# Tutorial -3

## Number Theory (Int. M.Sc 5th year)
## Primes and their distribution

① Give an example to show that the following conjecture is not true:
Every positive integer can be written in the form $p+a^2$, where $p$ is either a prime or 1, and $a \geq 0$

② Prove the following assertions

ⓐ Any prime of the form $3n+1$ is also of the form $6m+1$

ⓑ Each integer of the form $3n+2$ has a prime factor of this form.

ⓒ The only prime of the form $n^3-1$ is 7.

ⓓ The only prime $p$ for which $3p+1$ is a perfect square is $p=5$.

ⓔ The only prime of the form $n^2-4$ is 5.

③ If $p \geq 5$ is a prime number, show that $p^2+2$ is composite.

④ ⓐ Given that $p$ is a prime and $p \mid a^n$, prove that $p^n \mid a^n$.

ⓑ If $\gcd(a,b)=p$, a prime, what are the possible values of $\gcd(a^2,b^2)$, $\gcd(a^2,b)$ and $\gcd(a^3,b^2)$?

⑤ Establish the following statements:

ⓐ Every integer of the form $n^4+4$, with $n>1$, is composite.

(b) Any integer of the form $8^n + 1$, where $n \geq 1$ is composite.

(c) Each integer $n > 11$ can be written as the sum of two composite numbers.

(6) Find all prime numbers that divide $50!$

(7) Find a prime which can be expressed as $x^7 - 1$ where $x$ is an integer.

(8) (a) An unanswered question is whether there are infinitely many primes that are 1 more than a power of 2, such as $5 = 2^2 + 1$. Find two more of these primes.

(b) There exist infinitely many primes of the form $n^2 + 1$; for example, $257 = 16^2 + 1$. Exhibit five more primes of this type.

(9) If $p \neq 5$ is an odd prime, prove that either $p^2 - 1$ or $p^2 + 1$ is divisible by 10.

(10) Find the prime factorization of the integers 1234, 10140 and 36000.