4 SPECIAL FUNCTIONS

4.1 Introduction :

In this chapter, we shall see some special integral functions, such as Gamma function, Beta function, Error function etc. There are some functions which are most conveniently defined in the form of integrals, such functions are known as Integral functions. The application of these functions is in the evaluation of definite integrals.

4.2 The Gamma Function :

The improper integral defined by

$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx, \quad n > 0$ is called the Gamma function of n. (It is convergent for all $n > 0$.)

Note : (1) The definite integral which is defined over infinite interval or if the integral becomes infinite within the interval of integration, is known as improper integral.

(2) If f is continuous on the interval $[a, \infty)$, then the improper integral can be written in the form

$$\int_a^\infty f(x)\, dx = \lim_{l \to \infty} \int_a^l f(x)\, dx$$

is said to converge if the limit exists, otherwise it is said to diverge.

(3) Gamma function is used in probability theory, kinetic theory of gases, statistical mechanics, problems in condensed matter physics involving Fermi-Dirac and Einstein-Bose statistics; vartiety of problems in finding areas, volumes, moments of intertial and so on. It was original discovered by L. Euler in 1729.

4.3 Properties of Gamma Function :

1. **Gamma function has the recurrence relation $\Gamma n+1 = n \Gamma n$**

Proof : We have

$$\Gamma n+1 = \int_0^\infty x^n e^{-x}\, dx$$

$$= \lim_{k \to \infty} \int_0^k x^n e^{-x}\, dx$$

$$= \lim_{k \to \infty} \left\{ \left[-x^n e^{-x} \right]_0^k - \int_0^k n x^{n-1} (-e^{-x})\, dx \right\}$$

$$= \lim_{k \to \infty} \left\{ \left[-k^n e^{-k} \right] + n \int_0^k x^{n-1} e^{-x}\, dx \right\}$$

$$= n \int_0^\infty x^{n-1} e^{-x}\, dx \qquad \left(\because \lim_{k \to \infty} (-k^n e^{-k}) = 0 \right)$$

$$= n \Gamma n$$

2. $\lceil 1 = 1$

Proof : We have $\lceil n = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\therefore \lceil 1 = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1$

3. **When n is a positive integer $\lceil n+1 = n!$**

Proof : We have the recurrence relation

$$\lceil n+1 = n \lceil n$$

Thus, $\lceil n = (n-1) \lceil n-1$ (replacing n by $n-1$)

$$\lceil n-1 = (n-2) \lceil n-2$$

$$\vdots$$

$$\lceil 2 = 1 \lceil 1$$

$$\lceil 1 = 1$$

Multiplying above relations, we get

$$\lceil n+1 = n(n-1)(n-2) \quad \ldots\ldots (2.1)$$

$$= n!$$

4. **If we rewrite property (1) in the form**

$\lceil n = \dfrac{\lceil n+1}{n}$ then we have

$$\lceil n = \frac{\lceil n+1}{n} = \frac{\lceil n+2}{n(n+1)} = \frac{\lceil n+3}{n(n+1)(n+2)} = \ldots = \frac{\lceil n+k+1}{n(n+1)(n+2)\ldots(n+k)}$$

for $n \neq 0, -1, -2, \ldots$

from this we can find the values of Gamma function for negative $n \neq -1, -2, -3, \ldots$

Thus we can define Gamma function completely as follows :

$$\lceil n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0$$

$$= \frac{\lceil n+1}{n}, \quad n < 0 \text{ and } n \neq -1, -2, -3, \ldots$$

For example, $\lceil 6 = (6-1)! = 120$

$$\lceil -\tfrac{1}{2} = \frac{\lceil \frac{1}{2}}{-\frac{1}{2}} = -2 \lceil \tfrac{1}{2}$$

$$\lceil \tfrac{3}{2} = \tfrac{1}{2} \lceil \tfrac{1}{2} \qquad (\because \lceil n = (n-1) \lceil (n-1))$$

The graph of the Gamma function is as follows :

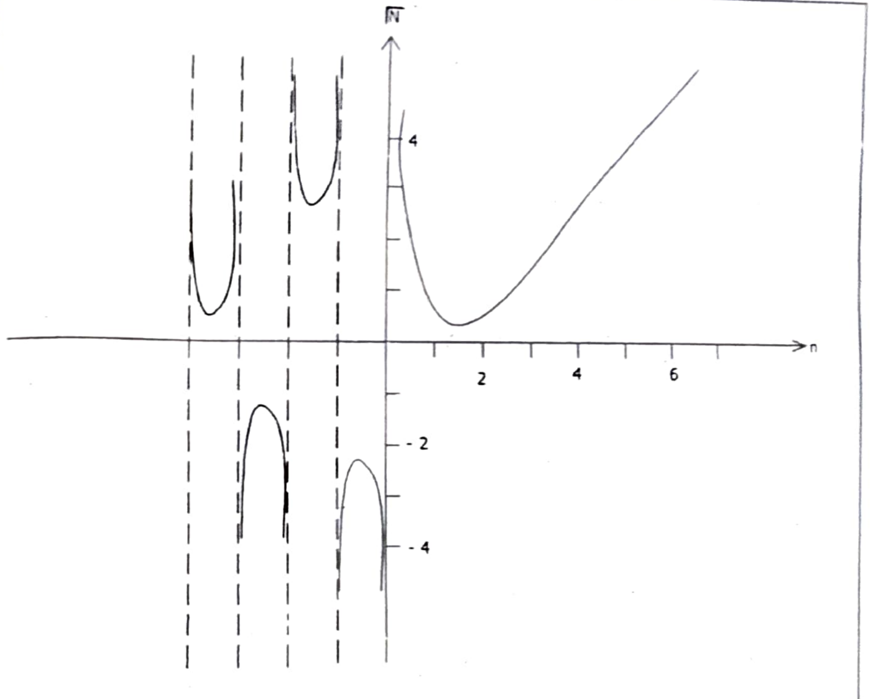

Fig 4.1

5. $\lceil n = 2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx$

Proof :

Put $x = t^{1/2} \Rightarrow dx = \frac{1}{2} t^{-1/2}\, dt$

$\therefore$ LHS $= 2 \int_0^\infty e^{-t}\, t^{\frac{2n-1}{2}}\, \frac{1}{2}\, t^{-1/2}\, dt$

$= \int_0^\infty e^{-t}\, t^{n-1}\, dt = \lceil n$

4.4 The Beta Function :

The improper integral $\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$, converges for $m > 0, n > 0$ is known as Beta function and is denoted by B (m, n).

4.5 Properties of Beta Function :

1. **B (m, n) = B (n, m). That is B function is symmetric.**

Proof : We have

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx.$$

Put $1 - x = y \Rightarrow -dx = dy$

$\therefore$ When $x = 0 \rightarrow y = 1, \quad x = 1 \rightarrow y = 0$

$$\therefore \quad B(m, n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1}\, dy$$

$$= B(n, m)$$

2. $$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Proof : We have

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Put $x = \dfrac{1}{1+y} \quad \Rightarrow dx = -\dfrac{1}{(1+y)^2}\, dy$

and $1 + y = \dfrac{1}{x} \Rightarrow y = \dfrac{1}{x} - 1$

$\therefore$ When $x = 0 \rightarrow y \rightarrow \infty$

and $x = 1 \rightarrow y = 0$

$$\therefore \quad B(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}} \left[1 - \frac{1}{1+y}\right]^{n-1} \left(-\frac{1}{(1+y)^2}\, dy\right)$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1+2+n-1}}\, dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

As $B(m, n) = B(n, m)$

We have $B(m, n) = \displaystyle\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

3. $$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Proof : We have $B(m, n) = \displaystyle\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0 \rightarrow \theta = 0$ and $x = 1 \rightarrow \theta = \pi/2$

$$\therefore \quad B(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta\, (1 - \sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

4.6 Relation Between Beta and Gamma Functions :

$$B(m, n) = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$$

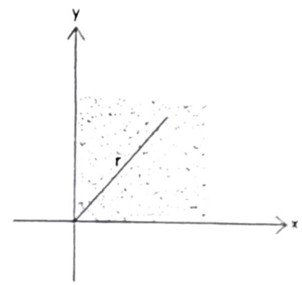
Fig 4.2

Proof : We know that

$\overline{|m} = 2\displaystyle\int_0^\infty e^{-x^2} x^{2m-1}\, dx$ and

$\overline{|n} = 2\displaystyle\int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$$\therefore \quad \overline{|m}\,\overline{|n} = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy$$

We use polar coordinates

$\therefore \quad 0 \le r \le \infty$

$0 \le \theta \le \pi/2$

$$\therefore \quad \overline{|m}\,\overline{|n} = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r^{2m-1} \cos^{2m-1}\theta\, r^{2n-1} \sin^{2n-1}\theta\; r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty \cos^{2m-1}\theta \sin^{2n-1}\theta\, r^{2m+2n-1} e^{-r^2}\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \left\{2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right\} d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, \overline{|m+n}\, d\theta$$

$$= \overline{|m+n}\; 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \quad \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}} = \int_0^{\pi/2} \cos^{2m-2}\theta \sin^{2n-2}\, 2\cos\theta\sin\theta\, d\theta$$

$$= \int_0^1 (1 - \sin^2\theta)^{m-1} (\sin^2\theta)^{n-1} (2\cos\theta\sin\theta)\, d\theta$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\, dx = B(m, n)$$

Note : We have $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

and $B(m, n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$

Put $m = n = \frac{1}{2}$

$$\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = 2\int_0^{\pi/2} d\theta$$

$$\left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

4.7 Duplication Formula :

$$\Gamma n\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\, \Gamma 2n \quad \text{or} \quad \Gamma(n+1)\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}}\, \Gamma(2n+1)$$

[It is useful formula to find the Gamma functions for values on n halfway between the integers]

Proof : We know that

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\Rightarrow B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2n}\theta \cos^{2n}\theta\, d\theta$$

$$= \frac{2}{2^{2n}}\int_0^{\pi/2} \sin^{2n} 2\theta\, d\theta$$

Put $2\theta = \phi \Rightarrow 2\, d\theta = d\phi$

$$\therefore B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{2}{2^{2n}}\int_0^{\pi} \sin^{2n}\phi\, \frac{d\phi}{2}$$

$$= \frac{2}{2^{2n}}\cdot\frac{1}{2}\cdot 2\int_0^{\pi/2} \sin^{2n}\phi\, d\phi$$

$$= \frac{2}{2^{2n}}\int_0^{\pi/2} \sin^{2n}\phi \cos^{0}\phi\, d\phi$$

$$= \frac{1}{2^{2n}}\, B\left(n+\frac{1}{2}, \frac{1}{2}\right)$$

$$\therefore \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} = \frac{1}{2^{2n}}\, \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\frac{1}{2}}{\Gamma(n+1)}$$

$$\Gamma(n+1)\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}}\, \Gamma(2n+1)$$

$$n\Gamma n\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}}\, 2n\Gamma 2n \qquad (\because \Gamma(n+1) = n\Gamma n)$$

$$\Gamma n\, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\, \Gamma 2n$$

Note : In duplication formula put $n = \frac{1}{4}$

$$\Gamma\frac{1}{4}\, \Gamma\frac{3}{4} = \frac{\sqrt{\pi}}{2^{-1/2}}\, \Gamma\frac{1}{2} = \sqrt{2}\pi$$

SOLVED EXAMPLES

1. **Prove that $B(m, n) = B(m+1, n) + B(m, n+1)$.**

Solution : RHS $= B(m+1, n) + B(m, n+1)$

$$= \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m\, \Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma m\,\Gamma n + \Gamma m\, n\Gamma n}{(m+n)\,\Gamma(m+n)} = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\left(\frac{m+n}{m+n}\right) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = B(m,n)$$

2. **Prove that $\Gamma n = \int_0^{\infty}\left[\log\frac{1}{x}\right]^{n-1} dx$**

Solution : Take $y = \log\frac{1}{x} \Rightarrow \frac{1}{x} = e^{y} \Rightarrow x = e^{-y}$

$\therefore\ dx = -e^{-y}\, dy$

$$\therefore \text{RHS} = \int_{\infty}^{0} y^{n-1}\,(-e^{-y}\, dy) = +\int_0^{\infty} e^{-y}\, y^{n-1}\, dy$$

$$= \Gamma n = \text{L.H.S.}$$

3. **Evaluate $\int_0^{\infty} e^{-ax}\, x^{n-1}\, dx$, where a, n are positive. Deduce that (a) $\int_0^{\infty} e^{-ax}\, x^{n-1} \cos bx\, dx = \dfrac{\Gamma n}{r^n}\cos n\theta$ (b) $\int_0^{\infty} e^{-ax}\, x^{n-1} \sin bx\, dx = \dfrac{\Gamma n}{r^n}\sin n\theta$ where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$.**

Solution : Put $y = ax \Rightarrow dy = a\, dx$

$$\therefore \int_0^\infty e^{-ax} x^{n-1}\,dx = \int_0^\infty e^{-y} \frac{y^{n-1}}{a^{n-1}} \frac{1}{a}\,dy$$

$$= \frac{1}{a^n}\int_0^\infty e^{-y} y^{n-1}\,dy = \frac{\Gamma n}{a^n}$$

Now replace a by $a+ib$, we get

$$\int_0^\infty e^{-(a+ib)x} x^{n-1}\,dx = \frac{\Gamma n}{(a+ib)^n}$$

$$\Rightarrow \int_0^\infty e^{-ax}(\cos bx - i\sin bx)\,x^{n-1}\,dx = \frac{\Gamma n}{(a+ib)^n}$$

Put $a = r\cos\theta$, $b = r\sin\theta$

$$\therefore\ r = \sqrt{a^2+b^2},\quad \theta = \tan^{-1}\frac{b}{a}$$

$$\therefore \int_0^\infty \{e^{-ax}\cos bx - i\,e^{-ax}\sin bx\}\,x^{n-1}\,dx = \frac{\Gamma n}{r^n(\cos n\theta - i\sin n\theta)}$$

$$= \frac{\Gamma n}{r^n}(\cos n\theta - i\sin n\theta)$$

Compare real and imaginary parts, we get

$$\int_0^\infty e^{-ax}\cos bx\, x^{n-1}\,dx = \frac{\Gamma n}{r^n}\cos n\theta$$

and $\int_0^\infty e^{-ax}\sin bx\, x^{n-1}\,dx = \frac{\Gamma n}{r^n}\sin n\theta.$

4. Evaluate $\int_0^1 x^4(1-\sqrt{x})^5\,dx$

Solution : Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore\ I = \int_0^1 t^8(1-t)^5\, 2t\,dt$$

$$= 2\int_0^1 t^9(1-t)^5\,dt$$

$$= 2B(10,6) = 2\frac{9!\,5!}{15!}$$

$$= \frac{2\cdot 9!\,1\cdot2\cdot3\cdot4\cdot5}{9!\,10\cdot11\cdot12\cdot13\cdot14\cdot15} = \frac{1}{15015}$$

5. Evaluate $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy$ where $p>0,\ q>0$.

Solution : Put $\log\frac{1}{y} = t$

$$\Rightarrow y = e^{-t} \Rightarrow dy = -e^{-t}\,dt$$

$$\therefore\ I = \int_\infty^0 e^{-t(q-1)}\, t^{p-1}(-e^{-t}\,dt)$$

$$= \int_0^\infty e^{-tq}\, t^{p-1}\,dt$$

Put $tq = y \Rightarrow q\,dt = dy$

$$\therefore\ I = \int_0^\infty e^{-y}\frac{y^{p-1}}{q^{p-1}}\frac{dy}{q}$$

$$= \frac{1}{q^p}\int_0^\infty e^{-y} y^{p-1}\,dy = \frac{\Gamma p}{q^p}$$

6. Evaluate $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt$ where $S>0$.

Solution : Put $st = x$

$$\Rightarrow s\,dt = dx$$

$$\therefore\ I = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^{-1/2}\frac{dx}{s}$$

$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-x} x^{-1/2}\,dx = \frac{1}{\sqrt{s}}\Gamma\frac{1}{2} = \sqrt{\frac{\pi}{s}}$$

7. Prove that $\int_0^\infty x^{2n-1}e^{-ax^2}\,dx = \frac{\Gamma n}{2a^n}$

Solution : Put $ax^2 = t \Rightarrow x = \frac{1}{\sqrt{a}}t^{1/2} \Rightarrow dx = \frac{1}{2\sqrt{a}\sqrt{t}}\,dt$

$$\therefore\ I = \int_0^\infty x^{2n-1} e^{-ax^2}\,dx$$

$$= \int_0^\infty \left(\frac{t}{a}\right)^{\frac{2n-1}{2}} e^{-t}\frac{1}{2\sqrt{a}\sqrt{t}}\,dt$$

$$= \frac{1}{2a^n}\int_0^\infty t^{n-1}e^{-t}\,dt = \frac{\Gamma n}{2a^n}$$

8. Prove that $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$

Solution : Put $x = t^{1/2} \Rightarrow dx = \frac{1}{2\sqrt{t}}\, dt$

$$\therefore \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \int_0^\infty t^{1/4}\, e^{-t}\, \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{-1/4}\, dt = \frac{1}{2}\Gamma\left(\tfrac{3}{4}\right)$$

and
$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \int_0^\infty e^{-1/4}\, e^{-t}\, \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2}\int_0^\infty t^{-3/4}\, e^{-t}\, dt = \frac{1}{2}\Gamma\left(\tfrac{1}{4}\right)$$

$\therefore$ From (1) and (2), we have

$$\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{1}{2}\Gamma\left(\tfrac{3}{4}\right) \cdot \frac{1}{2}\Gamma\left(\tfrac{1}{4}\right) = \frac{1}{4}\sqrt{2}\pi = \frac{\pi}{2\sqrt{2}}$$

9. Prove the recurrence relation for Beta function.

$$B(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\, B(m-1, n-1)$$

Solution : We know that

$$B(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma (m+n)}$$

$$= \frac{(m-1)\Gamma(m-1)\,(n-1)\,\Gamma(n-1)}{(m+n-1)\,(m+n-2)\,\Gamma(m+n-2)} \qquad (\because \Gamma n = (n-1)\,\Gamma(n-1))$$

$$= \frac{(m-1)\,(n-1)}{(m+n-1)\,(m+n-2)} \cdot \frac{\Gamma(m-n)\,\Gamma(n-1)}{\Gamma(m+n-2)}$$

$$= \frac{(m-1)\,(n-1)}{(m+n-1)\,(m+n-2)}\, B(m-1, n-1)$$

10. Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

Solution : We know that

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

$$\therefore \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \cos^{1/2}\theta\, \sin^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2}\, B\left(\tfrac{3}{4}, \tfrac{1}{4}\right)$$

$$= \frac{1}{2}\, \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma 1} = \frac{1}{2}\sqrt{2}\pi = \frac{\pi}{\sqrt{2}}$$

11. Evaluate $\int_0^1 (x \log x)^3\, dx$

Solution : Put $\log x = y \Rightarrow x = e^{-y}$

$\therefore \quad dx = -\, e^{-y}\, dy$

$$\therefore \quad I = -\int_\infty^0 e^{-3y}\,(-y)^3\, e^{-y}\, dy$$

$$= -\int_0^\infty e^{-4y}\, y^3\, dy$$

Put $4y = t \Rightarrow 4dy = dt$

$$\therefore \; I = -\int_0^\infty e^{-t}\, \frac{t^3}{4^3}\, \frac{dt}{4} = \frac{-1}{4^4}\int_0^\infty e^{-t}\, t^3\, dt$$

$$= -\frac{1}{256}\Gamma 4 = -\frac{3!}{256} = -\frac{3}{128}$$

12. Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$

Solution : We know that

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$\therefore \; I = \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$$

$$= \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$$

$$= B(5, 10) + B(10, 5)$$

$$= 2B(5, 10) = 2\,\frac{4!\,9!}{14!} = \frac{2\cdot 24\cdot 9!}{9!\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14} = \frac{1}{5005}$$

13. Prove that $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$

Solution : We know that

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore\ I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= B(9, 15) - B(15, 9) = 0$$

14. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\,\frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

Solution : Put $x = t^{1/n} \Rightarrow dx = \frac{1}{n}\,t^{\frac{1}{n}-1}\,dt$

$$\therefore\ I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 \frac{1}{\sqrt{1-t}}\,\frac{1}{n}\,t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{n}\,B\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{n}\,\frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

15. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Solution : Put $x = t^{1/4} \Rightarrow dx = \frac{1}{4}t^{-3/4}\,dt$

$$\therefore\ I = \int_0^1 (1-t)^{-1/2}\,\frac{1}{4}t^{-3/4}\,dt$$

$$= \frac{1}{4}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{4}B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

16. Evaluate $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$

Solution : Put $bx = at \Rightarrow b\,dx = a\,dt$

$$\therefore\ I = \int_0^\infty \left(\frac{at}{b}\right)^{m-1}\frac{1}{(a+at)^{m+n}}\,\frac{a\,dt}{b}$$

$$= \frac{a^{m-1}a}{b^{m-1}\,a^{m+n}\,b}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$= \frac{1}{a^n\,b^m}\,B(m,n)$$

17. Evaluate $\displaystyle\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1}\,dx$

Solution : Put $x = \cos 2\theta$

$\Rightarrow dx = -2\sin 2\theta\,d\theta$

$$\therefore\ I = 2\int_{\pi/2}^0 (1+\cos 2\theta)^{p-1}(1-\cos 2\theta)^{q-1}(-2\sin 2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} (2\cos^2\theta)^{p-1}\,(2\sin^2\theta)^{q-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2} 2^{p-1}\cos^{2p-2}\theta\;2^{q-1}\sin^{2q-2}\theta\,\sin\theta\,\cos\theta\,d\theta$$

$$= 4\cdot 2^{p+q-2}\int_0^{\pi/2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta$$

$$= 2^{p+q-1}\,B(p, q)$$

18. Evaluate $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

Solution : We use $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$$\therefore\ I = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right) \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \quad (\because \Gamma(n) = (n-1)\Gamma(n-1))$$

$$= \pi$$

19. Evaluate $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Solution : Put $x = t^{1/n} \Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$\therefore\ I = \int_0^1 (1-t)^{-1/n} \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-\frac{1}{n}} dt = \frac{1}{n} B\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(1)} = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{n}$$

20. Prove that $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ **$(n = 0, 1, 2, ...)$**

Hence fine $\Gamma\left(\frac{5}{2}\right)$

Solution : We have $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\therefore\ \Gamma\left(n + \frac{1}{2}\right) = \left(n + \frac{1}{2} - 1\right)\Gamma\left(n + \frac{1}{2} - 1\right)$$

$$= \left(n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\Gamma\left(n - \frac{3}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1}{2^n}\sqrt{\pi}$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^n (2n)(2n-2)\ldots 6 \cdot 4 \cdot 2}\sqrt{\pi}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^n \cdot 2^n\, n(n-1)\ldots 3 \cdot 2 \cdot 1}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^{2n}\, n!} = \frac{(2n)!\sqrt{\pi}}{4^n\, n!}$$

21. Prove that $\frac{B(m, n+1)}{n} = \frac{B(m+1, n)}{m} = \frac{B(m, n)}{m+n}$

Solution : $\frac{B(m, n+1)}{n} = \frac{\Gamma(m)\Gamma(n+1)}{n\Gamma(m+n+1)} = \frac{\Gamma(m)\, n\Gamma(n)}{n(m+n)\Gamma(m+n)} \quad (\because \Gamma(n+1) = n\Gamma(n))$

$$= \frac{\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} = \frac{B(m, n)}{m+n}$$

Similarly $\frac{B(m+1, n)}{m} = \frac{\Gamma(m+1)\Gamma(n)}{m\Gamma(m+n+1)} = \frac{m\Gamma(m)\Gamma(n)}{m\Gamma(m+n+1)}$

$$= \frac{\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} = \frac{B(m, n)}{m+n}$$

Hence the result.

22. Prove that $B(m, m)\, B\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}$

Solution : We know the duplication formula

$$\Gamma(n)\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

$$\therefore\ \text{LHS} = B(m, m)\, B\left(m + \frac{1}{2}, m + \frac{1}{2}\right)$$

$$= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \frac{\Gamma\left(m + \frac{1}{2}\right)\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(2m+1)} = \frac{1}{\Gamma(2m)\Gamma(2m+1)}\left\{\Gamma(m)\Gamma\left(m + \frac{1}{2}\right)\right\}^2$$

$$= \frac{1}{\Gamma(2m)\, 2m\Gamma(2m)}\left\{\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)\right\}^2 = \frac{1}{2m(\Gamma(2m))^2} \frac{\pi(\Gamma(2m))^2}{2^{4m-2}}$$

$$= \frac{\pi m^{-1}}{2^{4m-1}} = \text{RHS}$$

23. Evaluate $\int_0^\infty x^n e^{-a^2x^2}\,dx$,

$n > -1$. Deduce that $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}$

Solution : Put $a^2x^2 = t \Rightarrow x = \frac{t^{1/2}}{a}$

$$\Rightarrow dx = \frac{1}{2a}t^{-1/2}\,dt$$

$$\therefore\ I = \int_0^\infty \frac{t^{n/2}}{a^n} e^{-t} \frac{1}{2a} t^{-1/2}\,dt$$

$$= \frac{1}{2a^{n+1}} \int_0^\infty t^{\frac{n-1}{2}} e^{-t}\,dt = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right), \quad n > -1$$

We put $n = 0$

$$\therefore\ \int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2a}$$

Now, $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \int_{-\infty}^{0} e^{-a^2x^2}dx + \int_0^\infty e^{-a^2x^2}dx$

$$= I_1 + I_2$$

Put $x = -y$ in I_1

$$\therefore\ I_1 = \int_\infty^0 e^{-a^2y^2}(-dy) = \int_0^\infty e^{-a^2y^2}dy = I_2$$

$$\therefore\ \int_{-\infty}^{\infty} e^{-a^2x^2}dx = 2I_2 = 2\frac{\sqrt{\pi}}{2a} = \frac{\sqrt{\pi}}{a}$$

24. Prove that $B(m+2, n-2) = \dfrac{m(m+1)}{(n-1)(n-2)} B(m, n)$.

Solution : $B(m+2,\ n-2) = \dfrac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m+n)}$

$$= \frac{(m+1)\,\Gamma(m+1)\,(n-2)\,\Gamma(n-2)}{(n-2)\,\Gamma(m+n)} \quad (\because\ \Gamma(n+1) = n\Gamma(n))$$

$$= \frac{(m+1)\,m\,\Gamma(m)\,\Gamma(n-1)}{(n-2)\,\Gamma(m+n)}$$

$$= \frac{(m+1)\,m\,\Gamma(m)\,(n-1)\,\Gamma(n-1)}{(n-2)(n-1)\,\Gamma(m+n)}$$

$$= \frac{m(m+1)\,\Gamma(m)\,\Gamma(n)}{(n-1)(n-2)\,\Gamma(m+n)}$$

$$= \frac{m(m+1)}{(n-1)(n-2)} B(m, n)$$

25. Evaluate $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx$.

Solution : We use $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

$$\therefore\ \int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx = \frac{1}{2}B\left(2, \frac{7}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{15}{4}\right)} = \frac{1}{2}\frac{1!\,\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}\cdot\frac{7}{4}\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77}$$

26. Evaluate $\int_a^b (x-a)^m (b-x)^n\,dx$

Solution : Put $x - a = z \Rightarrow dx = dz$

$$\therefore\ I = \int_0^{b-a} z^m (b-a-z)^n\,dz$$

Put $z = (b-a)\,y \Rightarrow dz = (b-a)\,dy$

$$\therefore\ I = \int_0^1 (b-a)^m y^m (b-a)^n (1-y)^n (b-a)\,dy$$

$$= (b-a)^{m+n+1} \int_0^1 y^m (1-y)^n\,dy$$

$$= (b-a)^{m+n+1} \cdot B(m+1, n+1)$$

27. Prove that $\dfrac{B(m+1, n)}{B(m, n)} = \dfrac{m}{m+n}$

Solution : LHS $= \dfrac{B(m+1, n)}{B(m, n)} = \dfrac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$

$$= \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{m}{m+n} = \text{RHS}$$

28. Prove that $\dfrac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \sqrt{\pi}$

Solution : We know that

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$$
$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right) \ldots \tfrac{5}{2} \cdot \tfrac{3}{2} \cdot \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$
$$= \frac{(2n-1)(2n-3) \ldots 5 \cdot 3 \cdot 1 \cdot \sqrt{\pi}}{2^n}$$
$$\therefore \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{(2n-1)(2n-3) \ldots 5 \cdot 3 \cdot 1}{2^n\, n(n-1) \ldots 3 \cdot 2 \cdot 1} \sqrt{\pi}$$
$$= \frac{(2n-1)(2n-3) \ldots 5 \cdot 3 \cdot 1}{(2n)(2n-2) \ldots 6 \cdot 4 \cdot 2} \sqrt{\pi}$$

29. Evaluate $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx$

Solution : Put $c^x = t \Rightarrow x \log c = \log t$

$$\therefore (\log c)\,dx = \frac{1}{t}\,dt$$

When $x = 0 \Rightarrow t = 1$

$x \to \infty \Rightarrow t \to \infty$

$$\therefore I = \int_1^\infty \left(\frac{\log t}{\log c}\right)^c \frac{1}{t} \frac{dt}{t \log c}$$
$$= \frac{1}{(\log c)^{c+1}} \int_1^\infty \frac{1}{t^2} (\log t)^c\,dt$$

Put $\log t = y \Rightarrow t = e^y$

$\therefore\ dt = e^y\,dy$

$$\therefore I = \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-2y} y^c e^y\,dy$$
$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-y} y^c\,dy = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

30. Evaluate $\displaystyle\int_0^1 x^m (\log x)^n\,dx$, **where** n **is** a **+** ve **integer and** $m > 1$.

Solution : Put $\log x = -y \Rightarrow x = e^{-y}$

$\therefore\ dx = -e^{-y}\,dy$

When $x = 0 \Rightarrow y = \infty$

and $x = 1 \Rightarrow y = 0$

$$\therefore \int_0^1 x^m (\log x)^n\,dx = \int_\infty^0 e^{-my} (-y)^n (-e^{-y}\,dy)$$
$$= (-1)^n \int_0^\infty e^{-(m+1)y} y^n\,dt$$

Put $(m+1)y = t$

$\Rightarrow (m+1)\,dy = dt$

$$\therefore I = (-1)^n \int_0^\infty e^{-t} \frac{t^n}{(m+1)^n} \frac{dt}{(m+1)}$$
$$= \frac{(-1)^n}{(m+1)^{n+1}}\,\Gamma(n+1) = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$$

31. Evaluate $\displaystyle\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$.

Solution : Put $x^2 = \sin\theta$ in the first integral

$\therefore\ x = \sin^{1/2}\theta$

$$\Rightarrow dx = \frac{1}{2} \sin^{-1/2}\theta \cos\theta\,d\theta$$
$$\therefore \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \frac{1}{2} \sin^{-1/2}\theta \cos\theta\,d\theta$$
$$= \frac{1}{2} \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\,d\theta$$
$$= \frac{1}{4}\,B\left(\frac{3}{4}, \frac{1}{2}\right)$$

Now Put $x^2 = \tan\theta$ in the second integral

$$\therefore x = \tan^{1/2}\theta \Rightarrow dx = \frac{1}{2} \tan^{-1/2}\theta \sec^2\theta\,d\theta$$
$$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{1}{\sec\theta} \frac{1}{2} \tan^{-1/2}\theta \sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \tan^{-1/2}\theta \sec\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\,\frac{1}{\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sin^{1/2}\theta \cos^{1/2}\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{2^{1/2}}{\sin^{1/2}2\theta}\, d\theta$$

Put $2\theta = \phi \Rightarrow 2\, d\theta = d\phi$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\sqrt{2}}{2}\int_0^{\pi/2} \sin^{-1/2}\phi \,\frac{d\phi}{2}$$

$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{2}\, B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\therefore \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4}\, B\left(\frac{3}{4}, \frac{1}{2}\right) \frac{\sqrt{2}}{8}\, B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{32}\, \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\, \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{2}}{32}\, \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{\sqrt{2}}{8}\pi = \frac{\pi}{4\sqrt{2}}$$

32. Prove that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \frac{B(m, n)}{a^n (1+a)^m}$

Solution : Put $\dfrac{x}{a+x} = \dfrac{t}{a+1}$

$\therefore \quad x(a+1) = t(a+x) \quad \Rightarrow x(a+1-t) = at$

$$\Rightarrow x = \frac{at}{a+1-t}$$

$$\therefore dx = \frac{(a+1-t)\, a\, dt - at(-dt)}{(a+1-t)^2}$$

$$= \frac{a(a+1)\, dt}{(a+1-t)^2}$$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \int_0^1 \frac{\left(\frac{at}{a+1-t}\right)^{m-1}\left(1-\frac{at}{a+1-t}\right)^{n-1}}{\left(a+\frac{at}{a+1-t}\right)^{m+n}}\, \frac{a(a+1)}{(a+1-t)^2}\, dt$$

$$= \int_0^1 \frac{(at)^{m-1}(a+1-t-at)^{n-1}\, a(a+1)}{(a^2+a-at+at)^{m+n}}\, dt$$

$$= \int_0^1 \frac{a^{m-1}t^{m-1}(a+1)^{n-1}(1-t)^{n-1}\, a(a+1)}{a^{m+n}(1+a)^{m+n}}\, dt$$

$$= \frac{1}{a^n(a+1)^m}\int_0^1 t^{m-1}(1-t)^{n-1}\, dt$$

$$= \frac{B(m, n)}{a^n(a+1)^m}$$

33. Evaluate $\displaystyle\int_3^7 \sqrt[4]{(x-3)(7-x)}\, dx$

Solution : Put $x - 3 = y \Rightarrow dx = dy$

$$\therefore \quad I = \int_0^4 y^{1/4}(7-3-y)^{1/4}\, dy$$

$$\int_0^4 y^{1/4}(4-y)^{1/4}\, dy$$

Put $y = 4t \Rightarrow dy = 4dt$

$$\therefore \quad I = \int_0^1 4^{1/4}\, t^{1/4}\, 4^{1/4}\, (1-t)^{1/4}\, 4dt$$

$$= 8\int_0^1 t^{1/4}(1-t)^{1/4}\, dt$$

$$= 8B\left(\frac{5}{4}, \frac{5}{4}\right) = \frac{8\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)} = 8\,\frac{\left(\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{8\cdot 4}{4^2\cdot 3}\, \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}} = \frac{2}{3}\, \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}}$$

34. If $\mathbf{I}_n = \displaystyle\int_0^\infty e^{-x}x^{n-1}\, dx$ **for** $n > 0$, **find** $\dfrac{\mathbf{I}_{n+1}}{\mathbf{I}_n}$

Solution : We have

$$I_n = \int_0^\infty e^{-x}x^{n-1}\, dx = \Gamma(n)$$

$$\therefore \quad I_{n+1} = \Gamma(n+1) = n\Gamma(n) = nI_n$$

$$\therefore \quad \frac{I_{n+1}}{I_n} = n.$$

35. If $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$**, prove that**

$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$

Solution : We know that $B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Put $m + n = 1 \Rightarrow m = 1 - n$

$$\therefore \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

$$\therefore \Gamma n\,\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

Deduction : Put $y = t^{1/4} \Rightarrow dy = \frac{1}{4}t^{-3/4}\,dt$

$$\therefore \int_0^\infty \frac{dy}{1+y^4} = \int_0^\infty \frac{1}{1+t}\frac{1}{4}t^{-3/4}\,dt$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{-3/4}}{1+t}\,dt$$

$$= \frac{1}{4}\Gamma\tfrac{1}{4}\,\Gamma\left(1-\tfrac{1}{4}\right) \quad \text{(By above result)}$$

$$= \frac{1}{4}\Gamma\tfrac{1}{4}\,\Gamma\tfrac{3}{4} = \frac{1}{4}\sqrt{2}\pi = \frac{\pi}{2\sqrt{2}}$$

36. Prove that $\int_0^{\pi/2} \tan^n\theta\,d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2}$ $(-1 < n < 1)$

Solution : $\int_0^{\pi/2} \tan^n\theta\,d\theta = \int_0^{\pi/2} \sin^n\theta\cos^{-n}\theta\,d\theta$

$$= \frac{1}{2}B\left(\frac{n+1}{2}, \frac{1-n}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{1-n}{2}}{\Gamma 1}$$

$$= \frac{1}{2}\Gamma\frac{n+1}{2}\,\Gamma\left(1-\frac{n+1}{2}\right) = \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{1}{2}\frac{\pi}{\cos\frac{n\pi}{2}} = \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Here $n + 1 > 0$ and $1 - n > 0$

$\therefore n > -1$ and $n < 1$

$\therefore -1 < n < 1$

37. Establish Dirichlet's integral $\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$

Where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ **and** $x + y + z \le 1$.

Solution : We have

$0 \le z \le 1 - x - y$

$0 \le y \le 1 - x$

$0 \le x \le 1$

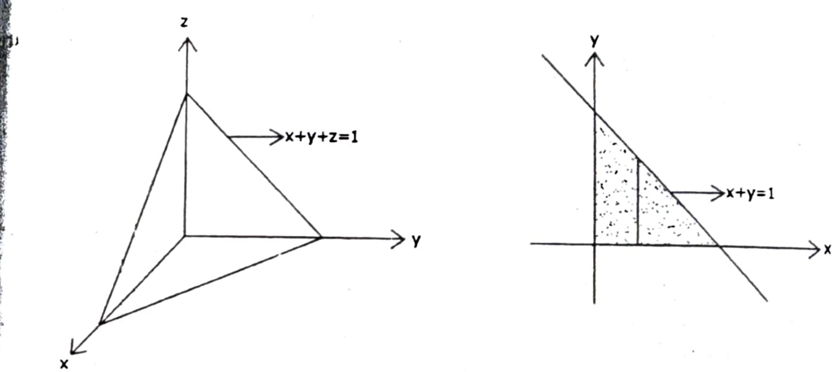

Fig 4.3

$$\therefore I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} x^{l-1}y^{m-1}\left[\frac{z^n}{n}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{n}\int_0^1\int_0^{1-x} x^{l-1}y^{m-1}(1-x-y)^n\,dy\,dx$$

$$= \frac{1}{n}\int_0^1 x^{l-1}\left\{\int_0^{1-x} y^{m-1}(1-x-y)^n\,dy\right\}dx$$

Put $y = (1-x)\,t \Rightarrow dy = (1-x)\,dt$

$$\therefore I = \frac{1}{n}\int_0^1 x^{l-1}\left\{\int_0^1 (1-x)^{m-1}t^{m-1}(1-x)^n(1-t)^n(1-x)\,dt\right\}dx$$

$$= \frac{1}{n}\int_0^1 x^{l-1}(1-x)^{m+n}\left\{\int_0^1 t^{m-1}(1-t)^n\,dt\right\}dx$$

$$= \frac{1}{n} B(m, n+1) \int_0^1 x^{l-1} (1-x)^{m+n}\, dx$$

$$= \frac{B(m, n+1)}{n} \cdot B(l, m+n+1)$$

$$= \frac{1}{n} \frac{\Gamma m\, \Gamma(n+1)}{\Gamma(m+n+1)} \cdot \frac{\Gamma l\, \Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma m\; n\Gamma n\, \Gamma l}{\Gamma(l+m+n+1)} = \frac{\Gamma m\, \Gamma n\, \Gamma l}{\Gamma(l+m+n+1)}$$

38. Prove that $\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1}\, dx\, dy = \frac{\Gamma m\, \Gamma n}{4a^m b^n}$**, where a, b, m, n are positive constants.**

Solution : We have

$$I = \int_0^\infty \int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy$$

$$= \int_0^\infty \left\{ \int_0^\infty e^{-ax^2} x^{2m-1}\, dx \right\} e^{-by^2}\, y^{2n-1}\, dy$$

But from example 7, we know that

$$\int_0^\infty e^{-ax^2} x^{2m-1}\, dx = \frac{\Gamma m}{2a^m}$$

$$\therefore\ I = \int_0^\infty \frac{\Gamma m}{2a^m} e^{-by^2} y^{2n-1}\, dy$$

$$= \frac{\Gamma m}{2a^m} \int_0^\infty e^{-by^2} y^{2n-1}\, dy$$

$$= \frac{\Gamma m}{2a^m} \cdot \frac{\Gamma n}{2b^n} = \frac{\Gamma m\, \Gamma n}{4a^m b^n}$$

EXERCISE

1. Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence show that $\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{n+2}{2}} \frac{\sqrt{\pi}}{2} = \int_0^{\pi/2} \cos^n\theta\, d\theta$

Evaluate $\int_0^1 (1-x^3)^{-1/2}\, dx$ $\left[\text{Ans. : } \frac{1}{3} B\left(\frac{1}{3}, \frac{1}{2}\right)\right]$

Evaluate $\int_0^\infty x^4 e^{-x}\, dx$ [Ans. : 24]

Evaluate $\int_0^\infty x^6 e^{-3x}\, dx$ $\left[\text{Ans. : } \frac{80}{243}\right]$

5. Evaluate $\int_0^1 x^4 (1-x)^3\, dx$ $\left[\text{Ans. : } \frac{1}{280}\right]$

6. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$ $\left[\text{Ans. : } \frac{\pi}{\sqrt{2}}\right]$

7. Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$ $\left[\text{Ans. : } \frac{1}{5005}\right]$

8. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\frac{1}{n}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

9. Prove that $B(n, m) = \frac{\sqrt{\pi}\, \Gamma n}{2^{2n-1} \Gamma\left(n+\frac{1}{2}\right)}$

10. Evaluate $\int_0^1 x^5 (1-x^3)^3\, dx$ $\left[\text{Ans. : } \frac{1}{60}\right]$

11. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n} \sqrt{\pi}$

12. Prove that $\int_0^a x^{n-1} (a-x)^{m-1}\, dx = a^{m+n-1} B(m, n)$ (**Hint :** Put $x = ay$)

13. Prove that $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}$

14. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{B(m, n)}{a^n (a+b)^m}$ $\left[\textbf{Hint. :}\ \text{Take } \frac{x}{a+bx} = \frac{z}{a+b}\right]$

15. Prove that $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} = 0,\ m > 0, n > 0.$

16. Prove that $n\, B(m+1, n) = m\, B(m, n+1)$

17. Prove that $\int_0^2 (8-x^3)^{-1/3}\, dx = \frac{1}{3} \Gamma\frac{1}{3} \Gamma\frac{2}{3}$

4.8 The Error Function :

The error function is defined and denoted by $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

Note : The error function occurs in statistics and different studies in physics, and heat conduction.

The complementary error function is defined and denoted by

$$erfc\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

4.9. Properties of Error Function :

1. $erf\,(0) = 0$

Proof : Put $x = 0$ in $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

$$\therefore \quad erf\,(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2} dt = 0$$

2. $erf\,(\infty) = 1$

Proof : We have $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

$$\therefore \quad erf\,(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt$$

But we know that $\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx$

$$\therefore\ erf\,(\infty) = \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

3. $erf\,(-x) = -\,erf\,(x)$

Proof : We have $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

$$\therefore \quad erf\,(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt$$

Put $t = -v \Rightarrow dt = -dv$

$$\therefore \quad erf\,(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} (-dv)$$

$$= \frac{-2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv = -\,erf\,(x).$$

4. $erf\,(x) = 1 - erfc\,(x)$

Proof : By definitions

$$erf\,(x) + erfc\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \int_0^x e^{-t^2} dt + \int_x^\infty e^{-t^2} dt \right\}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = erf\,(\infty) = 1$$

4.10 Geometrical Interpretation of Error Function :

The graph of $y = e^{-x^2}$ is given by

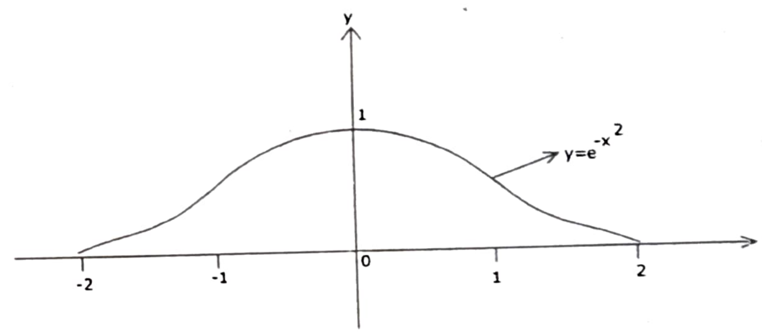

Fig 4.4

The graph of $y = e^{-x^2}$ is symmetrical about y–axis. Thus the area under the curve is given by

$$\text{Area} = \int_{-\infty}^\infty e^{-x^2} dx = 2\int_0^\infty e^{-t^2} dt$$

But $erf\,(\infty) = 1 \Rightarrow \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1$

$$\therefore \quad \text{Area} = \sqrt{\pi}$$

$$\therefore \quad \text{The half area} = \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

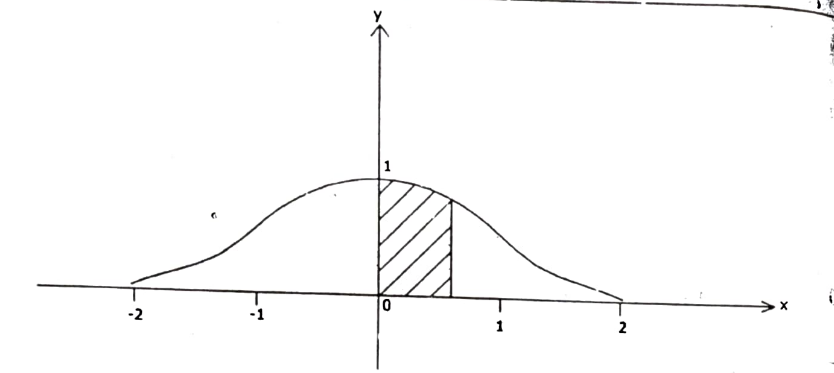

Fig 4.5

$\therefore$ The integral $\int_0^x e^{-t^2} dt$ denotes the area shaded, between $t = 0$ and $t = x$. And the value of error function tend to 1 as $x \to \infty$. The graph of $y = erf\,(x)$ is as follows :

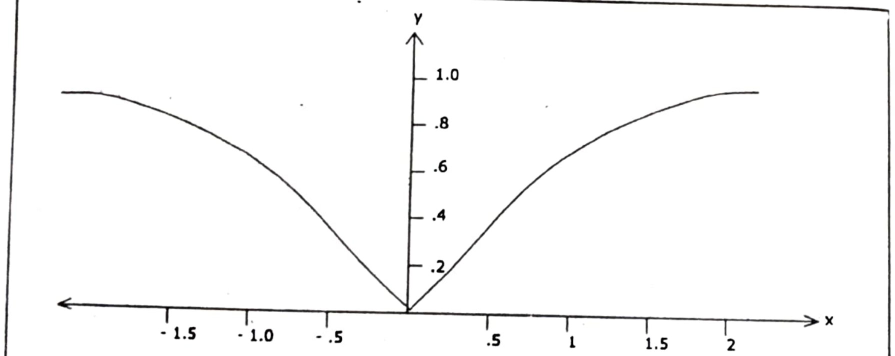

Fig 4.6

SOLVED EXAMPLES

1. Prove that $\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \{erf\,(b) - erf\,(a)\}$, $a < 0 < b$

Solution : $\int_a^b e^{-x^2} dx = \int_a^0 e^{-x^2} dx + \int_0^b e^{-x^2} dx$

$$= -\int_0^a e^{-x^2} dx + \int_0^b e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \left\{ -\frac{2}{\sqrt{\pi}} \int_0^a e^{-x^2} dx + \frac{2}{\sqrt{\pi}} \int_0^b e^{-x^2} dx \right\}$$

$$= \frac{\sqrt{\pi}}{2} \{erf\,(b) - erf\,(a)\}$$

2. Prove that $erfc\,(x) + erfc\,(-x) = 2$.

Solution : LHS $= erfc\,(x) + erfc\,(-x)$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_{-x}^\infty e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^{-\infty} e^{-y^2} (-dy) \quad \text{(Put } t = -y)$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \int_x^\infty e^{-t^2} dt + \int_{-\infty}^x e^{-y^2} dy \right\}$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^\infty 2 e^{-t^2} dt$$

$$= 2\, erf\,(\infty)$$

$$= 2 \qquad (\because\ erf\,(\infty) = 1)$$

3. Prove that $\int_0^\infty e^{-x^2 - 2bx} dx = \frac{\sqrt{\pi}}{2} e^{b^2} (1 - erf\,(b))$

Solution : $\int_0^\infty e^{-x^2-2bx} dx = \int_0^\infty e^{-(x^2+2bx+b^2)+b^2} dx$

$$= \int_0^\infty e^{-(x+b)^2} \cdot e^{b^2} dx = e^{b^2} \int_0^\infty e^{-(x+b)^2} dx$$

$$= e^{b^2} \int_b^\infty e^{-t^2} dt \quad \text{(Put } t = x + b)$$

$$= e^{b^2} \left[\int_0^\infty e^{-t^2} dt - \int_0^b e^{-t^2} dt \right]$$

$$= \frac{\sqrt{\pi}}{2} e^{b^2} [erf\,(\infty) - erf\,(b)]$$

$$= e^{b^2} \frac{\sqrt{\pi}}{2} [1 - erf\,(b)]$$

4. Prove that $\frac{d}{dx}[erfc\,(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$

Solution : $\frac{d}{dx}[erfc\,(ax)] = \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} \int_{ax}^{\infty} e^{-t^2} dt\right]$

We use the following formula for differentiation under the integral sign :

$$\frac{d}{d\alpha}\int_{g(\alpha)}^{h(\alpha)} f(x,\alpha)\,dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx + h'(\alpha) f[h(\alpha),\alpha] - g'(\alpha) f[g(\alpha),\alpha]$$

$$\therefore \frac{d}{dx}[erfc\,(ax)] = \frac{2}{\sqrt{\pi}}\left[\int_{ax}^{\infty}\left(\frac{\partial}{\partial x}e^{-t^2}\right)dt + \frac{d}{dx}(\infty)\,e^{-\infty} - \frac{d}{dx}(ax)\,e^{-a^2x^2}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + 0 - ae^{-a^2x^2}\right] = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

Similary, we can prove that

$$\frac{d}{dx}[erf\,(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$

5. Prove that $erf\,(x) = \frac{2}{\sqrt{\pi}}\left\{x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right\}$

Solution : We have $erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

We know that $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

$$\therefore \quad erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - t^2 + \frac{t^4}{2!} - \frac{t^7}{3!} + \ldots\right] dt$$

$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{3} + \frac{t^5}{2!5} - \frac{t^7}{3!7} + \ldots\right]_0^x$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]$$

6. Compute (i) ***erf*** **(0.3) (ii) erf (0.5)**

Solution : $erf\,(0.3) = \frac{2}{\sqrt{\pi}}\left[0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{10} - \frac{(0.3)^7}{42}\right]$ (approximation)

$= 0.3248$

Similarly, $erf\,(0.5) = 0.5204$

Prove that $\int_0^t erfc\,(ax)\,dx = t\,erfc\,(at) - \frac{e^{-a^2t^2}}{a\sqrt{\pi}} + \frac{1}{a\sqrt{\pi}}$

Solution : $\int_0^t erfc\,(ax)\,dx = \int_0^t erfc\,(ax)\cdot 1\,dx$

$$= [erfc\,(ax)\,x]_0^t - \int_0^t \frac{d}{dx}(erfc(ax)\cdot x\,dx$$

$$= erfc\,(at)\,t + \int_0^t \frac{2a}{\sqrt{\pi}} e^{-a^2x^2} x\,dx \qquad \left(\because \frac{d}{dx}(erf\,(ax)) = -\frac{2a}{\sqrt{\pi}} e^{-a^2x^2}\right)$$

$$= t\,erfc(at) - \frac{1}{a\sqrt{\pi}}\left[e^{-a^2x^2}\right]_0^t$$

$$= t\,erfc(at) - \frac{1}{a\sqrt{\pi}}\left(e^{-a^2t^2} - 1\right)$$

$$= t\,erfc(at) - \frac{1}{a\sqrt{\pi}} e^{-a^2t^2} + \frac{1}{a\sqrt{\pi}}$$

8. Prove that $\frac{d}{dx}\left[erf(\sqrt{x})\right] = \frac{e^{-x}}{2\sqrt{x}}$

Solution : $\frac{d}{dx}\left[erf(\sqrt{x})\right] = \frac{d}{dx}\left\{\int_0^{\sqrt{x}} e^{-t^2} dt\right\}$

$$= \int_0^{\sqrt{x}} \frac{\partial}{\partial x}\left(e^{-t^2}\right) dt + \left(\frac{d\sqrt{x}}{dx}\right) e^{-x} - \frac{d}{dx}(0)\,e^0$$

$$= 0 + \frac{1}{2\sqrt{x}} e^{-x} - 0$$

$$= \frac{e^{-x}}{2\sqrt{x}}$$

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