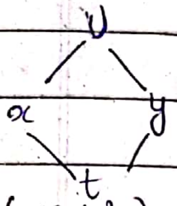
TUTORIAL - 5

EULER's AND MODIFIED EULER's THEORAM

Ans: $\left\{ \frac{3}{\sqrt{1-t^2}} \right\}$

1.) If $u = \sin^{-1}(x-y)$; $x = 3t$, $y = 4t^3$, then find Total derivative

$u = f(x,y)$ $x = f(t)$ $y = f(t)$



Total Derivative $= \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$

By chain rule

$$= \left(\frac{1}{\sqrt{1-(x-y)^2}}(1)\right)(3) + \left(\frac{1}{\sqrt{1-(x-y)^2}}(-1)\right)(12t^2)$$

$$= \frac{(3-12t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(9t^2+16t^6-24t^4)}}$$

$-16t^6+24t^4-9t^2+1 = (1-4t^2)^2(1-t^2)$

$$= \frac{3(1-2t)(1+2t)}{\sqrt{-16t^6+24t^4-9t^2+1}}$$

By observation $\{t=\pm 1\}$ are factors

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(16t^4+1-8t^2)}}$$

$16t^4 - 8t^2 + 1$

$t^2-1 \overline{) -16t^6+24t^4-9t^2+1}$

$-16t^6 + 16t^4$

$8t^4 - 9t^2$

$8t^4 - 8t^2$

$0 \quad t^2-1$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}} \frac{\cancel{(1-4t^2)}}{\cancel{(1-4t^2)}}$$

Ans $= \left[\frac{3}{\sqrt{1-t^2}}\right]$ Ans.

Ans.

8.> Find $\frac{dy}{dx}$, when $y^{x^y} = \sin(x)$ $\qquad \left(-\frac{(yx^{y-1}\log y - \cot x)}{x^y(\log x \cdot \log y + \frac{1}{y})} \right)$

9.> If $f(x,y) = 0$, then show that $q^3\left(\frac{d^2y}{dx^2}\right) = \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix}$

where $p = \frac{\partial f}{\partial x}$ $\quad q = \frac{\partial f}{\partial y}$ $\quad r = \frac{\partial f}{\partial x^2}$

$s = \frac{\partial^2 f}{\partial x . \partial y}$ $\quad t = \frac{\partial^2 f}{\partial y^2}$

10.> Find $\frac{dy}{dx}$ for $xe^y + \sin(xy) + y - \log 2 = 0$ at $(0, \log 2)$

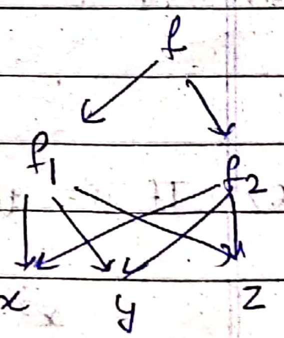
Also find $\frac{d^2y}{dx^2}$ at the same point. $\qquad (2 + \log 2,\ 8 + \log 16)$

(2) To prove $(y^2 - zx)\left(\frac{\partial u}{\partial x}\right) + (x^2 - zy)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

$u = f(x^2 + 2yz,\ y^2 + 2zx)$

let $f_1(x,y,z) = x^2 + 2yz$

$f_2(x,y,z) = y^2 + 2zx$



~~LHS~~: By Total Derivative (chain rule)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial f_1}\frac{\partial f_1}{\partial x} + \frac{\partial u}{\partial f_2}\frac{\partial f_2}{\partial x}$$

$$= \frac{\partial u}{\partial f_1}(2x) + \frac{\partial u}{\partial f_2}(2z(1)) \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial f_1}\frac{\partial f_1}{\partial y} + \frac{\partial u}{\partial f_2}\frac{\partial f_2}{\partial y}$$

$$= \frac{\partial u}{\partial f_1}(2z(1)) + \frac{\partial u}{\partial f_2}(2y) \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial f_1}\frac{\partial f_1}{\partial z} + \frac{\partial u}{\partial f_2}\frac{\partial f_2}{\partial z}$$

$$= \frac{\partial u}{\partial f_1}\left(2y(1)\right) + \frac{\partial u}{\partial f_2}\left(2x(1)\right) \quad -③$$

$Eq^n ① \times (y^2-zx) + Eq^n ② \times (x^2-zy) + Eq^n ③ (z^2-xy)$

LHS =

$$\frac{\partial u}{\partial f_1}\left(2x(y^2-zx) + 2z(x^2-zy) + 2y(z^2-xy)\right) +$$

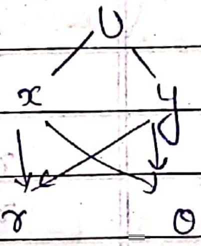
$$\frac{\partial u}{\partial f_2}\left(2z(y^2-zx) + 2y(x^2-zy) + 2x(z^2-xy)\right)$$

$$= \frac{\partial u}{\partial f_1}\left(\cancel{2xy^2} - \cancel{2zx^2} + \cancel{2zx^2} - \cancel{2z^2y} + \cancel{2yz^2} - \cancel{2xy^2}\right) +$$

$$\frac{\partial u}{\partial f_2}\left(\cancel{2zy^2} - \cancel{2z^2x} + \cancel{2yx^2} - \cancel{2zy^2} + \cancel{2xz^2} - \cancel{2x^2y}\right)$$

$$= 0 + 0 = 0 = RHS.$$

3.) $u = f(x, y)$ $\quad x = r\cos\theta \quad y = r\sin\theta$



i) prove $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

By Total Derivatives (chain rule)

$$\cancel{\frac{\partial u}{\partial x}} \quad \frac{\partial u}{\partial r} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right)$$

$$= \frac{\partial u}{\partial x}\left(\cos\theta\,(1)\right) + \frac{\partial u}{\partial y}\left(\sin\theta\,(1)\right) \quad -①$$

$$\frac{\partial u}{\partial \theta} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial \theta}\right)$$

$$= \frac{\partial u}{\partial x}\left(r(-\sin\theta)\right) + \frac{\partial u}{\partial y}\left(r\cos\theta\right) \quad -②$$

$$RHS = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 =$$

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$$= \left(\cos\theta\left(\frac{\partial u}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y}\right)\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial x}(-\sin(\theta))r + \frac{\partial u}{\partial y}(r\cos\theta)\right)^2$$

$$= \left(\cos\theta\left(\frac{\partial u}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y}\right)\right)^2 + \frac{\cancel{r^2}}{\cancel{r^2}}\left(-\frac{\partial u}{\partial x}\sin\theta + \cos\theta\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) + \cancel{2\sin\theta\cos\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}$$

$$\cancel{- 2\sin\theta\cos\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \text{LHS}$$

Hence proved

(ii) From previous question we know $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$

To prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)$

$$\frac{\partial u}{\partial r} = \cos\theta\left(\frac{\partial u}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y}\right) \quad -(1)$$

$$\frac{\partial^2 u}{\partial r^2} = 0 + 0 = 0 \quad -(2)$$

$$\cancel{\frac{\partial^2 u}{\partial \theta^2}} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \quad -(3)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{\partial u}{\partial x}(r)(\cos\theta) + r\frac{\partial u}{\partial y}(-\sin\theta) \quad -(4)$$

$$= (-1)r\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) \quad -(4)$$

$$\text{RHS} = 0 + \frac{1}{r}\left(\cos\theta\left(\frac{\partial u}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y}\right)\right) + \frac{1}{r^{\cancel{2}}}\left(-\cancel{r}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)\right)$$

$$= 0 + \frac{1}{r}\left(\cancel{\cos\theta\left(\frac{\partial u}{\partial x}\right)} + \cancel{\sin\theta\left(\frac{\partial u}{\partial y}\right)} - \cancel{\cos\theta\left(\frac{\partial u}{\partial x}\right)} - \cancel{\sin\theta\left(\frac{\partial u}{\partial y}\right)}\right)$$

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$= 0$

To prove: $\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right]$

$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)$

$Eq^n \; ① \, (\times) \; (-r\sin\theta) \;\; + \;\; Eq^n \; ③ \times (\cos\theta)$

$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)$

$$-r\sin\theta\left(\frac{\partial u}{\partial r}\right) + \cos\theta\left(\frac{\partial u}{\partial \theta}\right) = \left(-r\sin^2\theta + r\cos^2\theta\right)\left(\frac{\partial u}{\partial y}\right)$$

$$\left(\frac{\partial u}{\partial y}\right) = \frac{1}{r\cos(2\theta)}\left[\cos\theta\left(\frac{\partial u}{\partial \theta}\right) - r\sin\theta\left(\frac{\partial u}{\partial r}\right)\right]$$

$$\left[\frac{\partial^2 u}{\partial y^2} = 0\right] \quad — ④$$

$(Eq^n \, 1) \times (-r\cos\theta) + (Eq^n \, 3) \times (\sin\theta)$

$$-r\cos\theta\left(\frac{\partial u}{\partial r}\right) + \left(\sin\theta \frac{\partial u}{\partial \theta}\right) = r\left(-r\cos^2\theta + -r\sin^2\theta\right)\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{-r}\left[-r\cos\theta\left(\frac{\partial u}{\partial r}\right) + \sin\theta\frac{\partial u}{\partial \theta}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad — ⑤$$

From ④ & ⑤

$$LHS = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0 = RHS$$

Hence verified.

$\sin\theta \frac{\partial u}{\partial y}$

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4.) Verify Euler theorem $U = f(x,y)$ Homogenous function of 2 independent variable degree 'n'

$$x\left(\frac{\partial U}{\partial x}\right) + y\left(\frac{\partial U}{\partial y}\right) = n(U)$$

(i) $U = \sec^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Step 1: Check Homogeneous Eq^n or not

replace

$U = f(x,y)$ → degree $x \to xt$ $y \to yt$

$f(xt, yt) = t^{n} f(x,y)$

$$U(xt, yt) = \sec^{-1}\left(\frac{x\cancel{t}}{y\cancel{t}}\right) + \tan^{-1}\left(\frac{y\cancel{t}}{x\cancel{t}}\right)$$

$$= (t)^0\, U(x,y)$$

Degree '0' zero.

Acc to Euler's theorem,

To prove:- $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = (0)(U) = 0$

$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

$$\cancel{LHS = x\left(\frac{1}{|x/y|\sqrt{(x/y)^2-1}}\frac{(1)}{(y)}\right) + y\left(\frac{1}{(1+(y/x)^2)}\frac{(1)}{(x)}\right)}$$

$$\cancel{= \frac{y(\cancel{y})}{\sqrt{x^2-y^2}} + \frac{y(x)}{(x^2+y^2)}}$$

$$\frac{\partial U}{\partial x} = \frac{1}{|x/y|\sqrt{(x/y)^2-1}}\frac{(1)}{(y)} + \frac{1}{1+(y/x)^2}\left(\frac{-1}{x^2}\right)(y) \quad \text{—(i)}$$

$$\frac{\partial U}{\partial y} = \frac{1}{|x/y|\sqrt{(x/y)^2-1}}\left(\frac{-1}{y^2}\right)(x) + \frac{1}{1+(y/x)^2}\frac{(1)}{(x)} \quad \text{—(ii)}$$

$$\text{LHS} = \text{(i)} \times x + \text{(ii)} \times y$$

$$\left(\frac{x}{y} - \frac{x}{y}\right)$$

$$\Rightarrow \frac{1}{\left|\frac{x}{y}\right|\sqrt{\left(\frac{x}{y}\right)^2-1}} \; \frac{\cancel{(x-y)}}{(y)} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \; \cancel{(-x+y)} \left(\frac{-y}{x} + \frac{y}{x}\right)$$

$$= \cancel{\frac{y\,(x-y)}{x\sqrt{x^2-y^2}}\,(y)} + \cancel{\frac{(y-x)\,(x^2)}{(x^2+y^2)}}$$

$$= 0 + 0 \qquad = 0 \; (u)$$

$$= \boxed{\text{RHS}}$$

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(ii) $U = (x^2+y^2+z^2)^{-1/2}$

$x \to xt \quad y \to yt \quad z \to zt$

$U(xt, yt, zt) = (t^2(x^2+y^2+z^2))^{-1/2}$

$= (t^{-1}) \cdot \underbrace{(x^2+y^2+z^2)^{-1/2}}_{U(x,y,z)}$

degree $= (-1)$

Acc. to Euler

$$z\frac{\partial U}{\partial z} + x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = n(U), \quad n = -1$$

$$\text{LHS} = x\left(\frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}x)\right) + -y^2(x^2+y^2+z^2)^{-3/2}$$

$$+ -z^2(x^2+y^2+z^2)^{-3/2}$$

$$= (-1)(x^2+y^2+z^2) \times (x^2+y^2+z^2)^{-3/2}$$

$$= (-1)(x^2+y^2+z^2)^{-1/2}$$

$$= (-1)(U) = \text{RHS}.$$ Hence Euler Theorem Verified

(iii) $U = \frac{y}{z} + \frac{z}{x}$

$$U(tx, ty, tz) = \frac{y\cancel{t}}{z\cancel{t}} + \frac{z\cancel{t}}{x\cancel{t}}$$

$$= t^0 \; U(x,y,z)$$

degree = zero

$$z\frac{\partial U}{\partial z} + x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 0(U) = 0$$

$$\cancel{\text{LHS}=} \; \frac{\partial U}{\partial x} = 0 + \left(\frac{-z}{x^2}\right) \quad -(1)$$

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$$\frac{\partial u}{\partial y} = \frac{1}{z} \quad \text{—(ii)}$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x} \quad \text{—(iii)}$$

LHS $= (i) \times x + (ii) \times y + (iii) \times z$

$$= \cancel{\frac{-z}{x}} + \cancel{\frac{y}{z}} + \cancel{\frac{-y}{z}} + \cancel{\frac{z}{x}}$$

$= 0 = 0 \times u =$ RHS $\therefore$ Hence Euler Theorem Verified

5.) If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$ show $\Rightarrow$ $x \cdot u_x + y u_y = \frac{1}{2} \sin(2u)$

Let us use Modified Eulers theoram,

$$\tan(u) = \frac{x^2+y^2}{x-y}$$

Let $x \to xt$ $y \to yt$

$$\Rightarrow \frac{(xt)^2 + (yt)^2}{(xt - yt)}$$

$$= t^1 \left(\frac{x^2+y^2}{x-y}\right)$$

$[n = 1]$ degree

Modified Euler theoram

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

$f(u) = \tan(u)$

$f'(u) = \sec^2(u)$

$$x u_x + y u_y = (1) \frac{\tan(u)}{\sec^2(u)}$$

$$= \frac{1}{2}\left(\frac{2 \sin(u)}{\cancel{\cos(u)}} \cos^{\cancel{2}}(u)\right)$$

$$\frac{1}{2} \sin(2u) = \text{RHS}$$

[Hence proved]

6.) $u = \tan^{-1}(x^2 + 2y^2)$ then show :

$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = 2\sin(u)\cos(3u)$

$= t^{2}(x^2+2y^2)$

$= (xt)^2 + 2(yt)^2$ $\boxed{n=2}$

$\underbrace{\tan(u)}$ $= x^2 + 2y^2$ $\left.\begin{matrix} x \to xt \\ y \to yt \end{matrix}\right\}$

Let $V = \tan(u)$ $g(u) = n \dfrac{f(u)}{f'(u)}$

Acc. to corollary ② of Modified Euler,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$g(u) = n \dfrac{V}{(V')}$

$= (2) \dfrac{\tan(u)}{\sec^2(u)}$

$= (2) \dfrac{\sin(u)\, \cancel{\cos^2}(u)}{\cancel{\cos(u)}}$

$= \sin(2u)$

LHS = RHS $= g(u)\,[g'(u) - 1]$

$= \sin(2u)\,[\cancel{\sin(2u)}\ \cos(2u)\,(2) - 1]$

$= \sin(2u)\,[2(2\cos^2 u - 1) - 1]$

$= 2\sin(u)\cos(u)\,[4\cos^2(u) - 2 - 1]$

$= 2\sin(u)\,[4\cos^3(u) - 3\cos(u)]$

$[2\sin(u)\cos(3u)]$

(Hence Proved)

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7.) $(x)^y + (y)^x = ab$

Find $\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{(\partial f/\partial y)}$ (Implicit Differentiation)

$y = (2)^x$

$\frac{1}{y}(y) = \log(2)$

$f(x,y) = (x)^y + (y)^x - ab$

$\frac{\partial f}{\partial x} = (y)(x)^{y-1} + (y)^x \log(y) - (0)$

$\frac{\partial f}{\partial y} = (x)^y \log(x) + x(y)^{x-1} - 0$

Ans: $\frac{dy}{dx} = \frac{-(y(x)^{y-1} + (y)^x \log(y))}{((x)^y \log(x) + (x)((y)^{x-1}))}$

8.) $(y)^{x(y)} = \sin(x)$ $\frac{dy}{dx} = (?)$

$(x^y)\log(y) = \log(\sin x)$

$\cancel{y \log(x)} = \cancel{\log\left(\frac{\log(\sin x)}{\log(y)}\right)}$

$(x^y)\frac{1}{y}\left(\frac{dy}{dx}\right) + \cancel{\log(y)\,(y)(x)^{y-1}} = \cancel{\frac{1}{\sin(x)}\cos(x)}$

$= \cot(x)$

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