Basics of Mechanics

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Physics:
The most basic of all sciences!

- **Physics:** The "king" of all sciences!
- **Physics** = The study of structure of matter and energy and their interactions.
- Its Applications

Physics: General Discussion

- ***The Goal of Physics*** (& all of science): To quantitatively and qualitatively **describe the "world around us".**
- **Physics *IS NOT*** merely a collection of facts & formulas!
- **Physics *IS*** a **creative** activity!
- Physics ⟶ Observation ⟶ Explanation.
- Requires ***IMAGINATION!!***

Physics & Its Relation to Other Fields

- *The "Parent" of all Sciences!*
- The foundation for and is connected to *ALL* branches of *science and engineering*.
- Also useful in everyday life and in *MANY* professions
  - Chemistry
  - Life Sciences (Medicine also!!)
  - Architecture
  - Engineering
  - Various technological fields

The Nature of Science

- Physics is an *EXPERIMENTAL* science!
- Experiments & Observations:
  - Important first steps toward scientific theory.
  - It requires imagination to tell what is important
- Theories:
  - Created to *explain* experiments & observations. Will also make predictions
- Experiments & Observations:
  - Will tell if predictions are accurate.
- *No theory* can be absolutely verified
  - But a theory *CAN be proven false!!!*

Theory

- *Quantitative* (mathematical) *description* of experimental observations.
- Not just ***WHAT*** is observed but ***WHY*** it is observed as it is and ***HOW*** it works the way it does.
- Tests of theories:
  - Experimental observations:
    More experiments, more observation.
  - Predictions:
    Made before observations & experiments.

Model, Theory, Law

- **Model:** An analogy of a physical phenomenon to something we are familiar with.
- **Theory:** More detailed than a model. Puts the model into mathematical language.
- **Law:** Concise & **general** statement about **how nature behaves.** Must be verified by many, many experiments! Only a few laws.
  - Not comparable to laws of government!

- How does a new theory get accepted?
- Predictions agree better with data than old theory
- Explains a greater range of phenomena than old theory
- Example:
  - Aristotle believed that objects would return to a state of rest once put in motion.
  - Galileo realized that an object put in motion would stay in motion until some force stopped it.

The Structure of Physics

SUMMARY: THE STRUCTURE OF PHYSICS

	Low Speed $v << c$	High Speed $v < \sim c$
Large size >> atomic size	Classical Mechanics (Newton, Hamilton, Lagrange)	Special Relativity (Einstein)
Small size < ~ atomic size	Quantum Mechanics (Schrodinger, Heisenberg)	Relativistic Quantum Mechanics (Dirac)

Atomic Physics

Molecular Physics

Solid State Physics

Nuclear & Particle Physics

Quantum Field Theory (Feynman, Schwinger)

Quantum Electrodynamics (Photons, Weak Nuclear Force)

Quantum Chromodynamics (Gluons, Quarks, Leptons Strong Nuclear Force)

Mechanics

- The science of ***HOW*** objects move (behave) under ***given forces***.
- (Usually) Does not deal with the ***sources*** of forces. Answers the question: "Given the forces, how do objects move"?

Mechanics: "Classical" Mechanics

"Classical" Physics:

"Classical" ≡ ≈ Before the 20th Century

The foundation of pure & applied macroscopic physics & engineering!

– *Newton's Laws* + Boltzmann's Statistical Mechanics (& Thermodynamics):
 ≈ Describe most of macroscopic world!

– However, at high speeds ($v \sim c$) we need

Special Relativity: (Early 20th Century: 1905)

– Also, for small sizes (atomic & smaller) we need

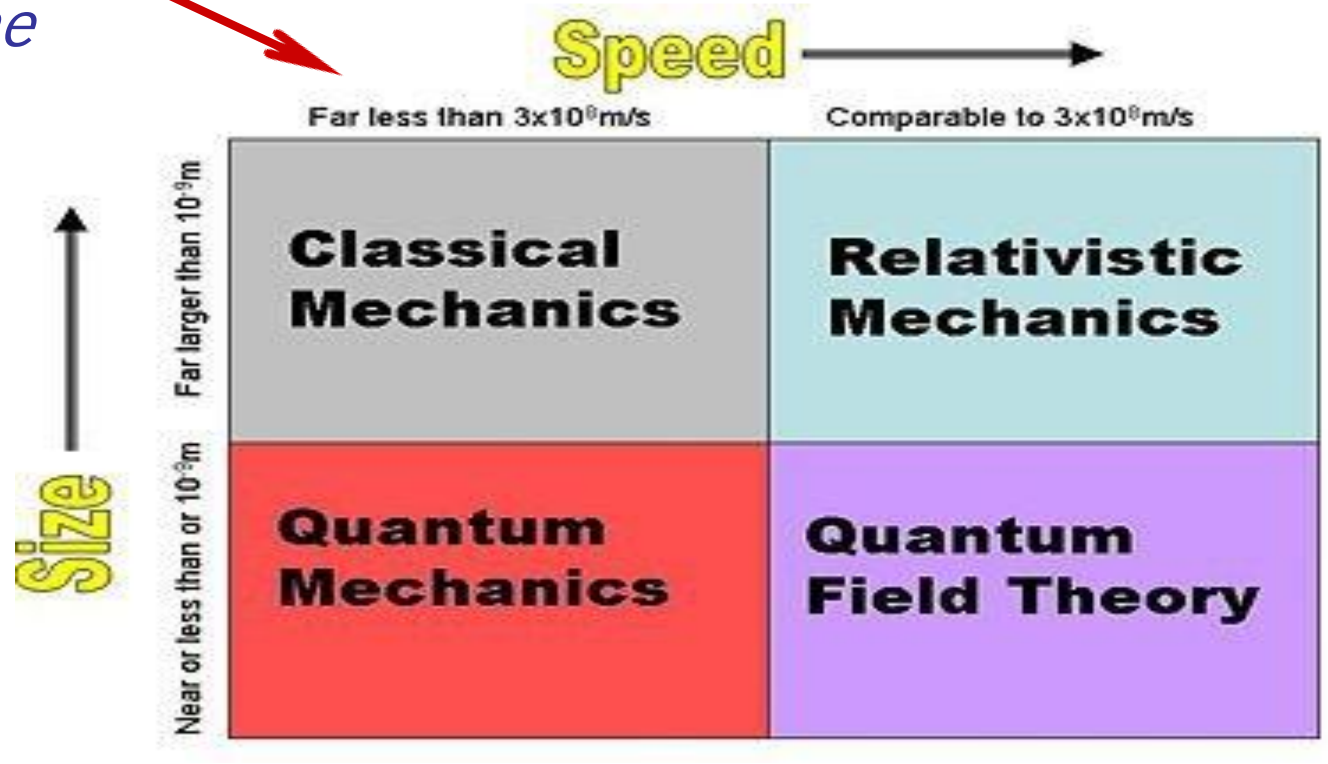
Quantum Mechanics: (1900 through ~ 1930)

"Classical" Mechanics: (17th & 18th Centuries) Still useful today!

"Classical" Mechanics

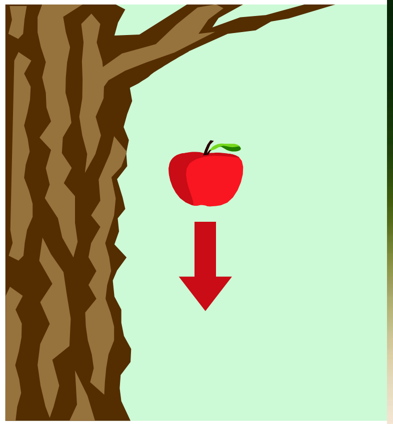
The mechanics in this course is *limited to macroscopic objects* moving at *speeds* v *much, much smaller than the speed of light* $c = 3 \times 10^8$ m/s. As long as $v \ll c$, our discussion will be valid.

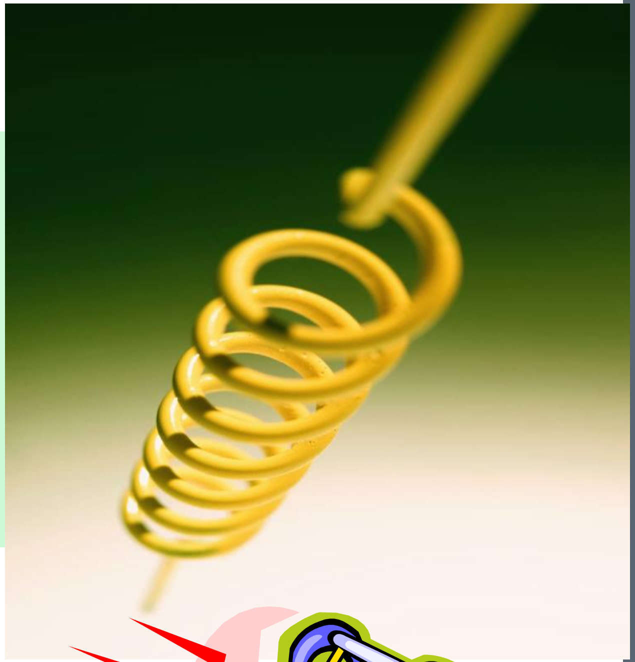
So, we will work exclusively in the gray region in the figure.

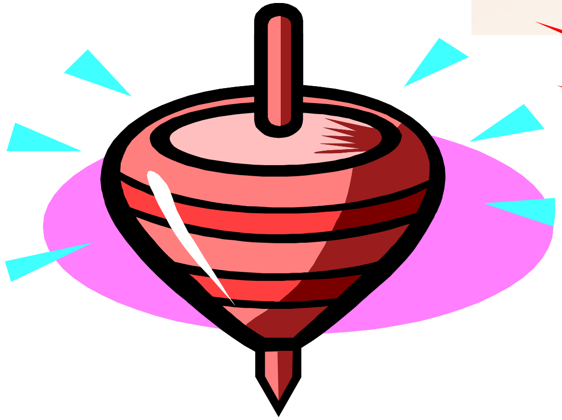


Classical Mechanics

- Kinematics – how objects move
  - Translational motion
  - Rotational motion
  - Vibrational motion
- Dynamics – Forces and why objects move as they do
  - Statics – special case - forces cause no motion







Classical Mechanics

- First began with Galileo (1584-1642), whose experiments with falling bodies (and bodies rolling on an incline) led to Newton's 1st Law.
- Newton (1642-1727) then developed his 3 laws of motion, together with his universal law of gravitation.
- Two additional, highly mathematical frameworks were developed by the French mathematician Lagrange (1736-1813) and the Irish mathematician Hamilton (1805-1865).
- Together, these three alternative frameworks by Newton, Lagrange, and Hamilton make up what is generally called Classical Mechanics.
- They are distinct from the other great forms of non-classical mechanics, Relativistic Mechanics and Quantum Mechanics, but both of these borrow heavily from Classical Mechanics.

Space and Time

- We live in a three-dimensional world, and for the purpose of this course we can consider space and time to be a fixed framework against which we can make measurements of moving bodies.
- Each point P in space can be labeled with a distance and direction from some arbitrarily chosen origin O. Expressed in terms of unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

- It is equivalent to write the vector as an ordered triplet of values $\mathbf{r} = (x,\ y,\ z)$
- We can also write components of vectors using subscripts

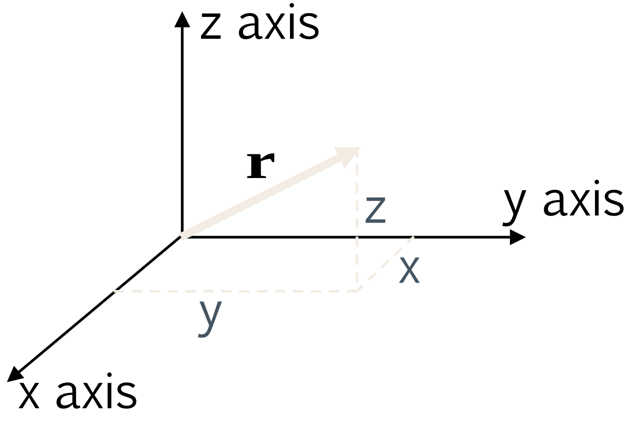
$$\mathbf{v} = (v_x,\ v_y,\ v_z) \qquad \mathbf{a} = (a_x,\ a_y,\ a_z)$$

Ways of writing vector notation

$$\mathbf{F} = m\mathbf{a}$$

$$\vec{F} = m\vec{a}$$

$$\underline{F} = m\underline{a}$$



Other Vector Notations

- You will be used to unit vector notation **i**, **j**, **k**, but we will follow the text and use the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ notation.
- At times, it is more convenient to use notation that makes it easier to use summation notation, so we introduce the equivalents:

$$r_1 = x, \quad r_2 = y, \quad r_3 = z$$

$$\mathbf{e}_1 = \hat{\mathbf{x}}, \quad \mathbf{e}_2 = \hat{\mathbf{y}}, \quad \mathbf{e}_3 = \hat{\mathbf{z}}$$

which allows us to write

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = \sum_{i=1}^{3} r_i\mathbf{e}_i$$

- In the above example, this form has no real advantage, but in other cases we will meet, this form is much simpler to use. The point is that we may choose any convenient notation, and you should become tolerant of different, but consistent forms of notation.

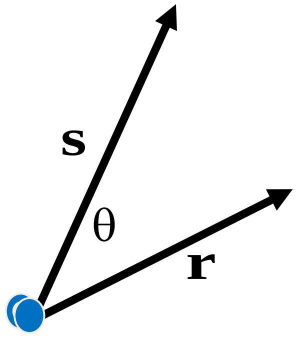
Vector Operations

- Sum of vectors $\mathbf{r} = (r_1, r_2, r_3);\ \mathbf{s} = (s_1, s_2, s_3)\quad \mathbf{r} + \mathbf{s} = (r_1 + s_1, r_2 + s_2, r_3 + s_3)$
- Vector times scalar $c\mathbf{r} = (cr_1, cr_2, cr_3)$
- Scalar product, or dot product

$$\mathbf{r} \cdot \mathbf{s} = rs\cos\theta$$

$$= r_1 s_1 + r_2 s_2 + r_3 s_3 = \sum_{n=1}^{3} r_n s_n$$

$$\mathbf{p} = \mathbf{r} \times \mathbf{s};\qquad |\mathbf{r} \times \mathbf{s}| = rs\sin\theta$$



- Vector product, or cross product

$$p_x = r_y s_z - r_z s_y$$
$$p_y = r_z s_x - r_x s_z$$
$$p_z = r_x s_y - r_y s_x$$

$$\mathbf{r} \times \mathbf{s} = \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix}$$

Differentiation of Vectors

› This course makes heavy use of Calculus (and differential equations, and other forms of advanced mathematics). In general, we will refresh your memory about the techniques you will need as they come up, but we will do so from a Physics perspective—only paying lip-service to the underlying mathematical proofs.

› What we need now is a simple form of something called Vector Calculus. As long as you remember that vectors are just triplets of numbers, and vector equations can be thought of as three separate equations, you will be fine.

› For now, consider only the derivative of the position vector $\mathbf{r}(t)$, which you should know gives the velocity $\mathbf{v}(t) = d\mathbf{r}(t)/dt$. Likewise, the derivative of the velocity (the second derivative of the position) gives the acceleration: $\mathbf{a}(t) = d\mathbf{v}(t)/dt = d^2\mathbf{r}(t)/dt^2$. Formally:

for scalars: $$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \quad \text{where} \quad \Delta x = x(t + \Delta t) - x(t)$$

for vectors: $$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \quad \text{where} \quad \Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

Differentiation of Vectors

- By the usual rules of differentiation, the derivative of a sum of vectors is

$$\frac{d}{dt}(\mathbf{r}+\mathbf{s}) = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt}$$

- derivative of a scalar times a vector follows the usual product rule

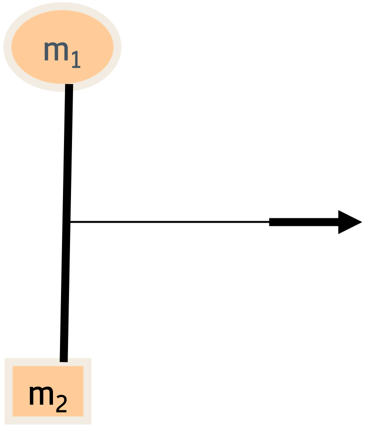
$$\frac{d}{dt}(f\mathbf{r}) = f\,\frac{d\mathbf{r}}{dt} + \frac{df}{dt}\mathbf{r}$$

- Also note: $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ so $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}$ $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

  implies that the unit vectors are constant (i.e. $\frac{d\hat{\mathbf{x}}}{dt} = \frac{d\hat{\mathbf{y}}}{dt} = \frac{d\hat{\mathbf{z}}}{dt} = 0$).

Mass and Force

- **What is the difference between mass and weight?**
- Mass has to do with inertial force (*ma*).
- Weight has to do with gravitational force (*mg*).
- In the first case, the mass is "resistance to changes in motion" while in the second case it is a rather mysterious "attractive property" of matter
- **Inertial balance:**



Allows measurement of inertial mass without mixing in gravitational force.

Point Mass (Particle)

- For now, we want to focus on the concept of a point mass, or particle. This is an approximation, which is worthwhile to look at carefully. It basically refers to a body that can move through space but has NO internal degrees of freedom (rotation, flexure, vibrations).
- Later we will talk about bodies as collections of particles, or a continuous distribution of mass, and in considering such bodies the laws of motion are considerably more complicated.
- Despite this being an approximation, the approximation is still useful in many cases, such as for elementary particles (protons, neutrons, electrons), or even planets and stars (sometimes).

Newton's Three Laws

› Law of Inertia

- In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.
- (An object in motion tends to remain in motion, an object at rest tends to remain at rest.)

› Force Law

- For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration: $\mathbf{F} = m\mathbf{a}$.

› Conservation of Momentum Law

- If particle 1 exerts a force $\mathbf{F}_{21}$ on particle 2, then particle 2 always exerts a reaction force $\mathbf{F}_{12}$ on particle 1 given by $\mathbf{F}_{12} = -\mathbf{F}_{21}$.
- (For every action there is an equal and opposite reaction.)

Aside: Dot Notation

- Dot Notation:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$$

- We will be using this dot notation extensively. It means differentiation with respect to time, t (only!).
- You may have seen “prime” notation, but if the differentiation is not with respect to time, it is NOT equivalent to y-dot.

$$y' = \frac{dy}{dx} \neq \dot{y}$$

- y-dot means dy/dt only.

Equivalence of First Two Laws

- The Law of Inertia and the Force Law can be stated in equivalent ways.
- Obviously, if $\mathbf{F} = m\mathbf{a}$, then in the absence of forces $\mathbf{F} = m\mathbf{a} = 0$

$$\frac{d\mathbf{v}}{dt} = 0 \Rightarrow \mathbf{v} = \mathbf{v}_o$$

- Thus, the velocity is constant (objects in motion tend to remain in motion) and could be zero (objects at rest tend to remain at rest).
- The second law can be rewritten in terms of momentum:

$$\mathbf{p} = m\mathbf{v}$$

- In Classical Mechanics, the mass of a particle is constant, hence

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a}$$

- So we can write $\mathbf{F} = \dot{\mathbf{p}}$.
- In words, forces cause a change in momentum, and conversely any change in momentum implies that a force is acting on the particle.

The Equation of Motion

- Newton's Second Law is the basis for much of Classical Mechanics, and the equation $\mathbf{F} = m\mathbf{a}$ has another name—the equation of motion.
- The typical use of the equation of motion is to write

$$m\mathbf{a} = \sum \text{Forces}$$

  where the right hand side lists all of the forces acting on the particle.
- In this text, an even more usual way to write it is:

$$m\ddot{\mathbf{r}} = \sum \text{Forces}$$

  which is perhaps an easier way to understand why it is called the equation of motion. This relates the position of the particle vs. time to the forces acting on it, and obviously if we know the position at all times we have an equation of motion for the particle.

Differential Equations

- Most of you should have had a course in differential equations by now, or should be taking the course concurrently.
- A differential equation is an equation involving derivatives, in this case derivatives of the particle position $\mathbf{r}(t)$.
- Consider the one-dimensional equation for the position $x(t)$ of a particle under a constant force:

$$\ddot{x}(t) = \frac{F_0}{m}$$

- This equation involves the second derivative (with respect to time) of the position, so to get the position we simply integrate twice:

$$\dot{x}(t) = \int \ddot{x}(t)dt = v_0 + \frac{F_0}{m}t$$

$$x(t) = \int \dot{x}(t)dt = x_0 + v_0 t + \frac{F_0}{2m}t^2$$

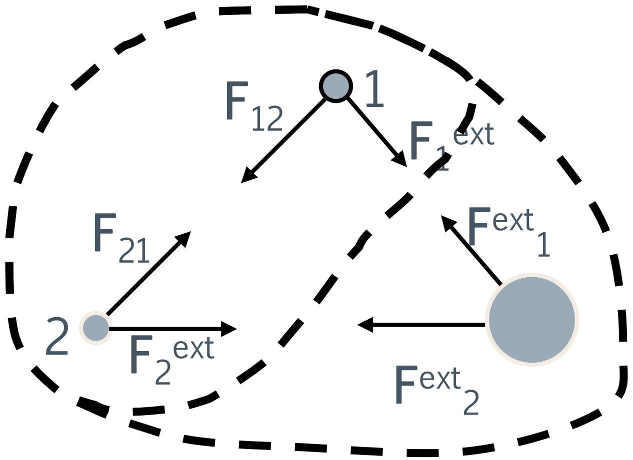
- This was so easy we did not actually need to know the theory of differential equations, but we will meet with lots of more complicated equations where the DiffEQ theory is needed, and will be introduced as needed.

Third Law and Conservation of Momentum

- Newton's first two laws refer to forces acting on a single particle. The Third Law, by contrast, explicitly refers to two particles interacting—the particle being accelerated, and the particle doing the forcing.
- Introduce notation $\mathbf{F}_{21}$ (F-on-by) to represent the force on particle 2 by particle 1. Then

Newton's Third Law

> If particle 1 exerts a force F_{21} on particle 2, then particle 2 always exerts a reaction force F_{12} on particle 1 given by $F_{12} = -\ F_{21}$.



$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_{21};$$
$$\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21};$$
$$\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21};$$
$$\dot{\mathbf{p}}_{ext} = \mathbf{F}_{ext} = \mathbf{F}^{\text{ext}}{}_1 + \mathbf{F}^{\text{ext}}{}_2; \quad \dot{\mathbf{P}} = 0$$

Multi-Particle Systems

› It should be fairly obvious how to extend this to systems of N particles, where N can be any number, including truly huge numbers like 10^{23}.

› Let α or β designate one of the particles. Both α and β can take any value $1, 2, \ldots, N$. The net force on particle α is then

$$\mathbf{F}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} = \dot{\mathbf{p}}_\alpha$$

where the sum runs over all particles except α itself (a particle does not exert a force on itself).

› The total force on the system of particles is just the sum of all of the $\dot{\mathbf{p}}_\alpha$

$$\dot{\mathbf{P}} = \sum_\alpha \dot{\mathbf{p}}_\alpha$$

$$\dot{\mathbf{P}} = \sum_\alpha \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} \boxed{= \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} = \mathbf{F}^{\text{ext}}}$$

› Each term $\mathbf{F}_{\alpha\beta}$ can be paired with $\mathbf{F}_{\beta\alpha}$:

$$\sum_\alpha \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} = \sum_\alpha \sum_{\beta > \alpha} \left(\mathbf{F}_{\alpha\beta} + \mathbf{F}_{\beta\alpha} \right) = 0$$

Law of Conservation of Linear Momentum

When the total external force on a system is zero, the total momentum of the system remains constant

Law of Conservation of Momentum (collisions):

The ***total (vector) momentum before a collision =***
the ***total (vector) momentum after a collision.***

$$\Rightarrow \quad p_{total} = p_A + p_B = (p_A)' + (p_B)' = \text{constant}$$

Or: $\Delta p_{total} = \Delta p_A + \Delta p_B = 0$

$p_A = m_A v_A, \quad p_B = m_B v_B,$ Initial momenta

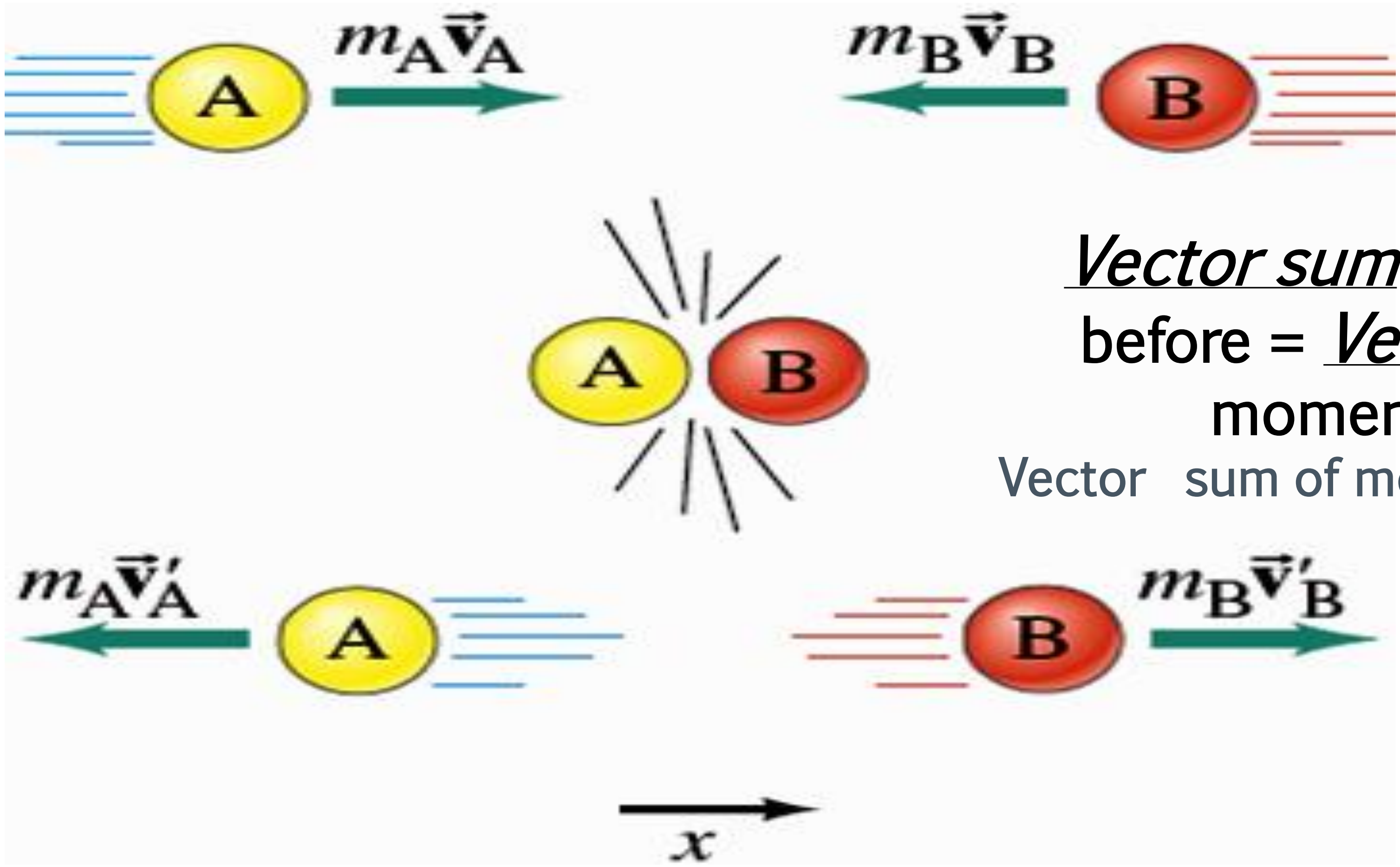
$(p_A)' = m_A(v_A)', \ (p_B)' = m_B(v_B)',$ Final momenta

$$\Rightarrow \quad m_A v_A + m_B v_B = m_A(v_A)' + m_B(v_B)'$$

Example: 2 billiard balls collide (zero external force)

$$m_A \vec{\mathbf{v}}_A + m_B \vec{\mathbf{v}}_B = m_A \vec{\mathbf{v}}'_A + m_B \vec{\mathbf{v}}'_B$$

The vector sum of the momenta is a constant!

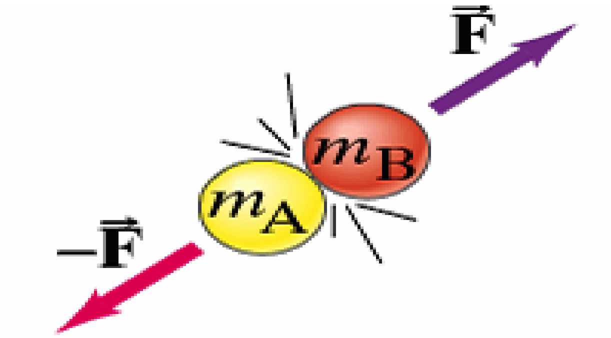


Vector sum of momenta before = *Vector sum* of momenta after!

Vector sum of momenta = constant!

Another brief *Proof*, using Newton's 2^{nd} & 3^{rd} Laws

Two masses, m_A & m_B in collision:
Internal forces: $F_{AB} = - F_{BA}$ by Newton's 3^{rd} Law



Newton's 2^{nd} Law:

The force on A, due to B, for a small time Δt:

$$F_{AB} = \Delta p_A/\Delta t = m_A[(v_A)' - v_A]/\Delta t$$

The force on B, due to A, for the same small Δt:

$$F_{BA} = \Delta p_B/\Delta t = m_B[(v_B)' - v_B]/\Delta t$$

Newton's 3^{rd} Law: $F_{AB} = - F_{BA} = F$

$$\Rightarrow \quad m_A[(v_A)' - v_A]/\Delta t = - m_B[(v_B)' - v_B]/\Delta t$$

or: $m_A v_A + m_B v_B = m_A(v_A)' + m_B(v_B)' \Rightarrow$ *Proven!*

So, for Collisions: $m_A v_A + m_B v_B = m_A(v_A)' + m_B(v_B)'$

Railroad Cars Collide: Conservation of Momentum

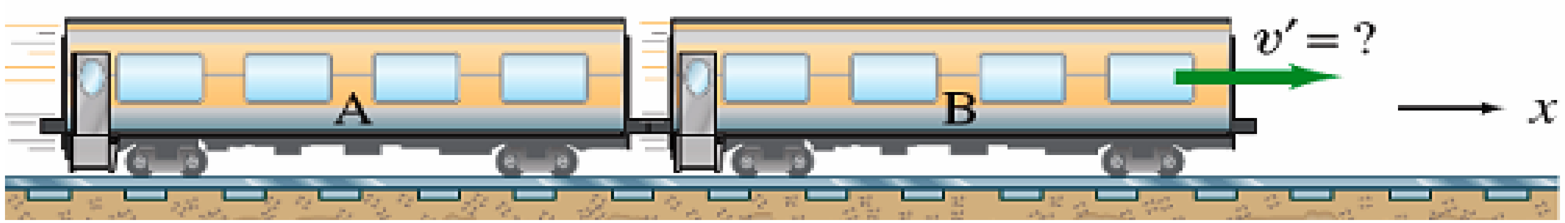
Simplest possible example!! Car A, mass $m_A = 10{,}000$ kg, is traveling at speed $v_A = 24$ m/s strikes car B (same mass, $m_B = 10{,}000$ kg), initially at rest ($v_B = 0$). The cars lock together after the collision. Calculate their speed v' immediately after the collision.

Conservation of Momentum in 1dimension

Initial Momentum = Final Momentum

Before Collision →

After Collision →



$$v_A = 0, (v_A)' = (v_B)' = v'$$

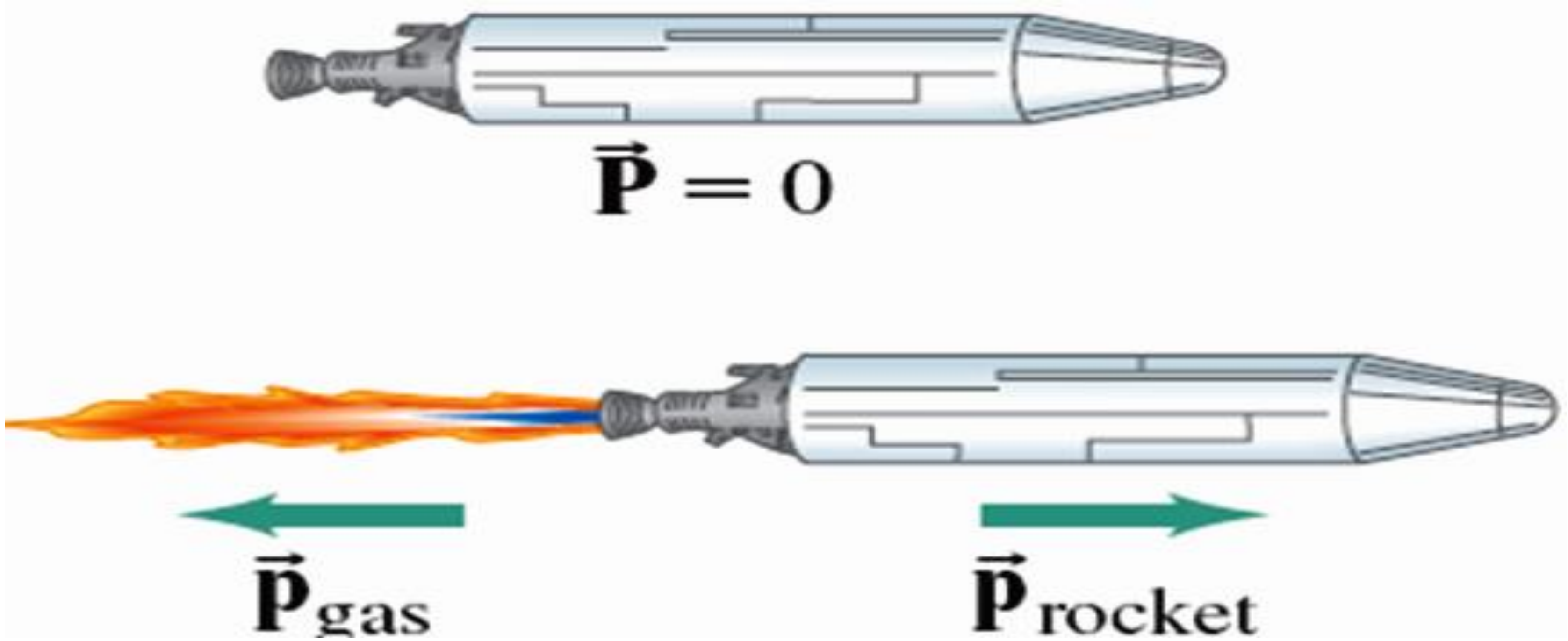
$$m_A v_A + m_B v_B = (m_A + m_{2B})v'$$

$$\Rightarrow \quad v' = [(m_A v_A)/(m_A + m_B)] = 12 \text{ m/s}$$

Rocket Propulsion

Momentum Before Take Off = Momentum After Take Off

Momentum conservation works for a rocket if we consider the rocket & its fuel to be one system, & we account for the mass loss of the rocket (**dm/dt**).



$$\vec{P} = 0 \quad = \quad \vec{p}_{rocket} \quad + \quad \vec{p}_{gas}$$

Rifle Recoil

Calculate the recoil velocity of a rifle, mass $m_R = 5$ kg, that shoots a bullet, mass $m_B = 0.02$ kg, at speed $v_B = 620$ m/s.

Momentum Before Shooting = Momentum After Shooting

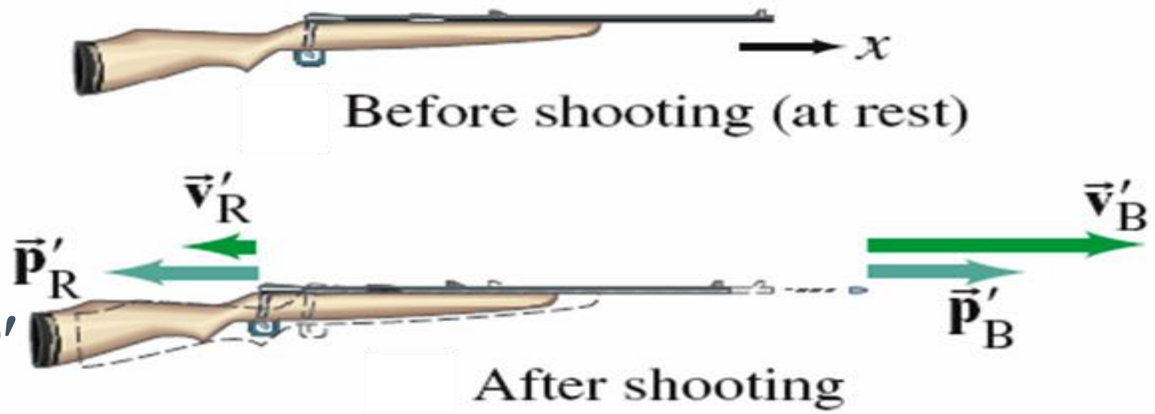
Momentum conservation works here if we consider rifle & bullet as one system

$m_B = 0.02$ kg, $m_R = 5.0$ kg

$(v_B)' = 620$ m/s

Conservation of Momentum

$$m_A v_A + m_B v_B = m_R (v_R)' + m_B (v_B)'$$



This gives:

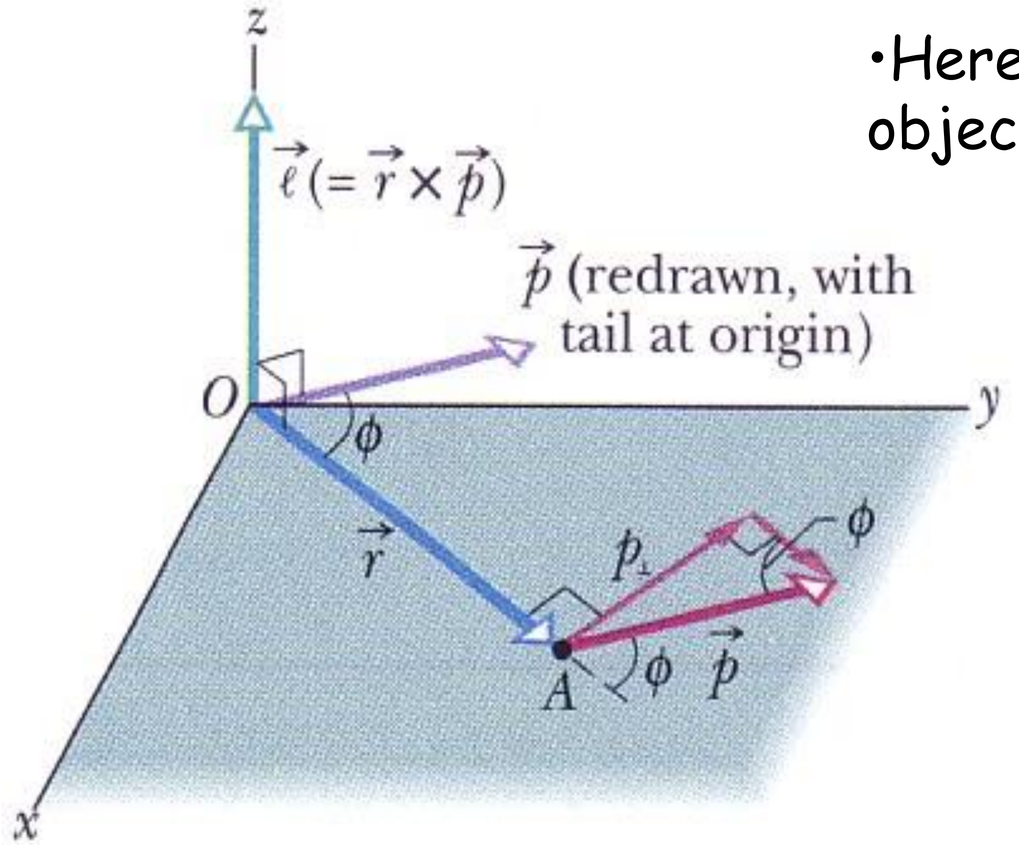
$$0 = m_B (v_B)' + m_R (v_R)'$$

$\Rightarrow$ $(v_R)' = -2.5$ m/s (to the left, of course!)

Torque and angular momentum

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition)}$$

Angular momentum $\vec{l}$ is defined as: $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$



- Here, p is the linear momentum mv of the object.

$$l = mvr\sin\phi$$

$$= rp_\perp = rmv_\perp$$

$$= r_\perp p = r_\perp mv$$

Newton's second law in angular form

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

Linear form

No surprise:

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

angular form

The vector sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum.

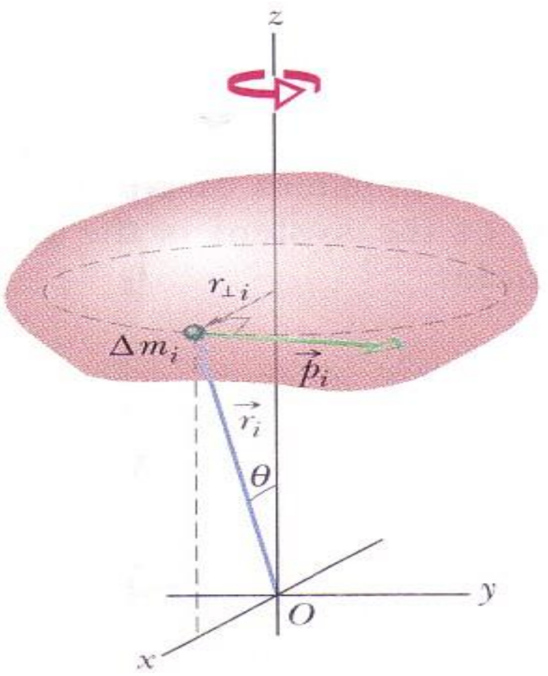
For a system of many particles, the total angular momentum is:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \cdots\cdots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{net,i} = \vec{\tau}_{net}$$

The net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum.

Angular momentum of a rigid body about a fixed axis



We are interested in the component of angular momentum parallel to the axis of rotation:

$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} m_i v_i r_{\perp i} = \int v r_{\perp} dm$$

$$= \int (r_{\perp}\omega) r_{\perp} dm = \omega \int r_{\perp}^2 dm = I\omega$$

In fact: $\vec{L} = I\vec{\omega}$

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

Conservation of angular momentum

It follows from Newton's second law that:

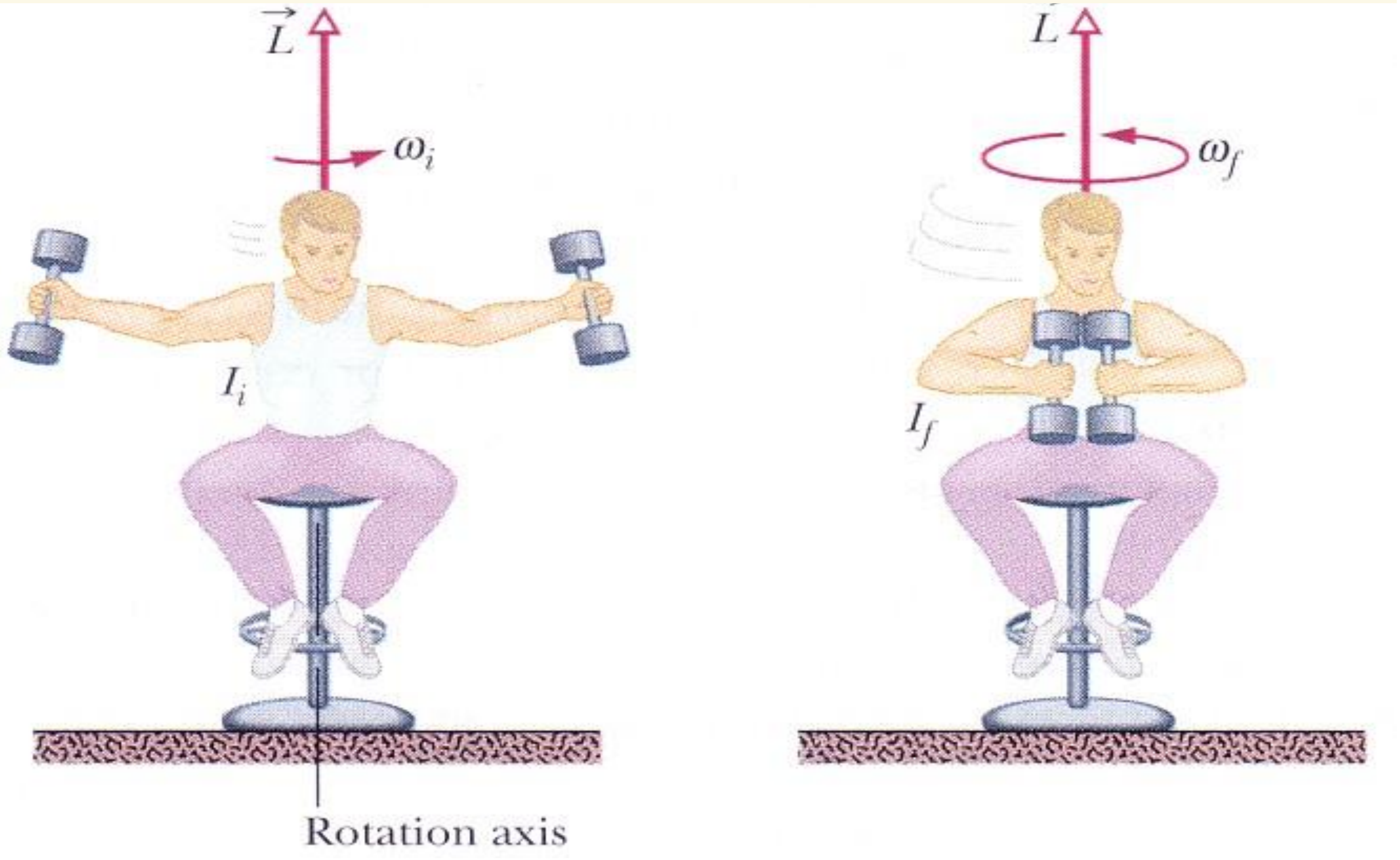
If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system.

$$\vec{L} = \text{a constant}$$

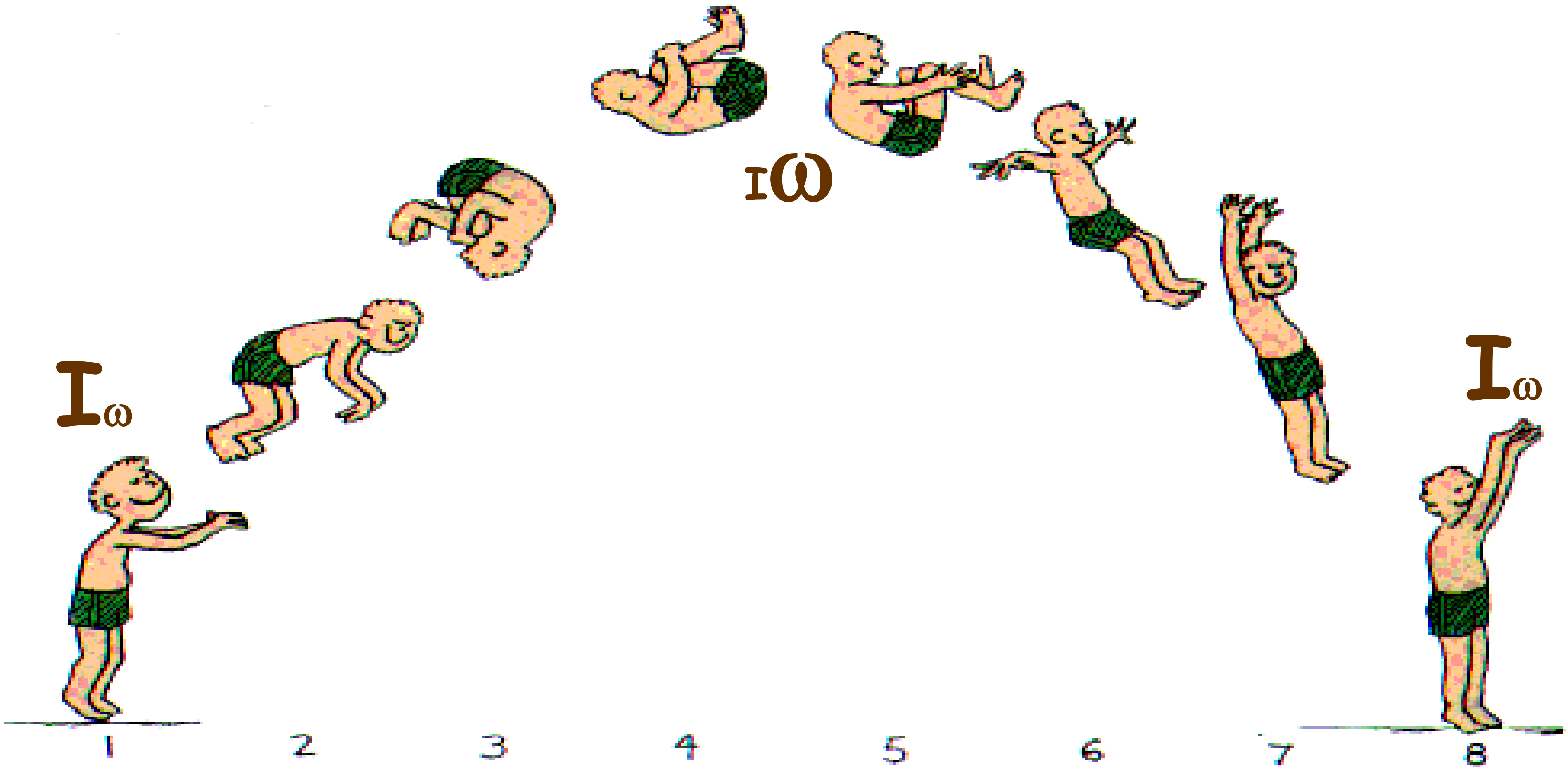
$$\vec{L}_i = \vec{L}_f$$

$$I_i\omega_i = I_f\omega_f$$

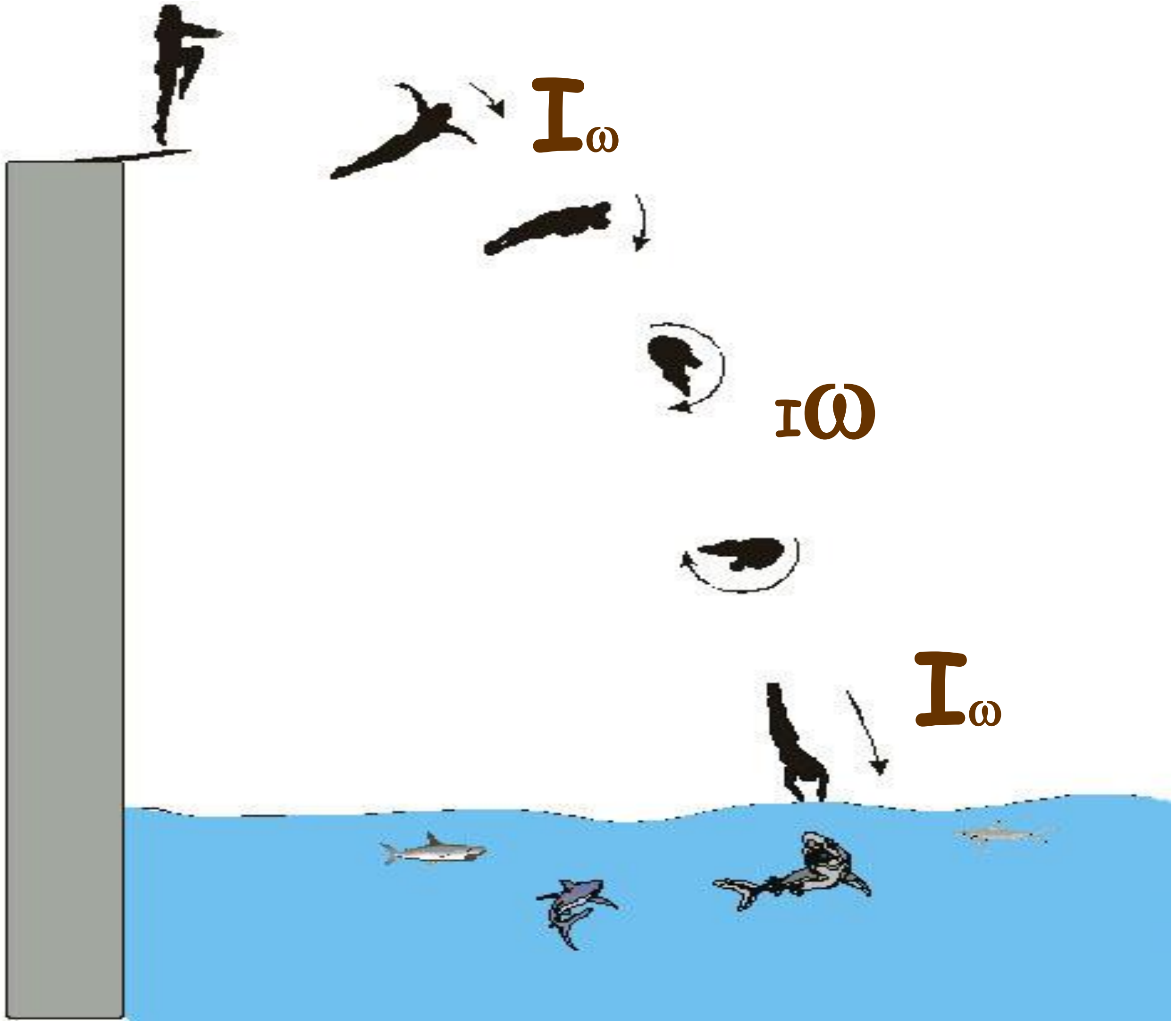
$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$



Conservation of angular momentum



High Diver

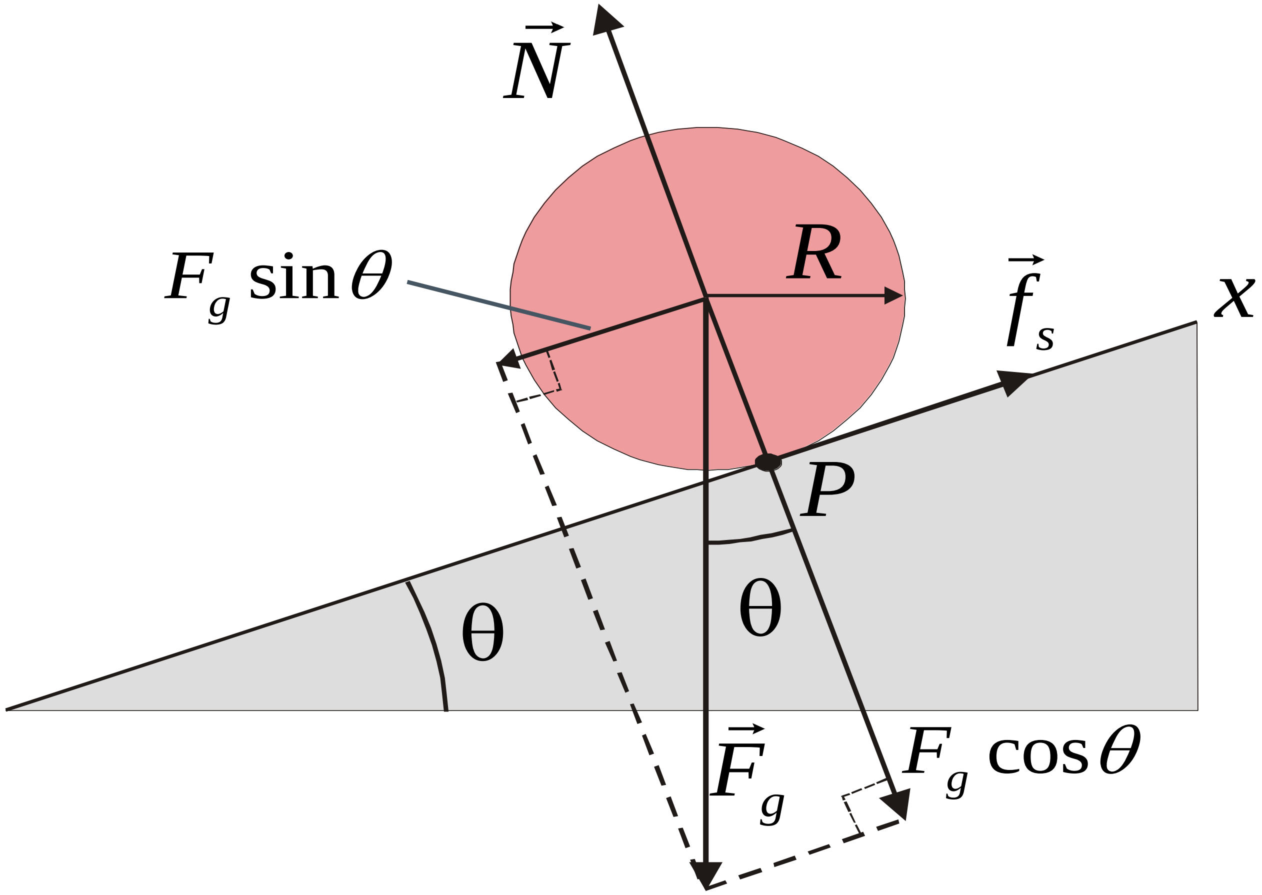


Conservation of angular momentum

$I\omega$

$I\omega$

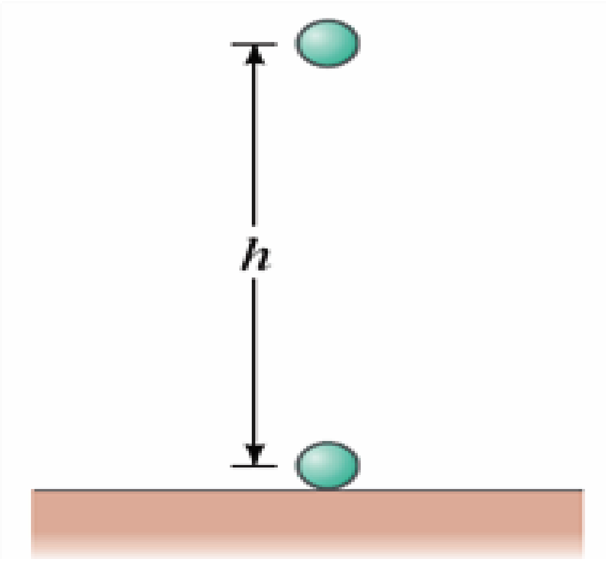
Rolling down a ramp

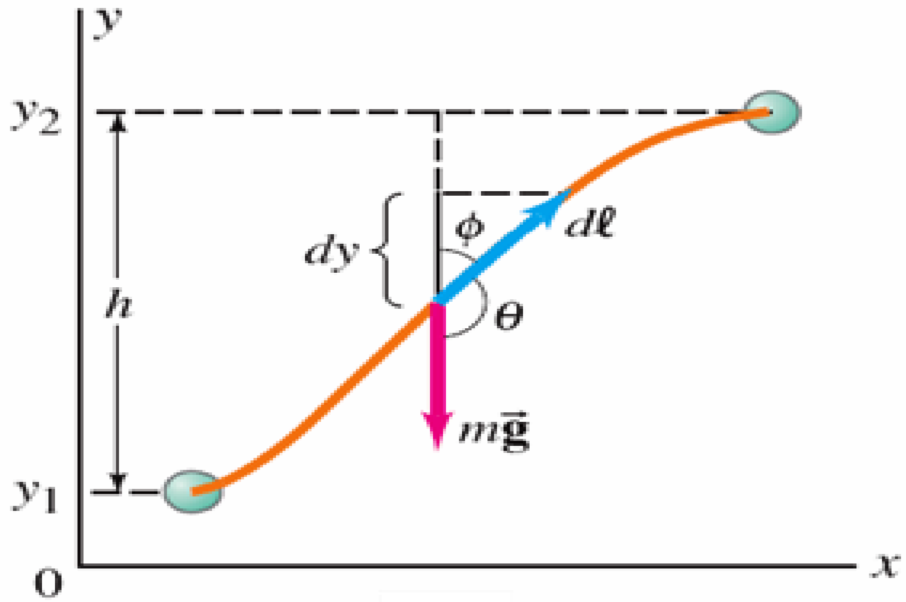


Conservative & Non-conservative Forces

Definition: A force is **conservative** if & only if *the work done by that force on an object moving from one point to another depends ONLY on the initial & final positions of the object, & is independent of the particular path taken.*

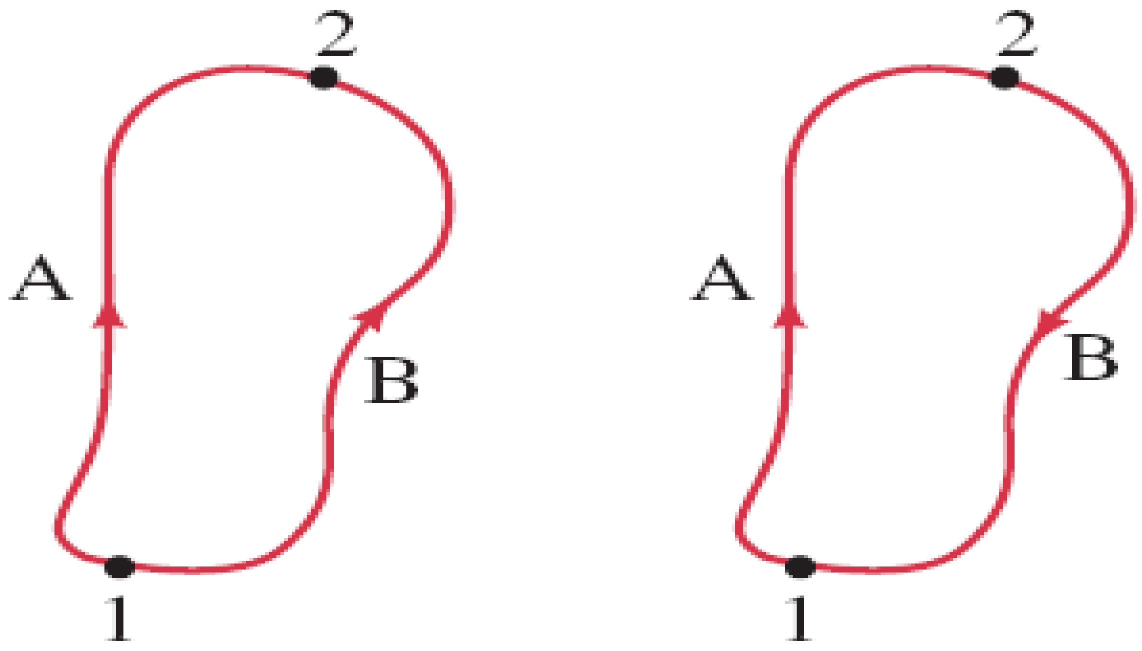
Example: gravity.



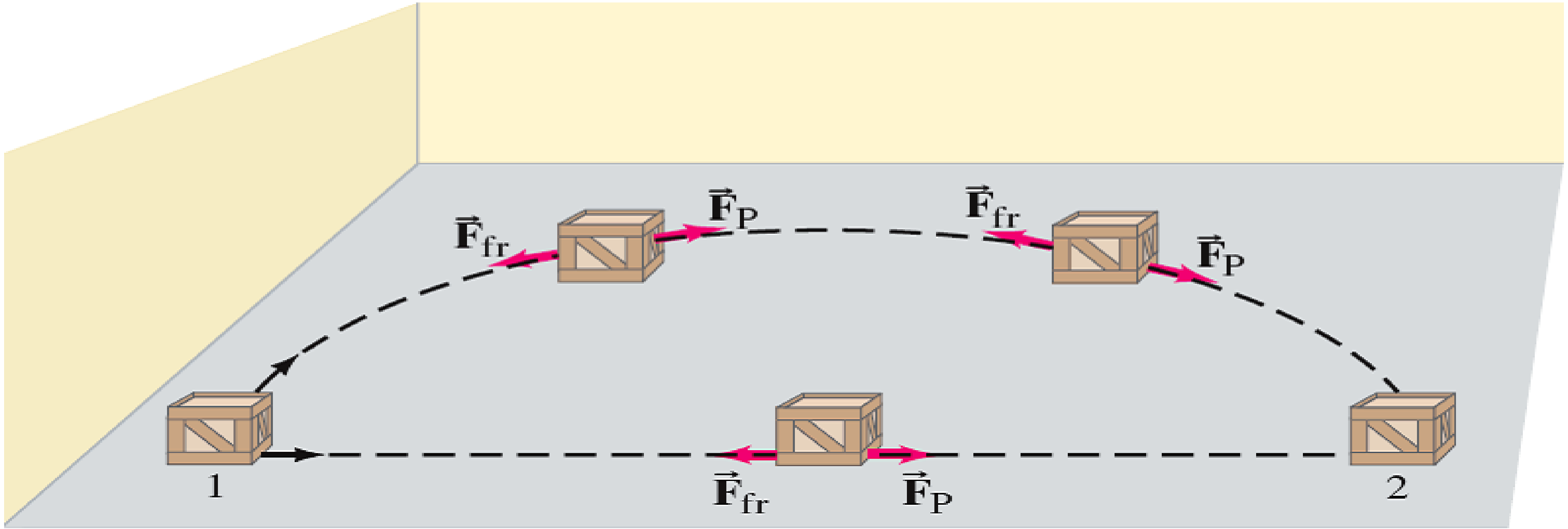


Conservative Force: Another definition:

A force is conservative if the net work done by the force on an object moving around any closed path is zero.



If **friction** is present, the work done depends not only on the starting & ending points, but also on the path taken. ***Friction is a Nonconservative Force!***



Friction is a Nonconservative Force.
The work done by friction depends on the path!

Potential Energy

A mass can have a *Potential Energy* due to its environment

Potential Energy (U) ≡

The energy associated with the position or configuration of a mass.

Examples of potential energy:

- A wound-up spring
- A stretched elastic band
- An object at some height above the ground

TABLE 8–1 Conservative and Nonconservative Forces

Conservative Forces	Nonconservative Forces
Gravitational	Friction
Elastic	Air resistance
Electric	Tension in cord
	Motor or rocket propulsion
	Push or pull by a person

Potential Energy:

Can only be defined for

Conservative Forces!

- *Potential Energy* (U) ≡ Energy associated with the position or configuration of a mass.

Potential work done!

Gravitational Potential Energy:

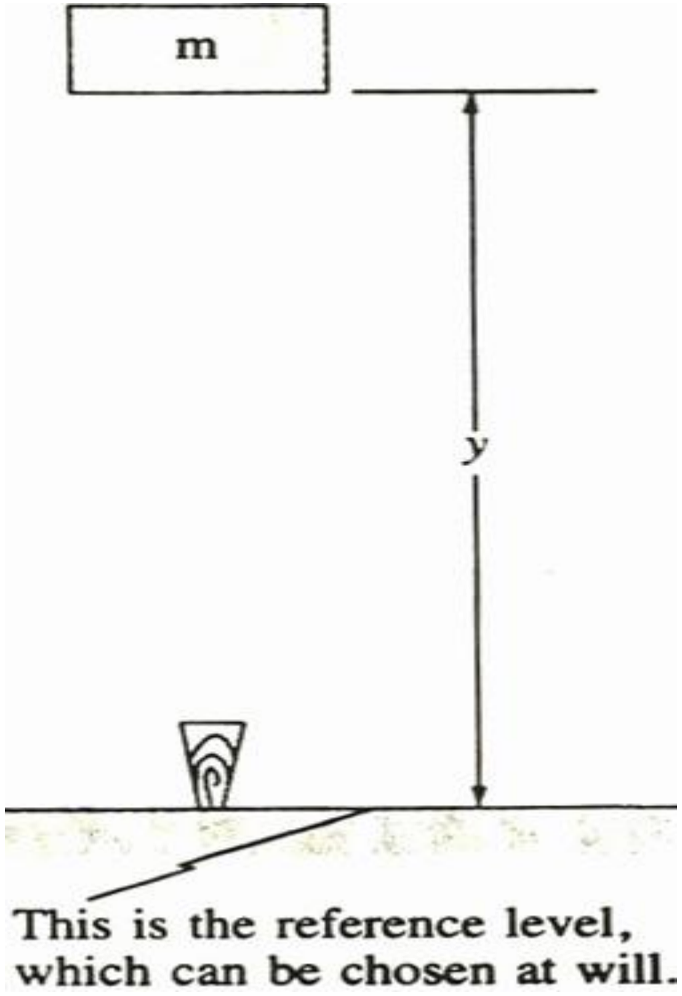
$$U_{grav} \equiv mgy$$

y = distance above Earth

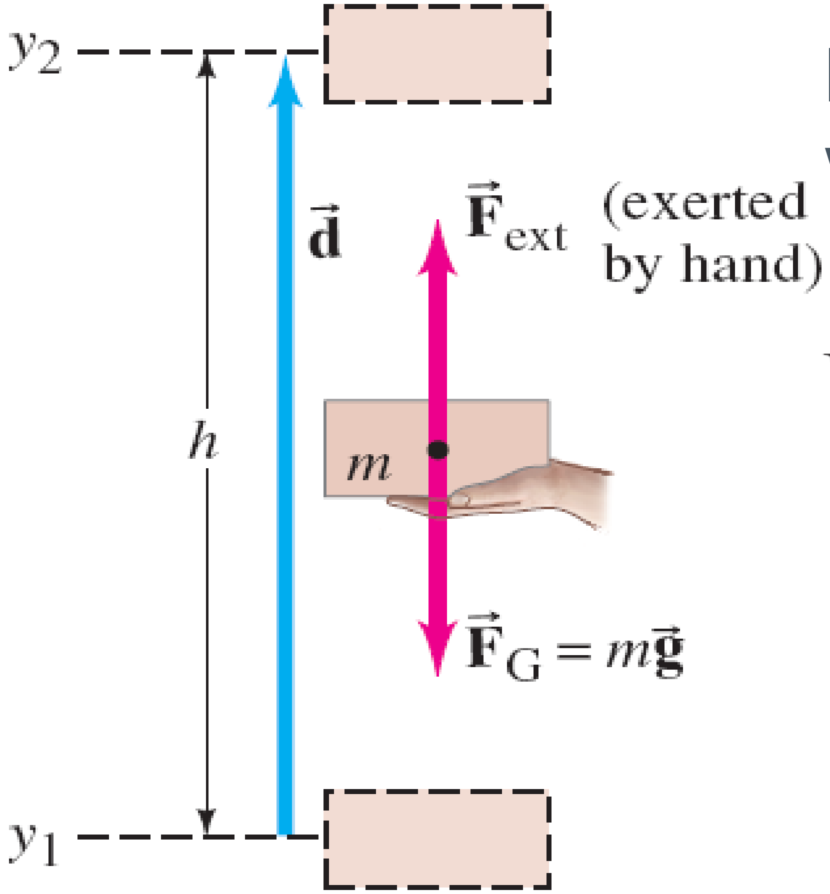
m has the *potential* to do work

mgy when it falls

(W = Fy, F = mg)



Gravitational Potential Energy



In raising a mass **m** to a height **h**, the work done by the external force is

$$W_{ext} = \vec{\mathbf{F}}_{ext} \cdot \vec{\mathbf{d}} = mgh\cos 0^\circ$$

$$= mgh = mg(y_2 - y_1)$$

So we *Define the Gravitational Potential Energy* at height **y** above some reference point as

$$U_{grav} = mgy$$

- Consider a problem in which the height of a mass above the Earth changes from y_1 to y_2:
- The ***Change in Gravitational Potential Energy*** is:

$$\Delta U_{grav} = mg(y_2 - y_1)$$

- The work done on the mass by gravity is: $W = \Delta U_{grav}$

y = distance above Earth

Where we choose $y = 0$ is ***arbitrary***, since we take the difference in 2 y's in calculating ΔU_{grav}

Of course, this <u>potential energy will be converted to kinetic energy if the object is dropped</u>.

Potential energy is a property of a system as a whole, not just of the object (because it depends on external forces).

If $U_{grav} = mgy$, from where do we measure y?

Doesn't matter, but we need to be consistent about this choice!

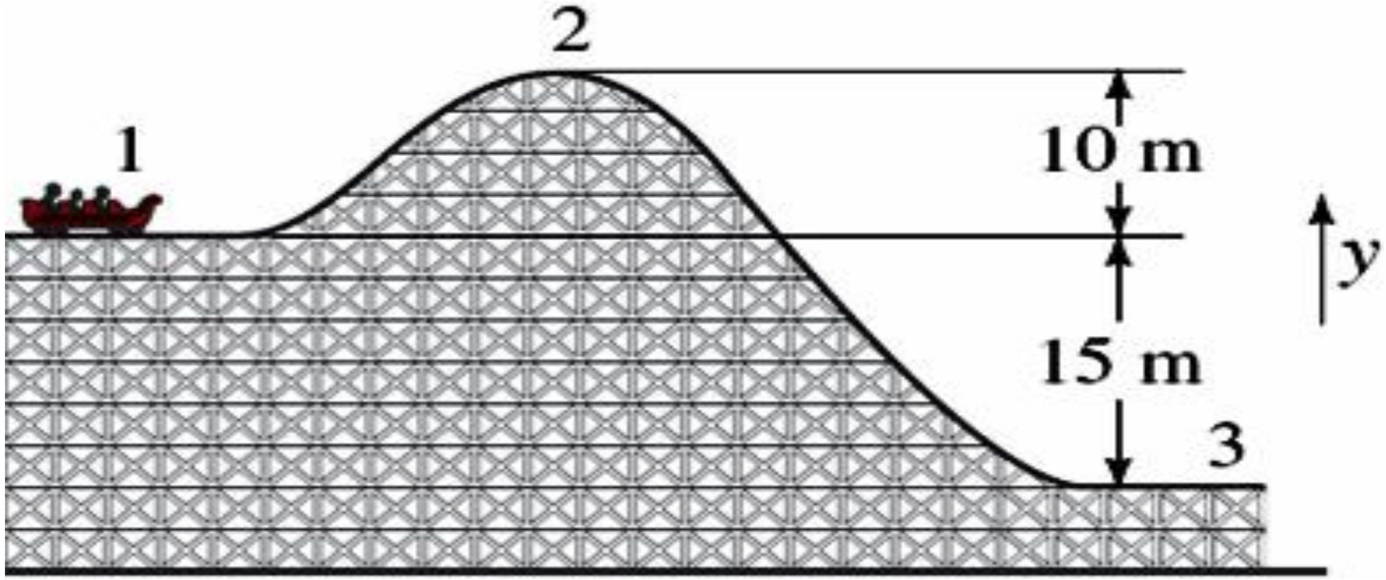
This is because only ***changes*** in potential energy can be measured.

Potential energy changes for a roller coaster

A roller-coaster car, mass m = 1000 kg, moves from point 1 to point 2 & then to point 3.

$\Delta U = mg\Delta y$

Depends only on differences Δy in vertical height!



a. Calculate the gravitational potential energy at points 2 & 3 relative to a point 1. (That is, take y = 0 at point 1.)

b. Calculate the change in potential energy when the car goes from aa point 2 to point 3.

c. Repeat parts a. & b., but take the reference point (y = 0) at point 3.

Work & Energy

- Particle is acted on by a total external force **F**. *Work* done *ON* particle in moving it from position **1** to position **2** in space is defined as line integral

(**ds** = differential path length, assume mass **m** = constant)

$$W_{12} \equiv \int \mathbf{F} \bullet d\mathbf{s} \qquad \text{(limits: from 1 to 2)}$$

- Newton's 2nd Law (& chain rule of differentiation): $\mathbf{F}\bullet d\mathbf{s} = (d\mathbf{p}/dt)\bullet(d\mathbf{r}/dt)\, dt$

$$= m(d\mathbf{v}/dt)\bullet\mathbf{v}\, dt = (½)m[d(\mathbf{v}\bullet\mathbf{v})/dt]\, dt$$

$$= (½)m(dv^2/dt)\, dt$$

Work-Energy Principle

$\Rightarrow \quad W_{12} = \int F \bullet ds \quad = (½)m\int[d(v^2)/dt]\ dt$

$= (½)m\int d(v^2)$ (limits: from 1 to 2)

Or: $W_{12} = (½)m[(v_2)^2 - (v_1)^2]$

› Kinetic Energy of Particle: $T \equiv (½)mv^2$

$\Rightarrow \quad W_{12} = T_2 - T_1 = \Delta T$

Total Work done = Change in kinetic energy

(Work-Energy Principle or Work-Energy Theorem)

Conservative Forces

- Special Case: Force F is such that the work W_{12} *does not depend on path* between points 1 & 2:

  F and the system are then ≡ *Conservative.*

- Alternative definition of conservative: Particle goes from point 1 to point 2 & back to point 1 (different paths, total path is closed). Work done is

$$W_{12} + W_{21} = \oint \mathbf{F} \bullet d\mathbf{s} = 0$$

Work done in closed path is zero

Because path independence means $W_{12} = - W_{21}$

Conservative Forces $\Rightarrow$ Potential Energy

- Consider $\mathbf{W_{12} = \int F \bullet ds}$ (limits: from 1 to 2)
- Conservative force **F** $\Rightarrow$ $\mathbf{W_{12}}$ is path independent.
  - Clearly, friction & similar forces are not conservative!
- *For conservative forces*, define a **Potential Energy** function **V(r)**. By definition:

$$\mathbf{W_{12} = \int F \bullet ds = V_1 - V_2 = - \Delta V}$$

Depends only on end points **1** & **2**

For conservative forces the total work done =

\- (the change in potential energy)

Potential Energy Function

- For conservative forces:

$$W_{12} = \int \mathbf{F} \bullet d\mathbf{s} = V_1 - V_2 = -\Delta V$$

- Vector calculus theorem. W_{12} path independent

$\Rightarrow$ $\mathbf{F}$ = gradient of some scalar function. That is this is satisfied if & only if the force has the form:

$$\mathbf{F} = -\nabla V(r) \quad \text{(minus sign by convention)}$$

For conservative forces, the force is the negative gradient of the potential energy (or potential).

- For conservative forces: $\mathbf{F} = -\nabla V(\mathbf{r})$.

$\Rightarrow$ Can write: $\mathbf{F \bullet ds} = -\nabla V(\mathbf{r}) \bullet \mathbf{ds} = -\mathbf{d}V$

$\Rightarrow$ $\mathbf{F} = -(\partial V/\partial s)$

- Physical (experimental) quantity is $\mathbf{F} = -\nabla V(\mathbf{r})$

$\Rightarrow$ **The zero of V(r) is arbitrary**

(since **F** is a derivative of **V(r)**!)

Energy Conservation

- For conservative forces only we had:

$$W_{12} = \int \mathbf{F} \bullet d\mathbf{s} = V_1 - V_2 \qquad \text{(independent of path)}$$

- In general, we had (Work-Energy Principle):

$$W_{12} = T_2 - T_1$$

- Combining $\Rightarrow$ *For conservative forces:*

$$V_1 - V_2 = T_2 - T_1 \quad \text{or} \quad \Delta T + \Delta V = 0$$

or
$$T_1 + V_1 = T_2 + V_2$$

or
$$E = T + V = \text{constant}$$

$E = T + V \equiv$ Total Mechanical Energy

(or just Total Energy)

- For conservative forces:

$$\Delta T + \Delta V = 0$$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant (conserved)}$$

Energy Conservation Theorem for a Particle:

If only conservative forces are acting on a particle, then the total mechanical energy of the particle, $E = T + V$, is conserved.

- Consider a **special case** where **F** is a function of both position **r** & time **t**: $\mathbf{F} = \mathbf{F}(\mathbf{r},t)$
- Further, suppose we can define a function $V(\mathbf{r},t)$ such that: $\mathbf{F} = -(\partial V/\partial s)$

$\Rightarrow$ Work done on particle in differential distance **ds** is still $\mathbf{F}\bullet d\mathbf{s} = -(\partial V/\partial s)ds$

However, in this case, ***cannot*** write $\mathbf{F}\bullet d\mathbf{s} = -dV$ since V is a function of *both time & space.* May still define a total mechanical energy $E = T + V$. However, E *is no longer conserved*! $E = E(t)$!!

(Conserved $\Rightarrow$ $dE/dt = 0$)

Mechanics of a System of Particles

Mechanics of a System of Particles

- Generalization to *many* (N) particle system:
  - Distinguish External & Internal Forces.
  - **Newton's 2nd Law** (eqtn. of motion), particle *i*:

$$\sum_j F_{ji} + F_i^{(e)} = (dp_i/dt) = p_i$$

$F_i^{(e)} \equiv$ Total external force on the i^{th} particle.

$F_{ji} \equiv$ Total (internal) force on the i^{th} particle due to the j^{th} particle.

- $F_{jj} = 0$ of course!!

$$\sum_j F_{ji} + F_i^{(e)} = (dp_i/dt) = p_i \qquad (1)$$

- *Assumption:* Internal forces F_{ji} obey Newton's 3rd Law: $F_{ji} = - F_{ij}$
  - ≡ The *"Weak" Law of Action and Reaction*
  - – Original form of the 3rd Law, but is not satisfied by all forces!

- Sum (1) over all particles in the system:

$$\sum_{i,j(\neq i)} F_{ji} + \sum_i F_i^{(e)} = \sum_i (dp_i/dt)$$

$$= d(\sum_i m_i v_i)/dt = d^2(\sum_i m_i r_i)/dt^2$$

Newton's 2nd Law for Many Particle Systems

- Rewrite as:

$$d^2(\sum_i m_i r_i)/dt^2 = \sum_i F_i^{(e)} + \sum_{i,j(\neq i)} F_{ji} \qquad (2)$$

$$\sum_i F_i^{(e)} \equiv \text{ total external force on system } \equiv F^{(e)}$$

$$\sum_{i,j(\neq i)} F_{ji} \equiv 0. \qquad \text{By } \textbf{Newton's 3}^{\textbf{rd}} \textbf{ Law:}$$

$F_{ji} = - F_{ij} \Rightarrow F_{ji} + F_{ij} = 0$ (*cancel pairwise*!)

- So, (2) becomes ($\mathbf{r}_i \equiv$ position vector of $\mathbf{m}_i$):

$$d^2(\sum_i m_i r_i)/dt^2 = F^{(e)} \qquad (3)$$

$\Rightarrow$ Only *external* forces enter Newton's 2nd Law to get the equation of motion of a many particle system!!

$$d^2(\sum_i m_i r_i)/dt^2 = F^{(e)} \qquad (3)$$

- Modify **(3)** by defining **R** ≡ mass weighted average of position vectors $\mathbf{r}_i$.

$$\mathbf{R} \equiv (\sum_i m_i \mathbf{r}_i)/(\sum_i m_i) \equiv (\sum_i m_i \mathbf{r}_i)/M$$

$M \equiv \sum_i m_i$ (total mass of particles in system)

R ≡ *Center of mass* of the system (schematic in Figure)

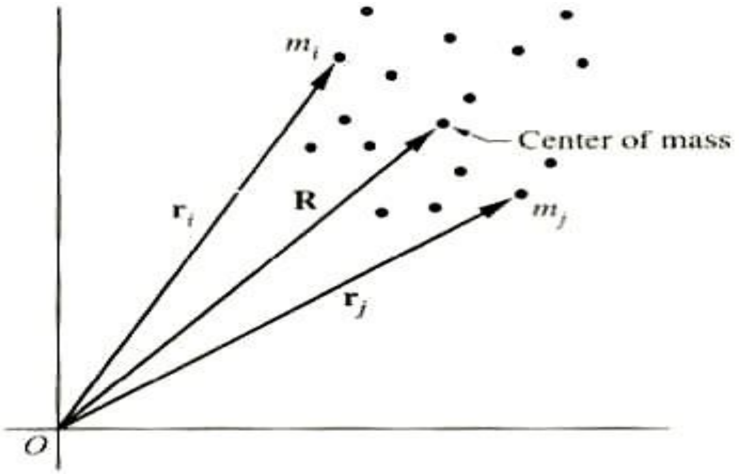


FIGURE 1.1 The center of mass of a system of particles.

⇒ (3) becomes:

$$M(d^2\mathbf{R}/dt^2) = M\mathbf{A} = M(d\mathbf{V}/dt) = (d\mathbf{P}/dt) = \mathbf{F}^{(e)} \quad (4)$$

Just like the eqtn of motion for mass M at position R under the force $F^{(e)}$!

$$M(d^2R/dt^2) = F^{(e)} \qquad (4)$$

⇒ Newton's 2nd Law for a many particle system: *The Center of Mass moves as if the total external force were acting on the entire mass of the system concentrated at the Center of Mass!*

- **Corollary:** Purely *internal* forces (assuming they obey Newton's 3rd Law) have no effect on the motion of the Center of Mass (CM).

Momentum Conservation

- $MR = (\sum_i m_i r_i)$. Consider: Time derivative (const M):

$M(dR/dt) = MV = \sum_i m_i[(dr_i)/dt] \equiv \sum_i m_i v_i \equiv \sum_i p_i \equiv P$

(total momentum = momentum of CM)

$\Rightarrow$ Using the definition of P, **Newton's 2nd Law** is:

$$(dP/dt) = F^{(e)} \quad (4)$$

- Suppose $F^{(e)} = 0$: $\Rightarrow$ $(dP/dt) = \dot{P} = 0$

$\Rightarrow$ P = constant (conserved)

Conservation Theorem for the Linear Momentum of a System of Particles:

If the total external force, $F^{(e)}$*, is zero, the total linear momentum,* P*, is conserved.*

Angular Momentum

- Angular momentum $\mathbf{L}$ of a many particle system (sum of angular momenta of each particle): $\mathbf{L} \equiv \sum_i[\mathbf{r}_i \times \mathbf{p}_i]$
- **Time derivative:** $\mathbf{L} = (d\mathbf{L}/dt) = \sum_i d[\mathbf{r}_i \times \mathbf{p}_i]/dt$

$$= \sum_i[(d\mathbf{r}_i/dt) \times \mathbf{p}_i] + \sum_i[\mathbf{r}_i \times (d\mathbf{p}_i/dt)]$$

$$(d\mathbf{r}_i/dt) \times \mathbf{p}_i = \mathbf{v}_i \times (m_i\mathbf{v}_i) = 0$$

$$\Rightarrow \quad (d\mathbf{L}/dt) = \sum_i[\mathbf{r}_i \times (d\mathbf{p}_i/dt)]$$

- **Newton's 2nd Law:** $(d\mathbf{p}_i/dt) = \mathbf{F}_i^{(e)} + \sum_{j(\neq i)} \mathbf{F}_{ji}$

  $\mathbf{F}_i^{(e)} \equiv$ Total external force on the *i th* particle

  $\sum_{j(\neq i)} \mathbf{F}_{ji} \equiv$ Total internal force on the *i th* particle due to interactions with all other particles (*j*) in the system

$$\Rightarrow \quad (d\mathbf{L}/dt) = \sum_i[\mathbf{r}_i \times \mathbf{F}_i^{(e)}] + \sum_{i,j(\neq i)}[\mathbf{r}_i \times \mathbf{F}_{ji}]$$

$$(dL/dt) = \sum_i [\mathbf{r}_i \times \mathbf{F}_i^{(e)}] + \sum_{i,j(\neq i)} [\mathbf{r}_i \times \mathbf{F}_{ji}] \qquad (1)$$

› Consider the 2nd sum & look at ***each particle pair*** (i,j). Each term $\mathbf{r}_i \times \mathbf{F}_{ji}$ has a corresponding term $\mathbf{r}_j \times \mathbf{F}_{ij}$. Take together & use Newton's 3rd Law:

$$\Rightarrow [\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij}] = [\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times (-\mathbf{F}_{ji})] = [(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}]$$

$(\mathbf{r}_i - \mathbf{r}_j) \equiv \mathbf{r}_{ij}$ = vector from particle j to particle i. (Figure)

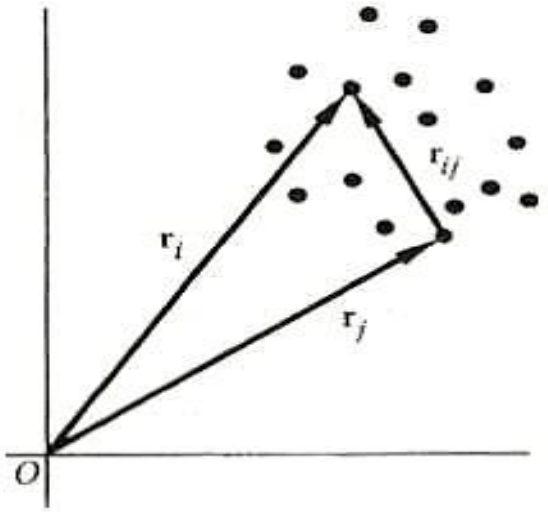


FIGURE 1.2 The vector r_{ij} between the ith and jth particles.

$$(dL/dt) = \sum_i[r_i \times F_i^{(e)}] + (½)\sum_{i,j(\neq i)}[r_{ij} \times F_{ji}] \qquad (1)$$

↑

To Prevent Double Counting!

- ***Assumption:*** Internal forces are ***Central Forces***: **Directed the along lines joining the particle pairs**
  (≡ ***The "Strong" Law of Action and Reaction***)

$\Rightarrow$ r_{ij} || F_{ji} for each (i,j) & $[r_{ij} \times F_{ji}] = 0$ for each (i,j)!

$\Rightarrow$ 2nd term in (1) is $(½)\sum_{i,j(\neq i)}[r_{ij} \times F_{ji}] = 0$

$$\Rightarrow \qquad (dL/dt) = \sum_i [r_i \times F_i^{(e)}] \qquad (2)$$

- Total external torque on particle *i*:

$$N_i^{(e)} \equiv r_i \times F_i^{(e)}$$

- (2) becomes:

$$(dL/dt) = N^{(e)}$$

(2ʹ)

$$N^{(e)} \equiv \sum_i [r_i \times F_i^{(e)}] = \sum_i N_i^{(e)}$$

= Total external torque on the system

$$(dL/dt) = N^{(e)} \qquad (2)$$

⇒ Newton's 2nd Law (rotational motion) for a many particle system: *The time derivative of the total angular momentum is equal to the total external torque.*

- Suppose $N^{(e)} = 0$: ⇒ $(dL/dt) = \dot{L} = 0$

  ⇒ L = constant (conserved)

Conservation Theorem for the Total Angular Momentum of a Many Particle System:

If the total external torque, $N^{(e)}$, *is zero, then* $(dL/dt) = 0$ *and the total angular momentum,* L, *is conserved.*

- $(dL/dt) = N^{(e)}$. **A vector equation!** Holds component by component. $\Rightarrow$ **Angular momentum conservation holds component by component.** For example, if $N_z^{(e)} = 0$, L_z is conserved.

- <u>**Linear & Angular Momentum Conservation Laws:**</u>
  - **Conservation of Linear Momentum** holds if internal forces obey ***the "Weak" Law of Action and Reaction***: Only Newton's 3rd Law $F_{ji} = - F_{ij}$ is required to hold!
  - **Conservation of Angular Momentum** holds if internal forces obey ***the "Strong" Law of Action and Reaction***: Newton's 3rd Law $F_{ji} = - F_{ij}$ holds, *PLUS* the forces **must be Central Forces**, so that $r_{ij} \parallel F_{ji}$ for each (i,j)!

    Valid for many common forces (gravity, electrostatic). Not valid for some (magnetic forces, etc.). See text discussion.

Center of Mass & Relative Coordinates

- More on angular momentum. Search for an **analogous relation** to what we had for linear momentum:

$$P = M(dR/dt) = MV$$

Want: Total momentum = Momentum of CM = Same as if entire mass of system were at CM.

- Start with total the angular momentum: $L \equiv \sum_i [r_i \times p_i]$
- $R \equiv$ CM coordinate (origin O). For particle *i* define:

$r_i' \equiv r_i - R$ = relative coordinate vector from CM to particle *i* (Figure)

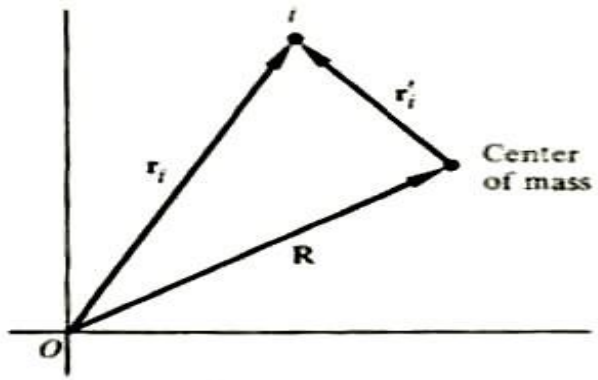


FIGURE 1.3 The vectors involved in the shift of reference point for the angular momentum.

- $r_i = r_i' + R$

  Time derivative: $(dr_i/dt) = (dr_i'/dt) + (dR/dt)$ or:

  $v_i = v_i' + V$, $V \equiv$ CM velocity relative to O

  $v_i' \equiv$ velocity of particle *i* relative to CM. Also:

  $p_i \equiv m_i v_i \equiv$ momentum of particle *i* relative to O

- Put this into angular momentum:

$$L = \sum_i [r_i \times p_i] = \sum_i [(r_i' + R) \times m_i(v_i' + V)]$$

Manipulation: (using $m_i v_i' = d(m_i r_i')/dt$)

$$L = R \times \sum_i (m_i) V + \sum_i [r_i' \times (m_i v_i')] +$$

$$\sum_i (m_i r_i') \times V + R \times d[\sum_i (m_i r_i')]/dt$$

- **Note:** $\sum_i (m_i r_i')$ defines the CM coordinate with respect to the CM & is thus zero!! $\sum_i (m_i r_i') \equiv 0$!

$\Rightarrow$ *The last 2 terms are zero!*

$$\Rightarrow \qquad L = R \times \sum_i(m_i)V + \sum_i[r_i' \times (m_i v_i')] \qquad (1)$$

- Note that $\sum_i(m_i) \equiv M$ = total mass & also

  $m_i v_i' \equiv p_i'$ = momentum of particle *i* relative to the CM

$$\Rightarrow \quad L = R\times(MV) + \sum_i[r_i' \times p_i'] = R\times P + \sum_i[r_i'\times p_i'] \qquad (2)$$

The total angular momentum about point O = the angular momentum of the motion of the CM + the angular momentum of motion about the CM

- (2) $\Rightarrow$ In general, **L** depends on the origin **O**, through the vector **R**. Only if the CM is at rest with respect to **O**, will the first term in (2) vanish. Then & only then will **L** be independent of the point of reference. Then & only then will **L** = angular momentum about the CM

Work & Energy

- The **work done by all forces** in changing the system from configuration 1 to configuration 2:

$$W_{12} \equiv \sum_i \int \mathbf{F}_i \bullet d\mathbf{s}_i \qquad \text{(limits: from 1 to 2)} \qquad (1)$$

As before: $\mathbf{F}_i = \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}$

$$\Rightarrow \qquad W_{12} = \sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i + \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i \qquad (2)$$

- Work with **(1)** first:
  - Newton's 2nd Law $\Rightarrow$ $\mathbf{F}_i = m_i(d\mathbf{v}_i/dt)$. Also: $d\mathbf{s}_i = \mathbf{v}_i dt$

    $\mathbf{F}_i \bullet d\mathbf{s}_i = m_i(d\mathbf{v}_i/dt) \bullet d\mathbf{s}_i = m_i(d\mathbf{v}_i/dt) \bullet \mathbf{v}_i dt$

    $= m_i \mathbf{v}_i d\mathbf{v}_i = d[(½)m_i(v_i)^2]$

$$\Rightarrow \qquad W_{12} = \sum_i \int d[(½)m_i(v_i)^2] \equiv T_2 - T_1$$

where $T \equiv (½)\sum_i m_i(v_i)^2$ = **Total System Kinetic Energy**

Work-Energy Principle

- $W_{12} = T_2 - T_1 = \Delta T$

  The total Work done = The change in kinetic energy

  (*Work-Energy Principle or Work-Energy Theorem*)

- Total Kinetic Energy: $T \equiv (½)\sum_i m_i(v_i)^2$
  - Another useful form: Use transformation to CM & relative coordinates: $v_i = V + v_i'$, $V \equiv$ CM velocity relative to O, $v_i' \equiv$ velocity of particle *i* relative to CM.

  $\Rightarrow \quad T \equiv (½)\sum_i m_i(V + v_i')\bullet(V + v_i')$

  $T = (½)(\sum_i m_i)V^2 + (½)\sum_i m_i(v_i')^2 \quad + V\bullet\sum_i m_i v_i'$

  Last term: $V\bullet d(\sum_i m_i r_i')/dt$. From the angular momentum discussion: $\sum_i m_i r_i' = 0 \Rightarrow$ The last term is zero!

$\Rightarrow$ **Total KE:** $\quad T = (½)MV^2 + (½)\sum_i m_i(v_i')^2$

Total KE

$$T = (½)MV^2 + (½)\sum_i m_i(v_i')^2$$

- *The total Kinetic Energy of a many particle system is equal to the Kinetic Energy <u>of</u> the CM plus the Kinetic Energy of motion <u>about</u> the CM.*

Work & PE

- 2 forms for work:

  $W_{12} = \sum_i \int \mathbf{F}_i \bullet d\mathbf{s}_i = T_2 - T_1 = \Delta T$ (just showed!) **(1)**

  $W_{12} = \sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i + \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i$ **(2)**

  Use **(2)** with ***Conservative Force Assumptions:***

1. **External Forces:** $\Rightarrow$ Potential functions $V_i(\mathbf{r}_i)$ exist such that (for each particle *i*): $\mathbf{F}_i^{(e)} = -\nabla_i V_i(\mathbf{r}_i)$
2. **Internal Forces:** $\Rightarrow$ Potential functions V_{ij} exist such that (for each particle pair *i,j*): $\mathbf{F}_{ij} = -\nabla_i V_{ij}$
3. **Strong Law of Action-Reaction:** $\Rightarrow$ Potential functions $V_{ij}(r_{ij})$ are functions only of distance

   $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ between *i* & *j* & the forces lie along line joining them (**Central Forces**!): $V_{ij} = V_{ij}(r_{ij})$

$\Rightarrow \quad \mathbf{F}_{ij} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) f(r_{ij})$

f is a scalar $\leftarrow$function!

› **Conservative external forces:**

$\Rightarrow \quad \sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i = - \sum_i \int \nabla_i V_i \bullet d\mathbf{s}_i = - \sum_i (V_i)_2 + \sum_i (V_i)_1$ Or:

$\sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i = (V^{(e)})_1 - (V^{(e)})_2$

Where: $V^{(e)} \equiv \sum_i V_i$ = total PE associated with external forces.

› **Conservative internal forces:** Write (sum over pairs)

$\Rightarrow \quad \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = (\tfrac{1}{2}) \sum_{i,j(\neq i)} \int [\mathbf{F}_{ji} \bullet d\mathbf{s}_i + \mathbf{F}_{ij} \bullet d\mathbf{s}_j]$

$= - (\tfrac{1}{2}) \sum_{i,j(\neq i)} \int [\nabla_i V_{ij} \bullet d\mathbf{s}_i + \nabla_j V_{ij} \bullet d\mathbf{s}_j]$

Note: $\nabla_i V_{ij} = - \nabla_j V_{ij} = \nabla_{ij} V_{ij}$ ($\nabla_{ij} \equiv$ grad with respect to $\mathbf{r}_{ij}$)

Also: $d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_{ij}$

$\Rightarrow \quad \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = - (\tfrac{1}{2}) \sum_{i,j(\neq i)} \int \nabla_{ij} V_{ij} \bullet d\mathbf{r}_{ij}$ do integral!!

$= - (\tfrac{1}{2}) \sum_{i,j(\neq i)} (V_{ij})_2 + (\tfrac{1}{2}) \sum_{i,j(\neq i)} (V_{ij})_1$

- **Conservative internal (Central!) forces:**

  $\sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = -(½)\sum_{i,j(\neq i)}(V_{ij})_2 + (½)\sum_{i,j(\neq i)}(V_{ij})_1$

  or: $\sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = (V^{(I)})_1 - (V^{(I)})_2$

  Where: $V^{(I)} \equiv (½)\sum_{i,j(\neq i)} V_{ij}$ = Total PE associated with internal forces.

- *For conservative external forces & conservative, central internal forces, it is possible to define a potential energy function for the system:*

$$V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (½)\sum_{i,j(\neq i)} V_{ij}$$

Conservation of Mechanical Energy

- *For conservative external forces & conservative, central internal forces:*
  - The total work done in a process is:

$$W_{12} = V_1 - V_2 = -\Delta V$$

with $V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (\tfrac{1}{2})\sum_{i,j(\neq i)} V_{ij}$

  - In general

$$W_{12} = T_2 - T_1 = \Delta T$$

Combining $\Rightarrow V_1 - V_2 = T_2 - T_1$ or $\Delta T + \Delta V = 0$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant}$$

$$E = T + V \equiv \text{Total Mechanical Energy}$$

(or just Total Energy)

Energy Conservation

$$\Delta T + \Delta V = 0$$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant (conserved)}$$

Energy Conservation Theorem for a Many Particle System:

If only conservative external forces & conservative, central internal forces are acting on a system, then the total mechanical energy of the system,

$E = T + V$, *is conserved.*

- Consider the potential energy:

$$V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (½)\sum_{i,j(\neq i)} V_{ij}$$

- 2nd term $V^{(I)} \equiv (½)\sum_{i,j(\neq i)} V_{ij} \equiv$ Internal Potential Energy of the System. This is generally non-zero & might vary with time.
  - Special Case: *Rigid Body:* System of particles in which distances $\mathbf{r}_{ij}$ are fixed (do not vary with time).

  $\Rightarrow$ $d\mathbf{r}_{ij}$ are all $\perp$ $\mathbf{r}_{ij}$ & thus to internal forces $\mathbf{F}_{ij}$

  $\Rightarrow$ $\mathbf{F}_{ij}$ do no work. $\Rightarrow$ $V^{(I)}$ = **constant**

  Since V is arbitrary to within an additive constant, we can ignore $V^{(I)}$ for rigid bodies only.

Thank You