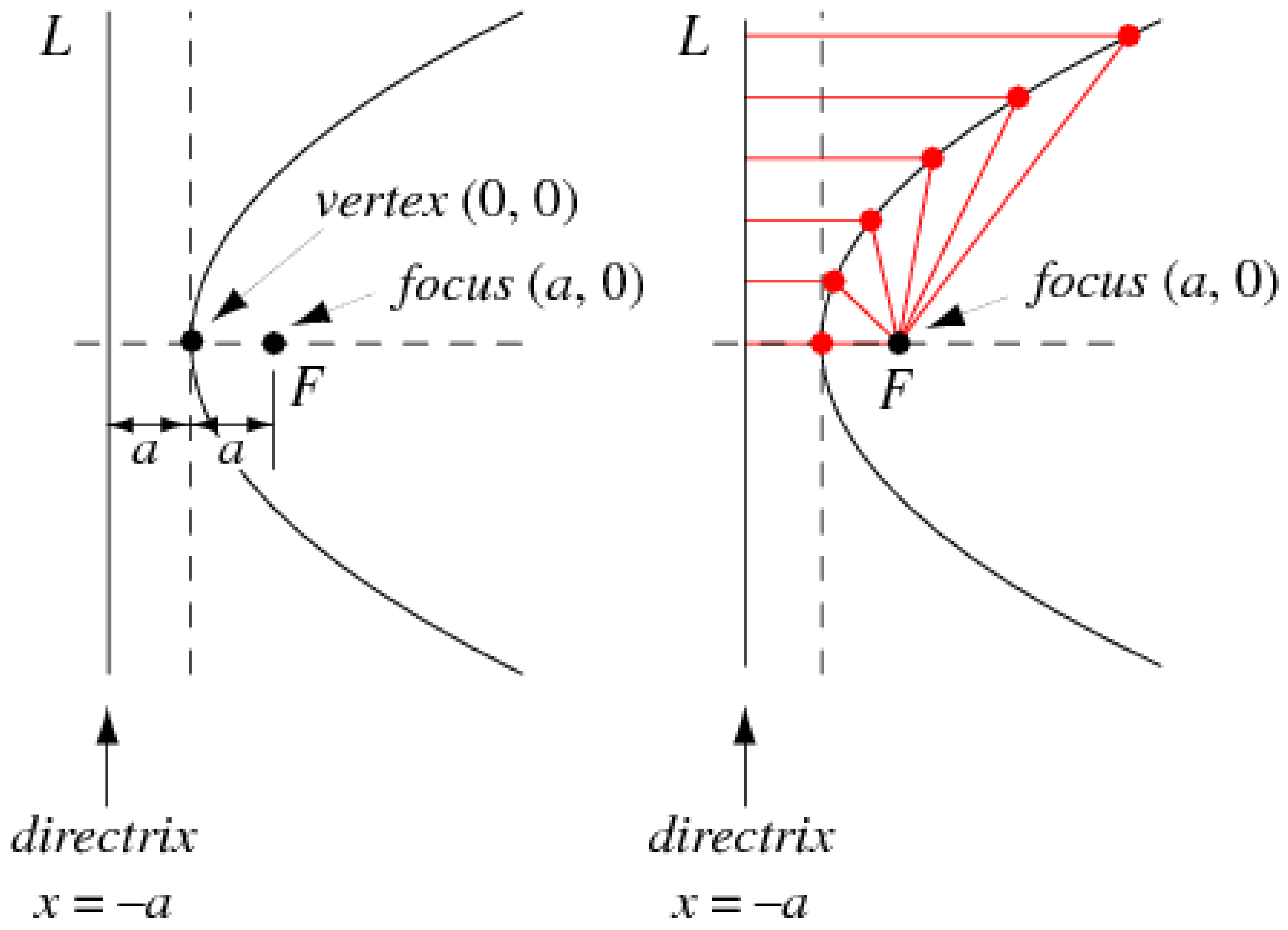
ASYMPTOTES



PARABOLAS

- Parabolas – a set of points in a plane that are equidistant from a focus and a fixed line – the directrix
- Directrix – the fixed straight line that together with the point known as the focus serves to define a parabola.

WHAT DOES IT LOOK LIKE?



ECCENTRICITY

- Eccentricity – a ratio of the distance from the focus and the distance from the directrix.
- Each shape has its own eccentricy: circle, parabolas, hyperbolas, and ellipses.
- Circle: $e = 0$
- Ellipse: $e = 0 < e < 1$
- Parabola: $e = 1$
- Hyperbola: $e > 1$

Definition: Eccentricity of an Ellipse

The **eccentricity** of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

where *a* is the semimajor axis, *b* is the semiminor axis, and *c* is the distance from the center of the ellipse to either focus.

What is the range of possible "*e*" values for an ellipse?

What happens when "*e*" is zero?

$$0 \le e < 1$$

→ A CIRCLE!!!