51.2 GAUSS ELIMINATION METHOD

In this method the unknowns of equations below are eliminated and the system is reduced to an upper triangular system. The unknowns are obtained by back substitution.

Let a system of simultaneous equations in n unknowns $x_1, x_2 .... x_n$ be

$$a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n = b_1 \qquad ...(1)$$
$$a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n = b_2 \qquad ...(2)$$
$$...$$
$$...$$
$$a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n = b_n \qquad ...(n)$$

Method to solve the above equations

Step 1. We eliminate x_1 from 2nd, 3rd ....nth equation with the help of the first equation

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n &= b_1 \\ a'_{22} x_2 + ... + a'_{2n} x_n &= b_2' \\ &...................................... \\ &...................................... \\ a'_{n2} x_2 + ... + a'_{nn} x_n &= b_n' \end{aligned}$$

Step 2. We again eliminate x_2 from 3rd, 4th..... nth equation with the help of second equation.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + ... a_{1n} x_n &= b_1 \\ a'_{22} x_2 + a'_{23} x_3 .... + a'_{2n} x_n &= b_2' \\ a''_{33} x_3 + ... + a''_{3n} x_n &= b_3'' \\ &. \\ &...................................... \\ &...................................... \\ a''_{n3} x_3 + ... + a''_{nn} x_n &= b_n'' \end{aligned}$$

In the third step we will eliminate x_3 and in fourth step x_4 and so on.

Finally the system of equations will be of the following form.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + ... a_{1n} x_n &= b_1 \\ a_{22} x_2 + ... + a'_{2n} x_n &= b_2' \\ &...................................... \\ &...................................... \\ c_{nn} x_n &= d_n \end{aligned}$$

The given system is reduced to the above form *i.e.* triangular form.

Backward Substitution

We first find out the value of x_n from the last equation, then substitute the value of x_n in the $(n-1)$th equation to get the value of x_{n-1}. Again substitute the value of x_{n-1} in $(n-2)$th equation to get the value of x_{n-2}. By this backward substitution we can find the values of all the unknowns.

Example 1. *Solve the following equations by using Gauss-elimination method* :

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 - 2x_3 &= -2 \\ x_1 - x_2 + x_3 &= 6 \end{aligned}$$

(*R.G.P.V. Bhopal, III Semester, June 2006*)

Example 2.3.10 Solve the linear system by Gauss elimination method.

$$\begin{aligned} y+z &= 2 \\ 2x+3z &= 5 \\ x+y+z &= 3 \end{aligned}$$

Solution: In this case, the augmented matrix is $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. The method proceeds along the following steps.

1. Interchange 1^{st} and 2^{nd} equation (or R_{12}).

$$\begin{aligned} 2x+3z &= 5 \\ y+z &= 2 \\ x+y+z &= 3 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

2. Divide the 1^{st} equation by 2 (or $R_1(1/2)$).

$$\begin{aligned} x+\tfrac{3}{2}z &= \tfrac{5}{2} \\ y+z &= 2 \\ x+y+z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

3. Add -1 times the 1^{st} equation to the 3^{rd} equation (or $R_{31}(-1)$).

$$\begin{aligned} x+\tfrac{3}{2}z &= \tfrac{5}{2} \\ y+z &= 2 \\ y-\tfrac{1}{2}z &= \tfrac{1}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

4. Add -1 times the 2^{nd} equation to the 3^{rd} equation (or $R_{32}(-1)$).

$$\begin{aligned} x+\tfrac{3}{2}z &= \tfrac{5}{2} \\ y+z &= 2 \\ -\tfrac{3}{2}z &= -\tfrac{3}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}.$$

5. Multiply the 3^{rd} equation by $\frac{-2}{3}$ (or $R_3(-\frac{2}{3})$).

$$\begin{aligned} x+\tfrac{3}{2}z &= \tfrac{5}{2} \\ y+z &= 2 \\ z &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last equation gives $z=1$, the second equation now gives $y=1$. Finally the first equation gives $x=1$. Hence the set of solutions is $(x,y,z)^t = (1,1,1)^t$, A UNIQUE SOLUTION.

Example 2.3.11 Solve the linear system by Gauss elimination method.

$$\begin{aligned} x+y+z &= 3 \\ x+2y+2z &= 5 \\ 3x+4y+4z &= 11 \end{aligned}$$

Solution: In this case, the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{bmatrix}$ and the method proceeds as follows:

1. Add -1 times the first equation to the second equation.

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \\ 3x+4y+4z &= 11 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 3 & 4 & 4 & 11 \end{bmatrix}.$$

2. Add -3 times the first equation to the third equation.

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \\ y+z &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

3. Add -1 times the second equation to the third equation

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the set of solutions is $(x,y,z)^t = (1, 2-z, z)^t = (1,2,0)^t + z(0,-1,1)^t$, with z arbitrary. In other words, the system has INFINITE NUMBER OF SOLUTIONS.

Example 2.3.12 Solve the linear system by Gauss elimination method.

$$\begin{aligned} x+y+z &= 3 \\ x+2y+2z &= 5 \\ 3x+4y+4z &= 12 \end{aligned}$$

Solution: In this case, the augmented matrix is $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{bmatrix}$ and the method proceeds as follows:

1. Add -1 times the first equation to the second equation.

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \\ 3x+4y+4z &= 12 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 3 & 4 & 4 & 12 \end{bmatrix}.$$

2. Add -3 times the first equation to the third equation.

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \\ y+z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

3. Add -1 times the second equation to the third equation

$$\begin{aligned} x+y+z &= 3 \\ y+z &= 2 \\ 0 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The third equation in the last step is

$$0x + 0y + 0z = 1.$$

This can never hold for any value of x, y, z. Hence, the system has NO SOLUTION.

51.3 GAUSS- JORDAN METHOD

This is modification of the Gauss elimination method.

By this method we eliminate unknowns not only from the equations below but also from the equations above. In this way the system is reduced to a diagonal matrix.

Finally each equation consists of only one unknown and thus, we get the solution. Here, the labour of backward substitution for finding the unknowns is saved.

Gauss-Jordan method is modification of Gauss elimination method.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x+5z=2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3}\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

Let's now perform row operations on this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2-6R_1}\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3-7R_1}\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row echelon form so we can stop. It corresponds to the augmented matrix of the following system.

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

We can express the solutions of this system as

$$x = \tfrac{7}{4}z, \quad y = \tfrac{1}{2} - \tfrac{1}{4}z.$$

Since there is no specific value for z, it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$\boxed{x = \tfrac{7}{4}t, \quad y = \tfrac{1}{2} - \tfrac{1}{4}t, \quad z = t}$$

Any value of the parameter t gives us a solution of the system. For example,

$$t = 4 \quad \text{gives the solution} \quad (x, y, z) = (7, -\tfrac{1}{2}, 4)$$

$$t = -2 \quad \text{gives the solution} \quad (x, y, z) = (-\tfrac{7}{2}, 1, -2).$$

Example 1.66

Solve $x + y + z = 1, \quad 2x - y + z = 4, \quad x - 2y - 3z = 0$
using the Gauss–Jordan method

Solution

$$[A\ B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 4 \\ 1 & -2 & -3 & 0 \end{array}\right] \underset{\substack{R_2 - 2R_1 \\ R_3 - R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & -3 & -4 & -1 \end{array}\right]$$

$$\underset{R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$$\underset{\substack{R_1 + \frac{1}{3}R_2 \\ -\frac{1}{3}R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 5/3 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\underset{\substack{R_1 - \frac{2}{3}R_3 \\ R_2 + R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{-\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution vector is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Example 1.67

Solve $x+y-z=3, \quad 3x+2y-2z=8, \quad 2x-y-3z=7$
using the Gauss–Jordan method

Solution

$$[A\ B]=\left[\begin{array}{ccc|c}1 & 1 & -1 & 3\\ 3 & 2 & -2 & 8\\ 2 & -1 & -3 & 7\end{array}\right] \begin{array}{c}R_2^{!}-3R_1\\ R_3-2R_1\end{array} \left[\begin{array}{ccc|c}1 & 1 & -1 & 3\\ 0 & -1 & 1 & -1\\ 0 & -3 & -1 & 1\end{array}\right]$$

$$R_3^{!}-3R_2 \left[\begin{array}{ccc|c}1 & 1 & -1 & 3\\ 0 & -1 & 1 & -1\\ 0 & 0 & -4 & 4\end{array}\right]$$

$$\begin{array}{c}R_1^{!}+R_2\\ R_2+\frac{1}{4}R_3\\ -\frac{1}{4}R_3\end{array} \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{array}\right]$$

$$(\overset{!}{-}1)R_2 \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{array}\right]$$

The solution vector is

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 0\\ -1\end{bmatrix}$$

Example 2. *Apply Gauss-Jordan method to solve the equations* :

$$\begin{aligned} x+y+z &= 9\\ 2x-3y+4z &= 13\\ 3x+4y+5z &= 40 \end{aligned}$$

(R.G.P.V. Bhopal, III Semester, Dec. 2007)

Solution. The following system of linear equations can be written in matrix form:

By using Gauss Jordan method we have

$$\begin{bmatrix}1 & 1 & 1\\ 2 & -3 & 4\\ 3 & 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 13\\ 40\end{bmatrix} \qquad \Rightarrow \begin{bmatrix}1 & 1 & 1\\ 0 & -5 & 2\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ -5\\ 13\end{bmatrix} \begin{array}{l}R_2 \to R_2-2R_1\\ R_3 \to R_3-3R_1\end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix} R_3 \to R_3 + \frac{1}{5} R_2 \qquad \Rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{5} \\ 0 & -5 & 2 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 12 \end{bmatrix} R_1 \to R_1 + \frac{1}{5} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -15 \\ 12 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{7}{12} R_3 \\ R_2 \to R_2 - \frac{5}{6} R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{5} R_2 \\ R_3 \to \frac{5}{12} R_3 \end{matrix}$$

Hence, $x = 1,\ y = 3,\ z = 5$

51.8 JACOBI'S METHOD

The method is illustrated by taking an example.

$$\text{Let} \quad \left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \qquad \text{... (1)}$$

After division by suitable constants and transposition, the equations can be written as

$$\left.\begin{aligned} x &= c_1 - k_{12}y - k_{13}z \\ y &= c_2 - k_{21}x - k_{23}z \\ z &= c_3 - k_{31}x - k_{32}y \end{aligned}\right\} \qquad \text{... (2)}$$

Let us assume $x = 0$, $y = 0$ and $z = 0$ as first approximation, substituting the values of x, y, z on the right hand side of (2), we get $x = c_1$, $y = c_2$, $z = c_3$. This is the second approximation to the solution of the equations.

Again substituting these values of x, y, z in (2) we get a third approximation.

The process is repeated till two successive approximations are equal or nearly equal.

Note. Condition for using the iterative methods is that the coefficients in the leading diagonal are large compared to the other. If these are not so, then on interchanging the equation we can make the leading diagonal dominant diagonal.

EXAMPLE 4.16

Starting with $(x_0, y_0, z_0) = (0, 0, 0)$ and using Jacobi method, find the next five iterations for the system

$$5x - y + z = 10$$
$$2x + 8y - z = 11$$
$$-x + y + 4z = 3.$$

Solution. The given equations can be written in the form

$$x = \frac{y - z + 10}{5},\ y = \frac{-2x + z + 11}{8}, \text{ and } z = \frac{x - y + 3}{4}, \text{ respectively.}$$

Therefore, starting with $(x_0, y_0, z_0) = (0, 0, 0)$, we get

$$x_1 = \frac{y_0 - z_0 + 10}{5} = 2$$

$$y_1 = \frac{-2x_0 + z_0 + 11}{8} = 1.375$$

$$z_1 = \frac{x_0 - y_0 + 3}{4} = 0.75.$$

The second iteration gives

$$x_2 = \frac{y_1 - z_1 + 10}{5} = \frac{1.375 - 0.75 + 10}{5} = 2.125$$

$$y_2 = \frac{-2x_1 + z_1 + 11}{8} = \frac{-4 + 0.75 + 11}{8} = 0.96875$$

$$z_2 = \frac{x_1 - y_1 + 3}{4} = \frac{2 - 1.375 + 3}{4} = 0.90625.$$

The third iteration gives

$$x_3 = \frac{y_2 - z_2 + 10}{5} = \frac{0.96875 - 0.90625 + 10}{5} = 2.0125$$

$$y_3 = \frac{-2x_2 + z_2 + 11}{8} = \frac{-4.250 + 0.90625 + 11}{8} = 0.95703125$$

$$z_3 = \frac{x_2 - y_2 + 3}{4} = \frac{2.125 - 0.96875 + 3}{4} = 1.0390625.$$

The fourth iteration yields

$$x_4 = \frac{y_3 - z_3 + 10}{5} = \frac{0.95703125 - 1.0390625 + 10}{5} = 1.98359375$$

$$y_4 = \frac{-2x_3 + z_3 + 11}{8} = \frac{-4.0250 + 1.0390625 + 11}{4} = 0.8767578$$

$$z_4 = \frac{x_3 - y_3 + 3}{4} = \frac{2.0125 - 0.95703125 + 3}{4} = 1.0138672,$$

whereas the fifth iteration gives

$$x_5 = \frac{y_4 - z_4 + 10}{5} = 1.9725781$$

$$y_5 = \frac{-2x_4 + z_4 + 11}{8} = \frac{-3.9671875 + 1.0138672 + 11}{8} = 1.005834963$$

$$z_5 = \frac{x_4 - y_4 + 3}{4} = \frac{1.98359375 - 0.8767578 + 3}{4} = = 1.02670898.$$

One finds that the iterations converge to (2, 1, 1).

Example 7. *Solve by Jacobi's method*

$$4x + y + 3z = 17$$
$$x + 5y + z = 14$$
$$2x - y + 8z = 12$$

Solution. The above equations can be written as

$$\left.\begin{aligned} x &= \frac{17}{4} - \frac{y}{4} - \frac{3z}{4} \\ y &= \frac{14}{5} - \frac{x}{5} - \frac{z}{5} \\ z &= \frac{3}{2} - \frac{x}{4} + \frac{y}{8} \end{aligned}\right\} \quad \text{... (1)}$$

On substituting $x = y = z = 0$ on the right hand side of (1), we get

$$x = \frac{17}{4},\ y = \frac{14}{5},\ z = \frac{3}{2}$$

Again substituting these values of x, y, z on R.H.S. of (1), we obtain

$$x = \frac{17}{4} - \frac{7}{10} - \frac{9}{8} = \frac{97}{40}$$

$$y = \frac{14}{5} - \frac{17}{20} - \frac{3}{10} = \frac{33}{20}$$

$$z = \frac{3}{2} - \frac{17}{16} + \frac{7}{20} = \frac{63}{80}$$

Again putting these values on R.H.S. of (1) we get next approximations.

$$x = \frac{17}{4} - \frac{33}{80} - \frac{189}{320} = \frac{1039}{320} = 3.25$$

$$y = \frac{14}{5} - \frac{97}{200} - \frac{63}{400} = \frac{863}{400} = 2.16$$

$$z = \frac{3}{2} - \frac{97}{160} + \frac{33}{160} = \frac{176}{160} = 1.1$$

Substituting, again, the values of x, y, z on R.H.S. of (1), we get

$$x = \frac{17}{4} - \frac{2.16}{4} - \frac{3(1.1)}{4} = 2.885$$

$$y = \frac{14}{5} - \frac{3.25}{5} - \frac{1.1}{5} = 1.93$$

$$z = \frac{3}{2} - \frac{3.25}{4} + \frac{2.16}{8} = 0.96$$

Repeating the process for $x = 2.885$, $y = 1.93$, $z = 0.96$, we have

$$x = \frac{17}{4} - \frac{1.93}{4} - \frac{3}{4} \times 0.96 = 4.25 - 0.48 - 0.72 = 3.05$$

$$y = \frac{14}{5} - \frac{2.885}{5} - \frac{0.96}{5} = 2.8 - 0.577 - 0.192 = 2.03$$

$$z = \frac{3}{2} - \frac{2.885}{4} + \frac{1.93}{8} = 1.5 - 0.721 + 0.241 = 1.02$$

This can be written in a table

Iterations	1	2	3	4	5	6
$x = \frac{17}{4} - \frac{y}{4} - \frac{3z}{4}$	0	$\frac{17}{4} = 4.25$	$\frac{97}{40} = 2.425$	$\frac{1039}{320} = 3.25$	2.885	3.05
$y = \frac{14}{5} - \frac{x}{5} - \frac{z}{5}$	0	$\frac{14}{5} = 2.8$	$\frac{33}{20} = 1.65$	$\frac{863}{400} = 2.16$	1.93	2.03
$z = \frac{3}{2} - \frac{x}{4} + \frac{y}{8}$	0	$\frac{3}{2} = 1.5$	$\frac{63}{80} = 0.7875$	$\frac{176}{160} = 1.1$	0.96	1.02

After 6th iteration $x = 3.05$, $y = 2.03$, $z = 1.02$

The actual values are $x = 3$, $y = 2$, $z = 1$ **Ans.**

EXAMPLE 1 *Applying the Jacobi Method*

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Because you do not know the actual solution, choose

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \qquad \text{Initial approximation}$$

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in Table 10.1.

TABLE 10.1

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Because the last two columns in Table 10.1 are identical, you can conclude that to three significant digits the solution is

$$x_1 = 0.186, \qquad x_2 = 0.331, \qquad x_3 = -0.423.$$

Example for Practice Purpose

Solve the following system by Gauss elimination method.

1. $x - y + z = 1,\ -3x + 2y - 3z = -6,\ 2x - 5y + 4z = 5$ **Ans.** $x = -2,\ y = 3,\ z = 6$

2. $x + 10y + z = 12,\ x + y + 10z = 12,\ 10x + y + z = 12$ **Ans.** $x = 1,\ y = 1,\ z = 1$

3. $x - 2y + 9z = 8,\ 2x - 8y + z = -5,\ 3x + y - z = 3$ **Ans.** $x = 1,\ y = 1,\ z = 1$

Solve the following system by Gauss Jordan method.

1. $2x - 6y + 8z = 24,\ 5x + 4y - 3z = 2,\ 3x + y + 2z = 16$ **Ans.** $x = 1,\ y = 3,\ z = 5$

2. $x + 2y + z = 8,\ 2x + 3y + 4z = 20,\ 4x + 3y + 2z = 16$ **Ans.** $x = 1,\ y = 2,\ z = 3$

3. $3x + 4y + 5z = 18,\ 2x - y + 8z = 13,\ 5x - 2y + 7z = 20$ **Ans.** $x = 3,\ y = 1,\ z = 1$

5. Use Jacobi's method to solve

$$\begin{aligned} 10x - 2y - 3z &= 205 \\ 2x - 10y + 2z &= -154 \\ 2x + y - 10z &= -120 \end{aligned}$$

Ans. $x = 32,\ y = 26,\ z = 21$

upto the end of sixth iteration.

6. Use Jacobi's method to solve

$$\begin{aligned} 5x + 2y + z &= 12 \\ x + 4y + 2z &= 15 \\ x + 2y + 5z &= 20 \end{aligned}$$

Ans. $x = 1.08,\ y = 1.95,\ z = 3.16$

upto the end of eighth iteration.