TUTORIAL - 1

LINEAR ALGEBRA : MATRICES

U19CS012 [BHAGYA VINOD RANA]

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Q1) Find the rank of the following matrices

(a)
$$\begin{array}{c|c} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \end{array}$$

Rank = 2

① $R_2 \leftarrow R_2 - R_1$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 2 & 6 & 5 \end{bmatrix}$$

② $R_1 \leftarrow 2 * R_1$
$$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & -1 \\ 2 & 6 & 5 \end{bmatrix}$$

③ $R_3 \leftarrow R_3 - R_1$
$$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

④ $R_3 \leftarrow R_3 - R_2$ && $R_1 \leftarrow R_1/2$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c|c} \begin{array}{c} R_1 \\ R_2 \\ \\ \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

No. of linearly independent rows = 2 = RANK.

(b)
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

Step ①: $R_2 \leftarrow R_2 - 3 * R_1$
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

Step ②: $R_3 \leftarrow R_3 - R_1$
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Step ③. $R_2 \leftarrow R_2 * 1/3$
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Step ④. $R_3 \leftarrow R_3 - R_2$
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of linearly independent rows = 2

∴ Rank of Matrix = 2

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(2)

(c)
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & 7 \end{bmatrix}$$

Rank = 3

Step 1: $R_2 \leftarrow R_2 - R_1/2$, $R_3 \leftarrow R_3 - \left(\frac{R_1}{2} \times 3\right)$, $R_4 \leftarrow R_4 - \left(\frac{R_4}{\cancel{2}} \times \cancel{6}^{3}\right)$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -5/2 & -3/2 & -7/2 \\ 0 & -7/2 & 9/2 & -1/2 \\ 0 & -6 & 3 & -4 \end{bmatrix}$$

Step 2: $R_3 \leftarrow R_3 - R_2 * \left(\frac{-2}{5}\right)\left(\frac{-7}{2}\right)$ $(R_2 7/5)$, $R_4 \leftarrow R_4 - R_2 * \left(\frac{-2}{5}\right)(-6)$ $(12 R_2/5)$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -5/2 & -3/2 & -7/2 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$$

$$\left\{ -4 + \frac{\cancel{12}^{6}}{5} \times \frac{7}{\cancel{2}} - \left(\frac{42-20}{5}\right) \right\}$$

Step 3: $R_4 \leftarrow R_4 - R_3$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & -5/2 & -3/2 & -7/2 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of linearly indepent rows = 3

∴ Hence Rank of Matrix = 3

(d)
$$\begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$$

No. of linearly independent rows = 2

Rank of given matrix = 2

Q2.> Reduce the following matrix to triangular form

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

① Eliminate elements in first column using R_1

$R_2 \leftarrow R_2 + 3R_1$ $R_3 \leftarrow R_3 + \frac{5}{3} R_1$

$$\begin{bmatrix} 3 & -4 & -5 \\ 0 & -11 & -11 \\ 0 & -11/3 & -22/3 \end{bmatrix}$$

② Eliminate the elements in second column using R_2

$R_3 \leftarrow R_3 - R_2/3$

$$\begin{bmatrix} 3 & -4 & -5 \\ 0 & -11 & -11 \\ 0 & 0 & -11/3 \end{bmatrix}$$

Triangular Form

No. of Linearly independent rows = 3

∴ Rank of Matrix = 3

Q3.> For the Matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find Non singular matrices P and Q such that PAQ is in Normal form. Hence find the rank of A.

In Normal form of Matrix, every row can have maximum of single one and rest are all zeros.

Identity Matrix

$A = I A I$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

① $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{row effect}} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{column effect}}$$

(2) $C_2 \leftarrow C_2 - C_1$ $\quad$ $C_3 \leftarrow C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $R_3 \leftarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) $C_3 \leftarrow C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

NORMAL FORM

Each Row = Atmost only 1

Now A is in Normal form with Rank 2 (No. of linearly independent rows) (R_1 & R_2)

[No. of non-zero rows in Normal form = RANK]

Comparing it PAQ

ANS:

$$\boxed{P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

4.Y Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by elementary row operations

$A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

(1) swap $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

(2) $R_3 \leftarrow R_3 - 3(R_1)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

(3) $(R_1 \leftarrow R_1 - 2 * R_2)$, $(R_3 \leftarrow R_3 + 5 * R_2)$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

(4) $R_2 \leftarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix} A$$

(5) $R_3 \leftarrow R_3/2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

Ans: $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

(6) $R_1 \leftarrow R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -4 & 3 & -1 \\ 2.5 & -1.5 & 0.5 \end{bmatrix}$$

$I = (A^{-1})(A)$

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5.) If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(i) Find A^{-1}

$$A = IA$$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

① $R_1 \leftarrow R1/3$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

② $R_2 \leftarrow R_2 - (2) R_1$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

③ $R_1 \leftarrow R_1 - R_2$ $\quad$ $R_3 \leftarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

④ $R_2 \leftarrow (-1) R_2$ $\quad$ $R_3 \leftarrow (-(3)) R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

⑤ $R_2 \leftarrow R_2 + (4/3) R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$I = A^{-1} A$$

ANS

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}$$

5.> (ii) Show that $A^3 = A^{-1}$

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$LHS = A^3 = A^2 \times A$

Let's calculate

$$A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 9-6 & -9+9-4 & 12-12+4 \\ 6-6+0 & -6+9\overset{-4}{\cancel{-3}} & 8-12\cancel{8}+4 \\ 0-2+0 & 0+3-1 & 0+-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = LHS = A^2 \times A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8+0 & -9+12-4 & 12-16+4 \\ -2 & 3 & -4 \\ -6+4+0 & \cancel{6}-\cancel{6}+3 & -\cancel{8}+\cancel{8}-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = RHS$$

Hence $A^3 = A^{-1}$, Proved.

SUBMITTED BY :

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BHAGYA RANA

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