Wave Function, Fourier Transform, Gaussian Wave packet and Heisenberg uncertainty relation

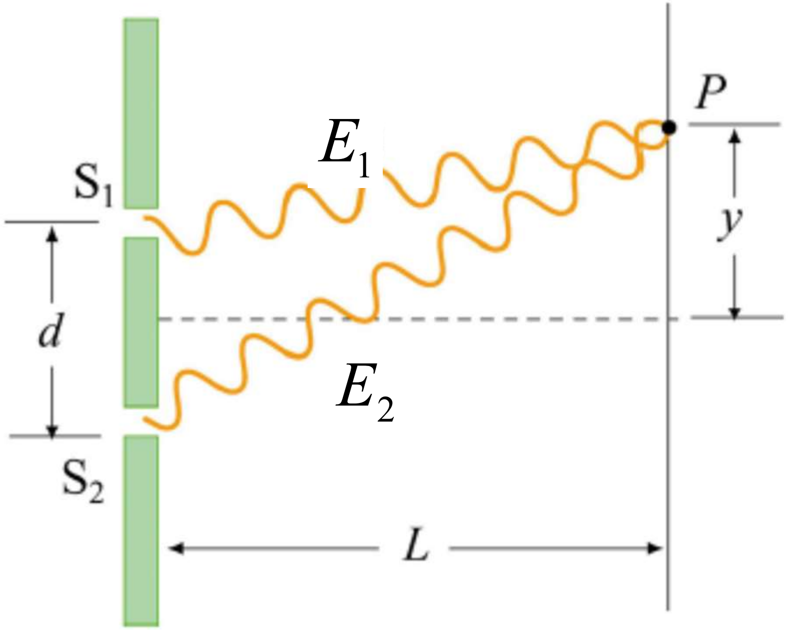
Young's Double Slit Experiment

Electric field at *P* from slit S_1 = E_1

Electric field at *P* from slit S_2 = E_2

Total electric field at point *P* =

$$E = E_1 + E_2$$



Intensity at P = $I = |E|^2 = |E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + E_1^* E_2 + E_1 E_2^*$

$$= I_1 + I_2 + 2\,\mathrm{Re}(E_1^* E_2)$$

Interference term

Double Slit Experiment with Electrons

$$E(x,y,z,t) \leftrightarrow \psi(x,y,z,t)$$

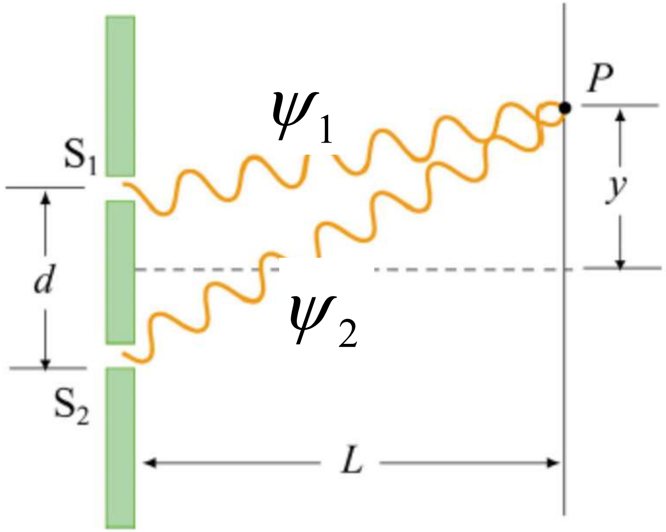
Electric field | Wave function

$$I = |E|^2 = E^* E$$

Intensity

$$P = |\psi|^2 = \psi^* \psi$$

Probability

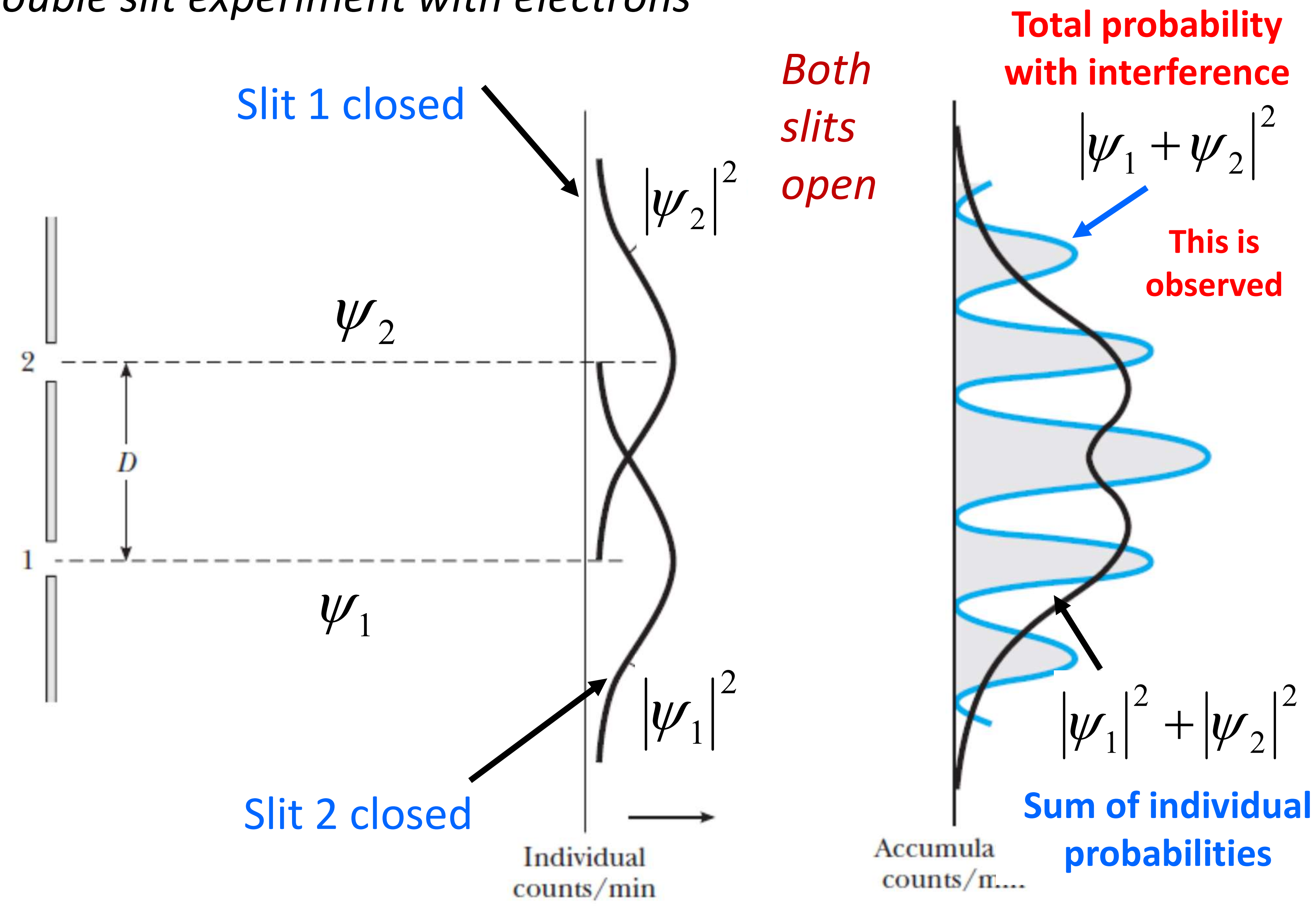


Probability of finding electron at P = $|\psi_1 + \psi_2|^2$

$$= |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}(\psi_1^* \psi_2)$$

$$= P_1 + P_2 + Interference$$

Double slit experiment with electrons



Key Points

Wave function $\psi(x,t)$

Probability $|\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$

Probability of finding a particle at x at time t.

Normalization $\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)dx = 1$

Average $\langle O \rangle = \int_{-\infty}^{\infty} \psi^*(x,t) O \psi(x,t)dx$

Superposition $\psi(x,t) = c_1\psi_1(x,t) + c_2\psi_2(x,t)$

What is Uncertainty?

Let us consider large number of measurements of x

Average

$$\langle x \rangle = \bar{x}$$

Variance

$$\begin{aligned}\sigma_x^2 &= \langle (x-\bar{x})^2 \rangle \\ &= \langle x^2 - 2x\bar{x} + \bar{x}^2 \rangle \\ &= \langle x^2 \rangle - 2\bar{x}\langle x \rangle + \bar{x}^2 = \langle x^2 \rangle - \bar{x}^2 \\ &= \langle x^2 \rangle - \langle x \rangle^2\end{aligned}$$

Standard deviation

σ_x *Measure of uncertainty*

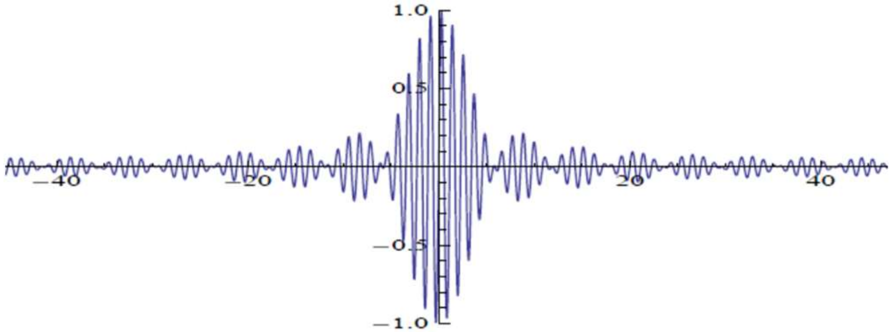
$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} \psi^*(x) x \psi(x) dx}{\int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx}$$

If wave function is normalized then

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x) dx = 1$$

Wave Packet and Fourier Integrals

Δx



$$\psi(x) = \sum_{i=1}^{n} a_i \sin(k_i x)$$

To form a true wave packet **that is zero everywhere outside a finite spatial range Δx**, requires adding together an **infinite number of harmonic waves** with continuously varying wavelengths and amplitudes

$$\psi(x) = \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

$$e^{ikx} = \cos kx + i \sin kx$$

Fourier integral

$$a(k) = \int_{-\infty}^{+\infty} \psi(x) e^{-ikx} dk$$

Fourier Transform (FT)

$$a(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \qquad a(k) = FT\,[f(x)]$$

$$f(x) = \int_{-\infty}^{+\infty} a(k) e^{ikx} dk \qquad f(x) = FT^{-1}\,[a(k)]$$

Inverse Fourier Transform

Connecting $f(x)$ and $a(p)$ since $k = p/\hbar$

$$b(\omega) = \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt \qquad b(\omega) = FT\,[g(t)]$$

$$g(t) = \int_{-\infty}^{+\infty} b(\omega) e^{i\omega t} d\omega \qquad g(t) = FT^{-1}\,[b(\omega)]$$

Connecting $g(t)$ and $b(E)$ since $E = \hbar\omega$

Note

Fourier transforms contain an additional multiplication factor

$$\frac{1}{\sqrt{2\pi}}$$

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

We do not write this factor since our interest is in the Fourier integral.

This is because we are anyway going to normalize the wave functions! We shall see this later.

Rectangular function

$$f(x) = A \quad for\, -T/2 \le x \le T/2$$

$$a(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

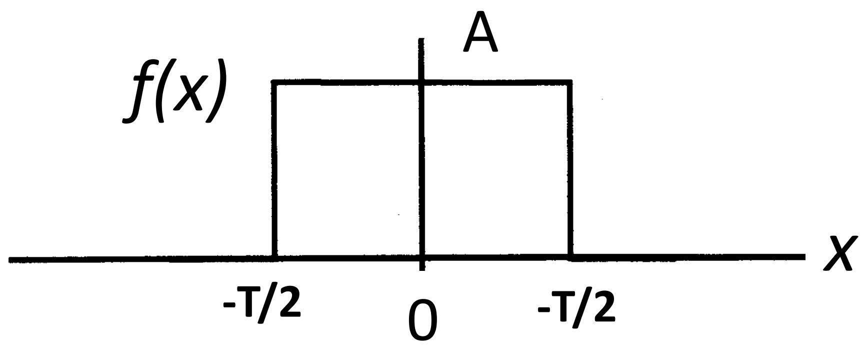
$$= \int_{-T/2}^{T/2} A e^{-ikx} dx$$

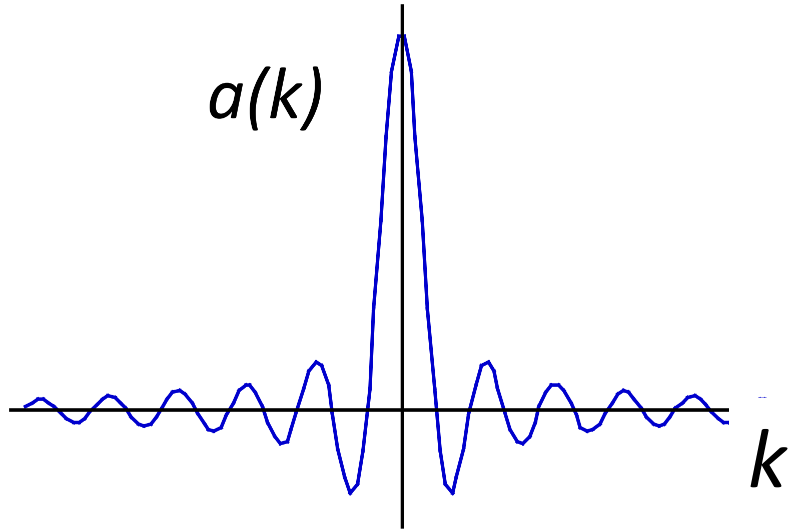
$$= \frac{A}{-ik}\left[e^{-ikx}\right]_{-T/2}^{+T/2}$$

$$= \frac{A}{(k/2)}\left[\frac{e^{ikT/2} - e^{-ikT/2}}{2i}\right]$$

$$= AT\frac{\sin(kT/2)}{(kT/2)}$$

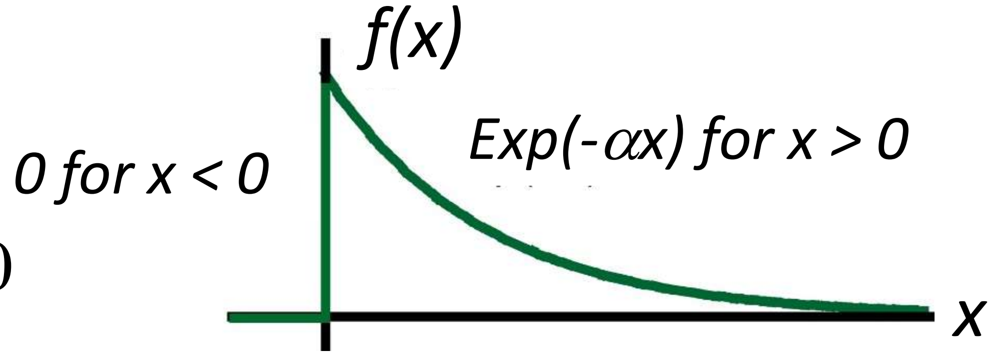
$$= AT\mathrm{sinc}(kT/2)$$





Note: *If T small, f(x) sharp, a(k) broad, and vice versa*

Exponential function



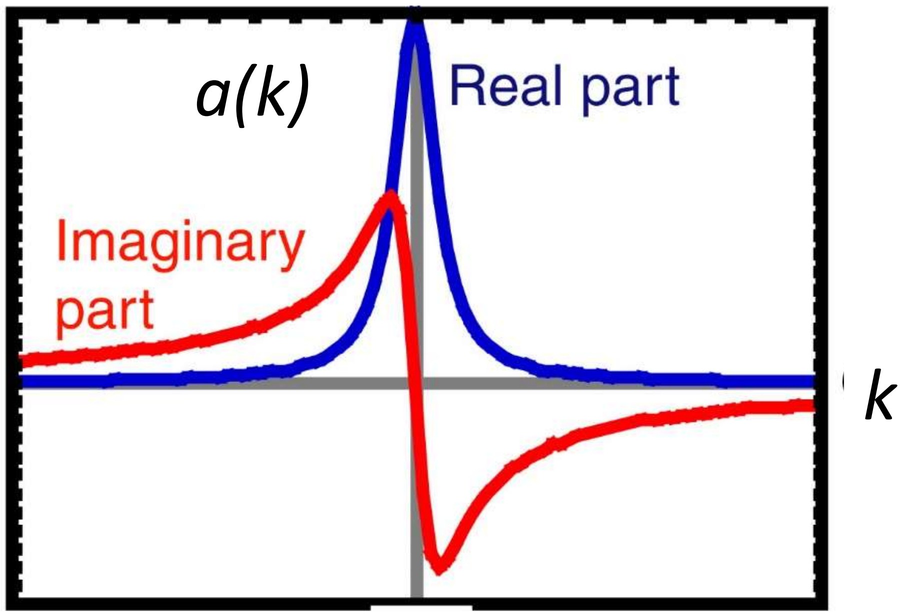
$$f(x) = e^{-\alpha x} \quad \alpha > 0,\ for\ x \geq 0$$

$$a(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \int_{0}^{\infty} e^{-\alpha x} e^{-ikx} dx = \int_{0}^{\infty} e^{-(\alpha + ik)x} dx$$

$$= \frac{1}{-(\alpha + ik)} \left[e^{-(\alpha + ik)x} \right]_0^\infty$$

$$= \frac{-1}{(\alpha + ik)} [0 - 1]$$

$$= \frac{1}{\alpha + ik}$$



Double sided exponential function

$$f(t) = e^{-\alpha|t|} \quad \alpha \geq 0$$

$$f(\omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-i\omega t} dt$$

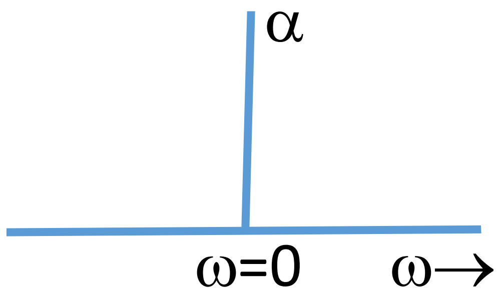
$$= \int_{-\infty}^{0} e^{\alpha t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-i\omega t} dt$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega}$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

Constant function

$$f(t)=\alpha \qquad f(\omega)=\alpha\int_{-\infty}^{\infty}e^{-i\omega t}dt \quad =\alpha\delta(\omega)$$



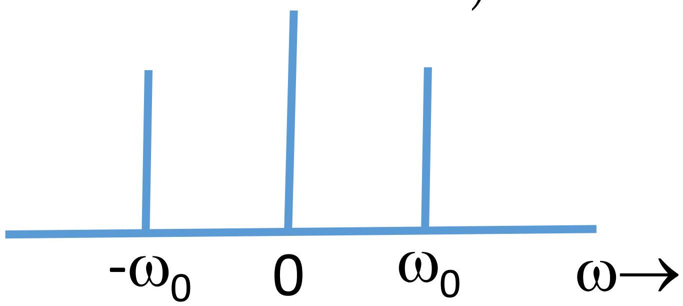
f(ω) zero everywhere except at ω=0

Oscillatory functions

$$f(t)=\cos\omega_0 t \qquad \cos\omega_0 t=\frac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)$$

$$f(\omega)=\frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)e^{-i\omega t}dt \;=\frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-i(\omega-\omega_0)t}+e^{-i(\omega+\omega_0)t}\right)dt$$

$$=\frac{1}{2}\delta(\omega-\omega_0)+\frac{1}{2}\delta(\omega+\omega_0)$$



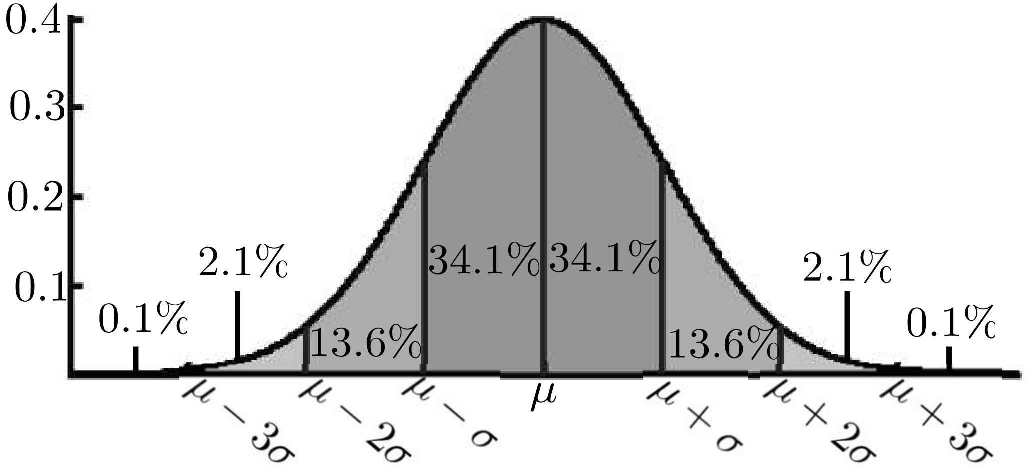
Exercise: $f(t)=\sin\omega_0 t$

Find $f(\omega)=\sin\omega_0 t$

Use $\sin\omega_0 t=\frac{1}{2i}\left(e^{i\omega_0 t}-e^{-i\omega_0 t}\right)$

Gaussian function

$$f(x) = A\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Width $\propto \sigma$

*μ specifies the position of the bell curve's central peak, σ specifies the standard deviation (**a measure of uncertainty**)*

Exercise-1: Normalization

Normalization of f(x) means

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$f(x) = A\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\int_{-\infty}^{\infty} Ae^{-(x-\mu)^2/2\sigma^2}dx = 1$$ Find A

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \qquad \alpha = \frac{1}{2\sigma^2}$$

$$\int_{-\infty}^{\infty} Ae^{-(x-\mu)^2/2\sigma^2}dx = A\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = A\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}dy$$

$$y = x - \mu$$

$$= A(2\sigma^2\pi)^{1/2} = 1$$

$$\therefore A = \frac{1}{\sqrt{2\pi\sigma}} \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$ is normalized.

$$i.e. \int_{-\infty}^{\infty} f(x)dx = 1$$

Exercise 2: Mean and variance

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{2\alpha^{3/2}}$$

$$f(x) = A\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

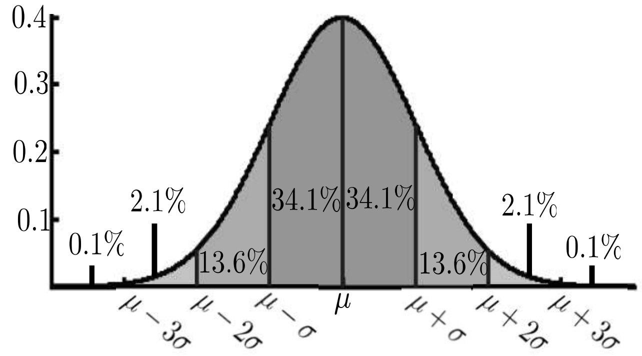
$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} x e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx} = \frac{\int_{-\infty}^{+\infty} (\mu + y) e^{-y^2/2\sigma^2} dy}{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy} = \mu$$

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{+\infty} x^2 e^{-(x-\mu)^2/2\sigma^2} dx}{\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx} = \frac{\int_{-\infty}^{+\infty} (\mu + y)^2 e^{-y^2/2\sigma^2} dy}{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy} = \mu^2 + \underbrace{\frac{\int_{-\infty}^{+\infty} y^2 e^{-y^2/2\sigma^2} dy}{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy}}_{\sigma^2}$$

$$= \mu^2 + \sigma^2$$

Results:

$$f(x) = A\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



$$\langle x \rangle = \mu \qquad \langle x^2 \rangle = \mu^2 + \sigma^2$$

$$\langle x^2 \rangle - \langle x \rangle^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Variance = (standard deviation)^2

Note the following examples

$$f(x) = A\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Is a Gaussian with <x>=0 and standard deviation σ

$$f(x) = A\exp\left(-\frac{x^2}{4\sigma^2}\right)$$

Is a Gaussian with <x>=0 and standard deviation $\sqrt{2}\sigma$

FT of a Gaussian Function

$$f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

A Gaussian centred at x=0

$$FT[f(x)] = a(k) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right)\exp(-ikx)dx$$

$$\boxed{\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}}$$

$$-\frac{x^2}{2\sigma^2} - ikx = -\frac{x^2}{2\sigma^2} - ikx + \frac{k^2\sigma^2}{2} - \frac{k^2\sigma^2}{2}$$

$$= -\underbrace{\left(\frac{x}{\sqrt{2}\sigma} + \frac{ik\sigma}{\sqrt{2}}\right)}_{y}{}^2 - \frac{k^2\sigma^2}{2}$$

$$a(k) = \exp\left(-\frac{k^2\sigma^2}{2}\right)\sqrt{2}\sigma\int_{-\infty}^{\infty}\exp(-y^2)dy = \sqrt{2\pi}\sigma\exp\left(-\frac{k^2\sigma^2}{2}\right)$$

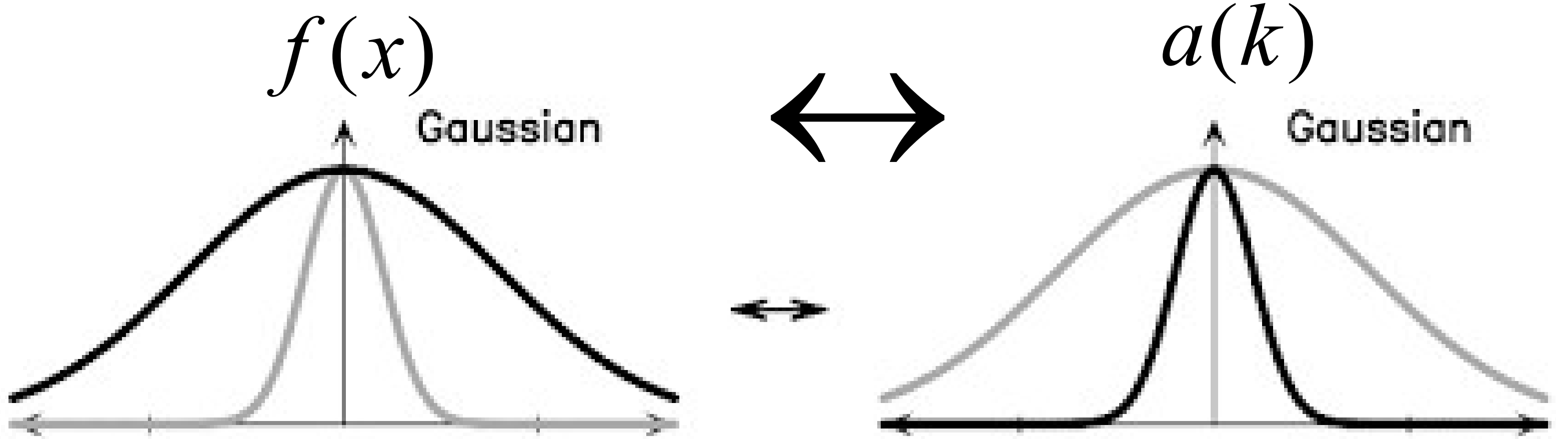
Fourier transform of a Gaussian is another Gaussian!
But with different width!!

$$f(x) = \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$ Width = σ_x

$$a(k) = \sqrt{2\pi}\sigma \exp\left(-\frac{k^2\sigma_x^2}{2}\right) = \sqrt{2\pi}\sigma \exp\left(-\frac{k^2}{2\sigma_k^2}\right)$$ Width = σ_k

$$\sigma_k = \frac{1}{\sigma_x}$$

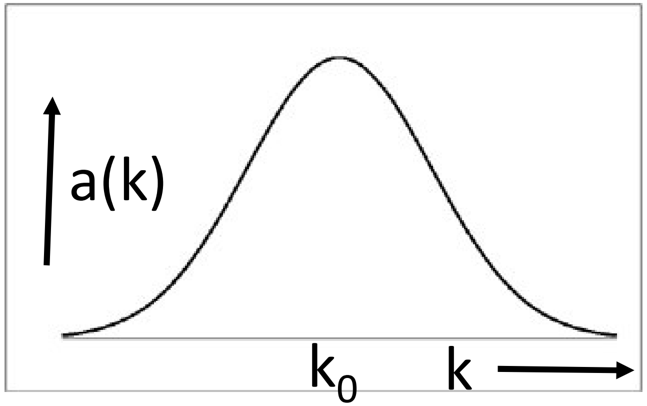
The reciprocal relation is at the origin of uncertainty relation. $\sigma_x \sigma_k = 1$



Gaussian Wave Packet

$$\psi(x) = \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

where a(k) is a Gaussian



$$\boxed{a(k) = A \exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2} \right)}$$

Gaussian wave packet is one where the amplitude function is a Gaussian, which is peaked at $k=k_0$ and has variance $(\sigma_k)^2$

a(k) is Gaussian $\Longleftrightarrow$ ***$\psi(x)$ is Gaussian***

$$\psi(x) = A\int_{-\infty}^{\infty}\exp\left[-\frac{(k-k_0)^2}{2\sigma_k^2}\right]\exp(ikx)dk$$

$$= A\int_{-\infty}^{\infty}\exp\left[-\frac{(k-k_0)^2}{2\sigma_k^2}\right]\exp[i(k-k_0)x]\exp(ik_0x)dk$$

$$= A\exp(ik_0x)\int_{-\infty}^{\infty}\exp\left[-\frac{\kappa^2}{2\sigma_k^2}+i\kappa x\right]d\kappa \qquad \kappa = k-k_0$$

$$-\frac{\kappa^2}{2\sigma_k^2}+ikx = -\frac{\kappa^2}{2\sigma_k^2}+ikx+\frac{\sigma_k^2x^2}{2}-\frac{\sigma_k^2x^2}{2} = -\left(\frac{\kappa}{\sqrt{2}\sigma_k}-\frac{i\sigma_k x}{\sqrt{2}}\right)^2-\frac{\sigma_k^2x^2}{2}$$

$$\psi(x) = A\exp(ik_0x)\exp\left(-\frac{\sigma_k^2x^2}{2}\right)\int_{-\infty}^{\infty}\exp\left[-\left(\frac{\kappa}{\sqrt{2}\sigma_k}-\frac{i\sigma_k\kappa}{\sqrt{2}}\right)^2\right]d\kappa$$

$$\psi(x) = A\exp(ik_0x)\exp\left(-\frac{\sigma_k^2x^2}{2}\right)\sqrt{2}\sigma_k\int_{-\infty}^{\infty}\exp\left(y^2\right)dy$$

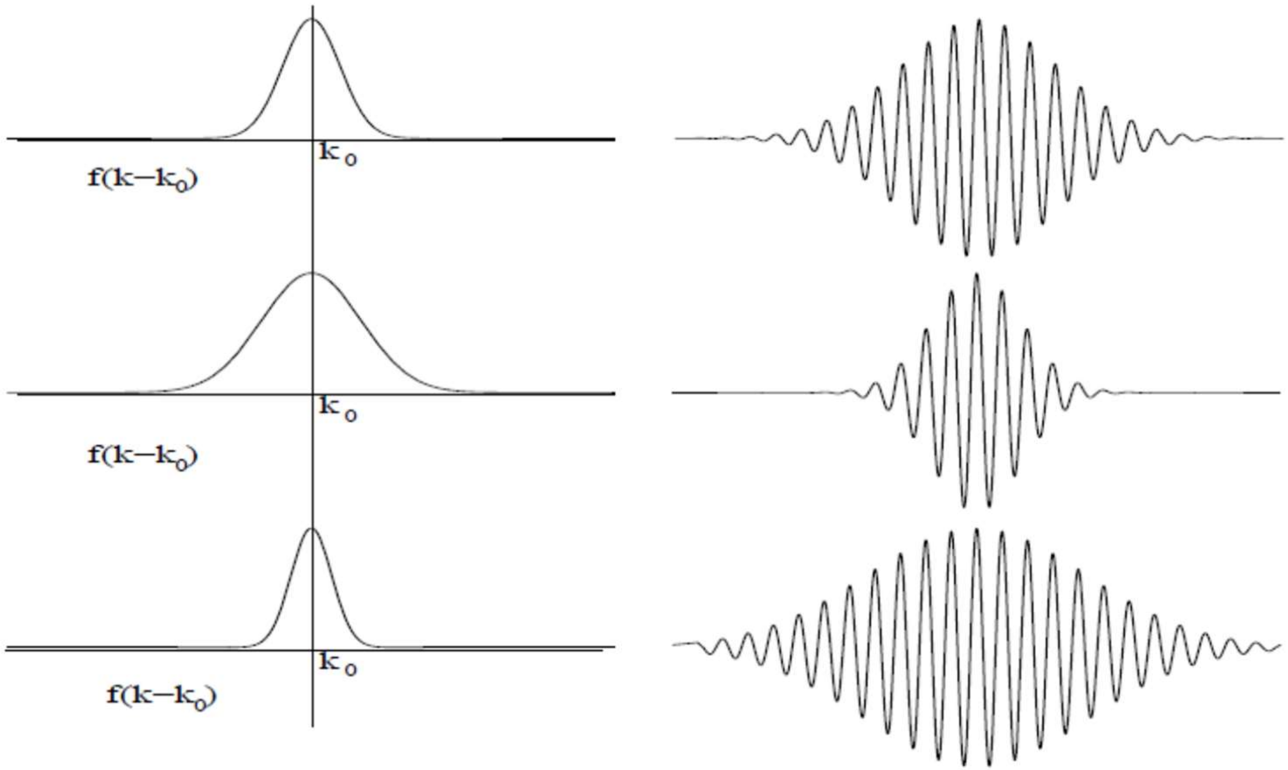
$$\boxed{\psi(x) = A\sqrt{2\pi}\sigma_k\exp\left(ik_0x\right)\exp\left(-\sigma_k^2x^2/2\right)}$$

$$a(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2}\right)$$

Variance: $=\sigma_k^2$

$$\psi(x) = \int_{-\infty}^{+\infty} a(k)e^{ikx}dk \quad = A\sqrt{2\pi}\sigma_k \exp(ik_0x)\exp\left(-\sigma_k^2 x^2/2\right)$$

Variance: $=1/\sigma_k^2$



Now be careful

$$a(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2}\right) \qquad \text{Variance: } = \sigma_k^2$$

$$\psi(x) = \int_{-\infty}^{+\infty} a(k)e^{ikx}dk \quad = A\sqrt{2\pi}\sigma_k \exp(ik_0x)\exp\left(-\sigma_k^2x^2/2\right)$$

$$\text{Variance: } \quad \sigma_x^2 = 1/\sigma_k^2$$

$$\sigma_x\sigma_k = 1 \Rightarrow \Delta x\Delta k = 1 \Rightarrow \Delta x\Delta p_x = \hbar$$

This uncertainty relation refers to the variances of a(k) and ψ(x)

The uncertainty is minimum for Gaussian wave packet, therefore, in general, $\Delta x\Delta p_x \geq \hbar$

Uncertainty Relation for the wave packet!

To get 'the' uncertainty relation, we need to calculate

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \qquad (\Delta k)^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle (k - k_0)^2 \rangle$$

In Quantum Physics, We define averages as follows:

Average $$\langle O \rangle = \int_{-\infty}^{\infty} \psi^*(x,t) O \psi(x,t) dx$$ **OR**

$$\langle O \rangle = \frac{\int_{-\infty}^{\infty} \psi^*(x,t) O \psi(x,t) dx}{\int_{-\infty}^{\infty} \psi^*(x,t) \psi(x,t) dx}$$

if wave function is not normalized.

Let us learn how to normalize $\psi(x)$

$$\psi(x) = A\sqrt{2\pi}\sigma_k \exp(ik_0 x)\exp\left(-\sigma_k^2 x^2/2\right)$$

$$= C\exp(ik_0 x)\exp(-\sigma_k^2 x^2/2)$$

Normalization of wave function means $\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = 1$

$$\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = C^2\int_{-\infty}^{\infty}e^{-\sigma_k^2 x^2}dx$$

$$\boxed{\int_{-\infty}^{\infty}\exp(-\sigma^2 x^2)dx = \frac{\sqrt{\pi}}{\sigma}}$$

$$= C^2\frac{\sqrt{\pi}}{\sigma} \qquad \therefore C^2\frac{\sqrt{\pi}}{\sigma} = 1 \quad \Rightarrow \quad C = \frac{\sqrt{\sigma}}{(\pi)^{1/4}}$$

$$\psi(x) = \frac{\sqrt{\sigma}}{(\pi)^{1/4}}\exp(ik_0 x)\exp(-\sigma_k^2 x^2/2)$$ **is normalized WF**

Given $a(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2}\right)$

$$\psi(x) = A\sqrt{2\pi}\sigma_k \exp(ik_0 x)\exp(-\sigma_k^2 x^2/2)$$

Calculate $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

$$\psi(x) = C\exp(ik_0 x)\exp(-\sigma_k^2 x^2/2)$$

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty} x\psi^*(x)\psi(x)dx}{\int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx} = \frac{C^2\int_{-\infty}^{\infty} x\exp(-\sigma_k^2 x^2)dx}{C^2\int_{-\infty}^{\infty}\exp(-\sigma_k^2 x^2)dx} = 0$$

since $\int_{-\infty}^{\infty} x\exp(-\sigma^2 x^2)dx = 0$

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 \psi^*(x)\psi(x)dx}{\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx} = \frac{C^2 \int_{-\infty}^{\infty} x^2 \exp(-\sigma_k^2 x^2)dx}{C^2 \int_{-\infty}^{\infty} \exp(-\sigma_k^2 x^2)dx}$$

$$= \frac{1}{2\sigma^2}$$

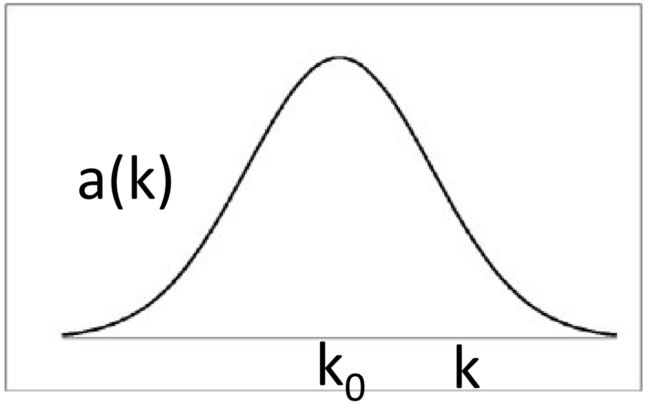
$$\boxed{\begin{aligned} \int_{-\infty}^{\infty} \exp(-\sigma^2 x^2)dx &= \frac{\sqrt{\pi}}{\sigma} \\ \int_{-\infty}^{\infty} x^2 \exp(-\sigma^2 x^2)dx &= \frac{\sqrt{\pi}}{2\sigma^3} \end{aligned}}$$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle - 0 = \frac{1}{2\sigma_k^2}$$

$$\Delta x = \frac{1}{\sqrt{2}\sigma_k}$$

Calculate $(\Delta k)^2 = \left\langle (k-k_0)^2 \right\rangle$

$$a(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2}\right)$$



$$(\Delta k)^2 = \frac{\int_{-\infty}^{\infty}(k-k_0)^2 a^*(k)a(k)dk}{\int_{-\infty}^{\infty} a^*(k)a(k)dk} = \frac{A^2\int_{-\infty}^{\infty}(k-k_0)^2 \exp\left[-\frac{(k-k_0)^2}{\sigma_k^2}\right]dk}{A^2\int_{-\infty}^{\infty}\exp\left(\frac{(k-k_0)^2}{\sigma_k^2}\right)dk}$$

$Let\ \ \kappa = k - k_0$

$$(\Delta k)^2 = \frac{\int_{-\infty}^{\infty}\kappa^2 \exp\left[-\frac{\kappa^2}{\sigma_k^2}\right]dk}{\int_{-\infty}^{\infty}\exp\left(-\frac{\kappa^2}{\sigma_k^2}\right)dk} = \frac{\sigma_k^2}{2}$$

$$\Delta k = \frac{\sigma_k}{\sqrt{2}}$$

Therefore, for a Gaussian wave packet, the uncertainties in x and k are

$$\Delta x = \frac{1}{\sqrt{2}\sigma_k} \qquad \Delta k = \frac{\sigma_k}{\sqrt{2}}$$

$$\Delta x \Delta k = \frac{1}{\sqrt{2}\sigma_k}\frac{\sigma_k}{\sqrt{2}} = \frac{1}{2} = \frac{\Delta x \Delta p}{\hbar}$$

$$\Delta x \Delta p = \frac{\hbar}{2}$$

The uncertainty product is minimum for Gaussian wave packet.

In General, $$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Heisenberg's Uncertainty Relation

Important points to note

- For a Gaussian wave packet

$$\psi(x) = \int_{-\infty}^{+\infty} a(k) e^{ikx} dk \qquad a(k) = A \exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2} \right)$$

σ_k is width of a(k) **$\sigma_x = 1/\sigma_k$ is width of ψ(x)**

$$\sigma_x \sigma_k = 1 \Rightarrow \Delta x \Delta k = 1 \Rightarrow \Delta x \Delta p_x = \hbar$$

- A Gaussian wave packet has minimum uncertainty
- Uncertainty relation for wave packet

$$\Delta x \Delta p_x \geq \hbar$$

On the other hand

When one calculates the uncertainties using relevant wave functions, one obtains

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$ ***Heisenberg's Uncertainty Relation***

Here Δx and Δp are uncertainties in the observables x and p.

$$\Delta x \Delta p_x \geq \hbar \qquad\qquad \Delta x \Delta p \geq \frac{\hbar}{2}$$

The factor 2

For a function, $a(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma_k^2}\right)$

Uncertainty is given by σ_k

When we obtain average, we use $a^*(k)a(k)$

$$a^*(k)a(k) = A\exp\left(-\frac{(k-k_0)^2}{\sigma_k^2}\right)$$

Uncertainty is given by $\sigma_k/\sqrt{2}$

The same factor 1/√2 comes in for uncertainty in x

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Also important is

$$\psi(k) = \int_{-\infty}^{+\infty} \psi(x) e^{-ikx} dx$$

$\psi(x)$ Is wave function in **coordinate representation**

$\psi(k)$ Is wave function in **k (momentum) representation**

$$\psi(x) = \int_{-\infty}^{+\infty} \psi(k) e^{ikx} dk$$

Wave functions in coordinate representation and momentum representation are related by Fourier transform.