Random Processes

.1 INTRODUCTION

In this chapter, we introduce the concept of a random (or stochastic) process. The theory of random processes was first developed in connection with the study of fluctuations and noise in physical systems. A random process is the mathematical model of an empirical process whose development is governed by probability laws. Random processes provides useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth, and management sciences.

.2 RANDOM PROCESSES

1. Defintion:

A *random process* is a family of r.v.'s $\{X(t), t \in T\}$ defined on a given probability space, indexed by the parameter t, where t varies over an index set T.

Recall that a random variable is a function defined on the sample space S . Thus, a random process $\{X(t), t \in T\}$ is really a function of two arguments $\{X(t, \zeta), t \in T, \zeta \in S\}$. For a fixed $t(=t_k)$, $X(t_k, \zeta) = X_k(\zeta)$ is a r.v. denoted by $X(t_k)$, as ζ varies over the sample space S. On the other hand, for a fixed sample point $\zeta_i \in S$, $X(t, \zeta_i) = X_i(t)$ is a single function of time t, called a *sample function* or a *realization* of the process. The totality of all sample functions is called an *ensemble.*

Of course if both ζ and t are fixed, $X(t_k, \zeta_i)$ is simply a real number. In the following we use the notation $X(t)$ to represent $X(t, \zeta)$.

B. Description of a Random Process:

In a random process $\{X(t), t \in T\}$, the index set T is called the *parameter set* of the random process. The values assumed by $X(t)$ are called *states*, and the set of all possible values forms the *state space* E of the random process. If the index set T of a random process is discrete, then the process is called a *discrete-parameter* (or *discrete-time*) process. A discrete-parameter process is also called a *random sequence* and is denoted by $\{X_n, n = 1, 2, \ldots\}$. If T is continuous, then we have a *continuous-parameter* (or *continuous-time*) process. If the state space E of a random process is discrete, then the process is called a *discrete-state* process, often referred to as a *chain*. In this case, the state space E is often assumed to be $\{0, 1, 2, \ldots\}$. If the state space E is continuous, then we have a *continuous-state* process.

A complex random process $X(t)$ is defined by

$$X(t) = X_1(t) + jX_2(t)$$

where $X_1(t)$ and $X_2(t)$ are (real) random processes and $j = \sqrt{-1}$. Throughout this book, all random processes are real random processes unless specified otherwise.

5.3 CHARACTERIZATION OF RANDOM PROCESSES

A. Probabilistic Descriptions:

Consider a random process $X(t)$. For a fixed time t_1, $X(t_1) = X_1$ is a r.v., and its cdf $F_X(x_1; t_1)$ is defined as

$$F_X(x_1; t_1) = P\{X(t_1) \le x_1\} \tag{5.1}$$

$F_X(x_1; t_1)$ is known as the *first-order distribution* of $X(t)$. Similarly, given t_1 and t_2, $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two r.v.'s. Their joint distribution is known as the *second-order distribution* of $X(t)$ and is given by

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\} \tag{5.2}$$

In general, we define the *nth-order distribution* of $X(t)$ by

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\} \tag{5.3}$$

If $X(t)$ is a discrete-time process, then $X(t)$ is specified by a collection of pmf's:

$$p_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) = x_1, \ldots, X(t_n) = x_n\} \tag{5.4}$$

If $X(t)$ is a continuous-time process, then $X(t)$ is specified by a collection of pdf's:

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{\partial x_1 \cdots \partial x_n} \tag{5.5}$$

The complete characterization of $X(t)$ requires knowledge of all the distributions as $n \to \infty$. Fortunately, often much less is sufficient.

B. Mean, Correlation, and Covariance Functions:

As in the case of r.v.'s, random processes are often described by using statistical averages.

The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)] \tag{5.6}$$

where $X(t)$ is treated as a random variable for a fixed value of t. In general, $\mu_X(t)$ is a function of time, and it is often called the *ensemble average* of $X(t)$. A measure of dependence among the r.v.'s of $X(t)$ is provided by its *autocorrelation function*, defined by

$$R_X(t, s) = E[X(t)X(s)] \tag{5.7}$$

Note that

$$R_X(t, s) = R_X(s, t) \tag{5.8}$$

and

$$R_X(t, t) = E[X^2(t)] \tag{5.9}$$

The *autocovariance function* of $X(t)$ is defined by

$$\begin{aligned} K_X(t, s) = \operatorname{Cov}[X(t), X(s)] &= E\{[X(t) - \mu_X(t)][X(s) - \mu_X(s)]\} \\ &= R_X(t, s) - \mu_X(t)\mu_X(s) \end{aligned} \tag{5.10}$$

It is clear that if the mean of $X(t)$ is zero, then $K_X(t, s) = R_X(t, s)$. Note that the *variance* of $X(t)$ is given by

$$\sigma_X{}^2(t) = \operatorname{Var}[X(t)] = E\{[X(t) - \mu_X(t)]^2\} = K_X(t, t) \tag{5.11}$$

If $X(t)$ is a complex random process, then its autocorrelation function $R_X(t, s)$ and autocovariance function $K_X(t, s)$ are defined, respectively, by

$$R_X(t, s) = E[X(t)X^*(s)] \tag{5.12}$$

and

$$K_X(t, s) = E\{[X(t) - \mu_X(t)][X(s) - \mu_X(s)]^*\} \tag{5.13}$$

where * denotes the complex conjugate.

.4 CLASSIFICATION OF RANDOM PROCESSES

If a random process $X(t)$ possesses some special probabilistic structure, we can specify less to characterize $X(t)$ completely. Some simple random processes are characterized completely by only the first- and second-order distributions.

A. Stationary Processes:

A random process $\{X(t), t \in T\}$ is said to be *stationary* or *strict-sense stationary* if, for all n and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n\}$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau) \tag{5.14}$$

for any τ. Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and $X(t)$ and $X(t + \tau)$ will have the same distributions for any τ. Thus, for the first-order distribution,

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x) \tag{5.15}$$

and

$$f_X(x; t) = f_X(x) \tag{5.16}$$

Then

$$\mu_X(t) = E[X(t)] = \mu \tag{5.17}$$

$$\text{Var}[X(t)] = \sigma^2 \tag{5.18}$$

where μ and σ^2 are contants. Similarly, for the second-order distribution,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1) \tag{5.19}$$

and

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1) \tag{5.20}$$

Nonstationary processes are characterized by distributions depending on the points $t_1, t_2, \ldots, t_n$.

B. Wide-Sense Stationary Processes:

If stationary condition (*5.14*) of a random process $X(t)$ does not hold for all n but holds for $n \le k$, then we say that the process $X(t)$ is *stationary to order k*. If $X(t)$ is stationary to order 2, then $X(t)$ is said to be *wide-sense stationary* (WSS) or *weak stationary*. If $X(t)$ is a WSS random process, then we have

1. $E[X(t)] = \mu$ (constant) (*5.21*)
2. $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$ (*5.22*)

Note that a strict-sense stationary process is also a WSS process, but, in general, the converse is not true.

C. Independent Processes:

In a random process $X(t)$, if $X(t_i)$ for $i = 1, 2, \ldots, n$ are independent r.v.'s, so that for $n = 2, 3, \ldots$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \prod_{i=1}^{n} F_X(x_i; t_i) \tag{5.23}$$

then we call $X(t)$ an *independent random process*. Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

D. Processes with Stationary Independent Increments:

A random process $\{X(t), t \ge 0)$ is said to have *independent increments* if whenever $0 < t_1 < t_2 < \cdots < t_n$,

$$X(0), X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent. If $\{X(t), t \geq 0)$ has independent increments and $X(t) - X(s)$ has the same distribution as $X(t + h) - X(s + h)$ for all $s, t, h \geq 0, s < t$, then the process $X(t)$ is said to have *stationary independent increments.*

Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments and assume that $X(0) = 0$. Then (Probs. 5.21 and 5.22)

$$E[X(t)] = \mu_1 t \tag{5.24}$$

where $\mu_1 = E[X(1)]$ and

$$\text{Var}[X(t)] = \sigma_1{}^2 t \tag{5.25}$$

where $\sigma_1{}^2 = \text{Var}[X(1)]$.

From Eq. (5.24), we see that processes with stationary independent increments are nonstationary. Examples of processes with stationary independent increments are Poisson processes and Wiener processes, which are discussed in later sections.

E. Markov Processes:

A random process $\{X(t), t \in T\}$ is said to be a *Markov process* if

$$P\{X(t_{n+1}) \leq x_{n+1} \mid X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n\} = P\{X(t_{n+1}) \leq x_{n+1} \mid X(t_n) = x_n\} \tag{5.26}$$

whenever $t_1 < t_2 < \cdots < t_n < t_{n+1}$.

A discrete-state Markov process is called a *Markov chain*. For a discrete-parameter Markov chain $\{X_n, n \geq 0\}$ (see Sec. 5.5), we have for every n

$$P(X_{n+1} = j \mid X_0 = i_0, X_i = i_1, \ldots, X_n = i) = P(X_{n+1} = j \mid X_n = i) \tag{5.27}$$

Equation (5.26) or Eq. (5.27) is referred to as the *Markov property* (which is also known as the *memoryless property*). This property of a Markov process states that the future state of the process depends only on the present state and not on the past history. Clearly, any process with independent increments is a Markov process.

Using the Markov property, the nth-order distribution of a Markov process $X(t)$ can be expressed as (Prob. 5.25)

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1; t_1) \prod_{k=2}^{n} P\{X(t_k) \leq x_k\} \mid X(t_{k-1}) = x_{k-1}\} \tag{5.28}$$

Thus, all finite-order distributions of a Markov process can be expressed in terms of the second-order distributions.

F. Normal Processes:

A random process $\{X(t), t \in T\}$ is said to be a *normal* (or *gaussian*) process if for any integer n and any subset $\{t_1, \ldots, t_n\}$ of T, the n r.v.'s $X(t_1), \ldots, X(t_n)$ are jointly normally distributed in the sense that their joint characteristic function is given by

$$\begin{aligned}\Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) &= E\{\exp j[\omega_1 X(t_1) + \cdots + \omega_n X(t_n)]\} \\ &= \exp\left\{j \sum_{i=1}^{n} \omega_i E[X(t_i)] - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_i \omega_k \text{Cov}[X(t_i), X(t_k)]\right\}\end{aligned} \tag{5.29}$$

where $\omega_1, \ldots, \omega_n$ are any real numbers (see Probs. 5.59 and 5.60). Equation (5.29) shows that a normal process is completely characterized by the second-order distributions. Thus, if a normal process is wide-sense stationary, then it is also strictly stationary.

G. Ergodic Processes:

Consider a random process $\{X(t), \ -\infty < t < \infty\}$ with a typical sample function $x(t)$. The time average of $x(t)$ is defined as

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt \tag{5.30}$$

Similarly, the time autocorrelation function $\bar{R}_X(\tau)$ of $x(t)$ is defined as

$$\bar{R}_X(\tau) = \langle x(t)x(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt \tag{5.31}$$

A random process is said to be *ergodic* if it has the property that the time averages of sample functions of the process are equal to the corresponding statistical or ensemble averages. The subject of *ergodicity* is extremely complicated. However, in most physical applications, it is assumed that stationary processes are ergodic.

.7 WIENER PROCESSES

Another example of random processes with independent stationary increments is a *Wiener process.*

DEFINITION 5.7.1

A random process $\{X(t), t \geq 0\}$ is called a Wiener process if

1. $X(t)$ has stationary independent increments.
2. The increment $X(t) - X(s)$ $(t > s)$ is normally distributed.
3. $E[X(t)] = 0$.
4. $X(0) = 0$.

The Wiener process is also known as the *Brownian motion process,* since it originates as a model for Brownian motion, the motion of particles suspended in a fluid. From Def. 5.7.1, we can verify that a Wiener process is a normal process (Prob. 5.61) and

$$E[X(t)] = 0 \tag{5.62}$$

$$\mathrm{Var}[X(t)] = \sigma^2 t \tag{5.63}$$

where σ^2 is a parameter of the Wiener process which must be determined from observations. When $\sigma^2 = 1$, $X(t)$ is called a *standard* Wiener (or standard Brownian motion) process.

The autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Wiener process $X(t)$ are given by (see Prob. 5.23)

$$R_X(t, s) = K_X(t, s) = \sigma^2 \min(t, s) \qquad s, t \geq 0 \tag{5.64}$$

DEFINITION 5.7.2

A random process $\{X(t), t \geq 0\}$ is called a *Wiener process with drift coefficient* μ if

1. $X(t)$ has stationary independent increments.
2. $X(t)$ is normally distributed with mean μt.
3. $X(0) = 0$.

From condition 2, the pdf of a standard Wiener process with drift coefficient μ is given by

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-\mu t)^2/(2t)} \tag{5.65}$$

Solved Problems

RANDOM PROCESSES

5.1. Let $X_1, X_2, \ldots$ be independent Bernoulli r.v.'s (Sec. 2.7A) with $P(X_n = 1) = p$ and $P(X_n = 0) = q = 1 - p$ for all n. The collection of r.v.'s $\{X_n, n \geq 1\}$ is a random process, and it is called a *Bernoulli process.*

(*a*) Describe the Bernoulli process.

(*b*) Construct a typical sample sequence of the Bernoulli process.

(*a*) The Bernoulli process $\{X_n, n \geq 1\}$ is a discrete-parameter, discrete-state process. The state space is $E = \{0, 1\}$, and the index set is $T = \{1, 2, \ldots\}$.

(*b*) A sample sequence of the Bernoulli process can be obtained by tossing a coin consecutively. If a head appears, we assign 1, and if a tail appears, we assign 0. Thus, for instance,

n	1	2	3	4	5	6	7	8	9	10	⋯
Coin tossing	H	T	T	H	H	H	T	H	H	T	⋯
x_n	1	0	0	1	1	1	0	1	1	0	⋯

The sample sequence $\{x_n\}$ obtained above is plotted in Fig. 5-3.

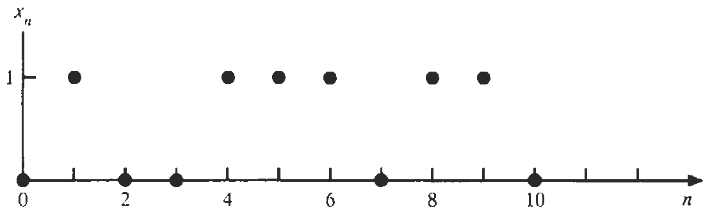

Fig. 5-3 A sample function of a Bernoulli process.

5.2. Let $Z_1, Z_2, \ldots$ be independent identically distributed r.v.'s with $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all n. Let

$$X_n = \sum_{i=1}^{n} Z_i \qquad n = 1, 2, \ldots \tag{5.66}$$

and $X_0 = 0$. The collection of r.v.'s $\{X_n, n \geq 0\}$ is a random process, and it is called the *simple random walk* $X(n)$ in one dimension.

(*a*) Describe the simple random walk $X(n)$.

(*b*) Construct a typical sample sequence (or realization) of $X(n)$.

(*a*) The simple random walk $X(n)$ is a discrete-parameter (or time), discrete-state random process. The state space is $E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and the index parameter set is $T = \{0, 1, 2, \ldots\}$.

(*b*) A sample sequence $x(n)$ of a simple random walk $X(n)$ can be produced by tossing a coin every second and letting $x(n)$ increase by unity if a head appears and decrease by unity if a tail appears. Thus, for instance,

n	0	1	2	3	4	5	6	7	8	9	10	⋯
Coin tossing		H	T	T	H	H	H	T	H	H	T	⋯
$x(n)$	0	1	0	−1	0	1	2	1	2	3	2	⋯

The sample sequence $x(n)$ obtained above is plotted in Fig. 5-4. The simple random walk $X(n)$ specified in this problem is said to be *unrestricted* because there are no bounds on the possible values of X_n.

The simple random walk process is often used in the following primitive gambling model: Toss a coin. If a head appears, you win one dollar; if a tail appears, you lose one dollar (see Prob. 5.38).

5.3. Let $\{X_n, n \geq 0\}$ be a simple random walk of Prob. 5.2. Now let the random process $X(t)$ be defined by

$$X(t) = X_n \qquad n \leq t < n + 1$$

(*a*) Describe $X(t)$.

(*b*) Construct a typical sample function of $X(t)$.

(*a*) The random process $X(t)$ is a continuous-parameter (or time), discrete-state random process. The state space is $E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and the index parameter set is $T = \{t, t \geq 0\}$.

(*b*) A sample function $x(t)$ of $X(t)$ corresponding to Fig. 5-4 is shown in Fig. 5-5.

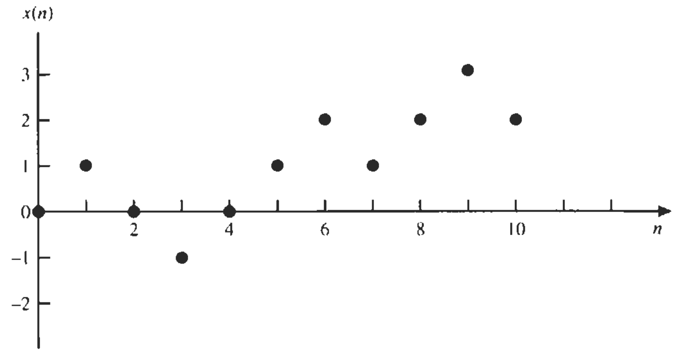

Fig. 5-4 A sample function of a random walk.

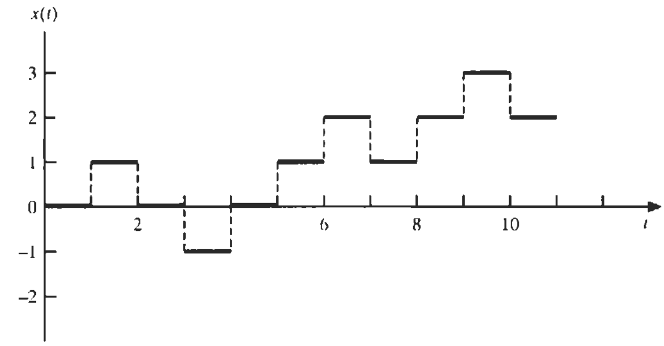

Fig. 5-5

5.4. Consider a random process $X(t)$ defined by

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

where ω is a constant and Y is a uniform r.v. over (0, 1).

(*a*) Describe $X(t)$.

(*b*) Sketch a few typical sample functions of $X(t)$.

(*a*) The random process $X(t)$ is a continuous-parameter (or time), continuous-state random process. The state space is $E = \{x: -1 < x < 1\}$ and the index parameter set is $T = \{t: t \geq 0\}$.

(*b*) Three sample functions of $X(t)$ are sketched in Fig. 5-6.

5.5. Consider patients coming to a doctor's office at random points in time. Let X_n denote the time (in hours) that the nth patient has to wait in the office before being admitted to see the doctor.

(*a*) Describe the random process $X(n) = \{X_n, n \geq 1\}$.

(*b*) Construct a typical sample function of $X(n)$.

(*a*) The random process $X(n)$ is a discrete-parameter, continuous-state random process. The state space is $E = \{x: x \geq 0)$, and the index parameter set is $T = \{1, 2, \ldots\}$.

(*b*) A sample function $x(n)$ of $X(n)$ is shown in Fig. 5-7.

CHARACTERIZATION OF RANDOM PROCESSES

5.6. Consider the Bernoulli process of Prob. 5.1. Determine the probability of occurrence of the sample sequence obtained in part (*b*) of Prob. 5.1.

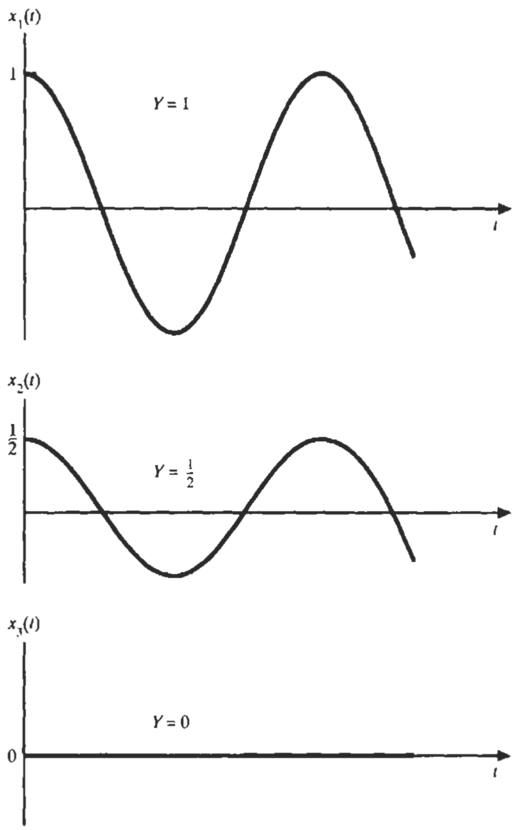

Fig. 5-6

Since X_n's are independent, we have

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_1)P(X_2 = x_2) \cdots P(X_n = x_n) \tag{5.67}$$

Thus, for the sample sequence of Fig. 5-3,

$$P(X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1, X_5 = 1, X_6 = 1, X_7 = 0, X_8 = 1, X_9 = 1, X_{10} = 0) = p^6q^4$$

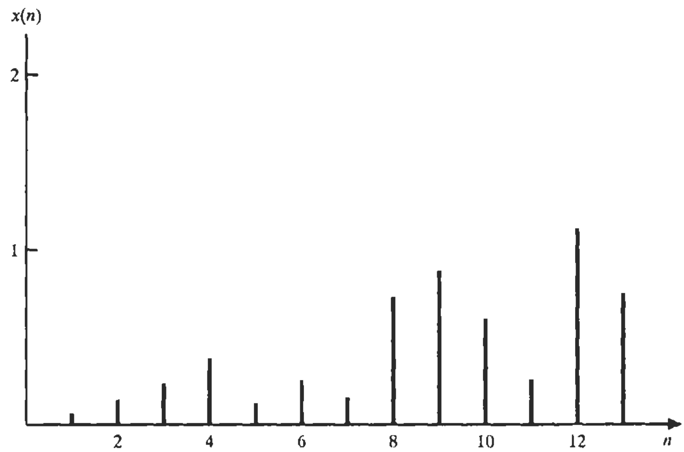

Fig. 5-7

5.7. Consider the random process $X(t)$ of Prob. 5.4. Determine the pdf's of $X(t)$ at $t = 0, \pi/4\omega, \pi/2\omega, \pi/\omega$.

For $t = 0$, $X(0) = Y \cos 0 = Y$. Thus,

$$f_{X(0)}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $t = \pi/4\omega$, $X(\pi/4\omega) = Y \cos \pi/4 = 1/\sqrt{2}\, Y$. Thus,

$$f_{X(\pi/4\omega)}(x) = \begin{cases} \sqrt{2} & 0 < x < 1/\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

For $t = \pi/2\omega$, $X(\pi/2\omega) = Y \cos \pi/2 = 0$; that is, $X(\pi/2\omega) = 0$ irrespective of the value of Y. Thus, the pmf of $X(\omega/2\omega)$ is

$$p_{X(\pi/2\omega)}(x) = P(X = 0) = 1$$

For $t = \pi/\omega$, $X(\pi/\omega) = Y \cos \pi = -Y$. Thus,

$$f_{X(\pi/\omega)}(x) = \begin{cases} 1 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

5.8. Derive the first-order probability distribution of the simple random walk $X(n)$ of Prob. 5.2.

The first-order probability distribution of the simple random walk $X(n)$ is given by

$$p_n(k) = P(X_n = k)$$

where k is an integer. Note that $P(X_0 = 0) = 1$. We note that $p_n(k) = 0$ if $n < |k|$ because the simple random walk cannot get to level k in less than $|k|$ steps. Thus, $n \geq |k|$.

Let N_n^+ and N_n^- be the r.v.'s denoting the numbers of $+1$s and -1s, respectively, in the first n steps. Then

$$n = N_n^+ + N_n^- \tag{5.68}$$

$$X_n = N_n^+ - N_n^- \tag{5.69}$$

Adding Eqs. (5.68) and (5.69), we get

$$N_n^+ = \tfrac{1}{2}(n + X_n) \tag{5.70}$$

Thus, $X_n = k$ if and only if $N_n^+ = \frac{1}{2}(n + k)$. From Eq. (5.70), we note that $2N_n^+ = n + X_n$ must be even. Thus, X_n must be even if n is even, and X_n must be odd if n is odd. We note that N_n^+ is a binomial r.v. with parameters (n, p). Thus, by Eq. (2.36), we obtain

$$p_n(k) = \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2} \qquad q = 1 - p \tag{5.71}$$

where $n \geq |k|$, and n and k are either both even or both odd.

5.9. Consider the simple random walk $X(n)$ of Prob. 5.2.

(*a*) Find the probability that $X(n) = -2$ after four steps.

(*b*) Verify the result of part (*a*) by enumerating all possible sample sequences that lead to the value $X(n) = -2$ after four steps.

(*a*) Setting $k = -2$ and $n = 4$ in Eq. (5.71), we obtain

$$P(X_4 = -2) = p_4(-2) = \binom{4}{1} pq^3 = 4pq^3 \qquad q = 1 - p$$

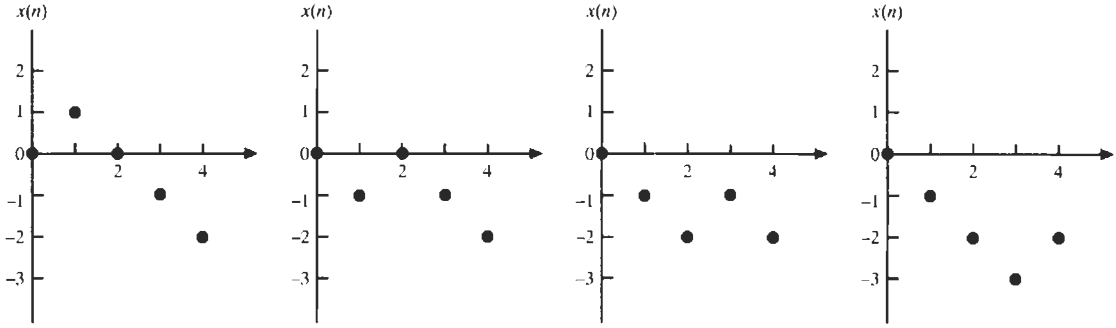

Fig. 5-8

(*b*) All possible sample functions that lead to the value $X_4 = -2$ after 4 steps are shown in Fig. 5-8. For each sample sequence, $P(X_4 = -2) = pq^3$. There are only four sample functions that lead to the value $X_4 = -2$ after four steps. Thus $P(X_4 = -2) = 4pq^3$.

5.10 Find the mean and variance of the simple random walk $X(n)$ of Prob. 5.2.

From Eq. (5.66), we have

$$X_n = X_{n-1} + Z_n \qquad n = 1, 2, \ldots \tag{5.72}$$

and $X_0 = 0$ and Z_n $(n = 1, 2, \ldots)$ are independent and identically distributed (iid) r.v.'s with

$$P(Z_n = +1) = p \qquad P(Z_n = -1) = q = 1 - p$$

From Eq. (5.72), we observe that

$$\begin{aligned} X_1 &= X_0 + Z_1 = Z_1 \\ X_2 &= X_1 + Z_2 = Z_1 + Z_2 \\ &\vdots \\ X_n &= Z_1 + Z_2 + \cdots + Z_n \end{aligned} \tag{5.73}$$

Then, because the Z_n are iid r.v.'s and $X_0 = 0$, by Eqs. (4.108) and (4.112), we have

$$E(X_n) = E\left(\sum_{k=1}^{n} Z_k\right) = nE(Z_k)$$

$$\text{Var}(X_n) = \text{Var}\left(\sum_{k=1}^{n} Z_k\right) = n \,\text{Var}(Z_k)$$

Now
$$E(Z_k) = (1)p + (-1)q = p - q \tag{5.74}$$

$$E(Z_k^2) = (1)^2 p + (-1)^2 q = p + q = 1 \tag{5.75}$$

Thus
$$\text{Var}(Z_k) = E(Z_k^2) - [E(Z_k)]^2 = 1 - (p - q)^2 = 4pq \tag{5.76}$$

Hence,
$$E(X_n) = n(p - q) \qquad q = 1 - p \tag{5.77}$$

$$\text{Var}(X_n) = 4npq \qquad q = 1 - p \tag{5.78}$$

Note that if $p = q = \frac{1}{2}$, then

$$E(X_n) = 0 \tag{5.79}$$

$$\text{Var}(X_n) = n \tag{5.80}$$

5.11. Find the autocorrelation function $R_X(n, m)$ of the simple random walk $X(n)$ of Prob. 5.2.

From Eq. (5.73), we can express X_n as

$$X_n = \sum_{i=0}^{n} Z_i \qquad n = 1, 2, \ldots \tag{5.81}$$

where $Z_0 = X_0 = 0$ and Z_i ($i \geq 1$) are iid r.v.'s with

$$P(Z_i = +1) = p \qquad P(Z_i = -1) = q = 1 - p$$

By Eq. (5.7),

$$R_X(n, m) = E[X(n)X(m)] = E(X_n X_m)$$

Then by Eq. (5.81),

$$R_X(n, m) = \sum_{i=0}^{n} \sum_{k=0}^{m} E(Z_i Z_k) = \sum_{i=0}^{\min(n, m)} E(Z_i^2) + \sum_{\substack{i=0 \\ i \neq k}}^{n} \sum_{k=0}^{m} E(Z_i)E(Z_k) \tag{5.82}$$

Using Eqs. (5.74) and (5.75), we obtain

$$R_X(n, m) = \min(n, m) + [nm - \min(n, m)](p - q)^2 \tag{5.83}$$

or

$$R_X(n, m) = \begin{cases} m + (nm - m)(p - q)^2 & m < n \\ n + (nm - n)(p - q)^2 & n < m \end{cases} \tag{5.84}$$

Note that if $p = q = \frac{1}{2}$, then

$$R_X(n, m) = \min(n, m) \qquad n, m > 0 \tag{5.85}$$

5.12. Consider the random process $X(t)$ of Prob. 5.4; that is,

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

where ω is a constant and Y is a uniform r.v. over (0, 1).

(a) Find $E[X(t)]$.

(b) Find the autocorrelation function $R_X(t, s)$ of $X(t)$.

(c) Find the autocovariance function $K_X(t, s)$ of $X(t)$.

(a) From Eqs. (2.46) and (2.91), we have $E(Y) = \frac{1}{2}$ and $E(Y^2) = \frac{1}{3}$. Thus

$$E[X(t)] = E(Y \cos \omega t) = E(Y) \cos \omega t = \tfrac{1}{2} \cos \omega t \tag{5.86}$$

(b) By Eq. (5.7), we have

$$\begin{aligned} R_X(t, s) &= E[X(t)X(s)] = E(Y^2 \cos \omega t \cos \omega s) \\ &= E(Y^2) \cos \omega t \cos \omega s = \tfrac{1}{3} \cos \omega t \cos \omega s \end{aligned} \tag{5.87}$$

(c) By Eq. (5.10), we have

$$\begin{aligned} K_X(t, s) &= R_X(t, s) - E[X(t)]E[X(s)] \\ &= \tfrac{1}{3} \cos \omega t \cos \omega s - \tfrac{1}{4} \cos \omega t \cos \omega s \\ &= \tfrac{1}{12} \cos \omega t \cos \omega s \end{aligned} \tag{5.88}$$

5.13. Consider a discrete-parameter random process $X(n) = \{X_n, n \geq 1\}$ where the X_n's are iid r.v.'s with common cdf $F_X(x)$, mean μ, and variance σ^2.

(a) Find the joint cdf of $X(n)$.

(b) Find the mean of $X(n)$.

(c) Find the autocorrelation function $R_X(n, m)$ of $X(n)$.

(d) Find the autocovariance function $K_X(n, m)$ of $X(n)$.

(a) Since the X_n's are iid r.v.'s with common cdf $F_X(x)$, the joint cdf of $X(n)$ is given by

$$F_X(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_X(x_i) = [F_X(x)]^n \tag{5.89}$$

(b) The mean of $X(n)$ is

$$\mu_X(n) = E(X_n) = \mu \qquad \text{for all } n \tag{5.90}$$

(c) If $n \neq m$, by Eqs. (5.7) and (5.90),

$$R_X(n, m) = E(X_n X_m) = E(X_n)E(X_m) = \mu^2$$

If $n = m$, then by Eq. (2.31),

$$E(X_n^2) = \text{Var}(X_n) + [E(X_n)]^2 = \sigma^2 + \mu^2$$

Hence,

$$R_X(n, m) = \begin{cases} \mu^2 & n \neq m \\ \sigma^2 + \mu^2 & n = m \end{cases} \tag{5.91}$$

(d) By Eq. (5.10),

$$K_X(n, m) = R_X(n, m) - \mu_X(n)\mu_X(m) = \begin{cases} 0 & n \neq m \\ \sigma^2 & n = m \end{cases} \tag{5.92}$$

CLASSIFICATION OF RANDOM PROCESSES

5.14. Show that a random process which is stationary to order n is also stationary to all orders lower than n.

Assume that Eq. (5.14) holds for some particular n; that is,

$$P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\} = P\{X(t_1 + \tau) \le x_1, \ldots, X(t_n + \tau) \le x_n\}$$

for any τ. Letting $x_n \to \infty$, we have [see Eq. (3.63)]

$$P\{X(t_1) \le x_1, \ldots, X(t_{n-1}) \le x_{n-1}\} = P\{X(t_1 + \tau) \le x_1, \ldots, X(t_{n-1} + \tau) \le x_{n-1}\}$$

and the process is stationary to order $n - 1$. Continuing the same procedure, we see that the process is stationary to all orders lower than n.

5.15. Show that if $\{X(t), t \in T\}$ is a strict-sense stationary random process, then it is also WSS.

Since $X(t)$ is strict-sense stationary, the first- and second-order distributions are invariant through time translation for all $\tau \in T$. Then we have

$$\mu_X(t) = E[X(t)] = E[X(t + \tau)] = \mu_X(t + \tau)$$

and hence the mean function $\mu_X(t)$ must be constant; that is,

$$E[X(t)] = \mu \text{ (constant)}$$

Similarly, we have

$$E[X(s)X(t)] = E[X(s + \tau)X(t + \tau)]$$

so that the autocorrelation function would depend on the time points s and t only through the difference $|t - s|$. Thus, $X(t)$ is WSS.

5.16. Let $\{X_n, n \ge 0\}$ be a sequence of iid r.v.'s with mean 0 and variance 1. Show that $\{X_n, n \ge 0\}$ is a WSS process.

By Eq. (5.90),

$$E(X_n) = 0 \text{ (constant)} \qquad \text{for all } n$$

and by Eq. (5.91),

$$R_X(n, n+k) = E(X_n X_{n+k}) = \begin{cases} E(X_n)E(X_{n+k}) = 0 & k \neq 0 \\ E(X_n^2) = \mathrm{Var}(X_n) = 1 & k = 0 \end{cases}$$

which depends only on k. Thus, $\{X_n\}$ is a WSS process.

5.17. Show that if a random process $X(t)$ is WSS, then it must also be covariance stationary.

If $X(t)$ is WSS, then

$$E[X(t)] = \mu \text{ (constant)} \qquad \text{for all } t$$
$$R_X(t, t+\tau)] = R_X(\tau) \qquad \text{for all } t$$

Now
$$K_X(t, t+\tau) = \mathrm{Cov}[X(t)X(t+\tau)] = R_X(t, t+\tau) - E[X(t)]E[X(t+\tau)]$$
$$= R_X(\tau) - \mu^2$$

which indicates that $K_X(t, t+\tau)$ depends only on τ; thus, $X(t)$ is covariance stationary.

5.18. Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t \qquad -\infty < t < \infty \tag{5.93}$$

where ω is constant and U and V are r.v.'s.

(a) Show that the condition

$$E(U) = E(V) = 0 \tag{5.94}$$

is necessary for $X(t)$ to be stationary.

(b) Show that $X(t)$ is WSS if and only if U and V are uncorrelated with equal variance; that is,

$$E(UV) = 0 \qquad E(U^2) = E(V^2) = \sigma^2 \tag{5.95}$$

(a) Now

$$\mu_X(t) = E[X(t)] = E(U) \cos \omega t + E(V) \sin \omega t$$

must be independent of t for $X(t)$ to be stationary. This is possible only if $\mu_X(t) = 0$, that is, $E(U) = E(V) = 0$.

(b) If $X(t)$ is WSS, then

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X^2$$

But $X(0) = U$ and $X(\pi/2\omega) = V$; thus

$$E(U^2) = E(V^2) = \sigma_X^2 = \sigma^2$$

Using the above result, we obtain

$$\begin{aligned} R_X(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\{(U \cos \omega t + V \sin \omega t)[U \cos \omega(t+\tau) + V \sin \omega(t+\tau)]\} \\ &= \sigma^2 \cos \omega\tau + E(UV) \sin(2\omega t + \omega\tau) \end{aligned} \tag{5.96}$$

which will be a function of τ only if $E(UV) = 0$. Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part (a) and Eq. (5.96), we have

$$\mu_X(t) = 0$$
$$R_X(t, t+\tau) = \sigma^2 \cos \omega\tau = R_X(\tau)$$

Hence, $X(t)$ is WSS.

5.19. Consider a random process $X(t)$ defined by

$$X(t) = U \cos t + V \sin t \qquad -\infty < t < \infty$$

where U and V are independent r.v.'s, each of which assumes the values -2 and 1 with the probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Show that $X(t)$ is WSS but not strict-sense stationary.

We have

$$E(U) = E(V) = \tfrac{1}{3}(-2) + \tfrac{2}{3}(1) = 0$$

$$E(U^2) = E(V^2) = \tfrac{1}{3}(-2)^2 + \tfrac{2}{3}(1)^2 = 2$$

Since U and V are independent,

$$E(UV) = E(U)E(V) = 0$$

Thus, by the results of Prob. 5.18, $X(t)$ is WSS. To see if $X(t)$ is strict-sense stationary, we consider $E[X^3(t)]$.

$$\begin{aligned} E[X^3(t)] &= E[(U \cos t + V \sin t)^3] \\ &= E(U^3) \cos^3 t + 3E(U^2 V) \cos^2 t \sin t + 3E(UV^2) \cos t \sin^2 t + E(V^3) \sin^3 t \end{aligned}$$

Now
$$E(U^3) = E(V^3) = \tfrac{1}{3}(-2)^3 + \tfrac{2}{3}(1)^3 = -2$$
$$E(U^2 V) = E(U^2)E(V) = 0 \qquad E(UV^2) = E(U)E(V^2) = 0$$

Thus
$$E[X^3(t)] = -2(\cos^3 t + \sin^3 t)$$

which is a function of t. From Eq. (*5.16*), we see that all the moments of a strict-sense stationary process must be independent of time. Thus $X(t)$ is not strict-sense stationary.

5.20. Consider a random process $X(t)$ defined by

$$X(t) = A \cos(\omega t + \Theta) \qquad -\infty < t < \infty$$

where A and ω are constants and Θ is a uniform r.v. over $(-\pi, \pi)$. Show that $X(t)$ is WSS.

From Eq. (*2.44*), we have

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Then
$$\mu_X(t) = \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0 \tag{5.97}$$

Setting $s = t + \tau$ in Eq. (5.7), we have

$$\begin{aligned} R_{XX}(t, t + \tau) &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta) \cos[\omega(t + \tau) + \theta]\, d\theta \\ &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} [\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]\, d\theta \\ &= \frac{A^2}{2} \cos \omega\tau \end{aligned} \tag{5.98}$$

Since the mean of $X(t)$ is a constant and the autocorrelation of $X(t)$ is a function of time difference only, we conclude that $X(t)$ is WSS.

5.21. Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

$$E[X(t)] = \mu_1 t \tag{5.99}$$

where $\mu_1 = E[X(1)]$.

Let
$$f(t) = E[X(t)] = E[X(t) - X(0)]$$

Then, for any t and s and using Eq. (*4.108*) and the property of the stationary independent increments, we have

$$\begin{aligned} f(t+s) &= E[X(t+s) - X(0)] \\ &= E[X(t+s) - X(s) + X(s) - X(0)] \\ &= E[X(t+s) - X(s)] + E[X(s) - X(0)] \\ &= E[X(t) - X(0)] + E[X(s) - X(0)] \\ &= f(t) + f(s) \end{aligned} \tag{5.100}$$

The only solution to the above functional equation is $f(t) = ct$, where c is a constant. Since $c = f(1) = E[X(1)]$, we obtain

$$E[X(t)] = \mu_1 t \qquad \mu_1 = E[X(1)]$$

5.22. Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

(*a*)
$$\operatorname{Var}[X(t)] = \sigma_1{}^2 t \tag{5.101}$$

(*b*)
$$\operatorname{Var}[X(t) - X(s)] = \sigma_1{}^2(t - s) \qquad t > s \tag{5.102}$$

where $\sigma_1{}^2 = \operatorname{Var}[X(1)]$.

(*a*) Let
$$g(t) = \operatorname{Var}[X(t)] = \operatorname{Var}[X(t) - X(0)]$$

Then, for any t and s and using Eq. (*4.112*) and the property of the stationary independent increments, we get

$$\begin{aligned} g(t+s) &= \operatorname{Var}[X(t+s) - X(0)] \\ &= \operatorname{Var}[X(t+s) - X(s) + X(s) - X(0)] \\ &= \operatorname{Var}[X(t+s) - X(s)] + \operatorname{Var}[X(s) - X(0)] \\ &= \operatorname{Var}[X(t) - X(0)] + \operatorname{Var}[X(s) - X(0)] \\ &= g(t) + g(s) \end{aligned}$$

which is the same functional equation as Eq. (*5.100*). Thus, $g(t) = kt$, where k is a constant. Since $k = g(1) = \operatorname{Var}[X(1)]$, we obtain

$$\operatorname{Var}[X(t)] = \sigma_1{}^2 t \qquad \sigma_1{}^2 = \operatorname{Var}[X(1)]$$

(*b*) Let $t > s$. Then

$$\begin{aligned} \operatorname{Var}[X(t)] &= \operatorname{Var}[X(t) - X(s) + X(s) - X(0)] \\ &= \operatorname{Var}[X(t) - X(s)] + \operatorname{Var}[X(s) - X(0)] \\ &= \operatorname{Var}[X(t) - X(s)] + \operatorname{Var}[X(s)] \end{aligned}$$

Thus, using Eq. (*5.101*), we obtain

$$\operatorname{Var}[X(t) - X(s)] = \operatorname{Var}[X(t)] - \operatorname{Var}[X(s)] = \sigma_1{}^2(t - s)$$

5.23. Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

$$\operatorname{Cov}[X(t), X(s)] = K_X(t, s) = \sigma_1{}^2 \min(t, s) \tag{5.103}$$

where $\sigma_1{}^2 = \operatorname{Var}[X(1)]$.

By definition (2.28),

$$\begin{aligned}\operatorname{Var}[X(t) - X(s)] &= E(\{X(t) - X(s) - E[X(t) - X(s)]\}^2) \\ &= E[(\{X(t) - E[X(t)]\} - \{X(s) - E[X(s)]\})^2] \\ &= E(\{X(t) - E[X(t)]\}^2 - 2\{X(t) - E[X(t)]\}\{X(s) - E[X(s)]\} + \{X(s) - E[X(s)]\}^2) \\ &= \operatorname{Var}[X(t)] - 2\operatorname{Cov}[X(t), X(s)] + \operatorname{Var}[X(s)]\end{aligned}$$

Thus,
$$\operatorname{Cov}[X(t), X(s)] = \tfrac{1}{2}\{\operatorname{Var}[X(t)] + \operatorname{Var}[X(s)] - \operatorname{Var}[X(t) - X(s)]\}$$

Using Eqs. (5.101) and (5.102), we obtain

$$K_X(t, s) = \begin{cases} \tfrac{1}{2}\sigma_1{}^2[t + s - (t - s)] = \sigma_1{}^2 s & t > s \\ \tfrac{1}{2}\sigma_1{}^2[t + s - (s - t)] = \sigma_1{}^2 t & s > t \end{cases}$$

or
$$K_X(t, s) = \sigma_1{}^2 \min(t, s)$$

where $\sigma_1{}^2 = \operatorname{Var}[X(1)]$.

5.24. (a) Show that a simple random walk $X(n)$ of Prob. 5.2 is a Markov chain.

(b) Find its one-step transition probabilities.

(a) From Eq. (5.73) (Prob. 5.10), $X(n) = \{X_n, n \geq 0\}$ can be expressed as

$$X_0 = 0 \qquad X_n = \sum_{i=1}^{n} Z_i \qquad n \geq 1$$

where Z_n ($n = 1, 2, \ldots$) are iid r.v.'s with

$$P(Z_n = k) = a_k \qquad (k = 1, -1) \qquad \text{and} \qquad a_1 = p \qquad a_{-1} = q = 1 - p$$

Then $X(n) = \{X_n, n \geq 0\}$ is a Markov chain, since

$$\begin{aligned}&P(X_{n+1} = i_{n+1} \mid X_0 = 0, X_1 = i_1, \ldots, X_n = i_n) \\ &\quad = P(Z_{n+1} + i_n = i_{n+1} \mid X_0 = 0, X_1 = i_1, \ldots, X_n = i_n) \\ &\quad = P(Z_{n+1} = i_{n+1} - i_n) = a_{i_{n+1} - i_n} = P(X_{n+1} = i_{n+1} \mid X_n = i_n)\end{aligned}$$

since Z_{n+1} is independent of $X_0, X_1, \ldots, X_n$.

(b) The one-step transition probabilities are given by

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} p & k = j + 1 \\ q = 1 - p & k = j - 1 \\ 0 & \text{otherwise} \end{cases}$$

which do not depend on n. Thus, a simple random walk $X(n)$ is a homogeneous Markov chain.

5.25. Show that for a Markov process $X(t)$, the second-order distribution is sufficient to characterize $X(t)$.

Let $X(t)$ be a Markov process with the nth-order distribution

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\}$$

Then, using the Markov property (5.26), we have

$$\begin{aligned}F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) &= P\{X(t_n) \leq x_n \mid X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_{n-1}) \leq x_{n-1}\} \\ &\quad \times P\{X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_{n-1}) \leq x_{n-1}\} \\ &= P\{X(t_n) \leq x_n \mid X(t_{n-1}) \leq x_{n-1}\} F_X(x_1, \ldots, x_{n-1}; t_1, \ldots, t_{n-1})\end{aligned}$$

Applying the above relation repeatedly for lower-order distribution, we can write

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_X(x_1, t_1) \prod_{k=2}^{n} P\{X(t_k) \leq x_k \mid X(t_{k-1}) \leq x_{k-1}\} \tag{5.104}$$

Hence, all finite-order distributions of a Markov process can be completely determined by the second-order distribution.

5.26. Show that if a normal process is WSS, then it is also strict-sense stationary.

By Eq. (5.29), a normal random process $X(t)$ is completely characterized by the specification of the mean $E[X(t)]$ and the covariance function $K_X(t, s)$ of the process. Suppose that $X(t)$ is WSS. Then, by Eqs. (5.21) and (5.22), Eq. (5.29) becomes

$$\Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) = \exp\left\{j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X(t_i - t_k)\omega_i \omega_k\right\} \tag{5.105}$$

Now we translate all of the time instants $t_1, t_2, \ldots, t_n$ by the same amount τ. The joint characteristic function of the new r.v.'s $X(t_i + \tau)$, $i = 1, 2, \ldots, n$, is then

$$\begin{aligned}
\Psi_{X(t_1+\tau)\cdots X(t_n+\tau)}(\omega_1, \ldots, \omega_n) &= \exp\left\{j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X[t_i + \tau - (t_k + \tau)]\omega_i \omega_k\right\} \\
&= \exp\left\{j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X(t_i - t_k)\omega_i \omega_k\right\} \\
&= \Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n)
\end{aligned} \tag{5.106}$$

which indicates that the joint characteristic function (and hence the corresponding joint pdf) is unaffected by a shift in the time origin. Since this result holds for any n and any set of time instants ($t_i \in T$, $i = 1, 2, \ldots, n$), it follows that if a normal process is WSS, then it is also strict-sense stationary.

5.27. Let $\{X(t), -\infty < t < \infty\}$ be a zero-mean, stationary, normal process with the autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & -T \le \tau \le T \\ 0 & \text{otherwise} \end{cases} \tag{5.107}$$

Let $\{X(t_i), i = 1, 2, \ldots, n\}$ be a sequence of n samples of the process taken at the time instants

$$t_i = i\frac{T}{2} \qquad i = 1, 2, \ldots, n$$

Find the mean and the variance of the sample mean

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X(t_i) \tag{5.108}$$

Since $X(t)$ is zero-mean and stationary, we have

$$E[X(t_i)] = 0$$

and
$$R_X(t_i, t_k) = E[X(t_i)X(t_k)] = R_X(t_k - t_i) = R_X\left[(k - i)\frac{T}{2}\right]$$

Thus
$$E(\hat{\mu}_n) = E\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right] = \frac{1}{n} \sum_{i=1}^{n} E[X(t_i)] = 0 \tag{5.109}$$

and
$$\begin{aligned}
\operatorname{Var}(\hat{\mu}_n) &= E\{[\hat{\mu}_n - E(\hat{\mu}_n)]^2\} = E(\hat{\mu}_n^2) \\
&= E\left\{\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right]\left[\frac{1}{n} \sum_{k=1}^{n} X(t_k)\right]\right\} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} E[X(t_i)X(t_k)] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} R_X\left[(k - i)\frac{T}{2}\right]
\end{aligned}$$

By Eq. (5.107),

$$R_X[(k-i)T/2] = \begin{cases} 1 & k = i \\ \frac{1}{2} & |k-i| = 1 \\ 0 & |k-i| > 2 \end{cases}$$

Thus

$$\operatorname{Var}(\hat{\mu}_n) = \frac{1}{n^2}\left[n(1) + 2(n-1)(\tfrac{1}{2}) + 0\right] = \frac{1}{n^2}(2n-1) \tag{5.110}$$

Chapter 6

Analysis and Processing of Random Processes

6.1 INTRODUCTION

In this chapter, we introduce the methods for analysis and processing of random processes. First, we introduce the definitions of stochastic continuity, stochastic derivatives, and stochastic integrals of random processes. Next, the notion of power spectral density is introduced. This concept enables us to study wide-sense stationary processes in the frequency domain and define a white noise process. The response of linear systems to random processes is then studied. Finally, orthogonal and spectral representations of random processes are presented.

6.2 CONTINUITY, DIFFERENTIATION, INTEGRATION

In this section, we shall consider only the continuous-time random processes.

A. Stochastic Continuity:

A random process $X(t)$ is said to be *continuous in mean square* or *mean square (m.s.) continuous* if

$$\lim_{\varepsilon \to 0} E\{[X(t+\varepsilon) - X(t)]^2\} = 0 \tag{6.1}$$

The random process $X(t)$ is m.s. continuous if and only if its autocorrelation function is continuous (Prob. 6.1). If $X(t)$ is WSS, then it is m.s. continuous if and only if its autocorrelation function $R_X(\tau)$ is continuous at $\tau = 0$. If $X(t)$ is m.s. continuous, then its mean is continuous; that is,

$$\lim_{\varepsilon \to 0} \mu_X(t+\varepsilon) = \mu_X(t) \tag{6.2}$$

which can be written as

$$\lim_{\varepsilon \to 0} E[X(t+\varepsilon)] = E[\lim_{\varepsilon \to 0} X(t+\varepsilon)] \tag{6.3}$$

Hence, if $X(t)$ is m.s. continuous, then we may interchange the ordering of the operations of expectation and limiting. Note that m.s. continuity of $X(t)$ does not imply that the sample functions of $X(t)$ are continuous. For instance, the Poisson process is m.s. continuous (Prob. 6.46), but sample functions of the Poisson process have a countably infinite number of discontinuities (see Fig. 5-2).

B. Stochastic Derivatives:

A random process $X(t)$ is said to have a *m.s. derivative* $X'(t)$ if

$$\underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon} = X'(t) \tag{6.4}$$

where l.i.m. denotes *l*imit *i*n the *m*ean (square); that is,

$$\lim_{\varepsilon \to 0} E\left\{\left[\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right]^2\right\} = 0 \tag{6.5}$$

The m.s. derivative of $X(t)$ exists if $\partial^2 R_X(t, s)/\partial t\, \partial s$ exists (Prob. 6.6). If $X(t)$ has the m.s. derivative $X'(t)$, then its mean and autocorrelation function are given by

$$E[X'(t)] = \frac{d}{dt} E[X(t)] = \mu'_X(t) \tag{6.6}$$

$$R_{X'}(t, s) = \frac{\partial^2 R_X(t, s)}{\partial t\, \partial s} \tag{6.7}$$

Equation (6.6) indicates that the operations of differentiation and expectation may be interchanged. If $X(t)$ is a normal random process for which the m.s. derivative $X'(t)$ exists, then $X'(t)$ is also a normal random process (Prob. 6.10).

C. Stochastic Integrals:

A *m.s. integral* of a random process $X(t)$ is defined by

$$Y(t) = \int_{t_0}^{t} X(\alpha)\, d\alpha = \underset{\Delta t_i \to 0}{\text{l.i.m.}} \sum_i X(t_i)\, \Delta t_i \tag{6.8}$$

where $t_0 < t_1 < \cdots < t$ and $\Delta t_i = t_{i+1} - t_i$.

The m.s. integral of $X(t)$ exists if the following integral exists (Prob. 6.11):

$$\int_{t_0}^{t} \int_{t_0}^{t} R_X(\alpha, \beta)\, d\alpha\, d\beta \tag{6.9}$$

This implies that if $X(t)$ is m.s. continuous, then its m.s. integral $Y(t)$ exists (see Prob. 6.1). The mean and the autocorrelation function of $Y(t)$ are given by

$$\mu_Y(t) = E\left[\int_{t_0}^{t} X(\alpha)\, d\alpha\right] = \int_{t_0}^{t} E[X(\alpha)]\, d\alpha = \int_{t_0}^{t} \mu_X(\alpha)\, d\alpha \tag{6.10}$$

$$\begin{aligned} R_Y(t, s) &= E\left[\int_{t_0}^{t} X(\alpha)\, d\alpha \int_{t_0}^{s} X(\beta)\, d\beta\right] \\ &= \int_{t_0}^{t} \int_{t_0}^{s} E[X(\alpha)X(\beta)]\, d\beta\, d\alpha = \int_{t_0}^{t} \int_{t_0}^{s} R_X(\alpha, \beta)\, d\beta\, d\alpha \end{aligned} \tag{6.11}$$

Equation (6.10) indicates that the operations of integration and expectation may be interchanged. If $X(t)$ is a normal random process, then its integral $Y(t)$ is also a normal random process. This follows from the fact that $\Sigma_i X(t_i)\, \Delta t_i$ is a linear combination of the jointly normal r.v.'s. (see Prob. 5.60).

6.3 POWER SPECTRAL DENSITIES

In this section we assume that all random processes are WSS.

A. Autocorrelation Functions:

The autocorrelation function of a continuous-time random process $X(t)$ is defined as [Eq. (5.7)]

$$R_X(\tau) = E[X(t)X(t + \tau)] \tag{6.12}$$

Properties of $R_X(\tau)$:

1. $R_X(-\tau) = R_X(\tau)$ (6.13)
2. $|R_X(\tau)| \le R_X(0)$ (6.14)
3. $R_X(0) = E[X^2(t)] \ge 0$ (6.15)

Property 3 [Eq. (*6.15*)] is easily obtained by setting $\tau = 0$ in Eq. (*6.12*). If we assume that $X(t)$ is a voltage waveform across a 1-Ω resistor, then $E[X^2(t)]$ is the average value of power delivered to the 1-Ω resistor by $X(t)$. Thus, $E[X^2(t)]$ is often called the *average power* of $X(t)$. Properties 1 and 2 are verified in Prob. 6.13.

In case of a discrete-time random process $X(n)$, the autocorrelation function of $X(n)$ is defined by

$$R_X(k) = E[X(n)X(n+k)] \tag{6.16}$$

Various properties of $R_X(k)$ similar to those of $R_X(\tau)$ can be obtained by replacing τ by k in Eqs. (*6.13*) to (*6.15*).

B. Cross-Correlation Functions

The cross-correlation function of two continuous-time jointly WSS random processes $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \tag{6.17}$$

Properties of $R_{XY}(\tau)$:

1. $R_{XY}(-\tau) = R_{YX}(\tau)$ (*6.18*)
2. $|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$ (*6.19*)
3. $|R_{XY}(\tau)| \le \frac{1}{2}[R_X(0) + R_Y(0)]$ (*6.20*)

These properties are verified in Prob. 6.14. Two processes $X(t)$ and $Y(t)$ are called (*mutually*) *orthogonal* if

$$R_{XY}(\tau) = 0 \qquad \text{for all } \tau \tag{6.21}$$

Similarly, the cross-correlation function of two discrete-time jointly WSS random processes $X(n)$ and $Y(n)$ is defined by

$$R_{XY}(k) = E[X(n)Y(n+k)] \tag{6.22}$$

and various properties of $R_{XY}(k)$ similar to those of $R_{XY}(\tau)$ can be obtained by replacing τ by k in Eqs. (*6.18*) to (*6.20*).

C. Power Spectral Density:

The *power spectral density* (or *power spectrum*) $S_X(\omega)$ of a continuous-time random process $X(t)$ is defined as the Fourier transform of $R_X(\tau)$:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\, d\tau \tag{6.23}$$

Thus, taking the inverse Fourier transform of $S_X(\omega)$, we obtain

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}\, d\omega \tag{6.24}$$

Equations (*6.23*) and (*6.24*) are known as the *Wiener-Khinchin relations*.

Properties of $S_X(\omega)$:

1. $S_X(\omega)$ is real and $S_X(\omega) \ge 0$. (*6.25*)

2. $S_X(-\omega) = S_X(\omega)$ (6.26)

3. $E[X^2(t)] = R_X(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$ (6.27)

Similarly, the power spectral density $S_X(\Omega)$ of a discrete-time random process $X(n)$ is defined as the Fourier transform of $R_X(k)$:

$$S_X(\Omega) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-j\Omega k} \tag{6.28}$$

Thus, taking the inverse Fourier transform of $S_X(\Omega)$, we obtain

$$R_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\Omega)e^{j\Omega k}\, d\Omega \tag{6.29}$$

Properties of $S_X(\Omega)$:

1. $S_X(\Omega + 2\pi) = S_X(\Omega)$ (6.30)
2. $S_X(\Omega)$ is real and $S_X(\Omega) \geq 0$. (6.31)
3. $S_X(-\Omega) = S_X(\Omega)$ (6.32)
4. $E[X^2(n)] = R_X(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} S_X(\Omega)\, d\Omega$ (6.33)

Note that property 1 [Eq. (6.30)] follows from the fact that $e^{-j\Omega k}$ is periodic with period 2π. Hence it is sufficient to define $S_X(\Omega)$ only in the range $(-\pi, \pi)$.

D. Cross Power Spectral Densities:

The *cross power spectral density* (or *cross power spectrum*) $S_{XY}(\omega)$ of two continuous-time random processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\, d\tau \tag{6.34}$$

Thus, taking the inverse Fourier transform of $S_{XY}(\omega)$, we get

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\, d\omega \tag{6.35}$$

Properties of $S_{XY}(\omega)$:

Unlike $S_X(\omega)$, which is a real-valued function of ω, $S_{XY}(\omega)$, in general, is a complex-valued function.

1. $S_{XY}(\omega) = S_{YX}(-\omega)$ (6.36)
2. $S_{XY}(-\omega) = S_{XY}^*(\omega)$ (6.37)

Similarly, the cross power spectral density $S_{XY}(\Omega)$ of two discrete-time random processes $X(n)$ and $Y(n)$ is defined as the Fourier transform of $R_{XY}(k)$:

$$S_{XY}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XY}(k)e^{-j\Omega k} \tag{6.38}$$

Thus, taking the inverse Fourier transform of $S_{XY}(\Omega)$, we get

$$R_{XY}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XY}(\Omega)e^{j\Omega k}\, d\Omega \tag{6.39}$$

Properties of $S_{XY}(\Omega)$:

Unlike $S_X(\Omega)$, which is a real-valued function of ω, $S_{XY}(\Omega)$, in general, is a complex-valued function.

1. $S_{XY}(\Omega + 2\pi) = S_{XY}(\Omega)$ *(6.40)*
2. $S_{XY}(\Omega) = S_{YX}(-\Omega)$ *(6.41)*
3. $S_{XY}(-\Omega) = S_{XY}^*(\Omega)$ *(6.42)*

6.12. Let $X(t)$ be the Wiener process with parameter σ^2. Let

$$Y(t) = \int_0^t X(\alpha)\, d\alpha$$

(*a*) Find the mean and the variance of $Y(t)$.

(*b*) Find the autocorrelation function of $Y(t)$.

(*a*) By assumption 3 of the Wiener process (Sec. 5.7), that is, $E[X(t)] = 0$, we have

$$E[Y(t)] = E\left[\int_0^t X(\alpha)\, d\alpha\right] = \int_0^t E[X(\alpha)]\, d\alpha = 0 \tag{6.121}$$

Then

$$\begin{aligned}\operatorname{Var}[Y(t)] = E[Y^2(t)] &= \int_0^t \int_0^t E[X(\alpha)X(\beta)]\, d\alpha\, d\beta \\ &= \int_0^t \int_0^t R_X(\alpha, \beta)\, d\alpha\, d\beta\end{aligned}$$

By Eq. (5.64), $R_X(\alpha, \beta) = \sigma^2 \min(\alpha, \beta)$; thus, referring to Fig. 6-3, we obtain

$$\begin{aligned}\operatorname{Var}[Y(t)] &= \sigma^2 \int_0^t \int_0^t \min(\alpha, \beta)\, d\alpha\, d\beta \\ &= \sigma^2 \int_0^t d\beta \int_0^\beta \alpha\, d\alpha + \sigma^2 \int_0^t d\alpha \int_0^\alpha \beta\, d\beta = \frac{\sigma^2 t^3}{3}\end{aligned} \tag{6.122}$$

(*b*) Let $t > s \geq 0$ and write

$$\begin{aligned}Y(t) &= \int_0^s X(\alpha)\, d\alpha + \int_s^t [X(\alpha) - X(s)]\, d\alpha + (t - s)X(s) \\ &= Y(s) + \int_s^t [X(\alpha) - X(s)]\, d\alpha + (t - s)X(s)\end{aligned}$$

Then, for $t > s \geq 0$,

$$\begin{aligned}R_Y(t, s) &= E[Y(t)Y(s)] \\ &= E[Y^2(s)] + \int_s^t E\{[X(\alpha) - X(s)]Y(s)\}\, d\alpha + (t - s)E[X(s)Y(s)]\end{aligned} \tag{6.123}$$

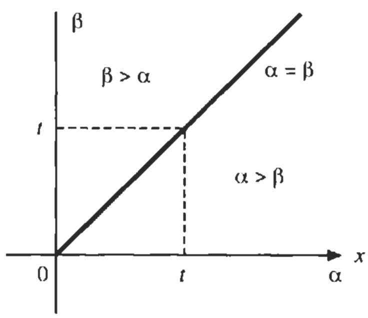

Fig. 6-3

Now by Eq. (6.122),

$$E[Y^2(s)] = \mathrm{Var}[Y(s)] = \frac{\sigma^2 s^3}{3}$$

Using assumptions 1, 3, and 4 of the Wiener process (Sec. 5.7), and since $s \le \alpha \le t$, we have

$$\begin{aligned}\int_s^t E\{[X(\alpha) - X(s)]Y(s)\}\, d\alpha &= \int_s^t E\left\{[X(\alpha) - X(s)] \int_0^s X(\beta)\, d\beta\right\} d\alpha \\ &= \int_s^t \int_0^s E\{[X(\alpha) - X(s)][X(\beta) - X(0)]\}\, d\beta\, d\alpha \\ &= \int_s^t \int_0^s E[X(\alpha) - X(s)]E[X(s) - X(0)]\, d\beta\, d\alpha = 0\end{aligned}$$

Finally, for $0 \le \beta \le s$,

$$\begin{aligned}(t - s)E[X(s)Y(s)] &= (t - s) \int_0^s E[X(s)X(\beta)]\, d\beta \\ &= (t - s) \int_0^s R_X(s, \beta)\, d\beta = (t - s) \int_0^s \sigma^2 \min(s, \beta)\, d\beta \\ &= \sigma^2(t - s) \int_0^s \beta\, d\beta = \sigma^2(t - s)\frac{s^2}{2}\end{aligned}$$

Substituting these results into Eq. (6.123), we get

$$R_Y(t, s) = \frac{\sigma^2 s^3}{3} + \sigma^2(t - s)\frac{s^2}{2} = \frac{1}{6}\sigma^2 s^2(3t - s)$$

Since $R_Y(t, s) = R_Y(s, t)$, we obtain

$$R_Y(t, s) = \begin{cases} \frac{1}{6}\sigma^2 s^2(3t - s) & t > s \ge 0 \\ \frac{1}{6}\sigma^2 t^2(3s - t) & s > t \ge 0 \end{cases} \tag{6.124}$$

POWER SPECTRAL DENSITY

6.13. Verify Eqs. (6.13) and (6.14).

From Eq. (6.12),

$$R_X(\tau) = E[X(t)X(t + \tau)]$$

Setting $t + \tau = s$, we get

$$R_X(\tau) = E[X(s - \tau)X(s)] = E[X(s)X(s - \tau)] = R_X(-\tau)$$

Next, we have

$$E\{[X(t) \pm X(t + \tau)]^2\} \ge 0$$

Expanding the square, we have

$$E[X^2(t) \pm 2X(t)X(t + \tau) + X^2(r + \tau)] \ge 0$$

or

$$E[X^2(t)] \pm 2E[X(t)X(t + \tau)] + E[X^2(t + \tau)] \ge 0$$

Thus

$$2R_X(0) \pm 2R_X(\tau) \ge 0$$

from which we obtain Eq. (6.14); that is,

$$R_X(0) \ge |R_X(\tau)|$$

6.14. Verify Eqs. (6.18) to (6.20).

By Eq. (6.17),

$$R_{XY}(-\tau) = E[X(t)Y(t-\tau)]$$

Setting $t - \tau = s$, we get

$$R_{XY}(-\tau) = E[X(s+\tau)Y(s)] = E[Y(s)X(s+\tau)] = R_{YX}(\tau)$$

Next, from the Cauchy-Schwarz inequality, Eq. (3.97) (Prob. 3.35), it follows that

$$\{E[X(t)Y(t+\tau)]\}^2 \le E[X^2(t)]E[Y^2(t+\tau)]$$

or

$$[R_{XY}(\tau)]^2 \le R_X(0)R_Y(0)$$

from which we obtain Eq. (6.19); that is,

$$|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$$

Now

$$E\{[X(t) - Y(t+\tau)]^2\} \ge 0$$

Expanding the square, we have

$$E[X^2(t) - 2X(t)Y(t+\tau) + Y^2(t+\tau)] \ge 0$$

or

$$E[X^2(t)] - 2E[X(t)Y(t+\tau)] + E[Y^2(t+\tau)] \ge 0$$

Thus

$$R_X(0) - 2R_{XY}(\tau) + R_Y(0) \ge 0$$

from which we obtain Eq. (6.20); that is,

$$R_{XY}(\tau) \le \tfrac{1}{2}[R_X(0) + R_Y(0)]$$

6.15. Two random processes $X(t)$ and $Y(t)$ are given by

$$X(t) = A\cos(\omega t + \Theta) \qquad Y(t) = A\sin(\omega t + \Theta)$$

where A and ω are constants and Θ is a uniform r.v. over $(0, 2\pi)$. Find the cross-correlation function of $X(t)$ and $Y(t)$ and verify Eq. (6.18).

From Eq. (6.17), the cross-correlation function of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t, t+\tau) &= E[X(t)Y(t+\tau)] \\ &= E\{A^2\cos(\omega t + \Theta)\sin[\omega(t+\tau) + \Theta]\} \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) - \sin(-\omega\tau)] \\ &= \frac{A^2}{2}\sin\omega\tau = R_{XY}(\tau) \end{aligned} \tag{6.125}$$

Similarly,

$$\begin{aligned} R_{YX}(t, t+\tau) &= E[Y(t)X(t+\tau)] \\ &= E\{A^2\sin(\omega t + \Theta)\cos[\omega(t+\tau) + \Theta]\} \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) + \sin(-\omega\tau)] \\ &= -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau) \end{aligned} \tag{6.126}$$

From Eqs. (6.125) and (6.126), we see that

$$R_{XY}(-\tau) = \frac{A^2}{2}\sin\omega(-\tau) = -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau)$$

which verifies Eq. (6.18).

6.16. Show that the power spectrum of a (real) random process $X(t)$ is real and verify Eq. (6.26).

From Eq. (6.23) and expanding the exponential, we have

$$\begin{aligned} S_X(\omega) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\,d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)(\cos\omega\tau - j\sin\omega\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau\,d\tau - j\int_{-\infty}^{\infty} R_X(\tau)\sin\omega\tau\,d\tau \end{aligned} \tag{6.127}$$

Since $R_X(-\tau) = R_X(\tau)$, $R_X(\tau)\cos\omega\tau$ is an even function of τ and $R_X(\tau)\sin\omega\tau$ is an odd function of τ, and hence the imaginary term in Eq. (6.127) vanishes and we obtain

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau\,d\tau \tag{6.128}$$

which indicates that $S_X(\omega)$ is real. Since $\cos(-\omega\tau) = \cos(\omega\tau)$, it follows that

$$S_X(-\omega) = S_X(\omega)$$

which indicates that the power spectrum of a real random process $X(t)$ is an even function of frequency.

6.17. Consider the random process

$$Y(t) = (-1)^{X(t)}$$

where $X(t)$ is a Poisson process with rate λ. Thus $Y(t)$ starts at $Y(0) = 1$ and switches back and forth from $+1$ to -1 at random Poisson times T_i, as shown in Fig. 6-4. The process $Y(t)$ is known as the *semirandom telegraph signal* because its initial value $Y(0) = 1$ is not random.

(a) Find the mean of $Y(t)$.

(b) Find the autocorrelation function of $Y(t)$.

(a) We have

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even} \\ -1 & \text{if } X(t) \text{ is odd} \end{cases}$$

Thus, using Eq. (5.55), we have

$$\begin{aligned} P[Y(t) = 1] &= P[X(t) = \text{even integer}] \\ &= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \cdots\right] = e^{-\lambda t}\cosh\lambda t \\ P[Y(t) = -1] &= P[X(t) = \text{odd integer}] \\ &= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \cdots\right] = e^{-\lambda t}\sinh\lambda t \end{aligned}$$

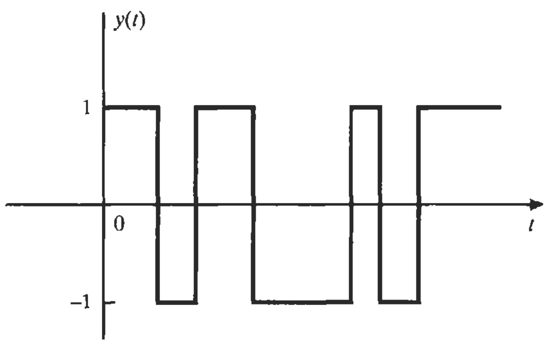

Fig. 6-4 Semirandom telegraph signal.

Hence
$$\mu_Y(t) = E[Y(t)] = (1)P[Y(t) = 1] + (-1)P[Y(t) = -1]$$
$$= e^{-\lambda t}(\cosh \lambda t - \sinh \lambda t) = e^{-2\lambda t} \tag{6.129}$$

(b) Similarly, since $Y(t)Y(t+\tau) = 1$ if there are an even number of events in $(t, t+\tau)$ for $\tau > 0$ and $Y(t)Y(t+\tau) = -1$ if there are an odd number of events, then for $t > 0$ and $t + \tau > 0$,

$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)]$$
$$= (1) \sum_{n \text{ even}} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} + (-1) \sum_{n \text{ odd}} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$$
$$= e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(-\lambda\tau)^n}{n!} = e^{-\lambda\tau} e^{-\lambda\tau} = e^{-2\lambda\tau}$$

which indicates that $R_Y(t, t+\tau) = R_Y(\tau)$, and by Eq. (6.13),

$$R_Y(\tau) = e^{-2\lambda|\tau|} \tag{6.130}$$

Note that since $E[Y(t)]$ is not a constant, $Y(t)$ is not WSS.

6.18. Consider the random process

$$Z(t) = AY(t)$$

where $Y(t)$ is the semirandom telegraph signal of Prob. 6.17 and A is a r.v. independent of $Y(t)$ and takes on the values ± 1 with equal probability. The process $Z(t)$ is known as the *random telegraph signal*.

(a) Show that $Z(t)$ is WSS.

(b) Find the power spectral density of $Z(t)$.

(a) Since $E(A) = 0$ and $E(A^2) = 1$, the mean of $Z(t)$ is

$$\mu_Z(t) = E[Z(t)] = E(A)E[Y(t)] = 0 \tag{6.131}$$

and the autocorrelation of $Z(t)$ is

$$R_Z(t, t+\tau) = E[A^2 Y(t)Y(t+\tau)] = E(A^2)E[Y(t)Y(t+\tau)] = R_Y(t, t+\tau)$$

Thus, using Eq. (6.130), we obtain

$$R_Z(t, t+\tau) = R_Z(\tau) = e^{-2\lambda|\tau|} \tag{6.132}$$

Thus, we see that $Z(t)$ is WSS.

(b) Taking the Fourier transform of Eq. (6.132) (see Appendix B), we see that the power spectrum of $Z(t)$ is given by

$$S_Z(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2} \tag{6.133}$$

6.19. Let $X(t)$ and $Y(t)$ be both zero-mean and WSS random processes. Consider the random process $Z(t)$ defined by

$$Z(t) = X(t) + Y(t)$$

(a) Determine the autocorrelation function and the power spectral density of $Z(t)$, (i) if $X(t)$ and $Y(t)$ are jointly WSS; (ii) if $X(t)$ and $Y(t)$ are orthogonal.

(b) Show that if $X(t)$ and $Y(t)$ are orthogonal, then the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.

(a) The autocorrelation of $Z(t)$ is given by

$$R_Z(t, s) = E[Z(t)Z(s)] = E\{[X(t) + Y(t)][X(s) + Y(s)]\}$$
$$= E[X(t)X(s)] + E[X(t)Y(s)] + E[Y(t)X(s)] + E[Y(t)Y(s)]$$
$$= R_X(t, s) + R_{XY}(t, s) + R_{YX}(t, s) + R_Y(t, s)$$

(i) If $X(t)$ and $Y(t)$ are jointly WSS, then we have

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

where $\tau = s - t$. Taking the Fourier transform of the above expression, we obtain

$$S_Z(\omega) = S_X(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_Y(\omega)$$

(ii) If $X(t)$ and $Y(t)$ are orthogonal [Eq. (*6.21*)],

$$R_{XY}(\tau) = R_{YX}(\tau) = 0$$

Then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) \tag{6.134a}$$

$$S_Z(\omega) = S_X(\omega) + S_Y(\omega) \tag{6.134b}$$

(*b*) Setting $\tau = 0$ in Eq. (*6.134a*), and using Eq. (*6.15*), we get

$$E[Z^2(t)] = E[X^2(t)] + E[Y^2(t)]$$

which indicates that the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.