Gauss's Theorem of Divergence

Statement. The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V div\, \vec{F} dw$$

Example 40. *State Gauss's Divergence theorem* $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint Div\, \vec{F}\, dv$ *where* S *is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *and* $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$.

Solution. Statement of Gauss's Divergence theorem is given in Art 24.8 on page 597.
Thus by Gauss's divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_v\int \nabla \cdot \vec{F}\, dv \qquad \text{Here } \vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$$

$$\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (3x\hat{i} + 4y\hat{j} + 5z\hat{k})$$

$$\nabla \cdot \vec{F} = 3 + 4 + 5 = 14$$

Putting the value of ∇ . F, we get

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_v\int 14 \, . \, dv \qquad \text{where } v \text{ is volume of a sphere}$$

$$= 14\, v$$

$$= 14\frac{4}{3}\pi\,(4)^3 = \frac{3584\,\pi}{3} \qquad \textbf{Ans.}$$

Example 41. *Evaluate* $\iint_S \vec{F} \cdot \hat{n}\, ds$ *where* $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ *and S is the surface of the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(*U.P., Ist Semester, 2009, Nagpur University, Winter 2003*)

Solution. By Divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_v\int (\nabla \cdot \vec{F})\, dv$$

$$= \iiint_v \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\, dv$$

$$= \iint_v\int \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dx\, dy\, dz$$

$$= \iint_v\int (4z - 2y + y)\, dx\, dy\, dz$$

$$= \iint_v \int (4z - y)\, dx\, dy\, dz = \int_0^1 \int_0^1 \left(\frac{4z^2}{2} - yz \right)_0^1 dx\, dy$$

$$= \int_0^1 \int_0^1 (2z^2 - yz)_0^1\, dx\, dy = \int_0^1 \int_0^1 (2 - y)\, dx\, dy$$

$$= \int_0^1 \left(2y - \frac{y^2}{2} \right)_0^1 dx = \frac{3}{2} \int_0^1 dx = \frac{3}{2} [x]_0^1 = \frac{3}{2}(1) = \frac{3}{2} \quad \textbf{Ans.}$$

Note: This question is directly solved as on example 14 on Page 574.

Example 42. *Find* $\iint \vec{F} \cdot \hat{n} \cdot ds$, *where* $\vec{F} = (2x + 3z)\,\hat{i} - (xz + y)\,\hat{j} + (y^2 + 2z)\,\hat{k}$ *and S is the surface of the sphere having centre (3, – 1, 2) and radius 3.*

(*AMIETE, Dec. 2010, U.P., I Semester, Winter 2005, 2000*)

Solution. Let V be the volume enclosed by the surface S.
By Divergence theorem, we've

$$\iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V div\, \vec{F}\, dv.$$

Now, $div\, \vec{F} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}[-(xz + y)] + \frac{\partial}{\partial z}(y^2 + 2z) = 2 - 1 + 2 = 3$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n} \cdot ds = \iiint_V 3\, dv = 3 \iiint_V dv = 3V.$$

Again V is the volume of a sphere of radius 3. Therefore

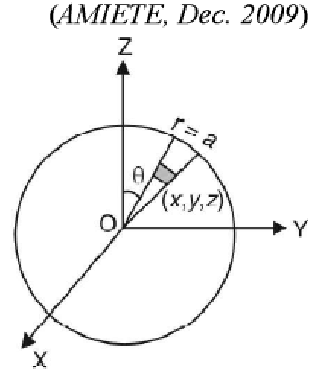
$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (3)^3 = 36\, \pi.$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n} \cdot ds = 3V = 3 \times 36\, \pi = 108\, \pi \quad \textbf{Ans.}$$

Example 43. *Use Divergence Theorem to evaluate* $\iint_S \vec{A} \cdot \vec{ds}$,

where $\vec{A} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*AMIETE, Dec. 2009*)



Solution. $\iint_S \vec{A} \cdot \vec{ds} = \iiint_v \text{div}\, \vec{A}\, dV$

$$= \iiint_v \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k})\, dV$$

$$= \iiint_v (3x^2 + 3y^2 + 3z^2)\, dV = 3 \iiint_v (x^2 + y^2 + z^2)\, dV$$

On putting $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, we get

$$= 3 \iiint_V r^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = 3 \times 8 \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^a r^4\, dr$$

$$= 24\, (\phi)_0^{\frac{\pi}{2}}\, (-\cos\theta)_0^{\frac{\pi}{2}} \left(\frac{r^5}{5} \right)_0^a = 24 \left(\frac{\pi}{2} \right) (-0 + 1) \left(\frac{a^5}{5} \right) = \frac{12\, \pi a^5}{5} \quad \textbf{Ans.}$$

Example 44. *Use divergence Theorem to show that*

$$\iint_S \nabla (x^2 + y^2 + z^2)\, d\vec{s} = 6\, V$$

where S is any closed surface enclosing volume V. (*U.P., I Semester, Winter 2002*)

Solution. Here $\nabla\ (x^2 + y^2 + z^2) = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 + z^2)$

$$= 2\,x\,\hat{i} + 2\,y\,\hat{j} + 2\,z\,\hat{k} = 2\,(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

$$\therefore \quad \iint_S \nabla (x^2 + y^2 + z^2)\cdot ds = \iint_S \nabla (x^2 + y^2 + z^2)\cdot \hat{n}\, ds$$

$\hat{n}$ being outward drawn unit normal vector to S

$$= \iint_S 2\,(x\hat{i} + y\,\hat{j} + z\hat{k})\cdot \hat{n}\, ds$$

$$= 2\iiint_V div\,(x\hat{i} + y\,\hat{j} + z\,\hat{k})\, d\,v \qquad ...(1)$$

(By Divergence Theorem)
(V being volume enclosed by S)

Now, $\quad$ div. $(x\,\hat{i} + y\,\hat{j} + z\hat{k}) = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \qquad ...(2)$$

From (1) & (2), we have

$$\iint \nabla (x^2 + y^2 + z^2)\cdot dS = 2\iiint_V 3\, dv = 6\iiint_V dv = 6\, V$$ **Proved.**

Example 45. *Evaluate* $\iint_S (y^2 z^2 \hat{i} + z^2 x^2 \hat{j} + z^2 y^2 \hat{k})\,\hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Solution. Let V be the volume enclosed by the surface S. Then by divergence Theorem, we have

$$\iint_S (y^2 z^2 \hat{i} + z^2 x^2 \hat{j} + z^2 y^2 \hat{k})\,\hat{n}\, dS = \iiint_V div\,(y^2 z^2 \hat{i} + z^2 x^2 \hat{j} + z^2 y^2 \hat{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2 z^2) + \frac{\partial}{\partial y}(z^2 x^2) + \frac{\partial}{\partial z}(z^2 y^2)\right] dV = \iint_V 2z\, y^2\, dV = 2\iint_V zy^2\, dV$$

Changing to spherical polar coordinates by putting

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta,\ dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore \qquad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta \cdot dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5 \sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

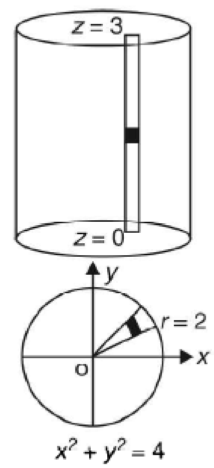
$$= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi}\sin^2\phi\cdot\frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\pi}{12} \qquad \textbf{Ans.}$$

Example 46. *Use Divergence Theorem to evaluate* $\iint_S \vec{F} \cdot d\vec{S}$ *where* $\vec{F} = 4\,x\hat{i} - 2\,y^2\hat{j} + z^2\hat{k}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$.

(*A.M.I.E.T.E., Summer 2003, 2001*)

Solution. By Divergence Theorem,



$$\iint_S \vec{F} \cdot dS = \iiint_V div\,\vec{F}\,dV$$

$$= \iiint_V \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot (4\,x\hat{i} - 2\,y^2\hat{j} + z^2\hat{k})\,dV$$

$$= \iiint_V (4 - 4\,y + 2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_0^3 (4 - 4y + 2z)dz = \iint dx\,dy\,[4z - 4yz + z^2]_0^3$$

$$= \iint (12 - 12y + 9)\,dx\,dy = \iint (21 - 12y)\,dx\,dy$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$

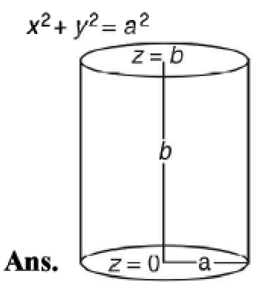
$$= \iint (21 - 12\,r\sin\theta)\,r\,d\theta\,dr = \int_0^{2\pi} d\theta \int_0^2 (21r - 12r^2\sin\theta)\,dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{21\,r^2}{2} - 4r^3\sin\theta\right]_0^2 = \int_0^{2\pi} d\theta\,(42 - 32\sin\theta) = (42\,\theta + 32\cos\theta)_0^{2\pi}$$

$= 84\,\pi + 32 - 32 = 84\,\pi$ **Ans.**

Example 47. *Apply the Divergence Theorem to compute* $\iint \vec{u} \cdot \hat{n}\,ds$, *where s is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = \hat{i}x - \hat{j}y + \hat{k}z$.

Solution. By Gauss's Divergence Theorem



$$\iint \vec{u} \cdot \hat{n}ds = \iiint_V (\nabla \cdot \vec{u})\,dv$$

$$= \iiint_V \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot (\hat{i}x - \hat{j}y + \hat{k}z)dv$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dv = \iiint_V (1 - 1 + 1)\,dv$$

$= \iiint_V dv = \iiint_V dx\,dy\,dz$ = Volume of the cylinder $= \pi\,a^2b$ **Ans.**

Example 48. *Apply Divergence Theorem to evaluate* $\iiint_V \vec{F}\,.\,\hat{n}\,ds$, *where* $\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = a^2$ *bounded by the planes* $z = 0$ *and* $z = b$.

(*U.P. Ist Semester, Dec. 2006*)

Solution. We have,

$$\vec{F} = 4x^3\hat{i} - x^2y\,\hat{j} + x^2z\hat{k}$$

$$\therefore \quad div\,\overline{F} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot (4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k})$$

$$= \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(-x^2y) + \frac{\partial}{\partial z}(x^2z) = 12x^2 - x^2 + x^2 = 12\,x^2$$

Now, $$\iiint_V div\, \vec{F}\, dV = 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\, dz\, dy\, dx$$

$$= 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\, (z)_0^b\, dy\, dx = 12\, b\int_{-a}^{a} x^2 (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\, dx$$

$$= 12\, b\int_{-a}^{a} x^2 . 2\sqrt{a^2-x^2}\, dx \qquad = 24\, b\int_{-a}^{a} x^2\sqrt{a^2-x^2}\, dx$$

$$= 48\, b\int_{0}^{a} x^2\sqrt{a^2-x^2}\, dx \qquad \text{[Put } x = a\sin\theta,\ dx = a\cos\theta\, d\theta\text{]}$$

$$= 48\, b\int_{0}^{\pi/2} a^2\sin^2\theta\, a\cos\theta\, a\cos\theta\, d\theta$$

$$= 48\, ba^4\int_0^{\pi/2}\sin^2\theta\cdot\cos^2\theta\, d\theta = 48\, ba^4\, \frac{\left|\frac{3}{2}\right.\left|\frac{3}{2}\right.}{2\left|3\right.}$$

$$= 48\, ba^4\, \frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 2} = 3\ b\ a^4\pi$$ **Ans.**

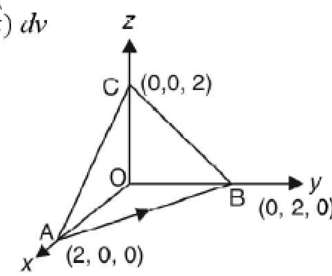
Example 49. *Evaluate surface integral* $\iint \vec{F}\cdot\hat{n}\, ds$, *where* $\vec{F} = (x^2 + y^2 + z^2)\ (\hat{i} + \hat{j} + \hat{k})$, *S is the surface of the tetrahedron x = 0, y = 0, z = 0, x + y + z = 2 and n is the unit normal in the outward direction to the closed surface S.*

Solution. By Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\, \vec{F}\cdot dv$$

where S is the surface of tetrahedron $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})\, dv$$



$$= \iiint_V (2x + 2y + 2z)\, dv$$

$$= 2\iiint_V (x + y + z)\, dx\, dy\, dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\int_0^{2-x-y}(x + y + z)\, dz$$

$$= 2\int_0^2 dx\int_0^{2-x} dy\left(xz + yz + \frac{z^2}{2}\right)_0^{2-x-y}$$

$$= 2\int_0^2 dx \int_0^{2-x} dy \left(2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2}\right]$$

$$= 2\int_0^2 dx \left[2xy - x^2 y - x y^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6}\right]_0^{2-x}$$

$$= 2\int_0^2 dx \left[2x(2-x) - x^2(2-x) - x(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\int_0^2 \left(4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$= 2\left[-\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2 = 2\left[\frac{8}{3} - \frac{16}{12} + \frac{16}{24}\right] = 4$$ **Ans.**

Example 50. *Use the Divergence Theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

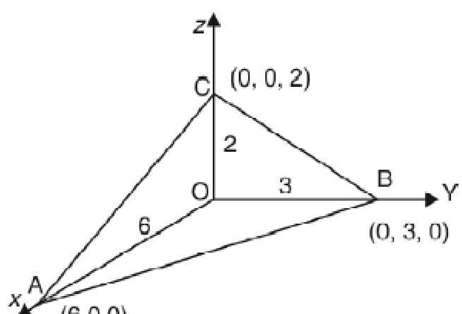
where S is the portion of the plane $x + 2y + 3z = 6$ which lies in the first Octant.

(*U.P., I Semester, Winter 2003*)

Solution. $\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy)$

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where S is a closed surface bounding a volume V.



$\therefore$ $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$

$$= \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz = 3\iiint_V dx\,dy\,dz$$

= 3 (Volume of tetrahedron $OABC$)

$$= 3\left[\left(\frac{1}{3}\text{ Area of the base } \Delta\, OAB\right) \times \text{height } OC\right]$$

$$= 3\left[\frac{1}{3}\left(\frac{1}{2} \times 6 \times 3\right) \times 2\right] = 18$$ **Ans.**

Example 51. *Use Divergence Theorem to evaluate :* $\iint (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *over the surface of a sphere radius a.* (*K. University, Dec. 2009*)

Solution. Here, we have

$$\iint_S [x\,dy\,dz + y\,dx\,dz + z\,dx\,dy]$$

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz \quad = \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint_V (1 + 1 + 1)\,dx\,dy\,dz = 3 \text{ (volume of the sphere)}$$

$$= 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$ **Ans.**

Example 52. *Using the divergence theorem, evaluate the surface integral* $\iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dy\,dx)$ *where* $S : x^2 + y^2 + z^2 = 4.$

(*AMIETE, Dec. 2010, UP, I Sem., Dec 2008*)

Solution. $\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy)$

$$= \iiint_v \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where S is closed surface bounding a volume V.

$$\therefore \quad \iint_S (yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy)$$

$$= \iiint_v \left(\frac{\partial (yz)}{\partial x} + \frac{\partial (zx)}{\partial y} + \frac{\partial (xy)}{\partial z}\right) dx\,dy\,dz = \iiint_v (0 + 0 + 0)\,dx\,dy\,dz$$

$$= 0$$ **Ans.**

Example 57. *Verify the Gauss divergence Theorem for*

$\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$ *(U.P., I Semester, Compartment 2002)*

Solution. We have

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[(x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z$$

$$\therefore \text{ Volume integral} = \iiint_V \nabla \cdot \vec{F}\, dV = \iiint_V 2\,(x + y + z)\, dV$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x + y + z)\, dx\, dy\, dz = 2\int_0^a dx \int_0^b dy \int_0^c (x + y + z)\, dz$$

$$= 2\int_0^a dx \int_0^b dy \left(xz + yz + \frac{z^2}{2}\right)_0^c = 2\int_0^a dx \int_0^b dy \left(cx + cy + \frac{c^2}{2}\right)$$

$$= 2\int_0^a dx \left(c\,x\,y + c\,\frac{y^2}{2} + \frac{c^2 y}{2}\right)_0^b = 2\int_0^a dx \left(bcx + \frac{b^2 c}{2} + \frac{bc^2}{2}\right)$$

$$= 2\left[\frac{bc\,x^2}{2} + \frac{b^2\,cx}{2} + \frac{bc^2 x}{2}\right]_0^a = [a^2bc + ab^2c + abc^2]$$

$$= abc\,(a + b + c) \qquad \text{...(A)}$$

To evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds,$ where S consists of six plane surfaces.

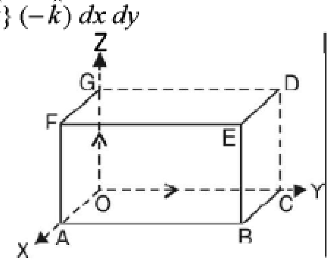
$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{OABC} \vec{F}\cdot\hat{n}\, ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\, ds + \iint_{OAFG} \vec{F}\cdot\hat{n}\, ds$$

$$+ \iint_{BCDE} \vec{F}\cdot\hat{n}\, ds + \iint_{ABEF} \vec{F}\cdot\hat{n}\, ds + \iint_{OCDG} \vec{F}\cdot\hat{n}\, ds$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\, ds = \iint_{OABC} \{(x^2 - yz)\hat{i} + (y^2 - xz)\,\hat{j} + (z^2 - xy)\hat{k}\}\,(-\hat{k})\, dx\, dy$$

$$= -\int_0^a\int_0^b (z^2 - xy)\, dx\, dy$$

$$= -\int_0^a\int_0^b (0 - xy)\, dx\, dy = \frac{a^2\, b^2}{4} \qquad \text{...(1)}$$



$$\iint_{DEFG} \vec{F}\cdot\hat{n}\, ds = \iint_{DEFG} \{(x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}\}(\hat{k})\, dx\, dy$$

$$= \int_0^a\int_0^b (z^2 - xy)\, dx\, dy = \int_0^a\int_0^b (c^2 - xy)\, dx\, dy$$

$$= \int_0^a \left[c^2 y - \frac{xy^2}{2}\right]_0^b dx = \int_0^a \left(c^2 b - \frac{xb^2}{2}\right) dx$$

$$= \left[c^2 bx - \frac{x^2 b^2}{4}\right]_0^a = abc^2 - \frac{a^2 b^2}{4} \quad ...(2)$$

S.No.	Surface	Outward normal	ds	
1	OABC	$-k$	$dx\,dy$	$z = 0$
2	DEFG	k	$dx\,dy$	$z = c$
3	OAFG	$-j$	$dx\,dz$	$y = 0$
4	BCDE	j	$dx\,dz$	$y = b$
5	ABEF	i	$dy\,dz$	$x = a$
6	OCDG	$-i$	$dy\,dz$	$x = 0$

$$\iint_{OAFG} \vec{F}\cdot\hat{n}\, ds = \iint_{OAFG} \{(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\}(-\hat{j})\, dx\, dz$$

$$= -\iint_{OAFG} (y^2 - zx)\, dx\, dz$$

$$= -\int_0^a dx \int_0^c (0 - zx)\, dz = \int_0^a dx \left(\frac{xz^2}{2}\right)_0^c = \int_0^a \frac{xc^2}{2}\, dx = \left[\frac{x^2 c^2}{4}\right]_0^a = \frac{a^2 c^2}{4} \quad ...(3)$$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\, ds = \iint \{(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\}\cdot\hat{j}\, dx\, dz = \iint_{BCDE} (y^2 - xz)\, dx\, dz$$

$$= -\int_0^a dx \int_0^c (b^2 - xz)\, dz = \int_0^a \left(b^2 z - \frac{xz^2}{2}\right)_0^c dx = \int_0^a \left(b^2 c - \frac{xc^2}{2}\right) dx$$

$$= \left[b^2 c x - \frac{x^2 c^2}{4}\right]_0^a = ab^2 c - \frac{a^2 c^2}{4} \quad ...(4)$$

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\, ds = \iint_{ABEF} \{(x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}\}\cdot\hat{i}\, dy\, dz$$

$$= \iint_{ABEF} (x^2 - yz)\, dy\, dz = \int_0^b dy \int_0^c (a^2 - yz)\, dz = \int_0^b dy \left(a^2 z - \frac{yz^2}{2}\right)_0^c$$

$$= \int_0^b \left(a^2 c - \frac{yc^2}{2}\right) dy = \left[a^2 cy - \frac{y^2 c^2}{4}\right]_0^b = a^2 bc - \frac{b^2 c^2}{4} \quad ...(5)$$

$$\iint_{OCDG} \vec{F}\cdot\hat{n}\, ds = \iint_{OCDG} \{(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}\}\cdot(-\hat{i})\, dy\, dz$$

$$= \int_0^b\int_0^c (x^2 - yz)\, dy\, dz = -\int_0^b dy \int_0^c (-yz)\, dz = -\int_0^b dy \left[\frac{-yz^2}{2}\right]_0^c$$

$$= \int_0^b \frac{yc^2}{2}\, dy = \left[\frac{y^2 c^2}{4}\right]_0^b = \frac{b^2 c^2}{4} \quad ...(6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint \vec{F}\cdot\hat{n}\,ds = \left(\frac{a^2 b^2}{4}\right)+\left(abc^2-\frac{a^2 b^2}{4}\right)+\left(\frac{a^2 c^2}{4}\right)+\left(ab^2 c-\frac{a^2 c^2}{4}\right)+\left(\frac{b^2 c^2}{4}\right)+\left(a^2 bc-\frac{b^2 c^2}{4}\right)$$

$$= abc^2 + ab^2c + a^2bc$$

$$= abc\,(a+b+c) \qquad \text{...(B)}$$

From (A) and (B), Gauss divergence Theorem is verified. **Verified.**

Example for Practice Purpose:

1. Use Divergence Theorem to evaluate $\iint_s (y^2z^2\hat{i}+z^2x^2\hat{j}+x^2y^2\hat{k}).\overline{ds}$, where S is the upper part of the sphere $x^2+y^2+z^2=9$ above xy- plane. **Ans.** $\frac{243\pi}{8}$

2. Evaluate $\iint_S (\nabla\times\vec{F}).ds$, where S is the surface of the paraboloid $x^2+y^2+z=4$ above the xy-plane and $\vec{F}=(x^2+y-4)\,\hat{i}+3\,xy\hat{j}+(2\,xz+z^2)\,\hat{k}$. **Ans.** $-4\,\pi$

3. Evaluate $\iint_s [xz^2\,dy\,dz+(x^2y-z^3)\,dz\,dx+(2\,xy+y^2z)\,dx\,dy]$, where S is the surface enclosing a region bounded by hemisphere $x^2+y^2+z^2=4$ above XY-plane. **Ans.** $\frac{64\pi}{5}$

4. Verify Divergence Theorem for $\vec{F}=x^2\hat{i}+z\hat{j}+yz\hat{k}$, taken over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$. **Ans.** $\frac{3}{2}$

5. Evaluate $\iint_S (2\,xy\,\hat{i}+yz^2\hat{j}+xz\,\hat{k})\cdot\vec{ds}$ over the surface of the region bounded by $x=0, y=0, y=3, z=0$ and $x+2\,z=6$ **Ans.** $\frac{351}{2}$