Recapitulate

Dispersion relation for free particle: $\omega(k)$

$$\lambda = h / p \qquad p = \hbar k$$

$$\omega = E/\hbar = \frac{\sqrt{p^2c^2 + m_0^2c^4}}{\hbar} \quad = c\sqrt{k^2 + \left(\frac{m_0 c}{\hbar}\right)^2}$$

$$\mathrm{v_p} = \frac{\omega}{k} = c\sqrt{1 + \left(\frac{m_0 c}{\hbar k}\right)^2}$$

$$\mathrm{v_g} = \left[\mathrm{v_p} + k\frac{d\mathrm{v_p}}{dk}\right]_{k_0} = c\left[1 + \left(\frac{mc}{\hbar k_0}\right)^2\right]^{-1/2} = \frac{c^2}{\mathrm{v_p}\big|_{k_0}} = \mathrm{v}$$

Nonrelativistic

$$E=\hbar\omega=\frac{p^2}{2m}=\frac{\hbar^2k^2}{2m} \Rightarrow \omega(k)=\left(\frac{\hbar}{2m}\right)k^2$$

$$\mathrm{v}_\mathrm{g}=\frac{\partial\omega(k)}{\partial \mathrm{k}}=\frac{\hbar k}{m}=\frac{p}{m}=\sqrt{\frac{2}{m}\frac{p^2}{2m}}=\sqrt{\frac{2E}{m}}$$

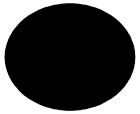
$$\mathrm{v}_\mathrm{g}^2=\frac{2E}{m} \Rightarrow \mathrm{E}=\frac{1}{2}m\mathrm{v}_\mathrm{g}^2 \Rightarrow \mathrm{v}_\mathrm{g}=\mathrm{v}$$

$$\mathrm{v}_\mathrm{p}=\frac{\omega}{k}=\frac{\hbar k}{2m} \Rightarrow \mathrm{v}_\mathrm{p}^2=\frac{\hbar^2k^2}{4m^2}=\frac{\hbar^2k^2}{2m}\frac{1}{2m}=\frac{E}{2m}$$

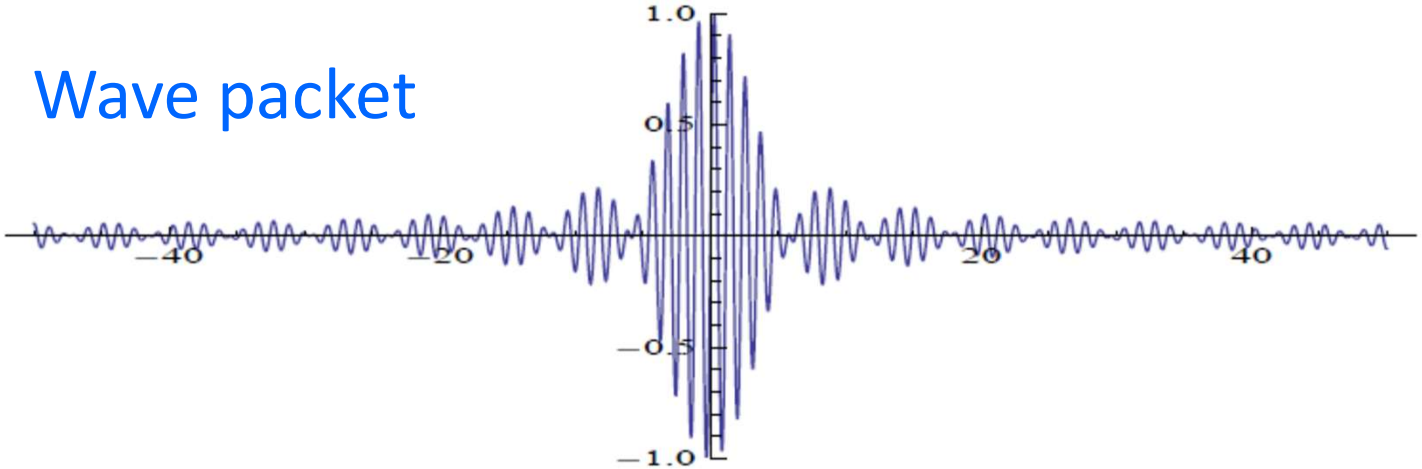
$$E=2m\mathrm{v}_\mathrm{p}^2 \Rightarrow \mathrm{v}_\mathrm{p}\neq\mathrm{v}$$

Heisenberg’s Uncertainty Relation

Particle



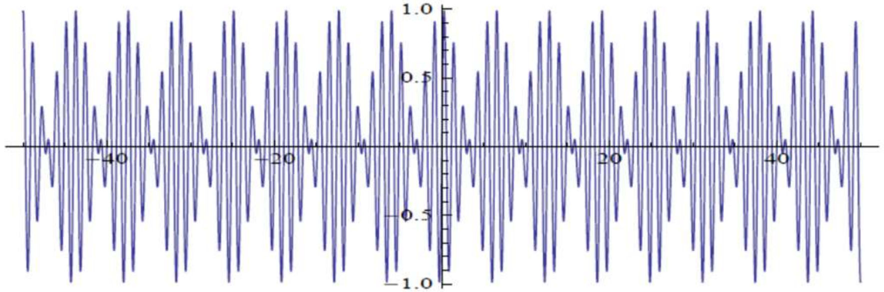
Wave packet



A wave packet is a group of waves with slightly different wavelengths interfering with one another in a way that the amplitude of the group (envelope) is non-zero in the neighbourhood of the particle.

A wave packet is localized; it is a good representation of a particle

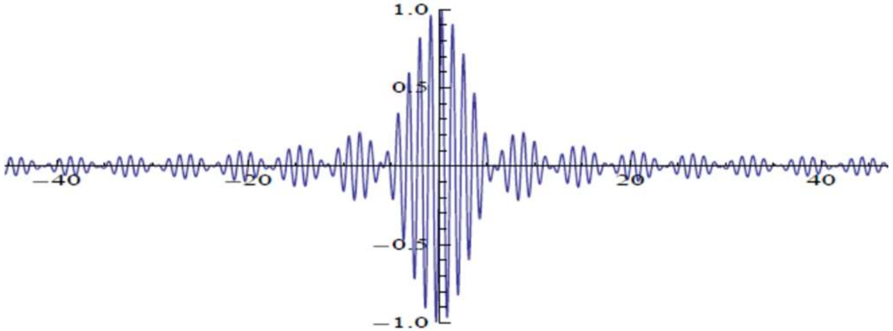
Example-1



[sin(5x) + sin(6x)]/2

An ideal wave has precise wavelength. Therefore k is precisely known. This implies wave extends from -∞ to +∞, which means the position is uncertain.

Δx



[sin(5x) + sin(5.0625x) + sin (5.125x) + sin(5.1875x) + sin (5.25x) + sin(5.3125x) + sin(5.375x) + sin(5.4375x) + sin(5.5x) + sin(5.5625x) + sin (5.625x) + sin(5.6875x) + sin (5.75x) + sin(5.8125x) + sin(5.875x) + sin(5.9375x) + sin(6x)]/17

Large range of wavelength means their wavelengths become more uncertain, but the position is made certain.

Δx = Uncertainty in position

Δk = Uncertainty in wavelength

$$\Delta x \Delta k \approx 1 \Rightarrow \Delta x \Delta p \approx \hbar$$

(Since $p = \hbar k$)

Smaller is the **spatial extent** Δx, **larger** is the **range of wavelengths** or wavenumbers, Δk needed to form a wave packet.

Similarly, if the time domain Δt is small, we require a wide spread of frequencies to form a group

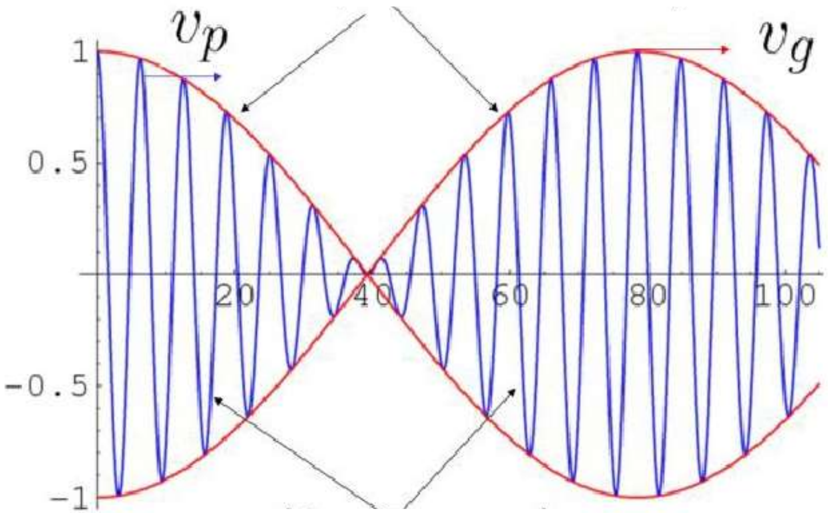
$$\Delta t \Delta \omega \approx 1 \Rightarrow \Delta t \Delta E \approx \hbar$$

(Since $E = \hbar \omega$)

$$\psi_1 + \psi_2 = \psi = A\sin(kx - \omega t) + A\sin[(k + \Delta k)x - (\omega + \Delta\omega)t]$$

$$= 2A\sin\left[\frac{(2k + \Delta k)x}{2} - \frac{(2\omega + \Delta\omega)t}{2}\right]\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$

$$\cos\left(\frac{\Delta kx}{2} - \frac{\Delta\omega t}{2}\right)$$



$$\sin(kx - \omega t)$$

$$\longleftarrow \Delta x \longrightarrow$$

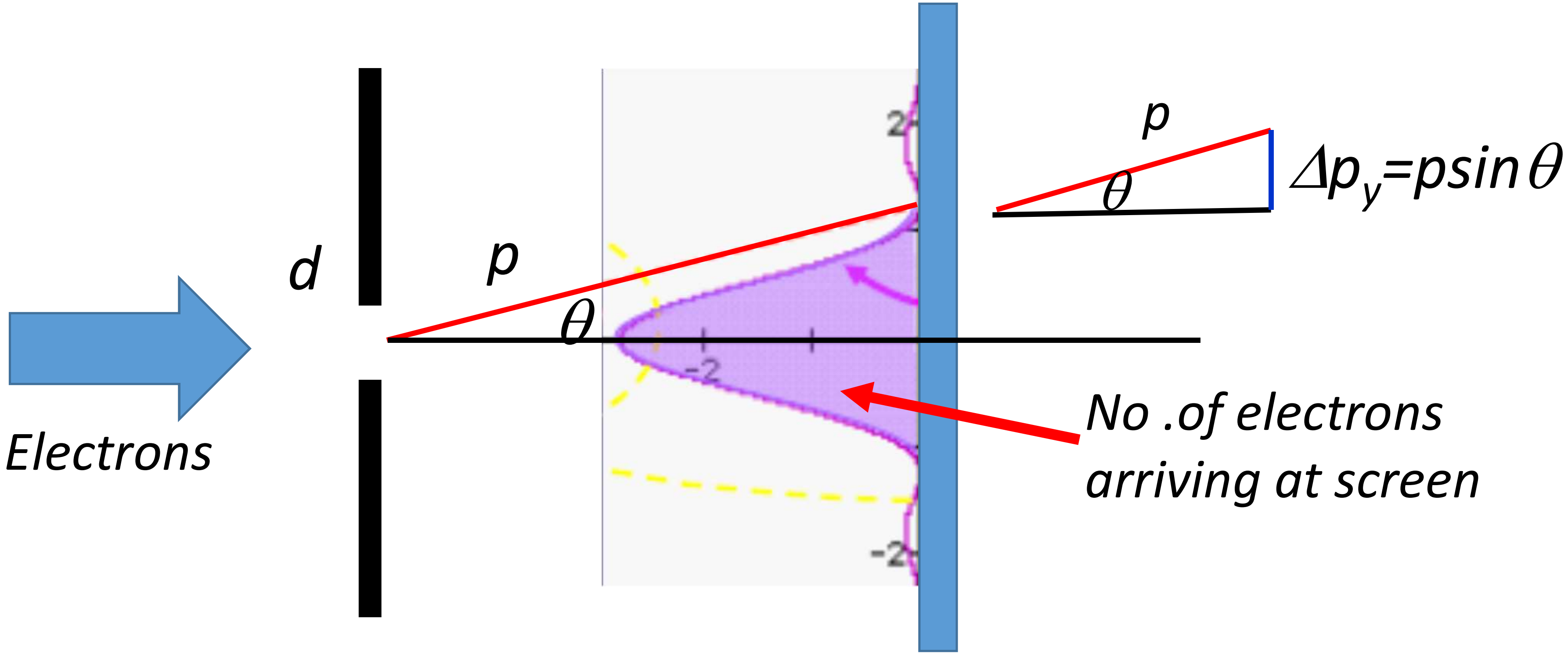
Δx = Distance between two consecutive minima

$$\frac{1}{2}\Delta k \Delta x = \pi \Rightarrow \Delta k \Delta x = 2\pi$$

$$\frac{1}{2}\Delta\omega \Delta t = \pi \Rightarrow \Delta\omega \Delta t = 2\pi$$

Example-2

Example-3 Single slit electron diffraction pattern



The condition for the first minimum: $\sin\theta = \lambda / d$ $\Rightarrow$ $\Delta y \approx d = \lambda / \sin\theta$

$$\Delta p_y \Delta y = p\sin\theta \frac{\lambda}{\sin\theta} = \lambda p = \frac{h}{p} p = h$$

Heisenberg's Uncertainty Relation

It is impossible to determine *simultaneously* with **unlimited precision** the **position and momentum** of a particle.

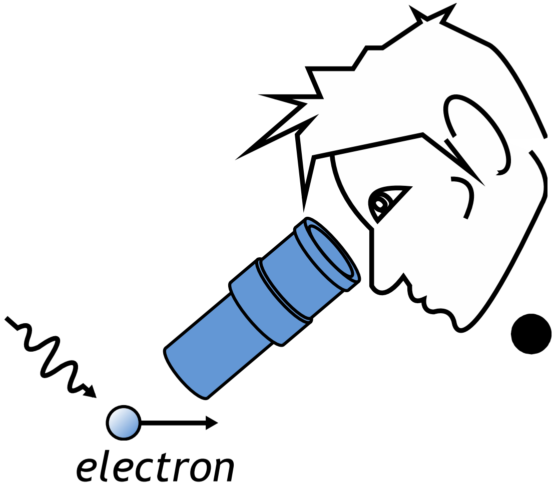
If a measurement of position is made with precision Δx and a simultaneous measurement of momentum in the x-direction is made with precision Δp_x, then the product of the two uncertainties can never be smaller than $\hbar / 2$

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Position-Momentum Uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Energy-Time Uncertainty relation



Measuring Position and Momentum of an Electron

- Shine light on electron and detect reflected light using a microscope

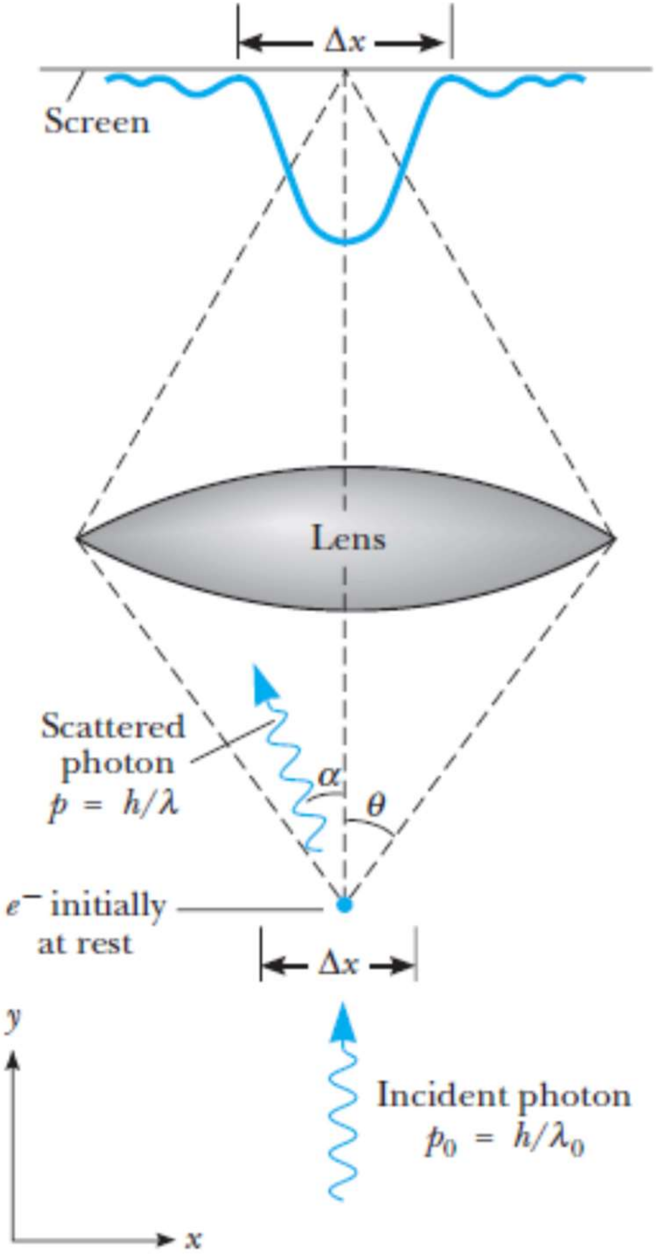
- Minimum uncertainty in position is given by wavelength of light

- So to determine position accurately, it is necessary to use light of short wavelength

- By Planck's law $E = hc/\lambda$, a photon of short wavelength has large energy. Thus it would impart a large 'kick' to the electron to make momentum uncertain.

- But to determine its momentum accurately, electron must only be given a small kick

- This means using light of long wavelength, which will make position uncertain!



The Heisenberg Microscope

The scattered photon is collected by the lens. Therefore it must be scattered within [-θ, +θ]

Momentum imparted to electron is in the range *[-hsinθ/λ, hsinθ/λ]*

$$\Delta p_x = 2h\sin\theta / \lambda$$

Uncertainty in the position (~ resolution of the microscope)

$$\Delta x = \lambda / 2\sin\theta$$

$$\Delta p_x \Delta x = h$$

Heisenberg's Uncertainty Relation

Position-Momentum Uncertainty relation

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Energy-Time Uncertainty relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Conjugate variables

$\{p_x, x\}$, $\{E, t\}$ are called *conjugate variables.*

$\{p_y, y\}$, $\{p_z, z\}$, are other examples of *conjugate variables.*

The conjugate variables cannot be measured (or known) to infinite precision simultaneously.

$$\Delta p_y \Delta y \geq \frac{\hbar}{2} \qquad \Delta p_z \Delta z \geq \frac{\hbar}{2}$$

What $\Delta p_x \Delta x \geq \frac{\hbar}{2}$ means

It sets the intrinsic lowest possible limits on the uncertainties in knowing the values of *p_x* and *x*, no matter how good an experiments is made.

These uncertainties are inherent in the physical world and have nothing to do with the skill of the observer

It is impossible to specify simultaneously and with infinite precision the linear momentum and the corresponding position of a particle

What $\Delta E \Delta t \geq \frac{\hbar}{2}$ **means**

If a system is known to exist in a state of energy *E* over a limited period Δ*t*, then this energy is uncertain by at least an amount *h*/(4$\pi\Delta$*t*)

therefore, the energy of an object or system can be measured with infinite precision (ΔE=0) only if the object of system exists for an infinite time (Δt→∞)

Because h is so small, these uncertainties are not observable in normal everyday situations (macroscopic physics)

$$\hbar = 1.054 \times 10^{-34} [\mathrm{J.s}]$$

A cricket ball

m = 100 g

v = 40 m/s

p= mv = 0.1 x 40 = 4 kg m/s

Suppose the momentum is measured to an accuracy of 1 percent,

Δp = 0.01 *p* = 4 x 10^{-2} kg m/s

The uncertainty in position is

$$\Delta x \geq \frac{h}{4\pi\Delta p} = 1.3 \times 10^{-33}\mathrm{m}$$

An electron

m = 9.11 x 10^{-31} kg

v = 40 m/s

p = mv = 3.6 x 10^{-29} kg m/s

Suppose the momentum is measured to an accuracy of 1 percent,

Δp = 0.01 *p* = 3.6 x 10^{-31} kg m/s

The uncertainty in position is

$$\Delta x \geq \frac{h}{4\pi\Delta p} = 1.4 \times 10^{-4}\mathrm{m}$$

Can an electron be found in the nucleus?

Size of the nucleus ~ 10^{-14} m

Let us take $\Delta x = 10^{-14}$ m

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{4.13\times10^{-15}\, eV.s}{2\pi\times1\times10^{-14}\, m} \times \frac{1\times10^{8}\, m}{c} = 2\times10^{7}\,\frac{eV}{c}$$

$$E^2 = p^2c^2 + (m_e c^2)^2 = (20\mathrm{MeV})^2 + (0.511\,\mathrm{MeV})^2 \approx (20\,\mathrm{MeV})^2$$

$$\Rightarrow \quad E = 20\,\mathrm{MeV}$$

$$KE = E - m_e c^2 = (20-0.511)\,\mathrm{MeV} \approx 19.5\,\mathrm{MeV}$$

This is too large!

Electrons emitted in beta decay have energies ~ 1 MeV or less!

Estimating size of an atom using uncertainty relation

Let an electron be confined in a spherical shell of radius a

Uncertainty in position: $\Delta x = a$

Uncertainty in momentum: $\Delta p = \hbar / a$

$$p \sim \Delta p = \hbar / a$$

$$E = KE + PE = \frac{p^2}{2m} + PE = \frac{\hbar^2}{2ma^2} - \frac{(e^2 / 4\pi\varepsilon_0)}{a}$$

$$\frac{\partial E}{\partial a} = 0 \quad \Rightarrow \quad a = \frac{\hbar^2}{m(e^2 / 4\pi\varepsilon_0)} \qquad E_{\min} = -\frac{m(e^2 / 4\pi\varepsilon_0)^2}{2\hbar^2}$$

A charged π meson has rest energy of 140 MeV and a lifetime of 26 ns. What is the energy uncertainty?

$E = m_0c^2 = 140$ **MeV,** $\Delta t = 26$ **ns.**

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{1.054\times 10^{-34}}{2\times 26\times 10^{-9}} J = 2.027\times 10^{-27} J$$

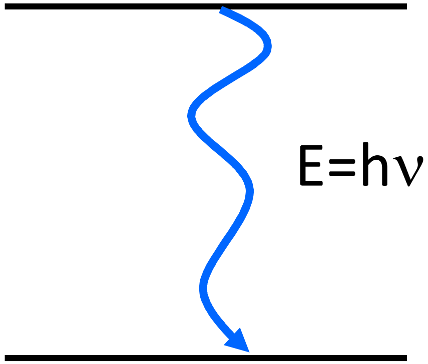
$$= 1.265\times 10^{-14}\, MeV$$

(1 MeV = 1.60218 x 10^{-13} ***J)***

$$\frac{\Delta E}{E} = \frac{1.265\times 10^{-14}}{140} = 9\times 10^{-17}$$

Radiative decay of atomic levels

Hydrogen Atom



n=2 Lifetime 0.625 ns

Photon energy = $13.6\left(\frac{1}{1^2}-\frac{1}{2^2}\right)\mathrm{eV}=10.2\,\mathrm{eV}$

n=1 Lifetime ∞

Uncertainty in energy due to finite lifetime of n=2 level:

$$\Delta E \geq \frac{\hbar}{2\Delta t}=\frac{1.054\times10^{-34}}{2\times0.625\times10^{-9}}J=0.843\times10^{-25}=0.53\times10^{-6}eV$$

$$\Delta E/E_p=\frac{0.53\times10^{-6}}{10.2}=0.52\times10^{-7}$$

Planck's Constant, Quantization and Matter Waves

- **Experiments supporting concept of quantization**: *Black Body radiation, Bohr Atom, photoelectric effect, Compton scattering*

- **They all involve Planck's constant (*h*)**

- **The appearance of *h* in a theory generally means quantum effect is taking place**

- **Due to smallness of *h* (6.626 x 10^{-34} J.s), quantum effects are easiest to be observed in experiments with microscopic systems (atomic and sub-atomic particles)**

Wave nature of a particle is also a quantum phenomenon

- The smallness of h in the relation ***(λ=h/p)*** makes wave characteristic of macroscopic particles hard to be observed

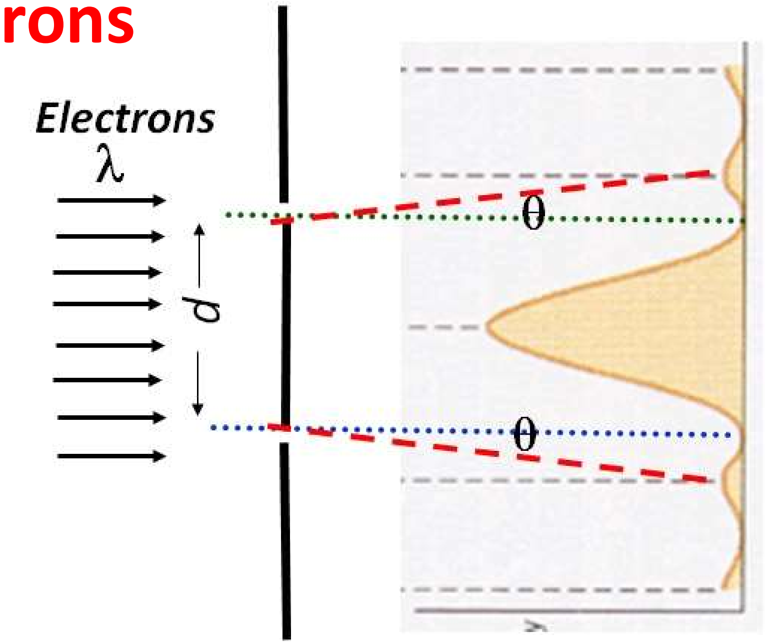
- The statement that when $h \rightarrow 0$, λ becomes vanishingly small means that the wave nature will becomes effectively ``shut-off'' and there would be loss of its wave nature whenever the relevant scale (e.g. the p of the particle) is too large in comparison with $h \sim 10^{-34}$ J.s

- The wave nature of a particle will only show up when the linear momentum scale p of the particle times the length dimension characterising the experiment (p x d) is comparable (or smaller) to the quantum scale of h

Double slit experiment with electrons

$$KE = m_e v^2 / 2 \qquad p = \sqrt{2 m_e KE}$$

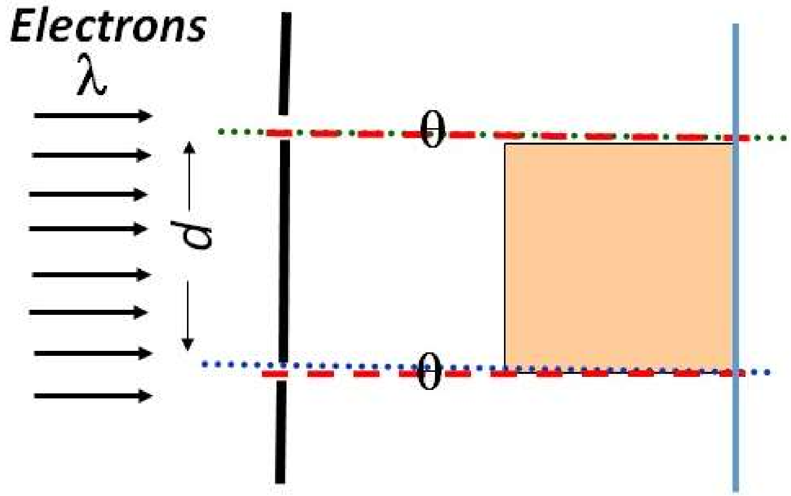
$$\lambda = h / \sqrt{2 m_e KE}$$



Resolution angle on interference pattern = $\theta = \lambda / d$

For non vanishing θ, interference pattern is seen (wave property).

$$\frac{\lambda}{d} \sim 1 \;\Rightarrow \frac{h}{pd} \sim 1 \Rightarrow h \sim pd$$



If $\theta \to 0$, no interference pattern (wave property not revealed)

$$\frac{\lambda}{d} << 1 \;\Rightarrow \frac{h}{pd} << 1 \Rightarrow h << pd$$

Important to note……..

- *h* characterises the scale at which quantum nature of particles starts to take over from macroscopic physics.

- Whenever *h* is not negligible compared to the characteristic scales of the experimental setup (= *p d* in the previous example), particle behaves like wave; whenever *h* is negligible compared to *pd*, particle behave like just a conventional particle.

Classical world

- The observer is **objective** and **passive**
- Physical events happen **independently** of whether there is an observer or not
- This is known as **objective reality**

Quantum world

- The observer is ***not*** objective and passive
- The act of observation changes the physical system irrevocably
- This is known as **subjective reality**

The Uncertainty Principle