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1 MULTIPLE INTEGRALS

1.1 Introduction :

In the application of calculus of integration, we know that integral of a function $y = f(x)$ over an interval $[a, b]$ is a limit of approximating sums.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\,\delta x_k \qquad \text{... ... ... (1)}$$

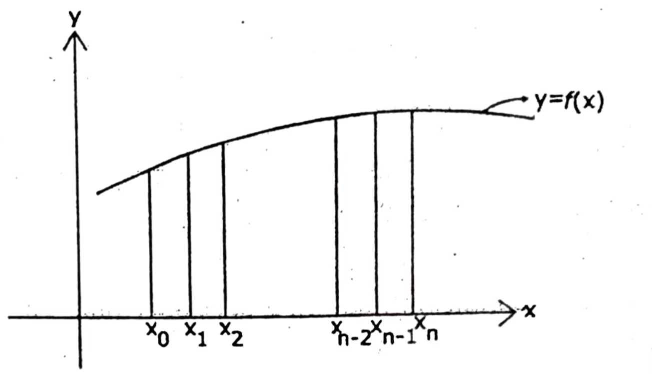


Fig 1.1.1

Where an interval $[a, b]$ is divided into n equal parts such that $a = x_0 < x_1 < \dots < x_n = b$ and $\delta x_k = x_k - x_{k-1}$ and $x_k^* \in [x_k, x_{k-1}]$. The above limit when $n \to \infty$, is exists if f is continuous or if f is bounded and finitely discontinuous in $[a, b]$. If $f(x) > 0$ then integrel (1) shows the area bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the X–axis.

In this chapter, we shall consider a function of two or more variables which yields multiple integrals. We recall here the concept of second order partial derivatives with respect to two variables, in which one variable is kept fixed when we differentiate w.r.t. another variable and vice–versa. Also we know that when the function is continuous then the two second ordered partial derivatives in different orders are same.

1.2 Double Integrals :

Let the function $z = f(x, y)$ be defined on some region R as shown in the figure 1.2.1.

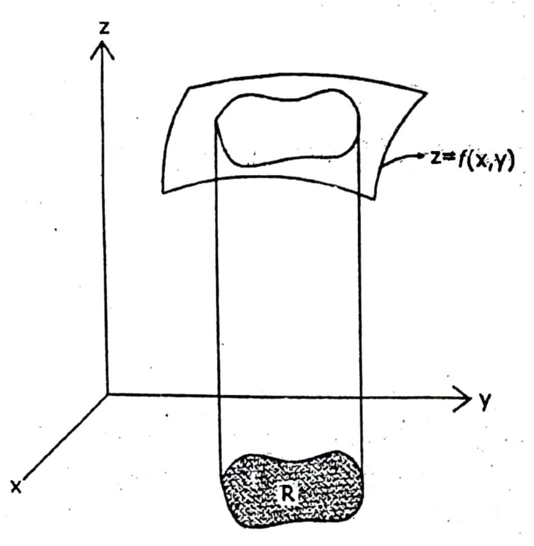


Fig 1.2.1

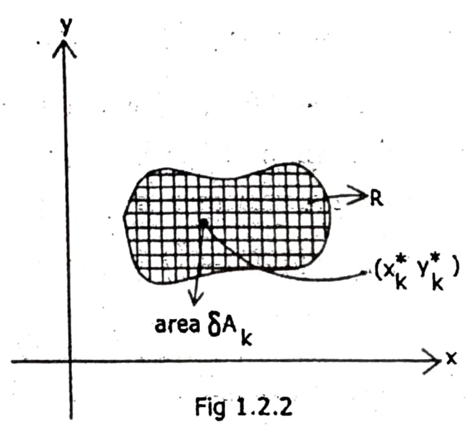


Fig 1.2.2

Divide the region R into small parts by drawing lines parallel to X–axis and Y–axis within the region R. These parts are in the form of rectangle of lengths δx and δy, except some of the parts at the boundaries of the region. Let the number of such rectangles be n, each of area $\delta A = \delta x \cdot \delta y$. Choose any point (x_k^*, y_k^*) in any area δA_k. If we take $n \to \infty$ then all the small parts within the region will approximately behave like rectangles. Then the following limit of sums over the region R :

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \delta A_k$$

is known as the double integral of $f(x, y)$ over R and can be written as

$$\iint_R f(x, y)\, dA. \qquad \dots\dots\dots (2)$$

If $f(x, y)$ is continuous on R and $f(x, y) \geq 0$ then (2), gives the **volume** of the solid bounded by the function over region R.

Note : For the existence of double integral, the continuity of $f(x, y)$ over R is a sufficient condition, but not a necessary one. The limit can exists for some discontinuous functions also.

1.2.1 Algebraic Properties :

It is evident from the case of limit that double integrals holds the following algebraic properties :

(a) $\iint_R k f(x, y)\, dA = k \iint_R f(x, y)\, dA$ (k is any constant.)

(b) $\iint_R [f(x, y) \pm g(x, y)]\, dA = \iint_R f(x, y) \pm \int g(x, y)\, dA$

(c) $\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(d) $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R.

(e) $\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

Where R_1 and R_2 are non-overlapping subregions of R such that $R_1 \cup R_2 = R$.

1.3 Determination of Limits of Integrals :

We have $dA = dx\, dy = dy\, dx$

$$\therefore \iint_R f(x, y)\, dA = \iint_R \left[\int f(x, y)\, dx\right] dy$$

$$= \int \left[\int f(x, y)\, dy\right] dx$$

The integral inside the bracket is known as inner integral which is always evaluated first, and the integral outside the bracket is known as outer integral which is evaluated last. A double integral is sometimes expressed in the form.

$\int dy \left[\int f(x, y)\, dx\right]$ or $\int dx \left[\int f(x, y)\, dy\right]$

in which case, we evaluate the integral in bracket first and then

$\int \left[\int f(x, y)\, dx\right] dy$ or $\int \left[\int f(x, y)\, dy\right] dx$

1.3.1 When R is a rectangle :

Suppose that $f(x, y)$ is defined on a rectangular region R defined by $R : a \leq x \leq b,\ c \leq y \leq d$.

If $f(x, y)$ is continuous on the rectangular region R then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

$$= \int_c^d \int_a^b f(x, y)\, dx\, dy$$

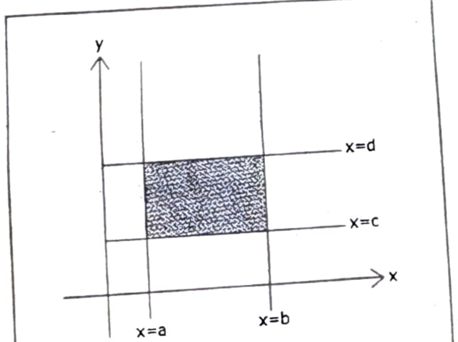


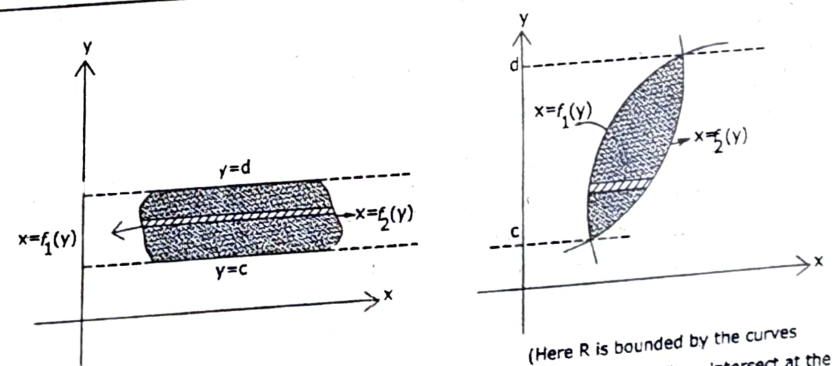
Fig 1.2.3

The above equality shows that the order of integration is immaterial, provided the limits of integration are changed accordingly. [The above result was derived by Guido Fubini (1879 – 1943)].

1.3.2 When R is a bounded Non rectangular Region :

(a) Inner limits as function of y (limits of x).

Consider the region R bounded by the boundaries as shown in the following figures :



(Here R is bounded by the curves $x=f_1(y)$, $x=f_2(y)$, $y=c$, $y=d$)

(Here R is bounded by the curves $x=f_1(y)$, $x=f_2(y)$ They intersect at the points $y=c$, $y=d$)

Fig 1.2.4

Draw a strip line parallel to X-axis, within the region R. See the end points of the strip which touch the region R on left to the curve $x = f_1(y)$, and on right side to the curve $x = f_2(y)$. Thus we get

$f_1(y) \leq x \leq f_2(y)$ [inner limits]

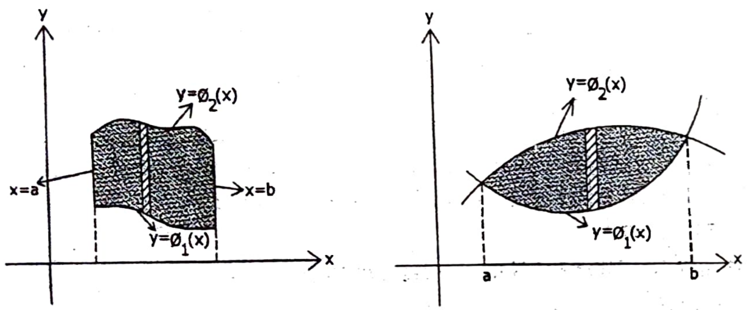
Now in the region R, move the strip line from lowest position ($y = c$) to the highest position ($y = d$). Thus we have

$c \leq y \leq d$. [outer limits]

$$\therefore \iint_R f(x, y)\, dA = \int_c^d \int_{f_1(x)}^{f_2(x)} f(x, y)\, dx\, dy$$

(b) Inner limits as function of x [Limits of y]

Consider the region R bounded by the boundaries as shown in the following figures :



(Here R is boundred by the curves $y = \phi_1(x)$, $y = \phi_2(x)$ and lines $x = a$, $x = b$)

(Here R is bounded by the curves $y = \phi_1(x)$, $y = \phi_2(x)$ They intersect at the points $x = a$, $x = b$)

Fig 1.2.5

Draw a strip line parallel to Y–axis, within the region R. See the end points of the strip which touch in the region R at below to the curve $y = \phi_1(x)$, and at above to the curve $y = \phi_2(x)$. Thus we have

$\phi_1(x) \leq y \leq \phi_2(x)$ [Inner limits]

Now in the region R, move the strip line from leftmost boundary ($x = a$) to the rightmost boundary ($x = b$). Thus we have

$a \leq x \leq b$ [outer limits]

$$\therefore \iint_R f(x, y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx$$

Note :

1. If we want to integrate first w.r.t. x, take a strip line parallel to X–axis. Y
2. If we want to integrate first w.r.t. y, take a strip line parallel to Y–axis. X
3. In both of the above types, the answer will be same.
4. The limits of outer integral are always constants, whereas the limits of inner integral may or may not be constants.
5. Integration w.r.t. one variable keeps the second variable constant.

SOLVED EXAMPLES

1. **Evaluate $\iint_R (4 - x - y)\, dA$ over the region R : $0 \leq x \leq 2$, $0 \leq y \leq 1$.**

Solution : Here the region R is a rectangle

$$I = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy$$

$$= \int_0^1 \left[4x - \frac{x^2}{2} - yx\right]_0^2 dy$$

$$= \int_0^1 \left[8 - \frac{4}{2} - 2y\right] dy$$

$$= \int_0^1 [6 - 2y]\, dy$$

$$= \left[6y - y^2\right]_0^1 = 6 - 1 = 5$$

OR $I = \int_0^2 \int_0^1 (4 - x - y)\, dy\, dx = 5$

2. **Evaluate $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\, dy\, dx$**

Solution : $I = \int_{\pi/2}^{\pi} [\sin xy]_1^2\, dx$

$$= \int_{\pi/2}^{\pi} [\sin 2x - \sin x]\, dx$$

$$= \left[-\frac{\cos 2x}{2} + \cos x\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{2} - 1 - \frac{1}{2} - 0 = -2.$$

3. **Evaluate $\iint_R x\sqrt{1 - x^2}\, dA$ where R : $0 \leq x \leq 1$, $2 \leq y \leq 3$**

Solution : Here R is a rectangle

$$\therefore I = \int_0^1 x\sqrt{1 - x^2}\, [y]_2^3\, dx$$

$$= \int_0^1 x\sqrt{1 - x^2}\, [y]_2^3\, dx$$

$$= \int_0^1 x\sqrt{1 - x^2}\, dx$$

$$= \left[\frac{(1-x^2)^{3/2}}{\frac{3}{2}(-2)}\right]_0^1 \qquad \left(\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1\right)$$

$$= 0 + \frac{1}{3} = \frac{1}{3}$$

4. Evaluate $\int_0^2 \int_0^{x^2} e^{y/x}\, dy\, dx$

Solution : $I = \int_0^2 \left[\frac{e^{y/x}}{\frac{1}{x}}\right]_0^{x^2} dx$

$$= \int_0^2 [x e^x - x]\, dx \qquad (\because \int uv\, dx = (u)(v_1) - (u')(v_2) + \ldots)$$

$$= [(x)(e^x) - (1)(e^x)]_0^2 - \left[\frac{x^2}{2}\right]_0^2$$

$$= 2e^2 - e^2 + 1 - 2 = e^2 - 1$$

5. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$

Solution : $I = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx \ = \int_0^1 x\sqrt{1-x^2}\,[y]_2^3\, dx$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\, dx\right]$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1})\right]_0^1 = \frac{\pi}{4} \log(\sqrt{2}+1)$$

6. Evaluate : $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

Solution : $I = \int_0^a \frac{y}{2} [x^2]_0^{\sqrt{ay}}\, dy$

$$= \frac{1}{2} \int_0^a y \cdot ay\, dy$$

$$= \frac{a}{2} \int_0^a y^2 dy = \frac{a}{2} \left[\frac{y^3}{3}\right]_0^a \quad = \frac{a}{2} \cdot \frac{a^3}{3} = \frac{a^4}{6}$$

7. Evaluate : $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$

Solution : $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2) - x^2}\, dx\, dy$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$\left(\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right)$$

$$= \int_0^a \frac{a^2-y^2}{2} \sin^{-1} 1\, dy$$

$$= \frac{\pi}{4} \int_0^a (a^2 - y^2)\, dy = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi a^3}{6}$$

8. Evaluate : $\int_0^1 \int_0^x e^{y/x}\, dx$

Solution : $I = \int_0^1 [x e^{y/x}]_0^x\, dx$

$$= \int_0^1 [x e - x]\, dx$$

$$= (e-1)\left[\frac{x^2}{2}\right]_0^1 = \frac{e-1}{2}$$

9. Evaluate : $\int_0^a \int_0^{\sqrt{ax-x^2}} (x^2 + y^2)\, dy\, dx$

Solution : $I = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{ax-x^2}} dx$

$$= \int_0^a \left[x^2 \sqrt{ax-x^2} + \frac{1}{3}(ax-x^2)^{3/2}\right] dx$$

$$= \int_0^{\pi/2} \left[a^2 \sin^4\theta\, a \sin\theta \cos\theta + \frac{a^3}{3} \sin^3\theta \cos^3\theta\right] 2a \sin\theta\, \cos\theta\, d\theta$$

(Put $x = a\sin^2\theta$)

$$= 2a^4 \int_0^{\pi/2} \left[\sin^6\theta\cos^2\theta + \frac{1}{3}\sin^4\theta\cos^4\theta\right] d\theta$$

$$= 2a^4 \int_0^{\pi/2} \left[\sin^6\theta - \sin^8\theta + \frac{1}{3}(\sin^4\theta - 2\sin^6\theta + \sin^8\theta)\right] d\theta$$

$$= 2a^4 \int_0^{\pi/2} \left[\frac{1}{3}\sin^4\theta + \frac{1}{3}\sin^6\theta - \frac{2}{3}\sin^8\theta\right] d\theta$$

$$= 2a^4\left[\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{3}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{2}{3}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 2a^4\, \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left[1 + \frac{5}{6} - \frac{35}{24}\right] = \frac{\pi a^4}{8}\cdot\frac{9}{24} = \frac{3\pi a^4}{64}$$

10. Evaluate : $\int_1^2 \int_0^x \dfrac{dA}{x^2+y^2}$

Solution : $I = \int_1^2 \int_0^x \dfrac{1}{x^2+y^2}\, dy\, dx = \int_1^2 \left[\dfrac{1}{x}\tan^{-1}\dfrac{y}{x}\right]_0^x$

$$= \int_1^2 \frac{1}{x}\,\frac{\pi}{4}\, dx = \frac{\pi}{4}\left[\log x\right]_1^2$$

$$= \frac{\pi}{4}\left[\log 2 - \log 1\right] = \frac{\pi}{4}\log 2$$

11. Evaluate $\iint\limits_R xy\, dA$, where R is the region bounded by x–axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$

Solution : Draw a figure. Take a vertical strip.

$\therefore$ The limits are

$0 \le y \le x^2/4a$

$0 \le x \le 2a$

$\therefore \quad I = \int_0^{2a} \int_0^{x^2/4a} xy\, dy\, dx$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \int_0^{2a} \frac{x}{2}\left[\frac{x^4}{16a^2}\right] dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{a^4}{3}$$

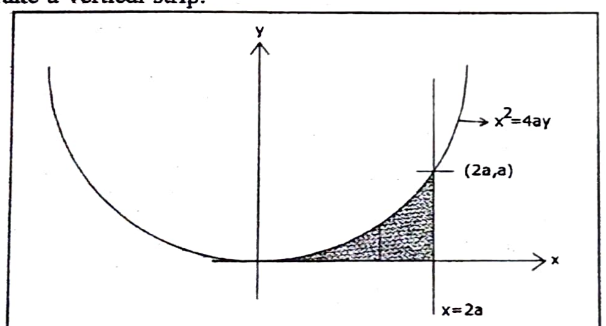


Fig 1.2.6

If we would take the strip parallel to X–axis then the limits are :

$2\sqrt{ay} \le x \le 2a$

$0 \le y \le a$

$$\therefore\ I = \int_0^a \int_{2\sqrt{ay}}^{2a} xy\, dx\, dy = \frac{a^4}{3}$$

12. Evaluate $\iint\limits_R xy\,(x+y)\ dA$ over the area between $y = x^2$ and $y = x$.

Solution : Draw a figure.

Take a strip parallel to Y–axis

$\therefore$ The limits are :

$x^2 \le y \le x$

$0 \le x \le 1$

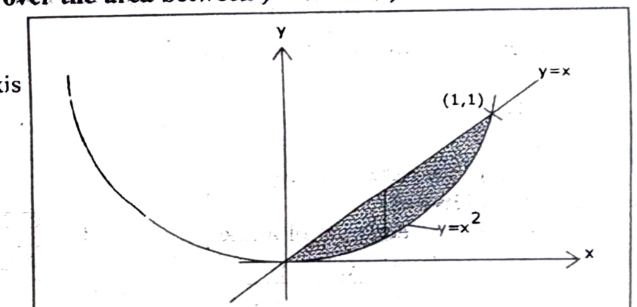


Fig 1.2.7

$\therefore \quad I = \int_0^1 \int_{x^2}^x xy\,(x+y)\, dy\, dx$

$$= \int_0^1 x\left[x\,\frac{y^2}{2} + \frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 x\left[\frac{x^3}{2} + \frac{x^3}{3} - \frac{x^5}{2} - \frac{x^6}{3}\right] dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 = \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{84 + 56 - 60 - 35}{15 \times 7 \times 8} = \frac{45}{15\cdot 7\cdot 8} = \frac{3}{56}$$

13. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

Solution : $I = \int_0^1 \left[x^2 y + \dfrac{y^3}{3}\right]_x^{\sqrt{x}}$

$$= \int_0^1 \left[\left(x^{5/2} + \frac{1}{3}x^{3/2}\right) - \left(x^3 + \frac{x^3}{3}\right)\right] dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{1}{3}x^{3/2} - \frac{4}{3}x^3\right] dx$$

$$= \left[\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{3}x^4\right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30 + 14 - 35}{105}$$

$$= \frac{9}{105}$$

$$= \frac{3}{35}$$

14. Evaluate $\iint\limits_R \sqrt{xy - y^2}\, dA$, where R is a triangle with vertices (0, 0), (10, 1) and (1, 1).

Solution : Draw the figure

Line OA : $y = x$

Line AB : $y = 1$

Line OB : $x = 10y$

Take a strip line parallel to X–axis.

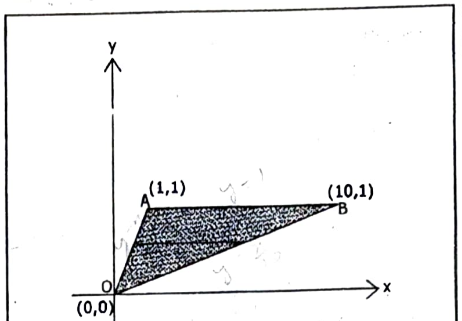
Thus the limits are :

$y \le x \le 10y$

$0 \le y \le 1$

equation of a line

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$



Fiq 1.2.8

$$\therefore\ I = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_0^1 \frac{2}{3y}\left[(xy - y^2)^{3/2}\right]_y^{10y} dy \qquad \left[\because \int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)}\right]$$

$$= \frac{2}{3}\int_0^1 \frac{1}{y}\left[(10y^2 - y^2)^{3/2} - (y^2 - y^2)^{3/2}\right] dy$$

$$= \frac{2}{3}\int_0^1 27y^2\, dy = \frac{2 \times 27}{3}\left[\frac{y^3}{3}\right]_0^1 = 6$$

Note : In the above example if we take strip line parallel to Y–axis then the limits will be :

R_1	R_2
$\frac{x}{10} \le y \le x$	$\frac{x}{10} \le y \le 1$
$0 \le x \le 1$	$1 \le x \le 10$

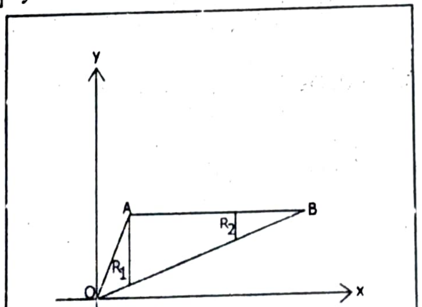


Fig 1.2.8a

15. Evaluate $\iint\limits_R x^2\, dA$, where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$.

Solution : Draw the figure take a vertical strip line.

$\therefore$ The limits are :

$$R_1 : \begin{cases} 0 \le y \le x \\ 0 \le x \le 4 \end{cases}$$

$$R_2 : \begin{cases} 0 \le y \le 16/x \\ 4 \le x \le 8 \end{cases}$$

$$I = \iint\limits_R x^2\, dA$$

$$= \int_0^4 \int_0^x x^2\, dy\, dx + \int_4^8 \int_0^{16/x} x^2 dy\, dx$$

$$= \int_0^4 x^2 [y]_0^x\, dx + \int_4^8 x^2\, [y]_0^{16/x}\, dx$$

$$= \int_0^4 x^3\, dx + \int_4^8 16x\, dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 + 8\left[x^2\right]_4^8$$

$$= 64 + 8\,[64 - 16] = 448$$

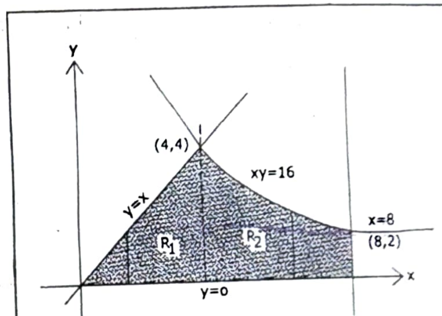


Fig 1.2.9

16. Evaluate $\iint\limits_R x^2 y^2\, dA$, where R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 1$.

Solution : Draw the figure. Take horizontal strip line.

$\therefore$ The limits are :

$0 \le x \le \sqrt{1 - y^2}$

$0 \le y \le 1$

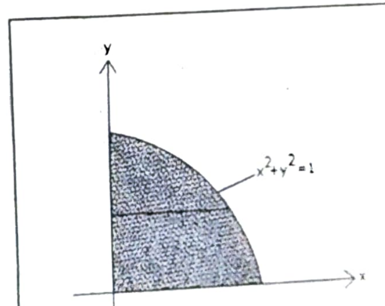


Fig 1.2.10

$$\therefore\ I = \int_0^1 \int_0^{\sqrt{1-y^2}} x^2 y^2\, dx\, dy$$

$$= \int_0^1 y^2 \left[\frac{x^3}{3}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{3}\int_0^1 y^2\, [1 - y^2]^{3/2}\, dy$$

Put $y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

when $y = 0 \rightarrow \theta = 0$

$y = 1 \rightarrow \theta = \pi/2$

$$\therefore\ I = \frac{1}{3}\int_0^{\pi/2} \sin^2\theta \cos^3\theta \cos\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta = \frac{1}{3}\int_0^{\pi/2} [\cos^4\theta - \cos^6\theta]\, d\theta$$

$$= \frac{1}{3}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{1}{3}\cdot\frac{3\pi}{16}\left[1 - \frac{5}{6}\right] = \frac{\pi}{16}\cdot\frac{1}{6} = \frac{\pi}{96}$$

17. **Evaluate $\iint_R (x^2+y^2)\,dA$ through the area enclosed by the curves $y = 4x$, $x + y = 3$, $y = 0$ and $y = 2$.**

Solution : Draw the figure.

Take horizontal strip

$\therefore$ The limits are :

$$\frac{y}{4} \le x \le 3 - y$$

$$0 \le y \le 2$$

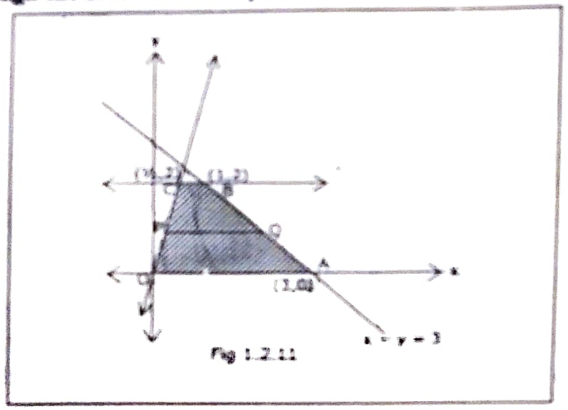


Fig 1.2.11

$$\therefore \quad I = \int_0^2 \int_{y/4}^{3-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]_{y/4}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{y^3}{3.64} - \frac{y^3}{4}\right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192}\,y^3\right] dy$$

$$= \left[-\frac{(3-y)^4}{12} + y^3 - \frac{241}{192}\,\frac{y^4}{4}\right]_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{48} + \frac{27}{4} = \frac{463}{48}$$

18. **Evaluate $\iint_R e^{2x+3y}\,dA$ where R is the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.**

Solution : Draw the figure.

Take a vertical strip

$\therefore$ The limits are :

$$0 \le y \le 1 - x$$

$$0 \le x \le 1$$

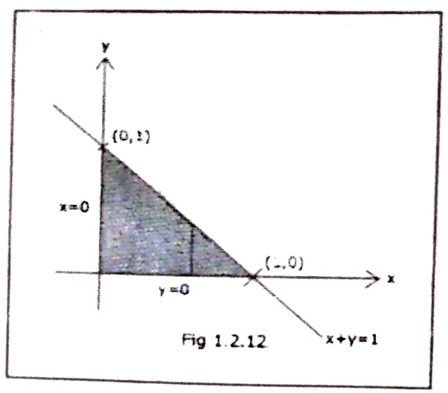


Fig 1.2.12

$$\therefore \quad I = \int_0^1 \int_0^{1-x} e^{2x+3y}\,dy\,dx$$

$$= \int_0^1 \frac{1}{3}\left[e^{2x+3y}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 \left[e^{3-x} - e^{2x}\right] dx$$

$$= \frac{1}{3}\left[-e^{3-x} - \frac{e^{2x}}{2}\right]_0^1 = \frac{1}{3}\left[-e^2 - \frac{e^2}{2} + e^3 + \frac{1}{2}\right]$$

$$= \frac{1}{3}\left[e^3 - \frac{3}{2}e^2 + \frac{1}{2}\right] = \frac{1}{6}\left[2e^3 - 3e^2 + 1\right]$$

19. **Evaluate $\iint_R y\,dA$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.**

Solution : Draw the figure.

Take horizontal line

$\therefore$ The limits are :

$$y^2/4 \le x \le 2\sqrt{y}$$

$$0 \le y \le 4$$

$$\therefore \quad I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} y\,dx\,dy$$

$$= \frac{48}{5} \quad \text{(After simple integration)}$$

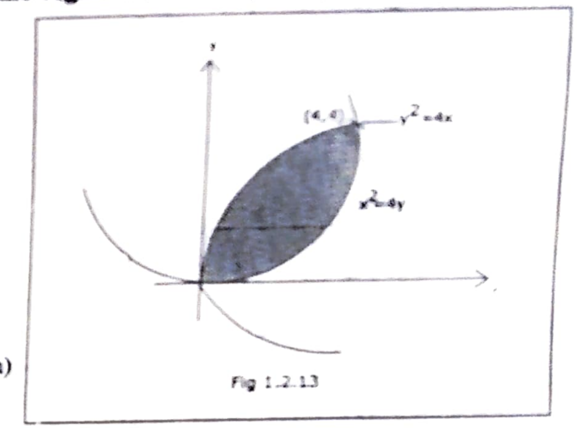


Fig 1.2.13

20. **Evaluate $\iint_R xy\,dA$, where R is given by $x^2 + y^2 - 2x = 0$, $y^2 = 2x$, $y = x$.**

Solution : Draw the figure $(x-1)^2 + y^2 = 1$. Take a vertical strip

$\therefore$ The limits are :

$$R_1 : \begin{cases} \sqrt{2x - x^2} \le y \le \sqrt{2x} \\ 0 \le x \le 1 \end{cases}$$

$$R_2 : \begin{cases} x \le y \le \sqrt{2x} \\ 1 \le x \le 2 \end{cases}$$

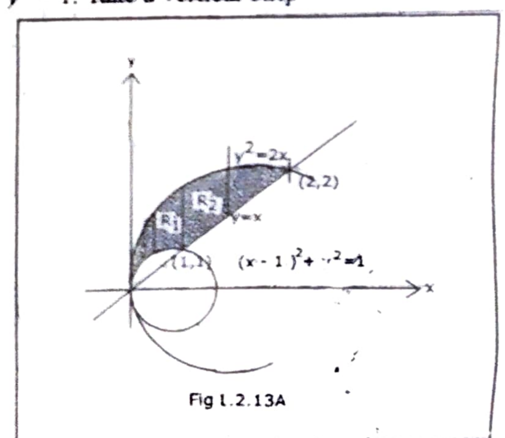


Fig 1.2.13A

$$\therefore \quad I = \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2 \int_x^{\sqrt{2x}} xy\,dy\,dx$$

$$= \int_0^1 \frac{x}{2}\left[y^2\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 \frac{x}{2}\left[y^2\right]_x^{\sqrt{2x}} dx$$

$$= \int_0^1 \frac{x}{2}\left[2x - 2x + x^2\right] dx + \int_1^2 \frac{x}{2}\left[2x - x^2\right] dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[2\,\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{7}{12}$$

21. Evaluate $\iint_R (2x - y^2)\, dA$, where R is the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$ and $y = 3$.

Solution : Draw the figure. Take a horizontal strip line.

$\therefore$ The limits are :

$1 - y \le x \le y - 1$

$1 \le y \le 3$

$$\therefore \quad I = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy$$

$$= \int_1^3 \left[x^2 - xy^2\right]_{1-y}^{y-1} dy$$

$$= -\frac{68}{3}$$

(After integration and simplification)

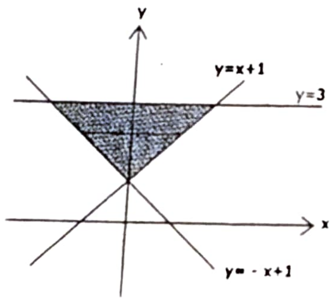


Fig 1.2.14

22. Evaluate $\iint_R \sqrt{4x^2 - y^2}\, dA$ over the triangular region given by $y = 0$, $y = x$ and $x = 1$.

Solution : Draw the figure.

Take horizontal strip line.

$\therefore$ The limits are :

$y \le x \le 1$

$0 \le y \le 1$

$$\therefore \quad I = \int_0^1 \int_y^1 \sqrt{4x^2 - y^2}\, dx\, dy$$

$$= 2\int_0^1 \int_y^1 \sqrt{x^2 - y^2/4}\, dx\, dy$$

$$= 2\int_0^1 \left[\frac{x}{2}\sqrt{x^2 - \frac{y^2}{4}} - \frac{y^2}{8}\log\left(x + \sqrt{x^2 - \frac{y^2}{4}}\right)\right]_y^1 dy$$

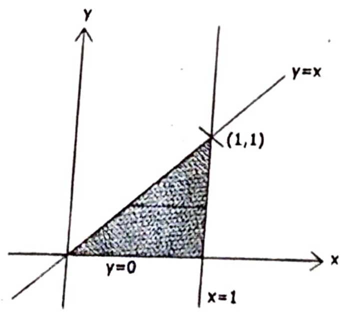


Fig 1.2.15

$$= \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \quad \text{[After integration and simplification]}$$

23. Evaluate $\iint_R (x + y)^2\, dA$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Draw the figure. Take a verticle strip line.

$\therefore$ The limits are :

$-\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$

$-a \le x \le a$

$$\therefore \quad I = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + 2xy + y^2)\, dy\, dx$$

$$= 2\int_{-a}^{a} \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2)\, dy\, dx$$

($\because$ y is an odd function.)

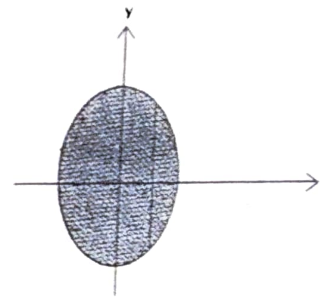


Fig 1.2.16

$$= 2\int_{-a}^{a} \left[x^2 y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[\frac{b}{a}x^2\sqrt{a^2 - x^2} + \frac{1}{3}\frac{b^3}{a^3}(a^2 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^{a} \left[\frac{b}{a}x^2\sqrt{a^2 - x^2} + \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\right] dx \quad (\because \text{ The integrand is even.})$$

$$= 4\int_0^{\pi/2} \left[a^3 b \sin^2\theta \cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right] d\theta \quad (\text{Put } x = a\sin\theta)$$

$$= 4\int_0^{\pi/2} \left[a^3 b \sin^2\theta - a^3 b \sin^4\theta + \frac{ab^3}{3}\cos^4\theta\right] d\theta$$

$$= 4\left[a^3 b\,\frac{1}{2}\cdot\frac{\pi}{2} - a^3 b\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}ab\,(a^2 + b^2)$$

24. Evaluate $\iint_R \sqrt{xy(1 - x - y)}\, dA$, where R is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Solution : Draw the figure. Take horizontal strip line

$\therefore$ The limits are :

$0 \le x \le 1 - y$

$0 \le y \le 1$

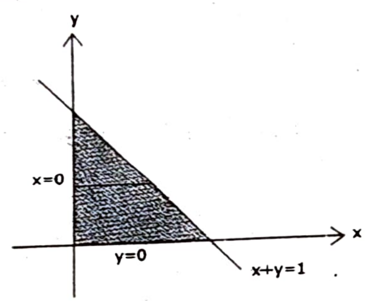


Fig 1.2.17

$$\therefore \quad I = \int_0^1 \int_0^{1-y} \sqrt{xy(1-x-y)}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \sqrt{y}\, \sqrt{(1-y)\,x - x^2}\, dx\, dy$$

Put $x = (1-y)\,t \rightarrow dx = (1-y)\, dt$

$$\therefore \quad I = \int_0^1 \int_0^1 \sqrt{y}\,(1-y)\,\sqrt{t-t^2}\,(1-y)\, dt\, dy$$

$$= \int_0^1 \int_0^{\pi/2} \sqrt{y}\,(1-y)^2 \sin\theta \cos\theta\; 2\sin\theta \cos\theta\; d\theta\, dy \qquad (\text{Put } t = \sin^2\theta)$$

$$= 2\int_0^1 \int_0^{\pi/2} (y^{1/2} - 2y^{3/2} + y^{5/2})\,(\sin^2\theta - \sin^4\theta)\, d\theta\, dy$$

$$= 2\left[\frac{y^{3/2}}{\frac{3}{2}} - 2\,\frac{y^{5/2}}{\frac{5}{2}} + \frac{y^{7/2}}{\frac{7}{2}}\right]_0^1 \left[\frac{1}{2}\,\frac{\pi}{2} - \frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right]$$

$$= 2\left[\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right] \frac{\pi}{4}\left[1 - \frac{3}{4}\right]$$

$$= \frac{2\pi}{105}$$

Note : If the limits of x and y are constants and integral is in the form $\int_a^b \int_e^d f(x)\, g(y)\, dy\, dx$ then this can be integrated separately in the form $\left[\int_0^b f(x)\, dx \cdot \int_c^d g(y)\, dy\right]$

EXERCISE – 1.1

- **Evaluate the following :**

1. $\iint_R (1 - 6x^2y)\, dA$ where $R : 0 \le x \le 2,\ -1 \le y \le 1$ **[Ans. : 4]**

2. $\int_0^{\log 2} \int_0^1 xy\, e^{xy^2}\, dy\, dx$ $\left[\textbf{Ans. :} \frac{1}{2}\,(1 - \log 2)\right]$

3. $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$ $\left[\textbf{Ans. :} \frac{21}{4} + e^4 - e^3\right]$

[Ans. : 1]

4. $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

[Ans. : 7/12]

5. $\int_0^1 \int_{x^2}^{x} (x^2 + 3y + 2)\, dA$

$\left[\textbf{Ans. :} \frac{67}{120}\right]$

6. $\int_0^1 \int_y^{y^2+1} x^2 y\, dx\, dy$

[Ans. : 3/35]

7. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dA$

[Ans. : 1.8882]

8. $\int_0^5 \int_0^{x^2} x\,(x^2 + y^2)\, dA$

9. Evaluate $\iint_R xy\, dA$ over the region in the positive quadrant for which $x + y \le 1$.

[Ans. : 1/24]

10. Evaluate $\iint_R x^2\, dA$, where R is the region bounded by the curves $y = x$ and $y = x^2$.

[Ans. : 1/20]

11. Evaluate $\iint_R xy\, dA$ over the region R bounded by $2y = x$, $y = \sqrt{x}$ $x = 2$ and $x = 4$.

[Ans. : 11/6]

12. Evaluate $\iint_R \frac{x}{\sqrt{1+y^2}}\, dA$, over the region R in the first quadrant enclosed by $y = x^2$,

$\left[\textbf{Ans. :} \frac{1}{2}(\sqrt{17} - 1)\right]$

$y = 4$ and $x = 0$.

13. Evaluate $\iint_R \frac{dA}{1+x^2}$ where R is the triangular region with vertices (0, 0), (1, 1) and

$\left[\textbf{Ans. :} \frac{\pi}{4} - \frac{1}{2}\log 2\right]$

(0,1).

14. Evaluate $\iint_R x \cos xy\, dA$ where R is the region enclosed by $x = 1, x = 2,\ y = \frac{\pi}{2}$ and

$\left[\textbf{Ans. :} -\frac{2}{\pi}\right]$

$y = \frac{2\pi}{x}$.

$\left[\textbf{Ans. :} \frac{41}{210}\right]$

15. Evaluate $\int_0^1 \int_{y^2}^{y} (1 + xy^2)\, dx\, dy$

16. Evaluate $\iint_R (5 - 2x - y)\, dA$, where R is given by $y = 0,\ x + 2y = 3,\ x = y^2$.

$\left[\textbf{Ans. :} \frac{214}{60}\right]$

1.4 Double Integrals in Polar Coordinates :

Let $f(r, \theta)$ be the function of polar coordinates r and θ, define on the region R bounded by the half lines $\theta = \alpha, \theta = \beta$ and the continuous curves $r = f_1(\theta)$, $r = f_2(\theta)$, where $\alpha \le \beta$ and $0 \le f_1(\theta) \le f_2(\theta)$. Then the region R is as shown in the figure.

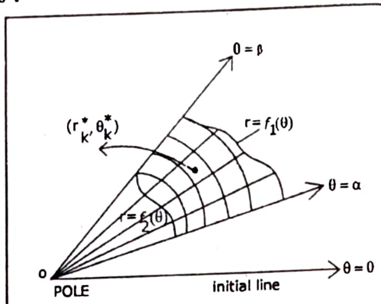


Fig 1.2.18

Divide the region R into small polar rectangles by drawing circular arcs within the region with centre 0, and the half lines between $\theta = \alpha$ and $\theta = \beta$. Let (r_k^*, θ_k^*) be the centre point in the polar rectangle, with area δA_k. If we increase the numbers of polar rectangles to be large then the following limit of sum of approximations,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\, \delta A_k$$

is known as double integral of a polar function over the region R, and can be written as

$$\iint_R f(r,\theta)\, dA.$$

Here δA_k is the area of a polar rectangle.

Suppose that the polar rectangle has central angle is $\delta\theta_k$ and a radius thickness is δr_k.

Then the radius of the inner boundary of δA_k is $r_k^* - \frac{1}{2}\delta r_k$, and that of the outer boundary of δA_k is $r_k^* + \frac{1}{2}\delta r_k$

$\therefore \quad \delta A_k$ = Area of larger sector – Area of smaller sector

$$= \frac{1}{2}\left(r_k^* + \frac{1}{2}\delta r_k\right)^2 \delta\theta_k - \frac{1}{2}\left(r_k^* + \frac{1}{2}\delta r_k\right)^2 \delta\theta_k$$

$$= r_k^*\, \delta r_k\, \delta\theta_k.$$

$\therefore$ In limiting sense,

$$dA = r\,dr\,d\theta$$

$$\therefore \quad \iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\,dr\,d\theta$$

Note : If $f(r,\theta)$ is continuous on R and $f(r,\theta) \ge 0$ for all (r,θ) in R then $\iint_R f(r,\theta)\, dA$ yields the volume of the solid formed by $f(r,\theta)$ under the region R.

1.4.1 Let the region R be as shown in the figure. Draw a strip line (Radial line) passing through the origin (pole) within the region R. See the end points of the strip line which touch the region R to the curves $r = f_1(\theta)$ and $r = f_2(\theta)$.

$\therefore \quad f_1(\theta) \le r \le f_2(\theta)$

Also the region lies between the lines $\theta = \alpha$ and $\theta = \beta$.

$\therefore \quad \alpha \le \theta \le \beta$

$\therefore$ Thus we have

$$\iint_R f(r,\theta)\, dA = \int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, d\theta$$

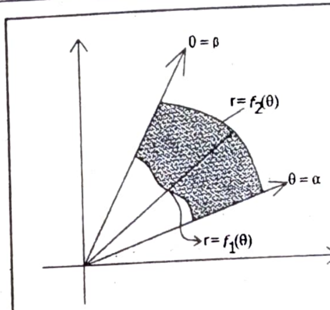


Fig 1.2.19

1.4.2 Change of Coordinates :

Sometimes the problem arise in cartesian form becomes very difficult to integrate, to determine the limits of integration or to draw figures. In this case the procedure of changing the coordinates (variables) becomes simple. Here we shall see change of variable in double integrals.

If a region S in the uv – plane is transformed into the region R in the xy–plane by the transformation $x = f(u, v)$, $y = g(u, v)$ then the function $F(x, y)$ defined on R can be thought of as a function $F(f(u, v), g(u, v))$ defined on S.

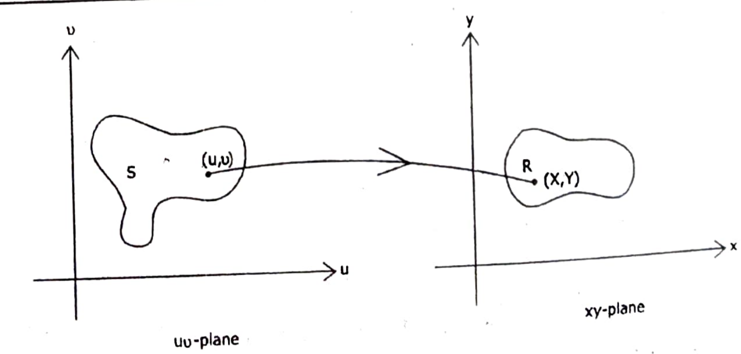


Fig 1.2.20

If the function $f(u, v)$ & $g(u, v)$ are continuous and posseses continuous derivatives then we have the following relation of integrals :

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(f(u,v),\, g(u,v))\,|J|\,du\,dv$$

Where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

is called the Jacobian of the transformation from the uv–plane to the xy–plane defined by the equations $x = f(u, v)$, $y = g(u, v)$

1.4.3 Changing into polar form :

Since we know that the relation between cartesian and polar coordinates is given by

$x = r\cos\theta, \quad y = r\sin\theta$

and $J = r$. Thus

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta$$

SOLVED EXAMPLES

1. **Evaluate** $\displaystyle\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\,dr\,d\theta$

Solution : $I = \displaystyle\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$

$= \dfrac{1}{3}\displaystyle\int_0^{\pi/2}[a^3 - a^3(1-\cos\theta)^3]\,d\theta$

$= \dfrac{a^3}{3}\displaystyle\int_0^{\pi/2}[1 - (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta]\,d\theta$

$= \dfrac{a^3}{3}\left[3\cdot 1 - 3\frac{1}{2}\frac{\pi}{2} + \frac{2}{3}\cdot 1\right]$

$= \dfrac{a^3}{3}\left[\dfrac{11}{3} - \dfrac{3\pi}{4}\right]$

$= \dfrac{a^3}{3}[44 - 9\pi]$

2. **Evaluate** $\iint_R r^3\,dr\,d\theta$ **over the area bounded between the circles** $r = 2\cos\theta$ **and** $r = 4\cos\theta$**.**

Solution : Draw the figure.

$\therefore$ The limits are :

$2\cos\theta \le r \le 4\cos\theta$

$-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

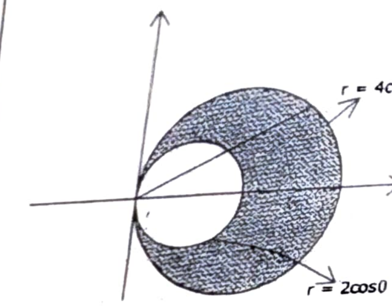


Fig 1.2.21

$\therefore\ I = \displaystyle\int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$

$= \displaystyle\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$

$= \dfrac{1}{4}\displaystyle\int_{-\pi/2}^{\pi/2}[256\cos^4\theta - 16\cos^4\theta]\,d\theta$

$= \dfrac{1}{4}\,240\displaystyle\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$

$= 2\times 60\displaystyle\int_0^{\pi/2}\cos^4\theta\,d\theta \qquad (\because \cos^4\theta$ is an even function.)

$= 120\left[\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}\right] = \dfrac{45\pi}{2}$

3. **Evaluate** $\displaystyle\int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2\sin\theta\,dr\,d\theta$

Solution : $I = \displaystyle\int_0^{\pi} 2\pi\sin\theta\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} d\theta$

$= \dfrac{2\pi}{3}\displaystyle\int_0^{\pi}\sin\theta\, a^3(1-\cos\theta)^3\,d\theta$

$= \dfrac{2\pi}{3}a^3\left[\dfrac{(1-\cos\theta)^4}{4}\right]_0^{\pi} \qquad \left(\because \int[f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}\right)$

$= \dfrac{2\pi a^3}{12}[2^4 - 0] = \dfrac{8\pi a^3}{3}$

4. **Evaluate** $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta$

Solution : $I = \displaystyle\int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$

$= \dfrac{1}{2}\displaystyle\int_0^{\pi/2}\cos\theta\sin^2\theta\,d\theta$

$= \dfrac{1}{2}\left[\dfrac{\sin^3\theta}{3}\right]_0^{\pi/2} = \dfrac{1}{6}$

5. **Evaluate** $\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta$

Solution : $I = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{1-\sin\theta} d\theta$

$$= \frac{1}{3}\int_0^{\pi} (\cos\theta\,(1-\sin\theta)^3\,d\theta$$

$$= \frac{1}{3}\left[-\frac{(1-\sin\theta)^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{9}[-1+1] = 0$$

6. **Evaluate** $\iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$ **, where R is a loop of** $r^2 = a^2\cos 2\theta$.

Solution : Draw the figure :

$\therefore$ The limits are

$0 \le r \le a\sqrt{\cos 2\theta}$

$\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$

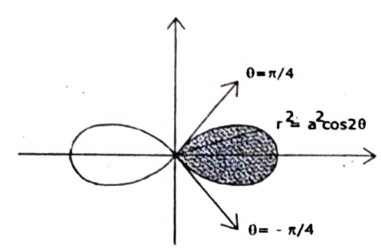


Fig 1.2.22

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{r^2+a^2}}\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left[\frac{(r^2+a^2)^{1/2}}{\frac{1}{2}}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[a\sqrt{1+\cos 2\theta} - a\right] d\theta$$

$$= a\int_{-\pi/4}^{\pi/4} \left[\sqrt{2}\cos\theta - 1\right] d\theta$$

$$= 2a\int_0^{\pi/4} \left[\sqrt{2}\cos\theta - 1\right] d\theta \qquad [\because f(\theta) = \sqrt{2}\cos\theta - 1 \text{ is an even function}]$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\,\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right)$$

7. **Evaluate** $\iint_R r^2\sin\theta\,dr\,d\theta$**, where R is the region of the circle** $r = 2a\cos\theta$**, lying above initial line.**

Solution : Draw the figure

$\therefore$ The limits are :

$0 \le r \le 2a\cos\theta$

$0 \le \theta \le \frac{\pi}{2}$

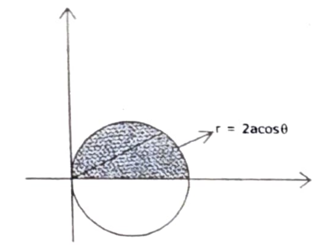


Fig 1.2.23

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} \sin\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} 8a^3\cos^3\theta\,\sin\theta\,d\theta$$

$$= \frac{8a^3}{3}\left[\frac{-\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{2a^3}{3}$$

8. **Evaluate** $\iint_R r\sin\theta\,dr\,d\theta$ **over the area of the cardioid** $r = a\,(1+\cos\theta)$ **above the initial line.**

Solution : Draw the figure

$\therefore$ The limits are :

$0 \le r \le a\,(1+\cos\theta)$

$0 \le \theta \le \pi$

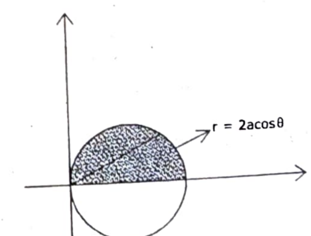


Fig 1.2.23

$$\therefore \quad I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\; a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\left[-\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi} = \frac{4a^2}{3}$$

9. Evaluate $\iint\limits_R \sin\theta\, dA$, where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$

Solution : Draw the figure

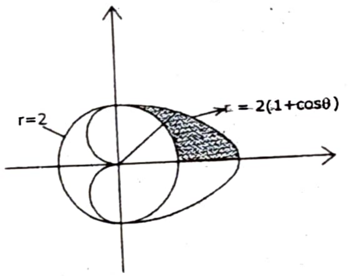


Fig 1.2.25

$\therefore$ The limits are :

$2 \le r \le 2(1 + \cos\theta)$

$0 \le \theta \le \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta\, r\, dr\, d\theta$$

$(\because dA = r\, dr\, d\theta)$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin\theta\, [(1+\cos\theta)^2 4 - 4]\, d\theta$$

$$= 2\int_0^{\pi/2} [\sin\theta\,(1+\cos\theta)^2 - \sin\theta]\, d\theta$$

$$= 2\left[\frac{-(1+\cos\theta)^3}{3} + \cos\theta\right]_0^{\pi/2} = \frac{8}{3}$$

10. Evaluate $\displaystyle\int_0^{\pi/2} \int_{a(1+\cos\theta)}^{a} r\, dr\, d\theta$

Solution : $\displaystyle I = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1+\cos\theta)}^{a} d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} [a^2 - a^2(1+\cos\theta)^2]\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} [1 - 1 - 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= -\frac{a^2}{2}\int_0^{\pi/2} [2\cos\theta + \cos^2\theta]\, d\theta$$

$$= -\frac{a^2}{2}\left[2 + \frac{1}{2}\cdot\frac{\pi}{2}\right] = -\frac{a^2}{2}\left[2 + \frac{\pi}{4}\right]$$

11. Evaluate $\iint\limits_R (x^2+y^2)\, x\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Solution : Draw the figure, we use polar coordinates.

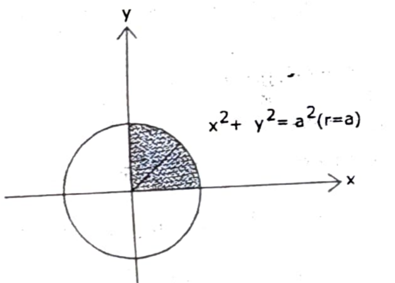


Fig 1.2.26

$\therefore$ The limits are :

$0 \le r \le a$

$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \int_0^a r^2 \cdot r\cos\theta \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^2 \cos\theta\, dr\, d\theta$$

$$= [\sin\theta]_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a = \frac{a^5}{5}$$

12. Evaluate $\iint\limits_R \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\, dA$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

Solution : Draw the figure. Here we use the polar coordinates.

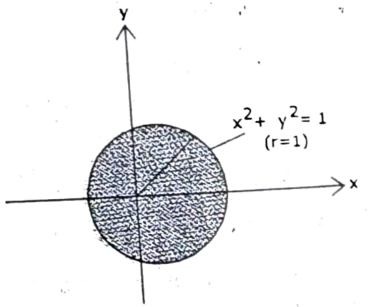


Fig 1.2.27

$\therefore$ The limits are :

$0 \le r \le 1$

$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 \frac{(1-r^2)r}{\sqrt{1-r^4}}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 \frac{r}{\sqrt{1-r^4}}\, dr\, d\theta - \int_0^{\pi/2} \int_0^1 \frac{r^3}{\sqrt{1-r^4}}\, dr\, d\theta$$

$$= I_1 - I_2$$

$$\therefore \quad I_1 = [\theta]_0^{\pi/2} \int_0^1 \frac{r}{\sqrt{1-r^4}}\, dr$$

$$= \frac{\pi}{2}\,\frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} \qquad (\text{Put } t = r^2)$$

$$= \frac{\pi}{4}\left[\sin^{-1} t\right]_0^1 = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

$$I_2 = [\theta]_0^{\pi/2}\left[-\frac{1}{4}\,\frac{(1-r^4)^{1/2}}{\frac{1}{2}}\right]_0^1 = \frac{\pi}{2}\left[+\frac{1}{2}\right] = \frac{\pi}{4}$$

$$\therefore \quad I = I_1 - I_2 = \frac{\pi^2}{8} - \frac{\pi}{4} = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right)$$

13. Evaluate $\iint\limits_R \sqrt{a^2 - x^2 - y^2}\; dx\,dy$ over the semi–circle $x^2 + y^2 = ax$ in the positive quadrant.

Solution : Draw the figure circle :

$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$. We use polar form of the integral.

$\therefore$ Circle $r = a\cos\theta$

$\therefore$ The limits are :

$0 \le r \le a\cos\theta$

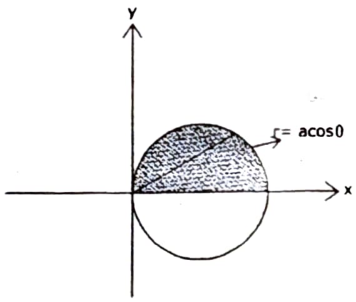
$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}\,\frac{(a^2 - r^2)^{3/2}}{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}[a^3\sin^3\theta - a^3]\,d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}\cdot 1 - \frac{\pi}{2}\right] = \frac{a^3}{18}(3\pi - 4)$$



Fic 1.2.28

14. Evaluate $\iint\limits_R (x^2 + y^2)\,dx\,dy$, where R is the region in the xy–plane bounded by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(b > a)$.

Solution : Draw the figure we use polar coordinates

$\therefore$ The limits are : $a \le r \le b$

$0 \le \theta \le 2\pi$

$$\therefore \quad I = \int_0^{2\pi}\int_a^b r^2 \cdot r\,dr\,d\theta$$

$$= [\theta]_0^{2\pi}\left[\frac{r^4}{4}\right]_a^b$$

$$= \frac{2\pi}{4}[b^4 - a^4] = \frac{\pi}{2}[b^4 - a^4]$$

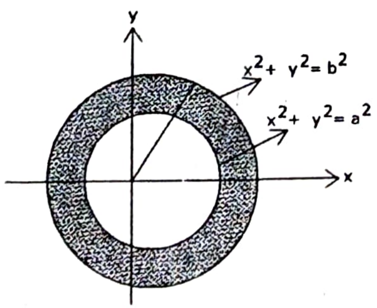


Fig 1.2.29

15. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

Solution : Draw the figure. We use polar coordinates.

$\therefore$ The limits are :

$0 \le r \le \infty$

$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= [\theta]_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty$$

$$= \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

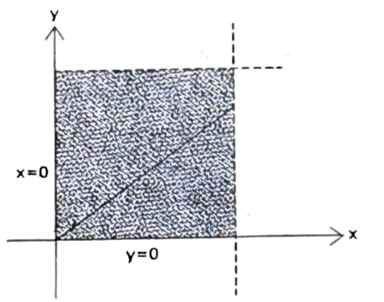


Fig 1.2.30

16. Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2\,dy\,dx$.

Solution : We convert it into polar form Draw the figure from

$y = 0,\ y = \sqrt{2ax - x^2},\ x = 0,\ x = 2a$

$\therefore$ The limits are :

$0 \le r \le 2a\cos\theta$

$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\cos^2\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos^2\theta\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\cos^2\theta\cdot 16a^4\cos^4\theta\,d\theta$$

$$= 4a^4\int_0^{\pi/2}\cos^6\theta\,d\theta = 4a^4\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi a^4}{8}$$

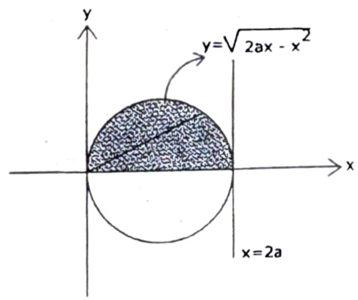


Fig. 1.2.31

17. Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$.

Solution : We change it into polar form.

We have $y = 0,\ y = \sqrt{2x - x^2}$

$x = 0,\ x = 2$

$\therefore$ The limits are :

$0 \le r \le 2\cos\theta$

$0 \le \theta \le \pi/2$

$$I = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\theta \cdot 4\cos^2\theta\, d\theta$$

$$= 2 \cdot \frac{2}{3} \cdot 1 = \frac{4}{3}$$

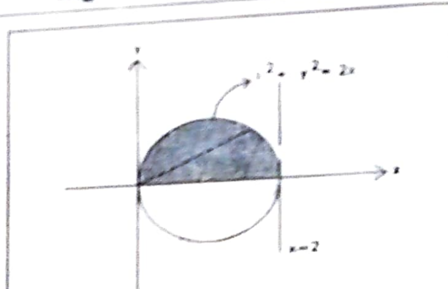


Fig 1.2.32

18. Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dA}{\sqrt{1-x^2-y^2}}$.

Solution : We convert it into polar form.

Draw the figure from

$x = 0,\ x = \sqrt{\frac{1}{2}(1-y^2)},\quad y = 0,\ y = 1$

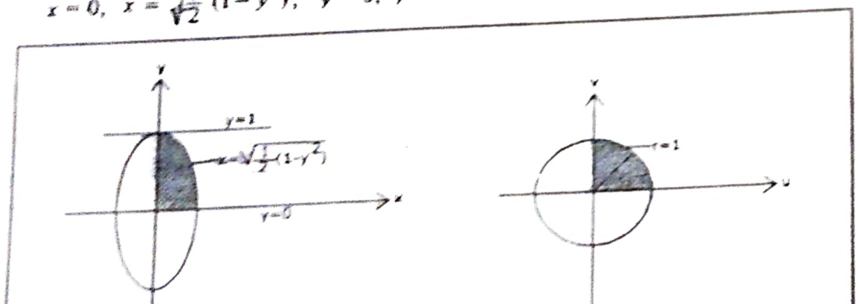


Fig 1.2.33

We have

$x^2 = \frac{1}{2}(1-y^2) \qquad \frac{x^2}{1} + \frac{y^2}{2} = \frac{1}{2}$

$\therefore\ \frac{x^2}{\frac{1}{2}} + \frac{y^2}{1} = 1$

$\therefore$ Take $x = \frac{1}{\sqrt{2}}u,\ y = v \Rightarrow u^2 + v^2 = 1$

$\therefore\ |J| = \frac{1}{\sqrt{2}} \qquad \left(\because \text{ For an ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow x = au,\ y = bv \Rightarrow J = ab\right)$

$dx\,dy = \frac{1}{\sqrt{2}}\,du\,dv$

Also take $u = r\cos\theta,\ v = r\sin\theta$

$$I = \int_0^{\pi/2}\int_0^1 \frac{1}{\sqrt{1-r^2}} \cdot \frac{1}{\sqrt{2}}\, r\,dr\,d\theta$$

$$= \frac{1}{\sqrt{2}}[\theta]_0^{\pi/2}\left[-\frac{1}{2}\frac{(1-r^2)^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} \cdot 1 = \frac{\pi}{2\sqrt{2}}$$

19. Evaluate $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$

Solution : We use polar coordinates

Draw the figure from

$x = y,\ x = a,\ y = 0,\ y = a$

$\therefore$ The limits are :

$0 \le r \le a\sec\theta$

$0 \le \theta \le \pi/4$

$$\therefore\ I = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r\cos\theta \cdot r\,dr\,d\theta}{r^2}$$

$$= \int_0^{\pi/4} \cos\theta\,[r]_0^{a\sec\theta}\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta \cdot a\sec\theta\,d\theta = \frac{a\pi}{4}$$

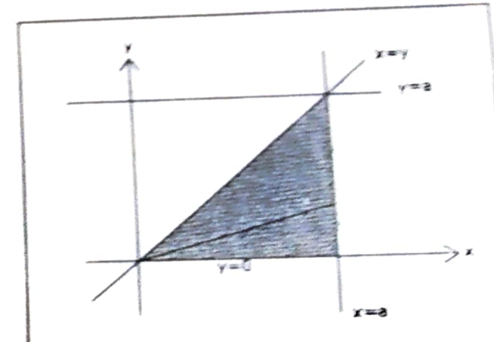


Fig. 1.2.34

20. Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\,dr\,d\theta$

Solution : $I = \int_0^{\pi/2}\left[\frac{1}{2}\frac{(r^2+a^2)^{-1}}{-1}\right]_0^{\infty} d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\left[0 + \frac{1}{a^2}\right]d\theta = \frac{1}{2a^2}[\theta]_0^{\pi/2} = \frac{\pi}{4a^2}$$

21. Evaluate $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}$

Solution : We use polar coordinates.

Draw the figure from

$x^2 = y^2/4a, \quad x = y, \quad y = 0, \quad y = 4a$

$y^2 = 4ax \rightarrow r = 4a \operatorname{cosec}\theta \cot\theta$

$\therefore$ The limits are :

$0 \le r \le 4a \operatorname{cosec}\theta \cot\theta$

$\frac{\pi}{4} \le \theta \le \pi/2$

$$\therefore \; I = \int_{\pi/4}^{\pi/2}\int_0^{4a \operatorname{cosec}\theta\cot\theta} \frac{r^2\cos 2\theta}{r^2}\, r\, dr\, d\theta$$

$(\because \cos^2\theta - \sin^2\theta = \cos 2\theta)$

$$= \int_{\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{4a\operatorname{cosec}\theta\cot\theta} \cos 2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2}[16a^2\operatorname{cosec}^2\theta\cot^2\theta]\,[\cos^2\theta - \sin^2\theta]\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}[\cot^4\theta - \cot^2\theta]\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}[\cot^2\theta(\operatorname{cosec}^2\theta - 1) - \cot^2\theta]\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}[\cot^2\theta\operatorname{cosec}^2\theta - 2(\operatorname{cosec}^2\theta - 1)]\,d\theta$$

$$= 8a^2\left[-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= 8a^2\left[0 + 0 + \pi + \frac{1}{3} - 2 - \frac{\pi}{2}\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

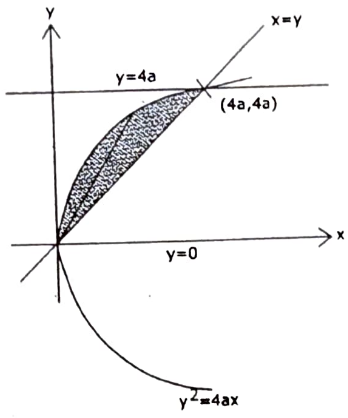


Fig 1.2.35

22. Transform the integral $\int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\, dr\, d\theta$ **to cartesian form and hence evaluate.**

Solution : Draw the figure from,

$r = 0, \quad r = a, \quad \theta = 0, \quad \theta = \pi$

Put $x = r\cos\theta, \quad y = r\sin\theta$

$\therefore \quad r^3\sin\theta\cos\theta\, dr\, d\theta$

$= (r\sin\theta)(r\cos\theta)(r\, dr\, d\theta)$

$= xy\, dA$

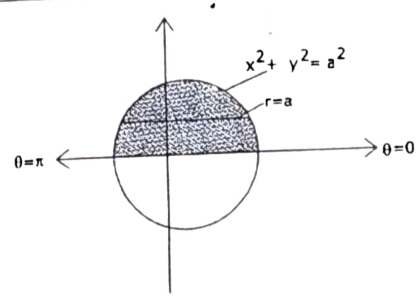


Fig 1.2.36

$$\therefore \; I = \iint_R xy\, dA$$

$$= \int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xy\, dx\, dy$$

$= 0$

$[\because x$ is an odd function.)

23. Evaluate $\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$

Solution : Changing into polar form.

Draw the figure from

$x = 0, \quad x = \sqrt{a^2-y^2}, \quad y = 0, \quad y = a$

$\therefore$ The limits are :

$0 \le r \le a$ and $0 \le \theta \le \pi/2$

$$\therefore \; I = \int_0^{\pi/2}\int_0^a r^2\sin^2\theta \cdot r \cdot r\, dr\, d\theta$$

$$= \left[\frac{1}{2}\cdot\frac{\pi}{2}\right]\left[\frac{r^5}{5}\right]_0^a = \frac{\pi a^5}{20}$$

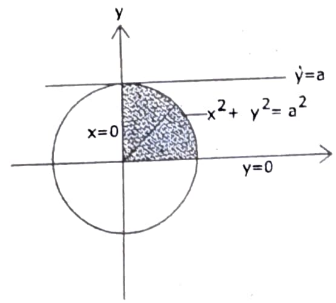


Fig 1.2.37

24. Evaluate $\iint_R (x+y)^2\, dx\, dy$ **where R is the parallelogram in the** xy**–plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation** $u = x + y, \; v = x - 2y$.

Solution : The transformation gives :

$(x, y) \rightarrow (u, v)$

$(1, 0) \rightarrow (1, 1)$

$(3, 1) \rightarrow (4, 1)$

$(2, 2) \rightarrow (4, -2)$

$(0, 1) \rightarrow (1, -2)$

Which gives a square in uv–plane

$$\therefore \; J' = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= -3 \Rightarrow J' = -\frac{1}{3}$$

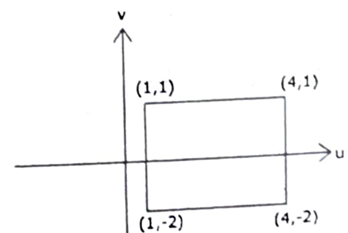


Fig 1.2.38

$\therefore \quad I = \int_{-2}^{1} \int_{1}^{4} u^2 \frac{1}{3}\, du\, dv$

$= \frac{1}{3} [v]_{-2}^{1} \left[\frac{u^3}{3}\right]_1^4 = \frac{1}{9} [1+2][64-1] = 21$

25. Evaluate $\iint_R \left(\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}\right)^{1/2} dx\,dy$. **Where R is the positive quadrant of the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Changing the variables.

Put $x = au,\ y = bv$

$\therefore$ Equation of ellipse becomes the equation of a circle $u^2 + v^2 = 1$

$\therefore$ $J = ab$

$\therefore$ Again take $u = r\cos\theta,\ v = r\sin\theta$

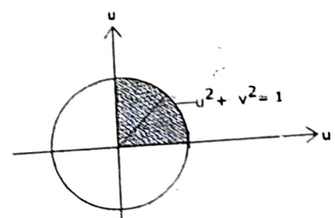


Fig 1.2.39

$\therefore \quad I = \iint_R \left(\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}\right) dA$

$= \iint_R \left(\frac{1-u^2-v^2}{1+u^2+v^2}\right) ab\, du\, dv$

$= ab \int_0^{\pi/2} \int_0^1 \left(\frac{1-r^2}{1+r^2}\right) r\, dr\, d\theta$

$= ab\, \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right)$

26. Evaluate $\iint_R (x+y)^2\, dx\, dy$ **over the area bounded by the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : Changing the variables.

Put $x = au,\ y = bu$

Also take $u = r\cos\theta,\ v = r\sin\theta$.

Draw the figure in uv-plane

$\therefore$ The limits are :

$0 \le r \le 1,\ 0 \le \theta \le 2\pi$

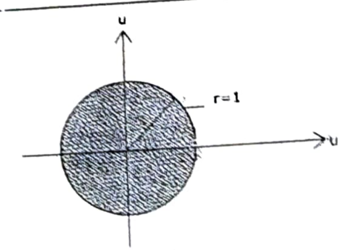


Fig 1.2.40

$\therefore \quad I = \int_0^{2\pi} \int_0^1 [ar\cos\theta + br\sin\theta]^2\, ab\, r\, dr\, d\theta \quad (\because J = ab)$

$= ab \int_0^{2\pi} \int_0^1 [a\cos\theta + b\sin\theta]^2 r^3\, dr\, d\theta$

$= ab \left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi} [a^2\cos^2\theta + 2ab\cos\theta\sin\theta + b^2\sin^2\theta]\, d\theta$

$= \frac{ab}{4}\, 4 \int_0^{\pi/2} [a^2\cos^2\theta + b^2\sin^2\theta]\, d\theta$

$\left(\because \int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \text{ if } f(a-x) = f(x)\right.$
$\left. = 0 \text{ if } f(a-x) = f(x)\right)$

$= ab\left[a^2 \frac{1}{2}\frac{\pi}{2} + b^2 \frac{1}{2}\frac{\pi}{2}\right] = \frac{ab\pi}{4}(a^2 + b^2)$

27. Evaluate $\iint_R e^{xy} dA$**, where R is the region enclosed by the lines** $u = \frac{1}{2}x$ **and** $y = x$ **and the hyperbolas** $y = \frac{1}{x}$ **and** $y = \frac{2}{x}$.

Solution : Changing the variable by using the transformation $u = \frac{y}{x},\ v = xy$

$\therefore \quad J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -2\frac{y}{x} = -2u$

$\therefore \quad |J| = \frac{1}{2u}$

Now the equation becomes, $u = \frac{1}{2},\ u = 1,\ v = 1,\ v = 2$

$\therefore \quad I = \int_1^2 \int_{1/2}^1 e^v \frac{1}{2u}\, du\, dv$

$= \frac{1}{2}[e^v]_1^2 [\log u]_{1/2}^1$

$= \frac{1}{2}[e^2 - e^1]\left[\log 1 - \log\frac{1}{2}\right] = \frac{1}{2}(e^2 - e)\log 2$

28. Evaluate $\iint_R \frac{\sin(x-y)}{\cos(x+y)} dA$**, where R is the triangular region enclosed by the lines** $y = 0,\ y = x,\ x + y = \pi/4$.

Solution : Changing the variables using the transformation

$u = x - y,\ v = x + y$

$$\therefore \quad J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \Rightarrow |J| = \frac{1}{2}$$

Now, $y = 0 \Rightarrow u = x$ and

$v = x \Rightarrow u = v$

$y = x \Rightarrow u = 0$

$x + y = \frac{\pi}{4} \Rightarrow v = \frac{\pi}{4}$

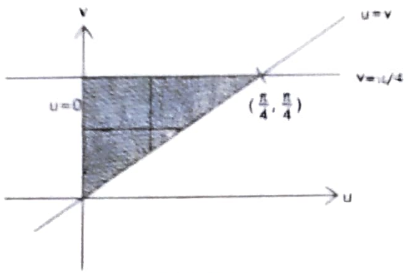


Fig 1.2.41

Draw the figure and take a horizontal strip line

$\therefore$ The limits are :

$0 \le u \le v$

$0 \le v \le \pi/4$

$$\therefore \quad I = \int_0^{\pi/4}\int_0^{v} \frac{\sin u}{\cos v}\,\frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos v}\,[-\cos u]_0^v\,dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos v}\,[-\cos v + 1]\,dv = \frac{1}{2}\int_0^{\pi/4}[-1 + \sec v]\,dv$$

$$= \frac{1}{2}\,[-v + \log\,(\sec v + \tan v)]_0^{\pi/4} = \frac{1}{2}\left[-\frac{\pi}{4} + \log\left(\sqrt{2} + 1\right)\right]$$

29. Evaluate $\iint\limits_R \sqrt{x+y}\;dx\,dy$, where R is the parallelogram bounded by the lines $x + y = 0$, $x + y = 1$, $2x - 3y = 0$ and $2x - 3y = 4$.

Solution : Changing the variables by using the transformation

$u = x + y,\ v = 2x - 3y$

$$\therefore \quad J' = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \Rightarrow |J| = \frac{1}{5}$$

Now the equations becomes

$u = 0,\ u = 1,\ v = 0,\ v = 4$

$$\therefore \quad I = \int_0^1\int_0^4 \sqrt{u}\,\frac{1}{5}\,dv\,du$$

$$= \frac{1}{5}\left[\frac{u^{3/2}}{3/2}\right]_0^1 [v]_0^4 = \frac{1}{5}\cdot\frac{2}{3}\cdot 4 = \frac{8}{15}$$

30. Evaluate $\iint\limits_R \frac{y-4x}{y+4x}\,dA$, where R is the region enclosed by the lines $y = 4x$, $y = 4x + 2$, $y = 2 - 4x$, $y = 5 - 4x$.

Solution : Changing the coordinates, by using the transformation $u = y - 4x,\ v = y + 4x$

$$\therefore \quad J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -4 & 1 \\ 4 & 1 \end{vmatrix} = -8$$

$$\therefore \quad |J| = \frac{1}{8}$$

Equations are : $u = 0,\ u = 2,\ v = 2,\ v = 5$.

$$\therefore \quad I = \int_0^2\int_2^5 \frac{u}{v}\,\frac{1}{8}\,dv\,du = \frac{1}{8}\left[\frac{u^2}{2}\right]_0^2 [\log v]_2^5$$

$$= \frac{1}{8}\cdot 2\,[\log 5 - \log 2] = \frac{1}{4}\log\frac{5}{2}$$

31. Evaluate $\iint\limits_R (x^2 + y^2)\,dA$, where R is the region lying in the first quadrant and bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$.

Solution : Changing the variables by using the transformation $u = x^2 - y^2,\ v = xy$

$$\therefore \quad J' = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2)$$

$$\therefore \quad |J| = \frac{1}{2(x^2 + y^2)}$$

The equations are : $u = 1,\ u = 9,\ v = 2,\ v = 4$

$$\therefore \quad I = \iint\limits_R (x^2 + y^2)\,\frac{1}{2(x^2 + y^2)}\,du\,dv$$

$$= \frac{1}{2}\iint\limits_R du\,dv = \frac{1}{2}\int_1^9\int_2^4 dv\,du = \frac{1}{2}\,[v]_2^4\,[u]_1^9$$

$$= \frac{1}{2}\cdot 2\cdot 8 = 8$$

32. Use an appropriate transformation to evaluate $\iint\limits_R e^{(y-x)/(y+x)}\,dA$, where R is the region in the first quadrant enclosed by the trapezoid with vertices (0, 1), (1, 0), (0, 4), (4, 0).

Solution : Put $u = y - x,\ v = y + x$

$$\therefore \quad J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \Rightarrow |J| = \frac{1}{2}$$