Lecture Note: Unit-II

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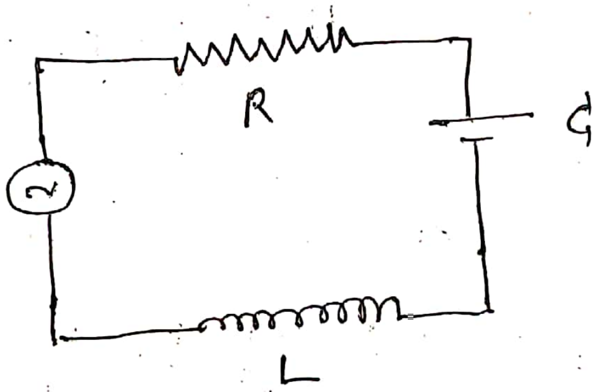
Model: IV — LCR Model —

* Problem statement
"Consider a series circuit consisting of an EMF source E, a resistor R, a capacitor C and a inductor L, Formulate a suitable differential Equation model and analyze it."



* Formulation of model:

Step-I (Identification of Variable)

Here the dependent variable is either the current i or the charge q and the independent variable is the time t.

Step-II (Assumption)

We assume that

(1) The characteristic parameters of the resistor, capacitor and inductor are constant

(ii) The flow of current in the closed circuit is given by the Kirchoff's voltage law

$$E_R + E_c + E_L = E \quad \text{—— (1)}$$

Step-III As we know that $E_R = R.i$, $E_L = L.\frac{di}{dt}$, $E_c = q/c$ and $i = \frac{dq}{dt}$

we get,

$$E_R + E_c + E_L = E$$

$$\Rightarrow R.i + q/c + L.\frac{di}{dt} = E \quad \text{—— (2)}$$

$$\Rightarrow L\frac{d^2q}{dt^2} + R.\frac{dq}{dt} + q/c = E \quad \text{—— (3)}$$

$$\left[\because i = \frac{dq}{dt}\right]$$

Again from eq. (2) we get,

$$R.\frac{di}{dt} + \frac{1}{c}\frac{dq}{dt} + L\frac{d^2i}{dt^2} = E' \quad \text{—— (4)}$$

$$\Rightarrow L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c}i = E' \quad \text{—— (5)}$$

$$\left[i = \frac{dq}{dt}\right]$$

Eq. (3) and (5) are second order differential equation with single dependent variable represent differential equation model for LCR network.

* **Analysis of Mathematical model:**

We analyze the LCR model in two steps. In Step-I we obtain the mathematical solution of the

Model and step-II the interpretation of the result obtained. ~~Due~~ Due to practical importance we consider following four cases.

(2)

Case-I LC current without voltage source

[$R=0$, $E(t)=0$]

Part (I)

From eq(5) we get,

$$L\frac{d^2i}{dt^2}+\frac{1}{c}i=0$$

$$\Rightarrow \frac{d^2i}{dt^2}+\frac{1}{LC}i=0$$

$$\Rightarrow \frac{d^2i}{dt^2}+\omega_0^2 i=0 \quad \text{——— (6)}$$

where $\omega_0^2=\frac{1}{LC}$

Solution of eq(10)

A.E. is

$$m^2+\omega_0^2=0$$

$$\Rightarrow m=\pm i\omega_0$$

Hence,

$$i(t)=(c_1\cos\omega_0 t+c_2\sin\omega_0 t) \quad \text{——— (7)}$$

Suppose $c_1=c\cos\alpha$, $c_2=c\sin\alpha$

$$\Rightarrow c=\sqrt{c_1^2+c_2^2}, \quad \alpha=\tan^{-1}\left(\frac{c_2}{c_1}\right)$$

$$\Rightarrow i(t)=c\cos\alpha\cos\omega_0 t+c\sin\alpha\sin\omega_0 t$$
$$=c[\cos\alpha\cos\omega_0 t+\sin\alpha\sin\omega_0 t)$$

$i(t)=c\cos(\omega_0 t-\alpha)$

Part-III

eq(8) described a simple harmonic motion of period T is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$$

It's frequency is $\frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$.

amplitude c and

phase angle α

Case-II LC - circuit with voltage source ($R=0$, $E(t) \neq 0$)

From eq(5) we get,

$$L\frac{d^2 i}{dt^2} + \frac{1}{C} i = E'(t)$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{1}{LC} i = \frac{E'(t)}{L}$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = \frac{E'(t)}{L} \quad \text{———} (9)$$

where $\omega_0^2 = \frac{1}{LC}$

Suppose $E(t) = E_0 \sin\omega t$, then eq(9) becomes

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = \frac{E_0 \cdot \omega \cos\omega t}{L}$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \omega_0^2 i = F_0 \cos\omega t, \quad \text{where } F_0 = \frac{E_0\omega}{L} \quad \text{———} (10)$$

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<u>Solution of eq. (10)</u>

The solution of eq. (10) is given by

$i(t) = i_c(t) + i_p(t)$ (3)

For $i_c(t)$

$$\frac{d^2 i}{dt^2} + \omega_0^2 i = 0$$

A.E $\quad m^2 + \omega_0^2 = 0$

$\Rightarrow i_c(t) = (c_1 \cos\omega t + c_2 \sin\omega t)$

suppose $c_1 = c\cos\alpha$, $c_2 = c\sin\alpha$ then

$i_c(t) = c\cos(\omega_0 t - \alpha)$ where $c = \sqrt{c_1^2 + c_2^2}$

$\alpha = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

For $i_p(t)$

$$i_p(t) = \frac{1}{(D^2+\omega_0^2)} F_0 \cos\omega t$$

$$= F_0 \frac{1}{D^2+\omega_0^2} \cos\omega t$$

when $\omega \neq \omega_0$ then

$$i_p(t) = F_0 \frac{1}{-\omega^2+\omega_0^2} \cdot \cos\omega t$$

$$\Rightarrow i_p(t) = \frac{F_0}{\omega_0^2-\omega^2} \cos\omega t \qquad (11)$$

When $\omega = \omega_0$ then

$$i_p(t) = F_0 \cdot t \cdot \frac{1}{2D} \cos\omega t \qquad \left[\frac{1}{D^2+a^2}\cos ax = x \cdot \frac{1}{2D}\cos ax\right.$$

$$= \frac{F_0 \cdot t}{2} \frac{\sin\omega t}{\omega}$$

$$i_p(t) = F_0\left(\frac{t}{2\omega}\right)\sin\omega t, \qquad (12)$$

Thus, the general solution is given by

$$i_g(t) = i_c(t) + i_p(t)$$

$$\Rightarrow \quad i(t) = c \cdot \cos(\omega_0 t - \alpha) + \frac{F_0 \cos \omega t}{\omega_0^2 - \omega^2} \quad [\omega \neq \omega_0] \quad \text{(13)}$$

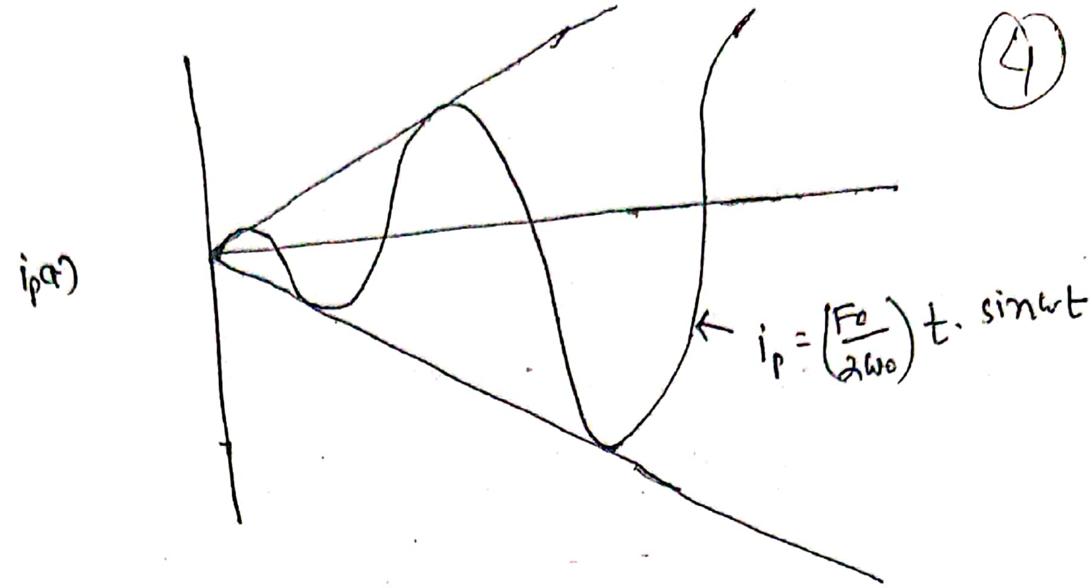
or

$$i(t) = c \cdot \cos(\omega_0 t - \alpha) + F_0 \cdot \left(\frac{t}{2\omega_0}\right) \sin \omega_0 t \quad \text{(14)}$$

Part-II Interpretation

from eq(13) it is clear that the resulting motion is the superposition of two oscillation one with natural circular frequency ω_0 and the other with external circular frequency ω.

from eq(14) it is clear that the resulting motion is the reinforcement of the natural vibrations of the system given by $\cos(\omega_0 t - \alpha)$ by externally impressed vibrations at the same frequency ω_0 but by every increasing amplitude given by $\left(\frac{F_0}{2\omega_0}\right) t \sin \omega_0 t$. The graph of $i_p(t)$ given below shows clearly how the amplitude of the oscillations theoretically ~~incr~~ increase without limit in the case $\omega = \omega_0$.



Case - III LCR - Network without voltage source $(E(t) = 0)$

From eq(5) we get.

$$L \frac{d^2 i}{dt^2} + R \cdot \frac{di}{dt} + \frac{1}{C} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + 2p \frac{di}{dt} + \omega_0^2 \, i = 0 \qquad (15)$$

where $2p = \frac{R}{L}$, $\omega_0^2 = \frac{1}{LC}$

solution of eq(15)

The auxiliary equation of

$$\frac{di^2}{dt^2} + 2p \frac{di}{dt} + \omega_0^2 \, i = 0 \text{ is}$$

$$m^2 + 2pm + \omega_0^2 = 0,$$

whose roots ~~are~~ m_1 and m_2 are given by

$m_1 = -p + \sqrt{p^2 - w_0^2}$ ~~cthos~~ and

$m_2 = -p - \sqrt{p^2 - w_0^2}$

we note that s_1, s_2 are real and distinct, repeated as complex conjugate according as

$$p^2 \gtreqless w_0^2$$

$$\Rightarrow \left(\frac{R}{2L}\right)^2 \gtreqless \frac{1}{LC} \qquad \left[\because p = \frac{R}{2L},\ w_0^2 = \frac{1}{LC}\right]$$

$$\Rightarrow \frac{R}{2L} \gtreqless \frac{1}{\sqrt{LC}}$$

$$\Rightarrow R \gtreqless 2\sqrt{L/C} \qquad (16)$$

Let us set $2\sqrt{L/C} = R_c$ and designate it as a critical path resistance.

<u>If</u> $\underline{R > R_c}$ [overdamped Case]

Then m_1 and m_2 are real and distinct root. Hence

$$\boxed{i(t) = A e^{m_1 t} + B e^{m_2 t}} \qquad (17)$$

where A and B are arbitrary Constant.

If $\underline{R = R_c}$ [critically damped case]

Then $m_1 = m_2 = -p$, Hence

$$\boxed{i(t) = (A + B t) e^{-pt}} \qquad (18)$$

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If $R < R_c$ (Underdamped case)

Then the roots are complex conjugate given by

$$m_1 = -p + i\sqrt{\omega_0^2 - p^2},$$
$$m_2 = -p - i\sqrt{\omega_0^2 - p^2}$$

Hence,

$$\cancel{i(t) = e^{-pt}\left(C_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right)}$$

$$i(t) = e^{-pt}\left(A\cos\left(\sqrt{\omega_0^2 - p^2}\right)t + B\sin\left(\sqrt{\omega_0^2 - p^2}\right)t\right);$$

$$i(t) = e^{-pt}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \quad \text{(19)}$$

where
$$\omega_1 = \sqrt{\omega_0^2 - p^2}$$
$$= \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

if we take

$$A = c\cos\alpha, \quad B = c\sin\alpha$$

$$\Rightarrow \quad i(t) = c\,e^{-pt}\left(\cos\alpha\cos\omega_1 t + \sin\alpha\sin\omega_1 t\right)$$

$$i(t) = c\,e^{-pt}\cos(\omega_1 t - \alpha) \quad \text{(20)}$$

where $c = \sqrt{A^2 + B^2}$

$\tan\alpha = \left(B/A\right) \Rightarrow \alpha = \tan^{-1}\left(B/A\right)$

(II) Interpretation

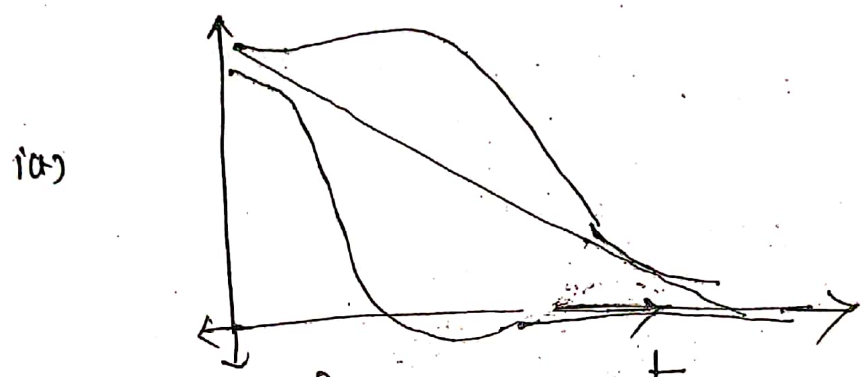
(1) $R > R_c$ (overdamped case)

From (17) it is clear that as $t \to \infty$

$i(t) \to 0$

Thus, the system settles to its equilibrium position without any oscillations.

We choose $i(0) = I_0$ a fixed positive numbers as the initial point and illustrate the effects of changing the initial slope $i'(0)$ in figure. We notice that in every case there would be oscillations which are damped out



fig

(2) $R = R_c$ [Critically damped case]

From eq(18) we note the following facts.

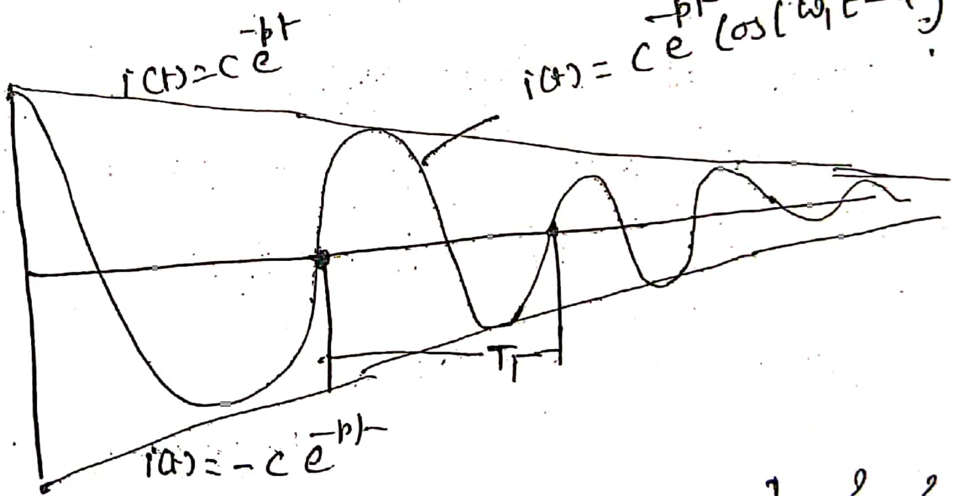
(a) As $t \to \infty$, $i(t) \to 0$

(b) since $e^{-pt} > 0$ and $(A+Bt)$ has almost the positive zero therefore the system passes through its equilibrium position almost once. The graph in this case is similar to that as in (1).

③ $R < R_c$ (6)

eq.(23) represents exponentially damped oscillation of the system about its equilibrium position. The graph of $i(t)$ lies between the curve $i(t) = c e^{-pt}$, and $i(t) = -c e^{-pt}$ (∵ Range of cos. is $[-1, 1]$.)

In this case the motion is said to be pseudoperiodic with ce^{-pt} as it's time varying amplitude, ω_1 as its circular frequency and $T_1 = \frac{2\pi}{\omega_1}$ as its pseudo-period.



Further we note that $\omega_1 < \omega_0$ [∵ $\omega_1^2 = \omega_0^2 - p^2$]

therefore $T_1 > T$

Thus, the damp exhibits the following three effects.

ⓐ It exponentially damps the oscillation according to the time varying amplitude.

ⓑ It shows the motion (since $\omega_1 < \omega_0$)

ⓒ it delays the motion.

Case-IV LCR Model with Voltage Source.

(I) From equation (5) we get,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E' \quad \text{— (21)}$$

suppose $E(t) = E_0 \sin\omega t$. then (21) becomes

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{E_0\omega}{L}\cdot\cos\omega t$$

$$\Rightarrow \frac{d^2i}{dt^2} + 2p\frac{di}{dt} + \omega_0^2 i = F_0\cos\omega t \quad \text{— (22)}$$

where $2p = R/L$, $\omega_0^2 = \frac{1}{LC}$, $F_0 = \frac{E_0\omega}{L}$

Solution of equation (22)

The solution of eq (22) is obtained as

$$i(t) = i_c(t) + i_p(t)$$

from the case (3)

$$i_c(t) = c\,e^{-pt}\cos(\omega_1 t - \alpha) \quad \text{— (23)}$$

For particular integral

$$i_p(t) = \frac{1}{(D^2 + 2pD + \omega_0^2)} F_0\cos\omega t$$

$$= F_0 \cdot \frac{1}{[2pD + \omega_0^2 - \omega^2]}\cos\omega t$$

$$= F_0 \frac{1}{(2pD - (\omega^2 - \omega_0^2))}\cos\omega t$$

$$= F_0 \frac{1}{(2pD - (w^2 - w_0^2)) \times (2pD + (w^2 - w_0^2))} (2pD + (w^2 - w_0^2)) \cos wt \quad (7)$$

$$= F_0 \frac{(2pD + (w^2 - w_0^2))}{[4p^2D^2 - (w^2 - w_0^2)^2]} \cos wt.$$

$$= \frac{F_0}{[4p^2(-w^2) - (w^2 - w_0^2)^2]} [-2pw \sin wt - (w_0^2 - w^2) \cos wt]$$

$$= \frac{F_0}{4p^2w^2 + (w^2 - w_0^2)} [2pw \sin wt + (w_0^2 - w^2) \cos wt] \quad (24)$$

Now, putting $2pw = \rho \cos\phi$ and

$w_0^2 - w^2 = -\rho \sin\phi$

in eq (24)

we get

$$i_p(t) = \frac{F_0}{4p^2w^2 + (w_0^2 - w^2)^2} \times [\rho \cos\phi \sin wt - \rho \sin\phi \cos wt]$$

$$i_p(t) = \frac{F_0}{4p^2w^2 + (w_0^2 - w^2)^2} \rho \sin(wt - \phi) \quad (25)$$

$$\Rightarrow \quad i_p(t) = I_0 \sin(wt - \phi) \quad (26.)$$

where $I_0 = \frac{F_0}{4p^2w^2 + (w_0^2 - w^2)^2}$

Hence, solution is

$$\boxed{i(t) = i_c(t) + i_p(t)}$$

II Interpretation

Notwithstanding the specific form of $i_c(t)$ that is, whether it is given by eq. (17), (18) or (21) we note that $i_c(t) \to 0$ at $t \to \infty$ thus $I_c(t)$ as a transient solution and gives the transient current which dies out with the passage of time.

From eq (26). it is clear that $i_p(t)$ represent the simple Harmonic motion of period $2/w$ and amplitude $E_0/2$.

The Expression for $i_p(t)$, being a sine term of constant amplitude, continue to contribute to the motion in periodic, oscillatory manner Thus $i_p(t)$ gives the stady periodic current.

Lecture Note: Unit-II

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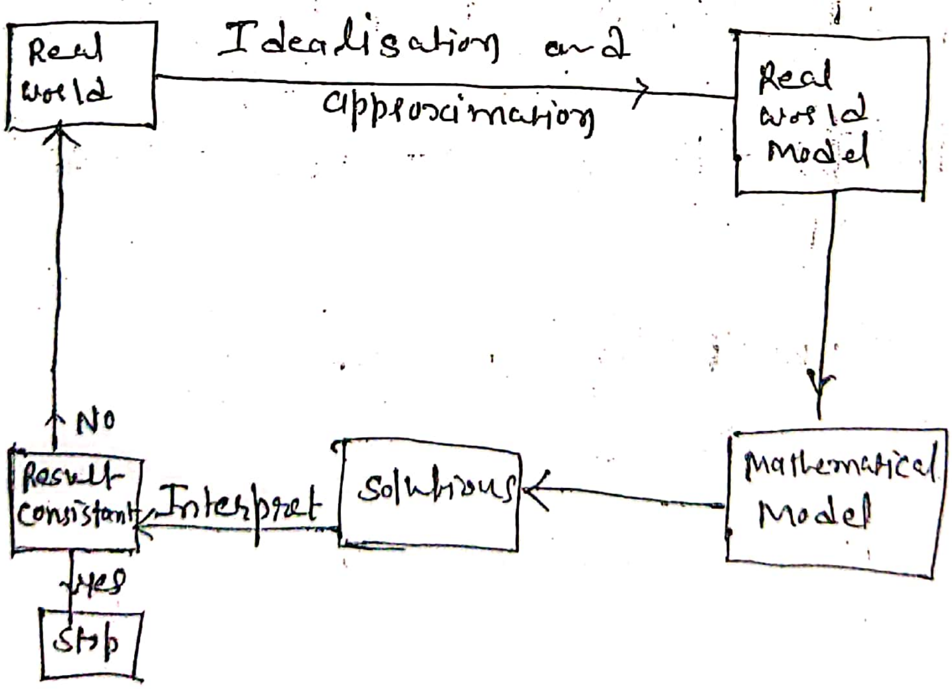
* Mathematical Model:

Mathematical Modelling essentially consist of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the ~~large~~ language of real world



Model-I Newton's Law of Cooling

According to Newton's Law of cooling, the rate of change of temperature T of a body is ~~and~~ is proportional to the difference between the temperature T of the body and temperature T_0 of the surrounding medium.

Let T be the temperature of the body.

and T_s be the temperature of the surrounding medium

Let $T(t)$ be the temperature at time t

and $T(t+\Delta t)$ be the temperature at time $t+\Delta t$

Hence,

$$(T(t+\Delta t) - T(t)) \propto (T-T_s).\Delta t + o(\Delta t)$$

$$\Rightarrow \frac{(T(t+\Delta t) - T(t))}{\Delta t} \propto (T-T_s) + o(\Delta t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \left[\frac{(T(t+\Delta t) - T(t))}{\Delta t}\right] \propto (T-T_s)$$

$$\Rightarrow \frac{dT}{dt} \propto (T-T_s)$$

$$\Rightarrow \frac{dT}{dt} = K(T-T_s) \qquad K<0$$

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solutions

$$\frac{dT}{T-T_s} = k\,dt$$

$$\Rightarrow \int \frac{dT}{T-T_s} = \int k\,dt + C_1$$

$$\Rightarrow \log(T-T_s) = kt + C_1$$

$$\Rightarrow T-T_s = e^{kt+C_1}$$

$$\Rightarrow T-T_s = A e^{kt} \quad \left[\because A = e^{C_1}\right] \quad \text{--- (1)}$$

let, at $t=0$ $T(0)=T_0$ be the initial condition

$$\Rightarrow T_0 - T_s = A e^{0}$$

$$\Rightarrow T_0 - T_s = A \quad \text{--- (2)}$$

from (1) and (2)

$$T - T_s = (T_0 - T_s) e^{kt}$$

$$\Rightarrow \boxed{T = T_s + (T_0 - T_s) e^{kt}} \qquad k<0$$

This shows that the excess of the temperature of the body over that of the surrounding medium decays exponentially

Example-1 According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of air. If the temperature of the air is 40°C and the substance cools from 80°C to 60°C in

Sol $T_s = 40\,^\circ C$

Hence, $\frac{dT}{dt} = K(T-40)$

$$T = T_s + (T_0 - T_s)e^{kt} \Rightarrow \cancel{T = T_s + (40 - T_s)e^{kt}}$$

$$T = 40 + (T_0 - 40)e^{kt}$$

at $t=0$ $T=80$

$\Rightarrow \cancel{80 = T_s + (40 - T_s)}$

$\Rightarrow 80 = 40 + (T_0 - 40)e^{0}$

$\Rightarrow \boxed{80 = T_0}$

when $t = 20$ $T = 60$

$\Rightarrow 60 = 40 + (80 - 40)\, e^{20k}$

$\Rightarrow 20 = 40\, e^{20k}$

$\Rightarrow \frac{1}{2} = e^{20k}$

$\Rightarrow 20k = \log(1/2)$

$\Rightarrow k = \frac{1}{20}[\log 1 - \log 2]$

$\Rightarrow k = -\frac{1}{20}\log 2$

at $t = 40$

$$T = 40 + (80 - 40)\, e^{(-\frac{1}{\cancel{20}}\log 2) \times \cancel{40}^{\,2}}$$

$$= 40 + (40)\, e^{-2\log 2}$$

$$= 40 + 40\, e^{\log 2^{-2}}$$

$T = 40 + 40 \times 2^{-2}$

$= 40 + \frac{40}{4}$

$= 40 + 10$

$\boxed{T = 50}$

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Example: Water at temperature 100 °c Cools in 10 (1'
minutes to 88 °c in a room of temperature 25 °
find the temperature of water after 20 minutes.

$T_0 = 100\ °c$

$T_s = 25\ °C$

$t = 10, \quad T = 88\ °c$

$\Rightarrow \quad T = T_s + (T_0 - T_s)\, e^{kt}$

$\Rightarrow \quad 88 = 25 + (100 - 25)\, e^{k10}$

$\Rightarrow \quad 88 - 25 = 75\, e^{k10}$

$\Rightarrow \quad \frac{63}{75} = e^{10k}$

$t = 20 \quad T = ?$

$\Rightarrow \quad T = T_s + (T_0 - T_s)\, e^{kt}$

$\Rightarrow \quad T = 25 + (100 - 25)\, e^{20k}$

$= 25 + 75\, (e^{10k})^2$

$T = 25 + 75\left(\frac{63}{75}\right)^2$

$\boxed{T = 77.92\ °C}$

Example If the temperature of the air is 30 °c and the substance cools from 100 °C to 70 °c in 15 minute, find when the temperature will be 40 °C [52.5 minute]

Example – A body where temperature T is initially 300 °C is placed in a large block of ice. Find its temperature at the end of 2 and 3 minutes?

Model – II Simple Compartment Model

Let a vessel contain a volume V of a solution with concentration c(t) of a substance at time t. Let a solution with constant concentration c' in an overhead tank enter the vessel at a constant rate R and after mixing ~~thoro~~ thoroughly with the solution in the vessel, let the mixture with concentration c(t) leave the vessel at the same rate R, so that the volume of the solution in the vessel remains V.

Using the principle of continuity

Let $c(t)$ be the concentration at t

and $c(t+\Delta t)$ be the concentration at $t+\Delta t$

we get,

$$V\,C(t+\Delta t) - V\,C(t) = Rq\,\Delta t - R\,C(t)\,\Delta t + O(\Delta t)$$

$$\Rightarrow V\left[\frac{C(t+\Delta t) - C(t)}{\Delta t}\right] = Rq - R\,C(t) + O(\Delta t)$$

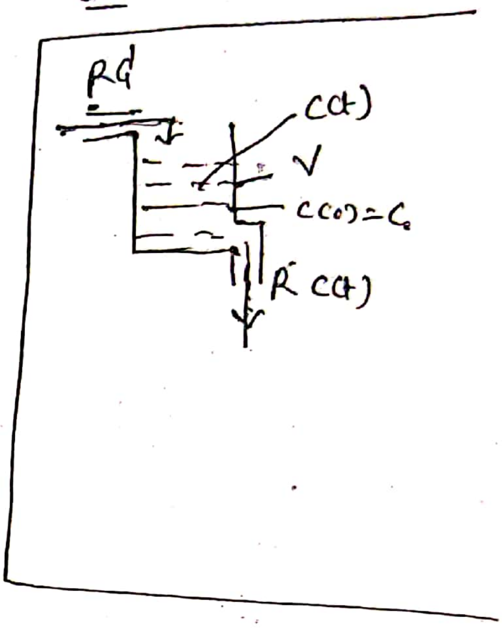
$$\Rightarrow V\left[\lim_{\Delta t\to 0}\frac{C(t+\Delta t) - C(t)}{\Delta t}\right] = Rq - R\,C(t)$$

$$\Rightarrow V\frac{dc}{dt} = Rq - R\,C(t)$$

$$\Rightarrow \frac{dc}{dt} = \frac{R}{V}q - \frac{R}{V}C(t)$$

$$\Rightarrow \boxed{\frac{dc}{dt} + \frac{R}{V}C(t) = \frac{R}{V}q}$$

Fig



~~At at $t=0$, $C(0)=C_0$~~

Solution

$$\text{I.F} = e^{\int \frac{R}{V}dt} = e^{\frac{R}{V}t}$$

Hence,

$$C(t)\,e^{\frac{R}{V}t} = \int \frac{R}{V}q\,e^{\frac{R}{V}t}dt + A$$

$$\Rightarrow C(t)\,e^{\frac{R}{V}t} = \frac{R}{V}q\left[\frac{e^{\frac{R}{V}t}}{\frac{R}{V}}\right] + A$$

$$\Rightarrow C(t)\,e^{\frac{R}{V}t} = q\,e^{\frac{R}{V}t} + A$$

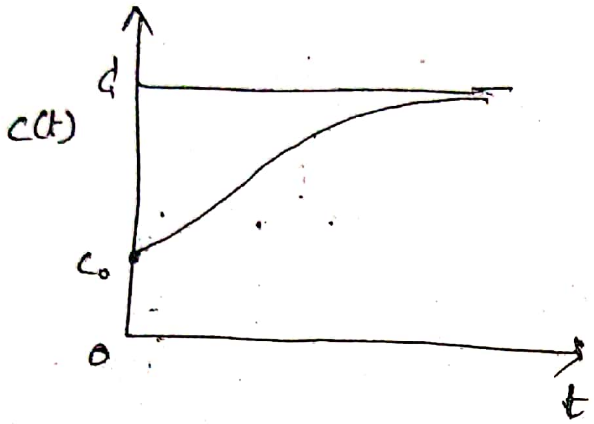
$$\Rightarrow C(t) = q + A\,e^{-\frac{R}{V}t} \quad \text{——— (1)}$$

at, at $t=0$, $C(0) = C_0$

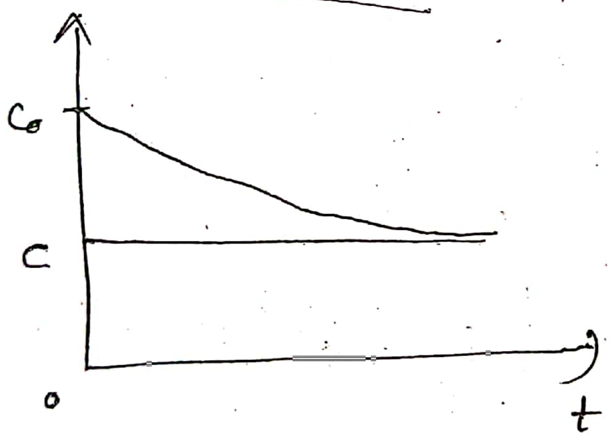
from ① and ② we get,

$$\boxed{C(t) = C_1 + [C_0 - C_1]\, e^{-\frac{R}{V}t}}$$

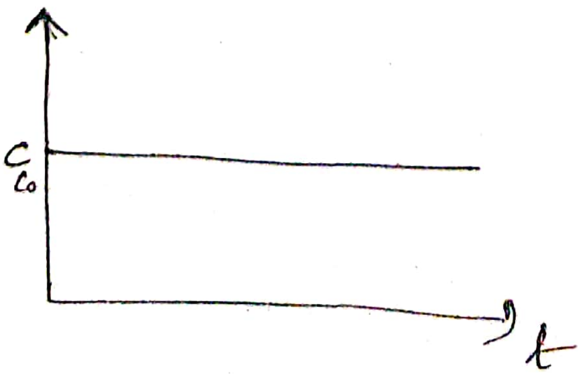
when ① $C_1 > C_0$ [Graphical Analysis]



② $C_1 < C_0$



③ $C = C_0$



Example:-1

A 2000 liters tank of water initially contains 20 kg of dessolved salt. A pipe brings salt solution with concentration 0.04 kg/Liter in to the tank at the rate 2 liters/second and a second pipe caries away the excess solution at the same rate. Calculate concentration of salt at any time with assumption that tank is well mixed.

Sol^n

$$V = 2000 \text{ liters}$$

$$C_0 = \frac{20}{2000} = 0.01 \text{ kg/liter}$$

$$R = 2 \qquad , \qquad G = 0.04 \text{ kg/liter}$$

$$C(t) = G + [C_0 - G] e^{-\frac{R}{V}t}$$

$$\Rightarrow C(t) = 0.04 + [0.01 - 0.04] e^{\frac{2}{2000}t}$$

$$\boxed{C(t) = 0.04 - 0.03\, e^{0.001t}}$$

Example:

A 8000 liter tank of water initially contains 5kg of dissolved potassium. A pipe brings a potassium solution with concentration 0.03 kg/liter in to the tank at the rate of 4 l/s and a second pipe carries away the excess solution with the same rate. Find the concentration of potassium at any time t in the tank

$\rightarrow$

$V = 8000$

$C_0 = 5/8000 = 0.000625$ kg/liter

$C' = 0.03$ kg/liter

$R = 4$

$$C(t) = C' + [C_0 - C']\,e^{-\frac{R}{V}t}$$

$$\Rightarrow \boxed{C(t) = 0.03 + [0.000625 - 0.03]\,e^{-\frac{4}{8000}t}}$$

~~Example~~ :

Example: A 4000 liter tank of water ~~init~~ initially contains ~~4~~ 4 kg of dissolved salt. A pipe brings a salt solution with concentration 0.02 kg/liter in to the tank at the rate of 2 l/s and a second pipe carries away the excess solution with the same rate. then calculate concentration

Model: III

Differential Equation Model for Epidemic Spread

Epidemic of infectious disease have always been a subject of great concern for the welfare of community. Here, we introduce and discussed there simple differential equation model for the spread of epidemic

(I) The SI Model (Susceptibles - Infective)

(II) The SIS Model (Susceptibles - Infective - Susceptibles)

(III) The SIR Model (Susceptibles - Infectives - Removed)

Problem Statement

" Consider a situation where a small group of people suffering with infectious disease is inserted in to a large population which is capable of catching diseases. Formulate a suitable differential Equation model and analyse it."

Step-I We partition the population N in to four mutually exclusive group.

(1) The susceptibles: (S): The person who are

catching the disease.

(2) The Latently Infected: (L):

Those person who are currently infected but not yet capable of transmitting the disease to other

(3) The infectives: (I):

Those person who are currently afected and capable of transmitting it to others

(4) The Removeds (R):

Those person who have had the diseases and either dead or have acquired permanent immunity from the disease or isolated from the community.

Here we consider, S, L, I, R are function of time because the ~~num~~ number of person in this group changes during the period of epidemics.

Step-II We assume that—

(1) S, L, I, R are continious variable though they are in fact integer value

$$S(t) + I(t) + L(t) + R(t) = N \quad \text{——— (1)}$$

(2) The rate of change of susceptible population

$$\frac{ds}{dt} = -\beta\, I(t) \cdot S(t) \quad \text{---} \ (2)$$

$$\forall\ t > 0$$

where β is positive constant called the infection rate

(3) The Latent period of disease is negligible that is

$$L(t) = 0 \quad \forall\ t > 0. \quad \text{---} \ (3)$$

(I) S-I Model

This is simplest model in which we make an additional assumption that no removal from the population is made during the epidemic

that is

$$R(t) = 0 \quad \forall\ t > 0 \quad \text{---} \ (4)$$

Hence, we get

$$S(t) + I(t) = N \quad \text{---} \ (5)$$

Assuming that the epidemic started with I_0 ie $t=0$, $I(0) = I_0$ infected person.

So we get, the following initial value problem

$$\frac{dI}{dt} = \beta I S \qquad I(0) = I_0$$

$$\Rightarrow \frac{dI}{dt} = \beta I (N - I), \quad I(0) = I_0 \quad \text{---} \ (6)$$

similarly If we are interested in number of suceptible the

$$\frac{ds}{dt} = -\beta I S \qquad s(0) = S_0$$

$$\Rightarrow \frac{ds}{dt} = -\beta S(N-S) \qquad S(0) = S_0 \quad \text{--- (7)}$$

Solution of SI-Model

From eq (6)

$$\frac{dI}{dt} = \beta I(N-I), \qquad I(0) = I_0$$

$$\Rightarrow \frac{dI}{I(N-I)} = \beta\, dt$$

$$\Rightarrow \frac{1}{N}\left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta \cdot dt$$

$$\Rightarrow \left[\frac{1}{N-I} + \frac{1}{I}\right] dI = \beta N\, dt$$

$$\Rightarrow -\log(N-I) + \log I = \beta N t + A$$

$$\Rightarrow \log\left(\frac{I}{N-I}\right) = \beta N t + A$$

$$\Rightarrow \frac{I}{N-I} = e^{\beta N t + A}$$

$$\Rightarrow \frac{I}{N-I} = c\, e^{\beta N t} \qquad [\because e^{A} = c] \quad \text{--- (8)}$$

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when $t=0$, $I(0)=I_0$

$$\Rightarrow \quad \frac{I_0}{N-I_0} = c' \qquad \text{——— (9)}$$

from (9) and (8) we get,

$$\frac{I}{N-I} = \frac{I_0}{N-I_0} e^{BNt} \qquad \text{——— (10)}$$

$$\Rightarrow \quad I = (N-I)\frac{I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow \quad I + I\left(\frac{I_0}{N-I_0} e^{BNt}\right) = N\left(\frac{I_0}{N-I_0} e^{BNt}\right)$$

$$\Rightarrow \quad \left[1 + \frac{I_0}{N-I_0} e^{BNt}\right] I = \frac{N \cdot I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow \quad \left[\frac{N - I_0 + I_0 e^{BNt}}{N-I_0}\right] I = N \cdot \frac{I_0}{N-I_0} e^{BNt}$$

$$\Rightarrow \quad I(t) = \frac{N \cdot I_0 e^{BNt}}{\left[N - I_0 + I_0 e^{BNt}\right]}$$

$$I(t) = \frac{N}{\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right]} \qquad \text{——— (11)}$$

Again $S(t) = N - I(t)$

$$= N - \frac{N}{\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right]} = \frac{N\left[1 + \left(\frac{N}{I_0} - 1\right) e^{-BNt}\right] - N}{[illegible]}$$

$$\therefore S(t) = \frac{N\left(\frac{N}{I_0}-1\right)e^{-BNt}}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]}$$

$$\Rightarrow S(t) = \frac{N\left(\frac{N}{I_0}-1\right)}{\left[e^{BNt}+\left(\frac{N}{I_0}-1\right)\right]} \quad \text{—— (12)}$$

<u>Analysis of SI model</u>

From eq. (11)

$$I(t) = \frac{N}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]}$$

$$\Rightarrow I'(t) = \frac{0 - N\left[0+\left(\frac{N}{I_0}-1\right)e^{-BNt}\times(-BN)\right]}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]^2}$$

$$= \frac{BN^2\left(\frac{N}{I_0}-1\right)e^{-BNt}}{\left[1+\left(\frac{N}{I_0}-1\right)e^{-BNt}\right]^2} \quad \text{—— (13)}$$

Again from eq (6)

$$I'(t) = BI(N-I) \quad \text{—— (14)}$$

$$\Rightarrow I''(t) = BI(0-I') + BI'(N-I)$$

$$= -BII' + BI'N - BII'$$

$$I''(t) = BI'N - 2BII'$$

$$I''(t) = BI'(N-2I) \quad \text{—— (15)}$$

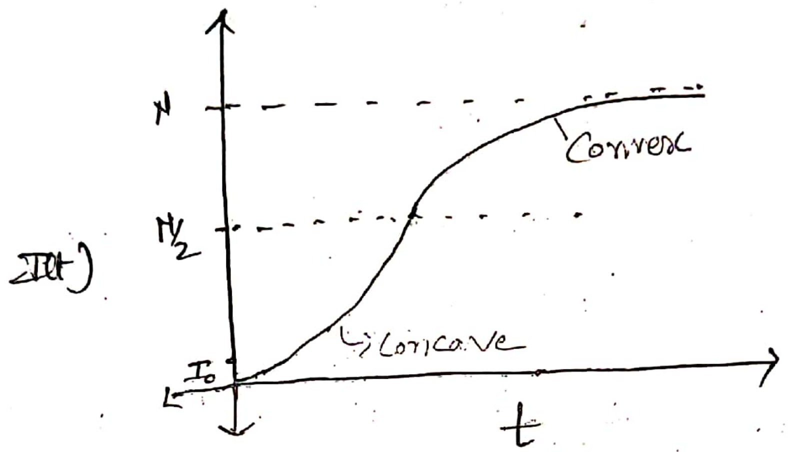
Again from (15)

$$I'''(t) = BI''(N-2I) + BI'(-2I')$$

$$I'''(t) = BI''(N-2I) - 2B(I')^2 \quad \text{———} (16)$$

From the above equation (11), (13), (14), (15) and (16) we make the following interpretation.

(I) Graph of $I(t)$ vs. t



If we take $\lim\limits_{t\to\infty} [I(t)] = \lim\limits_{t\to\infty}\left[\dfrac{N}{1+(\frac{N}{I_0}-1)e^{-BNt}}\right]$

$$= N$$

That means at long time interval all population are infected.

Again from eq. (15).

$$I''(t) > 0 \Rightarrow BI'(N-2I) > 0$$

$$\Rightarrow N-2I > 0$$

$$\Rightarrow N > 2I$$

$$\Rightarrow \frac{N}{2} > I$$

$$I''(t) < 0 \Rightarrow BI'(N-2I) < 0$$

$$\Rightarrow N-2I < 0$$

$$\Rightarrow N < 2I$$

$$\Rightarrow \frac{N}{2} < I$$

$$\Rightarrow I > N/2$$

and

$$I''(t) = 0 \Rightarrow BI'(N-2I) = 0$$

$$\Rightarrow N-2I = 0$$

$$\Rightarrow N = 2I$$

$$\Rightarrow I = \frac{N}{2}$$

Thus, the graph of $I(t)$ vs t is concave upward for $I < N/2$ and concave downward for $I > \frac{N}{2}$ and have inflexion point at $I = N/2$

(II) <u>Graph $I'(t)$ vs t</u>

The graph $I'(t)$ vs t is called the epidemic curve because $I'(t)$ being the rate of change of $I(t)$ measured the spread of epidemic.

From eq. (15)

$$I''(t) = 0 \quad \text{when} \quad I = N/2 \qquad (17)$$

and from eq (16) ~~and~~

$$[I'''(t)] < 0 \quad \text{when} \quad I = N/2$$

(17)

Equating (17) and (18) indicate that $I'(t)$ has maximum value at $I = N/2$.

From eq(10) we get,

$$\frac{I}{N-I} = \frac{I_0}{N-I_0} e^{BNt}$$

at $I = \frac{N}{2}$ we get t_{max}

$$\frac{\frac{N}{2}}{N-\frac{N}{2}} = \frac{I_0}{N-I_0} e^{BNt_{max}}$$

$$\Rightarrow 1 = \frac{I_0}{N-I_0} e^{BNt_{max}}$$

$$\Rightarrow e^{BNt_{max}} = \frac{N-I_0}{I_0}$$

$$\Rightarrow BNt_{max} = \log\left(\frac{N-I_0}{I_0}\right)$$

$$\Rightarrow t_{max} = \frac{1}{BN}\log\left(\frac{N-I_0}{I_0}\right) \quad \text{——— (19)}$$

Again $[I'(t)] = BI(N-I)$

at $I = N/2$

$$\Rightarrow [I'(t)]_{max} = B\frac{N}{2}\left(N-\frac{N}{2}\right)$$

$$= B\,N/2 \cdot N/2$$

$$[I'(t)]_{max} = BN^2 \quad \text{——— (20)}$$

Again from eq. (1)

$$I'(t) = \beta I(N-I)$$

$$= \beta \cdot \left[\frac{N}{1+(\frac{N}{I_0}-1)e^{-\beta Nt}}\right]\left[N - \frac{N}{1+(\frac{N}{I_0}-1)e^{-\beta Nt}}\right]$$

$$= \beta \cdot \left[\frac{N^2}{1+(\frac{N}{I_0}-1)e^{-\beta Nt}} - \frac{N^2}{[1+(\frac{N}{I_0}-1)e^{-\beta Nt}]^2}\right]$$

$$= \beta \left[\frac{N^2[1+(\frac{N}{I_0}-1)e^{-\beta Nt}] - N^2}{[1+(\frac{N}{I_0}-1)e^{-\beta Nt}]^2}\right]$$

$$= \beta \frac{[\cancel{N^2} + N^2(\frac{N}{I_0}-1)e^{-\beta Nt} - \cancel{N^2}]}{[1+(\frac{N}{I_0}-1)e^{-\beta Nt}]^2}$$

$$I'(t) = \frac{\beta N^2(\frac{N}{I_0}-1)e^{-\beta Nt}}{[1+(\frac{N}{I_0}-1)e^{-\beta Nt}]^2} \quad \text{--- (21)}$$

Now, at $t_{max}+\alpha$

$$I'(t_{max}+\alpha) = \frac{\beta N^2(\frac{N}{I_0}-1)e^{-\beta N(t_{max}+\alpha)}}{[1+(\frac{N}{I_0}-1)e^{-\beta N(t_{max}+\alpha)}]^2}$$

$$= \frac{\beta N^2(\frac{N}{I_0}-1)e^{-\beta N t_{max}} \cdot e^{-\beta N\alpha}}{}$$

$$= \frac{BN^2\left(\frac{N}{I_0}-1\right) e^{-BN \times \left(\frac{1}{BN}\log\left(\frac{N-I_0}{I_0}\right)\right)} \cdot e^{-BN\alpha}}{\left[1+\left(\frac{N}{I_0}-1\right) e^{-BN \times \left(\frac{1}{BN}\log\left(\frac{N-I_0}{I_0}\right)\right)} \cdot e^{-BN\alpha}\right]^2} \quad (18)$$

$$= \frac{BN^2\left(\frac{N-I_0}{I_0}\right) \times \left(\frac{N-I_0}{I_0}\right)^{-1} e^{-BN\alpha}}{\left[1+\left(\frac{N-I_0}{I_0}\right) \times \left(\frac{N-I_0}{I_0}\right)^{-1} \cdot e^{-BN\alpha}\right]^2}$$

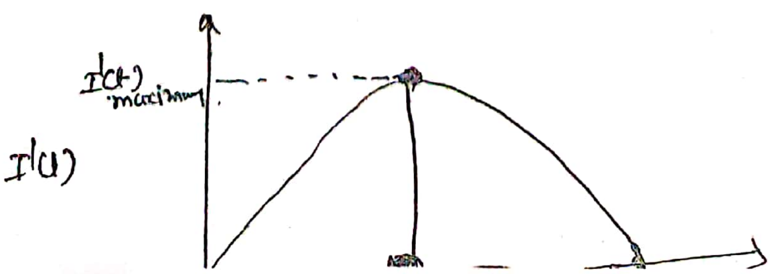
$$= \frac{BN^2 e^{-BN\alpha}}{\left[1+e^{-BN\alpha}\right]^2}$$

$$I'(t_{max}+\alpha) = \frac{BN^2}{\left[e^{\frac{BN\alpha}{2}}+e^{-\frac{BN\alpha}{2}}\right]^2} \quad (22)$$

similarly we get,

$$I'(t_{max}-\alpha) = \frac{BN^2}{\left[e^{\frac{BN\alpha}{2}}+e^{-\frac{BN\alpha}{2}}\right]^2} \quad (23)$$

from eq (22) and eq(23) we can say that the adoptive curve is symmetric about the ordinate.



* **SIS Model**

This is the slightly modified version of SI Model. Here we consider the infected individual can recover and becomes susceptibles at a rate λ, where λ is positive constant.

Then model becomes

$$\frac{dI}{dt} = \beta I (N - I) - \lambda I \quad , \quad I(0) = I_0$$

$$\Rightarrow \quad \frac{dI}{dt} = \beta I \left(N - I - \frac{\lambda}{\beta} \right).$$

$$\Rightarrow \quad \frac{dI}{dt} = \beta I \left(\left(N - \frac{\lambda}{\beta} \right) - I \right)$$

$$\Rightarrow \quad \frac{dI}{dt} = \beta I (a - I) \quad \text{where } a = N - \frac{\lambda}{\beta}$$

$$\Rightarrow \quad \frac{dI}{dt} = \beta I (a - I), \quad I(0) = I_0 \quad \text{(1)}$$

Solution of the model

$$\frac{dI}{I(a - I)} = \beta \cdot dt$$

$$\Rightarrow \quad \frac{1}{a} \left[\frac{1}{a - I} + \frac{1}{I} \right] = \beta \, dt$$

$$\Rightarrow \left[\frac{1}{a-I} + \frac{1}{I}\right] = a\beta . dt$$

$$\Rightarrow -\log(a-I) + \log I = a\beta t + A$$

$$\Rightarrow \log\left(\frac{I}{a-I}\right) = a\beta t + A$$

$$\Rightarrow \frac{I}{a-I} = e^{a\beta t + A} = e^{a\beta t} \cdot e^{A}$$

$$\Rightarrow \frac{I}{a-I} = c\, e^{a\beta t} \qquad [\because e^{A} = c] \quad \text{—— (2)}$$

at $t = 0$, $I(0) = I_0$

$$\Rightarrow \frac{I_0}{a-I_0} = c \quad \text{—— (3)}$$

From (2) and (3) we get

$$\frac{I}{a-I} = \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow I = (a-I) \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow I + I\left(\frac{I_0}{a-I_0}\right) e^{a\beta t} = a \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow \left[1 + \frac{I_0}{a-I_0} e^{a\beta t}\right] I = a \cdot \frac{I_0}{a-I_0} e^{a\beta t}$$

$$\Rightarrow \left[(a - I_0) + I_0 e^{a\beta t}\right] I = a I_0 \, e^{a\beta t}$$

$$\Rightarrow \left[(a - I_0) + I_0 e^{a\beta t}\right] I = a I_0 e^{a\beta t}$$

$$\Rightarrow \quad I(t) = \frac{a I_0 e^{a\beta t}}{\left[(a - I_0) + I_0 e^{a\beta t}\right]}$$

$$I(t) = \frac{a}{\left[1 + \left(\frac{a}{I_0} - 1\right) e^{-a\beta t}\right]} \qquad (4)$$

similarly, we can find $S(t)$. and analyze the model as SI

SIR - Model

This is the slightly modified ~~model~~ version of SI model. In this model we assume that the individuals are removed from the infective class at a rate γ which is proportional to the number of infectives.

Hence,

$$\frac{dR}{dt} = \gamma . I, \qquad \forall\ t > 0. \qquad (1)$$

This removal may be on account of death or permanent immunity or isolation from the community

In this model we have

$$S(t) + I(t) + R(t) = N \qquad \text{——— (2)}$$

and equations are

$$\frac{dI}{dt} = \beta I S - \gamma I \qquad \text{——— (3)}$$

$$\frac{dS}{dt} = -\beta I S \qquad \text{——— (4)}$$

$$\frac{dR}{dt} = \gamma I \qquad \text{——— (5)}$$

<u>Solution of the model</u>

From (4) and (5)

$$\frac{dS}{dR} = -\frac{\beta I S}{\gamma I} = -\frac{S}{\gamma/\beta}$$

$$\Rightarrow \frac{dS}{dR} = -\frac{S}{\rho} \qquad \left[\text{where } \rho = \frac{\gamma}{\beta}\right]$$

——— (6)

Again

$$\frac{dI}{dS} = \frac{dI}{dt} \times \frac{dt}{dS}$$

$$= (\beta S I - \gamma I) \times \frac{1}{(-\beta S I)}$$

$$= -1 + \frac{\gamma}{\beta S}$$

$$\Rightarrow \frac{dI}{dS} = -1 + \frac{\rho}{S} \quad \text{—— (7)}$$

$I(0) = I_0 \;,\; S(0) = S_0$

from (6)

$$\frac{dS}{S} = -\frac{dR}{\rho}$$

$$\Rightarrow \log S = -\frac{1}{\rho} R + A$$

$$\Rightarrow S = e^{-\frac{1}{\rho} R} \cdot e^{A}$$

$$\Rightarrow S = c\, e^{-\frac{1}{\rho} \cdot R} \quad \text{—— (8)} \quad [e^{A} = c]$$

at $t=0$, $S=S_0$

$$\Rightarrow S_0 = c \quad \text{—— (9)}$$

from (8) and (9)

$$\boxed{S = S_0\, e^{-\frac{1}{\rho} \cdot R}} \quad \text{—— (10)}$$

$$\Rightarrow \frac{S}{S_0} = e^{-\frac{1}{\rho} \cdot R}$$

$$\Rightarrow \log\left(\frac{S}{S_0}\right) = -\frac{1}{\rho} \cdot R$$

$$\Rightarrow \boxed{R = -\rho \cdot \log\left(\frac{S}{S_0}\right)}$$

Again, we know that

$$I(t) + S(t) + R(t) = N$$

$$\Rightarrow I(t) = N - S(t) - R(t)$$

$$\Rightarrow \boxed{I(t) = N - S - \rho \cdot \log\left(S/S_0\right)} \quad \text{(11)}$$