Now,

$(x, y) \to (u, v)$

$(0, 1) \to (1, 1)$

$(1, 0) \to (-1, 1)$

$(0, 4) \to (4, 4)$

$(4, 0) \to (-4, 4)$

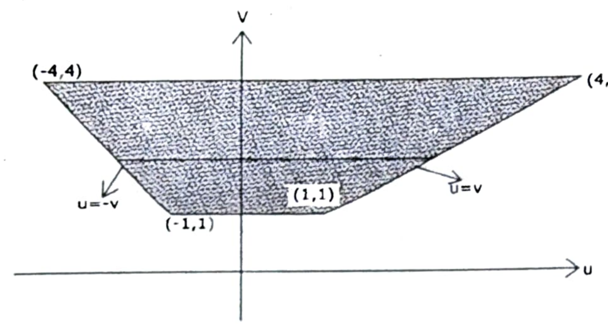

Fig 1.2.42

Draw the figure in uv–plane. Take a horizontal strip line.

$\therefore$ The limits are :

$-v \le u \le v$

$1 \le v \le 4$

$$\therefore \quad I = \int_1^4 \int_{-v}^{v} e^{u/v}\, \frac{1}{2}\, du\, dv = \frac{1}{2}\int_1^4 \left[\frac{e^{u/v}}{1/v}\right]_{-v}^{v} dv = \frac{1}{2}\int_1^4 v[e^1 - e^{-1}]\, dv$$

$$= \frac{1}{2}(e - e^{-1})\left[\frac{v^2}{2}\right]_1^4 = \frac{15}{4}(e - e^{-1})$$

EXERCISE : 1.2

1. Evaluate $\iint_R r^3 dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$\left[\textbf{Ans. : } \frac{45}{2}\pi\right]$

2. Evaluate $\int_{-\pi/2}^{\pi/2} \int_0^{a\sin\theta} r^2 dr\, d\theta$. **[Ans. : 0]**

3. Evaluate $\iint_R (x^2 + y^2)\, dA$ over the region in the positive quadrant for which $x + y \le 1$.

[Ans. : 1/6]

4. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$. $\left[\textbf{Ans. : } \frac{\pi a^3}{6}\right]$

5. Evaluate $\iint_R (a^2 - x^2 - y^2)\, dx\, dy$ over the semi-circle $x^2 + y^2 = ax$ in the positive quadrant. $\left[\textbf{Ans. : } \frac{5\pi a^4}{64}\right]$

6. Evaluate $\int_{-a}^{a} \int_{-\sqrt{a^2-x}}^{\sqrt{a^2-x^2}} dA$ **[Ans. : πa^2]**

7. Evaluate $\int_0^a \int_y^a \frac{x^2\, dA}{\sqrt{x^2+y^2}}$ $\left[\textbf{Ans. : } \frac{a^3}{3}\left[\log(\sqrt{2}+1)\right]\right]$

8. Evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx$. **[Ans. : $\pi/5$]**

9. Evaluate $\iint_R \sin[\pi(x^2 + y^2)]\, dA$ over the region bounded by the circle $x^2 + y^2 = 1$.

[Ans. : 2]

10. Use the transformation $u = xy$, $v = xy^4$ to find $\iint_R \sin(xy)\, dA$, where R is the region enclosed by the curves $xy = \pi$, $xy = 2\pi$, $xy^4 = 1$, $xy^4 = 2$. $\left[\textbf{Ans. : } -\frac{2}{3}\log 2\right]$

11. Use an appropriate transformation to find $\iint_R \sqrt{16x^2 + 9y^2}\, dA$, where R is the region enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$. [Hint : Put $x = 3u$, $y = 4v$] **[Ans. : 96π]**

12. Use an appropriate transformation to find $\iint_R \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) dA$, where R is the traingular region with vertices (0, 0), (2, 0), (1, 1).

$\left[\text{Hint : Put } u = \frac{x+y}{2},\ v = \frac{x-y}{2}\right]$ $\left[\textbf{\textit{Ans.}} : 1 - \frac{1}{2}\sin 2\right]$

13. Use an appropriate transformation to find $\iint_R \frac{x-y}{x+y}\, dA$, where R is the region enclosed by the lines $x - y = 0$, $x - y = 1$, $x + y = 3$ and $x + y = 1$.

[Hint : Put $u = x - y$, $v = x + y$] $\left[\textbf{Ans. : } \frac{1}{4}\log 3\right]$

1.5 Changing the Limits of Integration :

Sometimes the iterated integrals with given limits becomes more complicated. This can be made simple by changing the order or integration (limits). As we know that the double integral can be evaluated successively by integrating w.r.t. x first and then w.r.t y, or may be integrated in the reverse order. To change the limits always draw a figure from the given limits.

If it is given first to integrate w.r.t. x, then to change it consider a vertical strip line and determine the limits.

If it is given first to integrate w.r.t. y, then to change it consider a horizontal strip line and determine the limits

SOLVED EXAMPLES

1. Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing the order of integration.

Solution : Draw the figure from

$x = y,\ x = a,\ y = a,\ y = 0$

Take a strip line parallel to y–axis.

$\therefore$ The limits are :

$0 \le y \le x$

$0 \le x \le a$

$$\therefore\ I = \int_0^a \int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi a}{4}$$

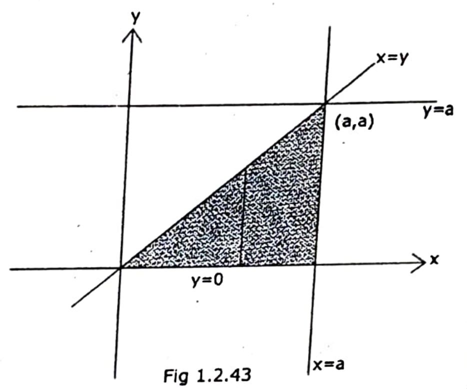

Fig 1.2.43

2. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dA$ by changing the order of integration.

Solution : Draw the figure from the limits $y = x,\ y \to \infty,\ x = 0,\ x \to \infty$

Take a horizontal strip line.

$\therefore$ The limits are :

$0 \le x \le y$

$0 \le y \le \infty$

$$\therefore\ I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,[x]_0^y\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = 1$$

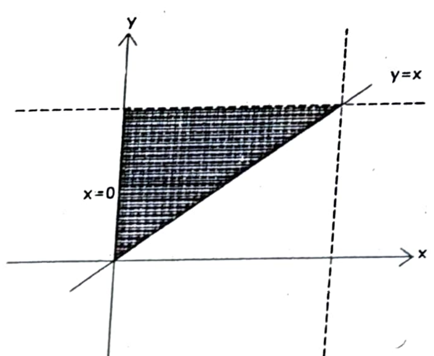

Fig 1.2.44

3. Evaluate $\int_1^4 \int_{\sqrt{y}}^2 (x^2+y^2)\,dA$ by changing the order of integration.

Solution : Draw the figure from the limits $x = \sqrt{y},\ x = 2,\ y = 1,\ y = 4$

Take a vertical strip line.

$\therefore$ The limits are :

$1 \le y \le x^2$

$1 \le x \le 2$

$$\therefore\ I = \int_1^2 \int_1^{x^2} (x^2+y^2)\,dy\,dx$$

$$= \int_1^2 \left[x^2 y + \frac{y^3}{3}\right]_1^{x^2} dx$$

$$= \int_1^2 \left[x^4 + \frac{x^6}{3} - x^2 - \frac{1}{3}\right] dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{x}{3}\right]_1^2$$

$$= \frac{1026}{105}$$

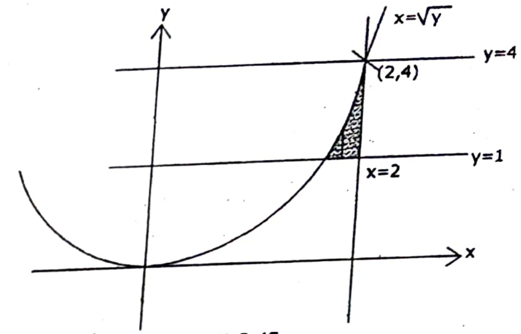

Fig 1.2.45

4. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = x^2,\ y = 2 - x,\ x = 0,\ x = 1$.

Take a horizontal strip line.

$\therefore$ The limits are :

[Here the region is divided in two parts R_1 & R_2]

$R_1 : 0 \le x \le \sqrt{y}$

$0 \le y \le 1$

$R_2 : 0 \le x \le 2 - y$

$1 \le y \le 2$

$$\therefore\ I = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_1^{2-y} xy\,dx\,dy$$

$$= \frac{3}{8}\quad \text{(After simple integrations)}$$

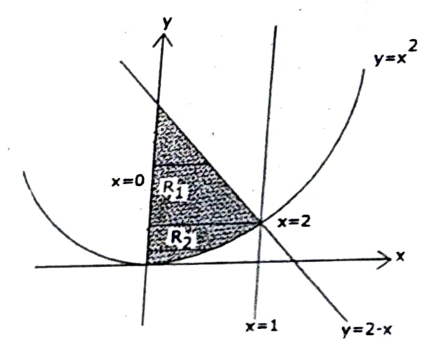

Fig 1.2.46

5. Evaluate $\int_0^\infty \int_0^\infty e^{-xy}\sin nx\,dA$ · show that $\int_0^\infty \frac{\sin nx}{x}\,dx = \frac{\pi}{2}$.

Solution : $I = \int_0^\infty \left[\frac{e^{-xy}}{n^2+y^2}\{-y\sin nx - n\cos nx\}\right]_0^\infty dx$

$$= \int_0^\infty \left[0 - \frac{1}{n^2+y^2}(-n)\right] dy = n\int_0^\infty \frac{1}{n^2+y^2}\, dy$$

$$= n\left[\frac{1}{n}\tan^{-1}\frac{y}{n}\right]_0^\infty = \frac{\pi}{2} \qquad \ldots\ldots\ldots (1)$$

If we change the order of integration, then

$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dy\, dx$$

$$= \int_0^\infty \sin nx \left[\frac{e^{-xy}}{-x}\right]_0^\infty dx = \int_0^\infty \frac{\sin nx}{x}\, dx \qquad \ldots\ldots\ldots (2)$$

$\therefore$ From (1) & (2), $\int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}$

6. Evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = x^2/4a$, $y = 2\sqrt{ax}$, $x = 0$, $x = 4a$

Take a horizontal strip line :

$\therefore$ The limits are :

$$\frac{y^2}{4a} \le x \le 2\sqrt{ay}$$

$$0 \le y \le 4a$$

$$\therefore\ I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\, dy$$

$$= \frac{16a^2}{3}$$

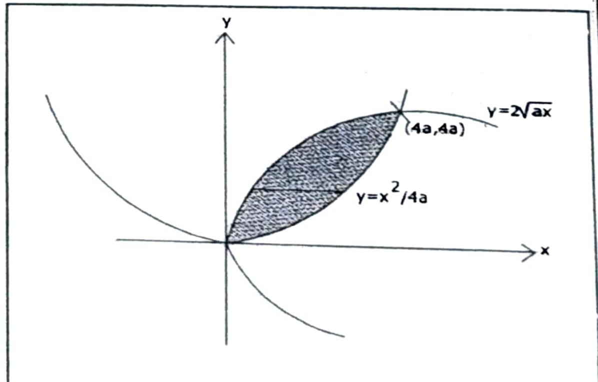

Fig 1.2.47

7. Evaluate $\int_0^a \int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}\, dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = 0$, $y = x$, $x = 0$, $x = a$.

Take a horizontal strip line.

$\therefore$ The limits are :

$$y \le x \le a$$

$$0 \le y \le a$$

$$\therefore\ I = \int_0^a \int_y^a \frac{\cos y}{\sqrt{(a-y)(a-x)}}\, dx\, dy$$

$$= \int_0^a \frac{\cos y}{\sqrt{a-y}} \left[\frac{(a-x)^{1/2}}{-1/2}\right]_y^a dy$$

$$= 2\int_0^a \frac{\cos y}{\sqrt{a-y}} \left[0 + \sqrt{a-y}\right] dy$$

$$= 2\int_0^a \cos y\, dy$$

$$= 2\left[\sin y\right]_0^a = 2\sin a$$

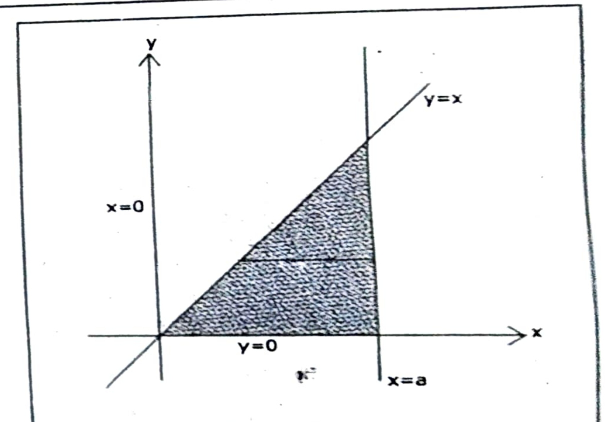

Fig 1.2.48

8. Evaluate $\int_0^1 \int_{4y}^4 e^{x^2}\, dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $x = 4y$, $x = 4$, $y = 0$, $y = 1$

Take a vertical strip line :

$\therefore$ The limits are :

$$0 \le y \le \frac{x}{4}$$

$$0 \le x \le 4$$

$$\therefore\ I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx$$

$$= \int_0^4 e^{x^2} \left[y\right]_0^{x/4} dx = \frac{1}{4}\int_0^4 xe^{x^2} dx$$

$$= \frac{1}{4}\left[\frac{e^{x^2}}{2}\right]_0^4 = \frac{1}{8}\left[e^{16} - 1\right]$$

$$\left(\because \int f'(x)\, e^{f(x)}\, dx = e^{f(x)}\right)$$

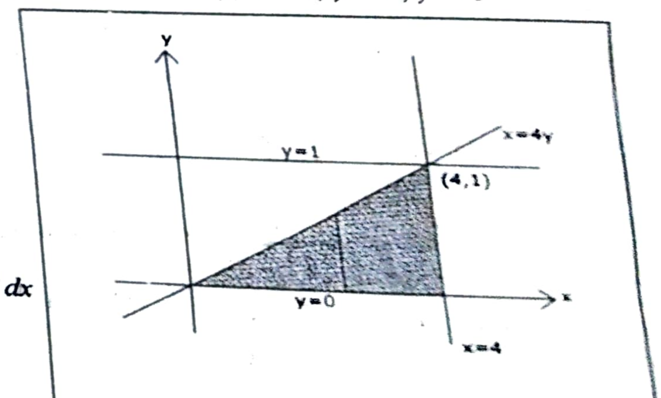

Fig 1.2.49

9. Evaluate $\int_0^{2a} \int_{x^2/4a}^{3a-x} (x^2 + y^2)\, dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = x^2/4a$, $y = 3a$ ...

Take a horizontal strip line.

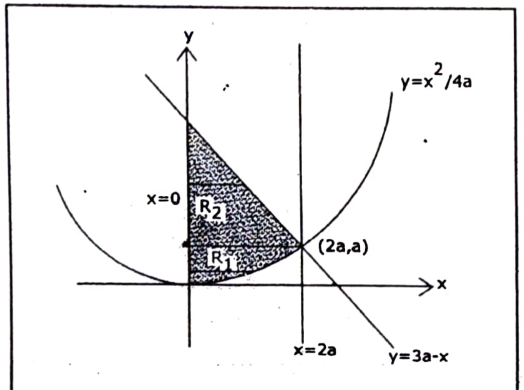

Fig 1.2.50

$\therefore$

R_1	R_2
$0 \le x \le 2\sqrt{ay}$	$0 \le x \le 3a-y$
$0 \le y \le a$	$a \le y \le 3a$

$$\therefore \quad I = \int_0^a \int_0^{2\sqrt{ay}} (x^2+y^2)\,dx\,dy + \int_a^{3a}\int_0^{3a-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^{2\sqrt{ay}} dy + \int_a^{3a} \left[\frac{x^3}{3}+xy^2\right]_0^{3a-y} dy$$

$$= \int_0^a \left[\frac{8}{3}a^{3/2}y^{3/2} + 2\sqrt{a}\,y^{5/2}\right] dy + \int_a^{3a}\left[\frac{(3a-y)^3}{3} + (3a-y)y^2\right] dy$$

$$= \left[\frac{8}{3}a^{3/2}\frac{y^{5/2}}{5/2} + 2\sqrt{a}\,\frac{y^{5/2}}{7/2}\right]_0^a + \left[\frac{(3a-y)^4}{-12} + 3a\frac{y^3}{3} - \frac{y^4}{4}\right]_a^{3a} = \frac{314}{35}a^4$$

10. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = x$, $y = \sqrt{2-x^2}$, $x = 0$, $x = 1$

Take a horizontal strip line.

Divide the region R in two parts R_1 and R_2.

$\therefore$ The limits are :

Over R_1 : $0 \le x \le y$, $0 \le y \le 1$

Over R_2 : $0 \le x \le \sqrt{2-y^2}$, $1 \le y \le \sqrt{2}$

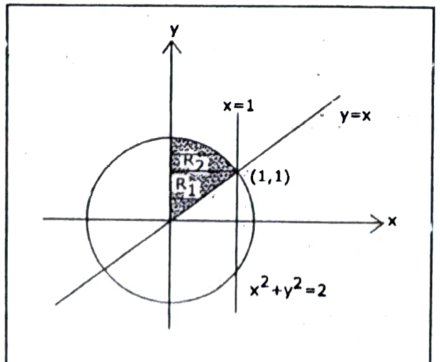

Fig 1.2.51

$$\therefore \quad I = \int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_0^1 \frac{1}{2}\left[\frac{(x^2+y^2)^{1/2}}{1/2}\right]_0^y dy + \int_0^{\sqrt{2}} \frac{1}{2}\left[\frac{(x^2+y^2)^{1/2}}{1/2}\right]_0^{\sqrt{2-y^2}}$$

$$= \int_0^1 \left[\sqrt{2}y - y\right] dy + \int_1^{\sqrt{2}} \left[\sqrt{2} - y\right] dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

. Evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$ by changing the order of integration.

Solution : Draw the figure from the given limits $x = 0$, $x = \frac{a}{b}\sqrt{b^2-y^2}$, $y = 0$, $y = b$

Take a vertical strip line :

$\therefore$ The limits are :

$0 \le y \le b\sqrt{1-\frac{x^2}{a^2}}$,

$0 \le x \le a$

$$\therefore \quad I = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} xy\,dy\,dx$$

$$= \int_0^a \frac{x}{2}\left[y^2\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{1}{2}\int_0^a x b^2\left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2} - \frac{a^4}{4a^2}\right] = \frac{a^2b^2}{8}$$

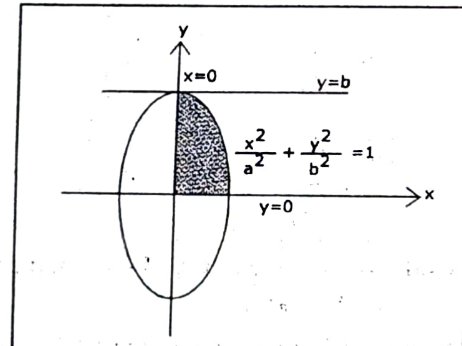

Fig 1.2.52

. Evaluate $\int_0^\infty \int_{-y}^{y} (y^2-x^2)\,e^{-y}\,dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $x = -y$, $x = y$, $y = 0$, $y \le \infty$.

Take vertical strip line.

Divide the region R into R_1 & R_2.

$\therefore$ The limits are :

Over R_1 : $-x \le y \le \infty$
$-\infty \le x \le 0$

Over R_2 : $x \le y < \infty$
$0 \le x < \infty$

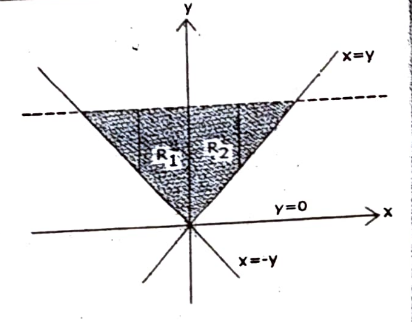

Fig 1.2.53

$$\therefore \quad I = \int_{-\infty}^{0}\int_{-x}^{\infty} (y^2 - x^2)\, e^{-y}\, dy\, dx + \int_{0}^{\infty}\int_{x}^{\infty} (y^2 - x^2)\, e^{-y}\, dy\, dx$$

$$= \int_{-\infty}^{0} \left[(y^2 - x^2)(-e^{-y}) - (2y)(e^{-y}) + (2)(-e^{-y})\right]_{-x}^{\infty} dx + \int_{0}^{\infty} \left[(y^2 - x^2)(-e^{-y}) - (2y)(e^{-y}) + (2)(-e^{-y})\right]_{x}^{\infty} dx$$

$$= \int_{-\infty}^{0} \left[-2xe^{x} + 2e^{x}\right] dx + \int_{0}^{\infty} \left[2xe^{-x} + 2e^{-x}\right] dx$$

$$= -2 \int_{-\infty}^{0} (x-1)e^{x}\, dx + 2 \int_{0}^{\infty} (x+1)e^{-x}\, dx$$

$$= -2\left[(x-1)(e^{x}) - (1)(e^{x})\right]_{-\infty}^{0} + 2\left[(x+1)(-e^{-x}) - (1)(e^{-x})\right]_{0}^{\infty}$$

$$= -2\,[-1-1] + 2\,[1+1] = 8$$

13. Evaluate $\displaystyle\int_{0}^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$**, by changing the order of integration.**

Solution : Draw the figure from the given limits

$x = a - \sqrt{a^2 - y^2}$, $x = a + \sqrt{a^2 - y^2}$,

$y = 0, \; y = a$

Take a vertical strip line.

$\therefore$ The limits are :

$0 \le y \le \sqrt{2ax - x^2}$

$0 \le x \le 2a$

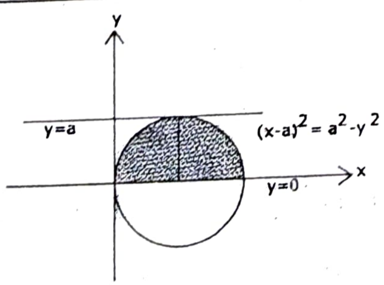

Fig 1.2.54

$$\therefore \quad I = \int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}} xy\, dy\, dx$$

$$= \int_{0}^{2a} x \left[\frac{y^2}{2}\right]_{0}^{\sqrt{2ax-x^2}} dx$$

$$= \frac{1}{2}\int_{0}^{2a} x(2ax - x^2)\, dx = \frac{1}{2}\left[2a\,\frac{x^3}{3} - \frac{x^4}{4}\right]_{0}^{2a}$$

$$= \frac{1}{2}\left[2a\,\frac{8a^3}{3} - \frac{16a^4}{4}\right] = \frac{2a^4}{3}$$

14. Evaluate $\displaystyle\int_{0}^{a}\int_{y^2/a}^{y} \frac{y\, dA}{(a-x)\sqrt{ax - y^2}}$ **by changing the order of integration.**

Solution : Draw the figure from the given limits $x = \dfrac{y^2}{a}$, $x = y$, $y = 0$, $y = a$

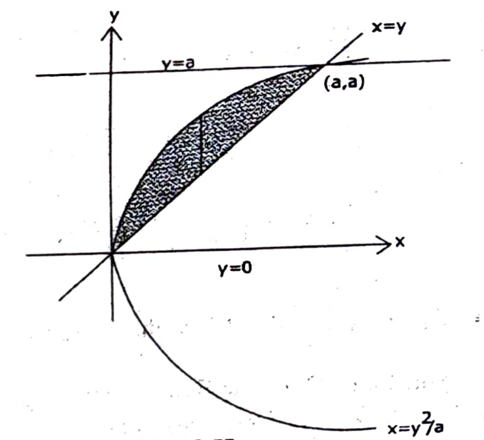

Fig 1.2.55

Take a vertical strip.

$\therefore$ The limits are :

$x \le y \le \sqrt{ax}$

$0 \le x \le a$

$$\because \quad I = \int_{0}^{a}\int_{x}^{\sqrt{ax}} \frac{y\, dy\, dx}{(a-x)\sqrt{ax - y^2}}$$

$$= \int_{0}^{a} \frac{1}{a-x}\left[\frac{(ax - y^2)^{1/2}}{\frac{1}{2}(-2)}\right]_{x}^{\sqrt{ax}} dx$$

$$= \int_{0}^{a} \frac{1}{a-x}\sqrt{ax - x^2}\; dx$$

$$= \int_{0}^{\pi/2} \frac{\sqrt{a^2\sin^2\theta - a^2\sin^4\theta}}{a - a\sin^2\theta}\, 2a\sin\theta\cos\theta\, d\theta \qquad (\text{Put } x = a\sin^2\theta)$$

$$= \int_{0}^{\pi/2} \frac{a\sin\theta\,\cos\theta}{a\cos^2\theta}\, 2a\sin\theta\cos\theta\, d\theta$$

$$= 2a\int_{0}^{\pi/2} \sin^2\theta\, d\theta = 2a\,\frac{1}{2}\,\frac{\pi}{2} = \frac{a\pi}{2}$$

15. Evaluate $\displaystyle\int_{0}^{1}\int_{0}^{y} (x^2 + y^2)\, dx\, dy + \int_{1}^{2}\int_{0}^{2-y} (x^2 + y^2)\, dx\, dy$ **by changing the order of integration.**

Solution : Draw the figure from the limits :

$I_n\ R_1 : x = 0,\ x = y,\ y = 0,\ y = 1$

$I_n\ R_2 : x = 0,\ x = 2 - y,\ y = 1,\ y = 2$

Take a horizontal strip line.

$\therefore$ The limits are :

$x \le y \le 2 - x$

$0 \le x \le 1$

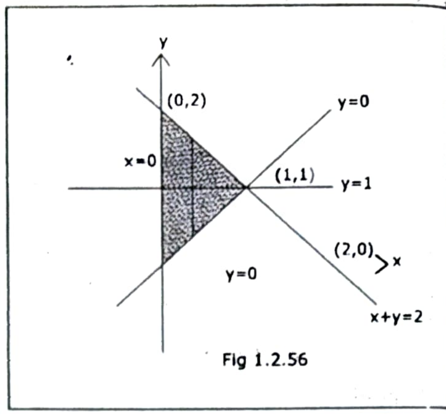

Fig 1.2.56

$$\therefore \quad I = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$$

$$= \int_0^1 \left[x^2(2 - x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$= \int_0^1 \left[2x^2 - \frac{7}{3}x^3 + \frac{(2-x)^3}{3}\right] dx$$

$$= \left[2\frac{x^3}{3} - \frac{7}{3}\frac{x^4}{4} - \frac{(2-x)^4}{12}\right]_0^1 = \frac{4}{3}$$

16. Evaluate $\int_0^1 \int_0^{\cos^{-1}x} x\, dA$ by changing the order of integration.

Solution : Draw the figure from the given limits $y = 0,\ y = \cos^{-1}x,\ x = 0,\ x = 1$

Take a horizontal strip line.

$\therefore$ The limits are :

$0 \le x \le \cos y$

$0 \le y \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \int_0^{\cos y} x\, dx\, dy$$

$$= \int_0^{\pi/2} \left[\frac{x^2}{2}\right]_0^{\cos y} dy$$

$$= \frac{1}{2} \int_0^{\pi/2} \cos^2 y\, dy$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

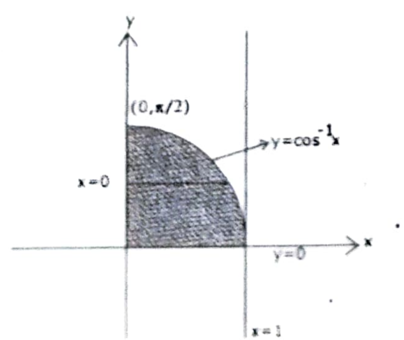

Fig 1.2.57

EXERCISE : 1.3

Evaluate the following by changing the order of integration :

1. $\int_0^a \int_{x^2/a}^{2a-x} xy\, dA$ $\left[\text{Ans. : } \frac{7a^4}{12}\right]$

2. $\int_0^a \int_0^{bx/a} x\, dA$ $\left[\text{Ans. : } \frac{a^2 b}{3}\right]$

3. $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} y^2\, dA$ $\left[\textit{Ans. : } \frac{(\pi+2)a^4}{32}\right]$

4. $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dA$ $\left[\text{Ans. : } \frac{1}{3}(e^8 - 1)\right]$

5. $\int_0^1 \int_{4x}^4 e^{-y^2}\, dA$ $\left[\text{Ans. : } \frac{1}{8}(1 - e^{-16})\right]$

1.6 Triple Integrals :

In article 1.2, we defined double integral of a function $f(x, y)$ defined over a bounded region R in the xy – plane. Here we consider a continuous function $f(x, y, z)$ defined over a bounded by a solid region D in three dimensional space. We divide a solid region D into small rectangular parallelopiped (called cells) by drawing planes parallel to the coordinate planes.

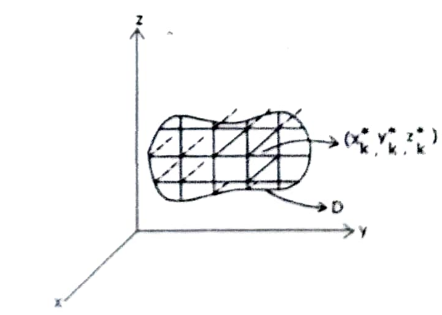

Fig 1.2.58

Consider any point (x_k^*, y_k^*, z_k^*) in one of the cells with volume $\delta V_k = \delta x_k, \delta y_k, \delta z_k$.

If we partition the solid region D into large number of such cells then the following limits of the approximation of sum.

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \delta V_k$$

is known as triple integral of $f(x, y, z)$ over D and can be written as :

$$\iiint_D f(x, y, z)\, dV$$

1.6.1 Algebraic Properties :

Triple integral holds following properties as in double integrals :

1. $\iiint\limits_D k\, f(x, y, z)\, dv = k \iiint\limits_D f(x, y, z)\, dv$ (k is any number)
2. $\iiint\limits_D [f(x, y, z) \pm g(x, y, z)]\, dv = \iiint\limits_D f(x, y, z)\, dv + \iiint\limits_D g(x, y, z)\, dv$
3. $\iiint\limits_D f(x, y, z)\, dv \geq 0$ if $f(x, y, z) \geq 0,\ \forall\ (x, y, z) \in D$.
4. $\iiint\limits_D f(x, y, z)\, dv \geq \iiint\limits_D g(x, y, z)\, dv$ if $f(x, y, z) \geq g(x, y, z)\ \forall\ (x, y, z) \in D$.
5. If the solid region D is partitioned into two subregion D_1 and D_2, then

$$\iiint\limits_D f(x, y, z)\, dv = \iiint\limits_{D_1} f(x, y, z)\, dv + \iiint\limits_{D_2} f(x, y, z)\, dv$$

1.6.2 Evaluation of Triple Integrals :

The triple integral is difficult to evaluate directly from its definition in the form of a limit. Thus we shall evaluate the triple integral by repeated single integrations.

1.6.2.1 When D is a solid region in the form of a rectangular parallelopiped (Box) then defined by the inequalities $a \leq x \leq b,\ c \leq y \leq d,\ e \leq z \leq f$

$$\iiint\limits_D f(x,y,z)\, dv = \int_a^b \int_c^d \int_e^f f(x,y,z)\ dz\, dy\, dx$$

Here the order of integration is immaterial provided the limits of integration can be changed accordingly.

That is $\int_a^b \int_e^f \int_c^d f(x, y, z)\, dy\, dz\, dx$

$$= \int_c^d \int_a^b \int_e^f f(x, y, z)\, dz\, dx\, dy$$

$$= \int_c^d \int_e^f \int_a^b f(x, y, z)\, dx\, dz\, dy$$

$$= \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

$$= \int_e^f \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz$$

1.6.2.2 Suppose that a solid region D in three dimensional space is bounded below by the surface $z = f_1(x, y)$, above by the surface $z = f_2(x, y)$ and on the side by a surface whose projection on xy–plane is the region, say R, as shown in the following figure.

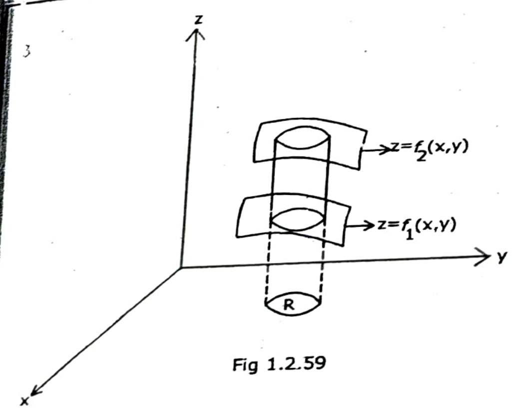

Fig 1.2.59

Here we shall determine first the limits of inner integral. Take a rectangular strip parallel to z–axis in the solid region D, which intersect lower and upper surfaces. Thus

$$f_1(x, y) \leq z \leq f_2(x, y)$$

To determine limits of outer double integral. Draw a separate figure showing region R, which is the projection of solid region D on xy–plane.

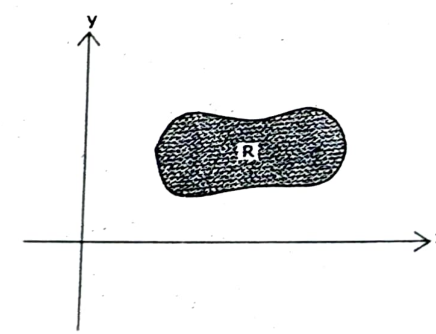

Fig 1.2.60

Determine the limits of x and y (in any order) by taking horizontal or vertical strip one, whichever we want in region R and using the same procedure as we discussed earlier. Thus we have

$$= \iiint\limits_D f(x,y,z)\, dV$$

$$= \iint\limits_R \left[\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz \right] dA$$

where $dA = dx\, dy = dy\, dx$

Note : Depending on the shape of the solid region D, sometimes it is better to evaluate first w.r.t. x or y by taking rectangular strip parallel to X–axis (or Y–axis) and then the limits of remaining double integral by taking the projection of the solid region D over yz–plane (or xz–plane) which is the region R in a plane. That is

$$\iiint\limits_D f(x,y,z)\, dV = \iint\limits_R \left[\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dx \right] dA$$

Where $dA = dy\, dz = dz\, dx$

SOLVED EXAMPLES

1. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

Solution : $I = \int_0^{\log 2}\int_0^{x} \left[e^{x+y+z}\right]_0^{x+\log y} dy\, dx$

$$= \int_0^{\log 2}\int_0^{x} \left[e^{x+y+x+\log y} - e^{x+y}\right] dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} \left[y e^{2x+y} - e^{x+y}\right] dy\, dx$$

$$= \int_0^{\log 2} \left[(y)\,(e^{2x+y}) - (1)\,(e^{2x+y}) - e^{x+y}\right]_0^x dx$$

$$= \int_0^{\log 2} \left[x e^{3x} - e^{3x} - e^{2x} + e^{2x} + e^x\right] dx$$

$$= \int_0^{\log 2} \left[(x-1)e^{3x} + e^x\right] dx$$

$$= \left[(x-1)\left(\frac{e^{3x}}{3}\right) - (1)\left(\frac{e^{3x}}{9}\right) + e^x\right]_0^{\log 2}$$

$$= (\log 2 - 1)\,\frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} + e^{\log 2} + \frac{1}{3} + \frac{1}{9} - 1$$

$$= (\log 2 - 1)\,\frac{8}{3} - \frac{8}{9} + 2 + \frac{1}{3} + \frac{1}{9} - 1$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

2. Evaluate $\int_0^{\pi}\int_0^{\pi/2}\int_0^{1} r^2 \sin\theta\, dr\, d\theta\, d\phi$

Solution : $I = [\phi]_0^{\pi}\, [-\cos\theta]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1$

$$= \frac{\pi}{3}$$

Note : When all the limits of integration are constants and the triple integral is in t[he] form

$$\iiint f(x)\, g(y)\, h(z)\, dz\, dy\, dx \text{ then it can be evaluated as}$$

$$\left[\int f(x)\,dx\right]\left[\int g(y)\,dy\right]\left[\int h(z)\,dz\right]$$

3. Evaluate $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z}}\, dz\, dy\, dx$

Solution : $I = \int_0^{1}\int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$

$$= \int_0^{1}\int_0^{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0\right] dy\, dx$$

$$= \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

4. Evaluate $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

Solution : $I = \int_0^{a}\int_0^{x} \left[e^{x+y+z}\right]_0^{x+y} dy\, dx$

$$= \int_0^{a}\int_0^{x} \left[e^{2(x+y)} - e^{x+y}\right] dy\, dx$$

$$= \int_0^{a} \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx = \int_0^{a} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3}{2}\,\frac{e^{2x}}{2} + e^x\right]_0^a = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{1}{8} + \frac{3}{4} - 1$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$$

5. Evaluate $\int_1^{e}\int_1^{\log y}\int_1^{e^x} \log z\, dz\, dx\, dy$

Solution : $\int_1^{e^x} \log z \cdot 1\, dz = [\log z \cdot z]_1^{e^x} - \int_1^{e^x} \frac{1}{z}\, z\, dz$

$$= xe^x - [z]_1^{e^x} = xe^x - e^x + 1$$

$$= (x-1)e^x + 1$$

$$\therefore \quad I = \int_1^{e}\int_1^{\log y} [(x-1)e^x + 1]\, dy\, dx$$

$$= \int_1^e \left[(x-1)(e^x) - (1)(e^x) + x\right]_1^{\log y} dy$$

$$= \int_1^e [(\log y - 1)\, y - y + \log y + e - 1]\, dy$$

$$= \int_1^e [(y+1)\log y - 2y + e - 1]\, dy$$

$$= \left[\log y \cdot \left(\frac{y^2}{2} + y\right)\right]_1^e - \int_1^e \frac{1}{y}\left(\frac{y^2}{2} + y\right) dy - [y^2]_1^e + (e-1)\,[y]_1^e$$

$$= \frac{e^2}{2} + e - \left[\frac{y^2}{4} + y\right]_1^e - (e^2 - 1) + (e-1)(e-1)$$

$$= \frac{e^2}{2} + e - \frac{e^2}{4} - e + \frac{1}{4} + 1 - e^2 + 1 + e^2 - 2e + 1$$

$$= \frac{1}{4}[e^2 - 8e + 13]$$

6. **Evaluate** $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz\, dz\, dy\, dx$

Solution : $I = \int_1^3 \int_{1/x}^1 xy \left[\frac{z^2}{2}\right]_0^{\sqrt{xy}} dy\, dx = \frac{1}{2}\int_1^3 \int_{1/x}^1 x^2y^2\, dy\, dx$

$$= \frac{1}{2}\int_1^3 x^2 \left[\frac{y^3}{3}\right]_{1/x}^1 dx = \frac{1}{6}\int_1^3 x^2\left[1 - \frac{1}{x^3}\right] dx$$

$$= \frac{1}{6}\int_1^3 \left[x^2 - \frac{1}{x}\right] dx = \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_1^3$$

$$= \frac{1}{6}\left[9 - \log 3 - \frac{1}{3}\right] = \frac{13}{9} - \frac{1}{6}\log 3$$

7. **Evaluate** $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2 - r^2}{a}} r\, dz\, dr\, d\theta$

Solution : $I = \int_0^{\pi/2} \int_0^{a\sin\theta} r\, [z]_0^{\frac{a^2-r^2}{a}}\, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^{a\sin\theta} r\left(\frac{a^2 - r^2}{a}\right) dr\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \left[\frac{a^4}{2}\sin^2\theta - \frac{a^4}{4}\sin^4\theta\right] d\theta$$

$$= \frac{1}{a}\left[\frac{a^4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^4}{4}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{1}{a}\,\frac{a^4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left[1 - \frac{3}{8}\right] = \frac{5\pi a^3}{64}$$

8. **Evaluate** $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$

Solution : $I = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\,(1 - x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} x\,(y - x^2y - y^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2} - x^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{1-x^2}{2} - \frac{x^2}{2}(1-x^2) - \frac{(1-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{1}{2} - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^4}{2} - \frac{1}{4} + \frac{x^2}{2} - \frac{x^4}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{1}{4} - \frac{x^2}{2} + \frac{x^4}{4}\right] dx = \frac{1}{2}\left[\frac{1}{4}\,\frac{x^2}{2} - \frac{1}{2}\,\frac{x^4}{4} + \frac{1}{4}\,\frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{8} - \frac{1}{8} + \frac{1}{24}\right] = \frac{1}{48}$$

9. **Evaluate** $\iiint_D xyz\ dV$**, where D is the region bounded by the positive octant of the sphere** $x^2 + y^2 + z^2 = a^2$.

Solution : Draw the figure

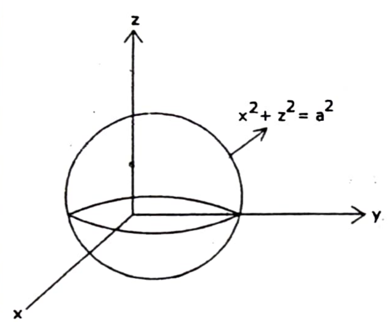

Fig 1.2.61

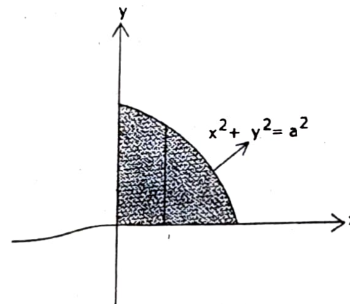

Fig 1.2.62

The limits are : $0 \le z \le \sqrt{a^2 - x^2 - y^2}$

$0 \le y \le \sqrt{a^2 - x^2}$

$0 \le x \le a$

$$\therefore \ I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$= \frac{a^6}{48} \quad \text{[Integrate as above example]}$$

10. Evaluate $\iiint\limits_D (x - 2y + z) \, dV$ where

$D : 0 \le x \le 1, \ 0 \le y \le x, \ 0 \le z \le x + y$

Solution : $I = \int_0^1 \int_0^x \int_0^{x+y} (x - 2y + z) \, dz \, dy \, dx$

$$= \int_0^1 \int_0^x \left[(x - 2y) \, z + \frac{z^2}{2} \right]_0^{x+y} dy \, dx$$

$$= \int_0^1 \int_0^x \left[(x - 2y)(x + y) + \frac{(x+y)^2}{2} \right] dy \, dx = \frac{8}{35} \quad \text{(After integration)}$$

11. Evaluate $\iiint\limits_D \dfrac{dV}{(x+y+z+1)^3}$, where D is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ (i.e. the coordinate planes) and $x + y + z = 1$.

Solution : Draw the figure

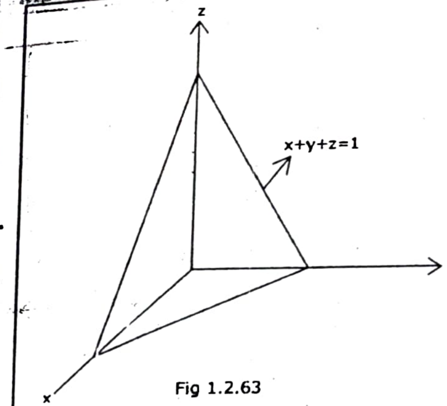

Fig 1.2.63

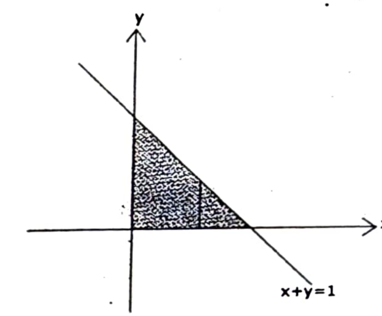

Fig 1.2.64

Thus the limits are $0 \le z \le 1 - x - y$

$0 \le y \le 1 - x$

$0 \le x \le 1$

$$\therefore \ I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z + 1)^{-3} dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy \, dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x + y + 1)^{-2} \right] dy \, dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y - \frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4}(1 - x) + \frac{1}{2} - (x + 1)^{-1} \right] dx$$

$$= -\frac{1}{2} \left[\frac{1}{4} \frac{(1-x)^2}{-2} + \frac{1}{2} x - \log(x + 1) \right]_0^1 = -\frac{1}{2} \left[\frac{1}{2} - \log 2 + \frac{1}{8} \right]$$

$$= -\frac{1}{2} \left[\frac{5}{8} - \log 2 \right]$$

. Evaluate $\iiint\limits_D (x^2 + y^2 + z^2) \, dV$, where G is the region bounded by the coordinate planes and $x + y + z = a$, $(a > 0)$

Solution : Draw the figure

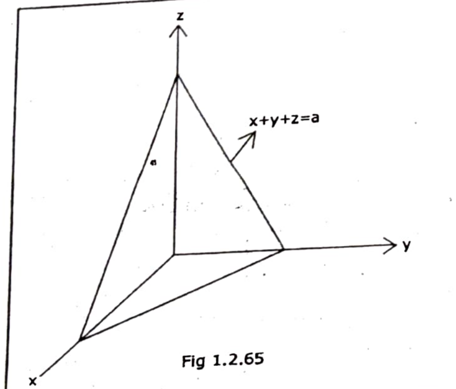

Fig 1.2.65

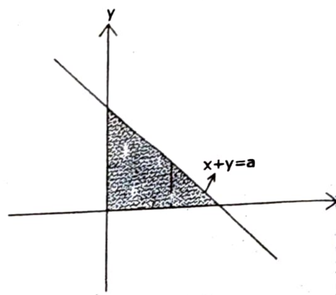

Fig 1.2.66

$\therefore$ The limits are : $0 \le z \le a-x-y$

$0 \le y \le a-x$

$0 \le x \le a$

$$\therefore \quad I = \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx = \int_0^a \int_0^{a-x} \left[(x^2+y^2)\, z + \frac{z^3}{3} \right]_0^{a-x-y} dy\, dx$$

$$= \int_0^a \int_0^{a-x} \left[(x^2+y^2)(a-x-y) + \frac{(a-x-y)^3}{3} \right] dy\, dx$$

$$= \int_0^a \int_0^{a-x} \left[ax^2 - x^3 + ay^2 - y^3 - x^2y - xy^2 + \frac{(a-x-y)^3}{12} \right] dy\, dx$$

$$= \int_0^a \left[ax^2y - x^3y + \frac{ay^3}{3} - \frac{y^4}{4} - \frac{x^2y^2}{2} - \frac{xy^3}{3} - \frac{(a-x-y)^4}{12} \right]_0^{a-x} dx$$

$$= \int_0^a \left[ax^2(a-x) - x^3(a-x) + \frac{x^2}{2}(a-x)^2 + \frac{a}{3}(a-x)^3 + \frac{(a-x)^4}{12} \right] dx$$

$$= \frac{a^5}{20}$$

EXERCISE 1.4

❑ **Evaluate the following triple integrals :**

1. $\int_0^2 \int_1^3 \int_1^2 xy^2z\, dz\, dy\, dx$ [Ans. : 2

2. $\int_0^1 \int_0^2 \int_1^2 x^2yz\, dz\, dy\, dx$ [Ans. :

$\left[\text{Ans. : } \frac{a^6}{48}\right]$

$\int_0^a \int_0^x \int_0^y xyz\, dz\, dy\, dx$ [Ans. : $(e-1)^3$]

$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx\, dy\, dz$ [Ans. : 0]

$\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x-y+z)\, dy\, dx\, dz$

Evaluate $\iiint_D z\,dv$, where D is the region in the first octant cut from the cylindrical solid $y^2+z^2 \le 1$ by the planes $y = x$ and $x = 0$. [Ans. : 1/8]

Evaluate $\iiint_D x^2yz\, dV$ throughout the volume bounded by the planes $x = 0, y = 0, z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. [Ans. : $a^3b^3c^2/2520$]

Triple Integrals in other space coordinate systems :

Till now, we have seen the evaluation of triple integrals in cartesian coordinates. In article we shall see such integrals in other special coordinate systems known as indrical and spherical coordinates.

1 Cylindrical Coordinates :

Cylindrical coordinates (r, θ, z) are useful in applications where an axis of symmetry urs.

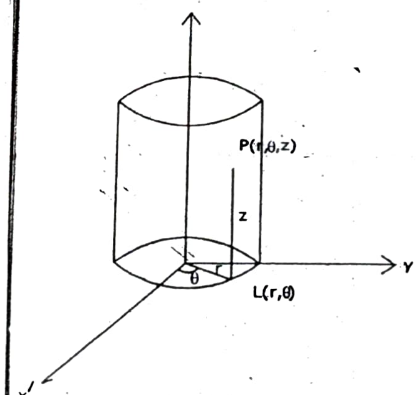

Fig 1.2.67

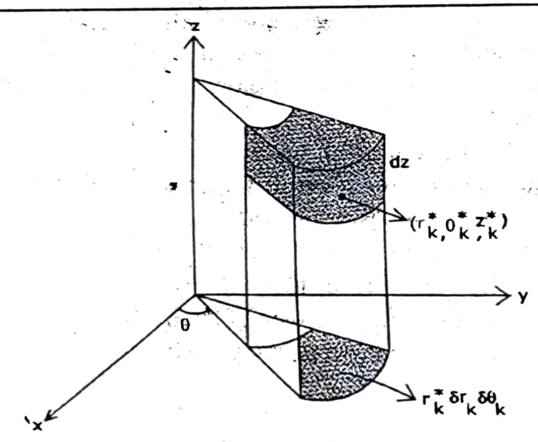

Fig 1.2.68

Consider a point P on the cylinder. If L is the projection of P on the xy-plane, then) are the useful polar coordinates of L. The cylindrical coordinates of P then merely ire the addition of the z–coordinate.

If we divide the solid region into subregions in space with respect to cylindrical dinates then the volume of such an element is taken to be

δV_k = (area of the base) (height)

$= r_k\, \delta r_k\, \delta\theta_k\, \delta z_k$

Consider any point $(r_k^*, \theta_k^*, z_k^*)$ in the sub volume. Then the following limit of s
when we increase the number of sub volumes largely.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\, \delta V_k$$

is known as triple integral in polar form and can be written as

$$\iiint_D f(r, \theta, z)\, dV$$

where $dV = r\, dr\, d\theta\, dz$

1.7.2 Spherical Coordinates :

Spherical coordinates (r, θ, ϕ) are useful in applications where a centre of symmetry occurs.

If P (r, θ, ϕ) is the point on a sphere then OP = r, θ is the angle which is measured from positive z–axis moving towards xy–plane, and ϕ is the angle made by the projection of OP on xy–plane which is OL and is measured from positive x–axis moving towards positive y axis.

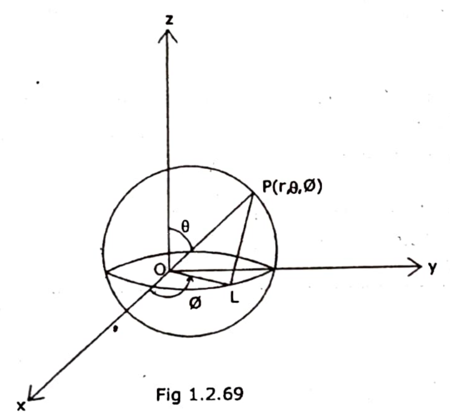

Fig 1.2.69

Divide the solid region D into small subregions with volume $\delta V_k = r_k^{*2} \sin \theta_k^*\, \delta r_k\, \delta\theta_k\, \delta\phi_k$. Considering any point $(r_k^*, \theta_k^*, \phi_k^*)$ in the subregion. If we consider the large number of such subregions then the following limit of sum.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, \phi_k^*)\, \delta V_k$$

is known as triple integral in spherical coordinates and can be written as

$$\iiint_D f(r, \theta, \phi)\, dV$$

where $dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

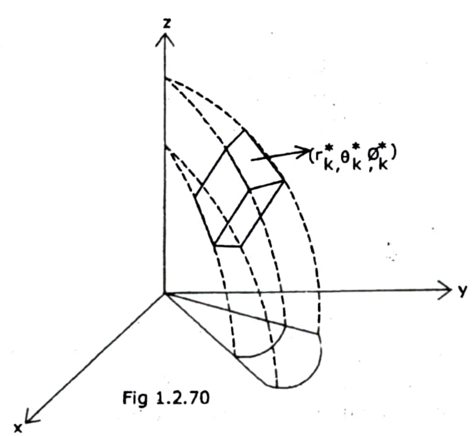

Fig 1.2.70

1.7.3 To determine the limits of Integration :

In cylindrical coordinates, first determine the limits of z by drawing a line parallel to z–axis which meets the lower and upper surfaces of the solid region. Then take the projection of the solid region over xy–plane and determine the limits of r and θ as we discussed in polar coordinates.

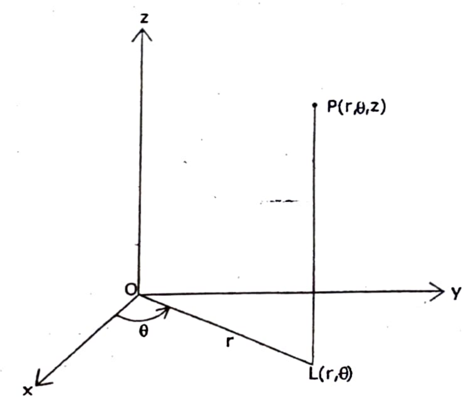

Fig 1.2.71

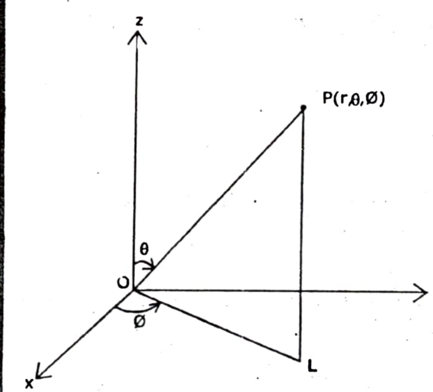

Fig 1.2.72

In spherical coordinates, draw an oblique line passing through origin which touch the solid region in interior surface r_1 and outer surface r_2 at the end points. Thus determine the limits of r first.

$\therefore\ r_1 \le r \le r_2$

Then determine the limits of θ. For this, move the oblique, line (OP) starting from positive z–axis, where $\theta = 0$, towards xy – plane upto which the region is covered. At last determine the limits of ϕ. For this, move the projected line (OL) of an oblique line (OP) starting from positive x–axis, where $\phi = 0$, towards positive y–axis upto which the projected region of the solid region is covered.

1.7.4 Changing the Variables :

As discussed in the section 1.4.2, the triple integral also become simple to evaluate in some certain cases by changing the coordinates.

If a solid region E in the uvw – space is transformed into the solid region D in the xyz space by the transformation $x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$ then the function

F (x, y, z) defined on D can be thought of as a function F $(f(u, v, w), g(u, v, w), h(u, v, w))$ defined on E.

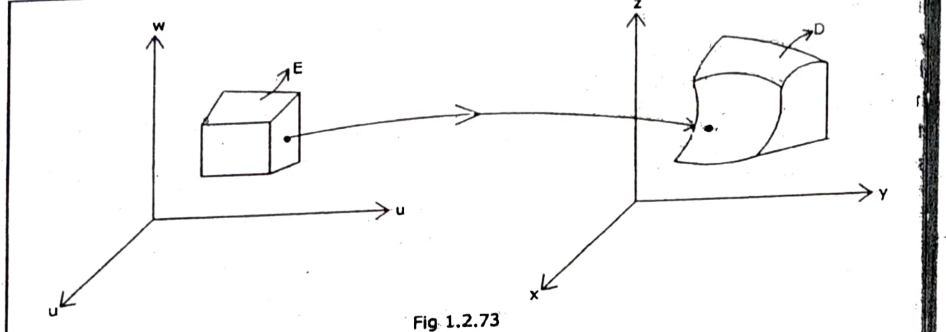

Fig 1.2.73

If the functions $f(u, v, w)$, $g(u, v, w)$ and $h(u, v, w)$ are continuous and possess continuous derivatives then we have the following relation of integrals :

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_D F[f(u, v, w),\ g(u, v, w),\ h(u, v, w)]\, |J|\, du\, dv\, dw$$

Where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

is called the Jacobian of the transformation from the uvw – space to the xyz – space defined by the equations $x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$.

1.7.5 Changing into Cylindrical Coordinates :

We know that the relation between cartesian and cylindrical coordinates is given by

$x = r\cos\theta,\ \ y = r\sin\theta,\ z = z$ and $J = r$

$$\therefore\ \iiint_D F(x,y,z)\, dx\, dy\, dz = \iiint_E F(r\cos\theta,\ r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

1.7.6 Changing into Spherical Coordinates :

We know that the relation between cartesian and spherical co–ordinates is given by

$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta$

$$\left[r = \sqrt{x^2+y^2+z^2},\ \theta = \cos^{-1}\left(\frac{z}{r}\right),\ \phi = \tan^{-1}\left(\frac{y}{x}\right)\right]$$

and $J = r^2\sin\theta$

$$\therefore\ \iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_D F(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

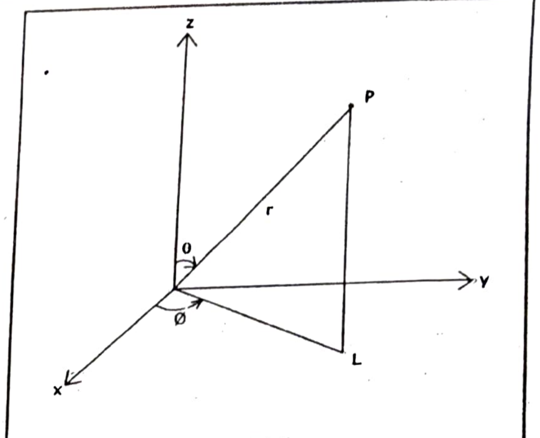

Fig 1.2.74

SOLVED EXAMPLES

1. **Evaluate $\iiint_D z(x^2+y^2+z^2)\, dV$ through the volume of the cylinder $x^2 + y^2 = a^2$ intercepted by the planes $z = 0$, $z = h$.**

Solution : Draw the figure changing to cylinderical coordinates (r, θ, z), we get the limits :

$0 \le z \le h$

$0 \le r \le a$

$0 \le \theta \le 2\pi$

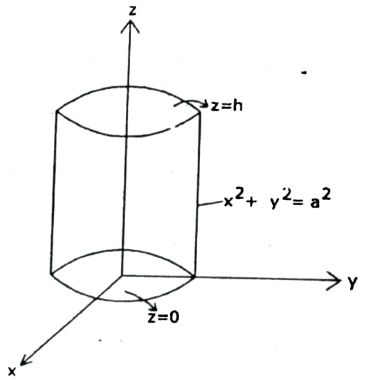

Fig 1.2.75

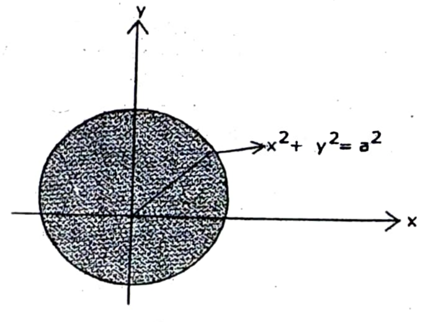

Fig 1.2.75A

$$\therefore \quad I = \int_0^{2\pi}\int_0^a\int_0^h z(r^2+z^2)\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a r\left[r^2\frac{z^2}{2}+\frac{z^4}{4}\right]_0^h dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a r\left[\frac{h^2}{2}r^2+\frac{h^4}{4}\right] dr\, d\theta = [\theta]_0^{2\pi}\left[\frac{h^2}{2}\frac{r^4}{4}+\frac{h^4}{4}\frac{r^2}{2}\right]_0^a = 2\pi\left[\frac{h^2a^4}{8}+\frac{h^4a^2}{8}\right]$$

$$= \frac{\pi h^2a^2}{4}\,[a^2+h^2]$$

2. **Evaluate** $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

Solution : Changing into spherical coordinates (r, θ, ϕ).

Draw the figure from the limits

$z = 0,\ z = \sqrt{1-x^2-y^2}$,

$y = 0,\ y = \sqrt{1-x^2}$,

$x = 0,\ x = 1$

$\therefore$ The limits are :

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$0 \le \phi \le \frac{\pi}{2}$

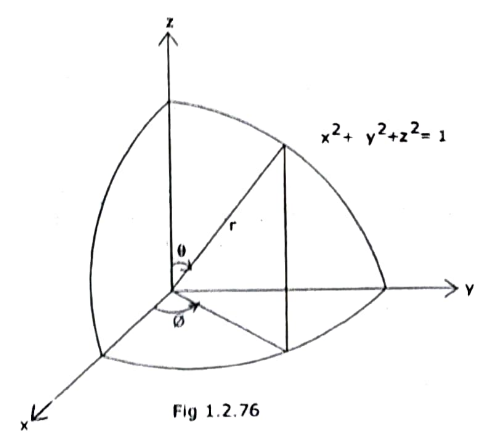

Fig 1.2.76

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}}$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2}\sin\theta\, d\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$$= [\phi]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\left[-\frac{r}{2}\sqrt{1-r^2}+\frac{1}{2}\sin^{-1}r\right]_0^1$$

$$\left(\because \int \frac{x^2}{\sqrt{1-x^2}}\, dx = -\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\right.$$

$$= \frac{\pi}{2}\cdot 1\cdot \frac{1}{2}\frac{\pi}{2} = \frac{\pi^2}{8}$$

Note :

(1) For the whole volume of the sphere

$x^2+y^2+z^2 = a^2;\ 0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

or $0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$

(2) For the hemisphere above xy-plane,

$0 \le r \le a,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$

Evaluate $\iiint_D \sqrt{x^2+y^2}\, dV$**, where D is the sdid bounded by the surfaces** $x^2+y^2 = z^2,\ z = 0,\ z = 1$**.**

Solution : Draw the figure. Here use cylindrical coordinates (r, θ, z)

$\therefore$ The limits are :

$\sqrt{x^2+y^2} \le z \le 1$

i.e. $r \le z \le 1$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\therefore \quad I = \int_0^{2\pi}\int_0^1\int_r^1 r\cdot r\,dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2\,[z]_r^1\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2(1-r)\, dr\, d\theta$$

$$= [\theta]_0^{2\pi}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1$$

$$= 2\pi\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{\pi}{6}$$

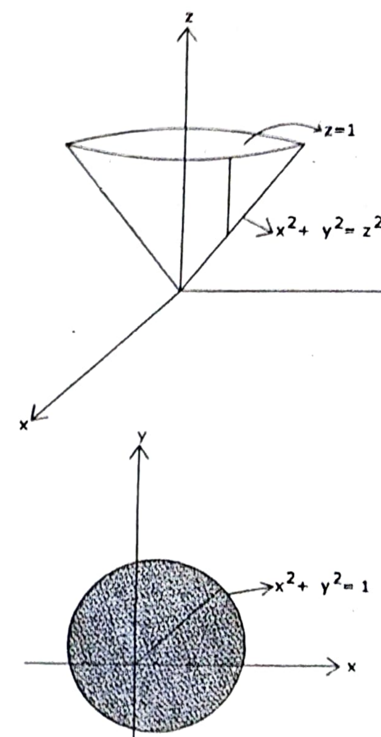

Fig 1.2.77

Evaluate $\iiint_D (x^2+y^2+z^2)\,dV$ **over the volume of the sphere** $x^2+y^2+z^2 = 1$

Solution : Draw the figure. Here we use spherical coordinates (r, θ, ϕ).

$\therefore$ The limits are :

$0 \le r \le 1$
$0 \le \theta \le \pi$
$0 \le \phi\ 2\pi$

$$\therefore \quad I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= [\phi]_0^{2\pi}\ [-\cos\theta]_0^{\pi} \left[\frac{r^5}{5}\right]_0^1$$

$$= 2\pi \cdot 2 \cdot \frac{1}{5} = \frac{4\pi}{5}$$

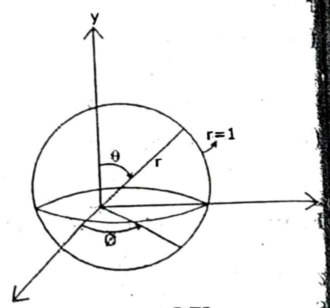

Fig. 1.2.78

5. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{a^2-x^2-y^2} x^2 \, dz\, dy\, dx$

Solution : We use cylinderical coordinates.

Draw the figure from the limits $z = 0, z = a^2 - x^2 - y^2$

$y = 0, \ y = \sqrt{a^2 - x^2}$
$x = 0, \ x = a$
$x^2 + y^2 = -(z - a^2)$

$\therefore$ The limits are :
$0 \le z \le a^2 - r^2$
$0 \le r \le a$
$0 \le \theta \le \pi/2$

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{a}\int_0^{a^2-r^2} r^2 \cos^2\theta \, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} r^3 \cos^2\theta\ [z]_0^{a^2-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{a} r^3 \cos^2\theta\ (a^2 - r^2)\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \, d\theta \cdot \int_0^{a} r^3 (a^2 - r^2)\, dr$$

$$= \frac{1}{2}\,\frac{\pi}{2} \cdot \left[a^2\,\frac{r^4}{4} - \frac{r^6}{6}\right]_0^a$$

$$= \frac{\pi}{4}\left[\frac{a^6}{4} - \frac{a^6}{6}\right] = \frac{\pi}{36}\, a^6$$

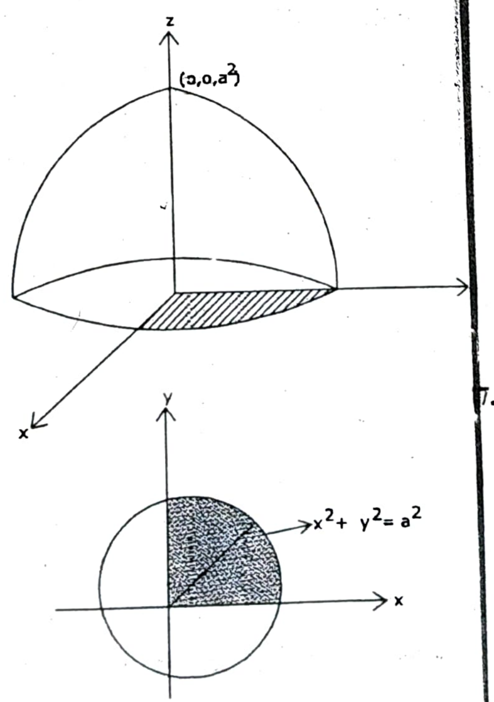

Fig 1.2.79

6. Evaluate $\displaystyle\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}} \, dz\, dy\, dx$

Solution : Here we use spherical coordinates. Draw the figure from the limits.

$z = 0, \ z = \sqrt{1 - x^2 - y^2}\,,$
$y = 0, \ y = \sqrt{1 - x^2}\,,$
$x = -1, \ x = 1$

$\therefore$ The limits are :
$0 \le r \le 1$
$0 \le \theta \le \pi/2$
$0 \le \varphi \le \frac{\pi}{2}$

$$\therefore \ I = \int_0^{\pi}\int_0^{\pi/2}\int_0^{1} e^{-r^3}\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= [\phi]_0^{\pi}\ [-\cos\theta]_0^{\pi/2} \left[\frac{e^{-r^3}}{-3}\right]_0^1$$

$$= \frac{\pi \cdot 1}{3} \cdot (-e^{-1} + 1) = \frac{\pi}{3}\,(1 - e^{-1})$$

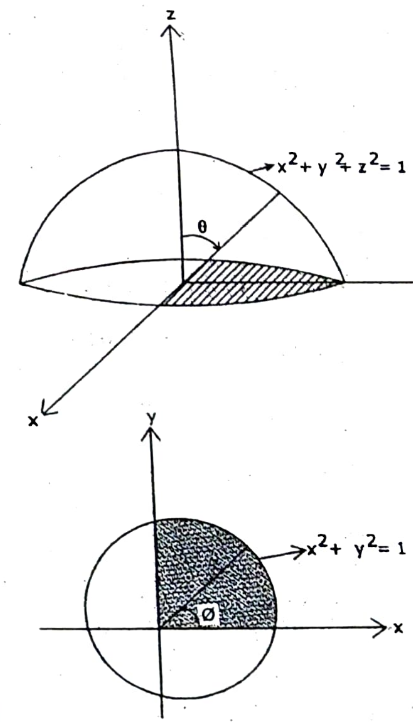

Fig 1.2.80

7. Use an appropriate transformation to evaluate $\iiint_D \sqrt{xyz}\ dV$ **where D is the region in the first octant enclosed by the hyperbolic cylinders** $xy = 1, xy = 2, yz = 1, yz = 3, xz = 1, xz = 4$**.**

Solution : Put $u = xy, \ \ v = yz, \ \ w = xz$

$$\therefore \quad J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} = 2xyz$$

$$\therefore \quad J = \frac{1}{2\,xyz}$$

$\therefore$ The equations becomes
$u = 1, u = 2, v = 1, v = 3, z = 1, z = 4$

$$\therefore \quad I = \int_1^4 \int_1^3 \int_1^2 \sqrt{xyz} \, \frac{1}{2xyz} \, du \, dv \, dw$$

$$= \frac{1}{2} \int_1^4 \int_1^3 \int_1^2 (xyz)^{-1/2} \, du \, dv \, dw$$

$$= \frac{1}{2} \int_1^4 \int_1^3 \int_1^2 [xy \, yz \, zx]^{-1/4} \, du \, dv \, dw$$

$$= \frac{1}{2} \int_1^4 \int_1^3 \int_1^2 [uvw]^{-1/4} \, du \, dv \, dw$$

$$= \frac{1}{2} \left[\frac{w^{3/4}}{3/4}\right]_1^4 \left[\frac{v^{3/4}}{3/4}\right]_1^3 \left[\frac{u^{3/4}}{3/4}\right]_1^2$$

$$= \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \, [2^{3/2} - 1] \, [3^{3/4} - 1] \, [2^{3/4} - 1]$$

$$= \frac{32}{27} [2^{3/2} - 1] \, [3^{3/4} - 1] \, [2^{3/4} - 1]$$

8. Evaluate $\iiint\limits_D x^2 dv$, where D is the region enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution : Put $x = au, \; y = bv, \; z = cw$

$$\therefore \quad J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

The equation becomes $u^2 + v^2 + w^2 = 1$

$$\therefore \quad I = \iiint\limits_E a^2 u^2 \, abc \; du \, dv \, dw$$

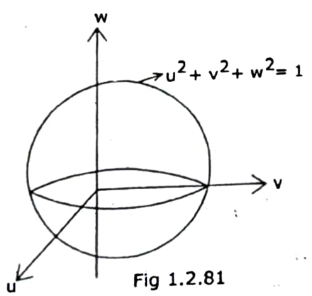

Fig 1.2.81

Now, we use spherical coordinates (r, θ, ϕ)

$\therefore$ The limits are :

$0 \le r \le 1$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$ $\qquad [u = r \sin\theta \cos\phi, \;\; v = r \sin\theta \sin\phi, \;\; w = r \cos\theta]$

$$\therefore \quad I = a^3 bc \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^2 \sin^2\theta \cos^2\phi \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= a^3 bc \int_0^{2\pi} \int_0^{\pi} \int_0^1 r^4 \sin^3\theta \cos^2\phi \, dr \, d\theta \, d\phi$$

$$= a^3 bc \int_0^{2\pi} \cos^2\phi \, d\phi \cdot \int_0^{\pi} \sin^3\theta \, d\theta \int_0^1 r^4 \, dr$$

$$= a^3 bc \; 4 \int_0^{\pi/2} \cos^2\phi \, d\phi \cdot 2 \int_0^{\pi/2} \sin^3\theta \, d\theta \cdot \left[\frac{r^5}{5}\right]_0^1$$

$$= \frac{8a^3 bc}{5} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{2}{3} \cdot 1 = \frac{4\pi a^3 bc}{15}$$

Evaluate $\iiint\limits_D \frac{dv}{(x^2 + y^2 + z^2)^{3/2}}$, where D is the region bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ $(a > b > 0)$.

Solution : Here we use spherical coordinates (r, θ, ϕ).

Draw the figure.

$\therefore$ The limits are :

$b \le r \le a$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$$\therefore \quad I = \int_0^{2\pi} \int_0^{\pi} \int_b^a r^{-3} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_b^a \frac{1}{r} \sin\theta \, dr \, d\theta \, d\phi$$

$$= [\phi]_0^{2\pi} \; [-\cos\theta]_0^{\pi} \; [\log r]_b^a$$

$$= 2\pi \cdot 2 \cdot (\log a - \log b)$$

$$= 4\pi \log \frac{b}{a}$$

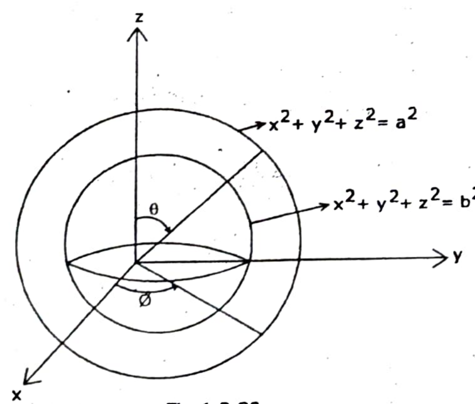

Fig 1.2.82

Prove that $\iiint\limits_D xyz \, (x^2 + y^2 + z^2)^{n/2}$, where D is the positive octant of the sphere

$x^2 + y^2 + z^2 = a^2$ is $\dfrac{a^{n+6}}{8(n+6)}$, where $n + 5 > 0$.

Solution : We use spherical coordinates

$x = r \cos\phi \sin\theta$

$y = r \sin\phi \sin\theta$

$z = r \cos\theta$

$\therefore$ The limits are :

$0 \le r \le a$

$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$

$xyz = r^3 \cos\phi \sin\phi \cos\theta \sin^2\theta$

$= \dfrac{r^3}{2} \sin 2\phi \cos\theta \sin^2\theta$

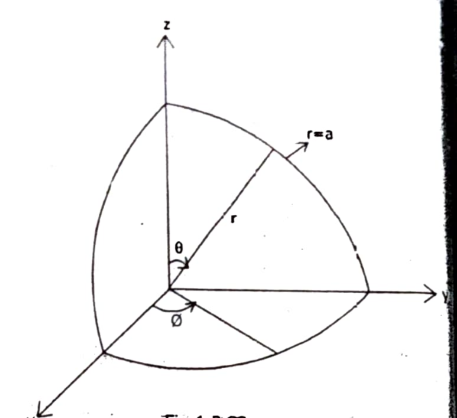

Fig 1.2.83

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \frac{r^3}{2}\sin 2\phi \cos\theta \sin^2\theta\, r^n\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^{n+5} \sin 2\phi \sin^3\theta \cos\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{2}\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\left[\frac{r^{n+6}}{n+6}\right]_0^{a}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{n+6}\cdot 2\cdot 1\cdot a^{n+6}$$

$$= \frac{a^{n+6}}{8(n+6)}$$

11. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}\, dv$, where D is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.

Solution : We use spherical coordinates (r, θ, ϕ)

$x = r \cos\phi \sin\theta$

$y = r \sin\phi \sin\theta$

$z = r \cos\theta$

$\therefore$ The limits are :

$0 \le r \le 3 \sec\theta$

$0 \le \theta \le \pi/4$

$0 \le \phi \le 2\pi$

$$I = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3\sec\theta} r\cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\theta \left[\frac{r^4}{4}\right]_0^{3\sec\theta} d\theta\, d\phi$$

$$= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4} 81 \sin\theta \cdot \sec^4\theta \, d\theta\, d\phi$$

$$= \frac{81}{4}\int_0^{2\pi}\int_0^{\pi/4} (\sec\theta \tan\theta) \sec^2\theta \, d\theta\, d\phi$$

$$= \frac{81}{4}[\phi]_0^{2\pi}\left[\frac{\sec^3\theta}{3}\right]_0^{\pi/4}$$

$$= \frac{27}{4}\cdot 2\pi \cdot 2\sqrt{2} = 27\pi\sqrt{2}$$

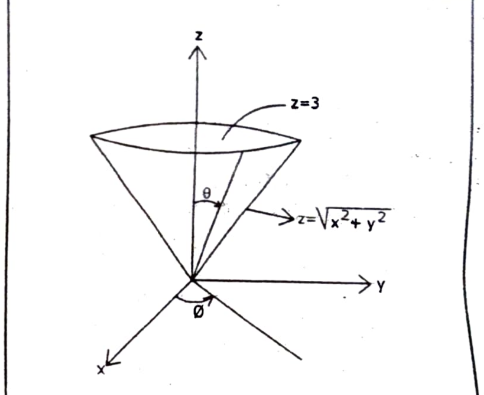

Fig 1.2.84

EXERCISE 1.5

Evaluate $\iiint\limits_D xyz\, dV$, where D is the solid region in the first octant bounded by $x^2 + y^2 + z^2 = 1$ **[Ans. : 1/48]**

Evaluate $\iiint\limits_D \dfrac{dV}{\sqrt{1-x^2-y^2-z^2}}$ taken over the volume of the sphere $x^2 + y^2 + z^2 = 1$, lying in the first octant. **$\left[\text{Ans. : } \dfrac{\pi^2}{8}\right]$**

Evaluate $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2 dz\, dy\, dx$ **$\left[\text{Ans. : } \dfrac{243\pi}{4}\right]$**

Evaluate $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

Use an appropriate transformation to evaluate $\iiint\limits_D (z-y)^2 xy\, dv$ over the region D enclosed by the surfaces $x = 1,\ x = 3,\ z = y,\ z = y + 1,\ xy = 2,\ xy = 4$

[Hint : Put $u = x$, $v = z - y$, $w = xy$] **[Ans. : 2log3]**

Evaluate $\iiint\limits_D \left(\dfrac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right) dv$, where D is the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

■