... eigen vector corresponding to $\lambda = 3$ is $k_2\,(1, 0, 0)$.
Similarly, the eigen vector corresponding to $\lambda = 5$ is $k_3\,(3, 2, 1)$.

2.14 PROPERTIES OF EIGEN VALUES

I. Any square matrix A and its transpose A′ have the same eigen values.

We have

$$(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$$

$$|(A - \lambda I)'| = |A' - \lambda I|$$

$$|A - \lambda I| = |A' - \lambda I| \qquad [\because |B'| = |B|]$$

$\therefore$ $|A - \lambda I| = 0$ if and only if $|A' - \lambda I| = 0$

i.e., λ is an eigen value of A if and only if it is an eigen value of A'.

II. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Let

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}$$ be a triangular matrix of order n.

Then $|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \dots\dots (a_{nn} - \lambda)$.

$\therefore$ Roots of $|A - \lambda I| = 0$ are $\lambda = a_{11}, a_{22}, \dots, a_{nn}$.

Hence the eigen values of A are the diagonal elements of A, *i.e.*, $a_{11}, a_{22}, \dots, a_{nn}$.

Cor. *The eigen values of a diagonal matrix are just the diagonal elements of the matrix.*

III. The eigen values of an idempotent matrix are either zero or unity.

Let A be an idempotent matrix so that $A^2 = A$. If λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X \qquad \dots(1)$$

$\therefore$ $A(AX) = A(\lambda X)$, *i.e.*, $A^2X = \lambda(AX)$

i.e. $AX = \lambda(\lambda X)$ $\qquad [\because A^2 = A \text{ and } AX = \lambda X$

$\therefore$ $AX = \lambda^2 X \qquad \dots(2)$

From (1) and (2), we get $\lambda^2 X = \lambda X$ or $(\lambda^2 - \lambda)X = 0$

or $\lambda^2 - \lambda = 0$ whence $\lambda = 0$ or 1.

Hence the result.

IV. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

[This property will be proved for a matrix of order 3, but the method will be capable of easy extension to matrices of any order.]

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \dots(i)$$

so that $$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \qquad \text{(On expanding)}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \dots \qquad \dots(ii)$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then $|A - \lambda I| = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

$$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \dots \qquad \dots(iii)$$

Equating the right hand sides of (*ii*) and (*iii*) and comparing coefficients of λ^2, we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$. Hence the result.

V. The product of the eigen values of a matrix A is equal to its determinant.

Putting $\lambda = 0$ in (*iii*), we get the result.

VI. If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.

If X be the eigen vector corresponding to λ, then $AX = \lambda X \qquad \dots(i)$

2.15 CAYLEY-HAMILTON THEOREM*

Every square matrix satisfies its own characteristic equation; *i.e.*, if the characteristic equation for the nth order square matrix A is

$$| A - \lambda I | = (-1)^n \lambda^n + k_1\lambda^{n-1} + \ldots + k_n = 0 \qquad \ldots(i)$$

then
$$(-1)^n A^n + k_1 A^{n-1} + \ldots + k_n = 0.$$

Let the adjoint of the matrix $A - \lambda I$ be P. Clearly, the elements of P will be polynomials of the $(n-1)$th degree in λ, for the cofactors of the elements in $| A - \lambda I |$ will be such polynomials.

$\therefore$ P can be split up into a number of matrices, containing terms with the same powers of λ, such that

$$P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_{n-1}\lambda + P_n \qquad \ldots(ii)$$

where $P_1, P_2, \ldots, P_n$ are all the square matrices of order n whose elements are functions of the elements of A.

Since the product of a matrix by its adjoint = determinant of the matrix × unit matrix.

$\therefore$
$$[A - \lambda I]P = | A - \lambda I | \times I$$

$\therefore$ by (i) and (ii),
$$[A - \lambda I]\,[P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \ldots + P_{n-1} \lambda + P_n]$$
$$= [(-1)^n \lambda^n + k_1 \lambda^{n-1} + \ldots + k_{n-1} \lambda + k_n]\, I.$$

Equating the coefficients of various powers of λ, we get $\qquad [\because \ IP_1 = P_1]$

$$-P_1 = (-1)^n I$$
$$AP_1 - P_2 = k_1 I,$$
$$AP_2 - P_3 = k_2 I,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AP_{n-1} - P_n = k_{n-1} I,$$
$$AP_n = k_n I.$$

Now pre-multiplying the equations by $A^n, A^{n-1}, \ldots, A, I$ respectively and adding, we get

$$(-1)^n A^n + k_1 A^{n-1} + \ldots + k_{n-1} A + k_n I = 0, \qquad \ldots(iii)$$

for the terms on the left cancel in pairs. This proves the theorem.

Cor. Another method of finding the inverse.

Multiplying (iii) by A^{-1}, we get

$$(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots + k_{n-1} I + k_n A^{-1} = 0$$

whence
$$A^{-1} = -\frac{1}{k_n}[(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots + k_{n-1} I].$$

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*See footnote on p.17. *William Rowan Hamilton* (1805–1865) an Irish mathematician who is known for his work i dynamics.

This result gives the inverse of A in terms of $\overline{n-1}$ powers of A and is considered as a practical method for the computation of the inverse of the large matrices. As a by-product of the computation, the characteristic equation and the determinant of the matrix are also obtained.

Example 2.45. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and find its inverse.*
Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A.* (*Bhopal, 2009*)

Solution. The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \qquad \text{...}(i)$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation (i), so that

$$A^2 - 4A - 5I = 0 \qquad \text{...}(ii)$$

Now

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Multiplying (ii) by A^{-1}, we get $A - 4I - 5A^{-1} = 0$

or

$$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$= \lambda + 5 \qquad \text{[By } (i)\text{]}$$

Hence $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5$, which is a linear polynomial in A.