Volume Integral

Introduction:

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral = $\iiint_V \vec{F}\, dv$

Example 15. *If* $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, *evaluate* $\iiint_V \vec{F}\, dv$ *where, v is the region bounded by the surfaces*

$$x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$$

Solution. $\iiint_V \vec{F}\, dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dz \quad = \int_0^2 dx \int_0^4 dy\, [z^2\,\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\, [4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2 y\hat{k}]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\, dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$ **Ans.**

Ex. 7.8.3 If F = $(2x^2 - 3z)i - 2xyj - 4xk$, evaluate $\iiint_V \nabla . F dv$ where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Sol: $\nabla . F = \frac{\partial}{\partial x}(2x^2 - 3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x) = 4x - 2x = 2x$

$$\therefore \iiint_V \nabla . F dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2xdzdydx = \int_{x=0}^{2}\int_{y=0}^{2-x}[\int_{z=0}^{4-2x-2y} 2xdz]dydx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2xz\Big|_{z=0}^{4-2x-2y} dydx = \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)dydx = \int_{x=0}^{2}[\int_{y=0}^{2-x}(8x-4x^2-4xy)dy]dx$$

$$= \int_{x=0}^{2} 8xy - 4x^2y - 2xy^2\Big|_{y=0}^{2-x} dx \quad = \int_{x=0}^{2}[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2]dx$$

$$= \int_{x=0}^{2}(8x - 8x^2 + 2x^3)dx = (4x^2 - \frac{8x^3}{3} + \frac{2x^4}{4})\Big|_0^2 \quad = 16 - \frac{64}{3} + 8 = \frac{8}{3}$$

Ex. 7.8.4 Evaluate $\iiint_V (\nabla . A)dv$ over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$, where **A** = $4xi - 2y^2j + z^2k$.

Sol: $\nabla . A = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\therefore \iiint_V (\nabla . A)dv = \iiint_V (4 - 4y + 2z)dv = \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}[\int_{z=0}^{3}(4 - 4y + 2z)dz]dydx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}(4z - 4yz + z^2)\Big|_0^3 \; dydx = \int_{x=-2}^{2}[\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}(21 - 12y)dy]dx$$

$$= \int_{x=-2}^{2}(21y - 6y^2)\Big|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx = \int_{x=-2}^{2} 42\sqrt{4-x^2}dx = 84\int_0^2 \sqrt{4-x^2}dx$$

$$= 84[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}(\frac{x}{2})]_0^2 = 84[0 + 2(\frac{\pi}{2}) - 0] = 84\pi$$

Ex. 7.8.5 Evaluate $\iiint_V \phi\, dv$ taken over the rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$ and $\phi = 2(x+y+z)$

Sol: $$\iiint_V \phi\, dv = \iiint_V 2(x+y+z)\, dv = \int_{x=0}^{a}\int_{y=0}^{b}\left[\int_{z=0}^{c} 2(x+y+z)\right] dy dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 2xz + 2yz + z^2\Big|_{z=0}^{c} \; dy dx = \int_{x=0}^{a}[\int_{y=0}^{b}(2cx + 2cy + c^2) dy] dx$$

$$= \int_{x=0}^{a} 2cxy + cy^2 + c^2 y\Big|_0^b \; dx = \int_{x=0}^{a}(2bcx + cb^2 + c^2 b) dx = bcx^2 + (b^2c + bc^2)x\Big|_0^a$$

$$= a^2bc + a(b^2c + bc^2) = abc(a+b+c)$$

Ex. 7.8.6 If $\phi = 4y + 2xz$, evaluate $\iiint_V \phi\, dv$ over the region in the first octant bounded by $x^2 + y^2 = 9$, $z = 0$, $z = 2$.

Sol: $$\iiint_V \phi\, dv = \iiint_V (4y + 2xz)\, dv$$

$$= \int_{x=0}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}}[\int_{z=0}^{z=2}(2xz + 4y) dz] dy dx = \int_0^3\int_0^{\sqrt{9-x^2}}(4yz + xz^2)\Big|_0^2 \; dy dx$$

$$= \int_0^3[\int_0^{\sqrt{9-x^2}}(8y + 4x) dy] dx = \int_0^3 (4y^2 + 4xy)\Big|_0^{\sqrt{9-x^2}} dx$$

$$= \int_0^3 [4(9-x^2) + 4x\sqrt{9-x^2}\,] dx = 108$$

Example for practice purpose

1. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \vec{F}\, dV$, where V is bounded by the plane $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}$

2. Evaluate $\iiint_V \phi\, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0, y = 0, z = 0$ **Ans.** 128

3. If $\vec{F} = (2x^2 - 3z)\,\hat{i} - 2xy\,\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla \times \vec{F}\,dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. **Ans.** $\frac{8}{3}(\hat{j} - \hat{k})$

4. Evaluate $\iiint_V (2x + y)\,dV$, where V is closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 2$ and $z = 0$. **Ans.** $\frac{80}{3}$

5. If $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, evaluate $\iiint \vec{F}\,dV$ over the region bounded by the surfaces $x = 0, y = 0, y = 6$ and $z = x^2, z = 4$. **Ans.** $(16\hat{i} - 3\hat{j} + 48\hat{k})$