Example 3.14 Let V be the set of all ordered pairs (x, y), where x, y are real numbers. Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ be two elements in V. Define the addition as

$$\mathbf{a} + \mathbf{b} = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and the scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Show that V is not a vector space. Which of the properties are not satisfied?

Solution Note that $(1, 1)$ is an element of V. From the given definition of vector addition, we find that

$$(x_1, y_1) + (1, 1) = (x_1, y_1).$$

and this is true only for the element $(1, 1)$. Therefore, the element $(1, 1)$ plays the role of $\mathbf{0}$ element as defined in property **4**. Now, there is no element in V for which $(\mathbf{a}) + (-\mathbf{a}) = \mathbf{0} = (1, 1)$, since

$$(x_1, y_1) + (-x_1, -y_1) = (-x_1^2, -y_1^2) \neq (1, 1).$$

Therefore, property **5** is not satisfied.

Now, let $\alpha = 1$, $\beta = 2$ be any two scalars. We have

$$(\alpha + \beta)(x_1, y_1) = 3(x_1, y_1) = (3x_1, 3y_1)$$

and $$\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2)$$

Therefore, $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property **7** is not satisfied.

Similarly, it can be shown that property **9** is not satisfied. Hence, V is not a vector space.

3.3.1 Subspaces

Let V be an arbitrary vector space defined under a given vector addition and scalar multiplication. A non-empty subset W of V, such that W is also a vector space under the same two operations of vector addition and scalar multiplication, is called a *subspace* of V. Thus, W is also closed under the two given algebraic operations on V. As a convention, the vector space V is also taken as a subspace of V.

Remark 8

To show that W is a subspace of a vector space V, it is not necessary to verify all the 10 properties as given in section 3.3. If it is shown that W is closed under the given definition of vector addition and scalar multiplication, then the properties **2, 3, 7, 8, 9 and 10** are automatically satisfied because these properties are valid for all elements in V and hence are also valid for all elements in W. Thus, we need to verify the remaining properties, that is, the existence of the zero element and the additive inverse in W.

and $\alpha(x_1, y_1) + \beta(x_1, y_1) = 1(x_1, y_1) + 2(x_1, y_1) = (x_1, y_1) + (2x_1, 2y_1) = (2x_1^2, 2y_1^2)$

Therefore, $(\alpha + \beta)(x_1, y_1) \neq \alpha(x_1, y_1) + \beta(x_1, y_1)$ and property **7** is not satisfied.

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Consider the following examples:

1. Let V be the set of n-tuples $(x_1\ x_2 \ldots\ x_n)$ in $\mathbb{R}^n$ with usual addition and scalar multiplication. Then
   - (i) W consisting of n-tuples $(x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_1 = 0$ is a subspace of V.
   - (ii) W consisting of n-tuples $(x_1\ \ x_2\ \ \ldots\ \ x_n)$ with $x_1 \geq 0$ is not a subspace of V, since W is not closed under scalar multiplication ($\alpha\mathbf{x}$, when α is a negative real number, is not in W).