Green's Theorem (For Plane)

Statement. *If* $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ *be continuous functions over a region R bounded by simple closed curve C in x – y plane, then*

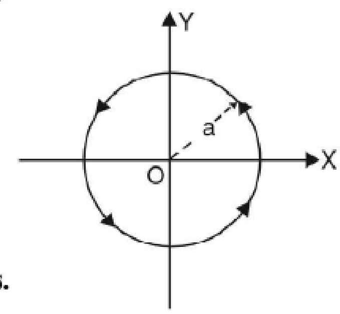
$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

Example 16. *A vector field* $\vec{F}$ *is given by* $\vec{F} = \sin y\hat{i} + x(1 + \cos y)\hat{j}$.
Evaluate the line integral $\int_C \vec{F}\cdot\vec{dr}$ *where C is the circular path given by* $x^2 + y^2 = a^2$.

Solution. $\vec{F} = \sin y\hat{i} + x(1 + \cos y)\hat{j}$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C [\sin y\hat{i} + x(1+\cos y)\hat{j}]\cdot(\hat{i}dx + \hat{j}dy) = \int_C \sin y\, dx + x(1+\cos y)\, dy$$

On applying Green's Theorem, we have



$$\oint_c (\phi\, dx + \psi\, dy) = \iint_s \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\, dy$$

$$= \iint_s [(1+\cos y) - \cos y]\, dx\, dy$$

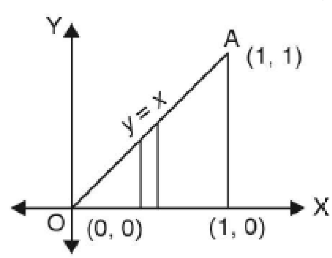
where s is the circular plane surface of radius a.

$$= \iint_s dx\, dy = \text{Area of circle} = \pi a^2. \quad \textbf{Ans.}$$

Example 17. *Using Green's Theorem, evaluate* $\int_c (x^2\, ydx + x^2\, dy)$, *where c is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1).*

(*U.P., I Semester, Winter 2003*)

Solution. By Green's Theorem, we have



$$\int_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\, dy$$

$$\int_c (x^2 y\, dx + x^2\, dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy = \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$

$$= \int_0^1 (2x - x^2)(x)\, dx = \int_0^1 (2x^2 - x^3)\, dx = \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{12}$$

Ans.

Example 18. *State and verify Green's Theorem in the plane for* $\oint (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ *where C is the boundary of the region bounded by* $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$.

Here the closed curve C consists of straight lines OB, BA and AO, where coordinates of A and B are (3, 0) and (0, – 2) respectively. Let R be the region bounded by C.

Then by Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy \qquad \text{...(1)}$$

$$= \iint_R (-6y + 16y)\, dx\, dy = \iint_R 10y\, dx\, dy$$

$$= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^{0} y\, dy = 10 \int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^{0} \qquad = -\frac{5}{9}\int_0^3 dx\, (2x-6)^2$$

$$= -\frac{5}{9}\left[\frac{(2x-6)^3}{3 \times 2}\right]_0^3 = -\frac{5}{54}(0+6)^3 \qquad = -\frac{5}{54}(216) = -20 \quad \text{...(2)}$$

Now we evaluate L.H.S. of (1) along OB, BA and AO.

Along OB, $x = 0$, $dx = 0$ and y varies form 0 to –2.

Along BA, $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2}\, dy$ and y varies from –2 to 0.

and along AO, $y = 0$, $dy = 0$ and x varies from 3 to 0.

L.H.S. of (1) = $\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$

$$= \int_{OB} [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy] + \int_{BA} [(3x^2 - 8y^2)\, dx + (4x - 6xy)\, dy]$$

$$+ \int_{AO} [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

$$= \int_0^{-2} 4y\, dy + \int_{-2}^{0} \left[\frac{3}{4}(6+3y)^2 - 8y^2\right]\left(\frac{3}{2}\, dy\right) + [4y - 3(6+3y)\, y]\, dy + \int_3^0 3x^2\, dx$$

$$= [2y^2]_0^{-2} + \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 12y^2 + 4y - 18y - 9y^2\right] dy + (x^3)_3^0$$

$$= 2[4] + \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 21y^2 - 14y\right] dy + (0 - 27)$$

$$= 8 + \left[\frac{9}{8}\frac{(6+3y)^3}{3 \times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27 = -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2\right]$$

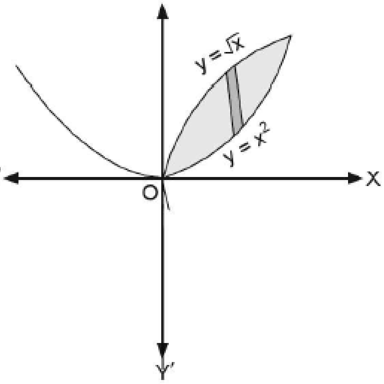
$$= -19 + 27 - 56 + 28 = -20 \qquad \text{...(3)}$$

With the help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Example 19. *Verify Green's Theorem in the plane for*

$$\oint_C \left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy$$

Where C is the boundary of the region defined by $y = \sqrt{x}$, and $y = x^2$ (*K.University, Dec. 2008*)



Solution. Here we have,

$$\oint_C \left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy$$

By Green's Theorem, we have

$$\int_C (\phi\, dx + \psi\, dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}\left(3x^2 - 8y^2\right)\right] dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (-6y + 16y)\, dx\, dy \quad = 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y\, dx\, dy \quad = 10 \int_0^1 dx \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} \quad = \frac{10}{2} \int_0^1 dx \left(x - x^4\right)$$

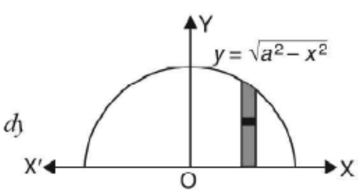
$$= 5 \left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 \quad = 5\left(\frac{1}{2} - \frac{1}{5}\right) \quad = 5\left(\frac{3}{10}\right) = \frac{3}{2}$$

Ans.

Example 20. *Apply Green's Theorem to evaluate* $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper half of circle* $x^2 + y^2 = a^2$.

Solution. $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$

By Green's Theorem, we've $\int_C (\phi\, dx + \psi\, dy) = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$



$$= \int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dx\, dy$$

$$= \int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} (2x + 2y)\, dx\, dy \; = \; 2 \int_{-a}^{a} dx \int_0^{\sqrt{a^2 - x^2}} (x + y)\, dy$$

$$= 2 \int_{-a}^{a} dx \left(xy + \frac{y^2}{2}\right)_0^{\sqrt{a^2 - x^2}} \quad = 2 \int_{-a}^{a} \left(x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right) dx$$

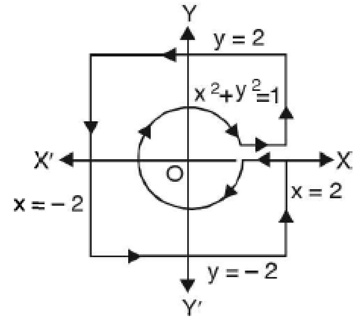
$$= 2 \int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx + \int_{-a}^{a} (a^2 - x^2)\, dx \qquad \left[\begin{array}{ll} \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx, & f \text{ is even} \\ \quad = 0, & f \text{ is odd} \end{array}\right]$$

$$= 0 + 2 \int_0^a (a^2 - x^2)\, dx \; = \; 2\left(a^2 x - \frac{x^3}{3}\right)_0^a = 2\left(a^3 - \frac{a^3}{3}\right) = \frac{4a^3}{3}$$

Ans.

Example 21. *Evaluate* $\oint_C -\dfrac{y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$, *where* $C = C_1 \cup C_2$ *with* $C_1 : x^2 + y^2 = 1$ *and* $C_2 : x = \pm 2, y = \pm 2$. (*Gujarat, I Semester, Jan 2009*)

Solution. $\oint_C -\dfrac{y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$



$$= \iint \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \right) dx\,dy$$

$$= \iint \left[\frac{(x^2+y^2)1 - 2x\,(x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)1 - 2y\,(y)}{(x^2+y^2)^2} \right] dx\,dy$$

$$= \iint \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right] dx\,dy$$

$$= \iint \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \right] dx\,dy = \iint \frac{0}{(x^2+y^2)^2}\,dx\,dy = 0$$

Ans.

AREA OF THE PLANE REGION BY GREEN'S THEOREM

Proof. We know that

$$\int_C M dx + N dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \qquad ...(1)$$

On putting $N = x \left(\dfrac{\partial N}{\partial x} = 1 \right)$ and $M = -y \left(\dfrac{\partial M}{\partial y} = 1 \right)$ in (1), we get

$$\int_C -y\,dx + x\,dy = \iint_A [1 - (-1)]\,dx\,dy = 2 \iint dx\,dy = 2A$$

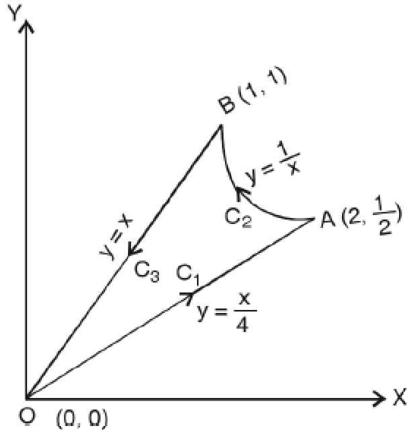
$$\text{Area} = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

Example 22. *Using Green's theorem, find the area of the region in the first quadrant bounded by the curves*

$$y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}$$

Solution. By Green's Theorem Area A of the region bounded by a closed curve C is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$



Here, C consists of the curves $C_1 : y = \frac{x}{4}$, $C_2 : y = \frac{1}{x}$

and $C_3 : y = x$ So

$$\left[A = \frac{1}{2}\oint_C = \frac{1}{2}\left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] = \frac{1}{2}(I_1 + I_2 + I_3)\right]$$

Along $\quad C_1 : y = \frac{x}{4},\ dy = \frac{1}{4}dx,\ x : 0 \text{ to } 2$

$$I_1 = \int_{C_1}(xdy - ydx) = \int_{C_1}\left(x\frac{1}{4}dx - \frac{x}{4}dx\right) = 0$$

Along $\quad C_2 : y = \frac{1}{x},\ dy = -\frac{1}{x^2}dx,\ x : 2 \text{ to } 1$

$$I_2 = \int_{C_2}(xdy - ydx) \quad = \int_2^1\left[x\left(-\frac{1}{x^2}\right)dx - \frac{1}{2}dx\right] = [-2\log x]_2^1 = 2\log 2$$

Along $\quad C_3 : y = x,\ dy = dx\ ;\ x : 1 \text{ to } 0\ ;$

$$I_3 = \int_{C_3}(xdy - ydx) = \int(xdx - xdx) = 0$$

$$A = \frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{2}(0 + 2\log 2 + 0) = \log 2$$ **Ans.**

Example for Practice Purpose

2. Verify Green's Theorem in plane for $\int_C (x^2 + 2xy)\,dx + (y^2 + x^3 y)\,dy$, where c is a square with the vertices P (0, 0), Q (1, 0), R (1, 1) and S (0, 1). **Ans.** $-\frac{1}{2}$

3. Verify Green's Theorem for $\int_c (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$ around the boundary c of the region $y^2 = 8x$ and $x = 2$.

4. Use Green's Theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where c is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy-plane. **Ans.** $\frac{4}{3}$

5. Apply Green's Theoem to evaluate $\int_c [(y - \sin x)\,dy + \cos x\,dy]$, where c is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2x}{\pi}$. **Ans.** $-\frac{\pi^2 + 8}{4\pi}$

6. Either directly or by Green's Therorem, evaluate the line integral $\int_c e^{-x} (\cos y\, dx - \sin y\, dy)$, where c is the rectangle with vertices (0, 0), (π, 0,), $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. **Ans.** $2\,(1 - e^{-\pi})$

9. Verify Green's Theorem for $\int_C \left[(xy + y^2)\, dx + x^2 dy\right]$ where C is the boundary by $y = x$ and $y = x^2$.
(*AMIETE, June 2010*) **Ans.** $-\frac{1}{20}$