7 CENTROID AND MOMENT OF INERTIA

7·1. Center of Gravity (G) : *It is a point of application of a* ***single*** *(equivalent)* ***resultant weight*** *of a body. The resultant is a replacement of a system of parallel forces of weights of number of particles.*

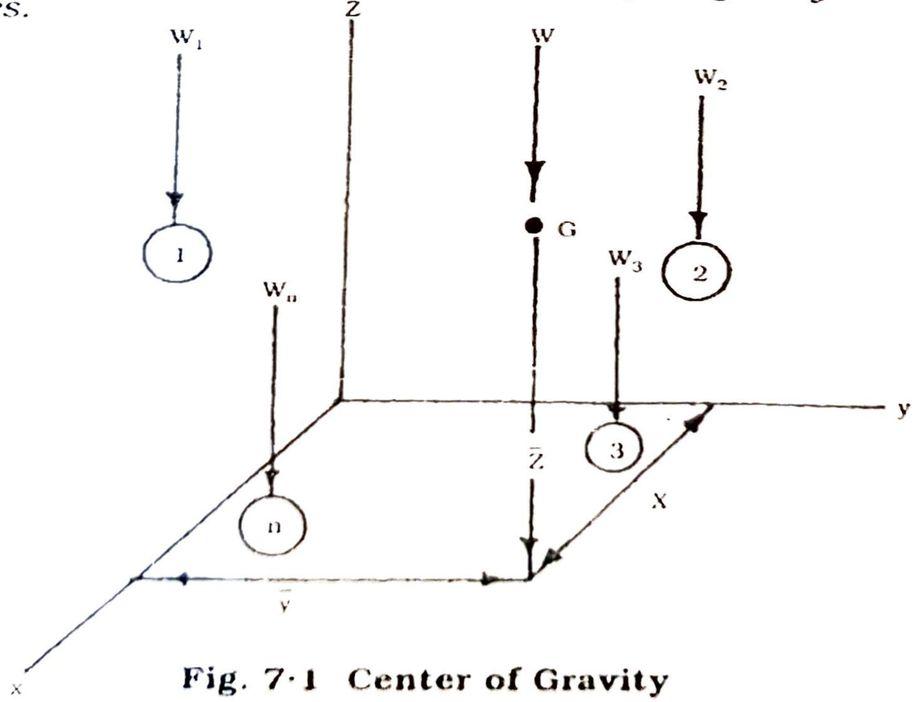


Fig. 7·1 Center of Gravity

In above figure, it is shown that there are **n** number of particles having different weights, say, W_1 to W_n. The Center of Gravity (G) is having coordinates say, $\bar{X}$, $\bar{Y}$, $\bar{Z}$.

Here, *Resultant Weight* : $W_R = \Sigma W = w_1 + w_2 + w_3 + \ldots$

and, *Moment about axis*, ΣM_y : $\bar{X}W = x_1 w_1 + x_2 w_2 + x_3 w_3 + \ldots$

ΣM_x : $\bar{Y}W = y_1 w_1 + y_2 w_2 + y_3 w_3 + \ldots$

Thus the coordinates of G ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) are :

$$\bar{X} = \frac{\Sigma\, xw}{\Sigma w}, \quad \bar{Y} = \frac{\Sigma\, yw}{\Sigma w}, \quad \bar{Z} = \frac{\Sigma\, zw}{\Sigma w}$$

The above equation is useful when the body can be divided in to several ***common shapes*,** such as **square, rectangle, triangle, circle** etc.

But if we increase the number of particles and simultaneously decrease the size of the each particle *(in case of* ***uncommon shapes*)**, *integration* is required rather than a discrete summation of the terms. The coordinates of G ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) will be

$$\bar{X} = \frac{\int x\, dw}{\int dw}, \quad \bar{Y} = \frac{\int y\, dw}{\int dw}, \quad \bar{Z} = \frac{\int z\, dw}{\int dw}$$

But, $w = \gamma v$

where, γ = specific weight of the body (Weight per unit volume)

v = volume of the body.

$$\bar{X} = \frac{\int_V x \gamma\, dv}{\int_V \gamma\, dv}, \quad \bar{Y} = \frac{\int_V y \gamma\, dv}{\int_V \gamma\, dv}, \quad \bar{Z} = \frac{\int_V z \gamma\, dv}{\int_V \gamma\, dv}$$

7.2 Center of Mass

7.2 Center of Mass : To study the problems concerning the *motion of* ***matter*** *under the* ***influence of force*** i.e. ***dynamics*,** it is necessary to locate a point called the *center of mass.*

Now $\gamma = \rho g$

where ρ = density = mass per unit volume.

g = acceleration of gravity which is constant for every particle.

$$\bar{X} = \frac{\int_V x \rho\, dv}{\int_V \rho\, dv}, \quad \bar{Y} = \frac{\int_V y \rho\, dv}{\int_V \rho\, dv}, \quad \bar{Z} = \frac{\int_V z \rho\, dv}{\int_V \rho\, dv}$$

Also for *common shapes*,

$$\bar{X} = \frac{\Sigma\, xm}{\Sigma m}, \quad \bar{Y} = \frac{\Sigma\, ym}{\Sigma m}, \quad \bar{Z} = \frac{\Sigma\, zm}{\Sigma m}$$

7.3 Centroid (C) : *The centroid is a point which defines the* ***geometrical center*** *of an object. If the material composing a body is* ***uniform*** *or* ***homogeneous*,** *the density or specific weight will be constant* throughout the body and therefore the same can be cancelled from the above equations. *Thus centroid is* **independent** *of* ***body's weight*.**

(1) Centroid of Line : The geometry of the object, such as a **thin rod** or **wire,** takes the form of a **line.** The coordinates of the centroid of line are.

For common shapes :

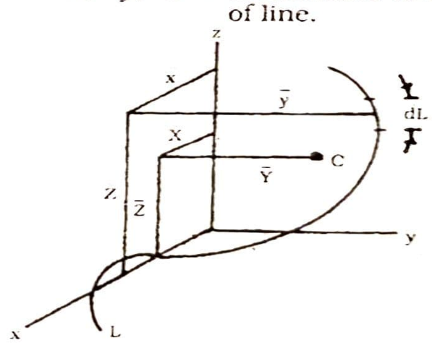
$$\bar{X} = \frac{\Sigma\, \bar{x}L}{\Sigma L}, \quad \bar{Y} = \frac{\Sigma\, \bar{y}L}{\Sigma L}, \quad \bar{Z} = \frac{\Sigma\, \bar{z}L}{\Sigma L}$$

For uncommon shapes :

$$\bar{X} = \frac{\int_L \bar{x}\, dL}{\int_L dL}, \quad \bar{Y} = \frac{\int_L \bar{y}\, dL}{\int_L dL}, \quad \bar{Z} = \frac{\int_L \bar{z}\, dL}{\int_L dL}$$

Where $\bar{X}$, $\bar{Y}$, $\bar{Z}$ = coordinates of centroid of the **complete line.**

$\bar{x}$, $\bar{y}$, $\bar{z}$ = coordinates of centroids of **small segments** of line.



The location of centroid of line may not be within the object, it can be outside the line.

ΣM_y : $\bar{X}L = \Sigma\, \bar{x}dL$

ΣM_x : $\bar{Y}L = \Sigma\, \bar{y}dL$

Fig. 7.2 Centroid of Line

(2) Centroid of Area :

In a similar manner, the centroid for the ***surface area*** of an object, such as a ***plate*** or ***shell*** can be found by subdividing the area in to differential elements and computing the ÔÒmoments" of these area elements about the coordinate axes. The coordinates of *centroid* of the *surface area* are.

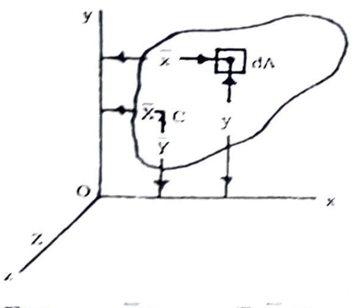
For common shapes :

$$\bar{X} = \frac{\Sigma\, \bar{x}A}{\Sigma A}, \quad \bar{Y} = \frac{\Sigma\, \bar{y}A}{\Sigma A}, \quad \bar{Z} = \frac{\Sigma\, \bar{z}A}{\Sigma A}$$

For uncommon shapes :

$$\bar{X} = \frac{\int_A x\,dA}{\int_A dA}\ ,\quad \bar{Y} = \frac{\int_A y\,dA}{\int_A dA}\ ,\quad \bar{Z} = \frac{\int_A z\,dA}{\int_A dA}$$

For **thin plate,** *we may consider* **two dimensions** *only i.e. only to find* $\bar{X}$ *and* $\bar{Y}$.



$\Sigma M_y :\quad \bar{X}A = \Sigma\, x dA$

$\Sigma M_x :\quad \bar{Y}A = \Sigma\, y dA$

Fig 7·3 Centroid of Area

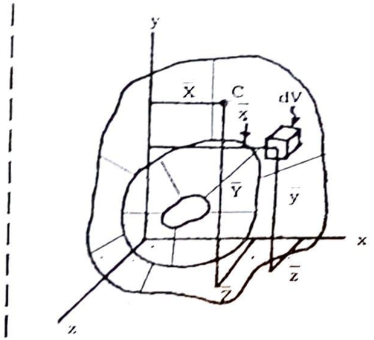


Fig 7·4 Centroid of Volume

(3) Centroid of Volume : The coordinates for the centroid C of the volume of space occupied by an object are,

For common shapes :

$$\bar{X} = \frac{\Sigma\, \bar{x}V}{\Sigma V}\ ,\quad \bar{Y} = \frac{\Sigma\, \bar{y}V}{\Sigma V}\ ,\quad \bar{Z} = \frac{\Sigma\, \bar{z}V}{\Sigma V}$$

For uncommon shapes :

$$\bar{X} = \frac{\int_V x\,dV}{\int_V dV}\ ,\quad \bar{Y} = \frac{\int_V y\,dV}{\int_V dV}\ ,\quad \bar{Z} = \frac{\int_V z\,dV}{\int_V dV}$$

7·4 Composite Bodies :

A composite body consists of a series of connected *"simpler" common)* shaped bodies, which may be ***rectangular, triangular, semi circular*** etc. Such a body can often be *sectioned or divided* in o composite parts and if the location of centroid of each of these *'composite parts"* are known, one can eliminate the need for ntegration for entire body. The centroid for composite lines, areas and volumes can be found by using the relations mentioned in section 7·3.

Centroid for common shapes are given in the following tables.

Table 7·1 Centroid of common shapes of *Lines*.

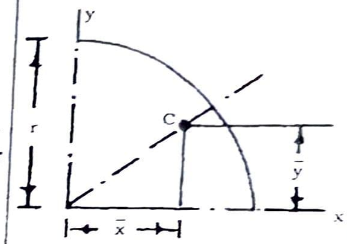
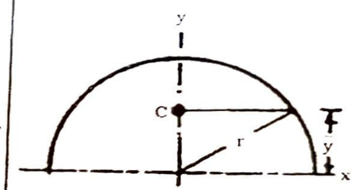
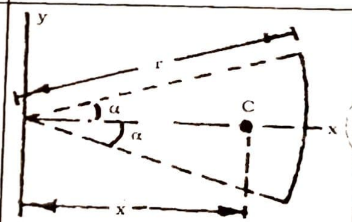
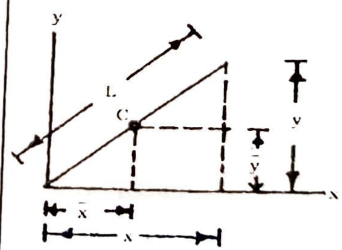
Shape	Diagram	$\bar{x}$	$\bar{y}$	Length
Quarter Circular Arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semi-Circular Arc		0	$\frac{2r}{\pi}$	πr
Arc of Circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$
Inclined Line		$\frac{x}{2}$	$\frac{y}{2}$	$L = (\sqrt{x^2+y^2})$

Table 7·2. Centroid of common shapes of *Areas*.

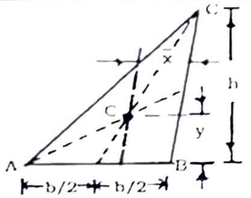
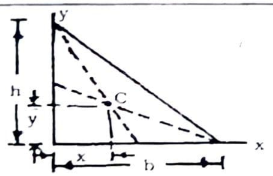
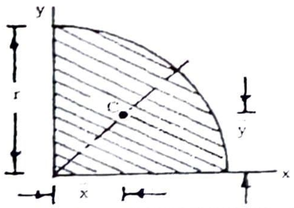
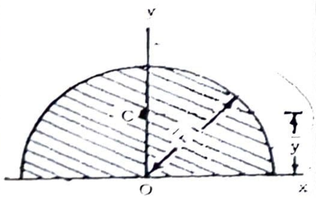
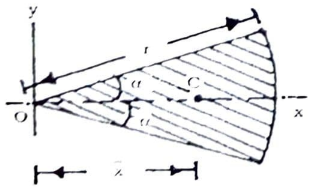
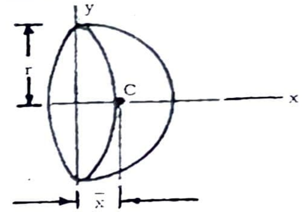
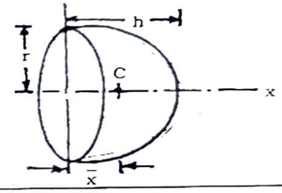
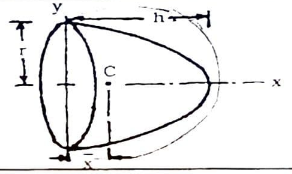
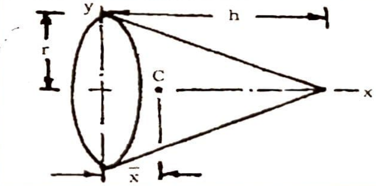
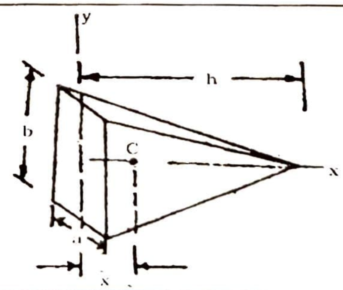
Shape	Diagram	$\bar{x}$	$\bar{y}$	Area
Triangular area		Away from BC by b/3	$\frac{h}{3}$	$\frac{bh}{2}$
- do -		$\frac{b}{3}$	$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-Circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semi-Circular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Circular sector		$\frac{2r \sin\alpha}{3\alpha}$	0	αr^2

Table 7·3 Centroid of common shapes of *Volumes* :

Shape	Diagram	$\bar{x}$	volume
Hemisphere		$\frac{3r}{8}$	$\frac{2}{3}\pi r^3$ $\frac{4\pi r^3}{3}$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi r^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi r^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi r^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}$ abh

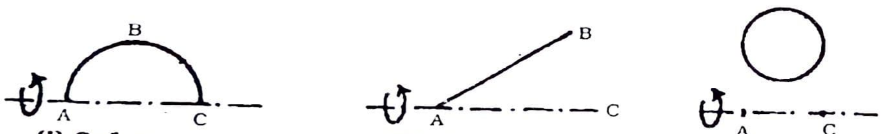
7.5 Theorems of Pappus - Guldinus :

Greek geometer Pappus(300 A.D.) had first formulated the theorems for surfaces and bodies of revolution, which were restated by Swiss mathematician Guldinus (1577-1643).

(1) Surface of Revolution : It is a surface which may be generated by *rotating a plane curve* about a fixed axis. Thus by rotating a **curve** or **line** about any axis, we get the **surface area** which is known as surface of revolution.

The surface of revolution are **three** :

(i) Surface of Sphere : It may be obtained by *rotating* a ***semicircular arc*** ABC about the diameter AC.



(i) Sphere **(ii) Cone** **(iii) Torus (ring)**
Fig 7.5

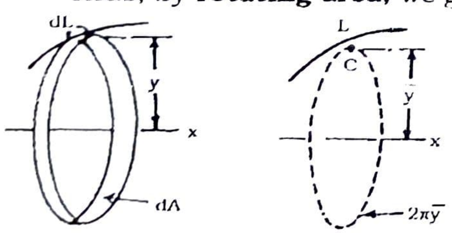
(ii) Surface of Cone : It is obtained by *rotating a* ***straight line*** AB about an axis AC.

(iii) Surface of Torus (Ring) : It can be obtained by *rotating the* ***circumference of a circle*** about a non-intersecting axis AC.

(2) Body of Revolution : It is a *body* which may be generated by ***rotating a plane area*** about a fixed axis. *A* ***solid sphere*** *may be obtained by rotating a* ***semicircular area****, a* ***cone*** *by rotating a* ***triangular*** *area and a* ***solid torus*** *by rotating a* ***full circular area.***

(i) Solid sphere **(ii) Solid cone** **(iii) Solid Torus**
Fig. 7.6

Thus, by **rotating area,** we get **volume** of body.



(a) **(b)**
Fig. 7.7

(1) THEOREM-1 : *The* ***area*** *of surface of revolution is equal to the* ***length*** *of the generating curve* ***times the distance travelled*** *by the* ***centroid*** *of the* ***curve*** *while the surface is being generated.*

$$\boxed{A = 2\pi\,\bar{y}L}$$

An element dL of line L is revolved about the x axis by 360° (i.e.2π). The area dA generated by the element dL is equal to 2π ydL.

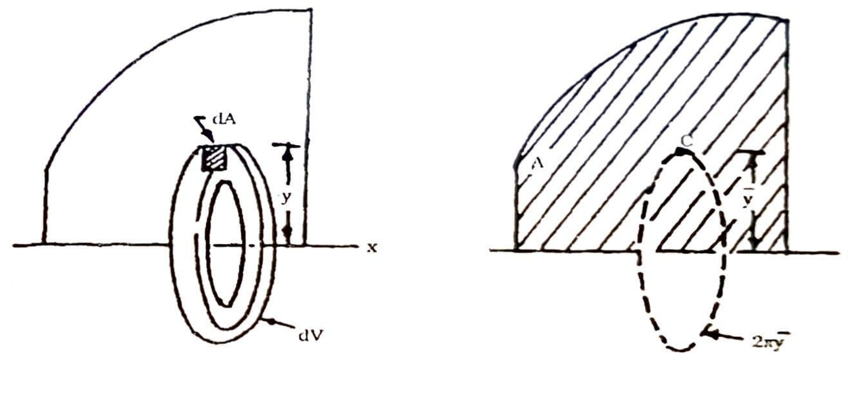
$\therefore$ Entire area generated by L $= A = \int 2\pi y\, dL$

$$\text{but } \int y dL = \bar{y}L$$

$$\therefore \boxed{A = 2\pi\,\bar{y}L}$$

(2) THEOREM-2 : *The* ***volume*** *of a body of revolution is equal to the generating* ***area times the distance travelled*** *by the* ***centroid*** *of the* ***area*** *while the body is being generated.*

$$\boxed{V = 2\pi\,\bar{y}A}$$



(a) **Fig. 7.8** **(b)**

An element dA of the area A which is revolved about the x axis.

The volume generated

by the element dA = dV= 2π y dA

$\therefore$ Entire volume generated by area A, $V = \int 2\pi\, y\, dA$

$$\text{but } \int y\, dA = \bar{y}A$$

$$\therefore \boxed{V = 2\pi\,\bar{y}A.}$$

7.6 Moment of Inertia of Areas :

It is shown in mechanics of materials that the internal forces in any section of the beam are distributed forces whose magnitudes dF = ky dA vary linearly with the distance y from an axis passing through the centroid of the section.

17

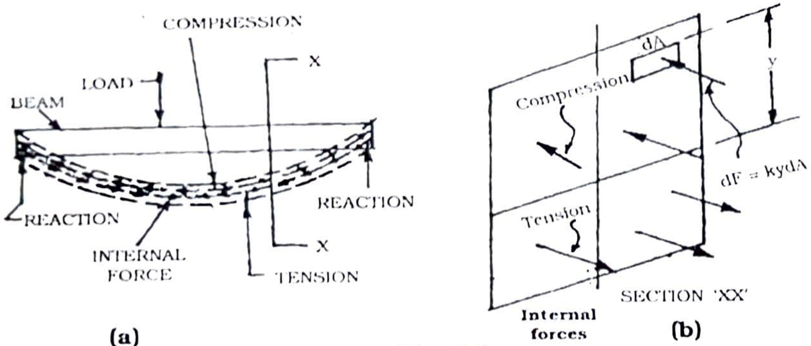


Fig 7·9

In above figure, a beam is shown having applied load at the center and reactions at ends. The deflected shape of the beam is shown with dotted lines having arrows for internal forces (compression in top and tension in bottum fibres). The cross section – XX is also shown with forces acting on it. The magnitude of elementary force dF varies linearly with the distance from the centroidal axis.

The magnitude of the resultant $\bar{R}$ of the elementary forces dF over the entire section is

$$R = \int dF = \int ky\,dA = k\int y\,dA$$

The integral $\int y\,dA$ is called ***first moment of the section*** *about x axis; it is equal to $\bar{y}A$ and is* **zero** *since the centroid of the section is located on the x axis.*

The system of forces dF thus reduces to a **couple.** *The magnitude of this couple must be equal to sum of moments $dM_x = y\,dF = ky^2 dA$. Integrating over the entire section,*

$$M = \int k\,y^2\,dA = k\int y^2\,dA.$$

The integral $\int y^2\,dA$ is called ***second moment*** *or* ***moment of inertia of area*** *with respect to the x axis and is denoted by I_x.*

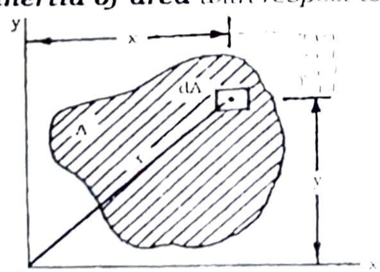


Fig. 7·10

Thus, for the entire area, the moment of inertia are

$$I_x = \int_A y^2\,dA$$

Similarly $$I_y = \int_A x^2\,dA$$

The second moment of area dA about the pole O or Z axis is called ***polar moment of inertia,*** *$dJ_o = r^2\,dA$.*

For the entire area the polar moment of inertia is

$$\boxed{J_O = \int_A r^2\,dA}$$

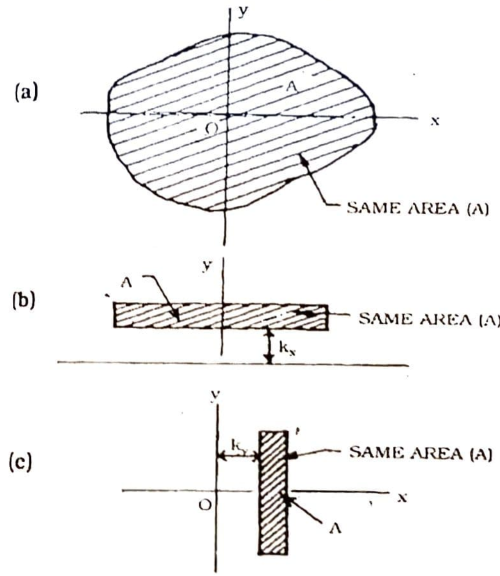
But, $r^2 = x^2 + y^2$

$$\therefore \quad \boxed{J_O = I_x + I_y}$$

Thus, polar moment of inertia (J_o) is the **sum of moment of inertia about x and y axes.**

7·7 Radius of Gyration of an Area :

Let us imagine an area to be concentrated into a thin strip parallel to x axis. If the area, A, thus concentrated, is to have the same moment of inertia with respect to the x axis, the strip should be placed at a distance k_x from the x axis, defined by the relation



$$I_x = k_x^2 A$$

and

$$\boxed{k_x = \sqrt{\frac{I_x}{A}}}$$

Similary,

$$I_y = k_y^2 A$$

and

$$\boxed{k_y = \sqrt{\frac{I_y}{A}}}$$

Fig. 7·11

The distance k_x and k_y is called as ***radius of gyration of the area*** *with respect to x and y axis respectively.*

Similarly,

$$J_o = k_o^2\,A \quad \text{and} \quad k_o = \sqrt{\frac{J_o}{A}}$$

and $$\boxed{k_o^2 = k_x^2 + k_y^2}$$

7·8 Parallel - Axis Theorem for M.I. of an Area :

If the moment of inertia for an area about an axis passing through its centroid is known, we can determine the moment of inertia of an area about a corresponding *parallel axis* using *the parallel - axis theorem.*

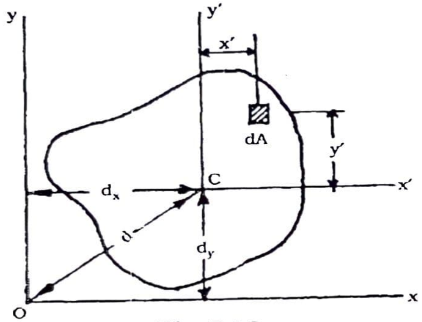


Fig 7·12

A differential element dA is located at an arbitrary distance y' from the **centroidal x' axis,** where the *fixed distance* between the parallel x and x' axes is d_y. Since moment of inertia of dA about the x axis is

$$dI_x = (y'+d_y)^2\, dA$$

Then for entire area

$$I_x = \int_A (y'+d_y)^2\, dA$$

$$= \int_A y'^2\, dA + 2\, d_y \int_A y'\, dA + d_y^2 \int_A dA$$

But, $\int y'\, dA = \bar{y} \int dA = 0$

($\therefore$ $\bar{y} = 0$. distance between centroid and centroidal axis is zero)

$$\therefore \quad \boxed{I_x = \bar{I}_{x'} + A d_y^2}$$

Similarly, $\boxed{I_y = \bar{I}_{y'} + A d_x^2}$

and, $\boxed{J_o = \bar{J}_c + A d^2}$

The theorem can be stated as, *"the moment of inertia of an area about any axis is equal to the moment of inertia of an area about the centroidal axis parallel to the axis plus the product of the area and the square of the perpendicular distance between the axes."*

7·9. Moment of Inertia of a Rectangular, Triangular and Circular Area :

(1) M.I. of Rectangular Area :

Take an arbitrary strip of area dA having depth dy.

$dA = b\, d_y$. Now $dI_x = y^2\, dA$

$$dI_x = y^2 b\, dy$$

$$\therefore I_x = \int_0^h b y^2\, d_y = \frac{1}{3} bh^3, \quad \boxed{I_x = \frac{bh^3}{3}}$$

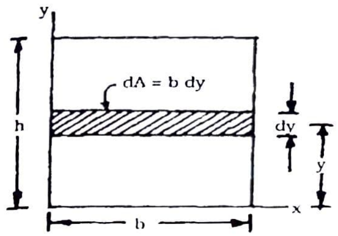


Fig. 7·13

This I_x is the moment of inertia about **base axis.**

Using parallel axis theorem,

$$I_x = \bar{I}_{x'} + A\, d_y^2$$

in which **x' is the centroidal axis** and **x is any parallel axis.**

$$\therefore \quad \bar{I}_{x'} = I_x - A d_y^2$$

$$= \frac{bh^3}{3} - bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{12}. \quad \boxed{\bar{I}_{x'} = \frac{bh^3}{12}}$$

Thus moment of inertia of Rectangular Area

about its centroidal axis ($\bar{I}_{x'}$) $= \dfrac{bh^3}{12}$

whereas about its base axis (I_x) $= \dfrac{bh^3}{3}$

Similarly, $\boxed{\bar{I}_{y'} = \frac{hb^3}{12}}$ and $\boxed{I_y = \frac{hb^3}{3}}$

$$\bar{J}_c = \bar{I}_{x'} + \bar{I}_{y'} = \frac{bh}{12}(h^2 + b^2) \quad \boxed{\bar{J}_c = \frac{bh}{12}(h^2 + b^2)}$$

(2) M.I. of Triangular Area :

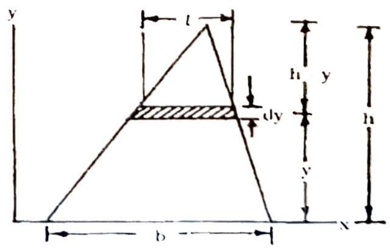


Fig. 7·14

Here, take elementary strip of depth dy at a distance y from base axis.

$$dA = l\,dy \qquad \therefore dI_x = y^2 dA$$

From similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad \therefore l = b\,\frac{h-y}{h} \quad \text{and} \quad dA = b\,\frac{h-y}{h}\,dy$$

Now

$$I_x = \int y^2 dA = \int_0^h y^2 b\,\frac{h-y}{h}\,dy = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h \qquad \therefore \boxed{I_x = \frac{bh^3}{12}}$$

and, $\bar{I}_{x'} = I_x - A\,d_y^2$

$$= \frac{bh^3}{12} - \frac{bh}{2}\left(\frac{h}{3}\right)^2 \qquad \therefore \boxed{\bar{I}_{x'} = \frac{bh^3}{36}}$$

(3) M.I. of Circular Area :

Here,
take elementary strip of width du at a distance u from centroid.

$$dA = 2\pi u\,du$$

$$\therefore dJ_o = u^2\,dA$$

$$= u^2\,(2\pi u du)$$

$$\text{and } J_o = \int dJ_o$$

$$= \int_0^r u^2\,(2\pi\,u\,du)$$

$$= 2\pi \int_0^r u^3\,du$$

$$\therefore \boxed{J_o = \frac{\pi}{2}\,r^4}$$

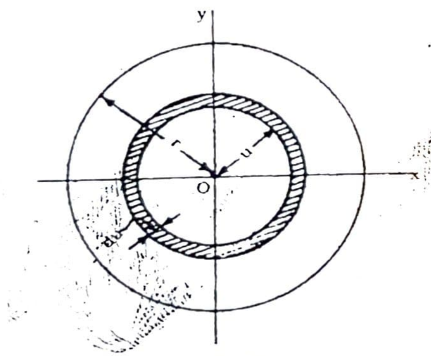


Fig 7·15

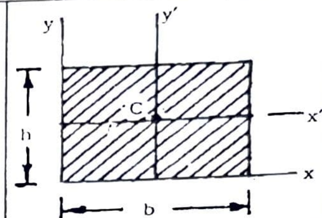
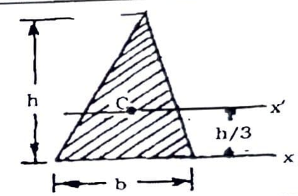
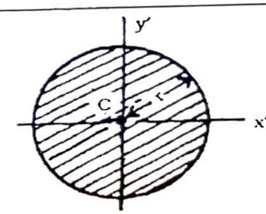
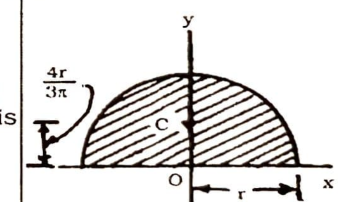
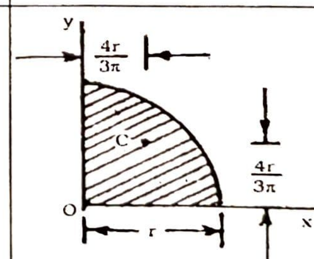
Now, $J_o = \bar{I}_{x'} + \bar{I}_{y'} = 2\,\bar{I}_{x'}$

$$\therefore \frac{\pi}{2}\,r^4 = 2\,\bar{I}_{x'} \qquad \therefore \boxed{\bar{I}_{x'} = \bar{I}_{y'} = \frac{\pi r^4}{4}}$$

7·10. Moments of Inertia for Composite Areas :

A composite area consists of a series of connected "simpler" area shapes, such as **semicircles, rectangles, triangles** etc. If the moment of inertia of each of these shapes is known or can be computed about a common axis, then the moment of inertia of the composite area equals the *algebraic sum* of the moments of inertia of all its composite parts.

Table 7·4. Moment of Inertia of common shapes :

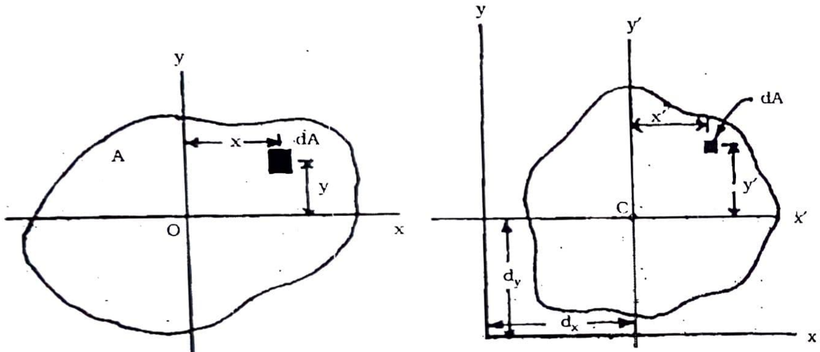
Rectangle x, y – base axis x′, y′ – centroidal axis		$\bar{I}_{x'} = \frac{bh^3}{12}$ $\quad I_x = \frac{bh^3}{3}$ $\bar{J}_c = \frac{bh}{12}(b^2 + h^2)$ $\bar{I}_{y'} = \frac{hb^3}{12}$ $\quad I_y = \frac{hb^3}{3}$
Triangle x – base axis x′ – centroidal axis		$\bar{I}_{x'} = \frac{bh^3}{36}$ $\quad I_x = \frac{bh^3}{12}$ $\frac{bh^3}{36}$
Circle x′, y′ – centroidal axis		$\bar{I}_{x'} = \frac{\pi r^4}{4}$ $\bar{J}_c = \frac{\pi r^4}{2}$ $\bar{I}_{y'} = \frac{\pi r^4}{4}$
Semicircle x – base axis y – centroidal axis		$I_x = \frac{\pi r^4}{8}$ *(about x axis)* $J_o = \frac{\pi r^4}{4}$ $I_y = \frac{\pi r^4}{8}$ *(about y axis)*
Quarter Circle x, y – base axis		$I_x = \frac{\pi r^4}{16}$ *(about x axis)* $J_o = \frac{\pi r^4}{8}$ $I_y = \frac{\pi r^4}{16}$ *(about y axis)*

7·11 Product of Inertia for an Area :

For the entire area, the *product of inertia* is

$$\boxed{I_{xy} = \int_A x\,y\,dA}$$

which is obtained by multiplying each element dA of an area A by its coordinates x and y and integrating over the area.



(i) (ii)

Fig. 7·16

Parallel- axis theorem for product of inertia has important application in determining the **product of inertia** of a ***composite area*** with respect to a set of x, y axes.

$$I_{xy} = \int (x' + d_x)(y' + d_y)\,dA$$

$$= \int_A x'\,y'\,dA + d_x \int_A y'\,dA + d_y \int_A x'\,dA + d_x\,d_y \int_A dA$$

Here $\int y'\,dA$ and $\int x'\,dA$ is zero

$$\therefore \quad \boxed{I_{xy} = \bar{I}_{x'y'} + A\,d_x d_y}$$

7×12. Moments of Inertia for an Area about Inclined Axes :

In structural mechanics, it is sometimes necessary to calculate the *moment and product of inertia I_u, I_v and I_{uv} or an area with respect to a set of inclined u and v axes when the values of θ, I_x, I_y and I_{xy} are known.*

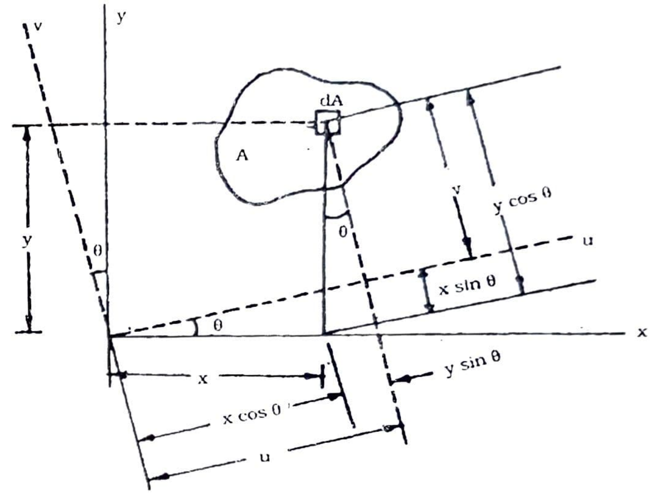


Fig. 7·17

In the above figure, the u and v axes are the inclined axes. Here, coordinates u and v of centroid of area dA are

$$u = x\cos\theta + y\sin\theta$$
$$v = y\cos\theta - x\sin\theta$$

and Moment of Inertia of dA about u and v axes are

$$dI_u = v^2 dA = (y\cos\theta - x\sin\theta)^2\,dA$$
$$dI_v = u^2 dA = (x\cos\theta + y\sin\theta)^2\,dA$$
$$\therefore \quad dI_{uv} = uv\,dA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA$$

Expanding & integrating, realizing that

$I_x = \int y^2 dA$

$I_y = \int x^2 dA$ and $I_{xy} = \int xy\,dA$,

we obtain

$$I_u = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$
$$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$
$$I_{uv} = I_x\sin\theta\cos\theta - I_y\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$

putting $\sin 2\theta = 2\sin\theta\cos\theta$

and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

We get moment of inertia about inclined axes u and v as :

$$\boxed{\begin{aligned} I_u &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \\ I_v &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \\ I_{uv} &= \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \end{aligned}}$$

and $J_O = I_u + I_v = I_x + I_y$

7·13 Principal Moment of Inertia :

It can be seen that I_u, I_v and I_{uv} depend upon the angle of inclination θ of the u, v axes. The orientation of u, v axes can be kept in such a way that about them, I_u and I_v are **maximum** and **minimum.**

The particular set of axes, about which I_u and I_v are maximum and minimum whereas $I_{uv} = 0$, is called ***principal axes*** *of the area* and moment of intertia with respect to these axes (I_u (max) and I_v (min) are called ***principal moment of inertia.*** This is useful in *structural mechanics.*

The angle $\theta = \theta_m$, which defines the orientation of the principal axes for the area, may be found by differentiating the equation of I_u in section 7·12 with respect to θ and setting to zero. Thus.

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right)\sin 2\theta - 2\,I_{xy}\cos 2\theta = 0$$

Therefore, at $\theta = \theta_m$,

$$\boxed{\tan 2\theta_m = \frac{-2I_{xy}}{I_x - I_y}}$$

This equation has two roots, θ_{m1} and θ_{m2}, which specify the **inclination** *of the* ***principal axes.***

Because of the nature of the tangent, the values $2\theta_{m1}$ and $2\theta_{m2}$ are 180° apart, so that θ_{m1} and θ_{m2} are 90° apart. *Thus two axes, perpendicular to each other are principal axes of the area about 0.*

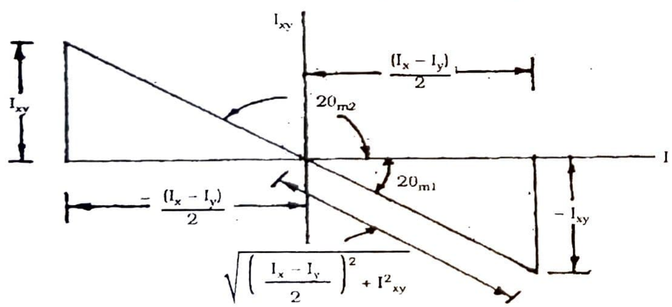


Fig 7·18

For θ_{m1} :

$$\sin 2\theta_{m1} = -I_{xy} / \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$$

$$\cos 2\theta_{m1} = \left(\frac{I_x - I_y}{2}\right) / \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$$

For θ_{m2} :

$$\sin 2\theta_{m2} = I_{xy} / \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$$

$$\cos 2\theta_{m2} = -\left(\frac{I_x - I_y}{2}\right) / \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$$

Substituting in I_u and I_v equations of section 7·12

$$\boxed{I_{\substack{max\\min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}}$$

I_{max} and I_{min} are the principal moment of inertia.

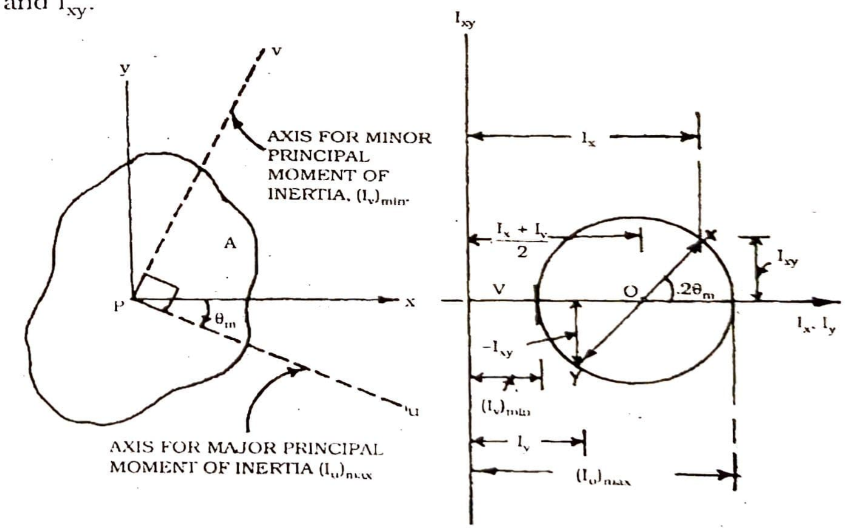
7·14 Mohr's Circle for Moment of Inertia :

*The above equations for I_u, I_v, I_{max} and I_{min} have a graphical solution that is convenient to use and **generally easy to remember.*** It is named after German engineer Otto Mohr (1835 - 1918).

The main purpose in using the circle is to have a convenient means for transforming I_x, I_y and I_{xy} in to principal moments of inertia, $(I_u)_{max}$ and $(I_v)_{min}$.

<u>Steps for Mohr's circle method :</u>

(1) Compute I_x, I_y, I_{xy} : Establish the x, y axes for an area, with the origin located at the point P of interest, and determine I_x, I_y and I_{xy}.



(a) **Fig 7·19** (b)

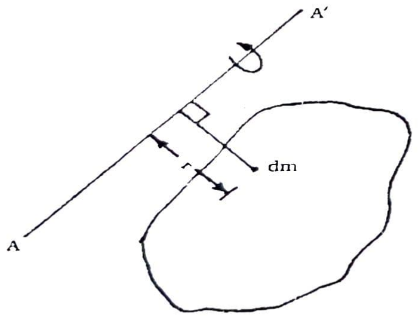
(2) Construct the Circle : Take I_x, I_y and I_{xy} [Fig 7·19 (b)] on rectangular coordinate system. The centre of circle will be located at $(I_x + I_y)/2$. Plot the "controlling point" x having coordinates (I_x, I_{xy}).

(3) Principal Moment of Inertia : The points where the circle intersects the abscissa give the values of $(I_u)_{max}$ and $(I_v)_{min}$. Notice that I_{uv} will be zero at these points.

(4) Principal axes : Measure angle $2\theta_m$ from radius OX to the positive I axis. Both the angle on the circle $(2\theta_m)$ and the angle on the area (θ_m) *must be measured in the same sense. (Fig 7·19 (b) and (a)). The axis for* $(I_v)_{min}$ *is perpendicular to the axis for* $(I_u)_{max}$

7·15 Mass Moment of Inertia :

The mass moment of inertia of a body is a *property* that measures the ***resistance of the body to angular acceleration.*** *It is used to study rotational motion.* It can be defined as the *integral of the "second moment" about an axis of all the differential- size elements of mass* ***dm*** *which compose the body.*



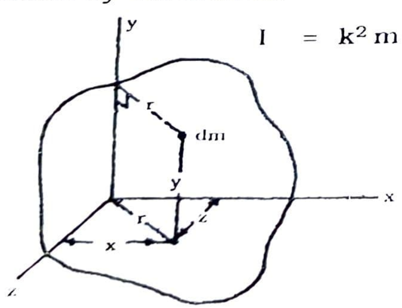
So,

$$\boxed{I = \int_m r^2\, dm}$$

Fig 7·20

The radius of gyration k of the body with respect to axis AA' is defined by the relation

$$I = k^2 m \qquad \text{or} \qquad k = \sqrt{\frac{I}{m}}$$



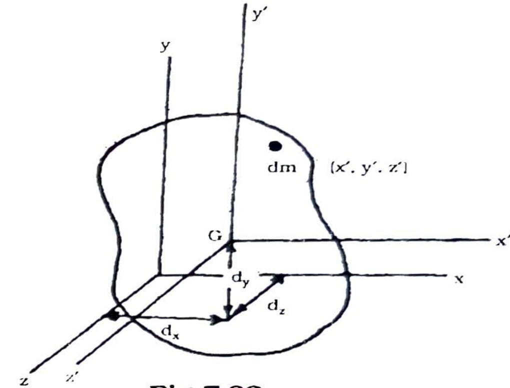
Here, $I_y = \int r^2 dm = \int (z^2 + x^2)\, dm$

Similarly,

$$\boxed{\begin{aligned} I_x &= \int (y^2 + z^2)\, dm \\ I_y &= \int (z^2 + x^2)\, dm \\ I_z &= \int (x^2 + y^2)\, dm \end{aligned}}$$

Fig 7·21

7·16. Parallel Axis Theorem for Mass Moment of Inertia :



Here
$x = x' + d_x$
$y = y' + d_y$
$z = z' + d_z$

Fig 7·22

If the moment of inertia of the body about an axis passing through the body's *mass center* is known, then the body's moment of inertia may be determined about any other ***parallel axis*** by using the parallel - axis theorem.

$$I_x = \int (y^2 + z^2)\, dm + \int [(y' + d_y)^2 + (z' + d_z)^2]\, dm$$

$$= \int (y'^2 + z'^2)\, dm + 2d_y \int y' dm + 2d_z \int z' dm + (d_y^2 + d_z^2) \int dm.$$

But $\int y' dm$ and $\int z'$ dm is zero

$\therefore$

Similarly,

$$\boxed{\begin{aligned} I_x &= \bar{I}_x' + m\,(d_y^2 + d_z^2) \\ I_y &= \bar{I}_y' + m\,(d_z^2 + d_x^2) \\ I_z &= \bar{I}_z' + m\,(d_x^2 + d_y^2) \end{aligned}}$$

and

$$\boxed{I = \bar{I} + md^2}$$

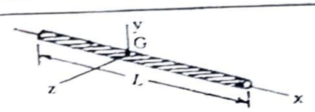
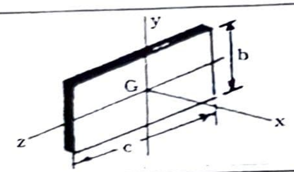
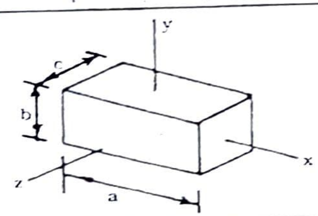
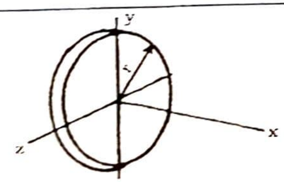
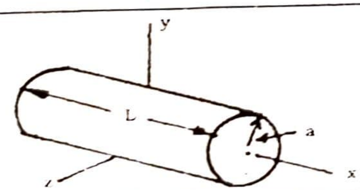
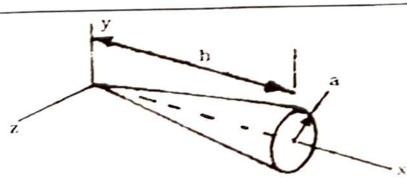
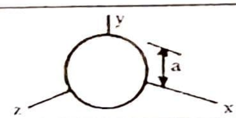
$$\boxed{k^2 = \bar{k}^2 + d^2}$$

7·17 Mass Moment of Inertia of Composite Bodies :

If a body is constructed from a number of simple shapes such as **disks, spheres, rods** etc. the mass moment of inertia of the body about any axis z can be determined by *adding algebraically* the moments of inertia of all the composite shapes computed about the z axis. The mass moment of inertia of a few common shapes are shown in Table 7·5.

ject

Table 7·5 Mass Moment of Inertia of Common Shapes

Slender Rod $I_y = I_z = \dfrac{mL^2}{12}$	
Thin Rectangular Plate $I_x = \dfrac{m}{12}(b^2 + c^2)$ $I_y = \dfrac{m}{12} mc^2, \quad I_z = \dfrac{mb^2}{12}$	
Rectangular Prism $I_x = \dfrac{m}{12}(b^2 + c^2)$ $I_y = \dfrac{m}{12}(c^2 + a^2)$ $I_z = \dfrac{m}{12}(a^2 + b^2)$	
Thin Disk $I_x = \dfrac{mr^2}{2}$ $I_y = I_z = \dfrac{mr^2}{4}$	
Circular Cylinder $I_x = \dfrac{ma^2}{2}$ $I_y = I_z = \dfrac{m}{12}(3a^2 + L^2)$	
Circular Cone $I_x = \dfrac{3}{10} ma^2$ (Here, y and z axes are **not** centroidal axes $I_y = I_z = \dfrac{3}{5} m \left(\dfrac{1}{4} a^2 + h^2\right)$	
Sphere $I_x = I_y = I_z = \dfrac{2ma^2}{5}$	

IMPORTANT EQUATIONS

(1) Center of Gravity : $\overline{X} = \dfrac{\int \overline{x} dw}{\int dw} \quad \overline{Y} = \dfrac{\int \overline{y} dw}{\int dw}$

(2) Centroid of Line : $\overline{X} = \dfrac{\int \overline{x} dL}{\int dL}$; $\overline{Y} = \dfrac{\int \overline{y} dL}{\int dL}$

(3) Centroid of Area : $\overline{X} = \dfrac{\int \overline{x} dA}{\int dA}$; $\overline{Y} = \dfrac{\int \overline{y} dA}{\int dA}$

(4) Centroid of volume : $\overline{X} = \dfrac{\int \overline{x} dv}{\int dv}$, $\overline{Y} = \dfrac{\int \overline{y} dv}{\int dv}$, $\overline{Z} = \dfrac{\int \overline{z} dv}{\int dv}$

(5) First moments : $\int \overline{x} dA, \int \overline{x} dL$ are called first moments.

(6) Pappus - Guldinus Theorems (I) $A = 2\pi \overline{y} L$
(II) $V = 2\pi \overline{y} A$

(7) Distributed Forces : $\Delta F = ky \Delta A$

(8) Moment of Inertia (second moment) : $I_x = \int y^2 dA$
$I_y = \int x^2 dA$

(9) Polar M.I. : $J_o = \int r^2 dA \qquad J_o = I_x + I_y$

(10) Radius of Gyration : $k_x = \sqrt{\dfrac{I_x}{A}}$, $k_y = \sqrt{\dfrac{I_y}{A}}$, $k_o = \sqrt{\dfrac{J_o}{A}}$

(11) Parallel - Axis Theorem : $I_x = \overline{I}_{x'} + Ad_y^2$ and $I_y = \overline{I}_{y'} + Ad_x^2$
$J_o = \overline{J}_c + Ad^2$

(12) Product of Inertia : $I_{xy} = \int xy dA$
$I_{xy} = \overline{I}_{x'y'} + d_x d_y A$

(13) M.I. about inclined axes (Rotation of axes)

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{uv} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

$$J_o = I_u + I_v = I_x + I_y$$

(14) Principal Axes : $\tan 2\theta_m = -\dfrac{2\,I_{xy}}{I_x - I_y}$

(15) Principal moment of Inertia : $I_{max,\,min} = \dfrac{I_x + I_y}{2} \pm \sqrt{\left(\dfrac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$

(16) Mass moment of Inertia : $I = \int r^2\,dm$, $I_x = \int (y^2 + z^2)\,dm$

$I_y = \int (x^2 + z^2)\,dm$

$I_z = \int (x^2 + y^2)\,dm$

(17) Radius of Gyration : $k = \sqrt{I/m}$

(18) Parallel axis theorem : $I_x = \overline{I}_{x'} + m\,(d_y^2 + d_z^2)$

$I_y = \overline{I}_{y'} + m\,(d_x^2 + d_z^2)$

$I_z = \overline{I}_{z'} + m\,(d_x^2 + d_y^2)$

$I = \overline{I} + md^2$

$k^2 = \overline{k}^2 + d^2$

SOLVED EXAMPLES

1. Locate the centroid of the **thin rod** bent in to the shape of a parabolic arc shown below.

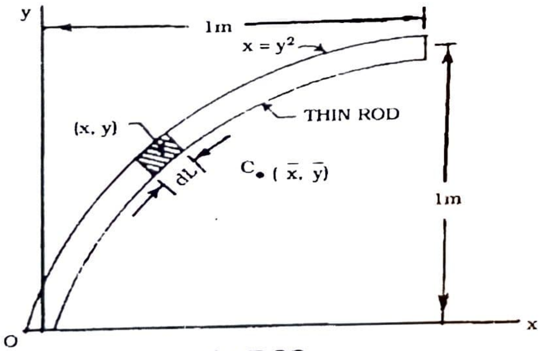


Fig 7·23

Thin rod is considered as **line** element. Take small sagment dL.

Length of the differential element

$$= \quad dL = \sqrt{(dx)^2 + (dy)^2} = \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

Since $\quad x = y^2$

$$\frac{dx}{dy} = 2y$$

$$\therefore\ dL = \sqrt{(2y)^2 + 1}\ dy = \sqrt{4y^2 + 1}\ dy$$

$$\text{Now}\ \overline{X} = \frac{\int_L \overline{x}\,dL}{\int_L dL} = \frac{\int_0^1 x\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy} = \frac{\int_0^1 y^2\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy}$$

$$= \frac{0{\cdot}739}{1{\cdot}479} = 0{\cdot}500$$

$$\boxed{\overline{X} = 0{\cdot}5\ \text{m}}$$

$$\text{Now}\ \overline{Y} = \frac{\int_L \overline{y}\,dL}{\int_L dL} = \frac{\int_0^1 y\sqrt{4y^2+1}\,dy}{\int_0^1 \sqrt{4y^2+1}\,dy} = \frac{0{\cdot}848}{1{\cdot}479} = 0{\cdot}573\ \text{m}$$

$$\boxed{\overline{Y} = 0{\cdot}573\ \text{m}}$$

2. Locate the centroid of the wire shown below.

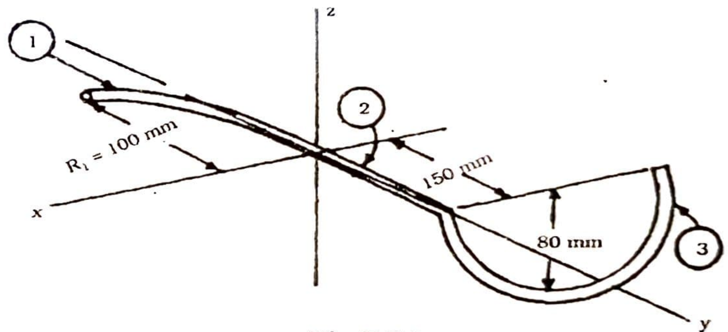


Fig 7·24

The wire can be divided into **three** segments.

Here, **segment 1** is quarter - circular arc in **x - y plane;** whereas **segment 3** is semi-circular arc **in x – z** plane.

Segment	L (mm)	$\overline{x}$ (mm)	$\overline{y}$ (mm)	$\overline{z}$ (mm)	$\overline{x}L$ (mm²)	$\overline{y}L$ (mm²)	$\overline{z}L$ (mm²)
(1) Quarter-circular arc	$\frac{\pi r}{2} = \frac{\pi(100)}{2}$ = 157·1	$100 - \frac{2r}{\pi}$ = 36·34	$-2r/\pi$ = – 63·66	0	5709	– 10,001	0
(2) straight rod	150	0	75	0	0	11,250	0
(3) Semicircular arc	πr = 251·33	- 80	150	$-\frac{2r}{\pi}$ = - 50·93	- 20,106·4	+ 37699·5	- 12800·2
	ΣL = 558·43				$\Sigma\,\overline{x}L$ = - 14,397·4	$\Sigma\,\overline{y}L$ = 38,948·5	$\Sigma\,\overline{z}L$ = - 12, 800·2

18

$$\therefore \quad \bar{X} = \frac{\Sigma\, xL}{\Sigma L} = \frac{-14,397.4}{558.43} = -25.78$$

$$\bar{Y} = \frac{\Sigma\, yL}{\Sigma L} = \frac{38,948.5}{558.43} = +69.75$$

$$\bar{Z} = \frac{\Sigma\, zL}{\Sigma L} = \frac{-12,800.2}{558.43} = -22.92$$

$$\boxed{\begin{aligned} \bar{X} &= -25.78 \text{ mm} \\ \bar{Y} &= 69.75 \text{ mm} \\ \bar{Z} &= -22.92 \text{ mm} \end{aligned}}$$

3. Locate the centroid of the shaded area bounded by the line $y = x$ and curve $y = x^2$ shown below.

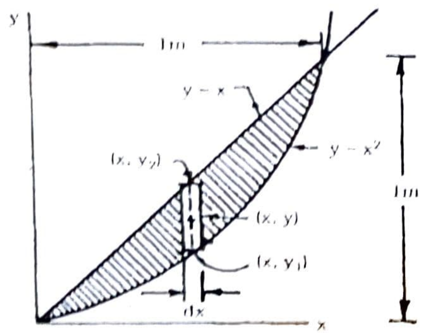


Fig 7.25

The area of differential element is $dA = (y_2 - y_1)\,dx$, and its centroid is (x, y)

$$\bar{X} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^1 x\,(y_2 - y_1)\,dx}{\int_0^1 (y_2 - y_1)\,dx} = \frac{\int_0^1 x(x - x^2)\,dx}{\int_0^1 (x - x^2)\,dx} = 0.5 \text{ m}$$

$$\bar{Y} = \frac{\int_A y\,dA}{\int_A dA} = \frac{\int_0^1 y\,(y_2 - y_1)\,dx}{\int_0^1 (y_2 - y_1)\,dx}$$

$$\text{Here,} \quad y = y_1 + \frac{y_2 - y_1}{2} = \frac{y_1 + y_2}{2} = \frac{x^2 + x}{2}$$

$$\therefore \quad \bar{Y} = \frac{\int_0^1 \frac{x^2 + x}{2}(x - x^2)\,dx}{\int_0^1 (x - x^2)\,dx} = \frac{(2/15)\,\frac{1}{2}}{1/6} = 0.4 \text{ m}$$

$$\boxed{\begin{aligned} \bar{X} &= 0.5 \text{ m} \\ \bar{Y} &= 0.4 \text{ m} \end{aligned}}$$

4. Locate the centroid of the plate area shown below :

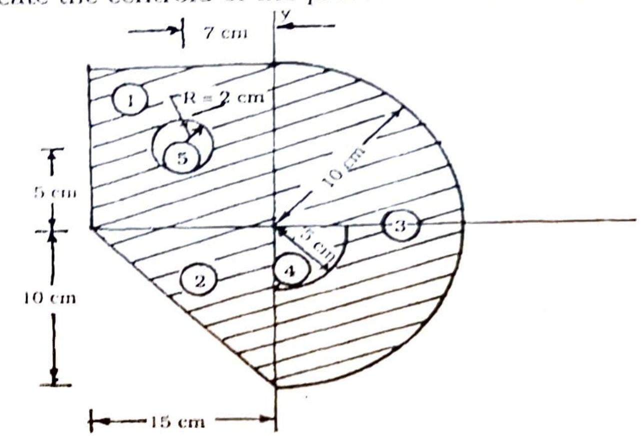


Fig 7·26

The complete area can be divided in to *five* segments.

Segment	Area (cm²)	$\bar{x}$ (cm)	$\bar{y}$ (cm)	$\bar{x}A$ (cm³)	$\bar{y}A$ (cm³)
(1) (+) Rectangular	$15 \times 10 = +150$	−7.5	+5.0	−1125.0	+750.0
(2) (+) Trianguler	$\frac{1}{2} \times 15 \times 10 = +75$	−5.0	−3.33	−375.0	−249.75
(3) (+) Semicircular	$\frac{\pi r^2}{2}$ + = 157.08	$4r/3\pi = +4.24$	0	+666.02	0
(4) (−) Quarter circular	$\frac{\pi r^2}{4} = -19.63$	$4r/3\pi = +2.12$	$-4r/3\pi = -2.12$	−41.62	+41.62
(5) (−) circular	$\pi r^2 = -12.57$	−7.0	+5.0	+87.99	−62.85
	$\Sigma A = 349.88$			$\Sigma\,\bar{x}A = -787.61$,	$\Sigma\,\bar{y}A = 479.02$

$$\therefore \quad \bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{-787.61}{349.88} = -2.25 \text{ cm} \qquad \boxed{\bar{X} = -2.25 \text{ cm}}$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{479.02}{349.88} = +1.37 \text{ cm} \qquad \boxed{\bar{Y} = +1.37 \text{ cm}}$$

5. Locate the centroid of the composite volume shown below.

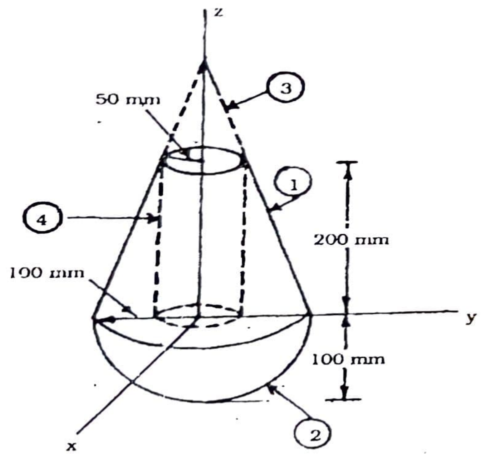


Fig 7·27

The composite can be divided in to *four* segments. (3) and (4) will be **negative** volumes.

Because of the symmetry, note that

$$\bar{X} = \bar{Y} = 0.$$

Segment	Volume (v) (cm)^3	$\bar{z}$ (cm)	$\bar{z}V$ (cm)^4
(1) (+) Cone	$\frac{1}{3}\pi r^2 h = +4188.8$ (h = 40 cm)	$\frac{h}{4} = +10.0$	+ 41888.0
(2) (+) Hemisphere	$\frac{2}{3}\pi r^3 = +2094.4$	$-\frac{3r}{8} = -3.75$	− 7854.0
(3) (−) Cone	$\frac{1}{3}\pi r^2 h = -523.6$ (h = 20 cm)	$(20 + \frac{h}{4}) = +25$	− 13090.0
(4) (−) Cylinder	$\pi r^2 h = -1570.8$ (h = 20 cm)	$\frac{h}{2} = +10.0$	− 15708.0
	Σv = 4188.8		**$\Sigma \bar{z}v$ = + 5236**

$$\therefore \quad \bar{z} = \frac{\Sigma \bar{z}v}{\Sigma v} = \frac{+5236}{4188.8} = +1.25 \text{ cm}$$

$$\boxed{\bar{Z} = 12.5 \text{ mm}}$$

6. **(a)** Determine the surface area of revolution of the length revolved about **x axis** as shown below.

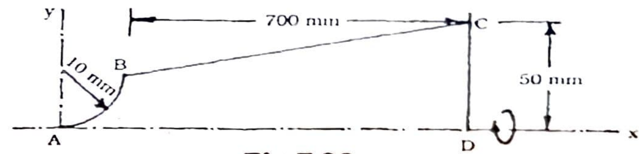


Fig 7·28

Segment	Length (L) (cm)	$\bar{y}$ (cm)	$2\pi\bar{y}L$ (cm^2)
AB	$\frac{\pi r}{2} = 1.57$	$1 - \frac{2r}{\pi} = +0.36$	3.55
BC	$\sqrt{70^2+4^2} = 70.11$	+ 3.0	1321.54
CD	5.0	+ 2.5	78.54

$$A = \Sigma 2\pi\bar{y}L = 1403.63$$

$$\boxed{\text{Surface Area } (A) = 1403.63 \text{ cm}^2}$$

(b) Determine the volume of revolution of the shaded area revolved about **y axis** as shown below.

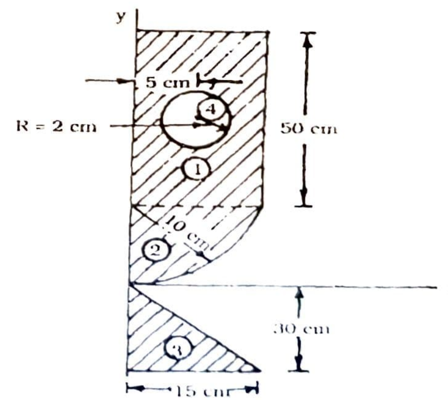


Fig 7·29

Segment	Area (cm²)	$\bar{x}$ (cm)	$2\pi \bar{x} A$ (cm³)
(1) (+) Rectangular	$50\times10 = +500$	$+5.0$	$+15707.96$
(2) (+) Quarter circular	$\frac{\pi r^2}{4} = \frac{\pi\times10^2}{4} = +78.5$	$\frac{4r}{3\pi} = +4.24$	$+2091.3$
(3) (+) Trigular	$\frac{bh}{2} = +225$	$\frac{+15}{3} = +5.0$	$+7068.58$
(4) (–) Circular	$-\pi r^2 = -\pi\times2^2 = -12.57$	$+5.0$	-394.9

$V = \Sigma 2\pi \bar{x}A = 25{,}262.74$ cm³

$$\boxed{\text{Volume of Revolution (v)} = 25{,}262.74 \text{ cm}^3}$$

7. Determine the moment of inertia of the shaded area shown below.

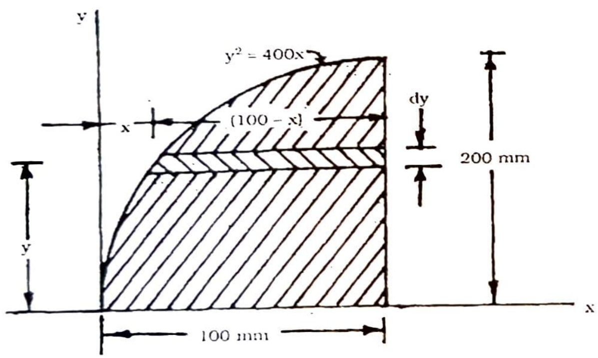


Fig 7·30

Area of elementary strip = $(100 - x)\,dy$

This strip is to be integrated along y axis from 0 to 200.

$$I_x = \int_A y^2 dA = \int_A y^2 (100-x)\,dy$$

$$= \int_0^{200} y^2\left(100 - \frac{y^2}{400}\right) dy = 100\int_0^{200} y^2\,dy - \frac{1}{400}\int_0^{200} y^4\,dy$$

$$= 107\times10^6 \text{ mm}^4.$$

$$I_y = \int_A x^2 dA$$

$$= \iint_A \left\{x + \left(\frac{100-x}{2}\right)\right\}^2 (100-x)\,dy \qquad \text{Now } x = \frac{y^2}{400}$$

$$= \int_0^{200} \left\{\frac{y^2}{400} + \left(\frac{100 - \frac{y^2}{400}}{2}\right)\right\}^2 \left(100 - \frac{y^2}{400}\right) dy$$

$$= 325.7 \times 10^6 \text{ mm}^4$$

$$\boxed{\begin{aligned} I_x &= 107 \times 10^6 \text{ mm}^4 \\ I_y &= 325.7 \times 10^6 \text{ mm}^4 \end{aligned}}$$

Note : It is better to take differential element strip parallel to y axis and integrating from 0 to 100 for finding out I_y.

8. Determine the moment of inertia of the shaded area with respect to the given X and Y axis.

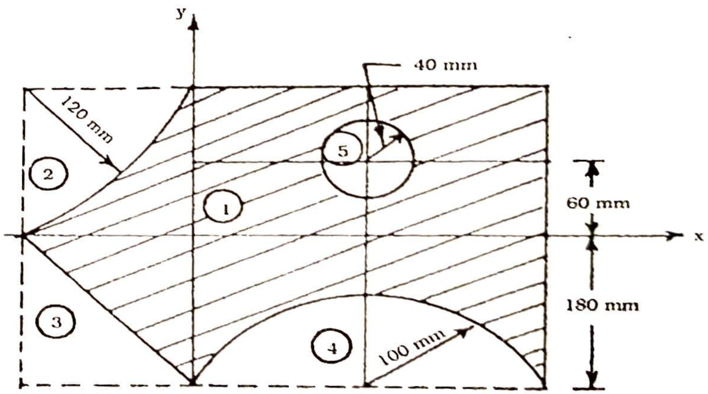


Fig 7·31

The above figure can be divided in to *five* segments- (1) is the complete rectangle minus (2), (3), (4) and (5). **For quarter circular area and semicircular area, we have equations of M.I. with respect to the base hence M.I. about individual centroidal axis will be determined by $I_{x'} = I_x - A d_y^2$, where I_x is about base.**

Segment	$\bar{I}_{x'}$ (cm⁴) @ centroidal axis	A (cm²)	d_y (cm)	$I_x = \bar{I}_{x'} + Ad_y^2$ (cm⁴)
(1) (+) Complete Rectangle	$\frac{bh^3}{12} = \frac{32 \times 30^3}{12}$ = 72000	bh=32×30 = 960	3·0	+ 80640·00
(2) (–) Quarter circular	$\frac{\pi r^4}{16} - A\left(\frac{4r}{3\pi}\right)^2$ = 1138	$\pi r^2/4$ = 113·1	$\left(12 - \frac{4r}{3\pi}\right)$ = 6·91	– 6538·31
(3) (–) Triangle	$\frac{1}{36} \times 12 \times 18^3$ = 1944	$\frac{1}{2} \times 12 \times 18$ = 108	$\frac{2}{3} \times 18$ = 12	– 17496·00
(4) (–) Semi circular	$\frac{\pi r^4}{8} - A\left(\frac{4r}{3\pi}\right)^2$ = 1097·57	$\pi r^2/2$ = 157·08	$18 - \frac{4r}{3\pi}$ = 13·76	– 30838·72
(5) (–) Circular	$\frac{\pi r^4}{4} = \frac{\pi \times 4^4}{4}$ = 201·06	πr^2 = 50·27	6·0	– 2010·78

Note : Radius of Gyration can be found by $k_x = \sqrt{\Sigma I_x / \Sigma A}$ = 6·69 cm. ∴ ΣI_x = 23756·19 cm⁴

Segment	$\bar{I}_{y'}$ (cm⁴) @ centroidal axis	A (cm²)	d_x (cm)	$I_y = \bar{I}_{y'} + Ad_x^2$ (cm⁴)
(1) (+) Complete Rectangle	$\frac{hb^3}{12} = \frac{30 \times 32^3}{12}$ = 81920	bh = 960	4·0	+ 97280·00
(2) (–) Quarter circular	$\frac{\pi r^4}{16} - A\left(\frac{4r}{3\pi}\right)^2$ = 1138	$\frac{\pi r^2}{4}$ = 113·1	6·91	6538·31
(3) (–) Triangle	$\frac{18 \times 12^3}{36}$ = 864	$\frac{1}{2} \times 18 \times 12$ = 108	$\frac{2}{3} \times 12$ = 8·0	– 7776·00
(4) (–) Semi circular	$\frac{\pi r^4}{8}$ = 3927	$\pi r^2/2$ = 157·08	10·0	– 19635·00
(5) (–) Circular	$\frac{\pi r^4}{4}$ = 201·06	πr^2 = 50·27	10·0	– 5228·06

$k_y = \sqrt{\Sigma I_y / \Sigma A}$ = 10·46 cm. ΣI_y = 58102·63 cm⁴

9. Determine the moment of inertia of the shaded area shown in **example 4 about the centroidal axes.**

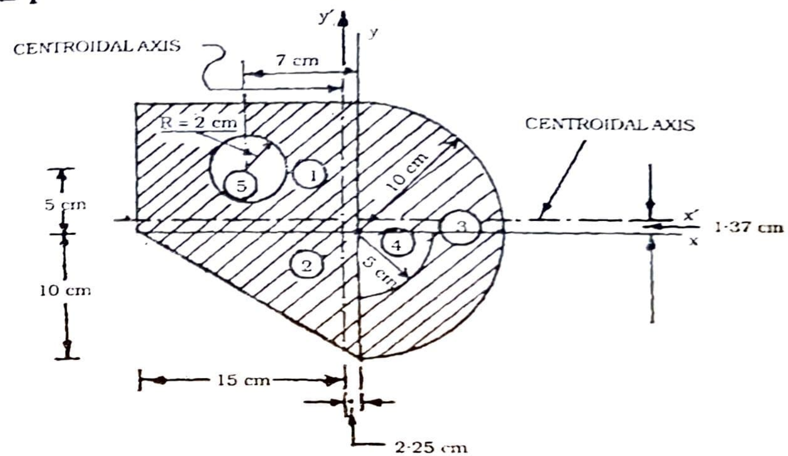


Fig. 7·32

The $\bar{x}$ and $\bar{y}$ (centroid of the **complete** shaded area) was found in example **4** as $\bar{x}$ = – 2·25 cm and $\bar{y}$ = 1·37 cm.

These centroidal axes are **not** the centroidal axes of **individual** segments.

Segment	$\bar{I}_{x'}$ (individual) (cm⁴)	A (cm²)	d_y (cm)	$I_{x'} = I_{centroidal}$ $\bar{I}_{x'} + Ad_y^2$ (cm⁴)
(1) (+) Rectangle	$\frac{bh^3}{12} = \frac{15 \times 10^3}{12}$ = 1250	150	5-1·37 = 3·63	+ 3226·54
(2) (+) Triangle	$\frac{bh^3}{36} = \frac{15 \times 10^3}{36}$ = 416·67	75	$\frac{10}{3}$ + 1·37 = 4·7	+ 2073·42
(3) (+) Semicircular	$\frac{\pi r^4}{8}$ = 3927	157·08	1·37	+ 4221·82
(4) (–) Quarter circular	$\frac{\pi r^4}{16} - A\left(\frac{4r}{3\pi}\right)^2$ = 34·32	19·63	$1·37 + \frac{4r}{3\pi}$ = 3·49	– 273·42
(5) (–) Circular	$\frac{\pi r^4}{4}$ = 12·57	12·57	5–1·37 = 3·63	– 178·20

$\Sigma I_{x'}$ = 9070·16 cm⁴

Segment	$I_{y'}$ (Individual) (cm^4)	A (cm^2)	d_x (cm)	$I_y = \bar{I}_{y'} + Ad_x^2$ (cm^4)
(1) (+) Rectangle	$\frac{hb^3}{12} = \frac{10 \times 15^3}{12} = 2812.5$	150	$7.5 - 2.25 = 5.25$	+ 6946.88
(2) (+) Triangle	$\frac{hb^3}{36} = \frac{10 \times 15^3}{36} = 937.5$	75	$5 - 2.25 = 2.75$	+ 1504.69
(3) (+) Semicircular	$\frac{\pi r^4}{8} - A\left(\frac{4r}{3\pi}\right)^2 = 1097.56$	157.08	$\frac{4r}{3\pi} + 2.25 = 6.49$	+ 7713.79
(4) (–) Quarter circular	$\frac{\pi r^4}{16} - A\left(\frac{4r}{3\pi}\right)^2 = 34.32$	19.63	$\frac{4r}{3\pi} + 2.25 = 4.37$	– 409.19
(5) (–) Circular	$\frac{\pi r^4}{4} = 12.57$	12.57	$7 - 2.25 = 4.75$	– 296.18

$$\Sigma I_{y'} = 15459.99 \text{ cm}^4$$

10. Determine **(a)** the principal axes of the section shown below about C, **(b)** the principal moments of inertia about C.

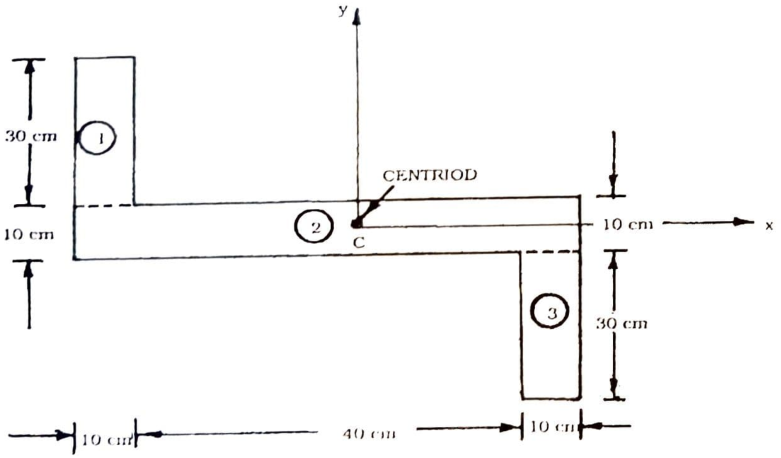


Fig 7·33

Segment	$\bar{I}_{x'}$	$\bar{I}_{y'}$	A	d_y	d_x	$I_x = \bar{I}_{x'} + Ad_y^2$	$I_y = \bar{I}_{y'} + Ad_x^2$	$I_{xy} = Ad_x d_y$
(1)	$\frac{10(30)^3}{12} = 22500$	$\frac{30(10)^3}{12} = 2500$	$30 \times 10 = 300$	20	25	142500	190000	150000
(2)	$\frac{60(10)^3}{12} = 5000$	$\frac{10(60)^3}{12} = 180000$	$60 \times 10 = 600$	0	0	5000	180000	0
(3)	$\frac{10(30)^3}{12} = 22500$	$\frac{30(10)^3}{12} = 2500$	$30 \times 10 = 300$	20	25	142500	190000	150000
						$\Sigma I_x = 290000$	$\Sigma I_y = 560000$	$\Sigma I_{xy} = 300000$

Here, $I_{xy} = \bar{I}_{x'y'} + Ad_x d_y$, but $\bar{I}_{x'y'} = 0$ of each rectangle about a set of x', y' axes pass through the individual rectangle's centroid due to symmetry.

(a) Principal axes : $\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{(2 \times 300000)}{290000 - 560000}$

$$= -2.22$$

$$2\theta_m = -65.8° \text{ and } (-65.8 + 180) = 114.2°$$

$$\therefore \quad \boxed{\theta_m = -32.9° \text{ and } \theta_m = 57.1°}$$

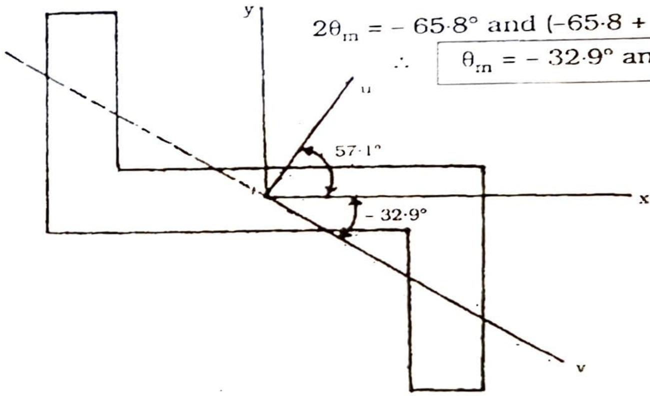


Fig 7·34

(b) Principal Moment of Inertia :

$$I_{\substack{max\\min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}}$$

$$= 425000 \pm 329000$$

$$\therefore \quad \boxed{(I_u)_{max} = 754000 \text{ cm}^4, \ (I_v)_{min} = 96000 \text{ cm}^4}$$

As by *inspection* most of the area is farthest away from selected u axis and hence $(I_u)_{max}$ is more than $(I_v)_{min}$. Thus u and v axes can be shown.

The above problem can be solved by using *Mohr's circle* after finding I_x, I_y and I_{xy}.

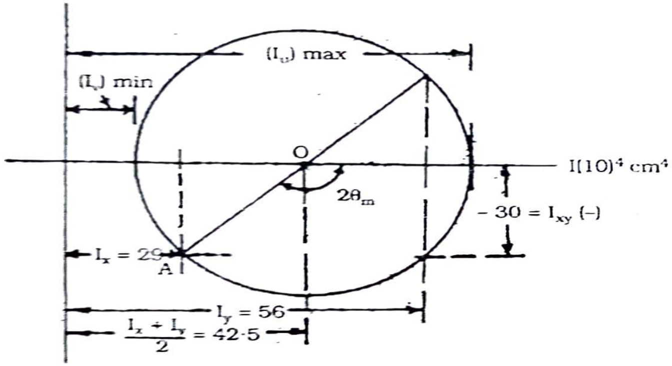


Fig. 7·35

Radius OA $= \sqrt{(42.5 - 29)^2 + (-30)^2} = 32.9 \times 10^4\ cm^4$

And measure

$2\theta_m = 114.2°$

$\therefore$ $\boxed{\theta_m = 57.1° \text{ and } -37.9°}$

$\boxed{(I_u)_{max} = 754000\ cm^4 \quad (I_v)_{min} = 96000\ cm^4}$

11. Determine the mass moments of inertia of the composite shown below with respect to the coordinate axes. (Density of steel = 7850 kg/m³ and density of Aluminium = 4000 kg/m³).

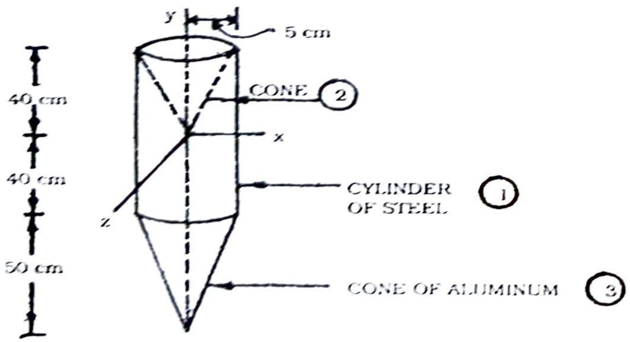


Fig. 7·36

Segment	Volume (v) (cm³)	Mass (m) ρv (kg)	$\bar{I}_x$ (kgcm²)	$\bar{I}_y$ (kgcm²)	$\bar{I}_z$ (kgcm²)	d_x (cm)	d_y (cm)	d_z (cm)	I_x (kgcm²)	I_y (kgcm²)	I_z (kgcm²)
(1) (+) Cylinder	$\pi r^2 h =$ 6283·2	0·007850 × 6283·2 = 49·32	$\frac{m}{12}(3a^2 + L^2)$ = 26612	$\frac{ma^2}{2}$ = 616·5	$\frac{m}{12}(3a^2 + L^2)$ = 26612	0	0	0	26612	616·5	26612
(2) (–) Cone	$\frac{1}{3}\pi r^2 h =$ 1047·2	0·007850 × 1047·2 = 8·22	-	-	-	-	-	-	$\frac{3m}{5}(\frac{a^2}{4} + h^2)$ = 7922 (–)	$\frac{3ma^2}{10}$ = 61·65 (–)	$\frac{3m}{5}(\frac{a^2}{4} + h^2)$ = 7922 (–)
(3) (+) Cone	$\frac{1}{3}\pi r^2 h =$ 1309	0·004 × 1309 = 5·24	$\frac{3m}{5}(\frac{a^2}{4} + h^2) - m(37.5)^2$ = 510·9	$\frac{ma^2}{2}$ = 65·5	$\frac{3m}{5}(\frac{a^2}{4} + h^2) - m(37.5)^2$ = 510·9	0	40 + 50/4 = 52·5	0	510·9 + 5·24 (52·5)² = 14954	65·5	510·9 + 5·24 (52·5)² = 14954

Mass Moment of Inertia of the composite

$$\left.\begin{aligned} I_x &= \Sigma I_x = 33644\ kg{\cdot}cm^2 \\ I_y &= \Sigma I_y = 620.35\ kg{\cdot}cm^2 \\ I_z &= \Sigma I_z = 33644\ kg{\cdot}cm^2 \end{aligned}\right\}$$

2. A rectangular hole is made in a triangular section as shown in figure. Determine the moment of inertia of the section about the x-axis passing through its centroid & also about the base BC.

(Pune University, Oct. '98) (8 marks)

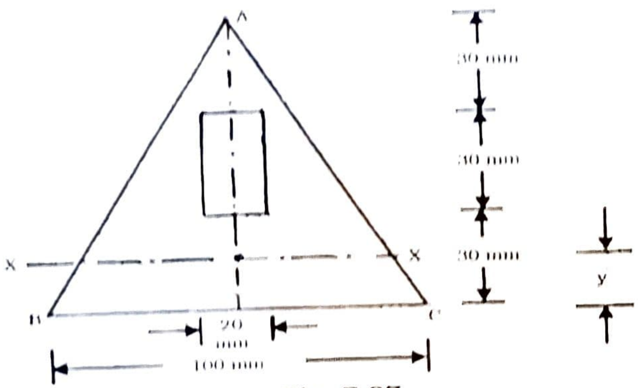


Fig. 7·37

Moment of Inertia @ base BC

$$= \frac{1}{12} \times 100 \times 90^3 - \left\{ \frac{1}{12} \times 20 \times 30^3 + (20 \times 30) \times (45)^2 \right\}$$

$$\boxed{I_{BC} = 4815000 \text{ mm}^4}$$

To determine Centroidal x axis, we have to find $\bar{y}$

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{\left\{\left(\frac{1}{2} \times 100 \times 90\right) \times (30)\right\} + \{(-20 \times 30) \times (45)\}}{\left(\frac{1}{2} \times 100 \times 90\right) - (20 \times 30)}$$

$= 27{\cdot}69$ mm.

Now, determine Moment of Inertia @ centroidal x axis I_{xx}

$$I_{xx} = \left\{ \frac{1}{36} \times 100 \times 90^3 + \left(\frac{1}{2} \times 100 \times 90\right) \times (30 - 27{\cdot}69)^2 \right\} - \left\{ \frac{1}{12} \times 20 \times 30^3 + (20 \times 30) \times (45 - 27{\cdot}69)^2 \right\}$$

$$\boxed{I_{xx} = 1824230 \text{ mm}^4}$$

13. Determine the moment of inertia of the area shown in figure with respect to its centroidal axes, XX and YY.

(Pune University, April - 1998) (8 marks)

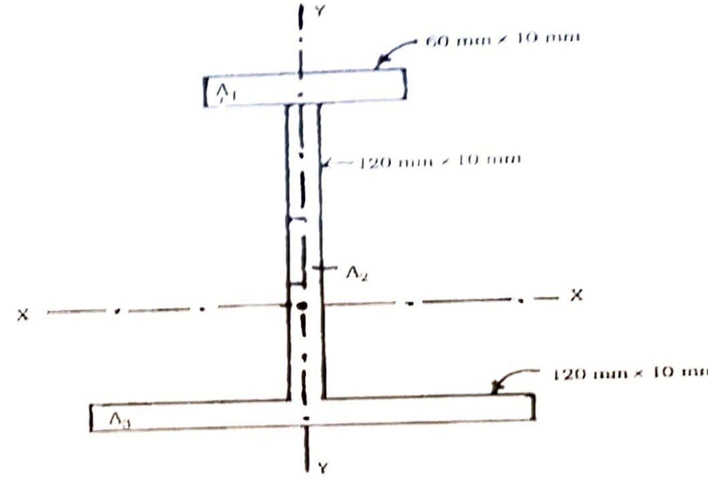


Fig. 7·38

Moment of Inertia @ centroidal YY axis

$$= I_{yy} = \frac{1}{12} \times 10 \times 60^3 + \frac{1}{12} \times 120 \times 10^3 + \frac{1}{12} \times 10 \times 120^3$$

$$\boxed{I_{yy} = 1630000 \text{ mm}^4}$$

To determine centroidal XX axis, we have to determine $\bar{y}$

$$\bar{y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{A_1 \bar{y}_1 + A_2 \bar{y}_2 + A_3 \bar{y}_3}{A_1 + A_2 + A_3}$$

$$= \frac{(60 \times 10) \times (10 + 120 + 5) + (10 \times 120)(10 + 60) + (120 \times 10) \times (5)}{60 \times 10 + 10 \times 120 + 120 \times 10}$$

$\therefore \quad \bar{y} =$ 57 mm from bottom

$=$ 83 mm from top.

Now, moment of inertia @ centroidal XX axis

$$= \frac{1}{12} \times 60 \times 10^3 + (60 \times 10) \times (83 - 5)^2 + \left\{ \frac{1}{12} \times 10 \times 120^3 + (10 \times 120) \times (70 - 57)^2 \right\} + \left\{ \frac{1}{12} \times 120 \times 10^3 + (120 \times 10) \times (57 - 5)^2 \right\}$$

$$\boxed{I_{xx} = 8553000 \text{ mm}^4}$$

14. A metal piece of uniform thickness is shown in figure. A hole of diameter 50 mm is to be drilled through the piece as shown by dotted line. Find the maximum distance 'd' of the centre of the hole from the vertical face, such that, when the piece is placed on horizontal floor as shown, tipping will not occur.

(Pune University, Oct. 1995) (8 marks)

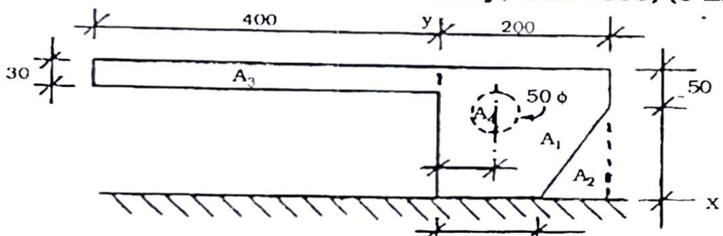


Fig. 7·39

To avoid tipping, the centroidal y axis and y axis shown in figure should coinside. Thus moment of weight of left portion about y axis and moment of weight of right portion will become same.

To have centroidal y axis just at the given y axis

$$\bar{X} = 0$$

$$\text{Now, } \bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A} = 0$$

$$\therefore \Sigma A\bar{x} = 0.$$

$$\Sigma A\bar{x} = A_1\bar{x}_1 - A_2\bar{x}_2 + A_3\bar{x}_3 + A_4\bar{x}_4 = 0$$

$$= (200 \times 180) \times 100 + (-\frac{1}{2} \times 80 \times 130)\ (120 + \frac{2}{3} \times 80)$$

$$+ (400 \times 30) \times (-200) + (-\pi \times 25^2) \times d = 0$$

$$\therefore \boxed{d = 152{\cdot}11 \text{ mm}}$$

15. The homogeneous wire ABCD is bent as shown and is attached to a hinge at C. Determine the length L for which portion AB is horizontal.

(Pune University, April - 1998) (8 marks)

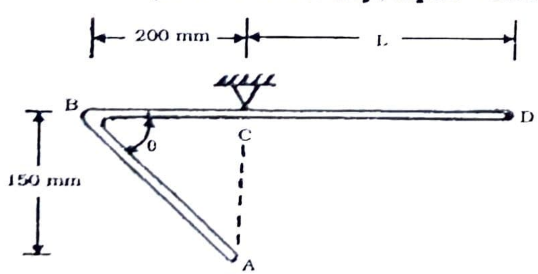


Fig. 7·40

$$\text{Here, } \theta = \tan^{-1}\frac{150}{200} = 36{\cdot}87°$$

$$\text{Length of AB} = \frac{150}{\sin\theta}$$

$$= \frac{150}{\sin 36{\cdot}87°} = 250 \text{ mm}$$

Keeping AB in horizontal position as shown here

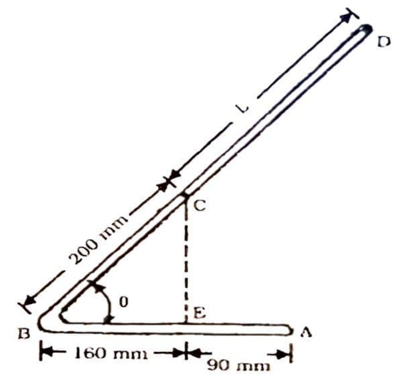


Fig. 7·41

$$BE = 200 \cos 36{\cdot}87°$$

$$= 160 \text{ mm}$$

$$EA = 90 \text{ mm}$$

To have the position of AB horizontal, CE should be perpendicular to AB and CE should also be the centroidal Y axis.

$$\therefore \bar{X} = 0, \frac{\Sigma L\bar{x}}{\Sigma L} = 0$$

$$\therefore \Sigma L\bar{x} = 0$$

$$\therefore AE(\bar{x}_1) + BE(\bar{x}_2) + BC(\bar{x}_3) + L(\bar{x}_4) = 0$$

$$90 \times 45 + 160 \times (-80) + 200 \times (-80) + L \times (\frac{L}{2}\cos 36{\cdot}87°) = 0$$

$$\therefore \boxed{L = 248{\cdot}75 \text{ mm}}$$

16. A thin wire of homogeneous material is bent to form an isosceles triangle as shown in figure. Determine the height of the triangle for which the centroid of the wire coincides with the centroid of the area enclosed by the wire.

(Pune University, April 1997) (6 marks)

19

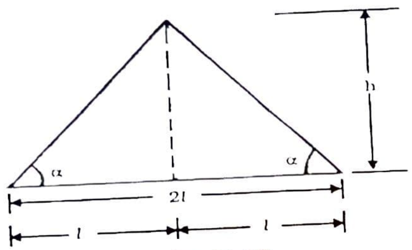


Fig. 7·42

Inclined side of the triangle $= \sqrt{h^2 + l^2}$

Now, centroidal X axis of the triangular area will be at h/3 from base.

$\bar{Y}$ of area $= h/3 = \bar{Y}$ of wire

$$\bar{Y} = \frac{h}{3} = \frac{\Sigma L\bar{y}}{\Sigma L}$$

$$\therefore \frac{h}{3} = \frac{2l \times 0 + 2\sqrt{h^2 + l^2} \times \frac{h}{2}}{2l + 2\sqrt{h^2 + l^2}}$$

$$\therefore \frac{h}{l} = \sqrt{3}, \quad \boxed{h = \sqrt{3}\, l}$$

$$\tan\alpha = \frac{h}{l} = \sqrt{3}$$

$$\therefore \boxed{\alpha = 60^\circ}$$

17. Obtain the coordinates of the centre of gravity of a thin uniform wire bent into the shape of 5 as shown in figure with reference to the x–y axes shown. **(Pune University, April - 1996) (8 marks)**

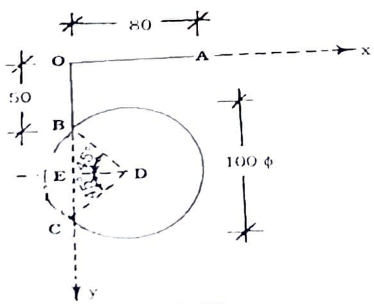


Fig. 7·43

Here $\cos 45^\circ = \frac{ED}{50}$

$\therefore$ ED $= 50 \times \cos 45^\circ$

$= 35{\cdot}36$ mm

BE $= 50 \times \sin 45^\circ$

$= 35{\cdot}36$ mm

Sr. No.	Line	Length (mm)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$L\bar{x}$ (mm²)	$L\bar{y}$ (mm²)
1.	OA	80	40	0	3200	0
2.	OB	50	0	25	0	1250
3.	Circle	$\pi \times 100$ $= 314{\cdot}16$	35·36	35·36 + 50 = 85·36	11108·7	26816·7
4.	Quarter Circular Arc BC (–)	$-\frac{\pi \times 100}{4}$ $= -78{\cdot}54$ **OR** $\frac{(2\alpha r)}{365{\cdot}62}$	$-\frac{8 \sin 45^\circ}{45^\circ}$ + 35·36 = – 45·02 + 35·36 = – 9·66	35·36 + 50 = 85·36	+ 758·7 –	6704·17
					15067·4	21362·53

$$\bar{x} = \frac{\Sigma L\bar{x}}{\Sigma L} = \frac{15067{\cdot}4}{365{\cdot}62} = 41{\cdot}21 \text{ mm}$$

$$\bar{y} = \frac{\Sigma L\bar{y}}{\Sigma L} = \frac{21362{\cdot}53}{365{\cdot}62} = 58{\cdot}43 \text{ mm}$$

18. Find position of centroid of a composite lamina, with respect to origin O as shown. **(Pune University) (8 marks)**

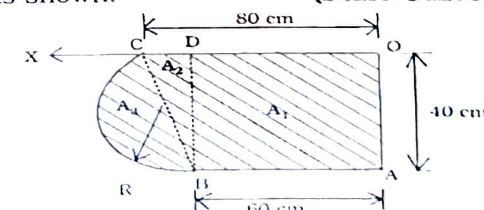


Fig. 7·44

Here, total area is divided into **three** parts; A_1, A_2 & A_3.

$$BC = \sqrt{40^2 + 20^2} = 44{\cdot}72 \text{ mm}$$

$$R = \frac{1}{2} BC = 22{\cdot}36 \text{ mm}$$

$$\text{Angle CBD} = \theta = \tan^{-1} 20/40 = 26{\cdot}56^\circ$$

Sr. No.	Segment	Area (mm^2)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$A\bar{x}$ (mm^3)	$A\bar{y}$ (mm^3)
1.	A_1 Rectangular (OABD)	$60 \times 40 = 2400$	30	20	72000	48000
2.	A_2 Triangular (BCD)	$\frac{1}{2} \times 20 \times 40 = 400$	$60 + \frac{20}{3} = 66{\cdot}67$	$\frac{40}{3} = 13{\cdot}33$	26668	5332
3.	A_3 Semicircular (BRC)	$\frac{1}{2} \times \pi \times 22{\cdot}36^2 = 785{\cdot}40$	$60 + \frac{20}{2} + \frac{4 \times 22{\cdot}36}{3\pi} \times \cos 26{\cdot}56^\circ = 78{\cdot}49$	$20 + \frac{4 \times 22{\cdot}36}{3\pi} \sin 26{\cdot}56^\circ = 23{\cdot}80$	61646·05	18692·52
	Total	3585·4		Total	148314·05	72024·52

$$\bar{x} = \frac{\Sigma A\bar{x}}{\Sigma A} = \frac{148314{\cdot}05}{3585{\cdot}4} = 41{\cdot}37 \text{ mm}$$

$$\bar{y} = \frac{\Sigma A\bar{y}}{\Sigma A} = \frac{72024{\cdot}52}{3585{\cdot}4} = 20{\cdot}09 \text{ mm}$$

THEORY RELATED QUESTIONS

1. **Define :** **(i)** Centre of Gravity **(ii)** Centre of Mass **(iii)** Centroid **(iv)** Radius of Gyration.
2. **State** **(i)** Pappus - Guldinus theorems.
   (ii) Parallel Axis Theorems for M.I. of area and masses.
3. **Explain :** Principal Axes and Principal Moments of Inertia.

EXERCISES

7·1 Locate the centroid of the wire (ABCDE) shown below.

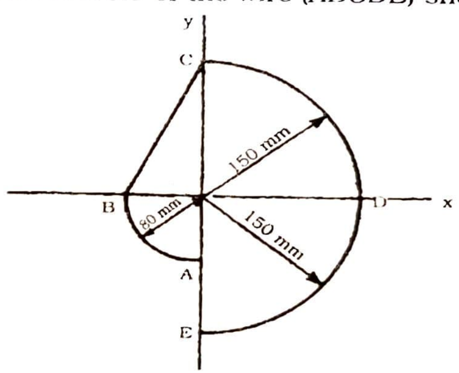


Fig 7·45

7·2 Locate the centroid of the shaded area shown below.

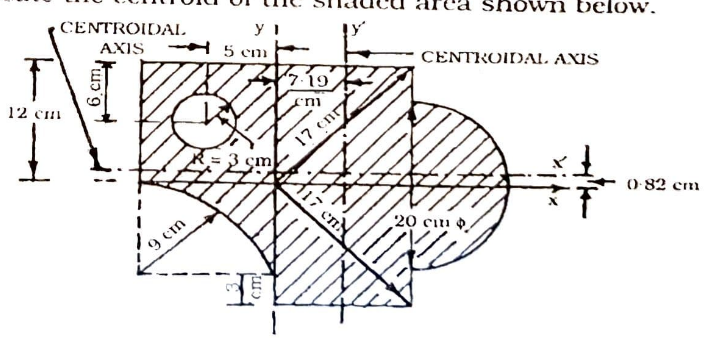


Fig 7·46

7·3 Locate the centroid of the volume shown below.

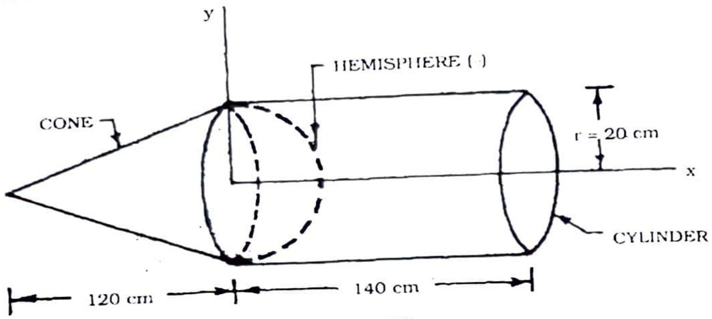


Fig. 7·47

7·4 Determine by direct integration the centroid of the area shown below.

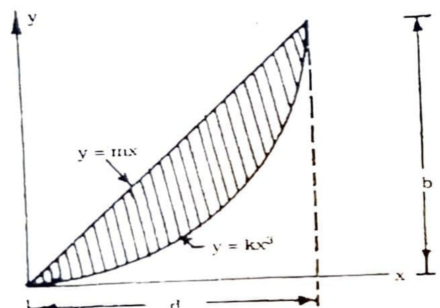


Fig 7·48

7·5 Determine the area of surface of revolution obtained by revolving a line ABCD about x axis.

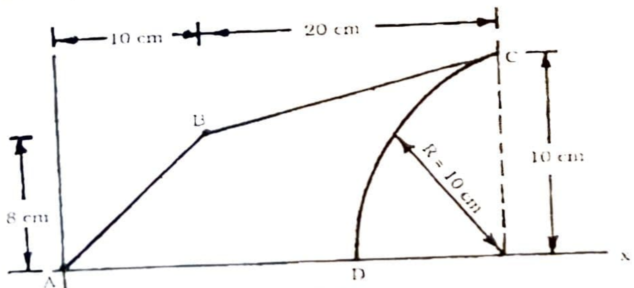


Fig. 7·49

7·6 Determine the volume of revolution obtained by revolving an area ABCDA shown in Fig 7·49.

7·7 Determine the moment of inertia of an area shown in Fig 7·46 about centroidal X and Y axis.

7·8 Determine (a) the principal axes of the section shown below about C (b) the principal moments of inertia about C.

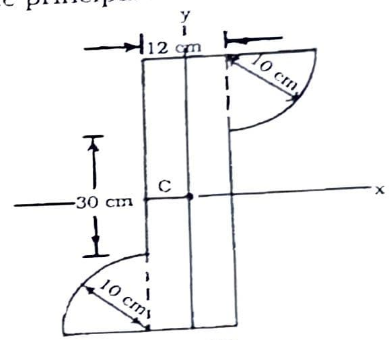


Fig. 7·50

7·9 Determine the mass moment of inertia of the composite shown below with respect to the coordinate axes.

(Density of composite = 6450 kg/m^3)

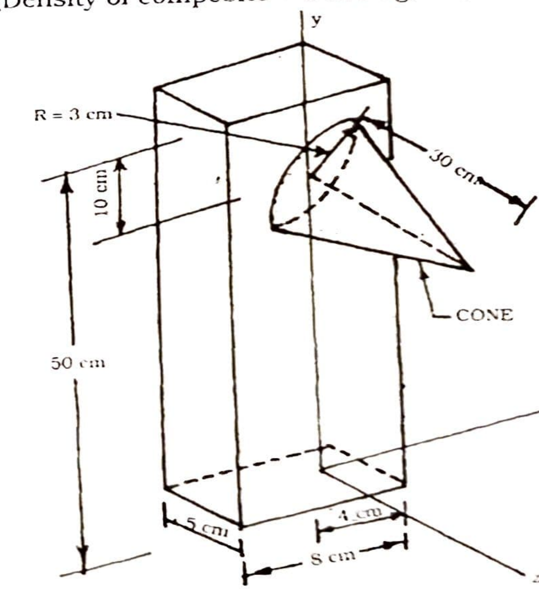


Fig 7·51

SOLUTIONS OF EXERCISES

7·1

Segment	L (mm)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{x}L$ (mm²)	$\bar{y}L$ (mm²)
AB	$\frac{\pi \times 80}{2}$	$-\frac{2 \times 80}{\pi}$	$-\frac{2 \times 80}{\pi}$	−6400	− 6400
BC	170	− 40	+ 75	− 6800	12750
CDE	$\pi \times 150$	$+\frac{2 \times 150}{\pi}$	0	45000	0

$\Sigma L = 766{\cdot}9$ $\qquad \Sigma\bar{x}L = 31800, \ \Sigma\bar{y}L = 6350$

$$\bar{x} = \frac{31800}{766{\cdot}9} = 41{\cdot}47 \text{ mm}$$

$$\bar{y} = \frac{6350}{766{\cdot}9} = 8{\cdot}28 \text{ mm}$$

7·2

Segment	Area (cm²)	$\bar{X}$ (cm)	$\bar{Y}$ (cm)	$\bar{X}A$ (cm³)	$\bar{Y}A$ (cm³)
(1) Left (+) rectangular	$+ 9 \times 21 = 189$	− 4·5	+ 1·5	− 850·5	+ 283·5
(2) Right (+) rectangular	$+ 12 \times 24 = 288$	+ 6·0	0	+ 1728	0
(3) Semi (+) circular	$+ \frac{\pi \times 10^2}{2} = 157{\cdot}08$	$+ (12 + \frac{4 \times 10}{3\pi})$	0	+ 2551·6	0
(4) Circular (−)	$- \pi \times 3^2 = - 28{\cdot}27$	− 5	+ 6	+ 141·35	− 169·62
(5) Quarter (−) circular	$- \frac{\pi \times 9^2}{4} = - 63{\cdot}62$	$- (9 - \frac{4 \times 9}{3\pi})$	$- (9 - \frac{4 \times 9}{3\pi})$	+ 329·6	+ 329·6

$\Sigma A = 542{\cdot}19, \ \Sigma\bar{X}A = + 3900{\cdot}05, \ \Sigma\bar{Y}A = + 443{\cdot}48$

$$\bar{X} = \frac{\Sigma\bar{X}A}{\Sigma A} = \frac{3900{\cdot}05}{542{\cdot}19} = 7{\cdot}19 \text{ cm}$$

$$\bar{Y} = \frac{\Sigma\bar{Y}A}{\Sigma A} = \frac{443{\cdot}48}{542{\cdot}19} = 0{\cdot}82 \text{ cm}$$

7·3

Segment	Volume (v) (cm³)	$\bar{X}$ (cm)	$\bar{X}V$ (cm⁴)
Cone	$+ \frac{1}{3} \pi \times 20^2 \times 120$	$- \frac{120}{4}$	− 1507964·5
Cylinder	$+ \pi \times 20^2 \times 140$	+ 70	+ 12315043
Hemispher	$- \frac{2}{3} \times \pi \times 20^3$	$+ \frac{3 \times 20}{8}$	− 125663·71
	$\Sigma V = 209439{\cdot}51$		$\Sigma \bar{X}V = + 10681415$

$\therefore \quad \bar{X} = 51 \text{ cm}, \ \bar{Y} = 0, \ \bar{Z} = 0.$

7·4 $$\bar{X} = \frac{\int_A \bar{x}\,dA}{\int_A dA} = \frac{\int_0^a x\,(y_2 - y_1)\,dx}{\int_0^a (y_2 - y_1)\,dx} = \frac{\int_0^a x\,(mx - kx^3)\,dx}{\int_0^a (mx - kx^3)\,dx}$$

$$= \frac{4a\,(5m - 3ka^2)}{15\,(2m - ka^2)}$$

$$\bar{Y} = \frac{\int_A \bar{y}\,dA}{\int_A dA} = \frac{\int_0^b y\,(y_2 - y_1)\,dx}{\int_0^b (y_2 - y_1)\,dx} = \frac{\int_0^b \frac{kx^3 + mx}{2}\,(mx - kx^3)\,dx}{\int_0^b (mx - kx^3)\,dx}$$

$$= \frac{2b\,(7m - 3kb^4)}{21(2m - kb^2)}$$

Here $y = y_1 + \frac{y_2 - y_1}{2} = \frac{y_1 + y_2}{2} = \frac{kx^3 + mx}{2}$

5

Segment	Length (L) (cm)	$\bar{y}$ (cm)	$2\pi \bar{y}L$ (cm²)
AB	12·81	4	321·95
BC	20·1	9	1136·63
CD	15·71	6·37	628·78

Surface Area = $\Sigma 2\pi \bar{y}L$ = 2087·36 cm²

·6

Segment	Area (A) cm²	$\bar{y}$ (cm)	$2\pi \bar{y}A$ (cm³)
Triangular (+)	40	2·67	+ 671·0
Rectangular (+)	200	5·0	+ 6283·2
Triangular (–)	– 20	9·33	– 1172·4
Quarter Circular (–)	– 78·54	4·24	– 2092·4

Volume = $\Sigma 2\pi \bar{y}A$ = 3689·4 cm³

·7

Segment	I_x' (cm⁴)	A(cm²)	d_y (cm)	$I_x' = \bar{I}_x' + Ad_y^2$ (cm⁴)
(1) Left (+) rectangular	$\frac{9 \times 21^3}{12}$	189	0·68	7033·1
(2) Right (+) rectangular	$\frac{12 \times 24^3}{12}$	288	0·82	14017·7
(3) Semi (+) circular	3927	157·08	0·82	4032·6
(4) Circular (–)	63·62	28·27	5·18	– 822·2
(5) Quarter circular (–)	1288 – 928·23	63·62	6·0	– 2650·1

$I_{centroial\ X}$ = **21611·1 cm⁴**

Segment	I_y' (cm⁴)	A (cm²)	d_x (cm)	$I_y' = \bar{I}_y' + Ad_x^2$ (cm⁴)
(1) Left (+) rectangular	$\frac{21 \times 9^3}{12}$	189	11·69	27103·8
(2) Right (+) rectangular	$\frac{24 \times 12^3}{12}$	288	1·19	3863·8

(3) Semi (+) circular	$\frac{\pi \times 10^4}{8}$ – 2829·4	157·08	9·05	13962·8
(4) Circular (–)	63·62	28·27	12·19	– 4264·4
(5) Quarter (–) circular	1288 – 928·23	63·62	12·37	– 10,094·7

$I_{centroidal\ y}$ = **30571·3 cm⁴**

7·8

Segment	$\bar{I}_x'$ (cm⁴)	$\bar{I}_y'$ (cm⁴)	A (cm²)	d_y (cm)	d_x (cm)	$I_x = \bar{I}_x + Ad_y^2$ (cm⁴)	$I_y = \bar{I}_y + ad_x^2$ (cm⁴)	$I_{xy} = Ad_xd_y$ (cm⁴)
Rectangle	125000	7200	600	0	0	125000	7200	0
Quarter circular	1630·2	1630·2	78·54	20·76	10·24	35479·2 × 2	9865·7 × 2	16696·2 × 2
						ΣI_x = **195958·4 cm⁴**	ΣI_y = **26931·4 cm⁴**	ΣI_{xy} = **33392·4 cm⁴**

(a) **Principal Axes :** $\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = \frac{-(2 \times 33392 \cdot 4)}{195958 \cdot 4 - 26931 \cdot 4}$

$= -0 \cdot 395$

$\therefore\ 2\theta_m = -21 \cdot 55°$ and $158 \cdot 45°$

$\therefore\ \theta_m = -10 \cdot 78°$ and $79 \cdot 2°$

(b) **Principal Moment of Inertia :**

$$I_{\substack{max \\ min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}}$$

$$= 111444 \cdot 9 \pm 84513 \cdot 7$$

$(I_u)_{max}$ = 195958·6 cm⁴, $(I_v)_{min}$ = 26931·2 cm⁴

7·9

Segment	Volume (v) cm^3	Mass (m) ρv (kg)	$\bar{I}_x'$ (kg cm^2)	$\bar{I}_y'$ (kg cm^2)	$\bar{I}_z'$ (kg cm^2)	d_x (cm)	d_y (cm)	d_z (cm)	I_x (kg cm^2)	I_y (kg cm^2)	I_z (kg cm^2)
Prism	$b \times d \times h = 2000$	$0{\cdot}00645 \times 2000 = 12{\cdot}9$	2714·4	95·7	2756·3	0	25	0	10776·9	95·7	10818·8
Cone	$\frac{1}{3} \pi r^2 h = 282{\cdot}74$	1·82	$\frac{3m}{5}\left(\frac{a^2}{4} + h^2\right) - m\,(22{\cdot}5)^2 = 63{\cdot}89$	$\frac{3m}{5}\left(\frac{a^2}{4} + h^2\right) - m(22{\cdot}5^2) = 63{\cdot}89$	$\frac{3}{10}\, mr^2 = 4{\cdot}91$	0	40	7·5	3078·3	166·27	2916·9
									I_x = 13855·2 kg cm^2	I_y = 261·97 kg cm^2	I_z = 13735·7 kg cm^2

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