

Partial Differential Equations

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17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z , the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables. The method is best illustrated by the following examples :

Example 17.1. Derive a partial differential equation (by eliminating the constants) from the equation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \quad \dots(i)$$

Solution. Differentiating (i) partially with respect to x and y , we get

$$2 \frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x} \frac{\partial z}{\partial x} = \frac{p}{x}$$

$$\text{and} \quad \frac{2 \partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y} \frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (i), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. Form the partial differential equations (by eliminating the arbitrary functions) from

(a) $z = (x + y) \phi(x^2 - y^2)$

(P.T.U., 2009)

(b) $z = f(x + at) + g(x - at)$ (V.T.U., 2009)

(c) $f(x^2 + y^2, z - xy) = 0$

(S.V.T.U., 2007)

Solution. (a) We have $z = (x + y) \phi(x^2 - y^2)$

Differentiating z partially with respect to x and y ,

$$p = \frac{\partial z}{\partial x} = (x + y) \phi'(x^2 - y^2) \cdot 2x + \phi(x^2 - y^2), \quad \dots(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y) \phi'(x^2 - y^2) \cdot (-2y) + \phi(x^2 - y^2) \quad \dots(ii)$$

From (i), $p - \frac{z}{x + y} = 2x(x + y) \phi'(x^2 - y^2)$

From (ii), $q - \frac{z}{x + y} = -2y(x + y) \phi'(x^2 - y^2)$

Division gives $\frac{p - z/(x + y)}{q - z/(x + y)} = -\frac{x}{y}$

i.e., $[p(x + y) - z]y + [q(x + y) - z]x$
i.e., $(x + y)(py + qx) - z(x + y) = 0$

Hence $py + qz = z$ is required equation.

(b) We have $z = f(x + at) + g(x - at)$...(i)

Differentiating z partially with respect to x and t ,

$$\frac{\partial z}{\partial x} = f'(x + at) + g'(x - at), \quad \frac{\partial^2 z}{\partial x^2} = f''(x + at) + g''(x - at) \quad \dots(ii)$$

$$\frac{\partial z}{\partial t} = af'(x + at) - ag'(x - at), \quad \frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at) = a^2 \frac{\partial^2 z}{\partial x^2} \quad [\text{By (ii)}]$$

Thus the desired partial differential equation is $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (i) is its solution.

(c) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f(u, v) = 0$.

Differentiating partially w.r.t. x and y , we have

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0$$

or $\frac{\partial f}{\partial u} (2x) + \frac{\partial f}{\partial v} (-y + p) = 0 \quad \dots(i)$

and $\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u} (2y) + \frac{\partial f}{\partial v} (-x + q) = 0 \quad \dots(ii)$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (i) and (ii), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 17.3. Find the differential equation of all planes which are at a constant distance a from the origin. (V.T.U., 2009 S ; Kurukshetra, 2006)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \quad \dots(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $l^2 + m^2 + n^2 = 1$ or $n = \sqrt{1 - l^2 - m^2}$

\therefore (i) becomes $lx + my + \sqrt{1 - l^2 - m^2} z = a$... (ii)

Differentiating partially w.r.t. x , we get

$$l + \sqrt{1 - l^2 - m^2} \cdot p = 0 \quad \dots (iii)$$

Differentiating partially w.r.t. y , we get

$$m + \sqrt{1 - l^2 - m^2} \cdot q = 0 \quad \dots (iv)$$

Now we have to eliminate l, m from (ii), (iii) and (iv).

From (iii), $l = -\sqrt{1 - l^2 - m^2} \cdot p$ and $m = -\sqrt{1 - l^2 - m^2} \cdot q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

$$\text{or} \quad (l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2 \text{ or } 1 - l^2 - m^2 = 1 - \frac{p^2 + q^2}{1 + p^2 + q^2} = \frac{1}{1 + p^2 + q^2}$$

$$\text{Also} \quad l = -\frac{p}{\sqrt{1 + p^2 + q^2}} \text{ and } m = -\frac{q}{\sqrt{1 + p^2 + q^2}}$$

Substituting the values of l, m and $1 - l^2 - m^2$ in (ii), we obtain

$$\frac{-px}{\sqrt{1 + p^2 + q^2}} - \frac{qy}{\sqrt{1 + p^2 + q^2}} + \frac{1}{\sqrt{1 + p^2 + q^2}} z = a$$

$$\text{or} \quad z = px + qy + a \sqrt{1 + p^2 + q^2} \text{ which is the required partial differential equation.}$$

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

$$1. z = ax + by + a^2 + b^2. \quad 2. (x - a)^2 + (y - b)^2 + z^2 = c^2. \quad (\text{Kottayam, 2005})$$

$$3. (x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha \quad (\text{Anna, 2009}) \quad 4. z = a \log \left\{ \frac{b(y - 1)}{1 - x} \right\} \quad (\text{J.N.T.U., 2002 S})$$

5. Find the differential equation of all spheres of fixed radius having their centres in the xy -plane. (Madras 2000 S)

6. Find the differential equation of all spheres whose centres lie on the z -axis. (Kerala, 2005)

Form the partial differential equations (by eliminating the arbitrary functions) from :

$$7. z = f(x^2 - y^2) \quad (\text{S.V.T.U., 2008}) \quad 8. z = f(x^2 + y^2) + x + y \quad (\text{Anna, 2009})$$

$$9. z = yf(x) + xg(y). \quad (\text{V.T.U., 2004}) \quad 10. z = x^2 f(y) + y^2 g(x). \quad (\text{Anna, 2003})$$

$$11. z = f(x) + e^y g(x). \quad 12. xyz = \phi(x + y + z).$$

$$13. z = f_1(x) f_2(y). \quad 14. z = e^{my} \phi(x - y). \quad (\text{P.T.U., 2002})$$

$$15. z = y^2 + 2f\left(\frac{1}{x} + \log y\right). \quad (\text{V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000})$$

$$16. z = f_1(y + 2x) + f_2(y - 3x). \quad (\text{Kurukshetra, 2005}) \quad 17. v = \frac{1}{r} [f(r - at) + F(r + at)].$$

$$18. z = xf_1(x + t) + f_2(x + t). \quad 19. F(xy + z^2, x + y + z) = 0. \quad (\text{V.T.U., 2006})$$

$$20. F(x + y + z, x^2 + y^2 + z^2) = 0. \quad (\text{S.V.T.U., 2007})$$

$$21. \text{ If } u = f(x^2 + 2yz, y^2 + 2zx), \text{ prove that } (y^2 - zx) \frac{\partial u}{\partial x} + (x^2 - yz) \frac{\partial u}{\partial y} + (z^2 - xy) \frac{\partial u}{\partial z} = 0.$$

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $f(x, y, z, a, b) = 0$... (1)

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a **particular integral**.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ , which is called the **general integral**.

The envelope of the family of surfaces (1), with parameters a and b , if it exists, is called a **singular integral**. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

(V.T.U., 2010)

Solution. Integrating twice with respect to x (keeping y fixed),

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} + 9x^2 y^2 - \frac{1}{2} \cos(2x - y) &= f(y) \\ \frac{\partial z}{\partial y} + 3x^3 y^2 - \frac{1}{4} \sin(2x - y) &= xf(y) + g(y).\end{aligned}$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3 y^3 - \frac{1}{4} \cos(2x - y) = x \int f(y) dy + \int g(y) dy + w(x)$$

The result may be simplified by writing

$$\int f(y) dy = u(y) \text{ and } \int g(y) dy = v(y).$$

Thus $z = \frac{1}{4} \cos(2x - y) - x^3 y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A \sin x + B \cos x$, where A and B are constants. Since z is a function of x and y , A and B can be arbitrary functions of y . Hence the solution of the given equation is $z = f(y) \sin x + \phi(y) \cos x$

$$\therefore \frac{\partial z}{\partial x} = f(y) \cos x - \phi(y) \sin x$$

$$\text{When } x = 0; z = e^y, \quad \therefore e^y = \phi(y). \quad \text{When } x = 0, \quad \frac{\partial z}{\partial x} = 1, \quad \therefore 1 = f(y).$$

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, for which $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\pi/2$.

(V.T.U., 2010 S)

Solution. Given equation is $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x , keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \dots(i)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$, $\therefore -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

\therefore (i) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y , keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \quad \dots(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$$\therefore 0 = 0 + 0 + g(x) \text{ or } g(x) = 0$$

$$[\because \cos(2n+1)\pi/2 = 0]$$

Hence from (ii), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

2. $\frac{\partial^2 z}{\partial x^2} = xy$

3. $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$

4. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

5. $\frac{\partial^3 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

6. $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as **Lagrange's Linear equation***, is of the form

$$Pp + Qq = R \quad \dots(1)$$

where P , Q and R are functions of x , y , z . This equation is called a **quasi-linear equation**. When P , Q and R are independent of z it is known as **linear equation**.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$

$\dots(2)$

where u , v are some functions of x , y , z .

Differentiating (2) partially with respect to x and y .

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0 \text{ and } \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get
$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \end{vmatrix} = 0$$

which simplifies to
$$\left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \right) p + \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right) q = \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \quad \dots(3)$$

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a , b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}}$$

$$\text{or} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots(4) \text{ [By virtue of (1) and (3)]}$$

The solutions of these equations are $u = a$ and $v = b$.

$\therefore \phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(i) form the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(ii) solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.

(iii) write the complete solution as $\phi(u, v) = 0$ or $u = f(v)$.

Example 17.7. Solve $\frac{y^2 z}{x} p + xzq = y^2$.

(Kottayam, 2005)

Solution. Rewriting the given equation as

$$y^2 zp + x^2 zq = y^2 x,$$

The subsidiary equations are $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$

The first two fractions give $x^2 dx = y^2 dy$.

Integrating, we get $x^3 - y^3 = a$...(i)

Again the first and third fractions give $x dx = z dz$

Integrating, we get $x^2 - z^2 = b$...(ii)

Hence from (i) and (ii), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. Solve $(mz - ny) \frac{\partial z}{\partial x} + (nx - lz) \frac{\partial z}{\partial y} = ly - mx$.

(V.T.U., 2010 ; S.V.T.U., 2009)

Solution. Here the subsidiary equations are $\frac{dx}{mz - ny} = \frac{dy}{mx - lz} = \frac{dz}{ly - mx}$

Using multipliers x, y , and z , we get each fraction = $\frac{xdx + ydy + zdz}{0}$

$\therefore xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$...(i)

Again using multipliers l, m and n , we get each fraction = $\frac{ldx + mdy + ndz}{0}$

$\therefore ldx + mdy + ndz = 0$ which on integration gives $lx + my + nz = b$...(ii)

Hence from (i) and (ii), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. Solve $(x^2 - y^2 - z^2) p + 2xyq = 2xz$.

(V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008)

Solution. Here the subsidiary equations are $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y} = \frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$...(i)

Using multipliers x, y and z , we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \quad \therefore \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

which on integration gives $\log(x^2 + y^2 + z^2) = \log z + \log b$

$$\text{or} \quad \frac{x^2 + y^2 + z^2}{z} = b \quad \dots(ii)$$

Hence from (i) and (ii), the required solution is $x^2 + y^2 + z^2 = zf(y/z)$.

Example 17.10. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. (P.T.U., 2009; Bhopal, 2008; S.V.T.U. 2007)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \quad \dots(i)$$

Using the multipliers $\frac{1}{x^2}$, $\frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \quad \dots(ii)$$

Hence from (i) and (ii), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. (Bhopal, 2008; V.T.U., 2006; Madras, 2000)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \dots(i)$$

$$\text{Each of these equations} = \frac{dx - dy}{x^2 - y^2 - (y-x)z} = \frac{dy - dz}{y^2 - z^2 - x(z-y)}$$

$$\text{i.e.,} \quad \frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\text{Integrating,} \quad \log(x-y) = \log(y-z) + \log c \quad \text{or} \quad \frac{x-y}{y-z} = c \quad \dots(ii)$$

$$\text{Each of the subsidiary equations (i)} = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)} \quad \dots(iii)$$

$$\text{Also each of the subsidiary equations} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \quad \dots(iv)$$

Equating (iii) and (iv) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $\int (xdx + ydy + zdz) = \int (x + y + z)d(x + y + z) + c'$

or $x^2 + y^2 + z^2 = (x + y + z)^2 + 2c'$ or $xy + yz + zx + c' = 0$... (v)

Combining (ii) and (v), the general solution is

$$\frac{x - y}{y - z} = f(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos(x + y) + q \sin(x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log(x + y)$.
8. $xp - yq = y^2 - x^2$ (J.N.T.U., 2002 S)
9. $(y + z)p - (z + x)q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$ (Bhopal, 2007)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$ (V.T.U., 2010 ; Anna, 2008)
12. $y^2p - xyq = x(z - 2y)$ (S.V.T.U., 2008)
13. $(y^2 + z^2)p - xyq + zx = 0$ (P.T.U., 2009 ; V.T.U., 2009)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$ (Kerala, 2005)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (i.e., equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. $f(p, q) = 0$, i.e., equations containing p and q only.

Its complete solution is $z = ax + by + c$... (1)

where a and b are connected by the relation $f(a, b) = 0$... (2)

[Since from (1), $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, which when substituted in (2) give $f(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. Solve $p - q = 1$. (Anna, 2009)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. Solve $x^2p^2 + y^2q^2 = z^2$. (Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \quad \dots (i)$$

and setting $\frac{dx}{x} = du, \frac{dy}{y} = dv, \frac{dz}{z} = dw$ so that $u = \log x, v = \log y, w = \log z$.

Then (i) becomes $\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$

i.e., $P^2 + Q^2 = 1$ where $P = \frac{\partial w}{\partial u}$ and $Q = \frac{\partial w}{\partial v}$.

Its complete solution is $w = au + bv + c$

...(ii)

where $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$.

\therefore (ii) becomes $w = au + \sqrt{1 - a^2}v + c$

or $\log z = a \log x + \sqrt{1 - a^2} \log y + c$ which is the required solution.

Form II. $f(z, p, q) = 0$, i.e., equations not containing x and y .

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$\therefore p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$ $q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a \frac{dz}{du}$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{dz}{du}, a \frac{dz}{du}\right) = 0 \text{ which is an ordinary differential equation of the first order.}$$

Rewriting it as $\frac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$$F(z, a) = u + b, \text{ or } x + ay + b = F(z, a) \text{ which is the desired complete solution.}$$

Thus to solve $f(z, p, q) = 0$,

(i) assume $u = x + ay$ and substitute $p = dz/du$, $q = a dz/du$ in the given equation;

(ii) solve the resulting ordinary differential equation in z and u ;

(iii) replace u by $x + ay$.

Example 17.14. Solve $p(1 + q) = qz$.

(Madras, 2000 S)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du} \left(1 + a \frac{dz}{du}\right) = az \frac{dz}{du} \text{ or } a \frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a dz}{az - 1} = \int du + b$$

or $\log(az - 1) = u + b$ or $\log(az - 1) = x + ay + b$

which is the required complete solution.

Example 17.15. Solve $q^2 = z^2 p^2 (1 - p^2)$.

(J.N.T.U., 2005 ; Kerala, 2005)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a \frac{dz}{du} \text{ and } q = \frac{\partial z}{\partial y} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

\therefore The given equation becomes $\left(\frac{dz}{du}\right)^2 = a^2 z^2 \left(\frac{dz}{du}\right)^2 \left\{1 - a^2 \left(\frac{dz}{du}\right)^2\right\}$...(i)

or $a^4 z^2 \left(\frac{dz}{du}\right)^2 = a^2 z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{a^2 z^2 - 1}}{a^2 z}$

Integrating, $\int \frac{a^2 z}{\sqrt{a^2 z^2 - 1}} dz = \int du + c \quad \text{or} \quad (a^2 z^2 - 1)^{1/2} = u + c$

i.e., $a^2 z^2 = (y + ax + c)^2 + 1$ [$\because u = y + ax$]

The second factor in (i) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. Solve $z^2(p^2 x^2 + q^2) = 1$.

(Bhopal, 2008 S)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2 \left[\left(x \frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1 \quad \dots(i)$$

Putting $X = \log x$, so that $x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1 \quad \dots(ii)$$

Let $u = X + ay$ and put $\frac{\partial z}{\partial X} = \frac{dz}{du}$ and $\frac{\partial z}{\partial y} = a \frac{dz}{du}$ in (ii), so that

$$z^2 \left[\left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 \right] = 1 \quad \text{or} \quad \sqrt{(1+a^2)} z dz = \pm du$$

Integrating, $\sqrt{(1+a^2)} z^2 = \pm 2u + b = \pm 2(X + ay) + b$

or $z^2 \sqrt{(1+a^2)} = \pm 2(\log x + ay) + b$

which is the complete solution required.

Form III. $f(x, p) = F(y, q)$, i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q .

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p , we get $p = \phi(x)$

and solving for q , we get $q = \psi(y)$

Since $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy$

$\therefore dz = \phi(x) dx + \psi(y) dy$

Integrating, $z = \int \phi(x) dx + \int \psi(y) dy + b$

which is the desired complete solution containing two constants a and b .

Example 17.17. Solve $p^2 + q^2 = x + y$.

(Bhopal, 2006; Madras, 2003)

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore p^2 - x = a$ gives $p = \sqrt{(a+x)}$

and $y - q^2 = a$ gives $q = \sqrt{(y-a)}$

Substituting these values of p and q in $dz = p dx + q dy$, we get

$$dz = \sqrt{(a+x)} dx + \sqrt{(y-a)} dy$$

\therefore integrating gives, $z = \frac{2}{3} (a+x)^{3/2} + \frac{2}{3} (y-a)^{3/2} + b$

which is the required complete solution.

Example 17.18. Solve $z^2(p^2 + q^2) = x^2 + y^2$.

(Bhopal, 2008)

Solution. The equation can be reduced to the above form by writing it as

$$\left(z \frac{\partial z}{\partial x} \right)^2 + \left(z \frac{\partial z}{\partial y} \right)^2 = x^2 + y^2 \quad \dots(i)$$

and putting $z dz = dZ$, i.e., $Z = \frac{1}{2} z^2$

$\therefore \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = z \frac{\partial z}{\partial x} = P$

and $\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = z \frac{\partial z}{\partial y} = Q$

\therefore (i) becomes $P^2 + Q^2 = x^2 + y^2$
 or $P^2 - x^2 = y^2 - Q^2 = a$, say.
 $\therefore P = \sqrt{(x^2 + a)}$ and $Q = \sqrt{(y^2 - a)}$.
 $\therefore dZ = Pdx + Qdy$ gives
 $dZ = \sqrt{(x^2 + a)} dx + \sqrt{(y^2 - a)} dy$
 Integrating, we have $Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}]$
 $+ \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$
 or $z^2 = x \sqrt{(x^2 + a)} + y \sqrt{(y^2 - a)} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$

which is the required complete solution.

Example 17.19. Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

(Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = P + Q$
 and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = P - Q$, where $P = \frac{\partial z}{\partial u}$, $Q = \frac{\partial z}{\partial v}$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2u = 1 - 4Q^2v = a \quad (\text{say})$$

$$P = \pm \frac{1}{2} \sqrt{\frac{a}{u}}, \quad Q = \pm \frac{1}{2} \sqrt{\frac{1-a}{v}}$$

$\therefore dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$
 $= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1-a}}{2} \frac{dv}{\sqrt{v}}$

Integrating, we have

$$z = \pm \sqrt{a} \sqrt{u} \pm \sqrt{1-a} \sqrt{v} + b$$

or

$$z = \pm \sqrt{a(x+y)} \pm \sqrt{(1-a)(x-y)} + b$$

which is the required complete solution.

Form IV. $z = px + qy + f(p, q)$: an equation analogous to the Clairaut's equation (§ 11.14).

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. Solve $z = px + qy + \sqrt{(1 + p^2 + q^2)}$.

(Anna, 2009)

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

\therefore Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.

2. $p^2 + q^2 = 1$.

(Osmania, 2000)

3. $z = p^2 + q^2$. (Anna, 2005 S ; J.N.T.U., 2002 S)

4. $p(1 - q^2) = q(1 - z)$

(Anna, 2006)

5. $yp + xq + pq = 0$.

6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.

9. $p^2 + q^2 = x^2 + y^2$. (Osmania, 2003)

8. $\sqrt{p} + \sqrt{q} = x + y$.

10. $z = px + qy + \sin(x + y)$.

11. $\sqrt{p} + \sqrt{q} = 2x$. (J.N.T.U., 2006)

12. $z = px + qy - 2\sqrt{pq}$.

13. $(x - y)(px - qy) = (p - q)^2$. [Hint. Use $x + y = u$, $xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

Since z depends on x and y , we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy \quad \dots(2)$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ $\dots(3)$ then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ , we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} = 0 \quad \dots(4)$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial x} = 0 \quad \dots(5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial y} = 0 \quad \dots(6)$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial y} = 0 \quad \dots(7)$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial p} \right) + \left(\frac{\partial f}{\partial z} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial p} \right) p + \left(\frac{\partial f}{\partial q} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q} \frac{\partial f}{\partial p} \right) \frac{\partial q}{\partial x} = 0 \quad \dots(8)$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial q} \right) + \left(\frac{\partial f}{\partial z} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial q} \right) q + \left(\frac{\partial f}{\partial p} \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p} \frac{\partial f}{\partial q} \right) \frac{\partial p}{\partial y} = 0 \quad \dots(9)$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial \phi}{\partial q} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p} \right) \frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial y} = 0 \quad \dots(10)$$

$$\text{i.e.,} \quad \left(-\frac{\partial f}{\partial p} \right) \frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial y} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial \phi}{\partial q} = 0 \quad \dots(11)$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q .

Example 17.21. Solve $(p^2 + q^2)y = qz$.

(V.T.U., 2007 ; Hissar, 2005)

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$

...(i)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $p^2 + q^2 = c^2$

...(ii)

Now to solve (i) and (ii), put $p^2 + q^2 = c^2$ in (i), so that $q = c^2 y/z$

Substituting this value of q in (ii), we get $p = c \sqrt{(z^2 - c^2 y^2)/z}$

Hence $dz = p dx + q dy = \frac{c}{z} \sqrt{(z^2 - c^2 y^2)} dx + \frac{c^2 y}{z} dy$

or

$$z dz - c^2 y dy = c \sqrt{(z^2 - c^2 y^2)} dx \quad \text{or} \quad \frac{\frac{1}{2} d(z^2 - c^2 y^2)}{\sqrt{(z^2 - c^2 y^2)}} = c dx$$

Integrating, we get $\sqrt{(z^2 - c^2 y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2 y^2$ which is the required complete integral.

Example 17.22. Solve $2xz - px^2 - 2qxy + pq = 0$.

(Rajasthan, 2006)

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$

...(i)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore dq = 0$ or $q = a$.

Putting $q = a$ in (i), we get $p = \frac{2x(z - ay)}{x^2 - a}$

$\therefore dz = p dx + q dy = \frac{2x(z - ay)}{x^2 - a} dx + a dy$ or $\frac{dz - a dy}{z - ay} = \frac{2x}{x^2 - a} dx$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or

$$z - ay = b(x^2 - a) \quad \text{or} \quad z = ay + b(x^2 - a)$$

which is the required complete solution.

Example 17.23. Solve $2z + p^2 + qy + 2y^2 = 0$.

(J.N.T.U., 2005 ; Kurukshetra, 2005)

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y} [-2z - 2y^2 - (a - x)^2]$$

$$\therefore dz = p dx + q dy = (a - x) dx - \frac{1}{y} [2z + 2y^2 + (a - x)^2] dy$$

Multiplying both sides by $2y^2$,

$$2y^2 dz + 4yz dy = 2y^2 (a - x) dx - 4y^3 dy - 2y(a - x)^2 dy$$

Integrating $2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.

2. $z^2 = pqxy$.

(Anna, 2009 ; V.T.U., 2004)

3. $1 + p^2 = qz$.

4. $pxy + pq + qy = yz$.

(J.N.T.U., 2006 ; Kurukshetra, 2006)

5. $p(p^2 + 1) + (b - z)q = 0$.

6. $q + xp = p^2$.

(Osmania, 2003)

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \quad \dots(1)$$

in which k 's are constants, is called a *homogeneous linear partial differential equation of the n th order with constant coefficients*. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\frac{\partial^r}{\partial x^r} = D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1 D^{n-1} D' + D' + \dots + k_n D'^n) z = F(x, y)$

or briefly $f(D, D')z = F(x, y) \quad \dots(2)$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and the *particular integral*.

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \dots(1)$

which in symbolic form is $(D^2 + k_1 DD' + k_2 D'^2)z = 0 \quad \dots(2)$

Its symbolic operator equated to zero, i.e., $D^2 + k_1 DD' + k_2 D'^2 = 0$ is called the *auxiliary equation (A.E.)*

Let its root be $D/D' = m_1, m_2$.

Case I. If the roots be real and distinct then (2) is equivalent to

$$(D - m_1 D')(D - m_2 D')z = 0 \quad \dots(3)$$

It will be satisfied by the solution of

$$(D - m_2 D')z = 0, \text{ i.e., } p - m_2 q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2 x = a \text{ and } z = b.$$

\therefore its solution is $z = \phi(y + m_2 x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1 D')z = 0, \text{ i.e., by } z = f(y + m_1 x)$$

Hence the complete solution of (1) is $z = f(y + m_1 x) + \phi(y + m_2 x)$.