MA212 LINEAR ALGEBRA AND STATISTICAL ANALYSIS U19CS012

TUTORIAL - II [BHAGYA RANA]

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1) The Random Variable X denotes the number of trials (Bernoulli) needed to obtain the first success. S.T. pmf $f(x) = (1-p)^{x-1} p$, $0<p<1$, $x = 1, 2, 3, \ldots$ Also S.T. $F(x) = 1 - q^{(x)}$

1) To get the success on first trial $f(x) = p$

to get success on second trial $f(x) = (1-p)\,p$

similarily, for third trial $f(x) = (1-p)^2\,p$

for i^{th} trial $= (1-p)^{i-1}\,(p)$

Hence,

The probability mass function $= \boxed{(1-p)^x\,p}$

Now $F(x) =$ cumulative probability

For first x terms,

$$p + p(1-p) + p(1-p)^2 + \cdots + p(1-p)^{x-1} = F(x)$$

$$\frac{p\left(1 - (1-p)^x\right)}{1-(1-p)} = \frac{\cancel{p}}{\cancel{p}}\left(1 - (1-p)^x\right)$$

$$= \boxed{(1 - q^x)}$$ cumulative function is required

2) Find the mean of Random Variable X, the number of trials needed to obtain a zero when generating a series of random digits.

2) The probability of getting 0 is $1/10$.

and probability of not getting 0 is $9/10$.

x	1	2	3	----	∞
$f(x)$	$1/10$	$9/10 \times 1/10$	$9/10 \times 9/10 \times 1/10$		

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$$E[X] = 1 \times \frac{1}{10} + 2 \times \frac{9}{10} \times \frac{1}{10} + 3 \times \frac{9}{10} \times \frac{9}{10} \times \frac{1}{10} + \cdots$$

let $E[X] = S$ (A.G.P)

$$\therefore \quad S = \frac{1}{10} + 2\left(\frac{9}{10^2}\right) + 3 \times \frac{9}{10^3} + \cdots \infty$$

$$- \left[\frac{9S}{10} = \frac{9}{10^2} + 2 \times \frac{9^2}{10^3} + \cdots \infty \right]$$

$$\frac{S}{10} = \frac{1}{10} + \frac{9}{10^2} + \frac{9^2}{10^3} + \cdots \infty$$

$$\frac{S}{10} = \frac{1/10}{(1 - 9/10)} \quad \} ①$$

$$\therefore \quad S = 10$$

$$\boxed{\therefore \quad E[X] = 10}$$

3> Find value of c, that makes $f(x) = ce^{-x}$, $x = 1, 2, 3, \ldots$ pdf.
Find moment generating function for X, using which find $E[X]$ & $E[X^2]$.

3.> For $f(x)$ to be probability density function,

$$\sum_{x=1}^{\infty} ce^{-x} = 1$$

$$c[e^{-1} + e^{-2} + e^{-3} + \cdots \infty] = 1$$

$$c \left[\frac{1/e}{1 - 1/e} \right] = 1$$

$$\frac{c}{(e-1)} = 1$$

$$\therefore \boxed{c = e - 1}$$

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$$S = E[X] = (e-1)\left[\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots \infty\right]$$

$$-\left(\frac{S}{e} = (e-1)\left[\frac{1}{e^2} + \frac{2}{e^3} + \cdots \infty\right]\right)$$

$$\frac{S\cancel{(e-1)}}{e} = \cancel{(e-1)}\left[\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots \infty\right]$$

$$\frac{S}{e} = \frac{(1/e)}{(1-1/e)}$$

$$\frac{S}{e} = \frac{1}{(e-1)}$$

$$S = \frac{e}{e-1}$$

$$\boxed{\therefore\ E[X] = \frac{e}{e-1}}$$

For $E[X^2] = S$

$$S = (e-1)\left[\frac{1}{e} + \frac{4}{e^2} + \frac{9}{e^3} + \cdots \infty\right]$$

$$-\left(\frac{S}{e} = (e-1)\left[\frac{1}{e^2} + \frac{4}{e^3} + \cdots \infty\right]\right)$$

$$\frac{S\cancel{(e-1)}}{e} = \cancel{(e-1)}\left[\frac{1}{e} + \frac{3}{e^2} + \frac{5}{e^3} + \frac{7}{e^4} + \cdots \infty\right]$$

$$\frac{S}{e} = \left[\frac{1}{e} + \frac{3}{e^2} + \frac{5}{e^3} + \frac{7}{e^4} + \cdots \infty\right]$$

$$-\left(\frac{S}{e^2} = \left[\frac{1}{e^2} + \frac{3}{e^3} + \frac{5}{e^4} + \cdots \infty\right]\right)$$

$$\frac{S}{e}\left[\frac{e-1}{e}\right] = \frac{1}{e} + \frac{2}{e^2} + \frac{2}{e^3} + \frac{2}{e^4} + \cdots$$

$$= \frac{1}{e} + 2\left[\frac{1/e^2}{1 - 1/e}\right]$$

$$\frac{S(e-1)}{e^2} = \frac{1}{e} + \frac{2}{e(e-1)}$$

$$\frac{S(e-1)}{e^2} = \frac{(e^2+e)}{\cancel{e}(e-1)}$$

$$\boxed{\therefore\ S = E[X^2] = \left(\frac{e}{e-1}\right)^2 (e+1)}$$

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4.> Suppose that X is Hypergeometric with N = 20, r = 3 & n = 5. What are possible values of X? What is E[X] & var X?

4.> P(x) for Hypergeometric function, N = 20, r = 3, n = 5

$$= \frac{{}^{r}C_{x} \; {}^{N}C_{n-x}}{{}^{N}C_{n}} \qquad x \in \{0, 1, 2, 3\}$$

∴ X =	0	1	2	3
P(X) =	$\frac{{}^{3}C_{0} \; {}^{20-3}C_{5}}{{}^{20}C_{5}}$	$\frac{{}^{3}C_{1} \; {}^{17}C_{4}}{{}^{20}C_{5}}$	$\frac{{}^{3}C_{2} \; {}^{17}C_{3}}{{}^{20}C_{5}}$	$\frac{{}^{3}C_{3} \; {}^{17}C_{2}}{{}^{20}C_{5}}$
	$\frac{6188}{15504}$	$\frac{7140}{15504}$	$\frac{2040}{15504}$	$\frac{136}{15504}$

*(calculation done in rough)

$\boxed{E[X] = 0.75}$

$\boxed{E[X^2] = 1.0657}$

$\text{Var } X = E[X^2] - (E[X])^2 = \boxed{0.5032}$

5.> Define Gamma random variable X with parameter α & β. Find E[X] variance X if $m_x(t) = (1 - \beta t)^{-\alpha}$, $t < 1/\beta$ is moment generating fn of X.

5.> A random variable X with density

$$\boxed{f(x) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}} \, x^{\alpha-1} \, e^{-x/\beta}} \qquad x > 0, \; \alpha > 0, \; \beta > 0$$

is said to have a gamma distribution with parameters α and β.

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$$M_X(t) = (1-\beta t)^{-\alpha}$$

$$E[X] = M_X'(0) = (-\alpha)(-\beta)\,(1-\beta t)^{-\alpha-1}$$
{ put $t=0$ }

$$\boxed{E[X] = \alpha\beta}$$

$$E[X^2] = M''_x(0) = (-\alpha)(-\alpha+1)\,\beta^2\,(1-\beta t)^{-\alpha-2}$$
$$= (\alpha^2+\alpha)\,(\beta^2)\,(1-\beta t)^{-\alpha-2}$$
{ Put $t=0$ }

$$E[X^2] = \boxed{(\alpha^2+\alpha)\,(\beta^2)}$$

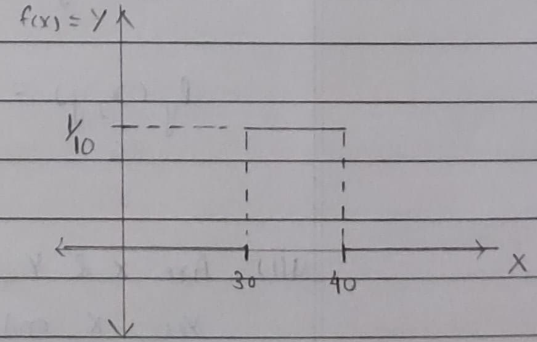
$$\text{Var } X = E[X^2] - (E[X])^2$$
$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \boxed{\alpha\beta^2}$$

ANS : $\boxed{E[X] = \alpha\beta \quad \& \quad \text{Var } X = \alpha\beta^2}$

6) Find density function & cumulative function for a random variable X distributed uniformly over (30, 40).

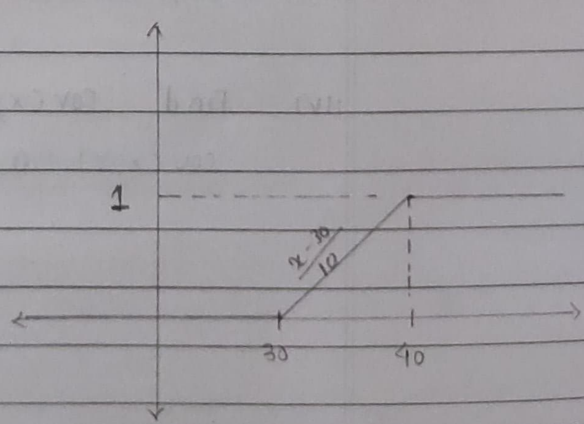
6) (A) Density Function

$$f(x) = \begin{cases} 1/10 & \underset{30}{a} \le x \le \underset{40}{b} \\ 0 & , \text{ otherwise} \end{cases}$$



(B) Cumulative Function

$$F(x) = \begin{cases} 0 & , \quad x < 30 \\ \dfrac{x-30}{10} & , \quad 30 \le x \le 40 \\ 1 & , \quad x > 40 \end{cases}$$



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7.> The joint density for (X,Y) is given by $f_{xy}(x,y) = \frac{1}{n^2}$, $x = 1,2,\dots n$, $y = 1,2,\dots n$

(i) Verify $f_{xy}(x,y)$ satisfies the conditions necessary to be density.

$Y \downarrow$ / $X \rightarrow$	1	2	3	....	n
1	$1/n^2$	$1/n^2$	$1/n^2$	---	
2	$1/n^2$	$1/n^2$	...		
3	$1/n^2$	:			
4	:				
n					

Probability density $= \sum_{x=1}^{n} \sum_{y=1}^{n} (1/n^2) = n^2/n^2 = \boxed{1}$

Hence it is probability density function.

(ii) Find Marginal densities of X & Y.

$$P_x(x,y) = \sum_{y=1}^{n} 1/n^2 = \boxed{1/n}$$

$$P_y(x,y) = \sum_{x=1}^{n} \frac{1}{n^2} = \boxed{1/n}$$

(iii) Are X & Y independent?

Yes, X and Y are independent.

(iv) Find cov(X,Y)

cov(X,Y) = 0, since X and Y are independent.

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8.) Economic conditions cause fluctuations in the prices of raw ~~material~~ commodities as well as in finished products. Let X denote the price paid for a barrel of crude oil by the initial carrier & let Y denote the price paid by the refinery purchasing the product from the carrier. Assume that joint density for (X,Y) is given by $f_{XY}(x,y) = c$, $20 < x < y < 40$. Answer the following:

(i) Find the value of c that makes this a joint density for a two-dimensional random variable.

(i) $f(x,y) = c \qquad 20 < x < y < 40$

$$\int_{x=20}^{40}\int_{y=x}^{40} c\,dx\,dy = 1$$

$$\int_{20}^{40} c(40-x)\,dx = 1$$

$$c\left[40x - \frac{x^2}{2}\right]_{20}^{40} = 1$$

$$c\left[40 \times 20 - \frac{40^2 - 20^2}{2}\right] = 1$$

$$c \times 200 = 1$$

$$\boxed{c = \frac{1}{200}}$$

(ii) Find the probability that the carrier will pay atleast \$25 per barrel and the refinery will pay atmost \$30 per barrel of oil.

$$\Rightarrow \int_{x=25}^{30}\int_{y=x}^{30} \frac{1}{200}\,dy\,dx = \int_{25}^{30} \frac{(30-x)}{200}\,dx = \frac{1}{200}\left[30(5) - \frac{(30^2-25^2)}{2}\right]$$

$$\{\underset{\text{atleast}}{25} < x < y < \underset{\text{atmost}}{30}\}$$

$$= \frac{1}{200}\left(150 - \frac{275}{2}\right) = \frac{25}{400} = \boxed{\frac{1}{16}}$$

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(iii) Find the probability that the price paid by the refinery exceeds that of the carrier by atleast \$10 per barrel.

(iii) $$\int_{20}^{30}\int_{x+10}^{40} \frac{1}{200}\,dx\,dy = \int_{20}^{30} \frac{(30-x)}{200}\,dx = \frac{1}{200}\left[30\times 40) - \frac{[50\times 10]}{2}\right]$$

$$= \boxed{\frac{1}{4}} = 0.25$$

(iv) Find the Marginal densities for X & Y.

$$f_{xy} = \int c\,dx\,dy = \int_{x=20}^{40}\int_{y=x}^{40} \frac{1}{200}\,dx\,dy$$

$$f_x(x) = \int_{y=x}^{40} \frac{1}{200}\,dx\,dy = \boxed{f_x(x) = \frac{40-x}{200}}$$

$$f_y(y) = \int_{x=20}^{y} \frac{1}{200}\,dx = \boxed{f_y(y) = \frac{y-20}{200}}$$

(v) Find the probability that the price paid by carrier is atleast \$25.

$$= \frac{1}{200}\int_{x=25}^{40}\int_{y=x}^{40} dx\,dy$$

$$= \int_{x=25}^{40} (40-x)\,dx = \frac{1}{200}\left[40\times 15 - \frac{(65\times 15)}{2}\right] = \frac{[1200-975]}{400}$$

$$= \frac{225}{400} = \boxed{\frac{9}{16}}$$

(vi) Find the probability that the price paid by refinery is almost 30 \$.

$$= \int_{x=20}^{30}\int_{y=x}^{30} c\,dx\,dy \Rightarrow \int_{20}^{30} \frac{1}{200}\times(30-x)\,dx = \left[\frac{30\times 10 - \left(\frac{30\times 30\times 10}{2}\right)}{200}\right]$$

$$= \frac{100}{400} = \boxed{0.25}$$

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(vii) Are X & Y independent? Explain.

$$f(x,y) = \frac{1}{200}$$

$$\left.\begin{array}{l} f(x) = 0.2 - 0.005x \\ f(y) = 0.005y - 0.2 \end{array}\right\} \quad f(x,y) \neq f(x) f(y)$$

Hence X & Y are Not Independent.

(viii) From a physical standpoint, should cov(X,Y) be +ve or -ve?

The covariance is -ve since increase in x will lead to decrease in y i.e. they are inversely related. ($x\uparrow$ $y\downarrow$)

(ix) Find E[X], E[Y], E[XY] & cov(X,Y)

(a) $$E[X] = \int_{x=20}^{x=40} \int_{y=x}^{40} \frac{x\,dy\,dx}{(200)}$$

$$= \int_{x=20}^{40} \left[\frac{xy}{200}\right]_x^{40}$$

$$= \int_{x=20}^{40} \left[\frac{40x}{200} - \frac{x^2}{200}\right] dx$$

$$= \left[\frac{40x^2}{400} - \frac{x^3}{600}\right]_{20}^{40}$$

$$= \frac{60 \times 20}{10} - \frac{560}{6} = \frac{160}{6} = \boxed{\frac{80}{3}} = \boxed{26.66}$$

(b) $$E[Y] = \int_{x=20}^{x=40} \int_{y=x}^{40} y \times \frac{1}{200}\, dx\, dy = \int_{x=20}^{40} \int_{y=x}^{40} \frac{y^2 dx}{(2 \times 200)} - \int_{20}^{40} \frac{(40-x)^2}{400} dx$$

$$= \left[\frac{-(40-x)^3}{3 \times 400}\right]_{20}^{40} = \frac{(20)^3}{1200} = \frac{20}{3}$$

$$= \boxed{6.66}$$

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(c) $E[XY] = \int_{x=20}^{40} \int_{y=x}^{40} \frac{xy}{200} \, dx \, dy$

$= \int_{x=20}^{40} \int_{y=x}^{40} \frac{xy^2}{400} \, dx = \int_{x=20}^{40} \frac{x(40-x)^2}{400} \, dx$

$= \int_{20}^{40} \frac{x(x^2 + 1600 - 80x)}{400} \, dx$

$= \int_{20}^{40} \left[\frac{x^3}{400} + \frac{1600x}{400} - \frac{80x^2}{400} \right] dx$

$= 1500 - 3733 + 2400$

$\approx (500/3) = \boxed{167}$

(d) $\text{cov}(X,Y) = E[XY] - E[X]\,E[Y]$

$= \frac{500}{3} - \frac{80}{3} \times \frac{20}{3}$

$= (1500 - 1600)/9$

$= \boxed{\frac{-100}{9}}$ [∴ Covariance is -ve]

(x) Find $E[Y-X]$

$E[Y-X] = E[Y] - E[X]$

$= \frac{20}{3} - \frac{80}{3} = \boxed{-20}$

-x-

SUBMITTED BY:

BHAGYA VINOD RANA

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