CHAPTER 4

Two-Body Central Force Problem

4.1. REDUCTION OF TWO-BODY CENTRAL FORCE PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

In this chapter, we plan to discuss the motion of two bodies under a mutual central force as an application of Lagrangian formulation. Consider a system of two particles of masses m_1 and m_2 whose instantaneous position vectors in an inertial frame with origin O_i are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively [Fig. 4.1]. Hence the vector distance of m_2 relative to m_1 is

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad ...(1)$$

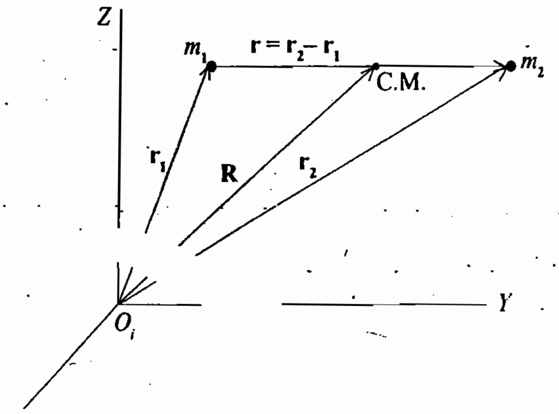

Fig. 4.1. Two-body problem : relative and centre of mass coordinates

The two masses are interacting via central force for which the potential energy for the system $V(r)$ is a function of scalar distance r only. The Lagrangian for the system is

$$L = T - V = \tfrac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - V(r) \qquad ...(2)$$

This system of two particles has six degrees of freedom and hence six idependent generalized coordinates are required to describe the state of the system. Instead of $\mathbf{r}_1$ and $\mathbf{r}_2$ (six) coordinates, we can choose the three components of the position vector of the centre of mass $\mathbf{R}$, and three components of the relative vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The position vector of the centre of mass is defined by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \qquad ...(3)$$

Solving (1) and (3), we get

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2 \mathbf{r}}{m_1 + m_2} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1 \mathbf{r}}{m_1 + m_2} \qquad ...(4)$$

Therefore $$\dot{\mathbf{r}}_1 = \dot{\mathbf{R}} - \frac{m_2 \dot{\mathbf{r}}}{m_1 + m_2} \quad \text{and} \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} + \frac{m_1 \dot{\mathbf{r}}}{m_1 + m_2} \qquad ...(5)$$

Hence $$L = \tfrac{1}{2} m_1 \left(\dot{\mathbf{R}} - \frac{m_2 \dot{\mathbf{r}}}{m_1 + m_2} \right)^2 + \tfrac{1}{2} m_2 \left(\dot{\mathbf{R}} + \frac{m_1 \dot{\mathbf{r}}}{m_1 + m_2} \right)^2 - V(r)$$

or $$L = \frac{1}{2}(m_1 + m_2)\,\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2 - V(r) \qquad ...(6)$$

We see that the three coordinates **R** (*i.e.*, X, Y, Z) are cyclic and hence corresponding linear momentum $(m_1 + m_2)\,\dot{\mathbf{R}}$ or $\dot{\mathbf{R}}$, the velocity of centre of mass, is constant. This means that the centre of mass is either at rest or moving with constant velocity. Obviously the Lagrange's equations of motion for three generalized coordinates **r** will not contain the terms in **R** and $\dot{\mathbf{R}}$. Hence to discuss the motion of the system, we can drop the first term in the Lagrangian, eq. (6). Thus

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r) \qquad ...(7)$$

where $\mu = (m_1 m_2)/(m_1 + m_2)$ is called the ***reduced mass*** of the two-particle system. The form of the Lagrangian, represented by eq. (7), is exactly the same as that of a particle with mass μ moving at a vector distance **r** from a centre O. This centre exerts a central force on the particle and is taken as the origin of the coordinate system [Fig. 4.2] and appears to be fixed. Thus the two-body central force problem has been reduced to the equivalent one-body problem.

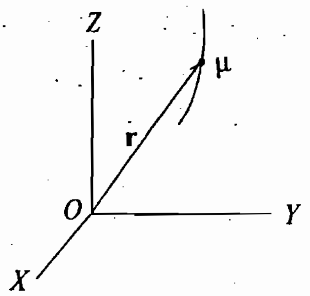

Fig. 4.2 : Reduction of two-body problem to one-body problem

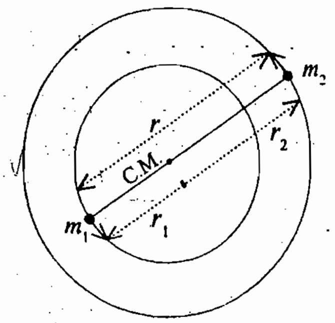

Fig.4.3

This is to be remembered that in fact, **r** is the vector distance of particle 2 relative to the particle 1, which is acting as the origin O, and therefore this origin O is moving with an acceleration in an inertial system. This means that the coordinate system attached to O is a non-inertial frame. However, one may not be aware of this fact just by having a look at the explicit form of the Lagrangian, represented by eq. (7). The actual paths of the two particles will depend upon the law of force, initial positions and initial velocities. One particular case may be of interest to mention. If force $F = \mu\omega^2 r$, the particle of mass μ will move on a circular path of radius r around O or m_2 particle will move on a circular path of radius r relative to m_1. However, in the inertial frame, fixed with the centre of mass, the two particles will move in circular orbits around their centre of mass with the same angular velocity. In this frame, r_1 and r_2 are the radii of their orbits and r is the distance between the two particles [Fig. 4.3].

In case, $m_1 >> m_2$, the reduced mass is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_2}{1 + \dfrac{m_2}{m_1}} \approx m_2 \text{ (say } m\text{)}$$

In such a case the problem becomes just one-particle problem. When high accuracy is not required, this approximation is good enough.

In general, if we are dealing with a two-particle central force problem, we need to solve the equivalent one-body problem, where a particle of mass m moves about a fixed centre of force with the Lagrangian, given by

$$L = \frac{1}{2} m\dot{r}^2 - V(r) \qquad ...(8)$$

wherever we need we may replace m by μ.

Examples of two-body problem are a planet-sun system, hydrogen atom (consisting of an electron revolving around a proton), positronium, any diatomic molecule like H_2, HCl etc.

Ex. 1. *Calculate the reduced mass of the following systems :*

Hydrogen atom, Positronium and H_2 *molecule.*

Solution : (*i*) In hydrogen atom, an electron of mass m revolves round a proton of much heavier mass M. The reduced mass of the hydrogen atom is given by

$$\mu_H = \frac{mM}{m + M} = m\left(1 + \frac{m}{M}\right)^{-1} = m\left(1 - \frac{m}{M}\right)$$

(using Bionomial expansion and neglecting higher order terms because $m/M << 1$)

But $$\frac{m}{M} = \frac{1}{1836}, \text{ hence } \mu_H = m\left(1 - \frac{1}{1836}\right)$$

Hence in case of hydrogen atom, if the energy, period etc. are calculated by assuming that an electron moves round a fixed proton, there will occur some error. To get the correct results m should be replaced by the reduced mass μ_H in the expression of energy, period etc. Actually, in case of hydrogen atom, the error will be very small, because

$$m\left(1 - \frac{1}{1836}\right) \cong m \text{ nearly.}$$

(*ii*) ***Positronium*** is a temporary combination of a positron and an electron similar to hydrogen atom. A positron is a particle which has mass equal to the electron mass but it has equal positive charge. The reduced mass of the positronium is given by

$$\mu_p = \frac{mm}{m + m} = \frac{m}{2}$$

(since m = mass of electron = mass of positron).

Thus the reduced mass of the positronium particle is one-half the mass of the electron.

(*iii*) H_2 molecule consists of two hydrogen atoms separated by a distance and bound together by electromagnetic forces. If M is the mass of each hydrogen atom, then the reduced mass of H_2 molecule is given by

$$\mu_{H_2} = \frac{MM}{M + M} = \frac{M}{2}.$$

Ex. 2. *Show that the spectral lines of positronium are arranged in the same pattern as in the case of atomic hydrogen spectrum but have nearly double the wavelengths.*

Solution : According to Bohr's theory, the frequencies of lines in the hydrogen spectrum are given by

$$\nu = \frac{2\pi\mu_H e^4}{h^3}\left(\frac{1}{n^2} - \frac{1}{p^2}\right) \qquad ...(i)$$

where n and p are integers ($p > n$) and μ is the reduced mass of the hydrogen atom.

Now, $\mu_H = m\,[1-(1/1836)] \cong m$, mass of the electron ...(*ii*)

Thus in the case of hydrogen spectrum, the frequencies are directly proportional to the mass of the electron m ($\nu \propto \mu$ or m). Positronium has the structure like hydrogen atom, the expression (*i*) will also provide the spectral lines, rediated by positronium, but the reduced mass of hydrogen atom $\mu_H \cong m$ is to be replaced by the reduced mass of the positronium $\mu_p = m/2$. Hence in case of positronium, we will get the spectral lines like that of hydrogen but their frequencies will be nearly half, *i.e.*, their wavelengths will be nearly double because

$$\frac{\nu_p}{\nu_H} = \frac{\mu_p}{\mu_H} = \frac{m/2}{m} = \frac{1}{2} \text{ or } \nu_p = \frac{\nu_H}{2} \qquad ...(iii)$$

Therefore,
$$\frac{c}{\lambda_p} = \frac{c}{2\lambda_H} \text{ or } \lambda_p = 2\lambda_H \qquad ...(iv)$$

where ν_H, ν_p and λ_H, λ_p are the frequencies and wavelengths of hydrogen and positronium respectively and c is the speed of light.

Thus we see that the *spectral lines of positronium are arranged in the same pattern as in case of hydrogen but have nearly double wavelengths.*

4.2. CENTRAL FORCE AND MOTION IN A PLANE

If a force acts on a particle in such a way that it is always directed towards or away from a fixed centre and its magnitude depends only upon the distance (r) from the centre, then this force is called ***central force***. Thus a central force is represented by

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}} = f(r)\,\mathbf{r}/r \qquad ...(9)$$

where $f(r)$ is a function of distance r only and $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector along $\mathbf{r}$ from the fixed centre. The force is attractive or repulsive, if $f(r) < 0$ or $f(r) > 0$ respectively.

A central force is always a conservative force and if $V(r)$ is the potential energy, then

$$f(r) = -\frac{\partial V}{\partial r} \text{ or } \mathbf{F} = -\frac{\partial V}{\partial r}\frac{\mathbf{r}}{r} \qquad ...(10)$$

The potential energy for central force depends only on the distance r and hence the system possesses spherical symmetry. Thus any rotation about a fixed axis will not have any effect on the solution and hence an angle coordinate for rotation about a fixed axis must be cyclic. This results in the conservation of angular momentum of the system *i.e.*,

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \text{constant (vector)} \qquad ...(11)$$

where $\mathbf{p}$ is linear momentum.

Taking dot product with $\mathbf{r}$ in eq. (11), we have

$$\mathbf{r} \cdot \mathbf{J} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = (\mathbf{r} \times \mathbf{r}) \cdot \mathbf{p} = 0 \qquad ...(12)$$

since in a scalar triple product the position of dot and cross are interchangeable and $\mathbf{r} \times \mathbf{r} = 0$.

Therefore, position vector $\mathbf{r}$ is always perpendicular to the constant $\mathbf{J}$ vector. This means that the ***motion of the particle under central force takes place in a plane*** and we can describe the instantaneous position of the particle in plane polar coordinates r and θ.

4.3. EQUATIONS OF MOTION UNDER CENTRAL FORCE AND FIRST INTEGRALS

Consider a particle of mass m moving about a fixed centre of force O. [Fig. 4.4]

The particle is moving under a central force

$$\mathbf{F} = f(r)\frac{\mathbf{r}}{r} = -\frac{\partial V}{\partial r}\frac{\mathbf{r}}{r},$$

where $V(r)$ is the potential energy.

Using polar coordinates (r, θ), the Lagrangian for the system can be written as

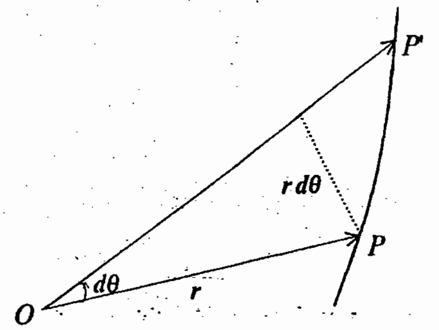

Fig. 4.4 : Area swept out by the radius vector in infinitesimal small time dt

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad ...(13)$$

In eq. (13), the Lagrangian L is independent of θ coordinate (*i.e.* $\partial L/\partial\theta = 0$) and hence θ is the cyclic coordinate. The canonical momentum p_θ corresponding to the coordinate θ is given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} \qquad ...(14)$$

which is the angular momentum.

Now, one of the equation of motion (Lagrange's equation for θ coordinate) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \text{ or } \frac{d}{dt}(mr^2\dot{\theta}) = 0 \qquad ...(15)$$

Integration of this equation gives one of the ***first integral*** of motion, i.e.,

$$mr^2\dot{\theta} = J \text{ (constant)} \qquad ...(16)$$

where J is the constant magnitude of the angular momentum and is conserved. Since m is a constant, we obtain from eq. (15)

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0 \text{ or } \frac{1}{2}r^2\dot{\theta} = h \text{ (constant)} \qquad ...(17)$$

The factor $\frac{1}{2}$ has been inserted above so that $\frac{1}{2}r^2\dot{\theta}$ represents the ***areal velocity*** (h), *i.e.*, the area swept out by the radius vector per unit time. It can be seen from Fig. 4.4 that the differential area dA swept out in time dt is

$$dA = \frac{1}{2}r\,(r\,d\theta) = \frac{1}{2}r^2 d\theta \text{ and so that } \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \text{ is the areal velocity.}$$

Thus from eq. (17) we see that the areal velocity is constant, when the motion is taking place under central force. This is in accordance with the well known *Kepler's second law of planetary motion*. In other words the conservation of angular momentum is equivalent to say that the areal velocity is constant.

Lagrange equation for r coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

From (13), $\quad \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $\dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - \dfrac{\partial V}{\partial r}$

Therefore, $$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad ...(18)$$

Putting from eq. (10), $\dfrac{\partial V}{\partial r} = -f(r)$ for central force, eq. (18) takes the form

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \qquad ...(19)$$

Also from (16), $\dot{\theta} = \dfrac{J}{mr^2}$, therefore

$$m\ddot{r} - \frac{J^2}{mr^3} = f(r) \qquad ...(20)$$

This is the second order differential equation in r coordinate only. Eq. (20) can also be written as

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{\partial V}{\partial r} = -\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{J^2}{mr^2}\right] - \frac{\partial V}{\partial r}$$

or $$m\ddot{r} = -\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{J^2}{mr^2} + V\right] \qquad ...(21)$$

Multiplying both sides of eq. (21) by $\dot{r}$, we get

$$m\ddot{r}\,\dot{r} = -\frac{\partial}{\partial r}\left[\frac{J^2}{2mr^2} + V\right]\dot{r}$$

or $$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right] = -\frac{d}{dt}\left[\frac{J^2}{2mr^2} + V\right] \text{ or } \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V\right) = 0 \qquad ...(22)$$

Integrating it, we get

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V = E, \text{ constant} \qquad ...(23)$$

Since from (16), $J = mr^2\dot{\theta}$, therefore

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = E, \text{ constant or } \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E, \text{ constant} \qquad ...(24)$$

where E represents the total energy.

Thus, the sum of kinetic and potential energy *i.e.*, total energy E, is constant. This is *the statement of conservation of energy*.

Eq. (23) or eq. (24) is known as another ***first integral*** of motion.

4.4. DIFFERENTIAL EQUATION FOR AN ORBIT

In case of a central force, we want to deduce the equation of the orbit whose solution can give us the radial distance (r) as function of θ.

The equation of motion for a particle of reduced mass m, moving under central force, can be written as [from eq. (20)]

$$m\ddot{r} - \frac{J^2}{mr^3} = f(r) \qquad \text{...(25)}$$

Now,
$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{J}{mr^2}\frac{dr}{d\theta} \quad \left[\text{as } \frac{J}{mr^2} = \dot{\theta} \text{ from eq. (16)}\right]$$

and
$$\ddot{r} = \frac{d}{dt}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right] = \frac{d}{d\theta}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right]\frac{d\theta}{dt} = \frac{J}{mr^2}\frac{d}{d\theta}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right]$$

Let
$$u = \frac{1}{r}, \text{ therefore, } \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Then
$$\ddot{r} = -\frac{J^2u^2}{m^2}\frac{d}{d\theta}\left[\frac{du}{d\theta}\right] = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Hence eq. (25) is
$$-\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{J^2u^3}{m} = f\left(\frac{1}{u}\right)$$

or
$$\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right) \qquad \text{...(26)}$$

This is the differential equation of an orbit, provided the force law $f(r) = f\left(\frac{1}{u}\right) = -\frac{\partial V}{\partial r}$ or the potential V is known.

4.5. INVERSE SQUARE LAW OF FORCE

Gravitational and coulomb force between two particles are the most important examples of central force. The force $f(r)$, as usual, is expressed as

$$f(r) = -\frac{Gm_1m_2}{r^2} \quad \text{(Newton's law of Gravitation)} \qquad \text{...(27)}$$

and
$$f(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \quad \text{(Coulomb's law)} \qquad \text{...(28)}$$

The general force law, governing eqs. (27) and (28), is the inverse square law of force, given by

$$f(r) = -\frac{K}{r^2} \qquad \text{...(29)}$$

If V is the potential, then

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{K}{r^2}$$

and its integration gives

$$V = -\frac{K}{r} \qquad \text{...(30)}$$

where the constant of integration is taken to be zero by assuming $V(r) = 0$ at infinite separation ($r = \infty$).

4.6. KEPLER'S LAWS OF PLANETARY MOTION AND THEIR DEDUCTION

The motion of planets has been a subject of much interest for astronomers from very early times. Kepler's laws of planetary motion are as follows :

(1) **The law of elliptical orbits :** Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.

(2) **The law of areas :** The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal times *i.e.*, the areal velocity of the radius vector is constant.

(3) **The harmonic law :** The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse.

Kepler's laws have been enunciated purely on the basis of observations, taken for the motion of the planets. These laws give us a simple and accurate description of their motions, but do not offer any explanation.

The planets move around the sun under the influence of gravitational force which is an inverse square law force. Hence we deduce the Kepler's laws of planetary motion around the sun on the basis of inverse square law of force.

4.6.1. Deduction of the Kepler's First Law

For $u = 1/r$, the inverse square law force $[f(r) = -K/r^2]$ is given by

$$f\left(\frac{1}{u}\right) = -Ku^2$$

Thus the differential equation (26) of the orbit can be expressed as

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{J^2u^2} Ku^2 \quad [\because f(r) = -Ku^2]$$

or
$$\frac{d^2u}{d\theta^2} + u - \frac{mK}{J^2} = 0 \qquad \text{...(31)}$$

Let
$$x = u - \frac{mK}{J^2}.$$

Then
$$\frac{d^2x}{d\theta^2} + x = 0 \qquad \text{...(32)}$$

which has the solution

$$x = A\cos(\theta - \theta') \qquad \text{...(33)}$$

where A and θ' are the constants of integration.

Since $x = u - \frac{mK}{J^2}$ and $u = \frac{1}{r}$, we can write eq. (33) as

$$\frac{1}{r} - \frac{mK}{J^2} = A\cos(\theta - \theta')$$

or
$$\frac{1}{r} = \frac{mK}{J^2} + A\cos(\theta - \theta') \qquad \text{...(34)}$$

or $$\frac{J^2/mK}{r} = 1 + \frac{J^2 A}{mK} \cos(\theta - \theta')$$

or $$\frac{l}{r} = 1 + e \cos(\theta - \theta') \qquad \text{...(35)}$$

where $$\frac{J^2}{mK} = l \text{ and } \frac{J^2 A}{mK} = e$$

It is easy to identify $e = \sqrt{1 + \dfrac{2EJ^2}{mK^2}}$ with the help of the two first integrals of motion*. The constant θ' appearing in eq. (33) is a constant of integration determined by initial conditions.

Thus, when a particle is moving under inverse square law of force, its orbit is represented by eq. (35). This is the general equation of a conic section with one focus at the origin and eccentricity e, given by

$$e = \sqrt{1 + \frac{2EJ^2}{mK^2}} \qquad \text{...(36)}$$

The magnitude of e decides the nature of the orbit, represented by eq. (35), as following :

Value of e	*Value of total energy E*	*Conic*
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	Ellipse
$e = 0$	$E = -\dfrac{mK^2}{2J^2}$	Circle

* Differentiating eq. (34), we get

$$-\frac{\dot{r}}{r^2} = -A \sin(\theta - \theta')\dot{\theta} = -A \sin(\theta - \theta')\frac{J}{mr^2} \quad \text{or,} \quad \dot{r} = \frac{AJ}{m}\sin(\theta - \theta')$$

where we have put $\dot{\theta} = J/mr^2$ from one of the first integrals. But from another first integral [eq. (23)]

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{K}{r} = E \qquad \left(\text{Here, } V(r) = -\frac{K}{r}\right)$$

Thus $$\frac{1}{2}m\frac{A^2J^2}{m^2}\sin^2(\theta - \theta') + \frac{J^2}{2m}\left[\frac{mK}{J^2} + A\cos(\theta - \theta')\right]^2 - K\left[\frac{mK}{J^2} + A\cos(\theta - \theta')\right] = E$$

or $$\frac{A^2J^2}{2m} = E + \frac{mK^2}{2J^2} \quad \text{or} \quad A = \frac{mK}{J^2}\sqrt{1 + \frac{2J^2E}{mK^2}}$$

Therefore, $$e = \frac{J^2 A}{mK} = \sqrt{1 + \frac{2EJ^2}{mK^2}}.$$

Thus the sign of the total energy E tells us about the nature of the orbit. If the total energy E is less than zero or negative, the orbit is an ellipse. This is actually the case for planetary motion, when a planet is moving around the sun under gravitational force. This is known as ***Kepler's first law**, according to which every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.* However, the result does not consider the effect of the presence of the other planets.

4.6.2. Deduction of Kepler's Second Law

According to this law, the radius vector drawn from the sun to a planet, sweeps out equal areas in equal times *i.e.*, *the areal velocity is constant in planetary motion.* A gravitational force is a central force and this law is the same, as given by the statement of conservation of angular momentum in a central force field in eq. (16) or eq. (17) *i.e.*,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m}, \text{ a constant} \quad ...(37)$$

The inverse square law force or gravitational force in planetary motion is a special case of central force. However, the constancy of areal velocity is a general property taking place under central force.

4.6.3. Deduction of Kepler's Third Law (*Period of motiom in an elliptical Orbit*)

For $e < 1$ or $E < 0$, the orbit is elliptical one, given by [eq. (35)] :

$$\frac{l}{r} = 1 + e \cos(\theta - \theta')$$

where $$l = \frac{J^2}{mK} \text{ and } e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$$

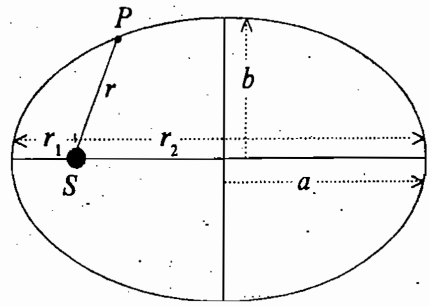

Fig. 4.5 : Motion of a planet around sun

When $\theta - \theta' = 0$ or $\cos(\theta - \theta') = 1$, the value of $r = r_1$ is minimum and when $\theta - \theta' = \pi$ or $\cos(\theta - \theta') = -1$, the value of $r = r_2$ is maximum [Fig. 4.5]. The apsidal distances r_1 and r_2 are known as *perihelion* and *aphelion* and are given by

$$\frac{l}{r_1} = 1 + e \text{ or } r_1 = \frac{l}{1+e} \quad ...(38)$$

and $$\frac{l}{r_2} = 1 - e \text{ or } r_2 = \frac{l}{1-e} \quad ...(39)$$

The semimajor axis (a) of the ellipse is one-half the sum of these two apsidal (turning) distances, *i.e.*,

$$a = \frac{r_1 + r_2}{2} = \frac{1}{2}\left[\frac{l}{1+e} + \frac{l}{1-e}\right] = \frac{l}{1-e^2} \quad ...(40a)$$

or $$a = -\frac{J^2}{mK} \cdot \frac{mK^2}{2EJ^2} = -\frac{K}{2E} \left[\text{from (36), } 1 - e^2 = -\frac{2EJ^2}{mK^2}\right] \quad ...(40b)$$

or $$E = -\frac{K}{2a} \quad ...(40c)$$

Thus in case of an elliptical orbit, the total energy depends solely on the major axis.

If T be the periodic time in which the particle or radius vector completes one revolution then the area of the orbit is obtained by using eq. (37) *i.e.*,

$$A = \int_0^A dA = \int_0^T \left(\frac{1}{2} r^2 \dot{\theta}\right) dt = \int_0^T \frac{J}{2m} dt = \frac{JT}{2m} \qquad ...(41)$$

But area of the ellipse $A = \pi ab$, ...(42)

where a and b are the semi-major and semi-minor axes of the ellipse respectively.

Equating (41) and (42), we get

$$T = \frac{2\pi\, abm}{J} \qquad ...(43)$$

But according to the property of the ellipse

$$b = a\sqrt{1-e^2} = a\sqrt{\frac{J^2}{mKa}} = a^{\frac{1}{2}} \frac{J}{\sqrt{mK}} \qquad ...(44)$$

because from (40 a), $a = \dfrac{l}{1-e^2} = \dfrac{J^2}{mK(1-e^2)}$ or $1-e^2 = \dfrac{J^2}{mKa}$.

Therefore, from eqs. (43) and (44), we obtain

$$T = \frac{2\pi am}{J} \cdot \frac{a^{1/2} J}{\sqrt{mK}} \text{ or } T = 2\pi\, a^{3/2} \sqrt{\frac{m}{K}} \qquad ...(45)$$

This gives the **periodic time** in an elliptical orbit.

Squaring both sides of eq. (45). we get

$$T^2 = 4\pi^2 a^3 \frac{m}{K} \text{ or } T^2 \propto a^3 \qquad ...(46)$$

Thus the square of period of revolution of a planet around the sun is proportional to the cube of semi-major axis of the elliptical orbit. This is known as ***Kepler's third law of planetary motion.***

It is to be reminded that the motion of a planet around the sun is a two body problem and hence in eq. (45), m is to be replaced by the reduced mass μ of the two-body system, *i.e.*,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad ...(47)$$

where m_1 may be taken to be the mass of the sun and m_2 that of the planet. Also, the force law is

$$f(r) = -\frac{G m_1 m_2}{r^2} = -\frac{K}{r^2}$$

so that $$K = G m_1 m_2 \qquad ...(48)$$

Therefore eq. (45) is

$$T = \frac{2\pi\, a^{3/2}}{\sqrt{G(m_1 + m_2)}} \text{ or } T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \qquad ...(49)$$

Thus we obtain a more correct statement of the third law *i.e.*, the square of the periods of various planets are proportional to the cube of their respective semi-major axes with different proportionality constants, because for each planet, proportionality constant is different. If we neglect the mass (m_2) of a planet compared to the mass of the sun (m_1), we obtain the same proportionality constant for all the planets *i.e.*,

$$T^2 = \frac{4\pi^2}{Gm_1} a^3 \qquad ...(50)$$

or

$$T^2 = 4\pi^2 \frac{m_2}{K} a^3 \quad [\because K = Gm_1 m_2] \qquad ...(51)$$

This is an approximate relation and in fact, this is Kepler's third law of planetary motion. For example, $m_2/m_1 = 0.1\%$ for Jupiter and $m_2/m_1 = 3\times10^{-4}\%$ for earth. This means that proportionality constant in (49) is different for different planets. However, in case of the Bohr model of atom, all revolving electrons are identical in mass and hence μ and K are the same for all orbits in an atom. This means that proportionality constant is the same for different electrons in different orbits. Therefore Kepler's third law is very much true for the electron orbits in the Bohr atom.

If an electron of charge $-e$ is moving around a nucleus of charge $+Ze$ (Z = atomic number), the central force acting on the electron is

$$F(r) = -\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r^2} = -\frac{K}{r^2} \quad \left(K = \frac{Ze^2}{4\pi\epsilon_0}\right)$$

If the total energy $E < 0$, the electron will move in an elliptical orbit with periodic time T, given by

$$T^2 = 4\pi^2 \frac{a^3\mu}{K} = \frac{4\pi^2 a^3 \mu}{Ze^2}(4\pi\epsilon_0)$$

or

$$T = \frac{4\pi}{e}\sqrt{\frac{\pi\epsilon_0 \mu a^3}{Z}}$$

where $\mu = \dfrac{mM}{m+M}$ (m = electronic mass and M = mass of nucleus).

4.7. STABILITY AND CLOSURE OF ORBIT UNDER CENTRAL FORCE

When a particle is moving under central force, total energy E is conserved and is given by [eq. (23)]

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V(r)$$

This energy integral can be reduced to a one dimensional motion in r, if an effective potential is defined by

$$V_e(r) = V(r) + \frac{1}{2}\frac{J^2}{mr^2} \qquad ...(52)$$

Hence

$$E = \frac{1}{2}m\dot{r}^2 + V_e(r) \qquad ...(53)$$

This equation suggests that the radial kinetic energy is $\frac{1}{2}m\dot{r}^2$ and the effective potential energy for the radial motion is $V_e(r)$. The later is composed of two parts : (*i*) $V(r)$, the actual potential energy, and (*ii*) $V_{cf}(r) = \frac{1}{2}\frac{J^2}{mr^2} = \frac{1}{2}mr^2\dot{\theta}^2$, the centrifugal energy for radial motion. Obviously the force acting on the particle is composed of a central force and a centrifugal force, because from eq. (19)

$$m\ddot{r} = f(r) + mr\dot{\theta}^2 \qquad ...(54)$$

Suppose the central force is attractive. For example, for gravitational force, $V(r) = -K/r$, which is negative, varying as $-1/r$ and plotted in Fig. 4.6. The centrifugal potential energy $V_{cf}(r)$ increases as $1/r^2$ for $r \to 0$. The plot of the resultant $V_e(r)$ can have a minimum with a finite negative value, thus allowing a range of bounded orbits. In fact, a motion is called bounded in r, if $\dot{r}$ vanishes at the extreme values of r, say $r = r_{max}$ and $r = r_{min}$. For a bounded motion, both of these bounds must exist. Thus for $r = r_{max}$ and $r = r_{min}$, from eq. (53)

$$E = V_e(r)$$

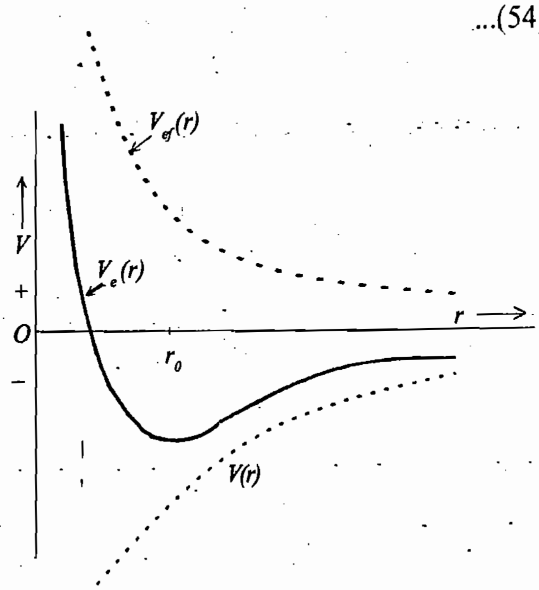

Fig. 4.6 : Effective potential for gravitational force $[V(r) = -K/r]$

Also from (53), we note that

$$E - V_e(r) = \frac{1}{2}m\dot{r}^2 \geq 0 \text{ for all } r.$$

Therefore for any possible radial motion, we must have

$$E \geq V_e(r) \qquad ...(55)$$

In other words, some part of the $V_e(r)$ curve must lie below the curve $V_e(r) = E$ for any radial motion.

The path of a particle moving under central force is called an **orbit**. The condition for stability in the radial motion is given by

$$\frac{dV_e}{dr} = 0 \text{ and } \frac{d^2V_e}{dr^2} > 0 \text{ at } r = r_0 \qquad ...(56)$$

If the potential energy function for the central force is of the form $V(r) = ar^{n+1}$, a being a constant, and centrifugal energy $V_{cf}(r) = br^{-2}$, ($b > 0$, a constant), then

$$V_e(r) = ar^{n+1} + br^{-2} \qquad ...(57)$$

The condition for stability in the radial motion is

$$\frac{dV_e}{dr} = 0 = (n+1)\,ar^n - 2br^{-3} \text{ at } r = r_0.$$

Thus $\qquad (n+1)\,a = 2br_0^{-(n+3)}$

Therefore $\qquad \left[\frac{d^2V_e}{dr^2}\right]_{r=r_0} = 2br_0^{-4}\,(n+3).$

Thus any **circular orbit** with $r = r_0$ under a central force is stable, if $(n + 3) > 0$ or $n > -3$ and the form of the central force is

$$f(r) = -\frac{\partial V}{\partial r} = -(n+1)\,ar^n \text{ or } f(r) = -Kr^n \qquad ...(58)$$

Further an orbit is said to be ***closed*** if the particle eventually retraces its path. The **stable** and **closed orbits** (circular and non-circular) are possible for $n = 1$ and $n = -2$ and the corresponding force laws are as follows :

For $n = 1$, $\quad f(r) = -Kr \quad$ *Hooke's law* ...(59)

For $n = -2$, $\quad f(r) = -K/r^2 \quad$ *Inverse square law* ...(60)

Thus the condition for bounded motion is that there is bounded region of r in which the total energy $E \geq V_e(r)$, effective potential energy. The condition for stability of circular orbit is $n > -3$ with the force law $f(r) = -Kr^n$. Further the orbits are closed only for the inverse square law of force ($n = -2$) of the Coulombian or Newtonian type and for the linear law of force ($n = 1$) of Hooke's type.

From eq. (23), we have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{J^2}{2mr^2}\right)}.$$

Now $$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{J}{mr^2} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{mr^2}{J}\dot{r}.$$

Thus $$\frac{dr}{d\theta} = \sqrt{\frac{2mr^4}{J^2}[E - V(r)] - r^2} \quad \text{or} \quad d\theta = \frac{dr}{\sqrt{\frac{2mr^4}{J^2}[E - V(r)] - r^2}}$$

Its solution is

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2m}{J^2}\left[E - V(r) - \frac{1}{r^2}\right]}} \quad \text{or} \quad \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{J^2}\left[E - V\left(\frac{1}{u}\right)\right] - u^2}} \qquad ...(61)$$

where $u = 1/r$.

If E, J and the form of the central force potential $V(r)$ are given, the orbit is fixed. u_0 and θ_0 refer to the starting point on the orbit. For $V(r) = ar^{n+1}$, the above integral can be directly integrated for $n = 1, -2, -3$. For $n = 5, 3, 0, -4, -5$ and -7, the results can be obtained in terms of elliptical integrals. The equation of the orbit is not obtainable in the closed form for other values of n.

Ex. 1. *Use Hamilton's equation to find the differential equation for planetary motion and prove that the areal velocity is constant.* **(Agra 1992)**

Solution : Lagrangian $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r}$

Hamiltonian $$H = p_r\dot{r} + p_\theta\dot{\theta} - L \quad [\because H = \sum_k p_k\dot{q}_k - L]$$

Here, $$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ and } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Therefore, $$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{K}{r} \quad \text{or,} \quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{K}{r}$$

Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r}, -\dot{p}_r = \frac{\partial H}{\partial r}, \dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ and } -\dot{p}_\theta = \frac{\partial H}{\partial \theta}$$

Therefore,
$$\dot{r} = \frac{p_r}{m}, \; -\dot{p}_r = -\frac{p_\theta^2}{mr^3} + \frac{K}{r^2}, \dot{\theta} = \frac{p_\theta}{mr^2} \text{ and } -\dot{p}_\theta = 0.$$

From last two equations, p_θ = constant $= mr^2\dot{\theta}$ or $\frac{1}{2} r^2\dot{\theta}$ = constant. This proves that the areal velocity is constant in planetary motion. From the first two equations

$$-m\ddot{r} = -\frac{p_\theta^2}{mr^3} + \frac{K}{r^2} \text{ or } -m\ddot{r} = -\frac{(mr^2\dot{\theta})^2}{mr^3} + \frac{K}{r^2}, \text{ whence } \ddot{r} = r\dot{\theta}^2 - \frac{K}{mr^2} \qquad ...(i)$$

Suppose
$$r^2\dot{\theta} = h \text{ and } u = \frac{1}{r}.$$

Then
$$\dot{\theta} = \frac{h}{r^2} = hu^2, \ddot{r} = -h^2u^2 \frac{d^2u}{d\theta^2} \text{ (as done earlier)}$$

Hence the equation of planetary motion (i) is

$$-h^2u^2 \frac{d^2u}{d\theta^2} = \frac{1}{u} h^2u^4 - \frac{K}{m}u^2 \text{ or } \frac{d^2u}{d\theta^2} + u = \frac{K}{mh^2}$$

which is the equation of planetary motion with $u = \frac{1}{r}$ and $V = -\frac{K}{r}$.

Ex. 2. *A particle of mass m moves under the action of central force whose potential is $V(r) = Kmr^3$ $(K > 0)$, then*

(*i*) *For what kinetic energy and angular momentum will the orbit be a circle of radius R about the origin ?*

(*ii*) *Calculate the period of circular motion.* **(Agra 1992)**

Solution : $V(r) = Kmr^3$

Therefore,
$$F = -\frac{\partial V}{\partial r} = -3Kmr^2$$

(*i*) For circular motion $F = -\dfrac{mv^2}{r} = -3Kmr^2$

Kinetic energy $\frac{1}{2}mv^2 = \frac{3}{2}Kmr^3$

Angular momentum $= mvr = mr\,(3Kr^3)^{1/2}$ [because $v^2 = 3Kr^3$]

(*ii*) Since $v = r\omega = \dfrac{2\pi r}{T}$, therefore from eq. $v^2 = 3Kr^3$, we have

$$\left[\frac{2\pi r}{T}\right]^2 = 3Kr^3 \text{ or } \frac{4\pi^2}{T^2} = 3Kr, \text{ whence } T = \frac{2\pi}{\sqrt{3Kr}}.$$

Ex. 3. *The eccentricity of the earth's orbit is 0.0167. Calculate the ratio of maximum and minimum speeds of the earth in its orbit.* **(Agra 1991)**

Solution : From eqs. (38) and (39), the ratio of the minimum value of r, *i.e.*, r_1 (perihelion) and the maximum value of r, *i.e.*, r_2 (aphelion) is given by

$$\frac{r_2}{r_1} = \frac{1+e}{1-e} \qquad ...(i)$$

However, the constancy of angular momentum [mvr] at the two apsidal (turning) points gives

$$mv_1r_1 = mv_2r_2 \text{ or } \frac{v_1}{v_2} = \frac{r_2}{r_1} \qquad ...(ii)$$

Obviously when the radius vector has the minimum value r_1, the speed of the planet (earth) v_1 will be maximum and when it is having maximum value r_2, the speed of the earth v_2 will be minimum. Thus the desired ratio is

$$\frac{v_1}{v_2} = \frac{r_2}{r_1} = \frac{1+e}{1-e} = \frac{1+0.0167}{1-0.0167} = \mathbf{1.03.}$$

Ex. 4. *The maximum and minimum velocities of a satellite are v_{max} and v_{min} respectively. Prove that the eccentricity of the orbit of the satellite is*

$$e = \frac{v_{max} - v_{min}}{v_{max} + v_{min}}.$$

(Agra 1998, 95, 93)

Solution : As shown in Ex. 3,

$$\frac{v_{max}}{v_{min}} = \frac{1+e}{1-e}, \quad \therefore \quad e = \frac{v_{max} - v_{min}}{v_{max} + v_{min}}.$$

Ex. 5. *A particle of mass m is observed to move in a spiral oribit given by the equation $r = C\theta$, where C is a constant. Is it moving in a central force field ? If it is so, find the force law.*

Solution : The differential equation of the orbit [eq.(26)] is given by

$$\frac{d^2u}{d\theta^2} + u = -\frac{mf\left(\frac{1}{u}\right)}{J^2u^2} \qquad ...(i)$$

Since $u = 1/r$ and $r = C\theta$ (given),

hence $$u = \frac{1}{C\theta} \qquad ...(ii)$$

Differentiating it, we get

$$\frac{du}{d\theta} = -\frac{1}{C\theta^2} \text{ and } \frac{d^2u}{d\theta^2} = \frac{2}{C\theta^2} = 2C^2u^3 \qquad ...(iii)$$

Substituting in eq. (i), we get

$$2C^2u^3 + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

Therefore, $$f\left(\frac{1}{u}\right) = -\frac{J^2}{m}(u^3 + 2C^2u^5) \text{ or } f(r) = -\frac{J^2}{m}\left[\frac{1}{r^3} + \frac{2C^2}{r^5}\right] \qquad ...(iv)$$

Thus the particle is moving under a central force field and the force law is given by eq. (iv).

Ex. 6. *A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of the distance.*

(Rohilkhand 1983; Meerut 83)

Solution : Let a be the radius of circle, described by the particle P [Fig. 4.7]. If (r, θ) are the polar coordinates of P, then

$$r = 2a\cos\theta \qquad \text{...}(i)$$

Differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

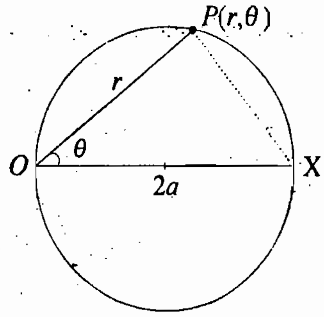

Fig. 4.7

Here, $u = \frac{1}{r} = \frac{1}{2a\cos\theta}$ or $u = \frac{\sec\theta}{2a}$.

and hence $\frac{du}{d\theta} = \frac{1}{2a}\sec\theta\tan\theta$ and $\frac{d^2u}{d\theta^2} = \frac{1}{2a}[\sec^3\theta + \sec\theta\tan^2\theta]$

Now,
$$f\left(\frac{1}{u}\right) = -\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -\frac{J^2u^2}{2am}[\sec\theta + \sec^3\theta + \sec\theta\tan^2\theta]$$

$$= -\frac{J^2u^2}{2am}[\sec\theta + \sec^3\theta + \sec\theta(\sec^2\theta - 1)]$$

$$= -\frac{J^2u^2}{m}\cdot\frac{2\sec^3\theta}{2a} = -\frac{J^2u^2}{m}\cdot 8a^2u^3 = -\frac{8J^2a^2}{m}\cdot\frac{1}{r^5}$$

Thus
$$f(r) = -\frac{8J^2a^2}{m}\cdot\frac{1}{r^5} \text{ or } f(r) \propto \frac{1}{r^5}$$

Thus the force varies as the inverse fifth power of the distance.

Ex. 7. *A particle, moving in a central force field located at $r = 0$, describes a spiral $r = e^{-\theta}$. Prove that the magnitude of force is inversely proportional* to r^3.

Solution : Here, $r = e^{-\theta}$ and therefore, $u = \frac{1}{r} = e^{\theta}$

Differential equation of orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

Substituting $u = e^{\theta}$, we get

$$e^{\theta} + e^{\theta} = -\frac{m}{J^2}e^{-2\theta} f\left(\frac{1}{u}\right) \text{ or } f\left(\frac{1}{u}\right) = -2\frac{J^2}{m}e^{3\theta}$$

or
$$f(r) = -\frac{2J^2}{m}\frac{1}{r^3} \text{ i.e., } f(r) \propto \frac{1}{r^3}.$$

Ex. 8. *A particle describes a conic* $r = \frac{p}{1+e\cos\theta}$, *where p and e involve constant quantities. Show that the force under which the particle is moving is a central force. Deduce the force law.* **(Agra 1992)**

Solution : Since the particle is moving on a curved path in a plane about a centre of force, its angular momentum remains constant *i.e.,*

$$J = mvr = mr^2\dot{\theta} = \text{constant}$$

Therefore,
$$r^2\dot{\theta} = \frac{J}{m} = \text{a constant} \quad ...(i)$$

The particle is moving on a curved path, hence it is obvious from Newton's first law of motion that a force is acting on it. Consequently an acceleration is acting on the particle. Resolving this acceleration into its two components, along and perpendicular to the radius vector, we have

(1) Component f, along the radius vector, *i.e.*, the radial acceleration of the planet is given by

$$f = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad ...(ii)$$

(2) Component f', perpendicular to the radius vector, *i.e.*, the transverse acceleration of the planet is given by

$$f' = \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$$

But from eq. (*i*), $r^2\frac{d\theta}{dt}$ = a constant.

Hence $f' = 0$

Thus the planet has no transverse acceleration and only the radial acceleration is acting on it *i.e.*, the force on the planet is directed towards the centre *i.e.*, it is a central force.

From eq. (*i*) we have

$$\frac{d\theta}{dt} = \frac{J}{mr^2} = \frac{J}{m}u^2 \quad ...(iii)$$

where $u = 1/r$.

Now, $\frac{dr}{dt} = \frac{d}{dt}\frac{1}{u} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$ or $\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{Ju^2}{m}$ or $\frac{dr}{dt} = -\frac{J}{m}\frac{du}{d\theta}$

$$\therefore \quad \frac{d^2r}{dt^2} = -\frac{J}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} \quad ...(iv)$$

Substituting the values of $\frac{d\theta}{dt}$ and $\frac{d^2r}{d\theta^2}$ from eqs. (*iii*) and (*iv*) in eq.(*ii*), we get

$$f = -\frac{J^2u^2}{m}\left[u + \frac{d^2u}{d\theta^2}\right] \quad ...(v)$$

The equation of the conic is

$$r = \frac{p}{1+e\cos\theta} \text{ or } \frac{p}{r} = 1+e\cos\theta \text{ or } pu = 1+e\cos\theta \quad ...(vi)$$

Differentiating eq. (*vi*) twice with respect to θ, we have

$$p\frac{d^2u}{d\theta^2} = -\,e\cos\theta \qquad ...(vii)$$

Adding eq.(*vi*) and (*vii*), we get

$$p\left[u + \frac{d^2u}{d\theta^2}\right] = 1, \text{ hence } u + \frac{d^2u}{d\theta^2} = \frac{1}{p}$$

Substituting this value of $u + \frac{d^2u}{d\theta^2}$ in eq. (*v*), we get

$$f = -\frac{J^2u^2}{mp} = -\frac{J^2}{mp}\cdot\frac{1}{r^2}$$

Let $\frac{J^2}{mp} = k$ (a constant).

Then, $f = -\frac{k}{r^2}$ *i.e.*, $f \propto -\frac{1}{r^2}$ ...(*viii*)

Thus, the acceleration and hence the force acting on the planet is inversely proportional to the square of its distance from the sun. Negative sign indicates that the force is one of attraction.

4.8. ARTIFICIAL SATELLITES

We have studied the motion of a planet and its orbit around the sun. In fact, a body which revolves constantly round a comparatively much larger body is said to be satellite. We know that the earth and other planets revolve round the sun in their specified orbits. The moon revolves round the earth and the planets Jupiter and Saturn have six and nine moons respectively revolving around them. All these are the examples of *natural satellites*. Each one of these satellites is attracted by its primary with a force, given by Newton's law of gravitation.

Scientists have also been able to placed man-made satellites, revolving round the earth or sun. They are called **artificial satellites**. The theory discussed above for the orbits and planetary motion is valid for the discussion of satellites.

An artificial satellite of the earth is a body, place in a stable orbit around the earth with the help of multistage rocket. In order to launch a satellite in a stable orbit, first it is necessary to take the satellite to the altitude h, where at the point P by some mechanism, it is given the necessary orbitting velocity, called the **insertion velocity** v_i (Fig. 4.8).

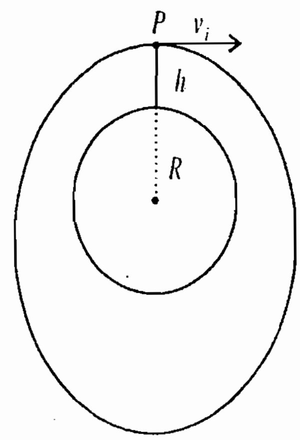

Fig. 4.8 : Elliptical path of a body projected horizontally from a height h above the earth's surface for $v_i^2 < 2GM/(R+h)$

The total energy of the satellite at P relative to the earth is given by

$$E = \frac{1}{2}mv_i^2 - \frac{GMm}{R+h} \qquad ...(62)$$

where m is the mass of the satellite and M that of the earth, having radius R.

The orbit will be an ellipse, a parabola or hyperbola, depending on whether E is negative, zero or positive. In each case, the centre of the earth is at one focus of the path. Therefore, the satellite will be moving in an elliptical orbit (Fig. 4.8), if

$$v_i^2 < \frac{2GM}{R+h} \qquad (63)$$

The total energy E determines the size or semi-major axis of the orbit. However the shape or eccentricity e of the orbit is determined by both total energy E and angular momentum J by the relation :

$$e = \sqrt{1+\frac{2EJ^2}{mK^2}} \qquad ...(64)$$

with $K = GMm$. For elliptical orbits, layer the angular momentum, the less elongated is the orbit (Fig. 4.9).

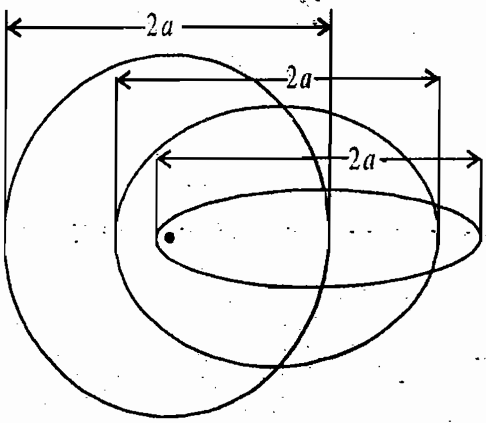

Fig. 4.9 : Elliptical orbits for different values of the angular momentum J with same enegry E; various orbits have the same focus and semi-major axis, but differing in eccentricity.

For circular orbit, the insertion velocity is found by equating the centripetal force mv^2/r to the gravitational force GMm/r^2 *i.e.*,

$$\frac{mv_i^2}{r} = \frac{GMm}{r^2} \text{ or } v_i^2 = \frac{GM}{r} = \frac{GM}{R+h} \qquad ...(65)$$

where $r = R + h$

Remember that for circular orbit $e = 0$, so that

$$1+\frac{2EJ^2}{mK^2} = 0,\ 1 + 2 \times \left(-\frac{K}{2a}\right)\times\frac{m^2 v_i^2 a^2}{mK^2} \text{ or } v_i^2 = \frac{GM}{R+h}$$

where $r = a = R + h$, $K = GMm$ and $J = mv\,(R + h)$.

For the circular obit at the height h above the earth's surface, the period of revolution is

$$T = \frac{2\pi r}{v_i} = \frac{2\pi(R+h)}{v_i} = \frac{2\pi(R+h)^{3/2}}{\sqrt{GM}} \qquad ...(66)$$

In Table 1, for some altitudes, we are presenting the values of insertion velocity v_i with corresponding period of revolution T.

TABLE 1

h (km)	v_i (km/s)	T
10	7·894	1 hr 25 min
320	7·714	1 hr 31 min
1600	7·068	1 hr 58 min
35880	3·070	24 hr

We see from the table that as the altitude increases, the insertion velocity decreases and the period of revolution increases.

Geosynchronous orbit : For a satellite, moving around the earth, if the period of revolution is equal to the period of earth's diuranal (one day) rotation, the orbit is said to be geosynchronous orbit. For such an orbit, the period must be 24 hours or more correctly, T = 23 hours 56 min. 4·099 sec.

Using Kepler's third law, the semi-major axis of the geosynchronous orbit is

$$a_s = \left(\frac{GM}{4\pi^2}\right)^{1/3} \times T^{2/3} = 42,164\cdot2 \text{ km} \qquad ...(67)$$

where M is the mass of the earth.

For a geosynchronous orbit, the eccentricity can have any value and the orbit can have any orientation with respect to the equator of the earth.

Geostationary Orbit : If the height of an artificial satellite at equator above the earth's surface is such that its period of revolution is exactly equal to the period of rotation of the earth, then the satellite would appear stationary over a point on earth's equator. Such a satellite is called **geostationary satellite** and its orbit is called **geostationary orbit**. Therefore for a geostationary satellite, we must have the orbit (i) *to be geosynchronous* (ii) *to be circular* and (iii) *to stay over the geographical equator of the earth.*

The height of geostationary satellite is

$$h = a_s - R = 35,786 \text{ km}. \qquad ...(68)$$

The geostatinary orbit is often called **parking orbit**. Artificial satellites used for telecasting are put in parking orbits.

Satellites in Space Exploration : The first man made earth satellite was launched by Russian scientists on October 4, 1957 and has come to be known as *Sputnik-1* (artificial satellite). This sputnik-1, being of the shape of a ball of diameter 58 cm. and weight 83·6 kgm., was placed in an orbit round the earth and made one full revolution in 96·2 minutes, attaining a speed of 8 km/sec. at a distance of 950 km. from the earth. A three stage rocket was used for this purpose. The rocket was launched vertically and a special device (incorporated in the system) enabled it to gradually curve away from its vertical path. As the fuel of first, second and third stage was exhausted, they dropped away one by one, finally leaving the satellite with a speed of 8 km./sec. to revolve round the earth. Part of the kinetic energy of the satellite was then decreased due to air friction and hence the radius of its path became smaller and smaller. In the denser layers of the atomosphere, it became too much hot and burnt away. Other satellites have been launched by Russian and American scientists. On January 2, 1959, a space-rocket was launched by Russians, which became an artificial planet of the sun, having a period of rotation 450 days.

A manned satellite, carrying Major Yuri Gagarin was placed into orbit for the first time by Russians on April 12, 1961. Americans succeded in putting a manned satellite, carrying Col. J. H. Glenn. in orbit in Feb. 1962. The far side of the moon, which was never seen from the earth, was first photographed by a television camera carried by a space rocket Lunik III launched by Russia in Oct. 1959. It revolved round the moon as well as the earth. In November 1969, American scientists launched Appolo-12 with three cosmonauts. Two of them actually landed their plane on the surface of the moon. After collecting some important informations about the surface of the moon, they came back to the earth after a journey of nearly 8 days. Since then a number of satellites have been launched for space exploration.

Uses of Artificial Satellites : Artificial satellites are used in the following :

(1) Distant transmission of radio and TV signals.

(2) To study upper regions of the atmosphere.

(3) High altitude satellites for astronomical observations (as the effects of atmosphere are not present).

(4) Weather forecasting.

(5) Eearth measurements (gravitation and magnetic fields).

Ex. 1. *An artificial satellite is revolving round the earth at a distance of 620 km. Calculate the orbital velocity and the period of revolution. Radius of earth is 6380 km and acceleration due to gravity at the surface of the earth is 9·8 m/sec^2.*

Sol. Radius of earth's satellite orbit $r = R + h$

= Radius of the earth + Distance of satellite from earth's surface.

= 6380 + 620 = 7000 km = 7×10^6 m/sec.

∴ Period of revolution $T = \frac{2\pi r}{R}\sqrt{\frac{r}{g}} = \frac{2\pi \times 7\times 10^6}{6380\times 10^3}\sqrt{\frac{7\times 10^6}{9{\cdot}8}}$ = **5775 sec.**

and orbital velocity $v = R\sqrt{\frac{g}{r}} = 6380 \times 10^3 \sqrt{\frac{9{\cdot}8}{7\times 10^6}}$ = **$7{\cdot}55 \times 10^3$ m/sec.**

Ex. 2. *Calculate the height of an equatorial satellite which is always seen over the same point of earth's surface. ($G = 6{\cdot}66 \times 10^{-11}$ S.I. units; $M = 5{\cdot}98 \times 10^{24}$ kg).*

Sol. Let the height of the equatorial satellite be h. The equatorial satellite is seen over the same point of earth's surface, *i.e.*, the angular velocity of satellite is the same as that the of the earth itself.

Hence, angular velocity of the satellite $\omega = \frac{2\pi}{24\times 60\times 60} = 7{\cdot}27 \times 10^{-5}$ sec^{-1}.

Also, $GMm/r^2 = m\omega^2 r$

or $r^3 = \frac{GM}{\omega^2} = \frac{6{\cdot}66\ 10^{-11}\times 5{\cdot}98\times 10^{24}}{(7{\cdot}27\times 10^{-5})^2} = 74{\cdot}74 \times 10^{21}$

whence, $r = 4{\cdot}21 \times 10^7$ m

Therefore, $h = r - R = 4{\cdot}21 \times 10^7 - 6{\cdot}4 \times 10^6 = 3{\cdot}57 \times 10^7$ m = **$3{\cdot}57 \times 10^4$ km.**

4.9. VIRIAL THEOREM

This theorem is very useful in a variety of problems in physics. Here, first we shall deduce the virial theorem and then use it to discuss the property of central force as a special case.

Let us consider a system of parcitcles with position vector $\mathbf{r}_i$ and applied force $\mathbf{F}_i$. According to Newton's second law, the equations of motion are

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \qquad ...(69)$$

We introduce a quantity λ ; defined by

$$\lambda = \sum_i \mathbf{p}_i \bullet \mathbf{r}_i \qquad ...(70)$$

where the sum is taken for all particles of the system.

The total time derivative of λ is

$$\frac{d\lambda}{dt} = \sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i + \sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i \qquad ...(71)$$

First term of eq. (71) can be written as

$$\sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T \qquad ...(72)$$

and second term [using eq. (72)] as

$$\sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i = \sum_i \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(73)$$

Hence,

$$\frac{d\lambda}{dt} = \frac{d}{dt}\left[\sum_i \mathbf{p}_i \bullet \mathbf{r}_i\right] = 2T + \sum_i \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(74)$$

The time average of eq. (74) over a time interval τ is obtained as

$$\frac{1}{\tau}\int_0^\tau \frac{d\lambda}{dt}\,dt = \overline{\frac{d\lambda}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i . \mathbf{r}_i} \text{ or } \frac{1}{\tau}[\lambda(\tau)-\lambda(0)] = \overline{2T} + \overline{\sum_i \mathbf{F}_i . \mathbf{r}_i} \qquad ...(75)$$

In case of periodic motion, τ is chosen as period and all coordinates repeat after this interval of time. In such a case, $\lambda(\tau) = \lambda(0)$ and then left hand side of eq. (75) vanishes, *i.e.*,

$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} = 0 \text{ or } \overline{T} = -\tfrac{1}{2}\overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} \qquad ...(76)$$

Eq. (76) is called the ***Virial theorem*** and the quantity $-\frac{1}{2}\overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i}$ is known as the ***Virial of Clausius.***

The theorem (76) stands also for non-periodic motions in the condition that the coordinates and velocities of all particles remain finite so there is an upper bound to the quantity λ. By taking τ large enough, left hand side of eq. (75) can be made as small as desired and consequently it may be reduced to zero. Thus in this case also eq. (76) is obtained.

In case of a particle, moving in a central force field,

$$\overline{T} = -\frac{1}{2}\overline{\mathbf{F}_i \bullet \mathbf{r}_i} = -\frac{1}{2}\overline{\left[-\frac{\partial V}{\partial r}\right]\hat{\mathbf{r}} \bullet \mathbf{r}} \text{ or } \overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r} r} \qquad ...(77)$$

where V is the potential energy.

If V is a function of the form

$$V = c\, r^{n+1}, \qquad ...(78)$$

then

$$\overline{T} = \frac{1}{2}\overline{c\,(n+1)r^n r} = \frac{n+1}{2}\overline{cr^{n+1}}$$

or

$$\overline{T} = \frac{n+1}{2}\overline{V} \qquad ...(79)$$

Further in case of ***inverse square law of force*** $n = -2$ and hence

$$\overline{T} = -\frac{\overline{V}}{2} \quad \text{or} \quad 2\overline{T} + \overline{V} = 0 \qquad ...(80)$$

This is a well known form, obtained by Virial theorem for c/r^2 type force.

4.10. SCATTERING IN A CENTRAL FORCE FIELD

The problem of scattering of particles at atomic scale in a central force field is extremely important in modern physics. For example, the scattering of a beam of protons or α-particles by the atomic nuclei of a target is an interesting problem in nuclear physics. Of course, the problem is quantum mechanical, but it can be dealt classically because the procedures for dealing the scattering problem in classical as well as in quantum mechanics are similar. Further a classical study of the scattering problem gives a chance to the beginner to learn the language of the problem.

Let us consider a uniform beam of particles incident on a centre of force. All the particles of the beam have the same mass and energy. The *intensity* of the beam I_0 is the number of particles crossing unit area per unit time normal to the direction of the beam. This I_0 is also called *flux density*. We assume that the force between an incident particle and the particle at the centre of force falls off to zero at large distances.

When a particle approaches the centre of force, it will interact [for example, an α-particle (+ve charge) will experience replusion from the positively charged nucleus] so that its path will deviate from the

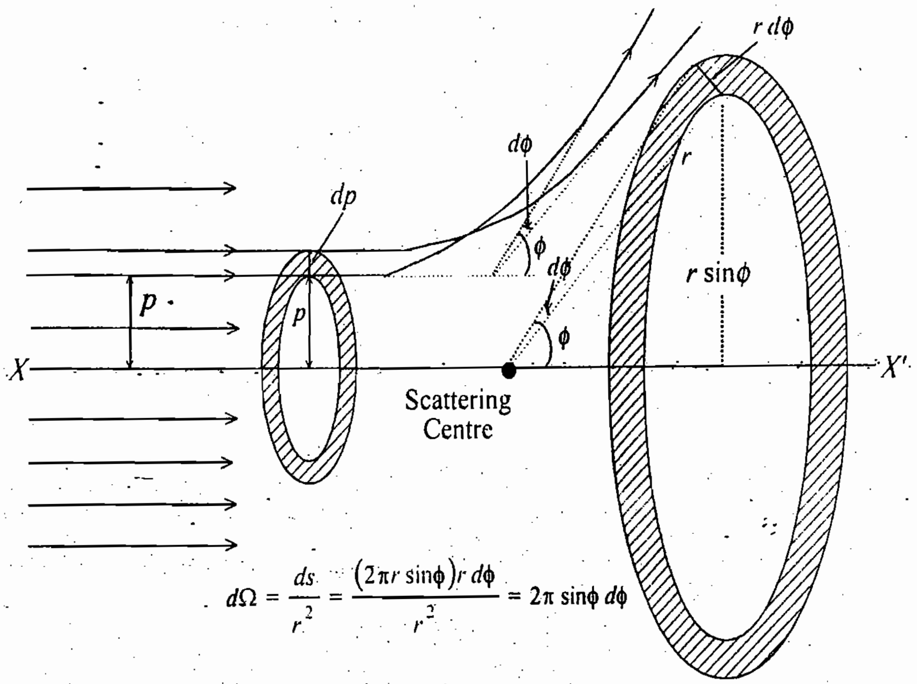

Fig. 4.10 : Scattering–impact parameter (p) and angle of scattering (ϕ)

incident straight line trajectory. After passing the centre of force, as the particle goes away, the force acting on it will decrease and finally at large distances, the force will become zero. This results again in the straight line motion but in general in a different direction and we say that the particle has been ***scattered.***

Scattering cross-section : Consider a uniform beam of particles, moving with a flux of I_0 particles per unit area towards a scattering centre (*e.g.*, an atom of a target). Imagine that the scattering centre presents an area $d\sigma$ perpendicular to the path of the beam such that whatever particles hit $d\sigma$ area are scattered into a solid angle $d\Omega$. Thus the number of particles, scattered into $d\Omega$ solid angle per second are $I_0\,d\sigma$. If $I(\Omega)$ (intensity of the scattered particles) is defined as the number of particles scattered in the direction Ω per unit solid angle per unit time, then the number of scattered particles in the small solid angle $d\Omega$ about Ω direction is given by

$$I_0 d\sigma = I(\Omega)\, d\Omega$$

or
$$\frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0} \qquad ...(81)$$

The quantity $\frac{d\sigma}{d\Omega} = \sigma(\Omega)$ is called ***differential scattering cross-section*** or simply scattering cross-section for scattering in Ω-direction *i.e.*,

$$\sigma(\Omega) = \frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0} \qquad ...(82)$$

Thus the differential scattering cross-section is the ratio of the number of the scattered particles per second per unit solid angle and the flux density of the incident particles.

The total cross-section (σ) is given by

$$\sigma = \int \sigma(\Omega)\, d\Omega = \int \frac{d\sigma}{d\Omega}\, d\Omega \qquad ...(83)$$

Scattering angle (ϕ) : The angle between the incident and scattered directions of the particle is called scattering angle and is denoted by ϕ.

Impact Parameter (p) : If we draw a perpendicular on the direction of the incident particle from the scattering centre, then the length of this perpendicular is known as impact parameter p.

As the force is central, there must be complete symmetry about the axis (XX') of the incident beam. The solid angle $d\Omega$ is given by*

$$d\Omega = 2\pi \sin\phi \, d\phi$$

where ϕ is the scattering angle. The incident particles crossing through area $d\sigma = 2\pi p \, dp$, lying between p and $p + dp$, are given by

$$I_0 \, d\sigma = I_0 2\pi p \, dp \qquad \text{...(84)}$$

These particles are scattered into the solid angle $d\Omega = 2\pi \sin\phi . d\phi$. But $I(\Omega) = I_0 \sigma(\Omega)$ is the number of particles scattered in the direction Ω per unit solid angle, hence the number of particles scattered into the solid angle $d\Omega$ are

$$I(\Omega) \, d\Omega = I_0 \, \sigma(\Omega) \, d\Omega$$

or

$$I(\Omega) \, d\Omega = I_0 \, \sigma(\phi) \, 2\pi \sin\phi \, d\phi \qquad \text{...(85)}$$

where $\sigma(\phi)$ represents the differential cross-section for the direction ϕ. Therefore by using eqs. (81) and (84), we get

$$I_0 \, 2\pi \, p dp = - I_0 \, \sigma(\phi) \, 2\pi \sin\phi \, d\phi$$

Negative sign is introduced, because an increase of p will decrease ϕ.

Thus
$$\sigma(\phi) = -\frac{p}{\sin\phi}\left[\frac{dp}{d\phi}\right] \qquad \text{...(86)}$$

This gives the dependence of differential cross-section on the scattering angle ϕ.

4.11. RUTHERFORD SCATTERING CROSS-SECTION

In the Rutherford scattering a positively charged particle of charge ze and mass m is scattered by a heavy nucleus N. The nucleus is assumed to be at rest during the collision. The charge on the nucleus is Ze, where Z is the atomic number. Suppose the positively charged particle is moving towards the heavy nucleus with initial velocity v_0. As the particle approaches the nucleus, the repulsive force $(Zze^2/4\pi\varepsilon_0 r^2)$ increases rapidily and the particle changes from a straight line path to a hyperbola ADB, having one focus at N as shown in Fig. 4.11. The asymptotes AO and BO to the hyperbola give the direction of the incident and scattered particle. The angle COB is the scattering angle ϕ and NM is the impact parameter p.

The charged particle is moving in a central force field, hence the equation of its path is given by eq. (34) *i.e.*,

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{...(87)}$$

---

* $$d\Omega = \frac{ds}{r^2} = \frac{r\sin\phi \, d\theta \, r \, d\phi}{r^2} = \sin\phi \, d\theta \, d\phi$$

For symmetry about XX' axis, $d\theta$ is to be integrated from 0 to 2π. In such a case,

$$d\Omega = 2\pi \sin\phi \, d\phi.$$

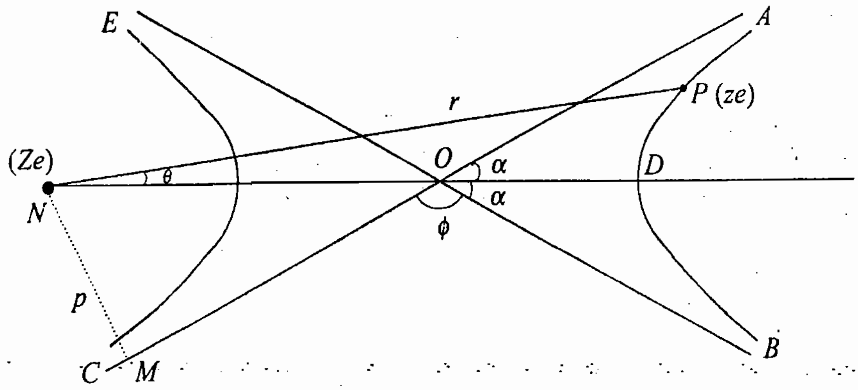

Fig. 4.11 : Scattering in a repulsive force field

where $l = J^2/mK$, $e = \sqrt{1 + \dfrac{2EJ^2}{mK^2}}$ and θ', the constant of integration, has been taken to be zero. Here the force F is given by

$$F = \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{r^2} = -\frac{K}{r^2}$$

Therefore, $$K = -\frac{Zze^2}{4\pi\varepsilon_0} \qquad ...(88)$$

Hence $$l = -\frac{J^2 4\pi\varepsilon_0}{Zze^2 m} \text{ and } e = \sqrt{1 + \frac{2EJ^2(4\pi\varepsilon_0)^2}{Z^2 z^2 e^4 m}} \qquad ...(89)$$

As the initial velocity of the particle is v_0, its total energy is given by

$$E = \frac{1}{2}mv_0^2, \text{ whence } mv_0 = \sqrt{2mE} \qquad ...(90)$$

According to the law of conservation of angular momentum,

$$mv_0 p = mr^2\dot{\theta} = J, \text{ whence } mv_0 = \frac{J}{p}$$

Therefore, $$\frac{J}{p} = \sqrt{2mE} \quad \text{or} \quad J = p\sqrt{2mE} \qquad ...(91)$$

This gives

$$e = \sqrt{1 + \frac{2E\left(p\sqrt{2mE}\right)^2(4\pi\varepsilon_0)^2}{mz^2Z^2e^4}} \text{ or } e = \sqrt{1 + \left(\frac{2Ep\,4\pi\varepsilon_0}{zZe^2}\right)^2} \qquad ...(92)$$

Obviously, $e > 1$, because $(2Ep/zZe^2)^2$ is a positive quantity. Hence eq. (87) represents the path of the charged particle as ***hyperbola***.

Since the hyperbolic path must be symmetric about the direction of the periapsis, the scattering angle ϕ is given by

$$\phi = \pi - 2\alpha \text{ or } \alpha = \frac{\pi}{2} - \frac{\phi}{2}. \qquad ...(93)$$

where α is the angle between the direction of the incoming asymptote and the periapsis direction (OD).

Further the asymptotic direction is that for which r is infinite (∞) and then $\theta \to \alpha$. Hence from eq. (87), we have

$$1+e\cos\alpha = 0 \text{ or } \cos\alpha = -\frac{1}{e} \text{ or } \cos\left(\frac{\pi}{2}-\frac{\phi}{2}\right) = -\frac{1}{e} \text{ or } \sin\frac{\phi}{2} = -\frac{1}{e}$$

Thus
$$\operatorname{cosec}\frac{\phi}{2} = -e$$

Squaring it, we get

$$\operatorname{cosec}^2\frac{\phi}{2} = e^2 \text{ or } 1+\cot^2\frac{\phi}{2} = 1+\left[\frac{2Ep\,4\pi\varepsilon_0}{zZe^2}\right]^2 \quad \text{[from (85)]}$$

whence
$$\cot\frac{\phi}{2} = \frac{2Ep\,(4\pi\varepsilon_0)}{zZe^2} \quad ...(94)$$

From which one can find the ***scattering angle*** ϕ.

Now from eq. (94), we get an expression for ***impact parameter*** :

$$p = \frac{zZe^2\cot\phi/2}{2E(4\pi\varepsilon_0)} \quad ...(95)$$

Differentiating it, we get

$$\frac{dp}{d\phi} = -\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\operatorname{cosec}^2\frac{\phi}{2} \quad ...(96)$$

Substituting the value of p and $dp/d\phi$ from eqs. (95) and (96) in eq. (86), we get

$$\sigma(\phi) = -\frac{zZe^2\cot\frac{\phi}{2}}{2E\,(4\pi\varepsilon_0)\,2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\left[-\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\right]\operatorname{cosec}^2\frac{\phi}{2}$$

or
$$\sigma(\phi) = \frac{1}{4}\left[\frac{zZe^2}{(4\pi\varepsilon_0)\,2E}\right]^2\operatorname{cosec}^4\frac{\phi}{2} \quad ...(97)$$

This is the well known expression for the ***Rutherford scattering cross-section***. Thus the scattering cross-section or the number of particles scattered per second along the direction ϕ are proportional to

(1) $\operatorname{cosec}^4\frac{\phi}{2}$,

(2) the square of the charge on the nucleus (Ze),

(3) the square of the charge on the particle (ze), and

(4) inversely proportional to the square of the initial kinetic energy E.

Thus if N_ϕ is the number of particles scattered along the angle ϕ per second, then one can represent

$$N_\phi = C\operatorname{cosec}^4\phi/2 \quad ...(98)$$

where C is a constant.

Ex. 1. *In Rutherford's experiment 10^5 α-particles are scattered at an angle of 2^0, calculate the number of α-particles scattered at an angle of 20^0* **(Agra 1991)**

Solution : The number of particles N_ϕ scattered per second at an angle ϕ is given by

$$N_\phi = C \operatorname{cosec}^4 \phi /2$$

Therefore, $\dfrac{N_{(20°)}}{N_{(2°)}} = \dfrac{\operatorname{cosec}^4 10°}{\operatorname{cosec}^4 1°} = \left(\dfrac{\sin 1°}{\sin 10°}\right)^4 = \dfrac{1}{10^4}$ (for smallθ, $\sin\theta \approx \theta$)

whence $N_{(20°)} = 10^5/10^4 =$ **10** (approximately).

Ex.2. *Show that for any repulsive central force, a formal solution for the angle of scattering can be expressed as :*

$$\phi = \pi + \int_0^{u_0} \frac{p\, du}{\sqrt{1 - \frac{V}{E} - p^2u^2}}$$

where V is the potential energy, $u = 1/r$ and u_0 corresponds to the turning point of the orbit. Write down the expression for ϕ for a force K/r^3.

Solution : For a central force,

$$\frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{J}{mr^2} + V = E \;, \therefore\; \dot{r} = \left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}$$

But $$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\cdot\frac{J}{mr^2}$$

Therefore, $$\frac{dr}{d\theta} = \frac{\dot{r}}{J/mr^2} = \frac{mr^2}{J}\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}$$

or $$d\theta = \frac{J dr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}} \quad \text{or} \quad \theta = \int \frac{J dr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}}$$

As $r = 1/u$, $dr = -du/u^2$.

Integrating from 0 to u_0 (turning point), we obtain

$$\theta = -\int_0^{u_0} \frac{p\, du}{\left(1 - \frac{V}{E} - p^2u^2\right)^{\frac{1}{2}}}$$

where we have used $J = p\sqrt{2mE}$.

Now, $$\phi = \pi - 2\theta = \pi + 2\int_0^{u_0} \frac{p\, du}{\left(1 - \frac{V}{E} - p^2u^2\right)^{1/2}}$$

For $F = K/r^3$; $V = K/2r^2$ and then

$$\phi = \pi + 2\int_0^{u_0} \frac{p\,du}{\left[1-\left(\frac{K}{2E}+p^2\right)u^2\right]^{1/2}}$$

Ex. 3. *Determine the differential scattering cross-section and the total scattering cross-section for the scattering of a particle by a rigid elastic sphere.*

Solution : Let the radius of the rigid elastic sphere be a. The impact parameter p for the particle under consideration is

$$p = a\sin\left(\frac{\pi}{2}-\frac{\phi}{2}\right) = a\cos\frac{\phi}{2}$$

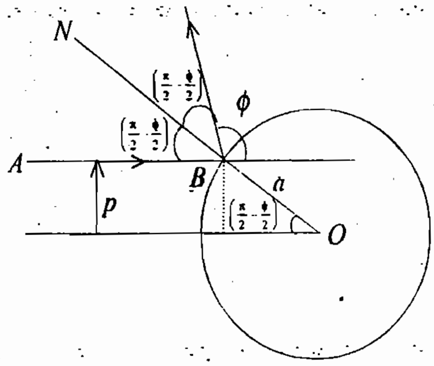

Fig. 4.12

Hence
$$\frac{dp}{d\phi} = -\frac{a}{2}\sin\frac{\phi}{2}$$

Differential scattering cross-section

$$\sigma(\phi) = -\frac{p}{\sin\phi}\frac{dp}{d\phi} = \frac{a\cos\frac{\phi}{2}\times\frac{a}{2}\sin\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = \frac{a^2}{4}$$

Total scattering cross-section

$$\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4}4\pi = \pi a^2.$$

Questions

1. How will you reduce the two-body problem into one-body problem ? Hence explain the concept of reduced mass. Give its two examples. Calculate reduced mass of the hydrogen atom and positronium. **(Agra 1998, 97, 94, 91)**
2. What do you mean by the reduced mass of a two-particle system. If the motion of the hydrogen nucleus is neglected, calculate the percentage error in the wavelength of any line in hydrogen spectrum determined by Bohr's theory.
3. Discuss a two-body problem reduced into a one-body problem. What is meant by equation of motion and first integrals ? Show that the areal velocity of a planet remains constant. **(Agra 2002)**
4. (*a*) Show that for a particle, moving under central force $f(r)$, the equation of the orbit is given by

$$\frac{d^2u}{d\theta^2}+u = -\frac{m^2}{l^2u^2}\,f\left(\frac{1}{u}\right)$$

where u = $1/r$ and l is the angular momentum. **(Meerut 2001, 1999)**

(*b*) Discuss the orbit if the force law is the attractive inverse square law. **(Meerut 1999)**

5. State and prove the Kepler's laws of planetary motion. **(Agra 2003, 01)**
6. Using the principle of classical mechanics, prove the law of conservauon angular momentum, Kepler's second law and derive the differential equation of the orbit in the central field force. **(Bundelkhand 1996)**

7. What are Kepler's laws of planetary motion ? Give the proof of Kepler's laws of planetary motion and hence deduce that the areal velocity is constant.
(Bundelkhand 1997, 95; Gorakhpur, 95; Agra 2003, 98, 95)
8. What is inverse square law force ? Derive Kepler's laws with its help. **(Meerut 1995; Agra 91)**
9. Derive the differential equation for the orbit of a particle moving under central force.
(Agra 1998, 92)
10. Derive the differential equation of an orbit in polar coordinates under central force. Investigate the motion of a particle under the attractive inverse square law. **(Agra 1990, 85)**
11. Obtain the differential equation for a particle undergoing a central force motion and use it to verify Kepler's laws of planetary motion. **(Agra 1997, 94, 93)**
12. Derive the equation for orbit of a particle moving under the influence of an inverse square central force field. Also calculate the time period of motion in elliptical orbit. **(Meerut 1994 ; Agra, 83)**
13. Show that in an elliptical orbit of a planet around the sun, the major axis solely depends on the total energy. Further prove that the periodic time in an elliptical orbit is given by

$$T = 2\pi a^{3/2} \Big/ \sqrt{G(M+m)}$$

where a is the semi-major axis, M the mass of the sun and m that of the planet.
14. (*a*) State and prove Virial theorem. **(Garwal 1999, 95)**
(*b*) When force is derivable from a potential, $V \propto r^{n+1}$, show that the average kinetic energy equals $\left(\frac{n+1}{2}\right)$ times the average polential energy. **(Garwal 1995)**
15. What is differential scattering cross-section ? Discuss the problem of scattering of charged particles by a Coulomb field and obtain Rutherford's formula for the differential scattering cross-section.
(Agra 2004, 1999, 97, 95, 94, 93, 92)
16. Discuss the scattering of α-particles under a central force field and hence obtain the expression for Rutherford scattering cross-section. **(Agra 1998)**
17. Discuss α-particle scattering in coulomb's field. **(Agra 2003, 02)**
18. What is the collision parameter ? Show that the number of particles scattered in the elementary solid angle is inversely proportional to the fourth power of the sign of the deflecting half angle.
(Garwal 1990)
19. If the particles of a system attract each other according to the inverse square law of force, prove that $2\overline{T} + \overline{V} = 0$, where $\overline{T}$ is the average kinetic energy and $\overline{V}$ the average potential energy.

Problems

[SET- I]

1. Calculate the reduced mass of CO and HCl molecules.
(atomic numbers of H, C, O and Cl atoms are 1, 12, 16 and 35.5 respectively; 1 amu $= 1.67 \times 10^{-27}$ kg).
Ans. $\mu(CO) = 1.15 \times 10^{-26}$ kg; $\mu(HCl) = 1.62 \times 10^{-27}$ kg.
2. Show that in rectangular coordinates the magnitude of a aeal velocity is

$$\tfrac{1}{2}(x\dot{y} - y\dot{x}).$$

3. Show that the differential equation describing the motion of a particle in a central field can be expressed as

$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E, \text{ where } h = r^2\dot{\theta}.$$

4. A particle moves in a central force field defined by $\mathbf{F} = -Kr^2\hat{\mathbf{r}}$. It starts from rest at a point on the circle $r = a$. Show that when it reaches the circle $r = b$, its speed will be $\sqrt{2K(a^3 - b^3)/3m}$.

5. The eccentricity of the earth's orbit is 0.0125. Calculate the ratio of the maximum and minimum speeds of the earth. **(Agra 1998, 95)**
Ans : 1.025

6. A satellite has its largest and smallest orbital speeds v_{max} and v_{min} respectively. If the satellite has a period T, then show that it moves in an elliptical orbit having major axis with length $\frac{T}{2\pi}\sqrt{v_{max}\,v_{min}}$.

7. A particle of mass m describes an elliptical orbit about a centre of attractive force at one of its focus given by $-k/r^2$, where k is a constant. Show that the speed v of the particle at any point of the orbit is given by

$$v^2 = \frac{k}{m}\left[\frac{2}{r} - \frac{1}{a}\right]$$

where a is the semimajor axis.

$\left[\textbf{Hint : } E = T(\text{K.E.}) + V = \frac{1}{2}mv^2 - \frac{k}{r}\right]$.

8. If a planet were suddenly stopped in its orbit, supposed to be circular, prove that it will fall into the sun in a time which is $\sqrt{2}/8$ times of its period.

9. A particle of mass m moves in a central force field defined by $F = -\frac{kr}{r^4}$. Show that if E is the total energy supplied to the particle, then its speed is given by $v = \sqrt{\frac{k}{mr^2} + \frac{2E}{m}}$

Show that for initial condition $E = 0$, the equation of orbit is $r = Ae^{\lambda\theta}$ where A and λ are constants.

10. A particle describes a leminiscate, given by $r^2 = a^2 \cos 2\theta$. Obtain the force law **(Gorakhpur 1991)**
Ans. $F(r) \propto 1/r^7$

11. An election of change $-e$ is moving around a nucleus of atomic number Z. Find the periodic time in case of elliptical orbit.

Ans. $T = \frac{4\pi}{e}\sqrt{\frac{\pi\varepsilon_0\mu\, a^3}{Z}}$

12. A satellite of radius a revolves in a circular orbit about a planet of radius b with period T. If the *shortest distance between the **satellites*** is c, show that the mass of the planet is $4\pi^2(a + b + c)^3/GT^2$.

13. A particle moves on a curve $r^n = a^n \cos n\theta$ under the influence of a central force. Find the law of force. **(Gorakhpur 1995)**
Ans : $F(r) \propto r^{-(2n+3)}$.

14. A negatively charged particle is moving under the Coulomb force of the nucleus. Deduce the nature of the orbit of the particle and periodic time. **(Agra 1983)**

Ans : Elliptical orbit, $T = 2\pi a^{3/2}\sqrt{\mu / K}$; $K = Ze^2/4\pi\varepsilon_0$.

[**Hint :** $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{(Ze)(-e)}{r^2} = -\dfrac{Ze^2}{4\pi\varepsilon_0 r^2}$ (Ze = charge on the nucleus) or $F = -K/r^2$

Eccentricity of the orbit $e = \sqrt{1+\dfrac{2EJ^2}{\mu K^2}} = \sqrt{1-\dfrac{J^2}{\mu Ka}}$, $\left(E = -\dfrac{K}{2a}\right)$

Therefore, $e < 1$, hence the orbit is elliptical with period $T = 2\pi a^{3/2}\sqrt{\mu / K}$.]

15. According to Yukawa's theory of nuclear forces, the attractive force between two nucleons inside a nucleus is given by the potential

$$V(r) = k\frac{e^{-\alpha r}}{r}.$$

Find the force law. Calculate the total energy E and angular momentum J, if the particle moves in a circle of radius r_0. Determine also the period of circular motion.

Ans : $f(r) = e^{-\alpha r}\left[\dfrac{k\alpha}{r}+\dfrac{k}{r^2}\right]$; $E = \dfrac{ke^{-\alpha r_0}}{2r_0}(1-\alpha r_0)$; $J = \left[-kmr_0 e^{-\alpha r_0}(1+\alpha r_0)\right]^{1/2}$;

$$T = 2\pi\left[-\frac{ke^{-\alpha r_0}}{mr_0^3}(1+\alpha r_0)\right]^{1/2}$$

16. A body of mass m is moving in a spiral orbit given by $r = r_0 e^{k\theta}$. Show that the force causing such an orbit is a central force and varies as r^{-3}.

17. A particle of mass m moves in an elliptical orbit under the action of an inverse square central force. If α be the ratio of the maximum angular velocity to the minimum angular velocity, show that the eccentricity of the ellipse e is given by $e = (\alpha - 1)/(\alpha + 1)$.

18. Prove that the product of the minimum and maximum speeds of a particle moving in an elliptical orbit is $(2\pi a / T)^2$.

19. If the orbit described under a central force be given by $r = a\,(1+\cos\theta)$ with centre at the origin, find the law of fource. **(Rohilkhand, 1984)**
Ans : $f(r) \propto 1/r^4$.

20. The first artificial satellite was circling round the earth at a distance of 900 km. Taking the radius of the earth equal to 6371 km. and mass $5{\cdot}983 \times 10^{27}$ gm. and G equal to $6{\cdot}66 \times 10^{8}$ C.G.S. units, find the velocity of satellite and its period of revolution.
Ans. Velocity = 7·40 km./sec.; Time period = 1 hour 42 minutes 52 sec.

21. An artificial satellite is going round the earth close to its surface. Calculate the time taken by it to complete one round. Take the radius of earth 6400 km. and g = 980 cm./sec^2. **(Punjab 1968)**
Ans. 5075 sec.

22. A sputnik revolves round the earth in a circular orbit of radius 13000 km. under the attraction of earth alone. Calculate the time-period of revolution of the sputnik in its orbit, assuming the radius of the earth to be 64000 km. (g = 980 cm./sec.^{2} at the surface of the earth.) **(Ranchi 1966)**
Ans. 6722 sec.

23. In Rutherford experiment, 10^3 α-particles are scattered at an angle of 4°, calculate the number of particles scattered at an angle of 14°. **(Agra 1995)**
Ans : 6.7 approx.

24. Determine the differential scattering cross-section for α-particles by Pb ($Z = 82$) nucleus, provided that the initial energy of α-particles is 11×10^{-13} Joule and scattering angle is 30° Calculate the value of impact parameter also.

$$\left(\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ S.I. unit}, e = 1.6 \times 10^{-19} \text{ coulomb}, \operatorname{cosec} 15^{\circ} = 3.86\right).$$

Ans : 1.8×10^{-26} m^2; 6.4×10^{-14} m.

25. A beam of particles of energy E encounters a spherical potential, given by $V(r) = K$ for $r < a$ and $V(r) = 0$ for $r > a$, where K is a constant. Show that the differential scattering cross-section is given by

$$\sigma(\Omega) = \left[\frac{a^2 f^2}{4\cos\frac{\phi}{2}}\right]\left[\frac{\left(f\cos\frac{\phi}{2}-1\right)\left(f-\cos\frac{\phi}{2}\right)}{\left(1+f^2-2f\cos\frac{\phi}{2}\right)^2}\right]+\frac{a^2}{4},$$

for $0 < \phi < 2\cos^{-1} f$ and is zero for $\phi > 2\cos^{-1} f$, where $f = [1-(K/E)]^{1/2}$.

26. For a particle moving under the action of a central force, the effective potential energy is given by

$$U(r) = -\frac{100}{r} + \frac{50}{r^2} \quad (MKS\ units)$$

Sketch roughly U as function of r and find the radius of circular motion. **(Mumbai 2001)**
Ans : $r = 1\ m$.

[SET- II]

1. A particle is describing a parabola about a centre of force which attracts according to the inverse square of the distance. If the speed of the particle is made one half without change of direction of motion when the particle is at one end of the latus rectum, prove that the new path is an ellipse with eccentricity $e = \sqrt{5/8}$.

2. The eccentricity of the earth's orbit is $e = 0.0167$. If the orbit is divided into two by the minor axis, show that the times spent in the two halves of the orbit are $\left(\frac{1}{2} \pm \frac{e}{\pi}\right)$ year. Also evaluate the difference in hours.

Ans : 93.2 hours.

3. Show that the velocity of a planet at any point of its orbit is the same asit would have been if it had fallen that point from rest at a distance from the sun equal to the length of the major axis.

4. Calculate the time in which a particle moving under inverse square law force describes the area $0 \le \theta \le \alpha$ of elliptical orbit.

$$\textbf{Ans :}\ t = \frac{J^3}{\mu K^2(1-e^2)^{3/2}}\left[2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}\right) - e\sqrt{1-e^2}\,\frac{\sin\alpha}{1+e\cos\alpha}\right]$$

5. A particle moves under the action of a central force and describes a curve $r = a/(1+\theta)^2$, where a is constant. What is the force law ? If when $\phi = 0$, the particle receives an impulse which reduces its radial velocity to zero and doubles its transverse velocity, show that its subsequent path is given by

$$\frac{3a}{2r} = 1 + \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}\phi\right)$$

6. A particle moves in a bounded orbit under an attractive inverse force. Prove that the time average of the kinetic energy is half the time average of the potential energy.

7. Find the differential scattering cross-section for the scattering of particles by the potential $V(r)$, where $V(r) = \alpha\left(\frac{1}{r} - \frac{1}{R}\right)$ for $r < R$ and $V(r) = 0$ for $r > R$.

Ans : $\dfrac{R^2}{4}\dfrac{(1+\gamma)}{\left(1+\gamma\sin^2\dfrac{\theta}{2}\right)^2}$, where $\gamma = \dfrac{4ER}{\alpha^2}(\alpha + RE)$.

Objective Type Questions

1. A particle is moving under central force about a fixed centre of force. Choose the correct statement :
(*a*) The motion of the particle is always on a circular path.
(*b*) Its angular momentum is conserved.
(*c*) Its kinetic energy remains constant.
(*d*) Motion of the particle takes place in a plane.
Ans : (*b*), (*d*).

2. Two particles of masses m and $2m$, interacting via gravitational force are rotating about common centre of mass with angular velocity ω at a fixed distance r. If the particle of mass $2m$ is taken as the origin O,
(*a*) the force between them can be represented as $F = \mu\omega^2 r$.
(*b*) in an inertial frame, fixed at the centre of mass, the origin is at rest.
(*c*) in the inertial frame, the origin O is moving on a circular path of radius $r/3$.
(*d*) in the inertial frame, the particle of mass m is moving on a circular path of radius $r/3$.
Ans : (*a*), (*c*).

3. A particle is moving on elliptical path under inverse square law force of the form $F(r) = -K/r^2$. The eccentricity of the orbit is
(*a*) a function of total energy.
(*b*) independent of total energy.
(*c*) a function of angular momentum.
(*d*) independent of angular momentum.
Ans : (*a*), (*c*).

4. The maximum and minimum velocities of a satellite are v_1 and v_2 respectively. The eccentricity of the orbit of the satellite is given by

(*a*) $e = \frac{v_1}{v_2}$ (*b*) $e = \frac{v_2}{v_1}$ (*c*) $e = \frac{v_1 - v_2}{v_1 + v_2}$ (*d*) $e = \frac{v_1 + v_2}{v_1 - v_2}$

Ans : (*d*).

5. Rutheford's differential scattering cross-section

(*a*) has the dimensions of area.

(*b*) has the dimensions of solid angle.

(*c*) is proportional to the square of the kinetic energy of the incident particle.

(*d*) is inversely proportional to cosec^4 (ϕ /2), where ϕ is the scattering angle.

Ans : (*a*).

6. Consider a comet of mass m moving in a parabolic orbit around the Sun. The closest distance between the comet and the Sun is b, the mass of the Sun is M and the universal gravitation constant is G..

(*i*) The angular momentum of the comet is

(*a*) $M\sqrt{Gmb}$ (*b*) $b\sqrt{GmM}$

(*c*) $G\sqrt{mMb}$ (*d*) $m\sqrt{2GMb}$ **(Gate 2004)**

Ans. (*d*)

[**Hint.** Equation of the parabolic orbit ($e = 1$)

$$\frac{l}{r} = 1 + \cos(\theta - \theta')$$

When the comet is closest to the Sun, $r = b$ and $\cos(\theta - \theta') = 1$. So that $l = 2b$, *i.e.*, $J^2/mK = 2b$ or $J = m\sqrt{2GMb}$ $(K = GMm)$].

(*ii*) Which one of the following is TRUE for the above system ?

(*a*) The acceleration of the comet is maximum when it is closest to the Sun.

(*b*) The linear momentum of the comet is a constant.

(*c*) The comet will return to the solar system after a specified period.

(*d*) The kinetic energy of the comet is a constant. **(Gate 2004)**

Ans. (*a*)

Short Answer Questions

1. Show that the motion of a particle under central force takes place in a plane.
2. What are first integrals ?
3. Calculate the reduced mass of H_2 modecule. Assume the mass of H atom = M.
   Ans. $M/2$.
4. What is differential scattering cross-section ?
5. In Rutherford's experiment, 10^4 particles are scattered at an angle of 2°, calculate the number of α - particles, scattered at an angle of 16°.
   Ans. 2.4 approx.
6. Discuss α-particle scattering in coulomb's field. **(Agra 2003, 02)**
7. Fill in the blanks :
   (*i*) The square of the period of revolution of the planet around the sun is proportional to the cube of the...................
   (*ii*) If e is the eccentricity of the earth's arbit, the ratio of maximum and minimum speeds of the planet is..................
   Ans. (*i*) Semi-major axis of the ellipse, (*ii*) $(1 + e) / (1 - e)$.

CHAPTER 5

Variational Principles

5.1. INTRODUCTION

Many variational principles have been proposed in physics and applied to deal with the problems. The main advantage of such principles lies in their extreme economy of expression. In sec. 2.11, we have made use of such a variational principle, namely Hamilton's principle, to deduce the Lagrange's equations. We discuss here the calculus of variation where the fundamental problem is to find the curve for which a given line integral has a stationary or extremum value. We shall see that the Hamilton's principle is just a special case of the general formulation.

5.2. THE CALCULUS OF VARIATIONS AND EULER-LAGRANGE'S EQUATIONS

Let us have a function $f(y, y', x)$ defined on a curve given by

$$y = y(x) \quad ...(1)$$

between two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Here, $y' = dy/dx$. We are interested in finding a particular curve $y(x)$ for which the line integral I of the function f between the two points

$$I = \int_{x_1}^{x_2} f(y, y', x)dx \quad ...(2)$$

has a stationary value.

Suppose that APB be the curve for which I is stationary. Now, consider a neighbouring curve $AP'B$ with the same end points A and B. The point $P(x, y)$ of the curve APB corresponds to the point $P'(x, y+\delta y)$ of the curve $AP'B$, keeping x-coordinate of the points fixed. *This defines a δ-variation of the curve.* The variation is arbitrary but small and may be expressed as

$$\delta y = \frac{\partial y}{\partial \alpha}\delta\alpha = \eta(x)\,\delta\alpha \quad ...(3)$$

where α is a parameter (independent of x) common to all points of the path and $\eta(x)$ is a function of x with the condition that

$$\delta y_1 = \delta y_2 = \eta(x_1) = \eta(x_2) = 0 \quad ...(4)$$

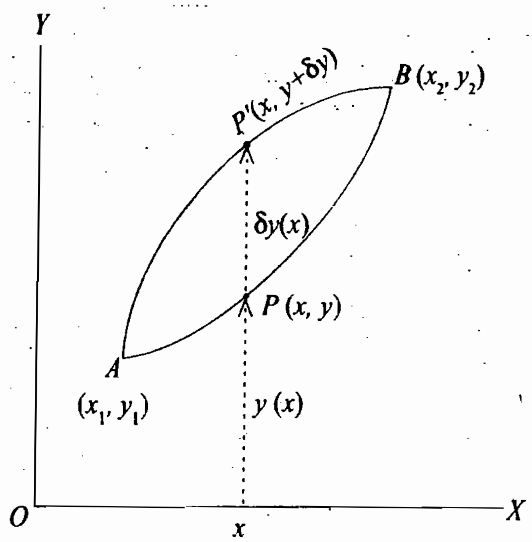

Fig. 5.1 : δ-variation

By choosing different $\eta(x)$, we may construct different varied paths.

The corresponding variation in y' is

$$\delta y' = \eta'(x)\,\delta\alpha \quad ...(5)$$

Now, the integral on the varied path is

$$I' = \int_{x_1}^{x_2} f(y + \delta y,\ y' + \delta y',\ x)\,dx$$

or
$$I' = \int_{x_1}^{x_2} f(y + \eta\,\delta\alpha,\ y' + \eta'\,\delta\alpha,\ x)\,dx \qquad ...(6)$$

Since the variation is small, the integral I' may be obtained by considering only first order terms in the Taylor expansion of the function f *i.e.*,

$$I' = \int_{x_1}^{x_2} [f(y, y', x) + \frac{\partial f}{\partial y}\eta\,\delta\alpha + \frac{\partial f}{\partial y'}\eta'\,\delta\alpha]\,dx \qquad ...(7)$$

Hence
$$\delta I = I' - I = \delta\alpha \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right) dx \qquad ...(8)$$

But
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'\,dx = \frac{\partial f}{\partial y'}\eta\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx$$

or
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'\,dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx \qquad [\text{as } \eta(x_1) = \eta(x_2) = 0]$$

Therefore,
$$\delta I = \delta\alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\,dx \qquad ...(9)$$

The condition that the integral I is stationary means that $\delta I = 0$, *i.e.*,

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\,dx = 0 \qquad ...(10)$$

As η is arbitrary, the integrand of (10) must be zero, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(11)$$

which is known as ***Euler-Lagrange equation.***

The result can easily be generalized to the case where f is a function of many independent variables y_k and their derivatives y'_k. However, y_k and y'_k are function of x. Then

$$\delta I = \delta \int_{x_1}^{x_2} f(y_1, y_2, \ldots, y_k, \ldots, y_n, y'_1, y'_2, \ldots, y'_k, \ldots y'_n, x)\,dx = 0 \qquad ...(12)$$

leads to the Euler-Lagrange equations

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_k}\right) = 0 \qquad ...(13)$$

where, $k = 1, 2, \ldots, n$.

It is to be pointed out that in most of the problems the stationary value of the integral is seen to be a minimum but occasionally maximum.

Hamilton's principle and Lagrange's equations : If we identify the Lagrangian to

$$L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \qquad \ldots(14)$$

to the function

$$f = f(y_1, y_2, \ldots y_n, y'_1, y'_2, \ldots y'_n, x)$$

with the transformations

$$x \to t,\ y_k \to q_k,\ y'_k \to \dot{q}_k,$$

then the integral $I \to S$ defines the action integral

$$S = \int_{x_1}^{x_2} L\, dt \qquad \ldots(15)$$

This integral is known as ***Hamilton's principal function.***

The variation

$$\delta S = \delta \int L\, dt = 0 \qquad \ldots(16)$$

corresponding to eq. (12) means that ***the motion of the system from time $t = t_1$ to time $t = t_2$ is such that the line integral* (15) *has a stationary value for the correct path of the motion.*** This is what is known as **Hamilton's principle.** The necessary condition for the Hamilton's principle ($\delta S = 0$) is given by Lagrange's equations of motion in place of Euler-Lagrange's equations, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \ldots(17)$$

where q_k are the generalized coordinates.

These coordinates are independent corresponding to the independent variables y_k and hence the constraints are holonomic.

Ex. 1. *Show that the shortest distance between two points in a plane is a straight line.*

(Garwal 1991; Meerut 99)

Solution : Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in X-Y plane. An element of length ds of any curve, say $AP'B$, passing through A and B points is given by

$$ds^2 = dx^2 + dy^2$$

or

$$ds = \sqrt{1 + y'^2}\, dx = 0 \qquad \ldots(i)$$

Total length of the curve from point A to the point B is given by

$$s = \int_A^B \sqrt{1 + y'^2}\, dx = \int_A^B f\, dx \qquad \ldots(ii)$$

where $f = \sqrt{1 + y'^2}$. The length of the curve s will be minimum, when $\delta s = 0$. This means that f should satisfy the Euler-Lagrange's equation, *i.e.*,

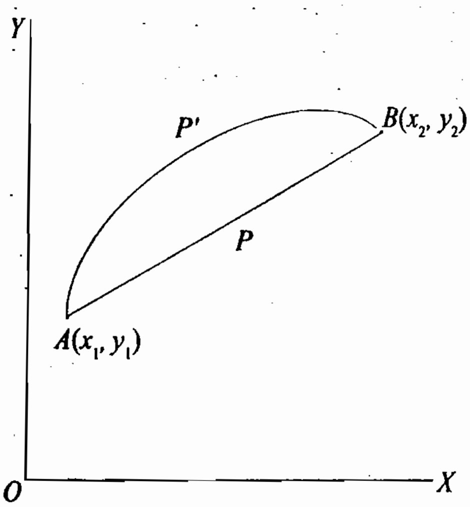

Fig. 5.2 : Shortest distance between two points in a plane.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots(iii)$$

Here, $\dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{\partial y'} = \dfrac{y'}{\sqrt{1+y'^2}}$

Therefore, eq. (*iii*) is

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \text{ or } \frac{y'}{\sqrt{1+y'^2}} = C, \text{ a constant}$$

or $$y'^2 = C^2(1+y'^2) \text{ or } y'^2(1-C^2) = C^2 \text{ or } y' = \frac{C}{\sqrt{1-C^2}} = a \text{ (constant)}$$

or $$\frac{dy}{dx} = a \qquad ...(iv)$$

Integrating it, we get

$$y = ax + b \qquad ...(v)$$

where b is a constant of intrgration. Eq. (v) represents a straight line. Therefore the shortest distance between any two points in a plane is a straight line. The constants of integration a and b can be determined by the condition that the straight line (v) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Ex. 2. *A particle of mass m falls a given distance z_0 in time $t_0 = \sqrt{2z_0/g}$ and the distance travelled in time t is given by $z = at + bt^2$, where constants a and b are such that the time t_0 is always the same. Show that the integration $\int_0^{t_0} L\,dt$ is an extremum for real values of the coefficients only when $a = 0$ and $b = g/2$.*

(Agra 1989)

Solution : Let the particle fall from O ($z = 0$) to P ($OP = z$) in time t.

Kinetic energy of the particle at P,

$$T = \frac{1}{2}m\dot{z}^2$$

Potential energy of the particle at P, $V = -mgz$

Hence $$L = T - V = \frac{1}{2}m\dot{z}^2 + mgz \qquad ...(i)$$

According to the Hamilton's principle

$$\delta\int_0^{t_0} L\,dt = 0$$

or $$\int_0^{t_0} L\,dt = \text{extremum, for which}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] - \frac{\partial L}{\partial z} = 0 \qquad ...(ii)$$

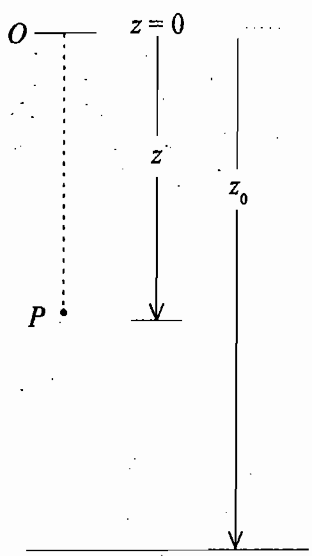

Fig. 5.3

is to be satisfied.

Here, $\dfrac{\partial L}{\partial \dot{z}} = m\dot{z}$ and $\dfrac{\partial L}{\partial z} = mg$. Hence eq. (*ii*) is

$$\frac{d}{dt}(m\dot{z}) - mg = 0 \text{ or } \ddot{z} = g \qquad ...(iii)$$

But $z = at + bt^2$ and therefore $\dot{z} = a + 2bt$ and $\ddot{z} = 2b$ ...(iv)

From (iii) and (iv), we get

$$2b = g \text{ or } b = g/2 \qquad \text{...(v)}$$

Also at $t = t_0$, $z = z_0$, we have

$$z_0 = at_0 + bt_0^2 \qquad \text{...(vi)}$$

But $$t_0 = \sqrt{\frac{2z_0}{g}} \text{ or } z_0 = \frac{1}{2}gt_0^2 \qquad \text{...(vii)}$$

Comparing (vi) and (vii) and putting $b = g/2$, we get

$$at_0 + \frac{g}{2}t_0^2 = \frac{1}{2}gt_0^2 \text{ or } at_0 = 0$$

Since $t_0 \neq 0$, therefore, $a = 0$.

Thus we find that $\int_0^{t_0} L\,dt$ is extremum, when $a = 0$, $b = g/2$.

Ex. 3. *We take a curve passing through the fixed points (x_1, y_1) and (x_2, y_2) and revolve it about Y-axis to form a surface of revolution. Find the equation of the curve for which the surface area is minimum.*

Solution : Let AB be the curve which passes through the fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$. The curve AB has been revolved about Y-axis to generate a surface. Consider a strip of the surface whose radius is x and breadth is $PP' = ds$, given by

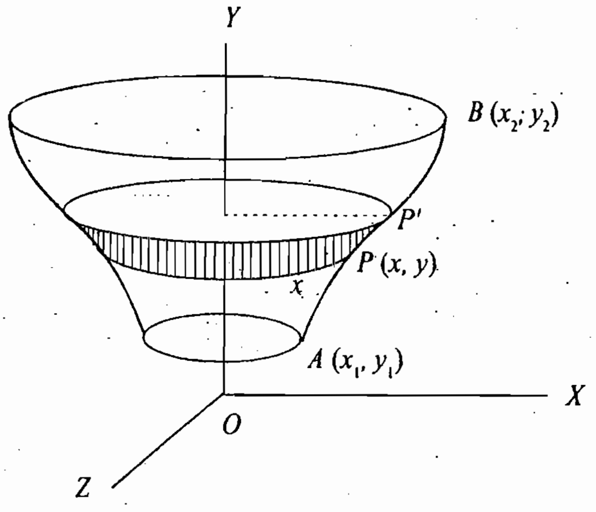

Fig. 5.4 : Minimum surface area of revolution

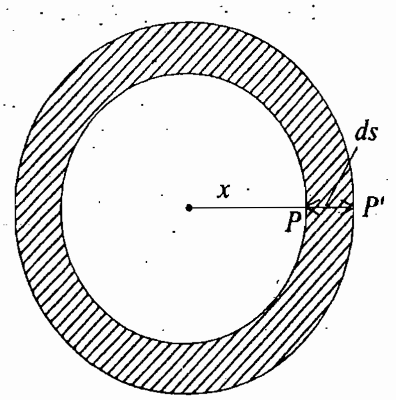

Fig. 5.5 : Circular strip of area $2\pi x\, ds$

$$ds^2 = dx^2 + dy^2 \text{ or } ds = \sqrt{1 + y'^2}\, dx$$

Area of the strip $dS = 2\pi x\, ds$ (Fig. 5.5)

$$= 2\pi x \sqrt{1 + y'^2}\, dx$$

Total area of revolution $S = 2\pi \int_A^B x\sqrt{1 + y'^2}\, dx$ ...(i)

This area will be minimum, strictly speaking extremum, if $\delta S = 0$, for which Euler-Lagrange equation is to be satisfied, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(ii)$$

where $f = x\sqrt{1+y'^2}$, when compared to eq. (2),

Here $$\frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial y'} = \frac{xy'}{\sqrt{1+y'^2}}$$

Substituting in eq. (*ii*), we have

$$\frac{d}{dx}\frac{xy'}{\sqrt{1+y'^2}} = 0 \text{ or } \frac{xy'}{\sqrt{1+y'^2}} = a \qquad ...(iii)$$

where a is constant of integration. Squaring (*iii*), we get

$$x^2y'^2 = a^2 + a^2y'^2 \text{ or } y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

Therefore, $$y = \int \frac{a}{\sqrt{x^2-a^2}}dx = a\cosh^{-1}\frac{x}{a} + b \qquad ...(iv)$$

where b is another constant of integration.

From (*iv*) we have

$$\cosh^{-1}\frac{x}{a} = \frac{y-b}{a} \text{ or } x = a\cosh\frac{y-b}{a} \qquad ...(v)$$

which is the equation of a catenary.

This is the equation of the curve for which the surface of revolution is minimum. The two constants a and b can be determined by the condition that the curve (v) passes through (x_1, y_1) and (x_2, y_2) points.

Ex. 4. Brachistochrone Problem. *A particle slides from rest at one point on a frictionless wire in a vertical plane to another point under the influence of the earth's gravitational field. If the particle travels in the shortest time, show that the path followed by it is a cycloid.* **(Kanpur 2003)**

Solution : Let the shape of wire be in the form of a curve OA. The particle starts to travel from O (0, 0) from rest and moves to A (x_1, y_1) under the influence of gravity on the frictionless wire.

Let v be the speed at P. Then in moving $PP' = ds$ element, the time taken will be ds/v. Therefore, total time taken by the particle in moving from the higher point O to the lower point A is

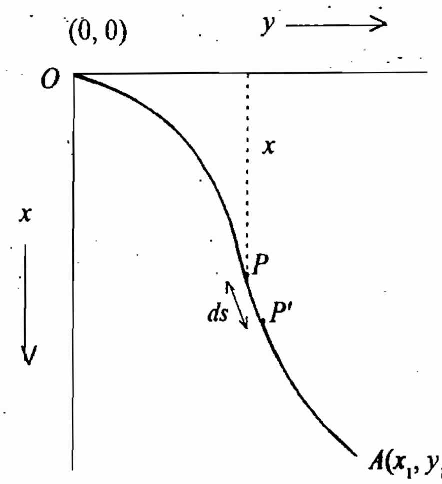

Fig. 5.6 : The brachistochrone problem

$$t = \int_0^A \frac{ds}{v} \qquad ...(i)$$

If the vertical distance of fall from O to P be x, then from the principle of conservation of energy

$\frac{1}{2}mv^2 = mgx$ or $v = \sqrt{2gx}$

Therefore, $$t = \int_0^A \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gx}} \quad \left[ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+y'^2}\right] \qquad ...(ii)$$

So that $f = \sqrt{\frac{1+y'^2}{2gx}}$ and for t to be minimum,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(iii)$$

Here, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}}$

Subsitituting in (iii), we get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}}\right) = 0 \text{ or } \frac{y'}{\sqrt{x}\sqrt{1+y'^2}} = C, \text{ constant}$$

or $$\frac{y'^2}{C^2} = x(1+y'^2) \text{ or } y'^2\left(\frac{1}{C^2} - x\right) = x \text{ or } y'^2 = \frac{x}{b-x}$$

(where $b = 1/C^2$, a constant).

or $$\frac{dy}{dx} = \sqrt{\frac{x}{b-x}}, \text{ or } y = \int \sqrt{\frac{x}{b-x}}\,dx + C', \text{ another constant} \qquad ...(iv)$$

Let $x = b\sin^2\theta$, then $dx = 2b\sin\theta\cos\theta\,d\theta$.

Therefore, $$y = \int \frac{\sin\theta}{\cos\theta} 2b\sin\theta\cos\theta\,d\theta + C'$$

$$= b\int 2\sin^2\theta\,d\theta + C' = b\int(1-\cos 2\theta)\,d\theta + C'$$

$$= b\left[\theta - \frac{\sin 2\theta}{2}\right] + C' = \frac{b}{2}[2\theta - \sin 2\theta] + C'$$

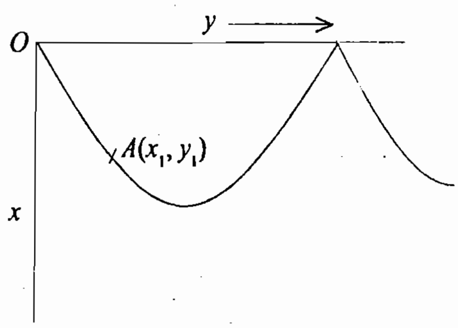

Fig. 5.7 : Cycloid

Thus the parametric equations of the curve are

$$x = b\sin^2\theta = \frac{b}{2}(1-\cos 2\theta) \text{ and } y = \frac{b}{2}(2\theta - \sin 2\theta) + C'$$

Since the curve passes through (0, 0), $C' = 0$. Therefore,

$$x = \frac{b}{2}(1-\cos 2\theta) \text{ and } y = \frac{b}{2}(2\theta - \sin 2\theta) \qquad ...(v)$$

Let $2\theta = \phi$ and $b/2 = a$. Then the parametric equations of the curve are

$$x = a\,(1 - \cos\phi) \text{ and } y = a\,(\phi - \sin\phi) \qquad \ldots(vi)$$

This represents a cycloid [Fig. 5.7]. The constant a can be determined because the curve passes through the point $A(x_1, y_1)$.

Ex. 5. *Apply variational principle to find the equation of one dimensional harmonic oscillator.* **(Agra 1988)**

Solution : The Lagrangian L for one dimensional harmonic oscillator is

$$L = T - V = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 \text{ or } L = f(x, \dot{x}, t) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

According to Hamilton's principle or variational principle $\int L\,dt$ or $\int f(x, \dot{x}, t)\,dt$ is extremum.

Euler-Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \frac{\partial f}{\partial x} = 0$$

Here $$\frac{\partial f}{\partial x} = -kx\ , \quad \frac{\partial f}{\partial \dot{x}} = m\dot{x}$$

Therefore, $$m\ddot{x} + kx = 0$$

which is the equation of motion for one-dimensional harmonic oscillator.

Ex. 6 . *Show that for a spherical surface, the geodesics are the great circles.* (*For a non-flat surface, the curves of extremal lengths are called geodesics.*) **(Rohilkhand 1999, 78)**

Solution : $ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$ or $ds = a\,d\theta\,\sqrt{1 + \sin^2\theta\,\phi'^2}$

According to the variational principle,

$$\delta s = \delta\int ds = \delta\int a\,d\theta\,\sqrt{1 + \sin^2\theta\,\phi'^2} = 0 \text{ or } \delta\int_{\theta_1,\phi_1}^{\theta_2,\phi_2} d\theta\,\sqrt{1 + \sin^2\theta\,\phi'^2} = 0$$

Here, $$f = \sqrt{1 + \sin^2\theta\,\phi'^2}\ ;\ \therefore \frac{\partial f}{\partial \phi} = 0 \text{ and } \frac{\partial f}{\partial \phi'} = \frac{\phi'\sin^2\theta}{\sqrt{1 + \sin^2\theta\,\phi'^2}}$$

Now, $$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \text{ or } \frac{\phi'\sin^2\theta}{\sqrt{1 + \sin^2\theta\,\phi'^2}} = C$$

or $$\phi' = \frac{C\,\mathrm{cosec}^2\theta}{\left(1 - C^2 - C^2\cot^2\phi\right)^{1/2}} = \frac{d\phi}{d\theta}\ , \ \therefore\ \phi = \alpha - \sin^{-1}(C'\cot\theta)$$

where α and C' are constants and these may be fixed by limits θ_1, ϕ_1 and θ_2, ϕ_2

$$C'\cot\theta = \sin(\alpha - \phi) \text{ or } C'r\cos\theta = r\sin(\alpha - \phi)\sin\theta$$

or $$C'r\cos\theta = \sin\alpha\, r\cos\phi\sin\theta - \cos\alpha\, r\sin\phi\sin\theta$$

or $$C'z = x\sin\alpha - y\cos\alpha$$

where we have transformed from spherical coordinates to cartesian coordinates.

The above equation represents a plane passing through the origin (0, 0, 0). This plane will cut the surface of the sphere in a great circle (whose centre is at the origin). This indicates that the shortest or longest distance between two points on the surface of the sphere is an arc of the circle with its centre at the origin.

5.3. DEDUCTION OF HAMILTON'S PRINCIPLE FROM D'ALEMBERTS PRINCIPLE

According to D'Alemberts principle [eq.(14), Chapter 2]

$$\sum_{i=1}^{N}(\mathbf{F}_i - \dot{\mathbf{p}}_i)\cdot\delta\mathbf{r}_i = 0$$

or $$\sum_i \dot{\mathbf{p}}_i\cdot\delta\mathbf{r}_i = \sum_i \mathbf{F}_i\cdot\delta\mathbf{r}_i \qquad ...(18)$$

Here $$\dot{\mathbf{p}}_i\cdot\delta\mathbf{r}_i = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i = \frac{d}{dt}(m_i\dot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i$$

$$= \frac{d}{dt}(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i) - m_i\dot{\mathbf{r}}_i\cdot\frac{d}{dt}(\delta\mathbf{r}_i) = \frac{d}{dt}(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i) - m_i\dot{\mathbf{r}}_i\cdot\delta\left(\frac{d\mathbf{r}_i}{dt}\right)$$

[because the δ-variation in the velocity corresponding to the virtual displacement $\delta\mathbf{r}_i$ is given by

$$\delta\left(\frac{d\mathbf{r}_i}{dt}\right) = \delta(\mathbf{v}_i) = \mathbf{v}_i(\mathbf{r}_i + \delta\mathbf{r}_i) - \mathbf{v}(\mathbf{r}_i) = \frac{d}{dt}(\mathbf{r}_i + \delta\mathbf{r}_i) - \frac{d}{dt}(\mathbf{r}_i) = \frac{d}{dt}(\delta\mathbf{r}_i)]$$

$$= \frac{d}{dt}[m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i] - \delta\left(\tfrac{1}{2}m_i\dot{\mathbf{r}}_i^2\right) \left[\because \delta(\dot{\mathbf{r}}_i^2) = \delta(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i) = 2\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i\right]$$

Thus the eq. (18) is

$$\frac{d}{dt}\left[\sum_i m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\right] - \delta\left(\sum_i \frac{1}{2}m_i\mathbf{v}_i^2\right) = \sum_i \mathbf{F}_i\cdot\delta\mathbf{r}_i \qquad ...(19)$$

If the forces are conservative,

$$\mathbf{F}_i = -\nabla_i V = -\frac{\partial V}{\partial x_i}\hat{\mathbf{i}} - \frac{\partial V}{\partial y_i}\hat{\mathbf{j}} - \frac{\partial V}{\partial z_i}\hat{\mathbf{k}},$$

and the virtual work done is

$$\sum_i \mathbf{F}_i\cdot\delta\mathbf{r}_i = -\sum_i\left[\frac{\partial V}{\partial x_i}\hat{\mathbf{i}} + \frac{\partial V}{\partial y_i}\hat{\mathbf{j}} + \frac{\partial V}{\partial z_i}\hat{\mathbf{k}}\right]\cdot\left[\delta x_i\hat{\mathbf{i}} + \delta y_i\hat{\mathbf{j}} + \delta z_i\hat{\mathbf{k}}\right]$$

$$= -\sum_i\left[\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right] = -\delta V \qquad ...(20)$$

Also $\sum_i \frac{1}{2}m_i v_i^2 = T$, represents the kinetic energy of the system. Hence eq. (19) is

$$\frac{d}{dt}\left[\sum_i m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\right] = \delta T - \delta V = \delta(T - V) \qquad ...(21)$$

Integrating it from $t = t_1$, to $t = t_2$ with the condition that $\delta\mathbf{r}_i(t_1) = \delta\mathbf{r}_i(t_2) = 0$ at the ends of the path, we get

$$\left[\sum_i m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \delta(T-V)\,dt \text{ or } \int_{t_1}^{t_2} \delta(T-V)\,dt = 0 \text{ or } \delta\int_{t_1}^{t_2}(T-V)\,dt = 0 \qquad \text{...(22)}$$

because $\delta\,[(T-V)\,dt] = \delta\,(T-V)\,dt + (T-V)\,\delta\,(dt) = \delta\,(T-V)\,dt\ [\because\ \delta\,(dt) = 0]$.

Putting $T - V = L$, we get the Hamilton's principle *i.e.*,

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \qquad \text{...(23)}$$

5.4. MODIFIED HAMILTON'S PRINCIPLE

According to Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \qquad \text{...(24)}$$

where $L = T - V = L(q_k, \dot{q}_k, t)$.

Equation (24) can be written in terms of Hamiltonian H by using the relation (25) of Chapter 3 *i.e.*,

$$H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Hence the Hamilton's principle (24) in the new form is obtained as

$$\delta\int_{t_1}^{t_2}\left(\sum_k p_k \dot{q}_k - H\right)dt = 0 \qquad \text{...(25)}$$

This is known as ***modified Hamilton's principle.***

In case of Hamilton's principle [eq. (24)] the path refers to configuration space and the variation of path allows for the variations in the generalized coordinates q_k at constant t. In the case of modified Hamilton's principle, the integral is to be evaluated over the path of the representative point of the system in phase space and q_k and p_k coordinates are to be treated as independent coordinates in the phase space. The δ-variation here implies independent variations of both the generalized and momenta coordinates q_k and p_k at constant t.

5.5. DEDUCTION OF HAMILTON'S EQUATIONS FROM MODIFIED HAMILTON'S PRINCIPLE (OR VARIATIONAL PRINCIPLE)

The δ-variation of q_k and p_k coordinates at constant t can be expressed in terms of a parameter α common to all points of the path of integration in phase space [(similar to eq.(3)] as

$$\delta q_k = \frac{\partial q_k}{\partial \alpha}\delta\alpha = \eta_k\,\delta\alpha \text{ and } \delta p_k = \frac{\partial p_k}{\partial \alpha}\delta\alpha = \eta'_k\,\delta\alpha \qquad \text{...(26)}$$

where η_k and η'_k are arbitrary subject to the conditions

$$\eta_k(t_1) = \eta_k(t_2) = \eta'_k(t_1) = \eta'_k(t_2) = 0 \qquad \text{...(27)}$$

Therefore, the δ-variation of the integral in (25) is

$$\delta\int_{t_1}^{t_2}\left[\sum_k p_k \dot{q}_k - H\right]dt = \int_{t_1}^{t_2}\sum_k\left[\left(\frac{\partial p_k}{\partial \alpha}\dot{q}_k + p_k\frac{\partial \dot{q}_k}{\partial \alpha}\right)\delta\alpha - \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial \alpha}\delta\alpha - \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \alpha}\delta\alpha\right]dt$$

$$= \delta\alpha \int_{t_1}^{t_2} \sum_k \left[\eta'_k \dot{q}_k + p_k \frac{\partial \dot{q}_k}{\partial \alpha} - \eta_k \frac{\partial H}{\partial q_k} - \eta'_k \frac{\partial H}{\partial p_k} \right] dt \qquad ...(28)$$

But
$$\int_{t_1}^{t_2} p_k \frac{\partial \dot{q}_k}{\partial \alpha} dt = \int_{t_1}^{t_2} p_k \frac{d}{dt}\left[\frac{\partial q_k}{\partial \alpha}\right] dt = \int_{t_1}^{t_2} p_k \frac{d\eta_k}{dt} dt$$

$$= \left[p_k \eta_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \eta_k dt = -\int_{t_1}^{t_2} \dot{p}_k \eta_k dt \quad [\because \eta_k(t_1) = \eta_k(t_2) = 0]$$

Also in view the modified Hamilton's principle [eq. (25)], the δ–variation of the integral must be zero. Therefore, we obtain from (28)

$$\delta\alpha \int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \eta'_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \eta_k \right] dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right] dt = 0 \qquad ...(29)$$

Since q_k and p_k are independent variables, the integral will be zero, when

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0 \text{ and } \dot{p}_k + \frac{\partial H}{\partial q_k} = 0 \text{ or } \dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad ...(30)$$

These are the desired ***Hamilton's canonical equations.***

5.6. DEDUCTION OF LAGRANGE'S EQUATIONS FROM VARIATIONAL PRINCIPLE FOR NON-CONSERVATIVE SYSTEMS (HOLONOMIC CONSTRAINTS)

We deduced Hamilton's principle from D'Alembert's principle for conservative forces. If the forces are not conservative, eq.(19) can be written as

$$\frac{d}{dt}\left[\sum_i m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i\right] = \delta T + \delta W \qquad ...(31)$$

where $\delta T = \delta \sum_i \frac{1}{2} m_i v_i^2$ and $\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i$ = virtual work done.

The integration of (31) from $t = t_1$ to $t = t_2$ with the condition $\delta \mathbf{r}_i(t_1) = \delta \mathbf{r}_i(t_2) = 0$ at the end points, we get

$$\int_{t_1}^{t_2} \delta[T + W] dt = 0 \quad \text{or} \quad \delta \int_{t_1}^{t_2} [T + W] dt = 0 \qquad ...(32)$$

Eq. (32) is known as ***extended Hamilton's principle.*** Here $\mathbf{F}_i$ are the non-conservative forces. We can write as in eq.(19) (Chapter 2)

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i,k} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_k G_k \delta q_k \qquad ...(33)$$

where G_k are the components of generalized force.

Thus the extended Hamilton's principle (32) gives

$$\delta\int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} \sum_k G_k\,\delta q_k\,dt = 0 \qquad ...(34)$$

Kinetic energy T in general is function of q_k and $\dot{q}_k$ and hence

$$\delta\int_{t_1}^{t_2} T(q_k,\dot{q}_k)\,dt = \int_{t_1}^{t_2} \delta T(q_k,\dot{q}_k)\,dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot{q}_k}\delta \dot{q}_k\right] dt$$

$$= \int_{t_1}^{t_2} \sum_k \frac{\partial T}{\partial q_k}\delta q_k\,dt + \sum_k \left[\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right]\delta q_k\,dt$$

$$= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right]\delta q_k dt \ \left[\because \delta q_k(t_1) = \delta q_k(t_2) = 0\right] \qquad ...(35)$$

Thus eq. (34) is

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + G_k\right]\delta q_k\,dt = 0 \qquad ...(36)$$

Since the constraints are holonomic, all δq_k are independent and hence the integral will vanish, if

$$\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + G_k = 0 \text{ or } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad ...(37)$$

These are the Lagrange's equations for holonomic and non-conservative system.

5.7. LAGRANGE'S EQUATIONS OF MOTION FOR NON-HOLONOMIC SYSTEMS (*LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS*)

In the derivation of Lagrange's equations from D'Alembert's principle or Hamilton's principle, we need the requirement of holonomic constraints at the final step, when the variations δq_k are considered to be independent of each other. In case of non-holonomic systems, the generalized coordinates are not independent of each other. However, we can treat certain types of non-holonomic systems for which the equations of constraint can be put in the form

$$\sum_k a_{lk}\,dq_k + a_{lt}\,dt = 0 \qquad ...(38)$$

These equations of constraints connect the differentials dq_k's by linear relations. For each l, there is one equation and we assume that there are m such equations for $l = 1, 2, ..., m$.

In case of δ-variation, the virtual displacements δq_k are taken at constant times and hence the m equations of constraints, consistent for virtual displacements, are

$$\sum_k a_{lk}\delta q_k = 0 \qquad ...(39)$$

Eq. (39) now can be used to reduce the number of virtual displacements to independent ones. The procedure applied for this purpose is called ***Lagrange's method of undetermined multipliers.***

If eq. (39) is valid, then the multiplication of this equation by λ_l, an undetermined quantity, yields

$$\lambda_l \sum_k a_{lk}\, \delta q_k = 0 \quad \text{or} \quad \sum_k \lambda_l a_{lk}\, \delta q_k = 0 \qquad ...(40)$$

where λ_l (1, 2,.. .,m) are undetermined quantities and they are functions in general of the coordinates and time. Summing eq. (40) over l and then integrating the sum with respect to time from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \sum_{k,l} \lambda_l\, a_{lk}\, \delta q_k\, dt = 0 \qquad ...(41)$$

We assume the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad ...(42)$$

to hold for the non-holonomic system. This implies that

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k\, dt = 0 \qquad ...(43)$$

Adding (41) and (43), we obtain

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_l \lambda_l a_{lk}\right] \delta q_k\, dt = 0 \qquad ...(44)$$

Still all δq_k's (k =1,2, ..., n) are not independent of each other. First $n - m$ of these δq_k's may be chosen independently and the last m of these δq_k's are then fixed by the eq. (39).

Till now the values of λ_l have not been specified. We choose the λ_l's such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{l=1}^{m} \lambda_k a_{lk} = 0 \qquad ...(45)$$

where $k = n - m + 1, n - m + 2, ..., n$. Thus eqs. (45) will determine m values of λ_l and then eq. (44) can be written as

$$\int_{t_1}^{t_2} \sum_{k=1}^{n-m} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{l=1}^{m} \lambda_l a_{lk}\right] \delta q_k\, dt = 0 \qquad ...(46)$$

where the δq_k's (k =1, 2, ..., $n - m$), involved, are independent ones. Therefore, for the integrand in (46) to vanish

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0 \qquad ...(47)$$

which are $n - m$ equations for k =1, 2, ..., $n - m$.

Adding eqs. (45) and (47), we get the complete set of the Lagrange's equations for the non-holonomic system, *i.e.*,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} \qquad ...(48)$$

where k = 1, 2, ...n.

Eq. (48) gives us n equations, but there are $n + m$ unknowns, n coordinates q_k and m Lagrange's multipliers. The remaining m unknown q_k's are determined from m equations of constraints (38), written in the following form of m first-order differential equations :

$$\sum_k a_{lk}\,\dot{q}_k + a_{lt} = 0 \qquad ...(49)$$

5.8. PHYSICAL SIGNIFICANCE OF LAGRANGE'S MULTIPLIERS λ_l

Suppose we remove the constraints on the system, but apply external forces G_k so that the motion of the system remains unchanged. Now, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = G_k \qquad ...(50)$$

Since the applied forces are equal to the forces of constraints, eqs. (48) and (50) must be identical, resulting

$$G_k = \sum_{l=1}^{n} \lambda_l\, a_{lk} \qquad ...(51)$$

Thus the generalized forces of constraints G_k have been identified to $\sum \lambda_l\, a_{lk}$. We observe that in such problems, we need not to eliminate the forces of the constraints but the procedure itself determines these forces by eq. (51).

Eq. (38) does not represent the most general type of nonholonomic constraints because it does not include equations of constraints in the form of inequalities. However, it includes holonomic constraints. Equation representing holonomic constraints is given by

$$f(q_1, q_2, ..., q_n, t) = 0 \qquad ...(52)$$

So that

$$\sum_{k=1}^{n} \frac{\partial f}{\partial q_k} dq_k + \frac{\partial f}{\partial t} dt = 0 \qquad ...(53)$$

This is identical in form to eq. (38) with the coefficients a_{lk} and a_{lt}, given by

$$a_{lk} = \frac{\partial f}{\partial q_k} \text{ and } a_{lt} = \frac{\partial f}{\partial t} \qquad ...(54)$$

Thus one can use Lagrange's method of undetermined multipliers for holonomic constraints when it is not easy to reduce all the q_k's to independent coordinates or we may be interested in knowing the force of constraints.

5.9. EXAMPLES OF LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

(1) ***Rolling hoop on an inclined plane*** : *Discuss the motion of a hoop rolling down an inclined plane without slipping. Find its acceleration and frictional force of constraint by using the method of undetermined multipliers.* **(Agra 1997, 91, 83; Rohilkhand 86)**

Solution : Let ϕ be the inclination of the inclined plane of length l with the horizontal. If a hoop of mass M and radius R is rolling down an inclined plane starting from a point O without slipping, then x and θ are two generalized coordinates and the equation of constraint is

$$x = R\theta \text{ or } dx = R d\theta \text{ or } dx - R d\theta = 0 \qquad ...(i)$$

As there is only one constraint equation, only one Lagrange's multiplier λ will be required. Here

$$a_{1x} dx + a_{1\theta}\, d\theta = 0$$

$$[\because \sum_{k=1}^{n} a_{lk}\, dq_k + a_{lt}\, dt = 0]$$

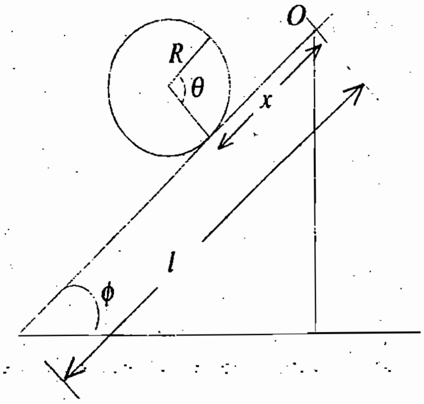

Fig. 5.8 : A hoop rolling down on inclined plane

So that $a_{1x} = 1 = a_x$ (say) and $a_{1\theta} = -R = a_\theta$ (say) ...(*ii*)

Kinetic energy of the hoop T = Kinetic energy of motion of centre of mass + Rotational kinetic energy about the centre of mass

$$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} MR^2\dot{\theta}^2$$

Potential energy of the hoop $V = Mg(l - x)\sin\phi$

Thus $$L = T - V = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} MR^2\dot{\theta}^2 - Mg(l-x)\sin\phi \qquad ...(iii)$$

Equations of motion for two coordinates x and θ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda a_x \qquad ...(iv)$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda a_\theta \qquad ...(v)$$

Here, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \frac{\partial L}{\partial x} = Mg\sin\phi, \frac{\partial L}{\partial \dot{\theta}} = MR^2\dot{\theta}, \frac{\partial L}{\partial \theta} = 0$ and also $a_x = 1$ and $a_\theta = -R$.

Therefore, $$M\ddot{x} - Mg\sin\phi = \lambda \qquad ...(vi)$$

and $$MR^2\ddot{\theta} = -R\lambda \qquad ...(vii)$$

But from (*i*) $\ddot{x} = R\ddot{\theta}$. Hence from eq. (*vii*), we get

$$M\ddot{x} = -\lambda \qquad ...(viii)$$

Substituting for λ in eq. (*vi*), we obtain

$$M\ddot{x} - Mg\sin\phi = -M\ddot{x} \text{ or } \ddot{x} = \frac{1}{2} g\sin\phi \qquad ...(ix)$$

This is the acceleration of the hoop along the inclined plane. Note that it is one half of the acceleration it would have in slipping down a frictionless inclined plane.

The force of constraint λ is [by using eq. (*viii*) and (*ix*)]

$$\lambda = -\frac{1}{2} Mg\sin\phi \qquad ...(x)$$

This gives the frictional force due to constraint which reduces the acceleration $\ddot{x} = g\sin\phi$ (when there is only slipping without friction) to $\ddot{x} = \frac{1}{2} g\sin\phi$ (when the hoop is rolling without slipping).

Note : It is to be remarked that if we take constraint equation as $R\,d\theta - dx = 0$, then the constraint force will be obtained as $\lambda = \frac{1}{2} Mg\sin\phi$ which bears +ve sign. Thus in such problems we obtain only the magnitude of the force of constraint.

(2) *Simple pendulum* : *Find the equation of motion and force of constraint in case of simple pendulum by using Lagrange's method of undetermined multipliers.*

Solution : Referring Fig. 5.9, the Lagrangian L is given by

$$L = \frac{1}{2} mr^2\dot{\theta}^2 + mgr\cos\theta \qquad ...(i)$$

where $V = -\, mgr\cos\theta$ with respect to position S.

The equation of constraint is

$$r = l \text{ or } dr = 0 \qquad ...(ii)$$

Here there is only one constraint equation, hence only one Lagrange's multiplier λ will be needed. There are two coordinates r and θ and the general constraint equation will be

$$a_r dr + a_\theta d\theta = 0 \qquad ...(iii)$$

$\therefore \qquad a_r = 1,\ a_\theta = 0$

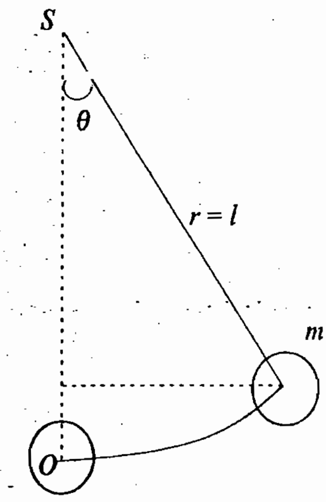

Fig. 5.9 : Simple pendulum

Equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\, a_r \qquad ...(iv)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda\, a_\theta \qquad ...(v)$$

Here, $\dfrac{\partial L}{\partial \dot{r}} = 0,\ \dfrac{\partial L}{\partial r} = mr\,\dot{\theta}^2 + mg\cos\theta,\ \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\ \dfrac{\partial L}{\partial \theta} = -\,mgr\sin\theta$

$$\therefore \qquad -\,mr\dot{\theta}^2 - mg\cos\theta = \lambda \qquad ...(vi)$$

$$mr^2\ddot{\theta} + mgr\sin\theta = 0 \qquad ...(vii)$$

where $\dot{r} = 0$ from (ii). Using $r = l$, (the constraint equation), equation of motion of simple pendulum is given by eq. (vii), *i.e.*,

$$l\ddot{\theta} + g\sin\theta = 0$$

For small θ, $\sin\theta = \theta$ and hence

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad ...(viii)$$

The force of constraint is

$$\lambda = -\,ml\dot{\theta}^2 - mg\cos\theta \qquad ...(ix)$$

which gives the force of constraint, *i.e.* tension $F = ml\dot{\theta}^2 + mg\cos\theta$ in magnitude.

5.10. Δ-VARIATION

In the δ-variation, the variation of the path allows for variations in the coordinate q_k at constant t and the varied path and the correct path have the same end points *i.e.*,

$$\delta q_k(t_1) = \delta q_k(t_2) = 0$$

Now, we introduce a new and more general type of variation of the path of a system, known as Δ-variation. In this variation, time as well as position coordinates are allowed to vary. At the end points of the path, the position coordinates are kept fixed, while changes in the time are allowed. The Δ-variation of a coordinate q_k is shown in Fig. 5.10 ; APB is the actual path and $A'P'B'$, the varied path. The end points of the path A and B after time Δt take the position A' and B' so that there is no change in the position coordinates, *i.e.*, $\Delta q_k(1) = \Delta q_k(2) = 0$. A point P on the actual path now goes over to the point P', on the varied path with the correspondence, given by

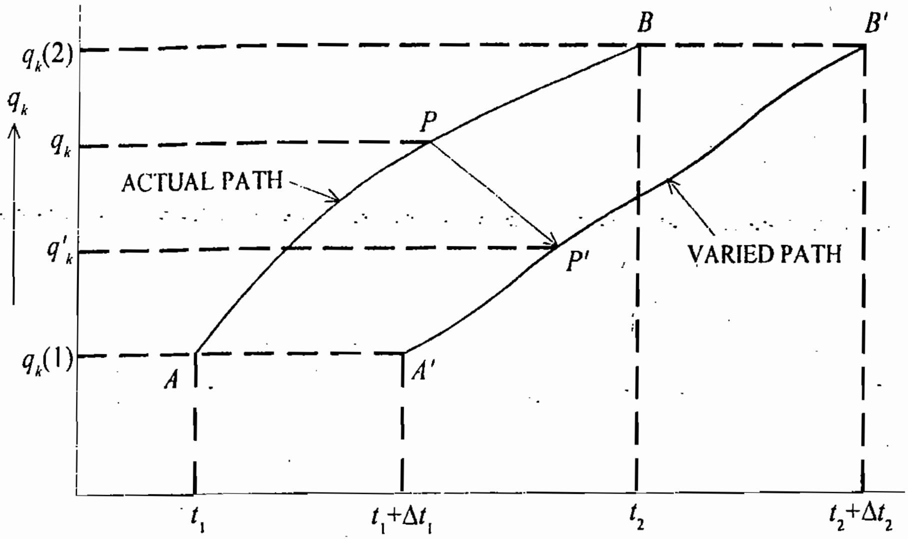

Fig. 5.10 : Δ-variation

$$q_k \rightarrow q'_k = q_k + \Delta q_k = q_k + \delta q_k + \dot{q}_k \Delta t \qquad ...(55)$$

where δ-variation has the same meaning as discussed earlier.

The Δ-variation of any function $f = f(q_k, \dot{q}_k, t)$ is given by

$$\Delta f = \sum_k \left[\frac{\partial f}{\partial q_k}\Delta q_k + \frac{\partial f}{\partial \dot{q}_k}\Delta \dot{q}_k\right] + \frac{\partial f}{\partial t}\Delta t = \sum_k \frac{\partial f}{\partial q_k}[\delta q_k + \dot{q}_k \Delta t] + \sum_k \frac{\partial f}{\partial \dot{q}_k}[\delta \dot{q}_k + \ddot{q}_k \Delta t] + \frac{\partial f}{\partial t}\Delta t$$

$$= \sum_k \left[\frac{\partial f}{\partial q_k}\delta q_k + \frac{\partial f}{\partial \dot{q}_k}\delta \dot{q}_k\right] + \left[\sum_k \left(\frac{\partial f}{\partial q_k}\dot{q}_k + \ddot{q}_k \frac{\partial f}{\partial \dot{q}_k}\right) + \frac{\partial f}{\partial t}\right]\Delta t$$

$$= \delta f + \Delta t \frac{df}{dt} \qquad ...(56)$$

Thus the Δ-operation is

$$\Delta = \delta + \Delta t \frac{d}{dt} \qquad ...(57)$$

5.11. PRINCIPLE OF LEAST ACTION

According to the principle of least action for a conservative system

$$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad ...(58)^*$$

* In the older literature, the integral in eq. (58) is usually called as action or action integral. The integral in Hamilton's principle is referred as action.

where the quantity $W = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt$ is sometimes called ***abbreviated action.***

Eq. (58) was established by Maupertuis (1668-1759) and therefore it is usually referred ***Maupertuis principle of least action.***

Proof : Let us consider Hamilton's principle function (or action integral) S, given by

$$S = \int_{t_1}^{t_2} L\, dt \qquad ...(59)$$

The Δ-variation of S is

$$\Delta S = \Delta \int_{t_1}^{t_2} L\, dt = \left[\delta + \Delta t \frac{d}{dt}\right] \int_{t_1}^{t_2} L\, dt$$

$$= \delta \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2} \Delta t\, d(L) = \delta \int_{t_1}^{t_2} L\, dt + [L\, \Delta t]_{t_1}^{t_2} = \int_{t_1}^{t_2} \delta L\, dt + [L \Delta t]_{t_1}^{t_2} \; [\because \delta(dt) = 0]$$

$$= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\right] dt + [L\, \Delta t]_{t_1}^{t_2} \qquad ...(60)$$

In the present case $\delta q_k \neq 0$ at the end points, hence $\delta \int_{t_1}^{t_2} L\, dt$ is not equal to zero. Now, according to Lagrange's equations, we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \text{ or } \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] \qquad ...(61)$$

Also
$$\delta \dot{q}_k = \frac{d}{dt}[\delta q_k] \qquad ...(62)$$

Using (61) and (62), the quantity in the first term of eq. (60) is

$$\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}[\delta q_k] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right] = \frac{d}{dt}[p_k\, \delta q_k] \qquad ...(63)$$

But in view of eq. (57)

$$\Delta q_k = \delta q_k + \Delta t \frac{dq_k}{dt} \text{ or } \delta q_k = \Delta q_k - \Delta t\, \dot{q}_k \text{ or } p_k \delta q_k = p_k \Delta q_k - p_k \dot{q}_k \Delta t \qquad ...(64)$$

Hence
$$\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \frac{d}{dt}[p_k \Delta q_k] - \frac{d}{dt}[p_k \dot{q}_k \Delta t] \qquad ...(65)$$

Thus equation (60) is

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt}[p_k \Delta q_k] - \frac{d}{dt}[p_k\, \dot{q}_k \Delta t]\right] dt + [L\, \Delta t]_{t_1}^{t_2}$$

$$= \sum_k \int_{t_1}^{t_2} [d\,(p_k \Delta q_k) - d\,(p_k \dot{q}_k \Delta t)] + [L\, \Delta t]_{t_1}^{t_2}$$

$$= \sum_k [p_k \Delta q_k]_{t_1}^{t_2} - \sum_k [p_k \dot{q}_k \Delta t]_{t_1}^{t_2} + [L\, \Delta t]_{t_1}^{t_2} \qquad ...(66)$$

As $\Delta q_k = 0$ at the end points, $[p_k \Delta q_k]_{t_1}^{t_2} = 0$.

Therefore equation (66) is

$$\Delta \int_{t_1}^{t_2} L dt = \left[\left(L - \sum_k p_k \dot{q}_k \right) \Delta t \right]_{t_1}^{t_2}$$

or $$\Delta \int_{t_1}^{t_2} L\, dt = -[H \Delta t]_{t_1}^{t_2} \qquad \left[\because H = \sum_k p_k \dot{q}_k - L \right] \qquad ...(67)$$

Now, if we restrict to systems for which $\partial H/\partial t = 0$ and to variations for which H remains constant (conservative systems), then

$$\Delta \int_{t_1}^{t_2} H\, dt = \int_{t_1}^{t_2} H d(\Delta t) = [H \Delta t]_{t_1}^{t_2} \qquad ...(68)$$

Substituting for $[H\Delta t]_{t_1}^{t_2}$ in eq. (67), we get

$$\Delta \int_{t_1}^{t_2} L\, dt = -\Delta \int_{t_1}^{t_2} H\, dt \quad \text{or} \quad \Delta \int_{t_1}^{t_2} [H + L]\, dt = 0$$

or $$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad \left[\because H = \sum_k p_k \dot{q}_k - L \right] \qquad ...(69)$$

This is what is known as ***principle of least action.***

The quantity $\int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = W$ is called ***Hamilton's characteristic function***. Hence the principle of least action can be stated as

$$\Delta W = \Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad ...(70)$$

5.12. OTHER FORMS OF PRINCIPLE OF LEAST ACTION

(1) For a conservative system, the Hamiltonian is constant and the potential energy is independent of time. So that

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$$

Therefore, $$\sum_k p_k \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T \qquad ...(71)$$

because $\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$ [eq. (30), Chapter 3].

Therefore the principle of least action (70) takes the form

$$\Delta \int_{t_1}^{t_2} 2T\, dt = 0 \quad \text{or} \quad \Delta \int_{t_1}^{t_2} T\, dt = 0 \qquad ...(72)$$

This is another form of principle of least action.

In case, if there is not any external force on the system, its kinetic energy T as well as total energy H will be conserved. Then the principle of least action (72) takes a special form, given by

$$\Delta \int_{t_1}^{t_2} dt = 0 \quad \text{or} \quad \Delta(t_2 - t_1) = 0 \qquad \text{...(73a)}$$

or $\qquad t_2 - t_1 = \text{an extremum.} \qquad$ ...(73b)

Thus we see that out of all possible paths between two points, the system moves along that particular path for which the time of transit is an extremum, provided that the kinetic energy along with total energy of the system remains constant. This form of principle of least action is the same as Fermat's principle in geometrical optics, which states that ***a ray of light travels between two points along such a path that the time taken is the extremum.***

(2) Jacobi's form of the principle of least action : When transformation equations do not involve time, the kinetic energy of a system can be expressed as a homogeneous quadratic function of the velocities [eq. (39), Chapter 2], *i.e.*,

$$T = \frac{1}{2} \sum_{kl} M_{kl}\, \dot{q}_k\, \dot{q}_l \qquad \text{...(74)}$$

We can construct a configuration space for which M_{kl} coefficients form the metric tensor and the element of path length $d\rho$ in this space is defined as

$$d\rho^2 = \sum_{k,l} M_{kl}\, dq_k\, dq_l \qquad \text{...(75)*}$$

So that $$[d\rho/dt]^2 = \sum_{k,l} M_{kl}\, \dot{q}_k\, \dot{q}_l \qquad \text{...(76)}$$

From eqs. (74) and (76), we get

$$T = \frac{1}{2}\left(\frac{d\rho}{dt}\right)^2 \qquad \text{...(77)}$$

whence $$dt = d\rho / \sqrt{2T} \qquad \text{...(78)}$$

Hence the principle of least action (72) is

$$\Delta \int_{t_1}^{t_2} T dt = \Delta \int_{t_1}^{t_2} \sqrt{2T} d\rho = 0 \qquad \text{...(79)}$$

But $H = T + V(q)$, total energy is constant for conservative system. Thus, the principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} \sqrt{2[H - V(q)]}\, d\rho = 0 \qquad \text{...(80)}$$

This is known as ***Jacobi's form of the least action principle***. This form of principle of least action is related with the path of the system point (in a curvilinear configuration space described by the metric tensor with elements M_{kl}) rather than with its motion in time.

(3) Principle of least action in terms of arc length of the particle trajectory : If the system contains only one particle of mass m, its kinetic energy is given by

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* This is similar to Riemannian space in which the path length is given by $ds^2 = \sum_{k,l} g_{k,l}\, dx_k\, dx_l$, where g_{kl} is the element of metric tensor.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{ds}{dt}\right]^2 \qquad ...(81)$$

where ds is the element of arc traversed by the particle in time dt.

From eq. (81), we obtain

$$dt = \sqrt{m/2T}\, ds$$

So that the principle of least action (72) can be written as

$$\Delta\int 2T\sqrt{\frac{m}{2T}}\, ds = 0 \text{ or } \Delta\int \sqrt{2mT}\, ds = 0$$

or

$$\Delta\int \sqrt{2m\,(H-V)}\, ds = 0 \qquad ...(82)$$

This equation represents the principle of least action in terms of arc length of the particle trajectory. Eq. (82) is similar to the Jacobi's form of the principle of least action.

Questions

1. What is δ-variation ? Show that the integral $I = \int_{x_1}^{x_2} f(y, y', x)\, dx$ is stationary, when $\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial y} = 0$ where $y' = \frac{dy}{dx}$.
2. Show that for a function $f = f(y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n, x)$, the integral
$$I = \int_{x_1}^{x_2} f\, dx \text{ will be extremum, if } \frac{d}{dx}\left(\frac{\partial f}{\partial y'_k}\right) - \frac{\partial f}{\partial y_k} = 0,$$
where $k = 1, 2, \ldots, n$ and $y_k' = dy_k/dx$. **(Meerut 1980)**
3. Obtain the Euler-Lagrange differential equation by variational method. **(Kanpur 2003)**
4. State Hamilton's principle and derive Lagrange's equations of motion from it. Discuss how the result will be modified if the forces are conservative. **(Agra 1992, 74 ; Meerut 81)**
5. Show that the path followed by a particle in sliding from one point to an other in the absence of friction in the shortest time is cycloid. **(Agra 1991, 89)**
6. What is variational principle ? Obtain Hamiltons equations from variational principle. **(Meerut 2001; Kanpur 1998; Rohilkhand 1999)**
7. What is meant by variational principle ? Prove that the equation of curve for which surface area of revolution is minimum, is a catenary $x = a\cosh(y-b)/a$ where a and b are constants. **(Meerut 1981)**
8. Deduce Hamilton's principle from D'Alembert's principle. Derive Lagrange's equations from it. **(Garwal 1999; Agra 99, 90)**
9. Derive the Euler-Lagrange's equations of motion using the calculus of variations and hence obtain Lagrange's equations of motion for a system of particles. **(Meerut 1999, 83; Agra 77)**
10. Derive Hamilton's equations from the variational principle. Deduce Hamilton's principle. How can this principle be used to find the equation of one-dimensional harmonic oscillator ? **(Agra 1988)**
11. Explain the method of Lagrange's undetermined multipliers in deriving the equation of motion for a conservative non-holonomic system from Hamilton's principle. Apply this method to the problem of a hoop rolling down an inclined plane without slipping. **(Agra 1991, 88)**

12. State and explain modified Hamilton's principle. Deduce Hamilton's equations by using this principle. **(Agra 1998, 97)**
13. What is Δ-variation ? Discuss how it differs from δ-variation. State and prove the principle of least action.
14. State and prove the principle of least action. **(Agra 2001, 1991, 90, Kanpur 98; Garwal 93)**
15. State Hamilton's principle of least action. Obtain. Hamilton's equations of motion from this principle. **(Gorakhpur 1996)**
16. (*a*) Deduce the principle of least action in the following form :

$$\Delta \int_{t_1}^{t_2} T\,dt = 0$$

where T is the kinetic energy. **(Kanpur 2002)**

(*b*) If the kinetic energy of the system is conserved, show that out of all the paths between two points, the system moves along that particular path for which the time of transit is an extremum.

17. Describe the principle of least action and deduce the Jacobi's form of the principle of least action. **(Rohilkhand 1999)**

Problems

[SET- I]

1. State and prove the brachistochrone problem. **(Kanpur 2001, 1998; Meerut 1991)**
2. A fixed volume of water is rotating in a cylinder with constant angular velocity. Find the curve of the water surface that will minimize the total potential energy of the water in the combined gravitational centrifugal field.

   Ans : Parabola.
3. Prove that the shortest distance between the points on the surface of a sphere is the arc of the great circle connecting them. **(Rohilkhand 1986)**
4. Find the plane curve of fixed peimeter and maximum area.

   Ans : Circle.
5. Apply variational principle to show that the path of projectile is parabola.
6. Use the variational principle to show that the shortest distance between two points in space is a straight line joining them. **(Meerut 1992)**
7. Apply the variational principles to deduce the equation for stable equilibrium configuration of a uniform heavy flexible string fixed between two points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ in the constant gravity field of the earth [Fig. 5.11].

   Ans : $y = a \cosh (b + x/a)$, catenary; a and b are fixed by the coordinates of the two ends *i.e.*, (x_1, y_1) and (x_2, y_2).

   [**Hint :** The condition of the minimum potential energy can be expressed as $\delta \int_A^B mg\, y\, ds = 0$, where $m\, ds$ is the mass of an element of length ds which is at a vertical height y. But $ds = dx\sqrt{1 + y'^2}$, therefore $\delta \int_A^B y\sqrt{1 + y'^2}\, dx = 0$.

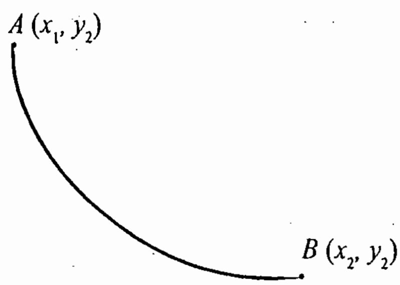

Fig. 5.11

Take $f = y\sqrt{1+y'^2}$ and apply Euler-Lagrange's equation.]

8. A particle moves on the frictionless inner surface of a cone of half angle α under the influence of gravity. Obtain the equations of motion.

9. A curve AB, having end points $A(x_1, y_1)$ and $B(x_2, y_2)$, is revolved about X-axis so that the area of the surface of revolution is a minimum. [Fig. 5.12]. Show that

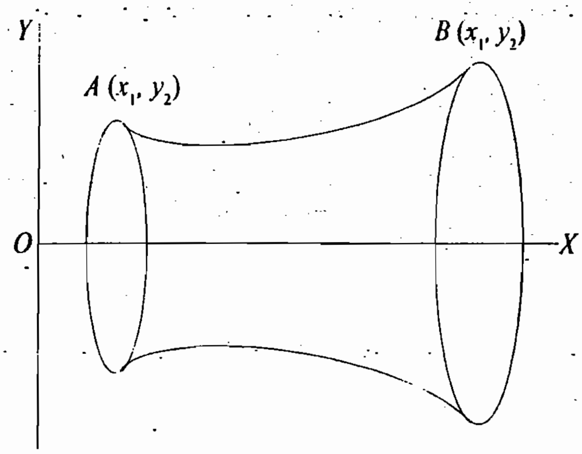

Fig. 5.12.

$$S = 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

Obtain the differential equation of the curve and prove that the curve represents a catenary.

Ans : $yy'' - y'^2 - 1 = 0$.

10. Two identical circular wires in contact are placed in a soap solution and then they are separated, resulting in the formation of a soap film. Explain why theshape of the surface of the soap film is related to the result of the above problem.

11. Discuss the motion of a disc that is rolling down an inclined plane without slipping. Find the acceleration and the force of constraint by using the method of undetermined multipliers.

Ans : $a = \frac{2}{3} g \sin\theta$; $\lambda = -\frac{1}{3} Mg \sin\theta$.

12. A sphere of radius a and mass m rests on the top of a fixed rough sphere of radius b. The first sphere is slightly displaced so that it rolls without slipping [Fig. 5.13]. Obtain the equation of motion for the rolling sphere.

Ans : $\ddot{\phi} = -\dfrac{5g}{7(a+b)}\sin\phi$

[**Hint :** $bd\phi = ad\psi$ or $bd\phi - ad\psi = 0$

Lagrangian of the rolling sphere is

$$L = \frac{1}{2}m(a+b)^2\dot{\phi}^2 + \frac{1}{2}I\omega^2 - mg(a+b)\cos\phi \text{ where } \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}ma^2\right)(\dot{\phi}+\dot{\psi})^2$$

Now, use $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\phi}}\right) - \dfrac{\partial L}{\partial\phi} = \lambda b$ and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\psi}}\right) - \dfrac{\partial L}{\partial\psi} = -\lambda a$.]

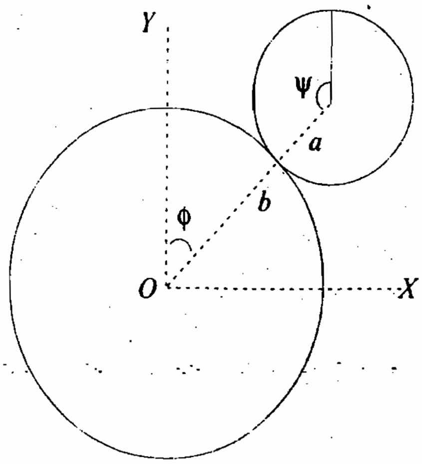

Fig. 5.13.

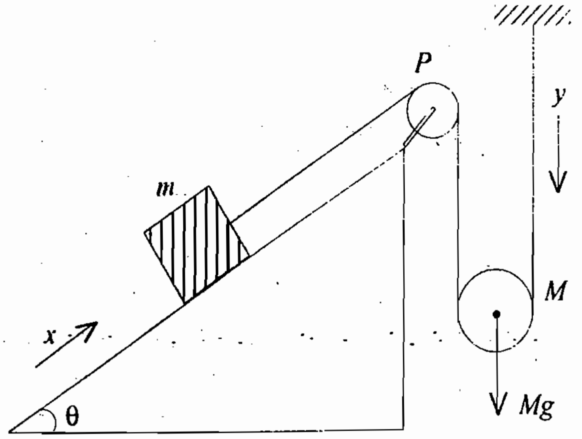

Fig. 5.14.

13. A block of mass m is pulled up as mass M moves down [Fig. 5.14]. The coefficient of friction between m and the incline is μ. Assume the pulley P to be smooth and the string inextensible. Use the Lagrange's method of undetermined multipliers to find the accelerations of m and M.

Ans : $\ddot{x} = 2\ddot{y}$; $\ddot{y} = (Mg - 2\mu\, mg\cos\theta - 2mg\sin\theta)/4(m+M)$

[**Hint :** $2dy = dx$ or $-dx + 2dy = 0$.]

[SET- II]

1. The Lagrangian of a free particle is given in the form

$$L = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m\left(\frac{ds}{dt}\right)^2 = \tfrac{1}{2} mg_{ik}\left(\frac{dx_i}{dt}\right)\left(\frac{dx_k}{dt}\right).$$

Use Lagrange's equations of motion to show that $\ddot{x}_i = \lambda_{ijk}\,\dot{x}_j\,\dot{x}_k = 0$

become the equations of motion, where $\lambda_{ijk} = \frac{1}{2} g_{il}\left[\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right]$ are the so called Riemann-Christoffel symbols. In the above equations, Einstein's summation convention for indices has been used.

2. Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume.

Objective Type Questions

1. Choose the correct statement/statements :

(*a*) In δ-variation, time as well as position coordinates are allowed to vary.

(*b*) In Δ-variation, time as well as position coordinates are allowed to vary.

(*c*) δ-variation does not involve time.

(*d*) Δ-variation does not involve time.

Ans : (*b*), (*c*).

2. In case of modified Hamilton's principle,

(*a*) the path refers to configuration space. (*b*) the path refers to phase space.

(*c*) $\int_{t_1}^{t_2} p_j \, .dq_j - \delta \int_{t_1}^{t_2} H\, dq_j = 0$ (*d*) $\delta\int_{t_2}^{t_1} (\sum_k p_k \dot{q}_k - H)dt = 0$

where the terms have usual meaning.

Ans : (*b*), (*d*).

3. According to the principle of least action

(*a*) $\Delta\int(\sum_k p_k\dot{q}_k - H)dt = 0$ (*b*) $\Delta\int\sum_k p_k\dot{q}_k\, dt = 0$

(*c*) $\Delta\int(H + L)\, dt = 0$ (*d*) $\int\sum_k p_k\dot{q}_k\, dt = 0$

Ans : (*b*), (*c*) .

4. The modified Hamilton's principle is given by

(*a*) $\delta\sum_j\int_{t_1}^{t_2} p_j dq_j - \delta\int_{t_1}^{t_2} H dt = 0$ (*b*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j + \int_{t_1}^{t_2} H\, dt = 0$

(*c*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j - \delta\int_{t_1}^{t_2} H\, dq_j = 0$ (*d*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j + \delta\int_{t_1}^{t_2} H\, dq_j = 0$

(Kanpur 2003)

Ans : (*a*) .

Short Answer Questions

1. What is δ-variation
2. Obtain the Euler-Lagrange differential equation by a variational method. **(Kanpur 2001)**
3. What is Brachistochrone problem. **(Kanpur 2003)**
4. What is extended Hamilton's principle ?
5. Using Hamilton's principle, obtain the modified Hamilton's principle. **(Kanpur 2001)**
6. What is Δ-variation ? Discuss how it differs from δ-variation. **(Kanpur 2002)**
7. Explain the principle of least action. **(Agra 2003, 02)**
8. Fill in the blanks :

   (*i*) Hamilton's principal function is....................

   (*ii*) The Δ - operation is $\Delta = \delta +$....................

Ans. (*i*) $S = \int_{t_1}^{t_2} L\, dt$, (*ii*) $\Delta t\,\dfrac{d}{dt}$