TUTORIAL - 2

29/01/2020

AUTOMATA AND FORMAL LANGUAGES

Mathematical Induction and Method of Contradiction

U19CS012 [BHAGYA VINOD RANA]

Q.1.) Prove the following by mathematical induction

1.) Statement: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$, $n \in N$

For any integer $n \geq 1$, let $P(n)$ be the statement

$$P(n): \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(A) Base Case: The statement $P(1)$ says that

$$LHS = \frac{1}{1.2} = \frac{1}{2}$$

$$RHS = \frac{n}{(n+1)} = \frac{1}{(1+1)} = \frac{1}{2}$$

$LHS = RHS$, Hence $P(1)$ is true.

(B) Inductive Step: Fix $k \geq 1$, and suppose $P(k)$ holds true

$$\boxed{\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} = \frac{k}{(k+1)}} \quad (1)$$

To show: $P(k+1)$ is also true, i.e.

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)}$$

$$LHS = \left\{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)}\right\} + \frac{1}{(k+1)(k+2)}$$

$$= \left(\frac{k}{k+1}\right) + \frac{1}{(k+1)(k+2)} \quad \text{(Using (1))}$$

$$= \frac{1}{(k+1)}\left(k + \frac{1}{(k+2)}\right) = \frac{1}{(k+1)} \frac{(k(k+2)+1)}{(k+2)}$$

$$LHS = \frac{(k^2+2k+1)}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)} = RHS$$

Hence, $P(k+1)$ hold's true.

Thus, by the principle of mathematical induction for all $n \geq 1$, $P(n)$ holds true.

Q.1.) 2.) Statement : $1.1! + 2.2! + 3.3! + \ldots + n.n! = (n+1)! - 1$, $n \in N$

For any integer $n \geq 1$, let $P(n)$ be the statement

$P(n)$: $1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$

(A) Base case : The statement $P(1)$ says that

$LHS = 1.1!$

$RHS = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

$LHS = RHS$, Hence $P(1)$ is true.

(B) Inductive step : Fix $k \geq 1$, and suppose $p(k)$ hold's true,

$$\boxed{1.1! + 2.2! + 3.3! + \ldots + k.k! = (k+1)! - 1} \quad \text{— (1)}$$

To show : $P(k+1)$ is also true, i.e.

$$1.1! + 2.2! + \ldots + k.k! + (k+1).(k+1)! = ((k+1)+1)! - 1$$

$$\begin{aligned} LHS &= \{1.1! + 2.2! + \ldots + k.k!\} + (k+1).(k+1)! \\ &= \{((k+1)! - 1)\} + (k+1)((k+1)!) \quad \text{using (1)} \\ &= (k+1)! - 1 + (k+1)(k+1)! \\ &= [((k+1)!)(1+k+1)] - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1 \quad [\because n.(n-1)! = n!] \\ &= ((k+1)+1)! - 1 = RHS \end{aligned}$$

$\therefore LHS = RHS$ $P(k+1)$ hold's true.

Thus, by principle of mathematical induction, for all $n \geq 1$, $p(n)$ holds true.

U19CS012

Q.1.) 3.) Statement $P(n)$:

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\cdots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2, \quad n\in N \quad (n \geq 1)$$

For any integer $n \geq 1$, let $P(n)$ be the statement

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2 \quad n \geq 1 \quad \{n \in I\}$$

(A) Base Case: The statement $P(1)$ says that

$$LHS = \left(1+\frac{(2n+1)}{n^2}\right) = \left(1+\frac{3}{1}\right) = 4$$

$$RHS = (n+1)^2 = (1+1)^2 = 4$$

$LHS = RHS$, Hence $P(1)$ is true.

(B) Inductive Step: Fix $K \geq 1$, and suppose $P(K)$ holds true,

$$\boxed{\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{(2K+1)}{K^2}\right) = (K+1)^2} \quad \text{— (1)}$$

To show: $P(K+1)$ is also true, i.e.

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\cdots\left(1+\frac{(2K+1)}{K^2}\right)\left(1+\frac{2(K+1)+1}{(K+1)^2}\right) = ((K+1)+1)^2$$

$$LHS = (K+1)^2\left(1+\frac{2(K+1)+1}{(K+1)^2}\right) \quad \text{using (1)}$$

$$= \cancel{(K+1)^2}\left(\frac{(K+1)^2+2(K+1)+1}{\cancel{(K+1)^2}}\right)$$

$$= (K^2+2K+1+2K+2+1)$$

$$= K^2+4K+4$$

$$= (K+2)^2 = ((K+1)+1)^2 = RHS$$

$\therefore$ $LHS = RHS$, $P(K+1)$ holds true

thus, by the principle of mathematical induction, for all $n \geq 1$, $P(n)$ holds true.

U19CS012

Q1.) 4.) Statement :

$1.2 + 2.2^2 + 3.2^3 + \dots + n.2^n = (n-1).2^{n+1} + 2, \; n \in N$

For any integer $n \geq 1$, let $P(n)$ be the statement

$P(n): \; 1.2 + 2.2^2 + 3.2^3 + \dots n.2^n = (n-1)2^{n+1} + 2 \quad \forall n \in N$

$n >= 1, n \in I$

Ⓐ Base Case : The statement $P(1)$ says that

LHS = $1.2 = 2$

RHS = $(1-1)\,2^{1+1} + 2$

$= 2$

LHS = RHS, Hence $P(1)$ is true.

Ⓑ Inductive step : Fix $k \geq 1$, and suppose $P(k)$ holds true

$$\boxed{1.2 + 2.2^2 + 3.2^3 \dots + k.2^k = (k-1)\,2^{k+1} + 2} \quad - (1)$$

To show : $P(k+1)$ is also true, i.e.

$$1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k + (k+1).2^{k+1} = ((k+1)-1)\,2^{(k+1)+1} + 2$$

$$\begin{aligned}
\text{LHS} &= \{1.2 + 2.2^2 + \dots + k.2^k\} + (k+1).2^{k+1} \\
&= (k-1)\,2^{k+1} + 2 + (k+1)\,2^{k+1} \\
&= 2^{k+1}(k-\cancel{1} + k+\cancel{1}) + 2 \\
&= (2k)(2^{k+1}) + 2 \\
&= (k)(2^{k+2}) + 2 \\
&= ((k+1)-1)(2^{(k+1)+1}) + 2 = \text{RHS}
\end{aligned}$$

$\therefore$ LHS = RHS, thus, by principle of mathematical induction,

for all $n \geq 1$, $P(n)$ holds true.

U19CS012

Q1 > 5> Prove by mathematical induction, that $x^{2n} - y^{2n}$ is divisible by $x+y$ for all natural numbers n.

5> Let $P(n)$ be the statement, $n \in N$ $(n >= 1, n \in I)$

$P(n)$: $x^{2n} - y^{2n} = (x+y) \times d$, where $d \in N$

① Base Case: For $n = 1$,

$$\begin{aligned} LHS &= x^{2\times1} - y^{2\times1} \\ &= x^2 - y^2 \\ &= (x+y)(x-y) \\ &= RHS \end{aligned} \qquad [(x-y) \in N]$$

$\therefore$ $P(n)$ is true for $n = 1$

② Inductive Case: Assume $P(K)$ is true,

$$\boxed{x^{2K} - y^{2K} = m(x+y)} \quad \text{where } m \in N \quad -①$$

We will prove that $p(K+1)$ is true.

$$\begin{aligned} LHS &= x^{2(K+1)} - y^{2(K+1)} \\ &= x^{2K+2} - y^{2K+2} \\ &= (x^{2K})x^2 - y^{2K}y^2 \\ &\Downarrow \\ &= (y^{2K} + m(x+y))x^2 - y^{2K}(y^2) \end{aligned}$$

using ② $[x^{2K} - y^{2K} = m(x+y)$
$x^{2K} = y^{2K} + m(x+y)]$

$$\begin{aligned} &= x^2(m(x+y)) + y^{2K}(x^2 - y^2) \\ &= x^2(m(x+y)) + y^{2k}(x+y)(x-y) \\ &= (x+y)[mx^2 + y^{2k}(x-y)] \\ &= (x+y) \times (r) \end{aligned}$$

where $r \in N$ & $r = mx^2 + y^{2k}(x-y)$

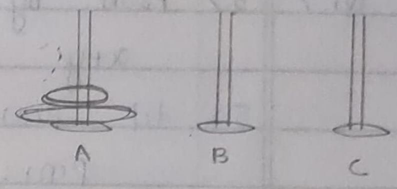
$\therefore$ $p(K+1)$ is true, whenever $p(K)$ is true.

$\therefore$ By principle of mathematical induction, $P(n)$ is true for n, where n is natural number.

U19CS012

⑥

Q1.> 6.> Prove using Mathematical Induction in standard Tower of Hanoi problem the number of moves to transfer all disk from is peg to last peg using second peg as intermediate is $2^n - 1$, $n \in N$



6.> $P(n)$

Claim: It takes $2^n - 1$ moves to move n disk from first peg to third peg.

Ⓐ Base case: For $n = 1$, it takes exactly one move.

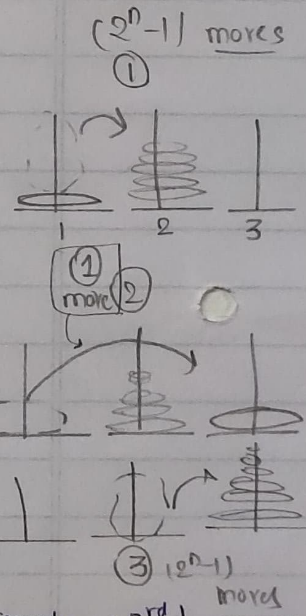
LHS = 1 move from peg ① to ③

RHS = $2^n - 1 = 2^1 - 1 = 1$

LHS = RHS, Hence $P(1)$ is true.

Ⓑ Inductive Hypothesis: Let suppose it takes $2^n - 1$ moves to move n disks — ①

We need to prove: The no. of moves for n+1 disks is $[2^{n+1} - 1]$



Step 1 :> First, we move top 'n' disks to the second peg (using third peg as intermediate)

∴ It takes $2^n - 1$ moves (using 1)

Step 2 :> Then, we move the last disk to third peg - it takes one move.

Step 3 :> We move n disks from the second peg to the third peg.

$$\therefore \text{Total number of moves} = LHS = \underset{(1\to 2)\ (\text{first } n)}{(2^n - 1)} + \underset{(1\to 3)\ (\text{last one})}{1} + \underset{(\text{second peg}\to 3^{rd})}{(2^n - 1)}$$

$$= 2 * (2^n - 1) + 1$$

∴ $P(n+1)$ also holds true.

$$= 2^{n+1} - 2 + 1 = [2^{n+1} - 1] = RHS$$

Thus, by principle of mathematical induction, for All $n \in N$ $P(n)$ hold's true.

U19CS012

2> Prove the following by contradiction

1> The square root of 7 is irrational.

Let us assume $\sqrt{7}$ is rational. Then, there exist co-prime positive integers a and b such that

$$\sqrt{7} = \frac{a}{b} \Rightarrow a = b\sqrt{7}$$

Squaring on both sides, we get $a^2 = 7b^2$

$\therefore$ a^2 is divisible by 7 and hence

a is also divisible by 7

So, we can write $a = 7p$, for some integer p.

Substituting for a, we get $49p^2 = 7b^2$

$$\Rightarrow b^2 = 7p^2$$

This means, $\therefore$ b^2 is divisible by 7 and hence

b is also divisible by 7

$\therefore$ a and b have at least one common factor ie. 7

But, this contradicts the fact that a and b are co-prime.

Thus, our supposition is wrong (no common factor other than 1)

Hence, $\sqrt{7}$ is irrational.

2> Show that following statement is true, by method of contradiction

if $x^5 + 16x = 0$ then $x = 0$

Let us ~~sup~~ assume $x \neq 0$ — (1) (Hypothesis)

multiplying both sides of with $(x^4 + 16)$ (+ve term)

$$x * (x^4 + 16) \neq 0 * (x^4 + 16)$$

$$x^5 + 16x \neq 0$$ (zero * (x) = zero)

But this contradicts the fact that $x^5 + 16x = 0$

~~$\therefore$ By method of~~ Continued ... →

p: if $x^5 + 16x = 0$, then x is 0.

Let us assume $x^5 + 16x = 0$ but $x \neq 0$

solving $x^5 + 16x = 0$

$x(x^4 + 16) = 0$

$x = 0$ OR $x^4 + 16 = 0$

$\boxed{x = 0}$ $\boxed{x^4 = -16}$ (X) $x \in R$

Hence only solution is $x = 0$. (square / even power of any number can't be negative) (Not possible)

but we take $x \neq 0$

Hence we get a contradiction. Hence our assumption was wrong.

$\therefore$ if $x^5 + 16x = 0$, then x is 0. is true $x \in R$.

Q2.> 3.> Using method of contradiction, prove that sum of an irrational number and a rational number is irrational.

3> Assume that a is rational, b is irrational, and a+b is rational.

Since 'a' and 'a+b' are rational, we can write them as fractions

Let $a = \frac{c}{d}$ —(1) and $a+b = \frac{m}{n}$ —(2)

Substituting $a = c/d$ in (2)

$$\frac{c}{d} + b = \frac{m}{n}$$

Let's subtract (c/d) from both sides,

$$b = \frac{m}{n} - \frac{c}{d} = \frac{m}{n} + \left(\frac{-c}{d}\right) = \text{Rational} = (?)$$

$\because$ Rational numbers are closed under addition, $\boxed{b = \left(\frac{m}{n} + \left(\frac{-c}{d}\right)\right)}$ is a rational number.

However, ~~at~~ the assumptions said that b is irrational, and b cannot be both rational and irrational. This is our contradiction.

So, it must be the case that the sum of a rational and an irrational is always irrational.

Q2> 4.> Prove that if $x > 3$, then $x^2 > 9$ using the method of contradiction ($x \in R$).

Given statement : If n is a real number with $n > 3$, then $n^2 > 9$

Let us assume that n is a real number with $n > 3$ ~~and~~ BUT $n^2 > 9$ is not true ie $n^2 < 9$

The $n > 3$ and n is a real number
Squaring both sides we obtain
$n^2 > (3)^2$
$\Rightarrow n^2 > 9$ which is a contradiction since we have assummed that $n^2 < 9$.

Thus the given statement is true.
If n is real number, with $n > 3$ then $n^2 > 9$.

—x—