Stoke's Theorem

Statement:

Surface integral of the component of curl $\overline{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically

$$\oint \vec{F} \cdot d\,\overline{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds,

Example 23. *Evaluate by Strokes theorem* $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ *where* C *is the curve* $x^2 + y^2 = 1,\ z = y^2$.

Solution. Here we have $\oint yz\,dx + zx\,dy + xy\,dz$

$$= \int (yz\hat{i} + zx\hat{j} + xy\hat{k}).(\hat{i}dx + \hat{j}dy + kdz)$$

$$= \oint F.dx$$

$$= \int curl\ F.\eta\ ds$$

$$\text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= (x - x)\ \hat{i} + (y - y)\ \hat{j} + (z - z)\ \hat{k}$$

$$= 0 \qquad = 0$$

Ans.

$$= 0$$

Example 24. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\,dx - yz^2 dy - y^2 z\,dz]$$

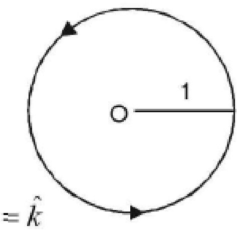
where c is the circle $x^2 + y^2 = 1$, *corresponding to the surface of sphere of unit radius.*

Solution. $\int_c [(2x - y)\, dx - yz^2 dy - y^2 z\, dz]$

$$= \int_c [(2x - y)\,\hat{i} - yz^2 \hat{j} - y^2 z\, \hat{k}] \cdot (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$$

By Stoke's theorem $\oint \vec{F} \cdot d\,\vec{r} = \iint_S \text{Curl}\, \vec{F} \cdot \hat{n}\, ds$ ...(1)

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$



$$= (-2\,yz + 2\,yz)\,\hat{i} - (0 - 0)\,\hat{j} + (0 + 1)\,\hat{k} = \hat{k}$$

Putting the value of curl $\vec{F}$ in (1), we get

$$\oint \vec{f} \,.\, d\,\vec{r} = \iint \hat{k} \cdot \hat{n}\, ds = \iint \hat{k} \cdot \hat{n} \frac{dx\, dy}{\hat{n} \cdot \hat{k}} = \iint dx\, dy = \text{Area of the circle} = \pi \left[\because ds = \frac{dx\, dy}{(\hat{n} \cdot \hat{k})} \right]$$

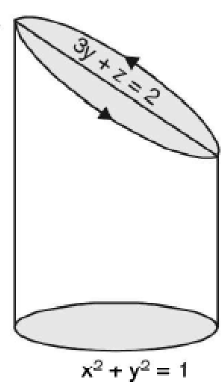
Example 25. *Evaluate* $\int_C \acute{F} \,.\, d\, \acute{r}$, *where* $F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *and C is the curve of intersection of the plane* $y + z = 2$ *and the cylinder* $x^2 + y^2 = 1$.

Solution. $\oint_C \vec{F} \,.\, \vec{dr} = \iint_S \text{curl}\, \vec{F} \,.\, \hat{n}\, ds = \iint_S \text{curl}\,(-y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k})\,\hat{n}\, ds$ ...(1)

$$F(x, y, z) = -y^2\,\hat{i} + x\,\hat{j} + z^2\,\hat{k}$$ *(By Stoke's Theorem)*

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - 0) + \hat{k}\,(1 + 2y) = (1 + 2y)\,\hat{k}$$



Normal vector = $\nabla\,\vec{F}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} \right)(y + z - 2) = \hat{j} + \hat{k}$$

Unit normal vector $\hat{n} = \dfrac{\hat{j} + \hat{k}}{\sqrt{2}}$

$$ds = \frac{dx\, dy}{\hat{\eta} \,.\, \hat{k}}$$

On putting the values of curl $\vec{F}, \hat{n}$ and ds in (1), we get

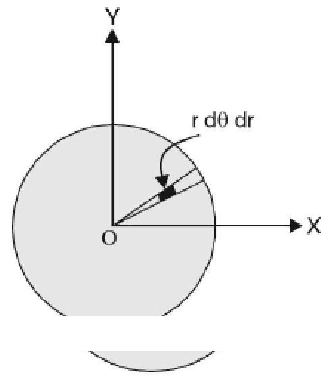
$$\int_C \vec{F} \cdot \vec{dr} = \iint_S (1+2y)\,\hat{k} \cdot \frac{\hat{j}+\hat{k}}{\sqrt{2}} \frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right) \cdot \hat{k}}$$

$$= \iint \frac{1+2y}{\sqrt{2}} \frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi} \int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi} \int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\sin\theta\right] d\theta$$

$$= \left[\frac{\theta}{2} - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi - \frac{2}{3} - 0 + \frac{2}{3}\right) = \pi \qquad \textbf{Ans.}$$



Example 26. *Apply Stoke's Theorem to find the value of*

$$\int_c (y\,dx + z\,dy + x\,dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$. (*Nagpur, Summer 2001*)

Solution. $\int_c (y\,dx + z\,dy + x\,dz)$

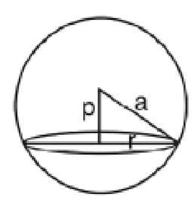
$$= \int_c (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \int_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot \vec{dr}$$

$$= \iint_S \text{curl}\,(y\hat{i} + z\hat{j} + x\hat{k}) \cdot \hat{n}\,ds \qquad \text{(By Stoke's Theorem)}$$

$$= \iint_S \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (y\hat{i} + z\hat{j} + x\hat{k}) \cdot \hat{n}\,ds = \iint_S -(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{n}\,ds \quad \text{...(1)}$$

where S is the circle formed by the intersection of $x^2 + y^2 + z^2 = a^2$ and *$x + z = a$.*

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+z-a)}{|\nabla\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{1+1}}$$



$$\therefore \qquad \hat{n} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$$

Putting the value of $\hat{n}$ in (1), we have

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{i}}{\sqrt{2}}+\frac{\hat{k}}{\sqrt{2}}\right)ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds \qquad \left[\text{Use } r^2 = R^2 - p^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}\right]$$

$$= \frac{-2}{\sqrt{2}}\iint_S ds = \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}$$

Ans.

Example 27. *Use Stoke's Theorem to evaluate* $\int_c \vec{v}\cdot\vec{dr}$, *where* $\vec{v} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$, *and c is the bounding curve of the hemisphere* $x^2 + y^2 + z^2 = 9$, $z > 0$, *oriented in the positive direction.*

Solution. By Stoke's theorem

$$\int_c \vec{v}\cdot\vec{dr} = \iint_S (\text{curl } \vec{v})\cdot\hat{n}\,ds = \iint_S (\nabla\times\vec{v})\cdot\hat{n}\,ds$$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = (0-0)\,\hat{i} - (z-0)\,\hat{j} + (y-2y)\,\hat{k} = -z\hat{j} - y\hat{k}$$

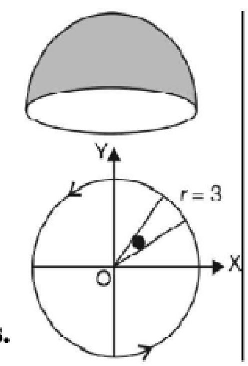
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

$$(\nabla\times\vec{v})\cdot\hat{n} = (-z\hat{j} - y\hat{k})\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3} = \frac{-yz-yz}{3} = \frac{-2yz}{3}$$

$$\hat{n}\cdot\hat{k}\,ds = dx\,dy \Rightarrow \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}.\,\hat{k}\;dx = dx\,dy \Rightarrow \frac{z}{3}ds = dx\,dy$$

$\therefore \qquad ds = \frac{3}{z}\,dx\,dy$



$$\iint_S (\nabla \times \vec{v})\cdot \hat{n}\,ds = \iint \left(\frac{-2\,yz}{3}\right)\left(\frac{3}{z}\,dx\,dy\right) = -\iint 2\,y\,dx\,dy$$

$$= -\iint 2r\sin\theta\, r\,d\theta\,dr = -2\int_0^{2\pi}\sin\theta\,d\theta\int_0^3 r^2\,dr$$

$$= -2\,(-\cos\theta)_0^{2\pi}\cdot\left[\frac{r^3}{3}\right]_0^3 = -2\,(-1+1)\,9 = 0 \quad \textbf{Ans.}$$

Example 28. *Evaluate the surface integral* $\iint_S \text{curl}\,\vec{F}\,.\,\hat{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$ and $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}\cdot$

Solution. $\quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$

Obviously $\qquad \hat{n} = \hat{k}.$

Therefore $\qquad (\nabla\times\vec{F})\cdot\hat{n} = (-\hat{i} - \hat{j} - \hat{k}).\,\hat{k} = -1$

Hence $\qquad \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S (-1)\,dx\,dy = -\iint_S dx\,dy$

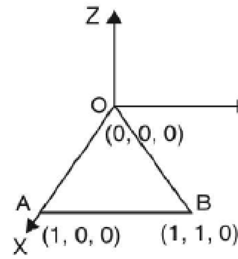
$= -\pi\,(1)^2 = -\pi. \qquad$ (Area of circle = $\pi\,r^2$) **Ans.**

Example 29. *Evaluate* $\oint_C \vec{F} \cdot \vec{dr}$ *by Stoke's Theorem, where* $\vec{F} = y^2\,\hat{i} + x^2\,\hat{j} - (x + z)\,\hat{k}$ *and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*

(*U.P., I Semester, Winter 2000*)

Solution. We have, curl $\vec{F} = \nabla \times \vec{F}$

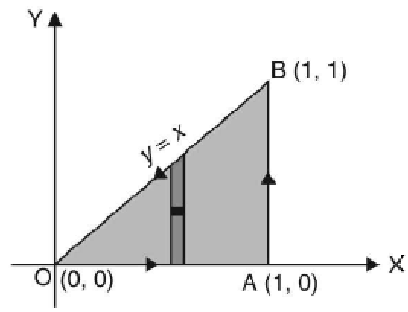
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0.\,\hat{i} + \hat{j}\,2\,(x - y)\,\hat{k}.$$



We observe that z co-ordinate of each vertex of the triangle is zero.

Therefore, the triangle lies in the xy-plane.

$\therefore \quad \hat{n} = \hat{k}$



$\therefore \quad$ curl $\vec{F} \cdot \hat{n} = [\hat{j} + 2\,(x - y)\hat{k}]\,.\,\hat{k} = 2\,(x - y).$

In the figure, only xy-plane is considered.

The equation of the line OB is $y = x$

By Stoke's theorem, we have

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\text{curl } \vec{F} \cdot \hat{n})\, ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2\,(x - y)\, dx\, dy = 2 \int_0^1 \left[xy - \frac{y^2}{2} \right]_0^x dx$$

$$= 2 \int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx = 2 \int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2\, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \qquad \textbf{Ans.}$$

Example 30. *Use the Stoke's Theorem to evaluate*

$$\int_C [(x + 2y)\, dx + (x - z)\, dy + (y - z)\, dz]$$

where c is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6) oriented in the anti-clockwise direction.

Solution.

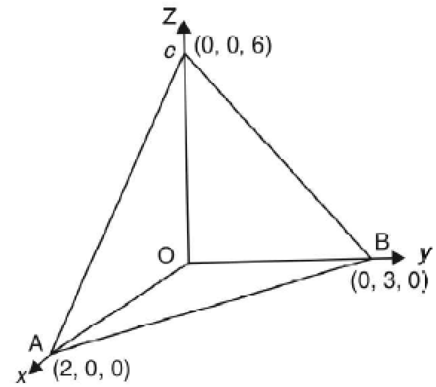
$$\int_C \vec{F} \cdot \vec{dr} = \int_C [(x + 2y)\, dx + (x - z)\, dy + (y - z)\, dz]$$

$$= \int_C [(x + 2y)\,\hat{i} + (x - z)\,\hat{j} + (y - z)\,\hat{k}].[\hat{i}\, dx + \hat{j} dy + \hat{k} dz]$$

$$\therefore \quad \vec{F} = (x+2y)\,\hat{i} + (x-z)\,\hat{j} + (y-z)\,\hat{k}$$

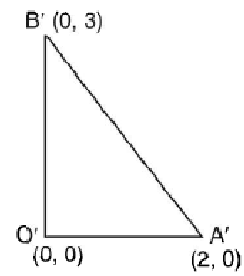
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y & x-z & y-z \end{vmatrix}$$

$$= (1+1)\,\hat{i} - (0-0)\,\hat{j} + (1-2)\,\hat{k} = 2\hat{i} - \hat{k}$$



S is the surface of the plane $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$,

$\hat{n}$ is the normal to the plane ABC.



$$\text{Normal Vector} = \nabla\phi = \left[\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right]\left[\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right]$$

$$= \frac{\hat{i}}{2} + \frac{\hat{j}}{3} + \frac{\hat{k}}{6} = \frac{1}{6}(3\hat{i} + 2\hat{j} + \hat{k})$$

$$\hat{n} = \frac{\frac{1}{6}(3\hat{i} + 2\hat{j} + \hat{k})}{\frac{1}{6}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$$

$$(\nabla \times \vec{F})\cdot\hat{n} = (2\hat{i} - \hat{k}).\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{14}}(6-1) = \frac{5}{\sqrt{14}}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F})\cdot\hat{n}\, ds$$

$$= \iint_s \frac{5}{\sqrt{14}}\, ds = \frac{5}{\sqrt{14}} \iint_R \frac{dx\, dy}{\hat{k}.\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})} = 5 \iint_R dx\, dy$$

where R is the projection of S on the $x\,y$-plane *i.e.* triangle OAB.

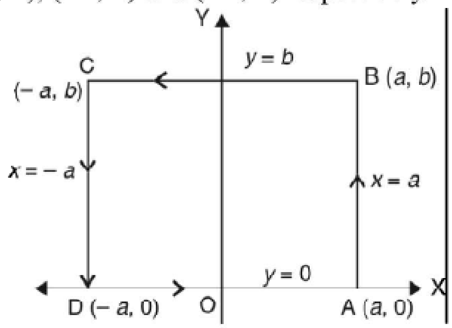
$= 5$. Area of triangle $OAB = \dfrac{5}{2}(2 \times 3) = 15$ **Ans.**

Example 31. *Evaluate* $\oint_C \vec{F}\cdot\vec{dr}$ *by Stoke's Theorem, where* $\vec{F}=(x^2+y^2)\,\hat{i}-2\,xy\,\hat{j}$ *and C is the boundary of the rectangle* $x=\pm\,a,\ y=0$ *and* $y=b$. (*U.P., I Semester, Winter 2002*)

Solution. Since the z co-ordinate of each vertex of the given rectangle is zero, hence the given rectangle must lie in the xy-plane.

Here, the co-ordinates of A, B, C and D are $(a, 0)$, (a, b), $(-\,a, b)$ and $(-\,a, 0)$ respectively.

$$\therefore \quad \text{Curl } \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -\,4\,y\,k$$



Here, $\hat{n}=\hat{k}$, so by Stoke's theorem, we've

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl } \overline{F}\cdot\hat{n}\,d\,s$$

$$= \iint_S (-\,4\,y\,\hat{k})\cdot(\hat{k})\,d\,x\,d\,y \;=\; -\,4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dx\,dy$$

$$= -\,4\int_{-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx \;=\; -\,2\,b^2\int_{-a}^{a} dx = -\,4\,a\,b^2$$

Ans.

Example 32. *Apply Stoke's Theorem to calculate* $\int_c 4\,y\,dx+2\,z\,dy+6\,y\,dz$ *where c is the curve of intersection of* $x^2+y^2+z^2=6\,z$ *and* $z=x+3$.

Solution.
$$\int_c \vec{F}\cdot\vec{dr} = \int_c 4\,y\,dx+2\,z\,dy+6\,y\,dz$$

$$= \int_c (4\,y\hat{i}+2\,z\hat{j}+6\,y\hat{k})\cdot(\hat{i}dx+\hat{j}dy+\hat{k}dz)$$

$$\vec{F} = 4\,y\hat{i}+2\,z\hat{j}+6\,y\hat{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4\,y & 2\,z & 6\,y \end{vmatrix} \begin{aligned} &= (6-2)\,\hat{i}-(0-0)\,\hat{j}+(0-4)\,\hat{k} \\ &= 4\,\hat{i}-4\,\hat{k} \end{aligned}$$

S is the surface of the circle $x^2+y^2+z^2=6z$, $z=x+3$, $\hat{n}$ is normal to the plane $x-z+3=0$

$$\hat{n} = \frac{\nabla\,\phi}{|\nabla\,\phi|} = \frac{\left(\hat{i}\,\dfrac{\partial}{\partial x}+\hat{j}\,\dfrac{\partial}{\partial y}+\hat{k}\,\dfrac{\partial}{\partial z}\right)(x-z+3)}{|\nabla\,\phi|} = \frac{\hat{i}-\hat{k}}{\sqrt{1+1}} = \frac{\hat{i}-\hat{k}}{\sqrt{2}}$$

$$(\nabla\times F)\cdot\hat{n} = (4\,\hat{i}-4\,\hat{k})\cdot\frac{\hat{i}-\hat{k}}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = 4\sqrt{2}$$

$$\int_c \vec{F}\cdot\vec{dr} = \iint_S (\text{curl } F)\cdot\hat{n}\, ds = \iint_S 4\sqrt{2}\,(dx\, dz) = 4\sqrt{2}\ \text{(area of circle)}$$

Centre of the sphere $x^2 + y^2 + (z-3)^2 = 9$, $(0, 0, 3)$ lies on the plane $z = x + 3$. It means that the given circle is a great circle of sphere, where radius of the circle is equal to the radius of the sphere.

Radius of circle = 3, Area = $\pi\,(3)^2 = 9\,\pi$

$$\iint_S (\nabla\times F)\cdot\hat{n}\, ds = 4\sqrt{2}(9\pi) = 36\sqrt{2}\,\pi$$ **Ans.**

Example 34. *Verify Stoke's theorem for the vector field* $\bar{F} = (2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$ *over the upper half of the surface* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on xy- plane.*

(*Nagpur University, Summer 2001*)

Solution. Let S be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary C or S is a circle in the xy plane of radius unity and centre O. The equation of C are $x^2 + y^2 = 1$, $z = 0$ whose parametric form is

$x = \cos t,\ y = \sin t,\ z = 0,\ 0 < t < 2\pi$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C [(2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}].[\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz]$$

$$= \int_C [(2x-y)\, dx - yz^2\, dy - y^2 z\, dz]$$

$$= \int_C (2x-y)\, dx, \quad \text{since on } C,\ z = 0 \text{ and } 2z = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\, dt = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt$$

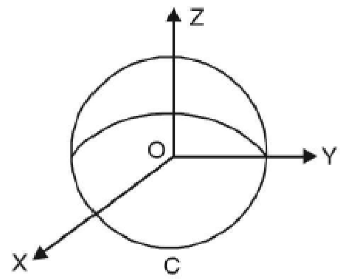
$$= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\, dt = \int_0^{2\pi}\left(-\sin 2t + \frac{1-\cos 2t}{2}\right) dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi \qquad \text{...(1)}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\,\hat{i} + (0-0)\,\hat{j} + (0+1)\,\hat{k} = \hat{k}$$

$$\text{Curl } \vec{F}\,.\,\hat{n} = \hat{k}\,.\,\hat{n} = \hat{n}\,.\,\hat{k}$$

$$\iint_S Curl\,\vec{F}\,.\,\hat{n}\,ds = \iint_S \hat{n}.\hat{k}\,ds = \iint_R \hat{n}.\hat{k}\,.\,\frac{dx}{\hat{n}}\,.\,\frac{dy}{\hat{k}}$$



Where R is the projection of S on xy-plane.

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = 4\left[\frac{1}{2}.\frac{\pi}{2}\right] = \pi \qquad ...(2)$$

From (1) and (2), we have

$\therefore \quad \int_C \vec{F}\,.\,\vec{dr} = \iint$ Curl $\overline{F}\,.\,\hat{n}\,ds$ which is the Stoke's theorem. **Ans.**

Example 36. *Verify Stoke's theorem for a vector field defined by* $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ *in the rectangular in xy-plane bounded by lines x = 0, x = a, y = 0, y = b.*

(*Nagpur University, Summer 2000*)

Solution. Here we have to verify Stoke's theorem $\int_C \vec{F}\,.\,\vec{dr} = \iint_S (\nabla \times \overline{F})\,.\,\hat{n}\,ds$

Where 'C' be the boundary of rectangle ($ABCD$) and S be the surface enclosed by curve C.

$$\vec{F} = (x^2 - y^2)\,\hat{i} + (2xy)\,\hat{j}$$

$$\vec{F}\,.\,\vec{dr} = [(x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}]\,.\,[\hat{i}\,dx + \hat{j}\,dy]$$

$$\Rightarrow \qquad \vec{F}\,.\,\vec{dr} = (x^2 + y^2)\,dx + 2xy\,dy \qquad ...(1)$$

Now,

$$\int_C \vec{F}\,.\,\vec{dr} = \int_{OA} F\,.\,\vec{dr} + \int_{AB} \vec{F}\,.\,\vec{dr} + \int_{BC} \vec{F}\,.\,\vec{dr} + \int_{CO} \vec{F}\,.\,\vec{dr} \qquad ...(2)$$

Along OA, put $y = 0$ so that $k\,dy = 0$ in (1) and $\vec{F}\,.\,\vec{d\,r} = x^2\,dx$,
Where x is from 0 to a.

$$\therefore \qquad \int_{OA} \vec{F}\,.\,\vec{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ...(3)$$

Along AB, put $x = a$ so that $dx = 0$ in (1), we get $\vec{F}\,.\,\vec{d\,r} = 2ay\,dy$
Where y is from 0 to b.

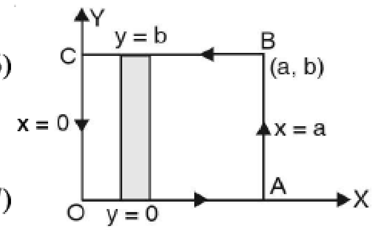
$$\therefore \qquad \int_{AB} \vec{F}\,.\,\vec{d\,r} = \int_0^b 2ay\,dy = [ay^2]_0^b = ab^2 \qquad ...(4)$$

Along BC, put $y = b$ and $dy = 0$ in (1) we get $\bar{F} \cdot \bar{dr} = (x^2 - b^2)\, dx$, where x is from a to 0.

$$\therefore \qquad \int_{BC} \vec{F} \cdot \vec{dr} = \int_a^0 (x^2 - b^2)\, dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = \frac{-a^3}{3} + b^2 a \qquad ...(5)$$

Along CO, put $x = 0$ and $dx = 0$ in (1), we get $\vec{F} \cdot \vec{dr} = 0$

$$\therefore \qquad \int_{CO} \vec{F} \cdot \vec{dr} = 0 \qquad ...(6)$$



Putting the values of integrals (3), (4), (5) and (6) in (2), we get

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2 \qquad ...(7)$$

Now we have to evaluate R.H.S. of Stoke's Theorem *i.e.* $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$

We have,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = (2y + 2y)\, \hat{k} = 4y\, \hat{k}$$

Also the unit vector normal to the surface S in outward direction is $\hat{n} = k$

($\because$ z-axis is normal to surface S)

Also in xy-plane $ds = dx\, dy$

$$\therefore \qquad \iint_S (\nabla \times \vec{F}) \cdot \hat{n}.\, ds = \iint_R 4y\, \hat{k} \cdot \hat{k}\, dx\, dy = \iint_R 4y\, dx\, dy.$$

Where R be the region of the surface S.

Consider a strip parallel to y-axis. This strip starts on line y = 0 (*i.e.* x-axis) and end on the line $y = b$, We move this strip from $x = 0$ (y-axis) to $x = a$ to cover complete region R.

$$\therefore \qquad \iint_S (\nabla \times \vec{F}) \cdot \hat{n}.\, ds = \int_0^a \left[\int_0^b 4y\, dy\right] dx = \int_0^a [2y^2]_0^b\, dx$$

$$= \int_0^a 2b^2\, dx = 2b^2\, [x]_0^a = 2ab^2 \qquad ...(8)$$

$\therefore$ From (7) and (8), we get

$\int_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$ and hence the Stoke's theorem is verified.

Example for Practice Purpose

1. Use the Stoke's Theorem to evaluate $\int_C y^2 dx + xy\, dy + xz\, dz$,

where C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the positive direction. **Ans.** 0

2. Evaluate the integral for $\int_C y^2 dx + z^2 dy + x^2 dz$, where C is the triangular closed path joining the points (0, 0, 0), (0, a, 0) and (0, 0, a) by transforming the integral to surface integral using Stoke's Theorem.

Ans. $\frac{a^3}{3}$.

4. Evaluate $\int_C \vec{F} \cdot \vec{d} R$ where $\vec{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$, C is the circl $x^2 + y^2 = 4$, $z = 1.5$ **Ans.** $\frac{19}{2}\pi$

6. Evaluate $\int_c \vec{F} \cdot dr$ by Stoke's Theorem for $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,k$ and C is the curve of intersection of $x^2 + y^2 = 1$ and $y = z^2$. **Ans.** 0

7. If $\vec{F} = (x - z)\,\hat{i} + (x^3 + yz)\,\hat{j} + 3xy^2\,\hat{k}$ and S is the surface of the cone $z = a - \sqrt{(x^2 + y^2)}$ above the xy-plane, show that $\iint_s \text{curl}\,\vec{F} \cdot dS = 3\,\pi\,a^4 / 4$.

8. If $\vec{F} = 3y\hat{i} - xy\hat{j} + yz2\hat{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$, show by using Stoke's Theorem that $\iint_s (\nabla \times \vec{F}) \cdot dS = 20\,\pi$.

5. Verify Stoke's Theorem for the vector field

$$\vec{F} = (2y + z)\,\hat{i} + (x - z)\,\hat{j} + (y - x)\,\hat{k}$$

over the portion of the plane $x + y + z = 1$ cut off by the co-ordinate planes.