11 MECHANICAL VIBRATIONS

11·1. Introduction : A ***mechanical vibration*** is the **periodic** ***motion*** *of a particle or a body which oscillates* ***about a position of equilibrium.*** *It generally results when a system is displaced from a position of stable equilibrium.* The system tends to return to this position under the action of ***restoring forces.*** The restoring forces may be either ***elastic forces,*** as in the case of a mass attached to a ***spring, or gravitational forces,*** as in the case of a **pendulum.**

Most vibrations in machines and structures are *undesirable* because of the *increased stresses* and *energy losses* which accompany them. They should therefore be eliminated or reduced as much as possible by appropriate design. The analysis of vibrations has become increasingly important in recent years owing to the current trend toward higher-speed machines and lighter structures. There is every reason to expect that this trend will continue and that an even *greater need for vibration analysis* will develop *in the future.*

11·2. Definitions of Common Terms :

(1) **Vibration :** It is the *periodic motion* of a body or system of connected bodies *displaced from a position of equilibrium.*

In general, there are *two types of vibration,* ***free and forced.***

(2) **Free Vibration :** When a *motion is maintained* by the *restoring forces* like *gravitational or elastic forces* only, the vibration is said to be *free vibration.* Swinging motion of a *pendulum* or the vibration of an *elastic rod* are the examples of free vibration.

(3) **Forced Vibration :** When an *external periodic* or *intermittent force* is applied to the system, the resulting motion is described as a *forced vibration.*

Both of these types of vibration may be either ***damped or undamped.***

(4) **Damped Vibrations :** When the effects of ***friction*** are ***considered,*** the vibrations are said to be damped *vibrations.* Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually damped.

(5) **Undamped Vibrations :** When the effects of ***friction*** may be ***neglected,*** the vibrations are said to be *undamped.* The undamped vibrations can *continue indefinitely* because frictional effects are neglected in the analysis. However, all vibrations are actually damped to some degree.

(6) **Single-degree - of - freedom System :** In this system, the bodies are **constrained to move** only **in one direction** hence *require only one coordinate* to specify completely the position of system at any time.

We limit our *present study* to the simpler types of vibrations of a body or a system of bodies with one *degree of freedom.*

(7) **Multi-degree - of - freedom System :** In this system, the bodies are ***free to move in any direction*** (more than one) which requires ***multy coordinates*** to specify completely the position of system at any time.

(8) **Period of the Vibration :** It is the **time interval** required for the system to *complete a full cycle of motion.*

(9) **Frequency of the Vibration :** It is the **number of cycles** completed *per unit of time,* which is also the *reciprocal of the period of vibration.*

The frequency of an undamped free vibration is known as **natural frequency.**

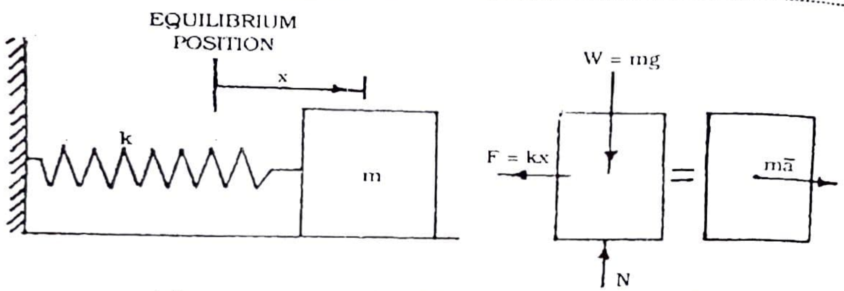
(10) **Amplitude of the Vibration :** It is the **maximum displacement** of the system *from its position of equilibrium.*

In ***undamped*** motion, the **amplitude** of the vibration **remains *constant,*** but it is affected by the magnitude of the damping forces in damped motion.

This chapter is limited to **undamped vibrations** only.

11·3. Undamped Free Vibrations of Particles. Simple Harmonic Motion :

The *simplest type* of vibrating motion is *undamped free vibration,* represented by the *model* shown below.



(a) Fig. 11·1 (b)

The block has a mass m and is attached to a spring having a stiffness k. Vibrating motion occurs when displacing the block a distance x from its equilibrium position and allowing spring to restore it to its original position. *Provided the supporting surface is smooth, oscillation will continue indefinitely.*

The time-dependent path of motion of the block may be determined by applying the *equation of motion* to the block when it is in the displaced position x.

From the *free-body diagram*,

$\overset{\rightarrow}{+} \Sigma F_x = ma_x;$ $\qquad -k\,x = m\,a$

$$\text{but, } a = \frac{dx^2}{dt^2} = \ddot{x}$$

$$\therefore \quad \boxed{m\,\ddot{x} + k\,x = 0} \qquad \text{—— (1)}$$

Here, same sign convention should be used for the acceleration $\ddot{x}$ and for the displacement x.

Using **constant p,**

$$p^2 = \frac{k}{m}$$

the eq. (1) can be written in **"standard form"** as

$$\boxed{\ddot{x} + p^2 x = 0} \qquad \text{—— (2)}$$

*The **motion** described by **this equation** is called **simple harmonic motion.***

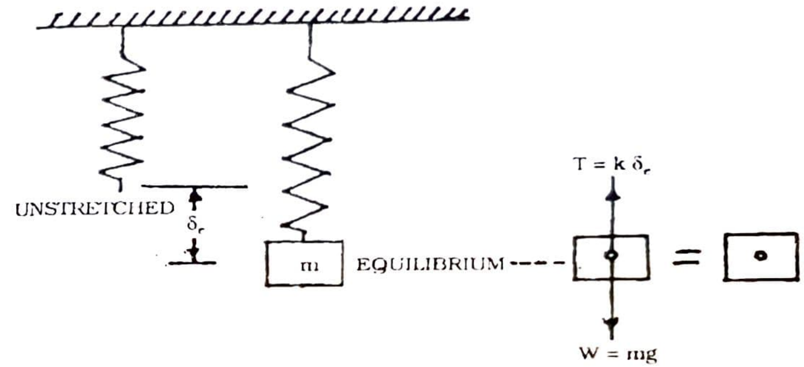
It is characterized by the fact that the *acceleration is proportional to the displacement and of opposite direction.*

Here,

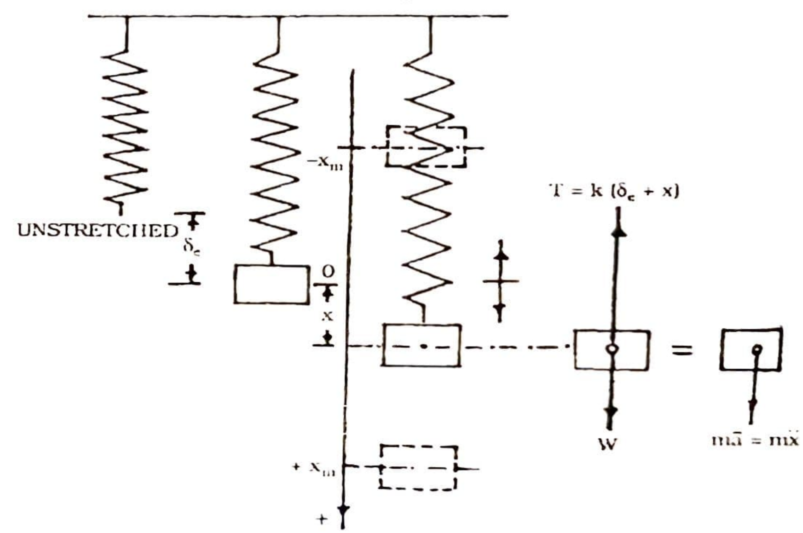
$$\boxed{p = \sqrt{\frac{k}{m}}} \qquad \text{—— (3)}$$

*The **constant p** is called the **circular frequency**,* expressed in rad/s.

Eq. (2) may also be obtained by *considering the **block to be suspended** as shown below.*



(a) Fig. 11·2 (b)



(a) Fig. 11·3 (b)

When the block is in *equilibrium*, the spring exerts an upward force of $T = W = k\,\delta_e$ on the block. Hence, when the block is displaced a distance x downward from this position, the magnitude of the spring force is

$$T = W + k\,x = k\,\delta_e + k\,x = k\,(\delta_e + x).$$

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Applying the equation of motion gives,

$+\downarrow \Sigma F_y = m a_y$, $\quad -k(\delta_e + x) + W = m\ddot{x}$

$\therefore \quad -k(\delta_e + x) + k\,\delta_e = m\ddot{x}$

$$\therefore \quad \boxed{m\ddot{x} + kx = 0} \qquad \text{——— (1)}$$

$$\text{and} \quad \boxed{\ddot{x} + p^2 x = 0} \qquad \text{——— (2)}$$

$$\text{where,} \quad \boxed{p = \sqrt{\frac{k}{m}}} \qquad \text{——— (3)}$$

The equation,

$$\ddot{x} + p^2 x = 0$$

is a *homogeneous, second-order, linear, differential* equation with constant coefficients.

The *general solution* of this equation is

$$x = A \sin pt + B \cos pt \qquad \text{——— (4)}$$

where A and B are the *constants of integration*.

The *velocity* and *acceleration* at time t can be obtained as

$$v = \dot{x} = Ap \cos pt - Bp \sin pt \qquad \text{——— (5)}$$

$$a = \ddot{x} = -Ap^2 \sin pt - Bp^2 \cos pt \qquad \text{——— (6)}$$

Eq. (6) and eq. (4) if substituted in to eq. (2), the eq. (2) is indeed satisfied and hence eq. (4) represents the *true solution* to eq. (2).

The *values* of the *constants A and B* depend upon the *initial conditions* of the motion.

For example,

let, at $t = 0$, $x = x_1$ and $v = v_1$

from eq. (4), $B = x_1$

and from eq. (5), $A = v_1/p$

Equation ***x = A sin pt + B cos pt*** *may also be expressed in terms of* **simple sinusoidal motion.**

Let

$$A = x_m \cos \phi \qquad \text{——— (7)}$$

$$\text{and} \quad B = x_m \sin \phi \qquad \text{——— (8)}$$

where x_m and ϕ are *new constants* to be determined *in place of* A and B.

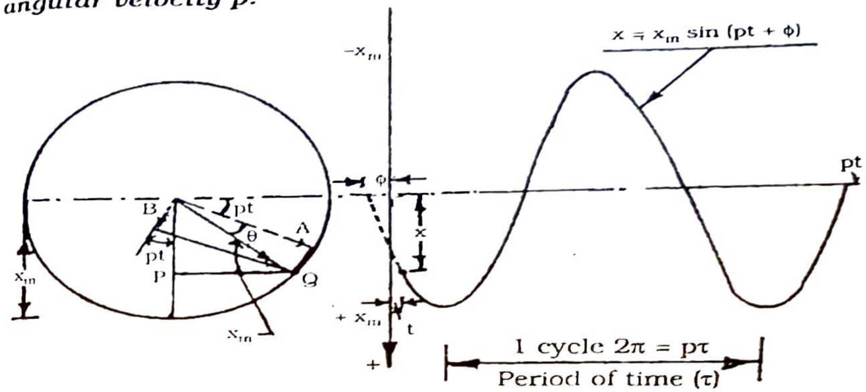
Substituting into eq. (4) yields,

$$x = x_m \cos\phi \sin pt + x_m \sin\phi \cos pt$$

Since $\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$, then

$$\boxed{x = x_m \sin(pt + \phi)} \qquad \text{——— (9)}$$

The simple harmonic motion of P along the x axis may be obtained by projecting on this axis the motion of a point Q discribing an auxiliary circle of radius x_m with a constant angular velocity p.



(a) **Fig. 11·4** (b)

Denoting by ϕ the angle formed by the vectors OQ and $\bar{A}$, we write

$$OP = OQ \sin(pt + \phi)$$

$$\therefore \quad \boxed{x = x_m \sin(pt + \phi)} \qquad \text{——— (9)}$$

$$x_m = \sqrt{A^2 + B^2} \qquad \text{——— (10)}$$

The **maximum displacement** (x_m) of the block from its equilibrium position is defined as the ***amplitude*** of vibration.

The *angular velocity* p of the point Q which describes the *auxiliary circle* is known as the ***circular frequency*** of the vibration and is measured in rad/s.

The *angle* ϕ which defines the initial position of Q on the circle is called ***phase angle.*** *It also* represents the amount by which the curve is displaced from the origin when $t = 0$.

$$\phi = \tan^{-1} \frac{B}{A} \qquad \text{——— (11)}$$

Note that the sine curve completes one cycle in time $t = \tau$ (tau) when $p\tau = 2\pi$.

$$\boxed{\textbf{Period of vibration} = \tau = \frac{2\pi}{p}} \qquad \text{——— (12)}$$

The *frequency* (f) of vibration is defined as the *number of cycles completed per unit of time*, which is the reciprocal of the period.

$$f = \frac{1}{\tau} = \frac{p}{2\pi} \qquad \text{——— (13)}$$

The unit of frequency is a frequency of 1 cycle per second, corresponding to a period of 1s. In terms of base units the unit of frequency is thus $1/s$ or s^{-1}. It is called a *heartz* (H_z) in the SI system of units.

From above equation it follows that a frequency of $1\ s^{-1}$ or 1 Hz corresponds to a circular frequency of 2π rad/s.

In problems involving *angular velocities* expressed in revolution per minute (rpm), we have

$$1\ r/min = \frac{1}{60}\ s^{-1} = \frac{1}{60}\ H_x$$

or

$$1\ r/min = \left(\frac{2\pi}{60}\right)\ rad/s.$$

If $x = x_m \sin(pt + \phi)$

then

$$v = \dot{x} = x_m\, p \cos(pt + \phi) \qquad \text{——— (14)}$$

$$a = \ddot{x} = -x_m\, p^2 \sin(pt + \phi) \qquad \text{——— (15)}$$

It may be noted that *maximum values of magnitudes of the velocity and acceleration* are

$$\boxed{\begin{aligned} v_m &= x_m\, p \\ a_m &= x_m\, p^2 \end{aligned}} \qquad \text{——— (16)}$$

11·4 Procedure For Analysis :

The *circular frequency* p of a rigid body or system of connected bodies having a single degree of freedom can be determined using the following procedure :

(1) Free-Body Diagram : Draw the *free-body diagram* of the body when the body is displaced by *asmallamount* from its *equilibrium* position. Locate the body with respect to its equilibrium position by using an appropriate *inertial coordinate x.*

The acceleration of the body's mass center a_G or the body's angular acceleration α should have a sense which is in the positive direction of the position coordinate. If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(M_P)_{eff}$ is to be used, then it may be beneficial to also draw the kinetic diagram since it graphically accounts for the components $m(a_G)_x$, $m(a_G)_y$, and $I_x\ \alpha$, and thereby makes it convenient for visualizing the terms needed in the moment sum $\Sigma(M_P)_{eff}$.

(2) Equation of Motion : Apply the *equation of motion* to relate the *elastic* or *gravitational restoring forces* and *couples* acting on the body to the body's acceleration motion.

(3) Kinematics : Express the body's accelerated motion in terms of the second time derivative of the position coordinate, x. Substitute this result into the equation of motion and determine p by rearranging the terms so that the resulting equation is of the form $\ddot{x} + p^2x = 0$.

Once the *circular frequency* p of the body is known, the *period of vibration* (τ), *natural frequency* (f) and other vibrating characteristics of the body (say a_m, v_m etc.) can be established using the equations mentioned before.

***Examples 1 to 5** given in "Solved Examples" are pertaining to this section.*

11·5. Application of the Principle of Conservation of Energy (Energy Methods) :

The *simple harmonic motion* of a body is due only to gravitational and eastic restoring forces acting on the body. Since these types of forces are *conservative*, it is possible to use the *conservation of energy equation* to obtain the body's *natural frequency or period of vibration.*

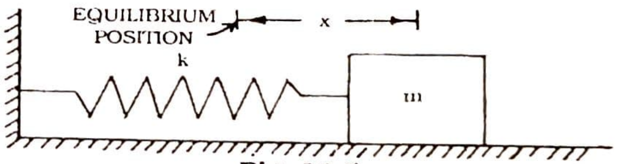


Fig. 11·5

When block is displaced an arbitrary distance x from the equilibrium position,

$$K.E. = T = \frac{1}{2}\ mv^2$$

and

$$P.E. = V = \frac{1}{2}\ kx^2$$

By the conservation of energy equation, it is necessary that

$$T + V = \text{constant}$$

$$\frac{1}{2}\ m\,\dot{x}^2 + \frac{1}{2}\ k\,x^2 = \text{constant} \qquad \text{——— (1)}$$

Differentiating w.r. to time,

$$m\,\dot{x}\,\ddot{x} + k\,x\,\dot{x} = 0$$

$$\dot{x}\,(m\,\ddot{x} + k\,x) = 0$$

But velocity $\dot{x} \neq 0$ always in a vibrating system

$$\therefore \quad \ddot{x} + p^2x = 0 \qquad \text{where,} \quad p = \sqrt{\frac{k}{m}}$$

which is the *equation of simple harmonic motion.*

If the energy equation is written for a *system of connected bodies*, the *natural frequency* or *the equation of motion* can also be determined by *time differentiation*. Here it is *not necessary* to dismember the system to account for reactive and connective forces which do no work. Also, by this method, the circular frequency p may be obtained *directly*.

For example, consider the total mechanical energy of the block and spring as shown above when the block is at its *maximum displacement*.

At maximum displacement, block is temporarily at rest.

Hence, K. E. = 0

and P.E. stored in spring = Maximum = $\frac{1}{2} kx^2_{max}$.

Thus eq (1) becomes,

$$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \text{constant}$$

$$\therefore \quad 0 + \frac{1}{2} kx^2_{max} = \text{constant}$$

At equilibrium position (when block passes through it)

$$\text{K.E.} = \text{maximum} = \frac{1}{2} m(\dot{x})^2_{max}$$

and P.E. = 0.

Thus eq. (1) becomes

$$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \text{constant}$$

$$\frac{1}{2} m(\dot{x})^2_{max} + 0 = \text{constant}$$

Since vibrating motion of block is *harmonic*, the *solution* for *displacement and velocity* may be written in the form of $x = C \sin (pt + \phi)$ and its time derivative, i.e.

$$x = C \sin (pt + \phi), \quad \dot{x} = Cp \cos (pt + \phi)$$

$$\text{so that} \quad x_{max} = C, \qquad \dot{x}_{max} = C\,p$$

Applying the *conservation of energy equation* (T + V = const.) yields,

$$V_{max} = T_{max}, \quad \frac{1}{2} kx^2_{max} = \frac{1}{2} m\dot{x}^2_{max} = \text{constant}$$

Solving for p yields

$$p = \sqrt{\frac{k}{m}}$$

which is identical to pervious equation.

11·6. Procedure For Analysis :

To determine the *circular frequency* p of a body or system of connected bodies *using the conservation of energy equation*, the following procedure should be adopted.

(1) Energy Equation : Draw the body when it is displaced by a *small amount* from its equilibrium position and define the location of the body from its equilibrium position by an appropriate position coordinate x.

Formulate the equation of energy for the body. T + V = constant in terms of the position coordinate.

Recall that, in general the *kinetic energy* must be account for both the *body's translational and rotational motion.*

$$T = \frac{1}{2} mv_G^2 + \frac{1}{2} I_G\omega^2$$

and that the *potential energy* is the sum of the gravitational and elastic potential energies of the body.

$$V = V_g + V_e$$

In particular, V_g should be measured. from a datum for which x = 0 (*equilibrium position*).

(2) Time Derivative : Take the *time derivative* of the energy equation and factor out the common terms. The resultant differential equation represents the *equation of motion* for the system.

The value of p is obtained after rearranging the terms in the "*standard form*"

$$\ddot{x} + p^2x = 0$$

Examples 6 and 7 *in "Solved Examples" are pertaining to this section.*

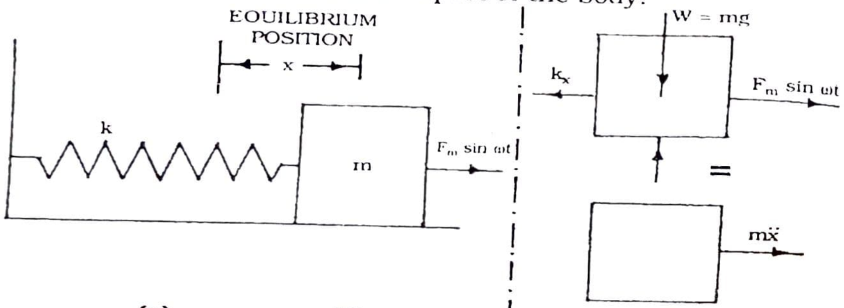
11·7. Undamped Forced Vibrations :

The most important vibrations from the point of view of engineering applications are the undamped forced vibrations of a system.

This vibrations occur when a system is subjected to a periodic force or when it is elastically connected to a support which has an alternating motion.

(A) Periodic Force :

A convenient "*model*" of block and spring is shown which represents the vibrational characteristics of a system subjected to a *periodic force* $F = F_m \sin \omega t$. *This force has a maximum magnitude of* F_m *and a forcing frequency* ω. This force may be an actual *external force applied to the body* or it may be a *centrifugal force* produced by the rotation of some unbalanced part of the body.



(a) **Fig. 11·6** (b)

x = displacement of body measured from its equilibrium position.

Applying the *equation of motion*,

$\overset{+}{\rightarrow}\Sigma F_x = ma_x$, $\quad F_m \sin \omega t - kx = m\ddot{x}$

$$\boxed{\ddot{x} + \frac{k}{m} x = \frac{F_m}{m} \sin \omega t} \qquad \text{——— (1)}$$

This equation is a nonhomogeneous second-order differential equation having right hand member different from zero.

The general solution consists of a complementary solution, x_{comp}, *plus a particular solution,* x_{parti}.

Complementary Solution : It is determined by setting the term on the right side of above equation (1) to zero and solving the resulting homogeneous equation.

The solution is

$$x_{comp} = A \sin pt + B \cos pt \qquad \text{——— (2)}$$

where, p = *circular frequency*

$= \sqrt{k/m}$.

Particular Solution : As the motion is *periodic*, the particular solution may be of the form

$$x_{parti} = x_m \sin \omega t \qquad \text{——— (3)}$$

where x_m is a constant.

Taking second time derivative and substituting into eq. (1)

$$-x_m \omega^2 \sin \omega t + \frac{k}{m}(x_m \sin \omega t) = \frac{F_m}{m} \sin \omega t$$

Hence, *amplitude*, $x_m = \dfrac{F_m/m}{\dfrac{k}{m} - \omega^2}$

$$\boxed{x_m = \frac{F_m/k}{1 - \left(\dfrac{\omega}{p}\right)^2}} \qquad \text{——— (4)}$$

Substituting in to eq. (3),

$$x_{parti} = \frac{F_m/k}{1 - \left(\dfrac{\omega}{p}\right)^2} \sin \omega t \qquad \text{——— (5)}$$

Therefore, the *general solution is*

$x = x_{comp} + x_{parti}$

$$\boxed{x = A \sin pt + B \cos pt + \frac{F_m/k}{1 - \left(\dfrac{\omega}{p}\right)^2} \sin \omega t} \qquad \text{——— (6)}$$

OR $\quad x = A \sin pt + B \cos pt + x_m \sin \omega t$

Here there are two superposed vibrations.

(1) $x_{comp} = A \sin pt + B \cos pt$, **(2)** $x_{parti} = x_m \sin \omega t$.

x_{comp}. *defines* ***free vibration***, *also called* ***transient vibration***, *since in actual practice, it will soon be damped out by friction forces.* The frequency of this vibration, called the *natural frequency* of the system, depends only upon the constant k of spring and mass m of body, and the constants A and B may be determined by the initial conditions.

x_{parti} *describe the forced vibration of the block caused by the applied force* $F = F_m \sin \omega t$.

This is also called ***steady state***, since it is the only vibration that remains. It is produced and maintained by the *impressed force or impressed support movement*. Its frequency is the *forced frequency* imposed by this force or movement and its amplitude x_m depends upon *frequency ratio* ω/p.

The *magnification factor* is a *ratio* of the *amplitude* x_m of the steady-state vibration to the *static deflection* P_m/k caused by the amplitude of the periodic force F_m.

$$\boxed{\textbf{Magnification Factor} = \frac{x_m}{F_m/k} = \frac{1}{1 - (\omega/p)^2}} \qquad \text{——— (7)}$$

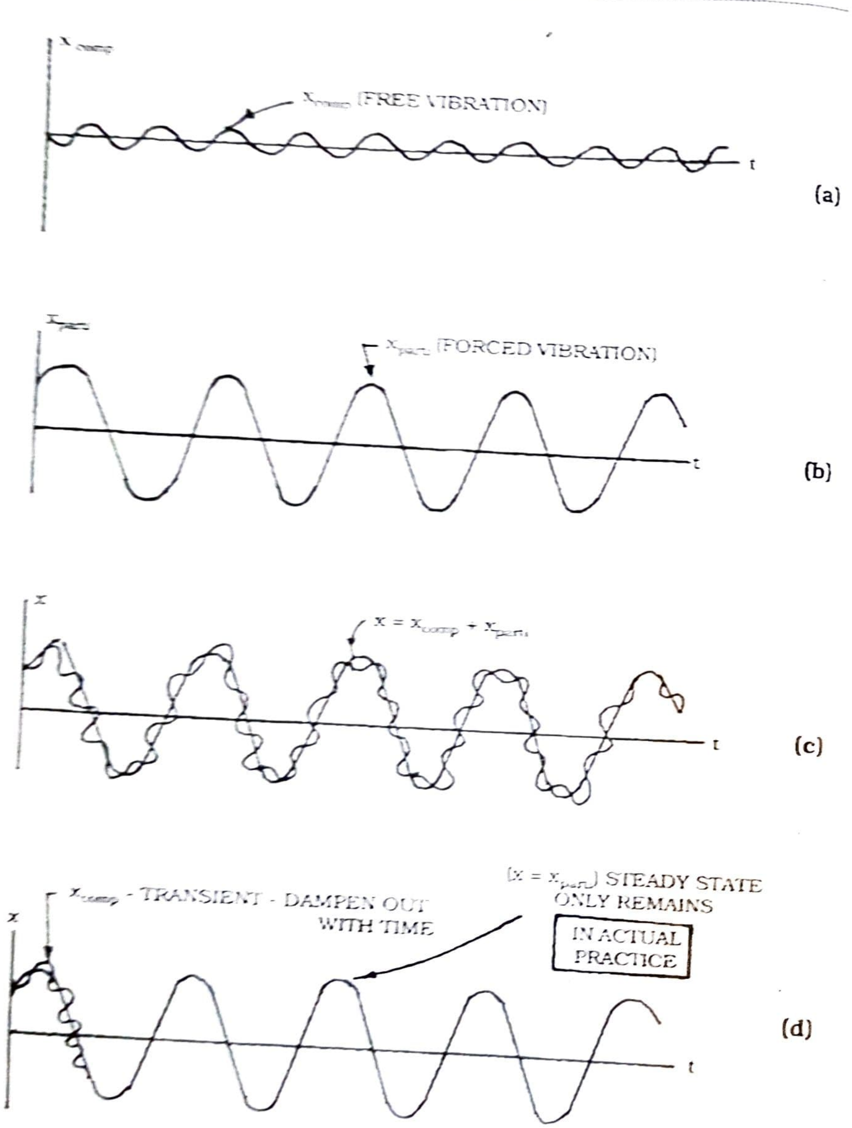


Fig. 11·7

The *magnification factor* has been plotted against the *frequency ratio* ω/p.

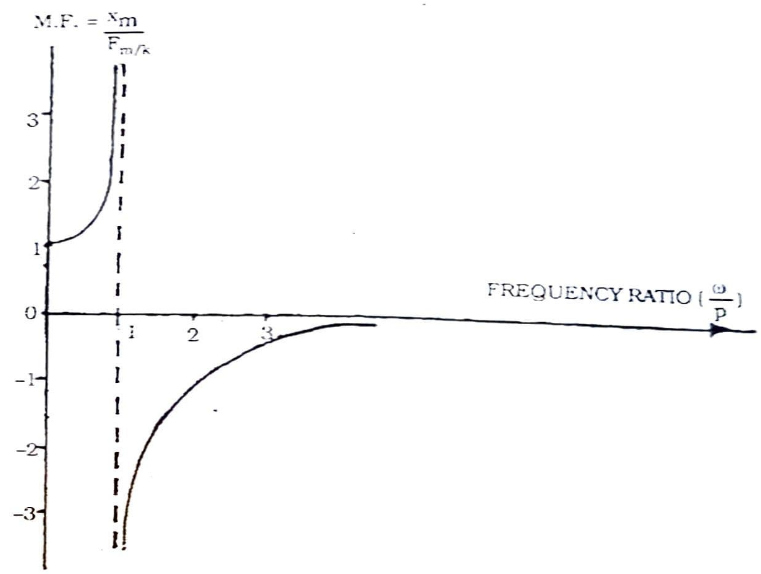


Fig. 11·8

Observations of the graph **M.F. v/s ω/p :**

(1) when ω = 0 or ω << p, M.F. = 1 → forced vibration is ***in phase*** with the impressed force or impressed support movement.

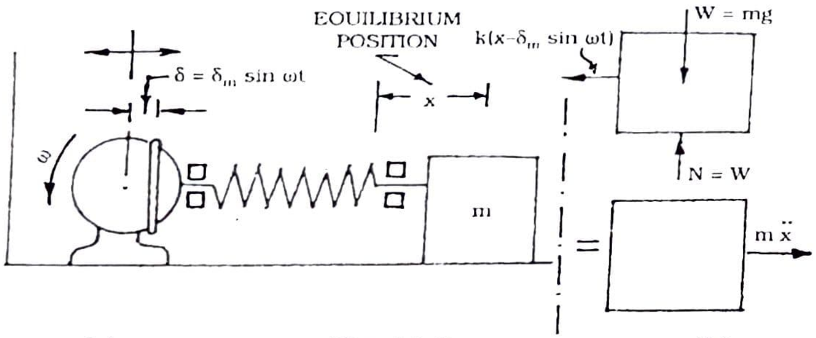
(2) when ω >> p, M.F. ≈ 0 → force vibration is ***180° out of phase*** with the impressed force. (block remains almost stationary).

(3) when ω ≈ p, ω/p ≈ 1 → (forced frequency = natural frequency) amplitude of the forced vibration becomes, too large. The impressed force or impressed support movement is said to be ***in resonance*** with the given system. Resonating vibrations can cause tremendous stress and rapid failure of parts, hence such a situation should be avoided.

(B) Periodic Support Displacement :

Forced vibrations can also arise from the periodic excitation of the suport of the system.

The model shown represents the periodic vibration of a block which is caused by harmonic movement $\delta = \delta_m \sin \omega t$ of the support.



(a) **Fig. 11·9** (b)

$\overset{+}{\rightarrow}\Sigma F_x = ma_x, \quad -k(x - \delta_m \sin \omega t) = m\ddot{x}$

$$\boxed{\ddot{x} + \frac{k}{m} x = \frac{k\delta_m}{m} \sin \omega t} \qquad \text{———— (8)}$$

Comparing with eq. (1) $\ddot{x} + \frac{k}{m} x = \frac{F_m}{m} \sin \omega t$,

this eq. (8) is identical provided F_m *is replaced by* $k\delta_m$.

If this substitution is made in to the solutions defined by eqns. (4) to (6), the results are appropriate for describing the motion of the block when subjected to the support displacement $\delta = \delta_m \sin \omega t$.

Examples 8 and 9 *in "Solved Examples" are pertaining to this section.*

IMPORTANT EQUATIONS

1. Undamped Free Vibrations :

(i) Equation of simple harmonic motion : "standard form"

$$\ddot{x} + p^2 x = 0$$

(ii) Circular frequency $= p = \sqrt{\frac{k}{m}}$

(iii) General solution of the eq. $\ddot{x} + p^2 x = 0$ is

$$x = A \sin pt + B \cos pt$$

A and B are *constants* which are to be found out from *initial conditions.*

(iv) Velocity at time t :

$$v = \dot{x} = A\,p \cos pt - B\,p \sin pt$$

(v) Acceleration at time t :

$$a = \ddot{x} = -Ap^2 \sin pt - Bp^2 \cos pt.$$

(vi) $x = A \sin pt + B \cos pt$ can be expressed in terms of *sinusoidal motion.*

$$x = x_m \sin (pt + \phi)$$

(vii) Amplitude of vibration $= x_m = \sqrt{A^2 + B^2}$

(viii) Phase Angle = ¯ = $\tan^{-1} \frac{B}{A}$

(ix) Period of vibration $= \tau = \frac{2\pi}{p}$

(x) Natural frequency $= f = \frac{1}{\tau} = \frac{p}{2\pi}$

(xi) Velocity $v = \dot{x} = x_m p \cos (pt + \phi)$

(xii) Acceleration $= a = \ddot{x} = -x_m p^2 \sin (pt + \phi)$

(xiii) Max. velo. $= v_m = x_m p$

Max. acceleration $= a_m = x_m p^2$

2. Application of Principle of Conservation of Energy (Energy Methods) :

(i) $T + V = \text{const}$

(ii) $T = \frac{1}{2} m\dot{x}^2$

(iii) $V = \frac{1}{2} kx^2$

(iv) $T_{max} = V_{max}$

(v) $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

3. Undamped Forced Vibrations :

(A) System subjected to Periodic Force, $F = F_m \sin \omega t$

(i) $F = F_m \sin \omega t$

where F_m = maximum magnitude of force

ω = forcing frequency

(ii) **"Standard form" of Vibration :**

$$\ddot{x} + \frac{k}{m} x = \frac{F_m}{m} \sin \omega t$$

(iii) **General solution of above eq. (ii) is**

$$x = x_{comp} + x_{parti}$$

(iv) $x_{comp} = A \sin pt + B \cos pt$

(v) $x_{parti} = x_m \sin \omega t$

where $x_m = \dfrac{F_m/k}{1-(\frac{\omega}{p})^2}$

$$x_{parti} = x_m = \frac{F_m/k}{1-(\frac{\omega}{p})^2} \sin \omega t$$

(vi) $x = x_{comp} + x_{parti}$

$$= A \sin pt + B \cos pt + \frac{F_m/k}{1-(\frac{\omega}{p})^2} \sin \omega t$$

(vii) x_{comp} – **free vibration** – transient vibration – it will soon be damped out by friction forces.

(viii) x_{parti} – **forced vibration** – steady state – it only remains – produced and maintained by impressed force or impressed support movement.

(ix) **Magnification Factor** = M.F.

$$= \frac{x_m}{F_m/k} = \frac{1}{1-(\omega/p)^2}$$

(x) when $\omega = o$ or $\omega << p$, M.F. = 1 → Forced vibration is *in phase* with the impressed force or support movement.

(xi) when $\omega >> p$, M.F. ≈ 0 → forced vibration is *180° out of phase* with the impressed force. (block remains almost stationary)

(xii) when $\omega \approx p$, $\frac{\omega}{p} \approx 1$ → impressed force or support movement is said to be in resonance with the system. Amplitude too large Causes trimendous stress and rapid failure of parts- such a situation should be avoided.

(forced frequency ≈ natural frequency)

(B) **System subjected to Periodic Support Displacement :**

(i) **Support movement** is $\delta = \delta_m \sin \omega t$

(ii) **"Standard form"** of motion is

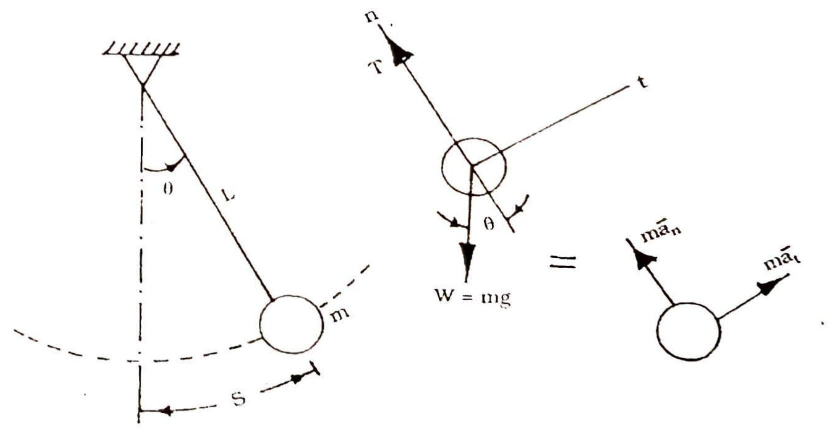
$$\ddot{x} + \frac{k}{m} x = \frac{k\delta_m}{m} \sin \omega t$$

(iii) All equations are identical as of system subjected to periodic force $F = F_m \sin \omega t$ but in all Fm is to be replaced by $k\delta_m$.

SOLVED EXAMPLES

1. Simple Pendulum : Most of the vibrations encountered in engineering applications may be represented by a simple harmonic motion. Many others, although of a different type, may be *approximated* by a simple harmonic motion, provided that their amplitude remains small. Here, let us consider the example of simple pendulum.

Determine the period of vibration for a simple pendulum consisting of a bob of mass m attached to a cord lengthl. *Neglect the size of the bob.*



(a) **Fig. 11·10** (b)

(i) **Free-Body Diagram :** Motion of the system will be related to the position coordinate (x =) θ.

When the bob is displaced by an angle θ, the *restoring force* acting on the bob is created by the *weight component* mgsin θ. Furthermore, a_t acts in the direction of *increasing s* (or θ)

(ii) Equation of motion :

$\Sigma F_t = ma_t$: $\quad -mg \sin\theta = ma_t$ ——— **(1)**

(iii) Kinematics : Now, $s = l\theta$

but $\quad a_t = \dfrac{d^2s}{dt^2} = \ddot{s}$

$\therefore \quad a_t = l\,\ddot{\theta}$ ——— **(2)**

$\therefore$ eq. (1) reduces to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad \text{——— (3)}$$

but, for *small displacement* $\sin\theta \approx \theta$

$$\therefore \quad \ddot{\theta} + \frac{g}{l}\,\theta = 0 \qquad \text{——— (4)}$$

Comparing this equation with $\ddot{x} + p^2x = 0$, which is the "standard form" for simple harmonic motion.

$$\therefore \quad p = \sqrt{g/l}$$

Hence, $\boxed{\text{Period of time reqd. for one swing} = \tau = \dfrac{2\pi}{p} = 2\pi\sqrt{l/g}}$

This interesting result, originally discovered by Galileo Galilei through experiment, indicates that the *period depends only on the length of the cord and not the mass of the pendulum bob.*

Further, displacement and velocity of the bob at a given instant can be found out as under.

Instead of $\quad x = A \sin pt + B \cos pt$

write $\quad \theta = -A \sin(\sqrt{g/l}\; t) + B \cos(\sqrt{g/l}\; t)$

and hence constants A and B can be found out by substituting *initial conditions.*

Then after, use equations

$$v = \dot{x} = Ap\cos pt - Bp\sin pt$$

$$a = \ddot{x} = Ap^2\sin pt - Bp^2\cos pt$$

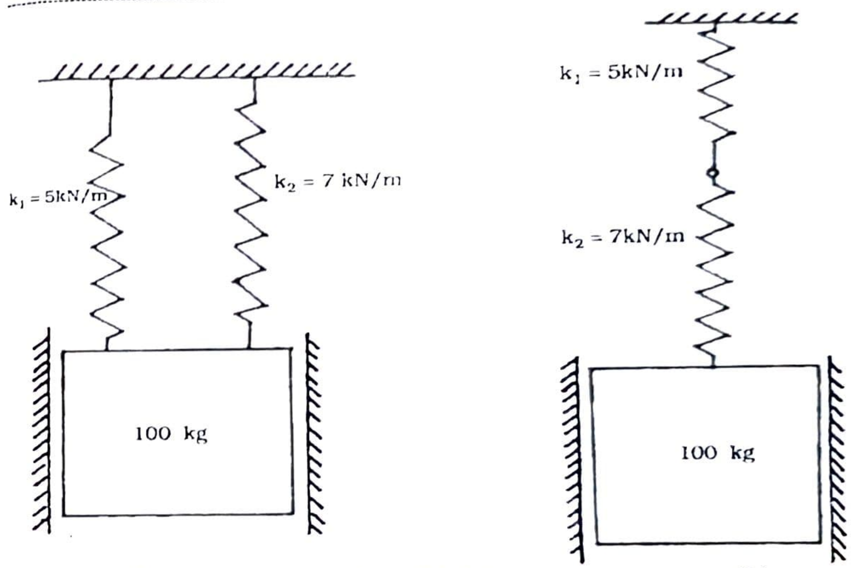
2. A 100-kg block moves between vertical guides as shown. The block is given vibration by intially pulling it down by 50 mm from its equilibrium position and released. The springs may be kept in series or parallel.

Determine for each arrangement

(i) the period of vibration.

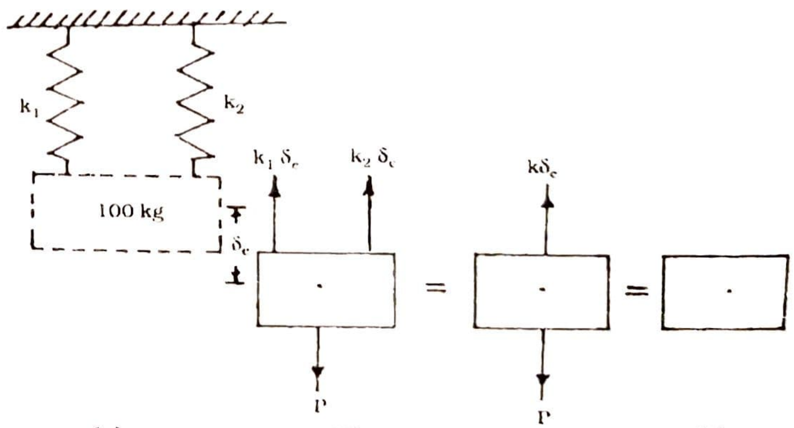
(ii) the maximum velocity of the block.

and **(iii)** the maximum acceleration of the block.



(a) **Fig. 11·11** (b)

(A) Springs in Parallel :



(a) **Fig. 11·12** (b)

Consatnt k of a single equivalent spring :

$$k = \frac{P}{\delta_e} = \frac{k_1\delta_e + k_2\delta_e}{\delta_e} = k_1 + k_2$$

$\therefore \quad \boxed{k = k_1 + k_2}$ ——— **For Parallel Combination.**

$\therefore \quad k = 5 kN/_m + 7 kN/_m$

$= 12 kN/_m = 12 \times 10^3 N/_m.$

Period of Vibration (τ) :

$$p^2 = \frac{k}{m} = \frac{12 \times 10^3 \text{ N/m}}{100 \text{ kg}}$$

$$\therefore \quad p = 10{\cdot}95 \text{ rad/s}$$

Now, $\tau = 2\pi/p = \dfrac{2\pi}{10{\cdot}95}$

$$\therefore \quad \boxed{\tau = 0{\cdot}574 \text{ s}}$$

Maximum Velocity : $v_m = x_m p$

$= (0{\cdot}050\text{m})\ (10{\cdot}95 \text{ rad/s})$

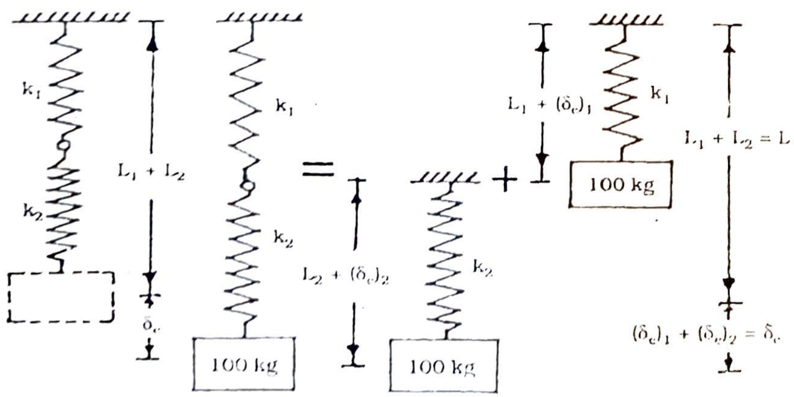
$$\boxed{v_m = 0{\cdot}55 \text{ m/s} \updownarrow}$$

Maximum Acceleration : $a_m = x_m p^2$

$= (0{\cdot}050\text{m})\ (10{\cdot}95 \text{ rad/s})^2$

$$\boxed{a_m = 6 \text{ m/s}^2 \updownarrow}$$

(B) Springs In Series :



(a) Fig. 11·13 (b)

Constant k of a single equivalent spring can be worked out as under.

Here $\delta_e = (\delta_e)_1 + (\delta_e)_2$

$$\frac{P}{k} = \frac{P}{k_1} + \frac{P}{k_2}$$

$$\therefore \quad \boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}} \quad \text{———— For Series Combination.}$$

$$\therefore \quad k = \frac{k_1 k_2}{k_1 + k_2}$$

$$= \frac{5 \times 7}{5 + 7} \quad \text{(kN/m)}$$

$$k = 2{\cdot}92 \text{ kN/m} = 2920 \text{ N/m}.$$

Period of Vibration : $p^2 = \dfrac{k}{m}$

$$= \frac{2920 \text{ (N/m)}}{100 \text{ kg}}$$

$$\therefore \quad p = 5{\cdot}4 \text{ rad/s}$$

$$\tau = \frac{2\pi}{p} = \frac{2\pi}{5{\cdot}4}$$

$$\therefore \quad \boxed{\tau = 1{\cdot}164 \text{ s}}$$

Maximum Velocity : $v_m = x_m p = (0{\cdot}050\text{m})\ (1{\cdot}164 \text{ rad/s})$

$$\boxed{v_m = 0{\cdot}058 \text{ m/s} \updownarrow}$$

Maximum Acceleration : $a_m = x_m p^2$

$= (0{\cdot}050 \text{ m})\ (1{\cdot}164 \text{ rad/s})^2$

$$\boxed{a_m = 0{\cdot}068 \text{ m/s}^2 \updownarrow}$$

3. The bent rod shown has a negligible mass and supports a 10-kg collar at its end. Determine the natural period of vibration for the system.

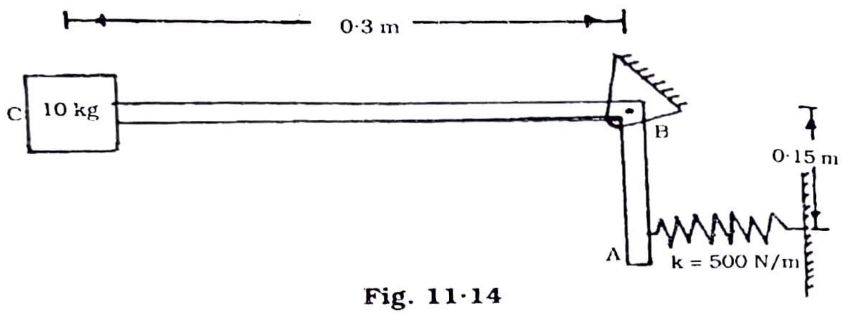


Fig. 11·14

(1) Free-Body and Kinetic Diagrams :

The rod is displaced by a small amount θ from the equilibrium position.

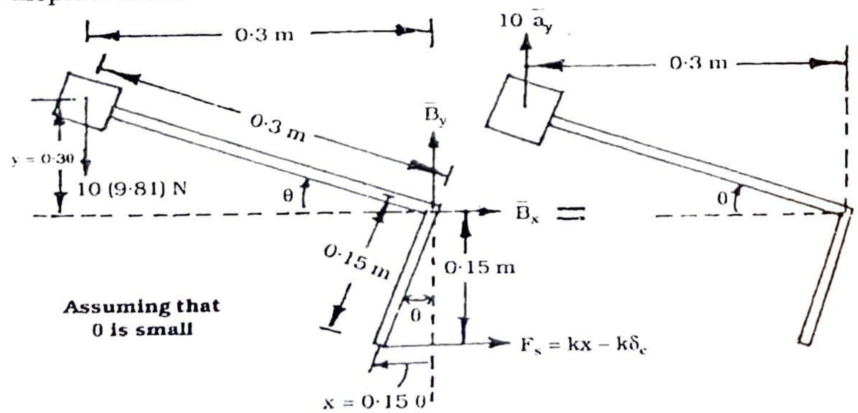
(i) Initial Compression in spring for statical equilibrium (when 10 kg collar is attached) $= k\delta_e$

(ii) Final Tension in spring due to displacement x $= kx$

(iii) Net force exerted by spring (when $x > \delta_e$) $= F_s = kx - k\delta_e$

To obtain "*standard form*" equation $\ddot{x} + p^2x = 0$ **or** $\ddot{\theta} + p^2\theta = 0$,

$10\ \bar{a}_y$ acts **upward** which is in accordance with positive θ displacement.



Free-Body Diagram
(a)

Kinetic Diagram
(b)

Fig. 11·15

(2) Equation of Motion : Since θ is small,

$$\circlearrowleft + \Sigma M_B = \Sigma (M_B)_{eff.} :$$

$$k\,x\,(0{\cdot}15) - k\delta_e\,(0{\cdot}15) + 10\,(9{\cdot}81)\,(0{\cdot}3) = -\,10\,a_y\,(0{\cdot}3)$$

but when $x = 0$, $k\delta_e\,(0{\cdot}15) = 10\,(9{\cdot}81)\,(0{\cdot}3)$ ——— **For static equilibrium.**

Therefore, above equation reduces to

$$kx\,(0{\cdot}15) = -\,10\,a_y\,(0{\cdot}3) \qquad \text{——— (1)}$$

(3) Kinematics : Since θ is small,

$$x = 0{\cdot}15\theta$$

and $y = 0{\cdot}3\,\theta$

Therefore $a_y = \ddot{y} = 0{\cdot}3\,\ddot{\theta}$ ——— **(2)**

Substituting eq. (2) in to eq. (1)

$$k\,(0{\cdot}15\,\theta)\,(0{\cdot}15) + 10\,(0{\cdot}3\,\ddot{\theta})\,(0{\cdot}3) = 0$$

$$500\,(0{\cdot}023\,\theta) + 0{\cdot}9\,\ddot{\theta} = 0$$

$$\ddot{\theta} + 12{\cdot}78\,\theta = 0$$

$\therefore$ $p^2 = 12{\cdot}78$

hence $p = 3{\cdot}57$ rad/s

The *natural period of vibration* $\tau = \dfrac{2\pi}{p} = \dfrac{2\pi}{3{\cdot}57}$

$\therefore$ $\boxed{\tau = 1{\cdot}76\text{ s}}$

4. A cylinder of mass 25 kg and radius 10 cm is suspended from a looped cord as shown. determine the period and frequency of vibration of the cylinder.

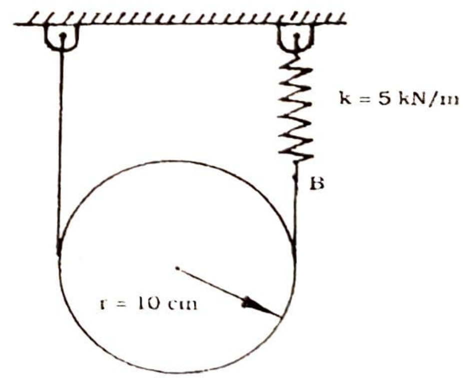
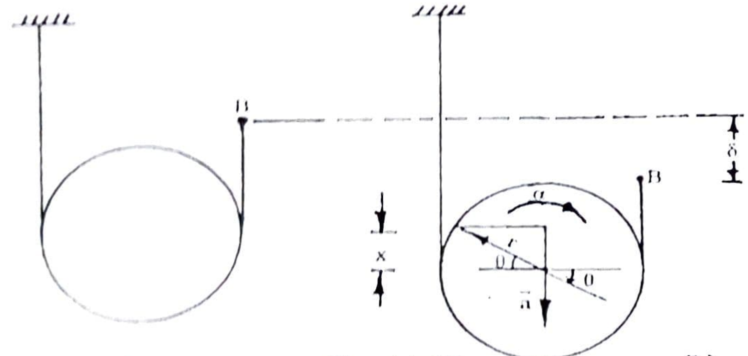


Fig. 11·16



(a) Fig. 11·17 (b)

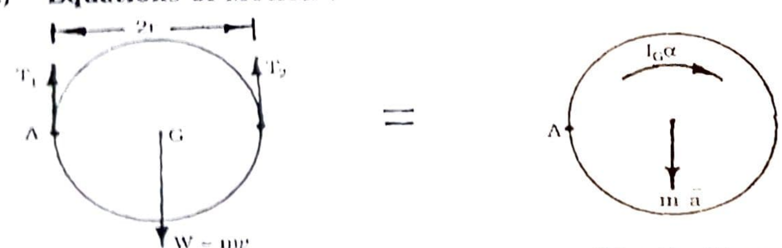
(1) Kinematics of Motion :

Relation between linear displacement and the acceleration of the cylinder in terms of the angular displacement θ :

$$\bar{x} = r\,\theta \qquad \delta = 2\,\bar{x} = 2r\theta$$

$$\alpha = \ddot{\theta} \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \bar{a} = r\ddot{\theta}\downarrow \qquad \text{———— (1)}$$

(2) Equations of Motion :



Free-Body Diagram **Kinetic Diagram**

(a) Fig. 11·18 (b)

$$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{eff} : mg(r) - T_2\,(2r) = m\,\bar{a}(r) + I_G\alpha \qquad \text{—— (2)}$$

When cylinder is in its position of equilibrium,

$$\text{tension in cord} = T_o = \frac{1}{2}\,W$$

For angular displacement θ

$$T_2 = T_o + k\delta = \frac{1}{2}\,W + k\delta$$

$$\therefore \quad T_2 = \frac{1}{2}\,mg + k\,(2r\theta) \qquad \text{———— (3)}$$

Substituting from (1) and (3) in to (2) and $I_G = \frac{1}{2}\,mr^2$

$$mgr - \left(\frac{1}{2}\,mg + 2kr\theta\right)(2r) = m\,(r\ddot{\theta})\,r + \frac{1}{2}\,mr^2\,\ddot{\theta}$$

$$\therefore \quad \ddot{\theta} + \frac{8k}{3m}\,\theta = 0 \qquad \text{————— } \textit{motion is simple harmonic.}$$

$$\text{Hence} \quad p^2 = \frac{8k}{3m} \quad \text{and} \quad p = \sqrt{\frac{8k}{3m}}$$

$$= \sqrt{\frac{8 \times 5000}{3 \times 25}}$$

$$= 23{\cdot}09 \text{ rad/s.}$$

$$\tau = \frac{2\pi}{p} = \frac{2\pi}{23{\cdot}09}$$

$$\therefore \quad \boxed{\tau = 0{\cdot}272 \text{ s}}$$

$$\text{and} \quad f = \frac{p}{2\pi} = \frac{23{\cdot}09}{2\pi}$$

$$\therefore \quad \boxed{f = 3{\cdot}67 \text{ s}^{-1}}$$

5. A 50 N block is suspended from a cord that passes over a 70N disk as shown. Determine the natural period of vibration for the system.

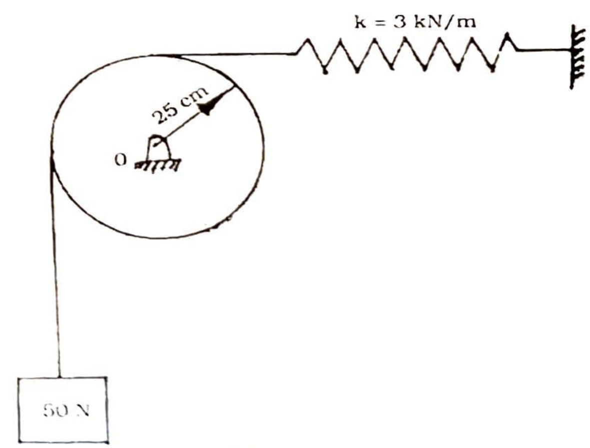
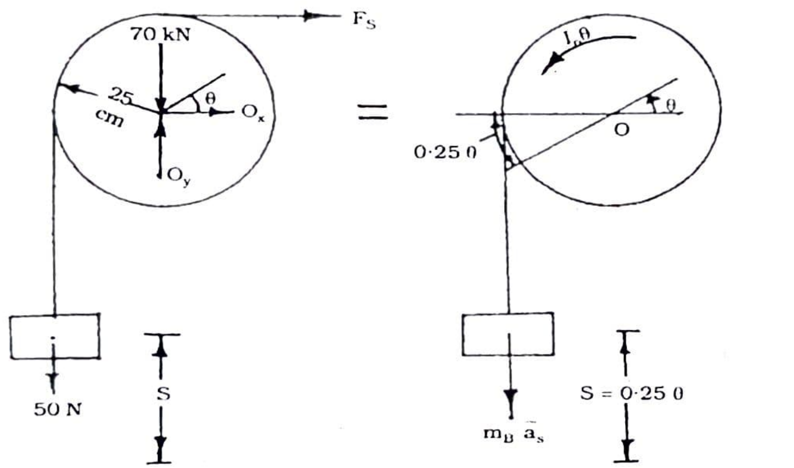


Fig. 11·19

(1) Free Body and Kinetic Diagrams :

Disk undergoes a rotation by the angle θ and block translates by an amount s. $I_o\theta$ acts in positive θ direction consequently $m_B\, a_s$ acts downward in the positive s direction.



Free-Body Diagram **(a)** **Fig. 11·20** **Kinetic Diagram** **(b)**

(2) Equation of Motion :

Summing moments about O, moment of O_x and O_y is zero and putting $I_o = 1/2\ mr^2$

$$\circlearrowleft + \Sigma M_o = \Sigma (M_o)_{eff} :$$

$$50\,(0{\cdot}25) - F_s\,(2{\cdot}25) = \frac{1}{2}\left(\frac{70}{9{\cdot}81}\right)(0{\cdot}25)2\,\ddot{\theta} + \left(\frac{50}{9{\cdot}81}\right) a_s(0{\cdot}25) \qquad \text{——— (1)}$$

(3) Kinematics :

Asmall positive displacement θ of the disk causes displacement of block by

$$s = 0{\cdot}25\,\theta$$

$$\therefore \quad a_s = \ddot{s} = 0{\cdot}25\,\ddot{\theta} \qquad \text{——— (2)}$$

when $\theta = 0$, $F_s = W_B$ for equilibrium. But for position θ,

$$F_s = W_B + k\,s$$

$$\therefore \quad F_s = 50 + (3000)\,(0{\cdot}25\,\theta) \qquad \text{——— (3)}$$

Substituting (2) and (3) in to (1).

$$50\,(0{\cdot}25) - (50 + 750\,\theta)\,(0{\cdot}25) = 0{\cdot}223\,\ddot{\theta} + (1{\cdot}27)\,(0{\cdot}25\,\ddot{\theta})$$

$$\therefore \quad \ddot{\theta} + 346{\cdot}58\,\theta = 0$$

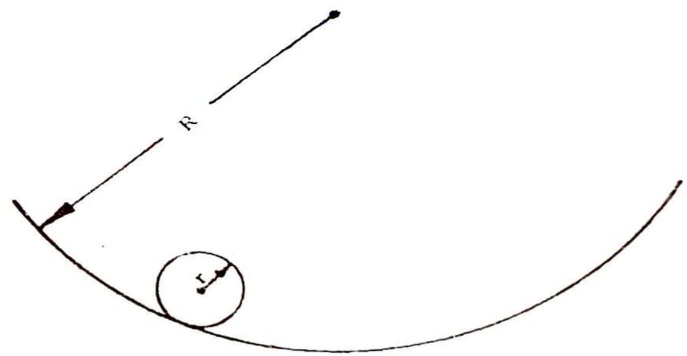
Hence, $p^2 = 346{\cdot}58$ $\quad\therefore\quad p = 18{\cdot}62$ rad/s

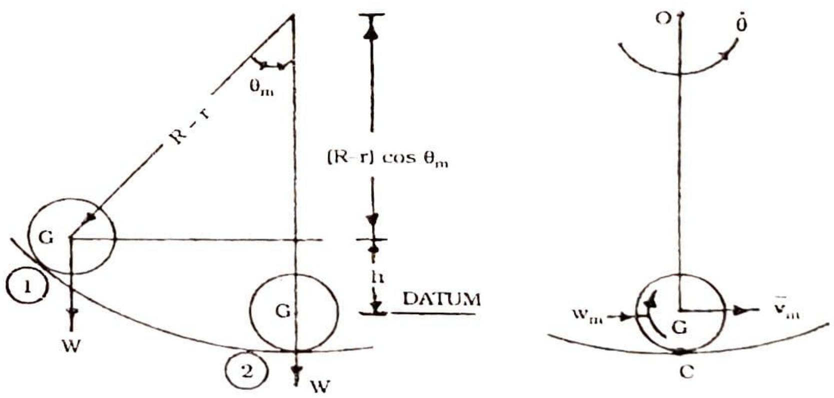
Therefore, *natural period of vibration* is

$$\tau = \frac{2\pi}{p} = \frac{2\pi}{18{\cdot}62} = 0{\cdot}337 \text{ s.}$$

$$\therefore \quad \boxed{\tau = 0{\cdot}337 \text{ s}}$$

6. Determine the period of small oscillations of a cylinder of radius r which rolls without slipping inside a curved surface of radius R.





(a) **Fig. 11·21** **(b)**

Let θ = angle between OG and vertical

At position (1), $\theta = \theta_m$

At position (2), $\theta = 0$

We apply *principle of conservation of energy* between position (1) and (2) as cylinder rolls without slipping.

Position (1) : Kinetic energy : $T_1 = 0$

Potential energy : $V_1 = Wh$

$= W(R - r)(1 - \cos\theta)$

For small oscillations,

$(1 - \cos\theta) = 2\sin^2(\theta/2) \approx \theta^2/2$

$$V_1 = W(R - r)\frac{\theta^2_m}{2}$$

Position 2 :

$\dot{\theta}_m$ = *angular velocity* of line OG

Point C is the instantaneous center of rotation of the cylinder

$\bar{v}_m = (R - r)\dot{\theta}_m$

and $\omega_m = \dfrac{\bar{v}_m}{r} = \dfrac{R-r}{r}\dot{\theta}_m$

Kinetic energy : $T_2 = \dfrac{1}{2} m \bar{v}_m^2 + \dfrac{1}{2} \bar{I}\omega_m^2$

$$= \frac{1}{2} m (R-r)^2\dot{\theta}_m^2$$

$$+ \frac{1}{2}\left(\frac{1}{2} mr^2\right)\left(\frac{R-r}{r}\right)^2\dot{\theta}_m^2$$

$$\therefore \quad T_2 = \frac{3}{4} m (R-r)^2\dot{\theta}_m^2$$

Potential energy : $V_2 = 0$

Not, $T_1 + V_1 = T_2 + V_2$

$$0 + W(R-r)\frac{\theta_m^2}{2} = \frac{3}{4} m (R-r)^2\dot{\theta}_m^2 + 0$$

But $\dot{\theta}_m = p\theta_m$

and $W = mg$

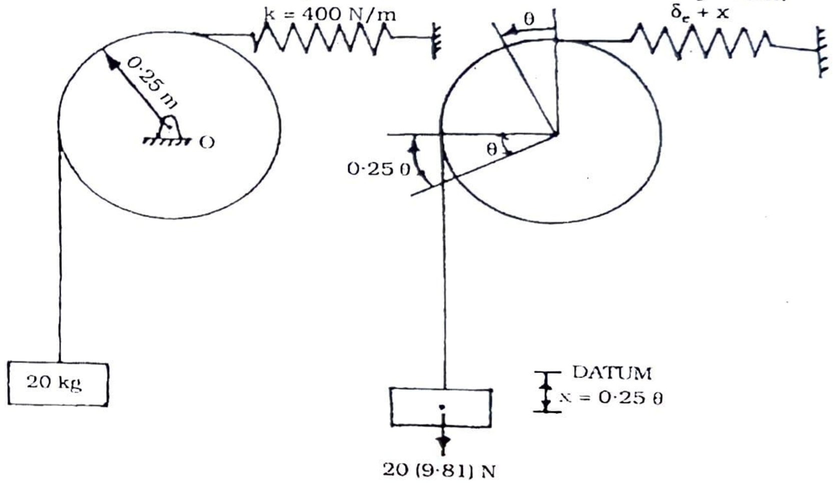
Therefore above equation becomes

$$mg(R-r)\frac{\theta_m^2}{2} = \frac{3}{4} m (R-r)^2 (p\theta_m)^2$$

$$\therefore \quad p^2 = \frac{2g}{3R-r}$$

Now $\tau = \dfrac{2\pi}{p}$ $\therefore$ $\boxed{\tau = 2\pi\sqrt{\dfrac{3R-r}{2g}}}$

7. A 20-kg block is suspended from a cord wrapped around a 10-kg disk as shown. If spring has a stiffness k = 400 N/m, determine the natural period of vibration for the system.



(a) **Fig. 11.22** (b)

Energy Equation :

Block is displaced by x from equilibrium position

Disk is displaced by θ from equilibrium position

And $x = 0.25\,\theta$

Kinetic energy of the system.

$$T = \frac{1}{2} m_b v_b^2 + \frac{1}{2} I_o\omega_d^2$$

$$= \frac{1}{2}(20)(0.25\,\dot{\theta})^2 + \frac{1}{2}\left[\frac{1}{2}(10)(0.25)^2\right](\dot{\theta})^2$$

$$= 0.781\,(\dot{\theta})^2$$

Datum is fixed at equilibrium position.

Spring stretches δ_e for equilibrium.

Potential energy, $V = \frac{1}{2} k (\delta_e + x)^2 - W x$

$$= \frac{1}{2} (400) (\delta_e + 0{\cdot}25\ \theta)^2 - 20\ (9{\cdot}81)\ (0{\cdot}25\ \theta)$$

Total energy of the system is,

$$T + V = 0{\cdot}781\ (\dot{\theta})^2 + 200\ (\delta_e + 0{\cdot}25\ \theta)^2 - 49{\cdot}05\ \theta$$

Time Derivative :

$$2\ (0{\cdot}781)\ (\dot{\theta})\ \ddot{\theta} + 2\ (200)\ (\delta_e + 0{\cdot}25\ \theta)\ (0{\cdot}25\ \dot{\theta}) - 49{\cdot}05\ \dot{\theta} = 0$$

But $\delta_e = \frac{20\ (9{\cdot}81)}{400} = 0{\cdot}491$ m

Writing the above equation in to "*Standard form*"

$$\ddot{\theta} + 16{\cdot}005\ \theta = 0$$

therefore $p = \sqrt{16} = 4$ rad/s

Thus $\tau = \frac{2\pi}{p} = \frac{2\pi}{4}$

$\therefore$ $\boxed{\tau = 1{\cdot}57\ s}$

8. The instrument shown is rigidly attached to a platform P, which in turn is supported by four springs, each having a stiffness k = 100 N/m. Initially the platform is at rest when the floor is subjected to a displacement δ = 10 sin (8t) mm, where t is in seconds. If the instrument is constrained to move vertically, and the total mass of the instrument and platform is 25 kg, determine the vertical displace ment y of the platform, measured from the equilibrium position, as a function of time. what floor vibration is required to cause resonance ?

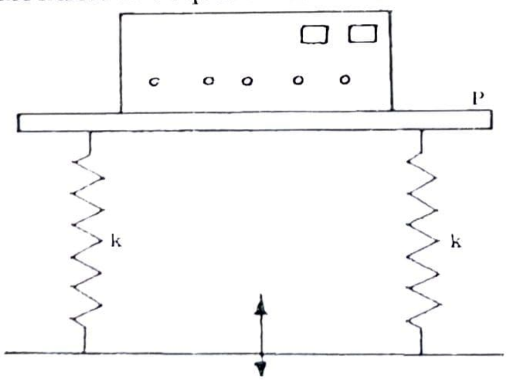


Fig. 11·23

As the induced vibration is caused by the displacement of the support, the motion is described by

$$\ddot{x} + \frac{k}{m} x = \frac{k\delta_m}{m} \sin \omega t$$

which is identical to

$$\ddot{x} + \frac{k}{m} x = \frac{F_m}{m} \sin \omega t$$

with F_m **replaced by** $k\delta_m$.

The solution of which is

$$x = A \sin pt + B \cos pt + \frac{\delta_m}{1 - \left(\frac{\omega}{p}\right)^2} \sin \omega t \qquad \text{——— (1)}$$

Here x will be measured in vertical direction or otherwise x may be replaced by y.

Here $\delta = \delta_m \sin \omega t = 10 \sin (8t)$ mm

so that, $\delta_m = 10$ mm, $\omega = 8$ rad/s

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\ (100)}{25}} = 12{\cdot}65 \text{ rad/s}$$

Amplitude of vibration

$$(y_p)_{max} = \frac{\delta_m}{1 - \left(\frac{\omega}{p}\right)^2}$$

$$= \frac{10}{1 - \left(\frac{8}{12{\cdot}65}\right)^2}$$

$$= 16{\cdot}67 \text{ mm}$$

Hence, above eq. (1) becomes

$$y = A \sin (12{\cdot}65t) + B\cos (12{\cdot}65\ t) + 16{\cdot}67 \sin (8t)$$

$$\therefore\ \dot{y} = A\ (12{\cdot}65) \cos (12{\cdot}65t) - B\ (12{\cdot}65) \sin (12{\cdot}65t) + 133{\cdot}3 \cos (8t)$$

Since at t = 0, y = 0 and $\dot{y} = 0$

then $0 = 0 + B + 0$ $\quad\therefore\ B = 0$

$0 = A\ (12{\cdot}65) - 0 + 133{\cdot}3$ $\quad\therefore\ A = -10{\cdot}6$

The vibrating motion is therefore described by

$$\boxed{y = -10{\cdot}6 \sin (12{\cdot}65\ t) + 16{\cdot}67 \sin (8t)}$$

Resonance will occur when the amplitude of vibration aused *by the floor displacement approaches infinity (means* hen *forced frequency ω approaches natural frequency).* From ·. (2) above, **resonance** requires

$$1 - (\omega/p)^2 = 0$$

$$\boxed{\omega = p = 12{\cdot}65 \text{ rad/s}}$$

. A 200 – kg motor is supported by four springs, each having a onstant of 125 kN/m. The unbalance of the rotor is equivalent to a ass of 50 g located 175 mm from the axis of rotation. Knowing at the motor is constrained to move vertically, determine **(a)** the peed in r/min at which resonance will occur **(b)** the amplitude of e vibration of the motor at a speed of 1500 r/min.

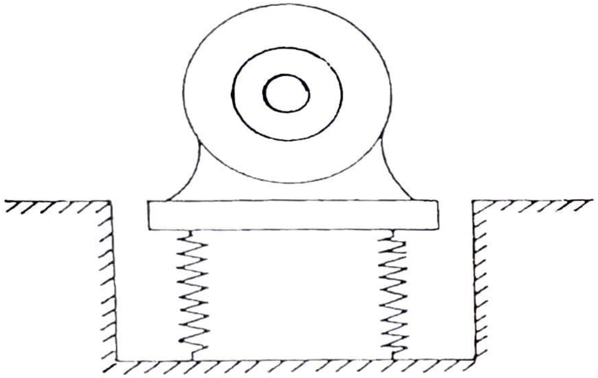


Fig. 11·24

a) Resonance Speed :

The resonance speed is equal to the *circular frequency* in r/min) of the free vibration of the motor.

$$\omega = p = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\ (125) \times 1000}{200}} = 50 \text{ rad/s} \times \frac{60}{2\pi}$$

$$\omega = 50\ \frac{60}{2\pi} = 477{\cdot}5 \text{ r/min}$$

$$\boxed{\text{Resonance speed} = 477{\cdot}5 \text{ r/min}}$$

b) Amplitude of vibration at 1500 r/min.

The *angular velocity* of the motor is

$$\omega = 1500 \text{ r/min} \times \frac{2\pi}{60} = 157{\cdot}1 \text{ rad/s}$$

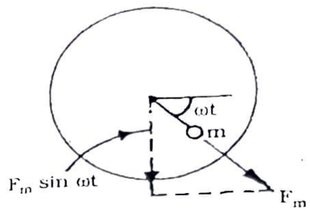


Fig. 11·25

The magnitude of the centrifugal force due to the unbalance of the rotor is

$$F_m = m\,a_n = m\,r\,\omega^2$$

$$= (0{\cdot}050 \text{ kg})\ (0{\cdot}175\text{m}) \times (157{\cdot}1 \text{ rad/s})^2$$

$$\therefore \quad F_m = 215{\cdot}95 \text{ N.}$$

The static deflection (δ_m) that would be caused by a constant load F_m is

$$\delta_m = \frac{F_m}{k} = \frac{215{\cdot}95}{4\ (125)\ 1000} = 4{\cdot}32 \times 10^{-4}\text{m}$$

Therefore,

$$x_m = \frac{F_m/k}{1-(\omega/p)^2} = \frac{4{\cdot}32 \times 10^{-4}}{1-\left(\frac{157{\cdot}1}{50}\right)^2}$$

$$\boxed{x_m = 4{\cdot}87 \times 10^{-5}\text{m}}$$

Since $\omega > p$, the vibration is 180° out of phase with the centrifugal force of the rotor.

10. A car of mass 1300 kg is constructed using a frame supported by four springs. Each spring has a force constant 20,000 N/m. If the combined mass of two people in a car is 160 kg, find the frequency of vibration when it is driven over a pot hole on the road. Also determine the period of exceution of two complete vibrations.

From angular frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20{,}000 \times 4}{(1300 + 160)}}$$

$$= 7{\cdot}40 \text{ rad/s}$$

∴ Frequency of vibration

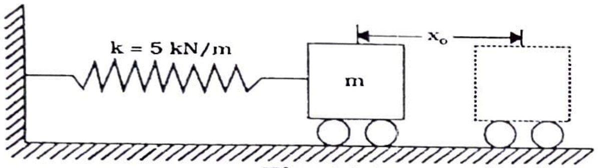
$$f = \frac{\omega_n}{2\pi} = \frac{7{\cdot}4}{2\pi} = \boxed{1{\cdot}18 \text{ Hz}}$$

Period of vibration = $T = 1/f = 0{\cdot}847$ second

Time for one complete vibration = 0·847 second

∴ $\boxed{\text{Time taken for two complete vibrations} = 1{\cdot}694 \text{ seconds}}$

11. A mass of 400 kg, shown in figure, is connected to a light spring whose force constant is 5 kN/m. It is free to oscillate on a horizontal frictionless track. If the mass is displaced 10 cm from equilibrium and released from rest, find **(a)** period of motion, **(b)** maximum speed of the mass, **(c)** maximum acceleration of the mass, **(d)** equations for displacement, speed and acceleration as function of time.



$$t = 0$$
$$A = x_o = 0 \cdot 1 \text{ m}$$
$$v_o = 0$$
$$x = A \cos \omega_n t$$

Fig. 11·26

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 1000}{400}} = 3 \cdot 53 \text{ rad/s}$$

(a) Period of motion : $T = \frac{2\pi}{\omega_n} = \frac{2\pi}{3 \cdot 53} = 1 \cdot 779$ s.

(b) Maximum speed of mass :

$$v_{max} = \omega_n A = 3 \cdot 53 \times 0 \cdot 1 = 0 \cdot 353 \text{ m/s}$$

(c) Maximum acceleration of mass :

$$a_{max} = \omega_n^2 A = (3 \cdot 53)^2 \times 0 \cdot 1 = 1 \cdot 246 \text{ m/s}^2$$

(d) Equations as a functionof time :

$$x = A \cos \omega_n t = 0 \cdot 1 \cos 3 \cdot 53 \, t$$
$$V = -\omega_n A \sin \omega_n t = -3 \cdot 53 \times 0 \cdot 1 \sin 3.53 \, t = -0 \cdot 353 \sin (3 \cdot 53 \, t)$$
$$a = -\omega_n^2 A \cos \omega_n t = -1 \cdot 246 \cos 3 \cdot 53 \, t$$

THEORY RELATED QUESTIONS

1. **Explain** in brief :
   (i) Free vibration
   (ii) Forced vibration
   (iii) Damped vibration
   (iv) Undamped vibration
   (v) Period of vibration
   (vi) Frequency of vibration
   (vii) Amplitude of vibration
   (viii) Natural andCcircular frequency
   (ix) Phase angle.
2. Derive the generalised equation of *simple harmonic motion.*
3. Derive the general solution of generalised equation of simple harmonic motion.
4. Derive the necessary expressions of the energy methods applied to simple harmonic motion.
5. Derive the generalised equation of *undamped forced vibration.*

6. Derive the general solution of the generalised equation of the undamped forced vibration.
7. Explain clearly steady - state and transient vibration with graphs.
8. Distinguish between the undamped forced vibration occurs due to **(i)** periodic force and **(ii)** periodic support displacement.

EXERCISES

11·1 A 3·5 kg collar is attached to a spring of constant 900 N/m and may slide without friction on a horizontal rod. If the collar is moved 80 mm from its equilibrium position and released, determine the maximum velocity and maximum acceleration of the collar during the resulting motion.

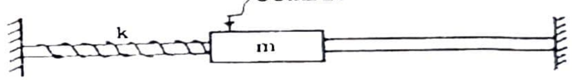


Fig. 11·27

11·2 A 2·5 kg collar is attached to a spring of constant k = 1000 N/m as shown. If the collar is given a displacement of 75 mm downward from its equilibrium position and released, determine **(a)** the time, required for the collar to move 50 mm upwards, **(b)** the corresponding velocity and acceleration of the collar.

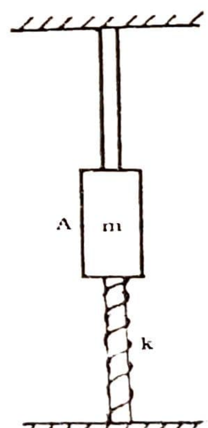


Fig. 11·28

.·3A 30 kg block is supported by the spring arrangement shown. the block is moved vertically downward from its equilibrium sition and released, determine (a) the period and frequency of : resulting motion, (b) the maximum velocity and acceleration of : block if the amplitude of the motion is 35 mm.

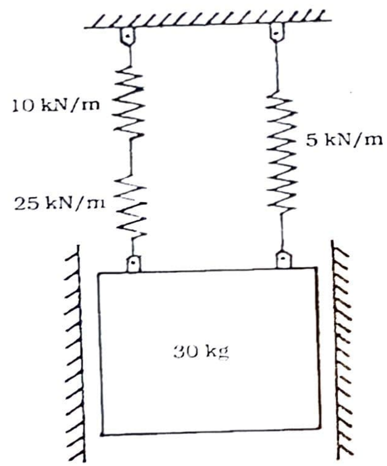


Fig. 11·29

.1·4When a 90 N weight is suspended from a spring, the spring is tretched a distance of 100 mm. Determine the natural frequency nd the period of vibration for a 40 N weight attached to the same pring.

1·5A 5 – kg uniform rod ABC is attached to two springs as shown. f end C is given a small displacement and released, determine the requency of vibration of the rod.

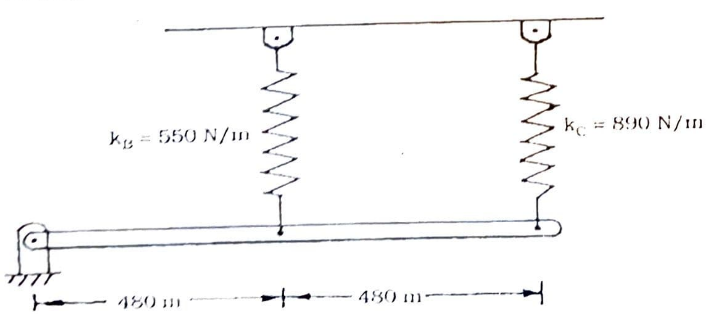


Fig. 11·30

11·6 The 10 – kg rod AB is attached to two 3·7 kg disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

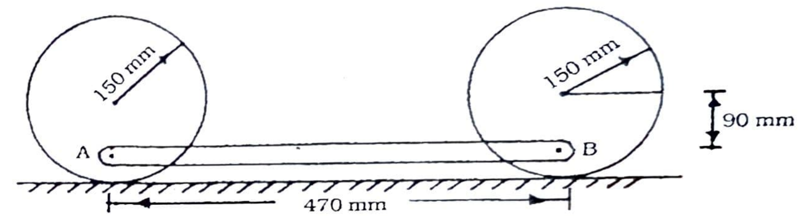


Fig. 11·31

11·7 If the disk has a mass of 10 kg, determine the natural frequency of vibration. The springs are originally unstretched.

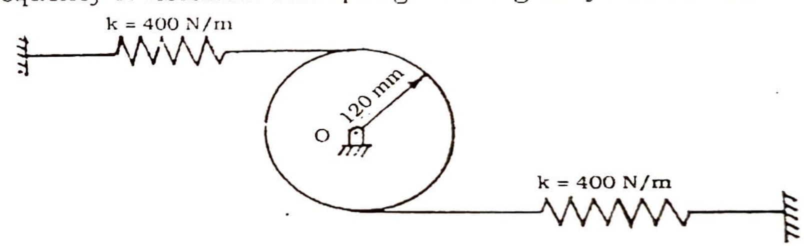


Fig. 11·32

11·8 Determine the period of vibration of the 4 kg sphere. *Neglect the mass of the rod and the size of the sphere.*

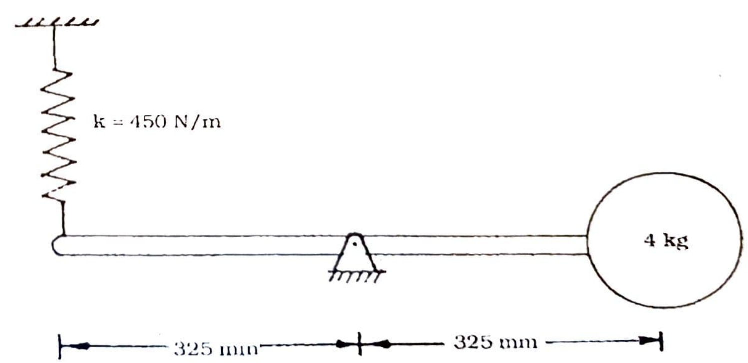


Fig. 11·33

11·9 A 5 – kg cylinder is suspended from a spring of constant 325 N/m and is acted upon by a vertical periodic force of magnitude $F = F_m \sin \omega t$, where $F_m = 15$ N. Determine the amplitude of the motion of the cylinder if **(a)** $\omega = 8$ rad/s **(b)** $\omega = 12$ rad/s.

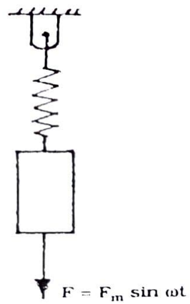


Fig. 11·34

11·10 A 100-kg motor is supported by a nest of springs having a total constant of 45 kN/m. The unbalance of the rotor is equivalent to a mass of 25g located 200 mm from the axis of rotation. Knowing that the motor is constrained to move vertically, determine **(a)** the speed (in r/min) at which resonance will occur, **(b)** the amplitude of the steady - state vibration of the motor at a speed of 750 r/min.

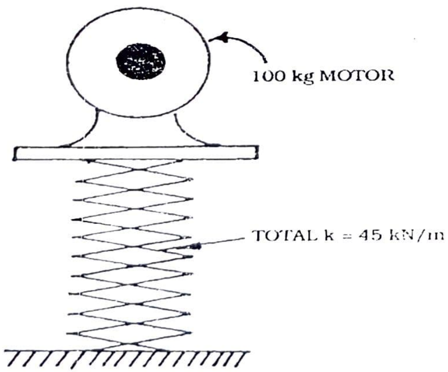


Fig. 11·35

11·11 The 90–N block is attached to a spring having a stiffness of 300 N/m. A force of $F = (27 \cos 2t)$ N, where t is in seconds, is applied to the block. Determine the maximum speed of the block after frictional forces cause the free vibrations to dampen out.

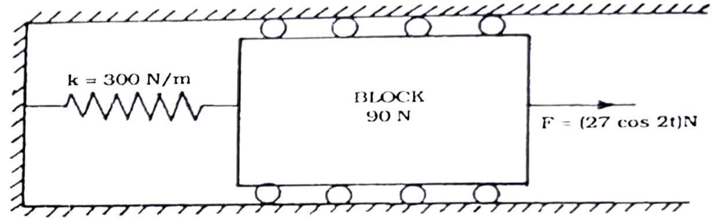


Fig. 11·36

11·12 A motor of mass 20 kg is supported by four springs, each of constant 45 kN/m. The motor is constrained to move vertically, and the amplitude of its motion is observed to be 2 mm at a speed of 1250 r/min. Knowing that the mass of the rotor is 4·5 kg, determine the distance between the mass center of the rotor and the axis of the shaft.

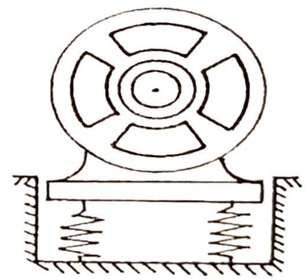


Fig. 11·37

SOLUSIONS OF EXERCISES

11·1 $p = \sqrt{\dfrac{k}{m}}$

$= \sqrt{\dfrac{900}{3{\cdot}5}}$

$= 16{\cdot}04$ rad/s

$v_m = x_m p$

$= 0{\cdot}08 \times 16{\cdot}04 = 1{\cdot}28$ m/s

$a_m = x_m p^2$

$= 0{\cdot}08 \times (16{\cdot}04)^2 = 20{\cdot}58$ m/s^2

11·2 $p = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{2.5}} = 20$ rad/s. Here, $A = 0$, $B = x_o = 0.075$ m

$x = B_o \cos pt = 0.075 \cos pt$, Now for $x = (0.075 - 0.050) = 0.025$ m

$0.025 = 0.075 \cos pt$

$pt = 1.231 \quad \therefore t = 0.062$ s

and $v = -Bp \sin pt = -1.42$ m/s $= 1.42$ m/s (↑)

$a = -Bp^2 \cos pt = -9.74$ m/s² $= 9.74$ m/s² (↑)

11·3 $k_s = \frac{k_1 k_2}{k_1 + k_2} = \frac{250}{35} = 7.143$ kN/m

$k_{total} = k_s + k = 7.143 + 5 = 12.143$ kN/m

Frequency (p) $= \sqrt{\frac{k}{m}} = \sqrt{\frac{12143}{30}} = 20.12$ rad/s

Max velo (v_m) $= x_m p = 0.035 \times 20.12 = 0.704$ m/s

Period of vibration $(\tau) = \frac{2\pi}{p} = 0.312$ s

Max acc. $(a_m) = x_m p^2 = 0.035\,(20.12)^2 = 14.17$ m/s²

11·4 $k = \frac{W}{\delta} = \frac{90}{0.1} = 900$ N/m

Natural Frequency $= p = \sqrt{\frac{k}{m}} = \sqrt{\frac{900}{40/9.81}} = 14.86$ rad/s

Period of Vibration $= \tau = \frac{2\pi}{p} = \frac{2\pi}{14.86} = 0.423$ s

11·5 $+ \Sigma M_A = \Sigma(M_k)_A$:

$k_B x_B (0.48) + k_C x_C (0.96) = -5\, a_y (0.48)$

Moment created by weight of rod and the moment created by the spring forces which is necessary to hold the rod in statical equilibrium are equal and opposite hence not considered in above equation.

Now, $x_B = 0.48\,\theta$, $x_C = 0.96\,\theta$ and $a_y = 0.48\,\ddot{\theta}$

Putting in above equation

$550\,(0.48\,\theta)(0.48) + 890\,(0.96\,\theta)(0.96) = -5\,(0.48\,\ddot{\theta}) \times (0.48)$

$1.152\,\ddot{\theta} + 946.94\,\theta = 0$

$\therefore \ddot{\theta} + 822\,\theta = 0 \qquad \therefore p^2 = 822$

$\therefore p = 28.67$ rad/s

11·6 $V_1 = Wh = 10 \times 9.81 \times 0.090\,(1 - \cos\theta)$

For small oscillations. $(1 - \cos\theta) = 2\sin^2\left(\frac{\theta}{2}\right) = \frac{\theta^2}{2}$

$\therefore V_1 = 4.415\,\theta^2$ and $T_1 = 0$.

$T_2 = \frac{1}{2} m_r v_r^2 + 2 \times \frac{1}{2} I_o \omega_d^2$

$= \frac{1}{2}(10)(0.15\,\dot{\theta})^2 + 2 \times \frac{1}{2}\left[\frac{1}{2}(3.7)(0.15)^2\right](\dot{\theta})^2$

$T_2 = 0.1545\,\theta^2$ and $V_2 = 0$

Conservation of Energy : $T_1 + V_1 = T_2 + V_2$

$4.415\,\theta^2 - 0.1545\,\theta^2 = 0$

since $\dot{\theta}_m = p\theta_m$, from above eq. $p = 5.35$ rad/s $= 0.85$ Hz

11·7 *Kinetic energy of the system* (T) $= \frac{1}{2} I_o \omega^2$

$= \frac{1}{2}\left[\frac{1}{2}(10)(0.12)^2\right]\dot{\theta}^2$

$= 0.036\,\dot{\theta}^2$

and *Potential energy* (V) $= 2 \times \frac{1}{2} kx^2 = 2 \times \frac{1}{2} \times 400\,(0.12\,\theta)^2$

$= 5.76\,\theta^2$

$\therefore$ Total energy of the system $= 0.036\,\dot{\theta}^2 + 5.76\,\theta^2$

Time Derivative : $0.072\,(\dot{\theta})\,\ddot{\theta} + 11.52\,\theta\,(\dot{\theta}) = 0$

$\therefore \ddot{\theta} + 160\,\theta = 0$

$\therefore p = 12.65$ rad/s $= 2.01$ Hz

11·8 *Kinetic energy of system* (T) $= \frac{1}{2} m_s v_s^2$

$= \frac{1}{2}(4)(0.325\,\dot{\theta})^2$

$= 0.21\,\dot{\theta}^2$

and *potential energy* (V) $= \frac{1}{2} kx^2$

$= \frac{1}{2} \times 450 \times (0.325\,\theta)^2$

$= 23.77\,\theta^2$

Moment due to sphere & spring forces for statical equilibrium are not considered.

$$T + V = 0{\cdot}21\,\dot{\theta}^2 + 23{\cdot}77\,\theta^2$$

Time derivative : $0{\cdot}42\,\dot{\theta}\,\ddot{\theta} + 47{\cdot}54\,\theta\,\dot{\theta} = 0$

$$\ddot{\theta} + 113{\cdot}2\,\theta = 0$$

$$p = 10{\cdot}64 \text{ rad/s}$$

Period of Vibration $= \dfrac{2\pi}{p} = \dfrac{2\pi}{10{\cdot}64} = 0{\cdot}59$ s

11·9 (a) $x_m = \dfrac{F_m/k}{1-(\omega/p)^2}$ where $p = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{325}{5}} = 8{\cdot}06$ rad/s

$$= \frac{15/325}{1-(8/8{\cdot}06)^2}$$

$$= 3{\cdot}12 \text{ m}$$

(b) $x_m = \dfrac{15/325}{1-(12/8{\cdot}06)^2}$

$$= 0{\cdot}038 \text{ m}$$

11·10 (a) Resonance speed $= p = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{45000}{100}}$

$$= 21{\cdot}21 \text{ rad/s} = 202{\cdot}5 \text{ rpm}$$

(b) Angular velocity $(\omega) = 750$ rpm $= 78{\cdot}56$ rad/s

Magni. of the centrifugal force due to the unbalance of the rotor is $P_m = ma_n = mr\omega^2$

$$= (0{\cdot}025)\,(0{\cdot}200)\,(78{\cdot}56)^2$$

$$= 30{\cdot}86 \text{ N}$$

The static deflection that would be caused by a constant load P_m is $P_m/k = \dfrac{30{\cdot}86}{45000} = 0{\cdot}000686$ m

Amplitude $(x_m) = \dfrac{P_m/k}{1-(\omega/p)^2} = \dfrac{0{\cdot}000686}{1-\left(\dfrac{78{\cdot}56}{21{\cdot}21}\right)^2}$

$$= -\,0{\cdot}000054 \text{ m}$$

$x_m = 0{\cdot}054$ mm (out of phase)

11·11 $p = \sqrt{\dfrac{300}{90/9{\cdot}81}} = 5{\cdot}72$ rad/s

$x_m = \dfrac{F_m/k}{1-(\omega/p)^2}$ Here $F_m \sin \omega t = 27 \cos 2t$

$\therefore\ F_m = 27,\ \omega = 2$

$$= \frac{27/300}{1-(2/5{\cdot}72)^2} = 0{\cdot}103$$

The free vibration represented by x_{comp} = Asin pt + B cos pt will soon be damped out by friction forces, where as $x_{parti} = x \sin\omega t$ representing the *steady – state vibration* will be maintained. Here

$$x = 0{\cdot}103 \cos 2t$$

$$\therefore\ \dot{x} = -\,0{\cdot}103 \times 2 \sin 2t$$

$$\therefore\ \dot{x}_{max} = 0{\cdot}206 \text{ m/s}$$

11·12 $p = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 45000}{20}} = 94{\cdot}87$ rad/s

$\omega = 1250$ rpm $= 130{\cdot}94$ rad/s

$x_m = 2$mm $= 0{\cdot}002$ m

$x_m = \dfrac{P_m/k}{1-(\omega/p)^2}$ $\therefore\ 0{\cdot}002 = \dfrac{P_m/4 \times 45000}{1-\left(\dfrac{130{\cdot}94}{94{\cdot}87}\right)^2}$

$\therefore\ P_m = 325{\cdot}79$ N

$P_m = ma_n = mr\omega^2\ \therefore\ (4{\cdot}5)r\,(130{\cdot}94)^2 = 325{\cdot}79$

$r = 0{\cdot}004$ m $= 4$ mm