Vector Differential Operator Del *i.e.* ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

GRADIENT OF A SCALAR FUNCTION

If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus,
$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\text{gard } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$\text{gard } \phi = \nabla \phi \qquad (\nabla \text{ is read del or nebla})$$

Example *If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point $(1, -2, -1)$.*

Solution.
$$\begin{aligned}
\text{grad } \phi &= \nabla\phi \\
&= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2) \\
&= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2) \\
&= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z) \\
\text{grad } \phi \text{ at } (1, -2, -1) &= \hat{i}(6)(1)(-2) + \hat{j}[(3)(1) - 3(4)(1)] + \hat{k}(-2)(-8)(-1) \\
&= -12\hat{i} - 9\hat{j} - 16\hat{k}
\end{aligned}$$

Ans.

Problem 13 : What is the greatest rate of increase of $\phi \equiv xyz^2$ at the point (1, 0, 3).

Solution : grad $\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$= \hat{i}\,yz^2 + \hat{j}\,xz^2 + \hat{k}\,2xyz$

$= \hat{i}\,0 + \hat{j}\,9 + \hat{k}\,0$ at (1, 0, 3)

$= 9\hat{j}$

Since we know the greatest rate of increase of $\phi = |\nabla\phi|$

$= \sqrt{(9)^2}$

= 9 Answer.

Example 3 :- Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{a}$ is a constant vector.

Proof : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Then $\vec{a}.\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$= a_1x + a_2y + a_3z$

Therefore $\nabla(\vec{a}.\vec{r}) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3$

$= \vec{a}$, hence proved.

Example 24. *If* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that :*

(*i*) *grad* $r = \dfrac{\vec{r}}{r}$ (*ii*) *grad* $\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$. (*Nagpur University, Summer 2002*)

Solution. (*i*) $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

$\therefore \quad 2r\dfrac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\bar{r}}{r}$$ **Proved.**

(*ii*) $$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{z}{r}\right) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = -\frac{\bar{r}}{r^3}$$ **Proved.**

$= -12\hat{i} - 9\hat{j} - 16\hat{k}$ **Ans.**

Example 8. *If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$ prove that grad u, grad v and grad w are coplanar vectors.* [*U.P., I Semester, 2001*]

Solution. We have,

$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy) = \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x)$$

[For vectors to be coplanar, their scalar triple product is 0]

$$\text{Now, grad } u.(\text{grad } v \times \text{grad } w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Since the scalar product of grad u, grad v and grad w are zero, hence these vectors are coplanar vectors. **Proved.**

Example 25. *Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.* (*K. University, Dec. 2008*)

Solution.

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$\left[r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\, f'(r)\frac{\partial r}{\partial z} = f'(r)\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= f'(r)\frac{xi + yj + zk}{r}$$

$$\nabla^2 f(r) = \nabla[\nabla f(r)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[f'(r)\frac{xi + yj + zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

$$= \left(f''(r)\frac{\partial r}{\partial x}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r1 - x\frac{\partial r}{\partial x}}{r^2} + \left(f''(r)\frac{\partial r}{\partial y}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y\frac{\partial r}{\partial y}}{r^2} +$$

$$\left(f''(r)\frac{\partial r}{\partial z}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r.1 - z\frac{\partial z}{\partial r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r-\frac{x^2}{r}}{r^2} + \left(f''(r)\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\frac{r-\frac{y^2}{r}}{r^2} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r-\frac{z^2}{r}}{r^2}$$

$$= \left(f''(r)\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\frac{r^2-x^2}{r^3} + \left(f''(r)\frac{y}{r}\right) + f'(r)\frac{r^2-y^2}{r^3} + \left(f''(r)\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\frac{r^2-z^2}{r^3}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{y^2+z^2}{r^3} + f''(r)\frac{y^2}{r^2} + f'(r)\frac{x^2+z^2}{r^3} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{x^2+y^2}{r^3}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2+z^2}{r^3} + \frac{z^2+x^2}{r^3} + \frac{x^2+y^2}{r^3}\right]$$

$$= f''(r)\frac{x^2+y^2+x^2}{r^2} + f'(r)\frac{2(x^2+y^2+z^2)}{r^3} = f''(r)\frac{r^2}{r^2} + f'(r)\frac{2r^2}{r^3}$$

$$= f''(r) + f'(r)\frac{2}{r}$$ **Ans.**

GEOMETRICAL MEANING OF GRADIENT, NORMAL

If a surface $\phi(x, y, z) = c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a *level surface* through P. *For example*, If $\phi(x, y, z)$ represents potential at the point P, then *equipotential surface* $\phi(x, y, z) = c$ is a *level surface*.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi + d\phi$.

Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\bar{r}$

$$\nabla\phi . d\bar{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \qquad ...(1)$$

If Q lies on the level surface of P, then $d\phi = 0$

Equation (1) becomes $\nabla\phi \,.\, dr = 0$. Then $\bar{\nabla}\phi$ is $\perp$ to $d\bar{r}$ (tangent).

Hence, $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$

Let $\nabla\phi = |\nabla\phi|\,\widehat{N}$, where $\widehat{N}$ is a unit normal vector. Let δn be the perpendicular distance between two level surfaces through P and R. Then the rate of change of ϕ in the direction of the normal to the surface through P is $\dfrac{\partial \phi}{\partial n}$.

$$\frac{d\phi}{dn} = \lim_{\delta n \to 0} \frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0} \frac{\nabla\phi . d\vec{r}}{\delta n}$$

$$= \lim_{\delta n \to 0} \frac{|\nabla\phi|\,\hat{N}.d\vec{r}}{\delta n} \qquad \left\{\begin{aligned} \hat{N}.\vec{\delta r} &= |\vec{N}||\vec{\delta r}|\cos\theta \\ &= |\vec{\delta r}|\cos\theta = \delta n \end{aligned}\right\}$$

$$= \lim_{\delta n \to 0} \frac{|\nabla\phi|\,\delta n}{\delta n} = |\nabla\phi|$$

$$\therefore \qquad |\nabla\phi| = \frac{\partial\phi}{\partial n}$$

Hence, gradient ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

Example 12 :- Find a unit vector which is perpendicular to the surface of the paraboloid of revolution.
$z = x^2+y^2$ at the point (1, 2, 5)

(B.P.S.C 1997)

Solution :- $\phi = x^2+y^2-z$

$\therefore$ grad $\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$= \hat{i}.2x + \hat{j}.2y - \hat{k}.1$

$= 2\hat{i} + 4\hat{j} - \hat{k}$ at point (1, 2, 5)

Hence unit normal = $\dfrac{\text{grad}\phi}{|\text{grad}\phi|}$

$= \dfrac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{4+16+1}}$

$= \dfrac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}$ Answer.

NORMAL AND DIRECTIONAL DERIVATIVE

(*i*) **Normal.** If $\phi(x, y, z) = c$ represents a family of surfaces for different values of the constant c. On differentiating ϕ, we get $d\phi = 0$

But $\qquad d\phi = \nabla\phi . d\vec{r} \quad$ so $\quad \nabla\phi . dr = 0$

The scalar product of two vectors $\nabla\phi$ and $d\vec{r}$ being zero, $\nabla\phi$ and $d\vec{r}$ are perpendicular to each other. $d\vec{r}$ is in the direction of tangent to the given surface.

Thus $\nabla\phi$ is a vector *normal* to the surface $\phi(x, y, z) = c$.

(*ii*) **Directional derivative.** The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi.\hat{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

$$\frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta\phi}{\delta r} \qquad \text{where, } \delta r = PQ$$

$\frac{\partial\phi}{\partial r}$ is called the *directional derivative* of ϕ at P in the direction of PQ.

Let a unit vector along PQ be $\hat{N}'$.

$$\frac{\delta n}{\delta r} = \cos\theta \Rightarrow \delta r = \frac{\delta n}{\cos\theta} = \frac{\delta n}{\hat{N}\cdot\hat{N}'} \qquad \text{...(1)}$$

Now
$$\frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0}\left[\frac{\delta\phi}{\dfrac{\delta n}{\hat{N}.\hat{N}'}}\right] = \hat{N}.\hat{N}'\frac{\partial\phi}{\partial n} \qquad \left[\text{From (1), } \delta r = \frac{\delta n}{\hat{N}.\hat{N}'}\right]$$

$$= \hat{N}'.\hat{N}\,|\nabla\phi| \quad = \hat{N}'.\nabla\phi \qquad (\because \hat{N}\,|\nabla\phi| = \nabla\phi)$$

Hence, $\frac{\partial\phi}{\partial r}$, directional derivative is the component of $\nabla\phi$ in the direction $\hat{N}'$.

$$\frac{\partial\phi}{\partial r} = \hat{N}'.\nabla\phi = |\nabla\phi|\cos\theta \le |\nabla\phi|$$

Hence, $\nabla\phi$ is the maximum rate of change of ϕ.

Example 10. *Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$.* (*M.U. 2008*)

Solution. Let $\phi(x, y, z) = xy^3z^2 - 4$

We know that $\nabla\phi$ is the vector normal to the surface $\phi(x, y, z) = c$.

$$\text{Normal vector} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Now
$$= \hat{i}\frac{\partial}{\partial x}(xy^3z^2) + \hat{j}\frac{\partial}{\partial y}(xy^3z^2) + \hat{k}\frac{\partial}{\partial z}(xy^3z^2)$$

$\Rightarrow$
$$\text{Normal vector} = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

Normal vector at $(-1, -1, 2) = -4\hat{i} - 12\hat{j} + 4\hat{k}$

Unit vector normal to the surface at $(-1, -1, 2)$.

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k}) \qquad \textbf{Ans.}$$

Example 9. *Find the directional derivative of $x^2y^2z^2$ at the point (1, 1, –1) in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.*

(*Nagpur University, Summer 2005*)

Solution. Let $\phi = x^2\, y^2\, z^2$

Directional Derivative of ϕ

$$= \nabla\phi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$\nabla\phi = 2xy^2z^2\,\hat{i} + 2yx^2z^2\,\hat{j} + 2zx^2y^2\,\hat{k}$$

Directional Derivative of ϕ at (1, 1, –1)

$$= 2(1)(1)^2(-1)^2\,\hat{i} + 2(1)(1)^2\,(-1)^2\,\hat{j} + 2(-1)(1)^2\,(1)^2\,\hat{k}$$

$$= 2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k} \qquad \ldots(1)$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = e^t\,\hat{i} + (\sin 2t + 1)\,\hat{j} + (1 - \cos t)\,\hat{k}$$

Tangent vector, $$\vec{T} = \frac{d\,\vec{r}}{d\,t} = e^t\,\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$$

$$\text{Tangent(at } t = 0) = e^0\,\hat{i} + 2\,(\cos 0)\,\hat{j} + (\sin 0)\,\hat{k} = \hat{i} + 2\,\hat{j} \qquad \ldots(2)$$

$$\text{Required directional derivative along tangent} = (2\,\hat{i} + 2\,\hat{j} - 2\,\hat{k})\,\frac{(\hat{i} + 2\,\hat{j})}{\sqrt{1+4}}$$

[From (1), (2)]

$$= \frac{2+4+0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$ **Ans.**

Example 11. *Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).* (*Nagpur University, Summer 2001*)

Solution. Rate of change of $\phi = \Delta\,\phi$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x\,y\,z) = \hat{i}\,yz + \hat{j}\,xz + \hat{k}\,xy$$

Rate of change of ϕ at (1, 1, 1) = $(\hat{i} + \hat{j} + \hat{k})$

Normal to the surface $\Psi = x^2y + y^2x + yz^2 - 3$ is given as -

$$\nabla\psi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}\,(2xy + y^2) + \hat{j}\,(x^2 + 2xy + z^2) + \hat{k}\,2yz$$

$$(\nabla\psi)_{(1,\,1,\,1)} = 3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$$

$$\text{Unit normal} = \frac{3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}}{\sqrt{9+16+4}}$$

$$\text{Required rate of change of } \phi = (\hat{i} + \hat{j} + \hat{k})\cdot\frac{(3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$ **Ans.**

Example 13. *Find the values of constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)\, x$, $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1, -1, 2)$.*

(*AMIETE, II Sem., Dec. 2010, June 2009*)

Solution. Here, we have

$$\lambda x^2 - \mu yz = (\lambda + 2)\, x \qquad ...(1)$$

$$4x^2\, y + z^3 = 4 \qquad ...(2)$$

Normal to the surface (1), $= \nabla\left[\lambda x^2 - \mu yz - (\lambda+2)x\right]$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[\lambda x^2 - \mu yz - (\lambda+2)x\right]$$

$$= \hat{i}\,(2\lambda x - \lambda - 2) + \hat{j}\,(-\mu z) + \hat{k}\,(-\mu y)$$

Normal at $(1, -1, 2) = \hat{i}\;(2\lambda - \lambda - 2) - \hat{j}\;(-2\mu) + \hat{k}\,\mu \qquad ...(3)$

$$= \hat{i}\;(\lambda - 2) + \hat{j}\;(2\mu) + \hat{k}\,\mu$$

Normal at the surface (2)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2 y + z^3 - 4)$$

$$= \hat{i}\;(8xy) + \hat{j}\;(4x^2) + \hat{k}\;(3z^2)$$

Normal at the point $(1, -1, 2) = -8\,\hat{i} + 4\,\hat{j} + 12\,\hat{k} \qquad ...(4)$

Since (3) and (4) are orthogonal so

$$\left[\hat{i}\,(\lambda-2) + \hat{j}\,(2\mu) + \hat{k}\,\mu\right].\left[-8\hat{i} + 4\hat{j} + 12\hat{k}\right] = 0$$

$$-8\,(\lambda-2) + 4\,(2\mu) + 12\mu = 0 \quad \Rightarrow \quad -8\lambda + 16 + 8\mu + 12\mu = 0$$

$$-8\lambda + 20\mu + 16 = 0 \quad \Rightarrow \quad 4(-2\lambda + 5\mu + 4) = 0$$

$$-2\lambda + 5\mu + 4 = 0 \quad \Rightarrow \quad 2\lambda - 5\mu = 4 \qquad ...(5)$$

Point $(1, -1, 2)$ will satisfy (1)

$\therefore \qquad \lambda(1)^2 - \mu(-1)(2) = (\lambda+2)\,(1) \Rightarrow \lambda + 2\mu = \lambda + 2 \Rightarrow \mu = 1$

Putting $\mu = 1$ in (5), we get

$$2\lambda - 5 = 4 \quad \Rightarrow \quad \lambda = \frac{9}{2}$$

Hence $\quad \lambda = \dfrac{9}{2}$ and $\mu = 1$ **Ans.**

2

Example 14. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (*Nagpur University, Summer 2002*)

Solution. Normal on the surface $(x^2 + y^2 + z^2 - 9 = 0)$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(1)

Normal on the surface $(z = x^2 + y^2 - 3) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3)$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$ ...(2)

Let θ be the angle between normals (1) and (2).

$$(4\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{16+4+16}\,\sqrt{16+4+1}\cos\theta$$

$$16 + 4 - 4 = 6\sqrt{21}\cos\theta \quad \Rightarrow \quad 16 = 6\sqrt{21}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{8}{3\sqrt{21}} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$ **Ans.**

Example 15. *Find the directional derivative of $\frac{1}{r}$ in the direction $\bar{r}$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.*
(*Nagpur University, Summer 2004, U.P., I Semester, Winter 2005, 2002*)

Solution. Here, $\phi(x, y, z) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

Now
$$\nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{i} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{j} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\hat{k}$$

$$= \left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2x\right\}\hat{i} + \left(-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2y\right)\hat{j} + \left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}2z\right\}\hat{k}$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{...(1)}$$

and $\hat{r}$ = unit vector in the direction of $x\hat{i} + y\hat{j} + z\hat{k}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad \text{...(2)}$$

So, the required directional derivative

$$= \nabla\phi.\hat{r} = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} \quad \text{[From (1), (2)]}$$

$$= \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2}$$ **Ans.**

Example 16. *Find the direction in which the directional derivative of* $\phi(x, y) = \dfrac{x^2 + y^2}{xy}$ *at* (*1, 1*) *is zero and hence find out component of velocity of the vector* $\bar{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$ *in the same direction at t = 1.* (*Nagpur University, Winter 2000*)

Solution. Directional derivative $= \nabla\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\left(\dfrac{x^2 + y^2}{xy}\right)$

$$= \hat{i}\left[\frac{xy.2x - (x^2 + y^2)y}{x^2y^2}\right] + \hat{j}\left[\frac{xy.2y - x(y^2 + x^2)}{x^2y^2}\right]$$

$$= \hat{i}\left[\frac{x^2y - y^3}{x^2y^2}\right] + \hat{j}\left[\frac{xy^2 - x^3}{x^2y^2}\right]$$

Directional Derivative at (1, 1) $= \hat{i}\,0 + \hat{j}\,0 = 0$

Since $(\nabla\phi)_{(1,1)} = 0$, the directional derivative of ϕ at (1, 1) is zero in any direction.

Again $\quad \bar{r} = (t^3 + 1)\hat{i} + t^2\hat{j}$

Velocity, $\quad \bar{v} = \dfrac{d\bar{r}}{dt} = 3t^2\hat{i} + 2t\hat{j}$

Velocity at $\quad t = 1$ is $= 3\hat{i} + 2\hat{j}$

The component of velocity in the same direction of velocity

$$= (3\hat{i} + 2\hat{j}).\left(\frac{3\hat{i} + 2\hat{j}}{\sqrt{9+4}}\right) = \frac{9+4}{\sqrt{13}} = \sqrt{13}$$ **Ans.**

Example 17. *Find the directional derivative of* $\phi(x, y, z) = x^2yz + 4xz^2$ *at* (*1, –2, 1*) *in the direction of* $2\hat{i} - \hat{j} - 2\hat{k}$. *Find the greatest rate of increase of* ϕ.

(*Uttarakhand, I Semester, Dec. 2006*)

Solution. Here, $\quad \phi(x, y, z) = x^2yz + 4xz^2$

Now, $\quad \nabla\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$\nabla\phi$ at $(1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\hat{i} + (1\times 1)\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k}$

$$= (-4+4)\hat{i} + \hat{j} + (-2+8)\hat{k} = \hat{j} + 6\hat{k}$$

Let $\quad \hat{a}$ = unit vector $= \dfrac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \dfrac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$

So, the required directional derivative at (1, –2, 1)

$$= \nabla\phi.\hat{a} = (\hat{j} + 6\hat{k}).\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left|\hat{j} + 6\hat{k}\right| = \sqrt{1+36}$

$$= \sqrt{37}$$ **Ans.**

Example 18. *Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).*

(*AMIETE, Dec. 2010, Nagpur University, Summer 2008, U.P., I Sem., Winter 2000*)

Solution. Directional derivative $= \bar{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

Directional Derivative at the point P (1, 2, 3) $= 2\hat{i} - 4\hat{j} + 12\hat{k}$ ...(1)

$$\overline{PQ} = \overline{Q} - \overline{P} = (5, 0, 4) - (1, 2, 3) = (4, -2, 1) \qquad ...(2)$$

Directional Derivative along $PQ = (2\hat{i} - 4\hat{j} + 12\hat{k}).\dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$ [From (1) and (2)]

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$ **Ans.**

Example 19. *Find the directional derivative of $\phi = 4\, e^{2x - y + z}$ at the point (1, 1, –1) in the directional towards the point (–3, 5, 6).* (*Nagpur University, Winter 2003, Summer 2000*)

Solution. Directional derivative $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) 4\, e^{2x-y+z}$$

$$= 4[\hat{i}\, 2e^{2x-y+z} - \hat{j}\, e^{2x-y+z} + \hat{k}\, e^{2x-y+z}] = 4[2\hat{i} - \hat{j} + \hat{k}]\, e^{2x-y+z}$$

Directional Derivative at (1, 1, –1)

$$= 4[2\hat{i} - \hat{j} + \hat{k}]\, e^{2-1-1} = 4[2\hat{i} - \hat{j} + \hat{k}] \qquad ...(1)$$

Direction of Directional Derivative

$$= (-3\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = -4\hat{i} + 4\hat{j} + 7\hat{k} \qquad ...(2)$$

Directional Derivative in the direction of $(-4\hat{i} + 4\hat{j} + 7\hat{k})$

$$= \left|(8\hat{i} - 4\hat{j} + 4\hat{k}).\frac{(-4\hat{i} + 4\hat{j} + 7\hat{k})}{\sqrt{16+16+49}}\right| \qquad \text{[From (1) and (2)]}$$

$$= \left|\frac{1}{9}[-32 - 16 + 28]\right| = \left|-\frac{20}{9}\right| = \frac{20}{9}$$ **Ans.**

Example 20. *For the function* $\phi(x, y) = \dfrac{x}{x^2+y^2}$, *find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at (0, 2).*

(*A.M.I.E.T.E., Winter 2002*)

Solution. Directional derivative = $\vec{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{x}{x^2+y^2} = \hat{i}\left(\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}\right) - \hat{j}\frac{x(2y)}{(x^2+y^2)^2}$$

$$= \hat{i}\frac{y^2-x^2}{(x^2+y^2)^2} - \hat{j}\frac{2xy}{(x^2+y^2)^2}$$

Directional derivative at the point (0, 2)

$$= \hat{i}\frac{4-0}{(0+4)^2} - \hat{j}\frac{2(0)(2)}{(0+4)^2} = \frac{\hat{i}}{4}$$

Directional derivative at the point (0, 2) in the direction $\overrightarrow{CA}$ *i.e.* $\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$$= \frac{\hat{i}}{4}\cdot\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) \qquad \left\{\begin{aligned}\overrightarrow{CA} = \overrightarrow{OB} + \overrightarrow{BA} &= \hat{i}\cos 30° + \hat{j}\sin 30° \\ &= \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)\end{aligned}\right\}$$

$$= \frac{\sqrt{3}}{8}$$

Ans.

Example 21. *Find the directional derivative of* $\vec{V}^2$, *where* $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, *at the point (2, 0, 3) in the direction of the outward normal to the sphere* $x^2+y^2+z^2 = 14$ *at the point (3, 2, 1).*

(*A.M.I.E.T.E., Dec. 2007*)

Solution. $V^2 = \vec{V}.\vec{V}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}).(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}) = x^2y^4 + z^2y^4 + x^2z^4$$

Directional derivative $= \vec{\nabla}V^2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2y^4z + 4x^2z^3)\hat{k}$$

Directional derivative at (2, 0, 3) $= (0 + 2\times 2\times 81)\hat{i} + (0+0)\hat{j} + (0 + 4\times 4\times 27)\hat{k}$

$$= 324\hat{i} + 432\hat{k} = 108(3\hat{i} + 4\hat{k}) \qquad \text{...(1)}$$

Normal to $x^2+y^2+z^2-14 = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14)$$

$$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Normal vector at (3, 2, 1) $= 6\hat{i} + 4\hat{j} + 2\hat{k}$ ...(2)

Unit normal vector $= \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \dfrac{2(3\hat{i}+2\hat{j}+\hat{k})}{2\sqrt{14}} = \dfrac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$ [From (1), (2)] Directional derivative along the normal = $108(3\hat{i} + 4\hat{k}).\dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$.

$$= \frac{108 \times (9+4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$ **Ans.**

Example 22. *Find the directional derivative of $\nabla(\nabla f)$ at the point (1, – 2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.* (*U.P., I Semester, Dec 2008*)

Solution. Here, we have

$$f = 2x^3 y^2 z^4$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4) = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Directional derivative of $\nabla(\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j}$$
$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Directional derivative at (1, – 2, 1) = $(48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k}$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Normal to$(xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

Normal at(1, – 2, 1) = $\hat{i} - 4\hat{j} + 2\hat{k}$

Unit Normal Vector = $\dfrac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}} = \dfrac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$

Directional derivative in the direction of normal

$$= (348\hat{i} - 144\hat{j} + 400\hat{k})\frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$ **Ans.**

Example 23. *If the directional derivative of* $\phi = a\,x^2 y + b\,y^2 z + c\,z^2 x$ *at the point* (1, 1, 1) *has maximum magnitude 15 in the direction parallel to the line* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, *find the values of a, b and c.* (*U.P. I semester, Winter 2001*)

Solution. Given $\phi = a\,x^2 y + b\,y^2 z + c\,z^2 x$

$$\therefore \quad \overline{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a\,x^2\,y + b\,y^2\,z + c\,z^2\,x)$$

$$= \hat{i}(2axy + cz^2) + \hat{j}(ax^2 + 2byz) + \hat{k}(by^2 + 2czx)$$

$$\overline{\nabla}\phi \text{ at the point } (1, 1, 1) = \hat{i}(2a+c) + \hat{j}(a+2b) + \hat{k}(b+2c) \qquad ...(1)$$

We know that the maximum value of the directional derivative is in the direction of $\overline{\nabla}\phi$.

$$i.e.\ |\nabla\phi| = 15 \Rightarrow (2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$$

But, the directional derivative is given to be maximum parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \text{ i.e., parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}. \qquad ...(2)$$

On comparing the coefficients of (1) and (2)

$$\Rightarrow \quad \frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1}$$

$$\Rightarrow \quad 2a + c = -2b - a \Rightarrow 3a + 2b + c = 0 \qquad ...(3)$$

and $2b + a = -2(2c + b)$

$$\Rightarrow \quad 2b + a = -4c - 2b \Rightarrow a + 4b + 4c = 0 \qquad ...(4)$$

Rewriting (3) and (4), we have

$$\left.\begin{array}{l} 3a + 2b + c = 0 \\ a + 4b + 4c = 0 \end{array}\right\} \Rightarrow \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \text{ (say)}$$

$$\Rightarrow \quad a = 4k, \quad b = -11k \quad \text{and} \quad c = 10k.$$

Now, we have

$$(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = (15)^2$$

$$\Rightarrow \quad (8k + 10k)^2 + (-22k + 4k)^2 + (20k - 11k)^2 = (15)^2$$

$$\Rightarrow \quad k = \pm\frac{5}{9}$$

$$\Rightarrow \quad a = \pm\frac{20}{9}, \quad b = \pm\frac{55}{9} \quad \text{and} \quad c = \pm\frac{50}{9} \qquad \textbf{Ans.}$$

Example 6 : Find the directional derivative of $\phi = xy + yz + zx$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$ at (1, 2, 0)

Solution :- Since we know

directional derivative = $\hat{a} \cdot \text{grad}\ \phi$

Now grad $\phi = \hat{i}\dfrac{\partial \phi}{\partial x} + \hat{j}\dfrac{\partial \phi}{\partial y} + \hat{k}\dfrac{\partial \phi}{\partial z}$

$= (y + z)\ \hat{i} + (z + x)\ \hat{j} + (x + y)\ \hat{k}$

$= 2\hat{i} + \hat{j} + 3\hat{k}$ at (1, 2, 0)

Also $\hat{a} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

$\therefore$ directional derivative $= \dfrac{1}{3}\ (\hat{i} + 2\hat{j} + 2\hat{k}).(2\hat{i} + \hat{j} + 3\hat{k})$

$= \dfrac{1}{2}(2+2+6)$

$= \dfrac{10}{3}$ Answer.

Problem 11 :- Find the directional derivative of a function $\phi = x^2\ y^3\ z^4$ at the point (2, 3, -1) in the direction making equal angles with the positive x, y, & z axis.

Solution :- Given $\phi = x^2y^3z^4$

Now grad $\phi = \hat{i}\dfrac{\partial \phi}{\partial x} + \hat{j}\dfrac{\partial \phi}{\partial y} + \hat{k}\dfrac{\partial \phi}{\partial z}$

$= 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k}$

If $\hat{a}$ be the unit vector in the required direction and α be the angle which $\hat{a}$ makes with the axes, then

$\hat{a} = (\cos\alpha)\,\hat{i} + (\cos\alpha)\,\hat{j} + (\cos\alpha)\,\hat{k}$

where $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

which gives $\cos\alpha = \dfrac{1}{\sqrt{3}}$

$\therefore \hat{a} = \dfrac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

$\therefore$ directional derivative = $\hat{a} \cdot \text{grad}\ \phi$

$= \dfrac{1}{\sqrt{3}}\ (\hat{i} + \hat{j} + \hat{k}).(2xy^2z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k})$

$= \dfrac{1}{\sqrt{3}}\ (2xy^3z^4 + 3x^2y^2z^4 + 4x^2y^3z^3)$

$= \dfrac{1}{\sqrt{3}}\ (108 + 108 - 432)$ at the point (2, 3, -1)

$= -\dfrac{216}{\sqrt{3}}$ Answer.

Example for Practice Purpose

EXERCISE 23.1

1. Evaluate grad ϕ if $\phi = \log(x^2 + y^2 + z^2)$ **Ans.** $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

2. Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 5$ at the point (0, 1, 2). **Ans.** $\dfrac{1}{\sqrt{5}}(\hat{j} + 2\hat{k})$

(*AMIETE, June 2010*)

3. Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point (1, –1, 1) in the direction of (3, 1, –1) (*A.M.I.E.T.E. Winter 2009, 2000*) **Ans.** $\dfrac{5}{\sqrt{11}}$

4. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero.

(*Nagpur Winter 2000*) **Ans.** $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

5. Find the directional derivative of the scalar function of $(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3). **Ans.** $\dfrac{27}{\sqrt{11}}$

6. The temperature of the points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move? **Ans.** $\dfrac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

7. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find grad ϕ at the point (1, –2, –1) **Ans.** $-(16\hat{i} + 9\hat{j} + 4\hat{k})$

8. Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point (2, –2, 3).

Ans. $\dfrac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

9. What is the greatest rate of increase of the function $u = xyz^2$ at the point (1, 0, 3)? **Ans.** 9

10. If θ is the acute angle between the surfaces $xyz^2 = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1) show that $\cos\theta = 3/7\sqrt{6}$.