SPECIAL RANDOM VARIABLES

Certain types of random variables occur over and over again in applications. In this chapter, we will study a variety of them.

5.1 THE BERNOULLI AND BINOMIAL RANDOM VARIABLES

Suppose that a trial, or an experiment, whose outcome can be classified as either a "success" or as a "failure" is performed. If we let $X = 1$ when the outcome is a success and $X = 0$ when it is a failure, then the probability mass function of X is given by

$$\begin{aligned} P\{X = 0\} &= 1 - p \\ P\{X = 1\} &= p \end{aligned} \tag{5.1.1}$$

where p, $0 \leq p \leq 1$, is the probability that the trial is a "success."

A random variable X is said to be a Bernoulli random variable (after the Swiss mathematician James Bernoulli) if its probability mass function is given by Equations 5.1.1 for some $p \in (0, 1)$. Its expected value is

$$E[X] = 1 \cdot P\{X = 1\} + 0 \cdot P\{X = 0\} = p$$

That is, the expectation of a Bernoulli random variable is the probability that the random variable equals 1.

Suppose now that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability $1 - p$, are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a *binomial* random variable with parameters (n, p).

The probability mass function of a binomial random variable with parameters n and p is given by

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots, n \tag{5.1.2}$$

where $\binom{n}{i} = n!/[i!(n-i)!]$ is the number of different groups of i objects that can be chosen from a set of n objects. The validity of Equation 5.1.2 may be verified by first noting that the probability of any particular sequence of the n outcomes containing i successes and $n-i$ failures is, by the assumed independence of trials, $p^i(1-p)^{n-i}$. Equation 5.1.2 then follows since there are $\binom{n}{i}$ different sequences of the n outcomes leading to i successes and $n-i$ failures — which can perhaps most easily be seen by noting that there are $\binom{n}{i}$ different selections of the i trials that result in successes. For instance, if $n = 5$, $i = 2$, then there are $\binom{5}{2}$ choices of the two trials that are to result in successes — namely, any of the outcomes

$$\begin{array}{lll}
(s, s, f, f, f) & (f, s, s, f, f) & (f, f, s, f, s) \\
(s, f, s, f, f) & (f, s, f, s, f) & \\
(s, f, f, s, f) & (f, s, f, f, s) & (f, f, f, s, s) \\
(s, f, f, f, s) & (f, f, s, s, f) &
\end{array}$$

where the outcome (f, s, f, s, f) means, for instance, that the two successes appeared on trials 2 and 4. Since each of the $\binom{5}{2}$ outcomes has probability $p^2(1-p)^3$, we see that the probability of a total of 2 successes in 5 independent trials is $\binom{5}{2}p^2(1-p)^3$. Note that, by the binomial theorem, the probabilities sum to 1, that is,

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = [p + (1-p)]^n = 1$$

The probability mass function of three binomial random variables with respective parameters (10, .5), (10, .3), and (10, .6) are presented in Figure 5.1. The first of these is symmetric about the value .5, whereas the second is somewhat weighted, or *skewed*, to lower values and the third to higher values.

EXAMPLE 5.1a It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

SOLUTION If X is the number of defective disks in a package, then assuming that customers always take advantage of the guarantee, it follows that X is a binomial random variable

EXAMPLE 5.1b The color of one's eyes is determined by a single pair of genes, with the gene for brown eyes being dominant over the one for blue eyes. This means that an individual having two blue-eyed genes will have blue eyes, while one having either two brown-eyed genes or one brown-eyed and one blue-eyed gene will have brown eyes. When two people mate, the resulting offspring receives one randomly chosen gene from each of its parents' gene pair. If the eldest child of a pair of brown-eyed parents has blue eyes, what is the probability that exactly two of the four other children (none of whom is a twin) of this couple also have blue eyes?

SOLUTION To begin, note that since the eldest child has blue eyes, it follows that both parents must have one blue-eyed and one brown-eyed gene. (For if either had two brown-eyed genes, then each child would receive at least one brown-eyed gene and would thus have brown eyes.) The probability that an offspring of this couple will have blue eyes is equal to the probability that it receives the blue-eyed gene from both parents, which is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$. Hence, because each of the other four children will have blue eyes with probability $\frac{1}{4}$, it follows that the probability that exactly two of them have this eye color is

$$\binom{4}{2}(1/4)^2(3/4)^2 = 27/128 \quad \blacksquare$$

5.3 THE HYPERGEOMETRIC RANDOM VARIABLE

A bin contains $N + M$ batteries, of which N are of acceptable quality and the other M are defective. A sample of size n is to be randomly chosen (without replacements) in the sense that the set of sampled batteries is equally likely to be any of the $\binom{N+M}{n}$ subsets of size n. If we let X denote the number of acceptable batteries in the sample, then

$$P\{X = i\} = \frac{\binom{N}{i}\binom{M}{n-i}}{\binom{N+M}{n}}, \qquad i = 0, 1, \ldots, \min(N, n)^* \tag{5.3.1}$$

Any random variable X whose probability mass function is given by Equation 5.3.1 is said to be a *hypergeometric* random variable with parameters N, M, n.

EXAMPLE 5.3a The components of a 6-component system are to be randomly chosen from a bin of 20 used components. The resulting system will be functional if at least 4 of its 6 components are in working condition. If 15 of the 20 components in the bin are in working condition, what is the probability that the resulting system will be functional?

SOLUTION If X is the number of working components chosen, then X is hypergeometric with parameters 15, 5, 6. The probability that the system will be functional is

$$\begin{aligned} P\{X \geq 4\} &= \sum_{i=4}^{6} P\{X = i\} \\ &= \frac{\binom{15}{4}\binom{5}{2} + \binom{15}{5}\binom{5}{1} + \binom{15}{6}\binom{5}{0}}{\binom{20}{6}} \\ &\approx .8687 \quad \blacksquare \end{aligned}$$

---

* We are following the convention that $\binom{m}{r} = 0$ if $r > m$ or if $r < 0$.

To compute the mean and variance of a hypergeometric random variable whose probability mass function is given by Equation 5.3.1, imagine that the batteries are drawn sequentially and let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th selection is acceptable} \\ 0 & \text{otherwise} \end{cases}$$

Now, since the ith selection is equally likely to be any of the $N + M$ batteries, of which N are acceptable, it follows that

$$P\{X_i = 1\} = \frac{N}{N+M} \tag{5.3.2}$$

Also, for $i \neq j$,

$$\begin{aligned} P\{X_i = 1, X_j = 1\} &= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\} \\ &= \frac{N}{N+M}\frac{N-1}{N+M-1} \end{aligned} \tag{5.3.3}$$

which follows since, given that the ith selection is acceptable, the jth selection is equally likely to be any of the other $N + M - 1$ batteries of which $N - 1$ are acceptable.

To compute the mean and variance of X, the number of acceptable batteries in the sample of size n, use the representation

$$X = \sum_{i=1}^{n} X_i$$

This gives

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \frac{nN}{N+M} \tag{5.3.4}$$

Also, Corollary 4.7.3 for the variance of a sum of random variables gives

$$\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) \tag{5.3.5}$$

Now, X_i is a Bernoulli random variable and so

$$\text{Var}(X_i) = P\{X_i = 1\}(1 - P\{X_i = 1\}) = \frac{N}{N+M}\frac{M}{N+M} \tag{5.3.6}$$

5.4 THE UNIFORM RANDOM VARIABLE

A random variable X is said to be uniformly distributed over the interval $[\alpha, \beta]$ if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

A graph of this function is given in Figure 5.4. Note that the foregoing meets the requirements of being a probability density function since

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} dx = 1$$

The uniform distribution arises in practice when we suppose a certain random variable is equally likely to be near any value in the interval $[\alpha, \beta]$.

The probability that X lies in any subinterval of $[\alpha, \beta]$ is equal to the length of that subinterval divided by the length of the interval $[\alpha, \beta]$. This follows since when $[a, b]$

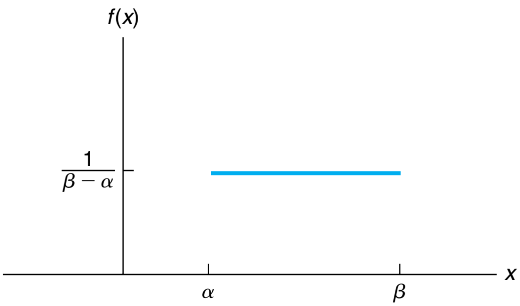

FIGURE 5.4 *Graph of $f(x)$ for a uniform $[\alpha, \beta]$.*

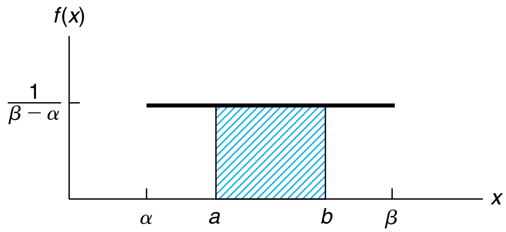

FIGURE 5.5 *Probabilities of a uniform random variable.*

is a subinterval of $[\alpha, \beta]$ (see Figure 5.5),

$$\begin{aligned} P\{a < X < b\} &= \frac{1}{\beta - \alpha} \int_a^b dx \\ &= \frac{b - a}{\beta - \alpha} \end{aligned}$$

EXAMPLE 5.4a If X is uniformly distributed over the interval $[0, 10]$, compute the probability that **(a)** $2 < X < 9$, **(b)** $1 < X < 4$, **(c)** $X < 5$, **(d)** $X > 6$.

SOLUTION The respective answers are **(a)** 7/10, **(b)** 3/10, **(c)** 5/10, **(d)** 4/10. ■

EXAMPLE 5.4b Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus;
(b) at least 12 minutes for a bus.

SOLUTION Let X denote the time in minutes past 7 A.M. that the passenger arrives at the stop. Since X is a uniform random variable over the interval $(0, 30)$, it follows that the passenger will have to wait less than 5 minutes if he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence, the desired probability for **(a)** is

$$P\{10 < X < 15\} + P\{25 < X < 30\} = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

Similarly, he would have to wait at least 12 minutes if he arrives between 7 and 7:03 or between 7:15 and 7:18, and so the probability for **(b)** is

$$P\{0 < X < 3\} + P\{15 < X < 18\} = \frac{3}{30} + \frac{3}{30} = \frac{1}{5} \quad ■$$

The mean of a uniform $[\alpha, \beta]$ random variable is

$$
\begin{aligned}
E[X] &= \int_\alpha^\beta \frac{x}{\beta - \alpha}\, dx \\
&= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} \\
&= \frac{(\beta - \alpha)(\beta + \alpha)}{2(\beta - \alpha)}
\end{aligned}
$$

or

$$
E[X] = \frac{\alpha + \beta}{2}
$$

Or, in other words, the expected value of a uniform $[\alpha, \beta]$ random variable is equal to the midpoint of the interval $[\alpha, \beta]$, which is clearly what one would expect. (Why?)

The variance is computed as follows.

$$
\begin{aligned}
E[X^2] &= \frac{1}{\beta - \alpha} \int_\alpha^\beta x^2 dx \\
&= \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)} \\
&= \frac{\beta^2 + \alpha\beta + \alpha^2}{3}
\end{aligned}
$$

and so

$$
\begin{aligned}
\operatorname{Var}(X) &= \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \left(\frac{\alpha + \beta}{2}\right)^2 \\
&= \frac{\alpha^2 + \beta^2 - 2\alpha\beta}{12} \\
&= \frac{(\beta - \alpha)^2}{12}
\end{aligned}
$$

EXAMPLE 5.4c The current in a semiconductor diode is often measured by the Shockley equation

$$
I = I_0(e^{aV} - 1)
$$

where V is the voltage across the diode; I_0 is the reverse current; a is a constant; and I is the resulting diode current. Find $E[I]$ if $a = 5$, $I_0 = 10^{-6}$, and V is uniformly distributed over $(1, 3)$.

5.5 NORMAL RANDOM VARIABLES

A random variable is said to be normally distributed with parameters μ and σ^2, and we write $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty^*$$

The normal density $f(x)$ is a bell-shaped curve that is symmetric about μ and that attains its maximum value of $1/\sigma\sqrt{2\pi} \approx 0.399/\sigma$ at $x =$

The normal distribution was introduced by the French mathematician Abraham de Moivre in 1733 and was used by him to approximate probabilities associated with binomial random variables when the binomial parameter n is large. This result was later extended by Laplace and others and is now encompassed in a probability theorem known as the central limit theorem, which gives a theoretical base to the often noted empirical observation that, in practice, many random phenomena obey, at least approximately, a normal probability distribution. Some examples of this behavior are the height of a person, the velocity in any direction of a molecule in gas, and the error made in measuring a physical quantity.

To compute $E[X]$ note that

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/2\sigma^2}dx$$

Letting $y = (x-\mu)/\sigma$ gives that

$$E[X-\mu] = \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ye^{-y^2/2}dy$$

But

$$\int_{-\infty}^{\infty} ye^{-y^2/2}dy = -e^{-y^2/2}\Big|_{-\infty}^{\infty} = 0$$

showing that $E[X-\mu] = 0$, or equivalently that

$$E[X] = \mu$$

Using this, we now compute $\mathrm{Var}(X)$ as follows:

$$\begin{aligned}\mathrm{Var}(X) &= E[(X-\mu)^2]\\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2e^{-(x-\mu)^2/2\sigma^2}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma^2y^2e^{-y^2/2}dy \qquad (5.5.1)\end{aligned}$$

With $u = y$ and $dv = ye^{-y^2/2}$, the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

yields that

$$\int_{-\infty}^{\infty} y^2e^{-y^2/2}dy = -ye^{-y^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-y^2/2}dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2/2}dy$$

Hence, from (5.5.1)

$$\begin{aligned}\mathrm{Var}(X) &= \sigma^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy\\ &= \sigma^2\end{aligned}$$

where the preceding used that $\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$ is the density function of a normal random variable with parameters $\mu = 0$ and $\sigma = 1$, so its integral must equal 1.

Thus μ and σ^2 represent, respectively, the mean and variance of the normal distribution.

A very important property of normal random variables is that if X is normal with mean μ and variance σ^2, then for any constants a and b, $b \neq 0$, the random variable $Y = a+bX$ is also a normal random variable with parameters

$$E[Y] = E[a + bX] = a + bE[X] = a + b\mu$$

and variance

$$\mathrm{Var}(Y) = \mathrm{Var}(a + bX) = b^2\mathrm{Var}(X) = b^2\sigma^2$$

To verify this, let $F_Y(y)$ be the distribution function of Y. Then, for $b > 0$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(a + bX \le y) \\ &= P\left(X \le \frac{y-a}{b}\right) \\ &= F_X\left(\frac{y-a}{b}\right) \end{aligned}$$

where F_X is the distribution function of X. Similarly, if $b < 0$, then

$$\begin{aligned} F_Y(y) &= P(a + bX \le y) \\ &= P\left(X \ge \frac{y-a}{b}\right) \\ &= 1 - F_X\left(\frac{y-a}{b}\right) \end{aligned}$$

Differentiation yields that the density function of Y is

$$f_Y(y) = \begin{cases} \frac{1}{b} f_X\left(\frac{y-a}{b}\right), & \text{if} \quad b > 0 \\ -\frac{1}{b} f_X\left(\frac{y-a}{b}\right), & \text{if} \quad b < 0 \end{cases}$$

which can be written as

$$\begin{aligned} f_Y(y) &= \frac{1}{|b|} f_X\left(\frac{y-a}{b}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma|b|} e^{-\left(\frac{y-a}{b}-\mu\right)^2/2\sigma^2} \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}\sigma|b|} e^{-(y-a-b\mu)^2/2b^2\sigma^2}$$

showing that $Y = a + bX$ is normal with mean $a + b\mu$ and variance $b^2\sigma^2$.

It follows from the foregoing that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with mean 0 and variance 1. Such a random variable Z is said to have a *standard*, or *unit*, normal distribution. Let $\Phi(\cdot)$ denote its distribution function. That is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \qquad -\infty < x < \infty$$

This result that $Z = (X - \mu)/\sigma$ has a standard normal distribution when X is normal with parameters μ and σ^2 is quite important, for it enables us to write all probability statements about X in terms of probabilities for Z. For instance, to obtain $P\{X < b\}$, we note that X will be less than b if and only if $(X - \mu)/\sigma$ is less than $(b - \mu)/\sigma$, and so

$$\begin{aligned} P\{X < b\} &= P\left\{\frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma}\right\} \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) \end{aligned}$$

Similarly, for any $a < b$,

$$\begin{aligned} P\{a < X < b\} &= P\left\{\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma}\right\} \\ &= P\left\{\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma}\right\} \\ &= P\left\{Z < \frac{b - \mu}{\sigma}\right\} - P\left\{Z < \frac{a - \mu}{\sigma}\right\} \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \end{aligned}$$

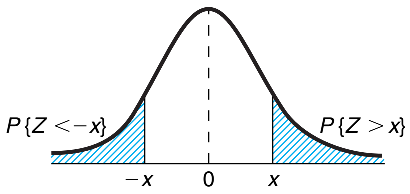

FIGURE 5.8 *Standard normal probabilities.*

It remains for us to compute $\Phi(x)$. This has been accomplished by an approximation and the results are presented in Table A1 of the Appendix, which tabulates $\Phi(x)$ (to a 4-digit level of accuracy) for a wide range of nonnegative values of x. In addition, Program 5.5a of the text disk can be used to obtain $\Phi(x)$.

While Table A1 tabulates $\Phi(x)$ only for nonnegative values of x, we can also obtain $\Phi(-x)$ from the table by making use of the symmetry (about 0) of the standard normal probability density function. That is, for $x > 0$, if Z represents a standard normal random variable, then (see Figure 5.8)

$$\begin{aligned}\Phi(-x) &= P\{Z < -x\}\\ &= P\{Z > x\} \quad \text{by symmetry}\\ &= 1 - \Phi(x)\end{aligned}$$

Thus, for instance,

$$P\{Z < -1\} = \Phi(-1) = 1 - \Phi(1) = 1 - .8413 = .1587$$

EXAMPLE 5.5a If X is a normal random variable with mean $\mu = 3$ and variance $\sigma^2 = 16$, find

(a) $P\{X < 11\}$;
(b) $P\{X > -1\}$;
(c) $P\{2 < X < 7\}$.

SOLUTION

(a)

$$\begin{aligned}P\{X < 11\} &= P\left\{\frac{X-3}{4} < \frac{11-3}{4}\right\}\\ &= \Phi(2)\\ &= .9772\end{aligned}$$

(b)

$$\begin{aligned}P\{X > -1\} &= P\left\{\frac{X-3}{4} > \frac{-1-3}{4}\right\}\\ &= P\{Z > -1\}\end{aligned}$$

$$
\begin{aligned}
&= P\{Z < 1\} \\
&= .8413
\end{aligned}
$$

(c)

$$
\begin{aligned}
P\{2 < X < 7\} &= P\left\{\frac{2-3}{4} < \frac{X-3}{4} < \frac{7-3}{4}\right\} \\
&= \Phi(1) - \Phi(-1/4) \\
&= \Phi(1) - (1 - \Phi(1/4)) \\
&= .8413 + .5987 - 1 = .4400 \quad \blacksquare
\end{aligned}
$$

EXAMPLE 5.5b Suppose that a binary message — either "0" or "1" — must be transmitted by wire from location A to location B. However, the data sent over the wire are subject to a channel noise disturbance and so to reduce the possibility of error, the value 2 is sent over the wire when the message is "1" and the value -2 is sent when the message is "0." If x, $x = \pm 2$, is the value sent at location A then R, the value received at location B, is given by $R = x + N$, where N is the channel noise disturbance. When the message is received at location B, the receiver decodes it according to the following rule:

$$
\begin{aligned}
&\text{if } R \geq .5, \text{ then "1" is concluded} \\
&\text{if } R < .5, \text{ then "0" is concluded}
\end{aligned}
$$

Because the channel noise is often normally distributed, we will determine the error probabilities when N is a standard normal random variable.

There are two types of errors that can occur: One is that the message "1" can be incorrectly concluded to be "0" and the other that "0" is incorrectly concluded to be "1." The first type of error will occur if the message is "1" and $2 + N < .5$, whereas the second will occur if the message is "0" and $-2 + N \geq .5$.

Hence,

$$
\begin{aligned}
P\{\text{error}|\text{message is "1"}\} &= P\{N < -1.5\} \\
&= 1 - \Phi(1.5) = .0668
\end{aligned}
$$

and

$$
\begin{aligned}
P\{\text{error}|\text{message is "0"}\} &= P\{N > 2.5\} \\
&= 1 - \Phi(2.5) = .0062 \quad \blacksquare
\end{aligned}
$$

EXAMPLE 5.5c The power W dissipated in a resistor is proportional to the square of the voltage V. That is,

$$
W = rV^2
$$

where r is a constant. If $r = 3$, and V can be assumed (to a very good approximation) to be a normal random variable with mean 6 and standard deviation 1, find

(a) $E[W]$;
(b) $P\{W > 120\}$.

SOLUTION

(a)

$$\begin{aligned} E[W] &= E[3V^2] \\ &= 3E[V^2] \\ &= 3(\text{Var}[V] + E^2[V]) \\ &= 3(1 + 36) = 111 \end{aligned}$$

(b)

$$\begin{aligned} P\{W > 120\} &= P\{3V^2 > 120\} \\ &= P\{V > \sqrt{40}\} \\ &= P\{V - 6 > \sqrt{40} - 6\} \\ &= P\{Z > .3246\} \\ &= 1 - \Phi(.3246) \\ &= .3727 \quad \blacksquare \end{aligned}$$

Another important result is that the sum of independent normal random variables is also a normal random variable. To see this, suppose that X_i, $i = 1, \ldots, n$, are independent, with X_i being normal with mean μ_i and variance σ_i^2. The moment generating function of $\sum_{i=1}^n X_i$ is as follows.

$$\begin{aligned} E\left[\exp\left\{t\sum_{i=1}^n X_i\right\}\right] &= E\left[e^{tX_1}e^{tX_2}\cdots e^{tX_n}\right] \\ &= \prod_{i=1}^n E\left[e^{tX_i}\right] \qquad \text{by independence} \\ &= \prod_{i=1}^n e^{\mu_i t + \sigma_i^2 t^2/2} \\ &= e^{\mu t + \sigma^2 t^2/2} \end{aligned}$$

where

$$\mu = \sum_{i=1}^n \mu_i, \qquad \sigma^2 = \sum_{i=1}^n \sigma_i^2$$

Therefore, $\sum_{i=1}^n X_i$ has the same moment generating function as a normal random variable having mean μ and variance σ^2. Hence, from the one-to-one correspondence between moment generating functions and distributions, we can conclude that $\sum_{i=1}^{n} X_i$ is normal with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

EXAMPLE 5.5d Data from the National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angeles is a normal random variable with a mean of 12.08 inches and a standard deviation of 3.1 inches.

(a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches.

(b) Find the probability that next year's precipitation will exceed that of the following year by more than 3 inches.

Assume that the precipitation totals for the next 2 years are independent.

SOLUTION Let X_1 and X_2 be the precipitation totals for the next 2 years.

(a) Since $X_1 + X_2$ is normal with mean 24.16 and variance $2(3.1)^2 = 19.22$, it follows that

$$\begin{aligned} P\{X_1 + X_2 > 25\} &= P\left\{\frac{X_1 + X_2 - 24.16}{\sqrt{19.22}} > \frac{25 - 24.16}{\sqrt{19.22}}\right\} \\ &= P\{Z > .1916\} \\ &\approx .4240 \end{aligned}$$

(b) Since $-X_2$ is a normal random variable with mean -12.08 and variance $(-1)^2(3.1)^2$, it follows that $X_1 - X_2$ is normal with mean 0 and variance 19.22. Hence,

$$\begin{aligned} P\{X_1 > X_2 + 3\} &= P\{X_1 - X_2 > 3\} \\ &= P\left\{\frac{X_1 - X_2}{\sqrt{19.22}} > \frac{3}{\sqrt{19.22}}\right\} \\ &= P\{Z > .6843\} \\ &\approx .2469 \end{aligned}$$

Thus there is a 42.4 percent chance that the total precipitation in Los Angeles during the next 2 years will exceed 25 inches, and there is a 24.69 percent chance that next year's precipitation will exceed that of the following year by more than 3 inches. ■

For $\alpha \in (0, 1)$, let z_α be such that

$$P\{Z > z_\alpha\} = 1 - \Phi(z_\alpha) = \alpha$$

That is, the probability that a standard normal random variable is greater than z_α is equal to α (see Figure 5.9).

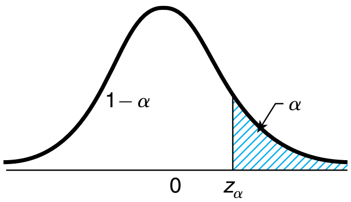

FIGURE 5.9 $P\{Z > z_\alpha\} = \alpha$.

The value of z_α can, for any α, be obtained from Table A1. For instance, since

$$\begin{aligned} 1 - \Phi(1.645) &= .05 \\ 1 - \Phi(1.96) &= .025 \\ 1 - \Phi(2.33) &= .01 \end{aligned}$$

it follows that

$$z_{.05} = 1.645, \qquad z_{.025} = 1.96, \qquad z_{.01} = 2.33$$

Program 5.5b on the text disk can also be used to obtain the value of z_α.

Since

$$P\{Z < z_\alpha\} = 1 - \alpha$$

it follows that $100(1-\alpha)$ percent of the time a standard normal random variable will be less than z_α. As a result, we call z_α the $100(1-\alpha)$ *percentile* of the standard normal distribution.

5.6 EXPONENTIAL RANDOM VARIABLES

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is said to be an *exponential* random variable (or, more simply, is said to be exponentially distributed) with parameter λ. The cumulative distribution function $F(x)$ of an exponential random variable is given by

$$\begin{aligned} F(x) &= P\{X \leq x\} \\ &= \int_0^x \lambda e^{-\lambda y}\, dy \\ &= 1 - e^{-\lambda x}, \qquad x \geq 0 \end{aligned}$$

The exponential distribution often arises, in practice, as being the distribution of the amount of time until some specific event occurs. For instance, the amount of time (starting from now) until an earthquake occurs, or until a new war breaks out, or until a telephone call you receive turns out to be a wrong number are all random variables that tend in practice to have exponential distributions (see Section 5.6.1 for an explanation).

The moment generating function of the exponential is given by

$$\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx \\
&= \frac{\lambda}{\lambda - t}, \qquad t < \lambda
\end{aligned}$$

Differentiation yields

$$\begin{aligned}
\phi'(t) &= \frac{\lambda}{(\lambda - t)^2} \\
\phi''(t) &= \frac{2\lambda}{(\lambda - t)^3}
\end{aligned}$$

and so

$$\begin{aligned}
E[X] &= \phi'(0) = 1/\lambda \\
\text{Var}(X) &= \phi''(0) - (E[X])^2 \\
&= 2/\lambda^2 - 1/\lambda^2 \\
&= 1/\lambda^2
\end{aligned}$$

Thus λ is the reciprocal of the mean, and the variance is equal to the square of the mean.

The key property of an exponential random variable is that it is memoryless, where we say that a nonnegative random variable X is *memoryless* if

$$P\{X > s + t | X > t\} = P\{X > s\} \qquad \text{for all } s, t \geq 0 \tag{5.6.1}$$

To understand why Equation 5.6.1 is called the *memoryless property*, imagine that X represents the length of time that a certain item functions before failing. Now let us consider the probability that an item that is still functioning at age t will continue to function for at least an additional time s. Since this will be the case if the total functional lifetime of the item exceeds $t + s$ given that the item is still functioning at t, we see that

$$
\begin{aligned}
&P\{\text{additional functional life of } t\text{-unit-old item exceeds } s\} \\
&\quad = P\{X > t + s | X > t\}
\end{aligned}
$$

Thus, we see that Equation 5.6.1 states that the distribution of additional functional life of an item of age t is the same as that of a new item — in other words, when Equation 5.6.1 is satisfied, there is no need to remember the age of a functional item since as long as it is still functional it is "as good as new."

The condition in Equation 5.6.1 is equivalent to

$$
\frac{P\{X > s + t, X > t\}}{P\{X > t\}} = P\{X > s\}
$$

or

$$
P\{X > s + t\} = P\{X > s\}P\{X > t\} \tag{5.6.2}
$$

When X is an exponential random variable, then

$$
P\{X > x\} = e^{-\lambda x}, \qquad x > 0
$$

and so Equation 5.6.2 is satisfied (since $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$). Hence, *exponentially distributed random variables are memoryless* (and in fact it can be shown that they are the only random variables that are memoryless).

EXAMPLE 5.6a Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000-mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery? What can be said when the distribution is not exponential?

SOLUTION It follows, by the memoryless property of the exponential distribution, that the remaining lifetime (in thousands of miles) of the battery is exponential with parameter $\lambda = 1/10$. Hence the desired probability is

$$
\begin{aligned}
P\{\text{remaining lifetime} > 5\} &= 1 - F(5) \\
&= e^{-5\lambda} \\
&= e^{-1/2} \approx .604
\end{aligned}
$$

However, if the lifetime distribution F is not exponential, then the relevant probability is

$$P\{\text{lifetime} > t + 5|\text{lifetime} > t\} = \frac{1 - F(t + 5)}{1 - F(t)}$$

where t is the number of miles that the battery had been in use prior to the start of the trip. Therefore, if the distribution is not exponential, additional information is needed (namely, t) before the desired probability can be calculated. ■

For another illustration of the memoryless property, consider the following example.

EXAMPLE 5.6b A crew of workers has 3 interchangeable machines, of which 2 must be working for the crew to do its job. When in use, each machine will function for an exponentially distributed time having parameter λ before breaking down. The workers decide initially to use machines A and B and keep machine C in reserve to replace whichever of A or B breaks down first. They will then be able to continue working until one of the remaining machines breaks down. When the crew is forced to stop working because only one of the machines has not yet broken down, what is the probability that the still operable machine is machine C?

SOLUTION This can be easily answered, without any need for computations, by invoking the memoryless property of the exponential distribution. The argument is as follows: Consider the moment at which machine C is first put in use. At that time either A or B would have just broken down and the other one — call it machine 0 — will still be functioning. Now even though 0 would have already been functioning for some time, by the memoryless property of the exponential distribution, it follows that its remaining lifetime has the same distribution as that of a machine that is just being put into use. Thus, the remaining lifetimes of machine 0 and machine C have the same distribution and so, by symmetry, the probability that 0 will fail before C is $\frac{1}{2}$. ■

*5.7 THE GAMMA DISTRIBUTION

A random variable is said to have a gamma distribution with parameters (α, λ), $\lambda > 0$, $\alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}\, dx \\ &= \int_0^\infty e^{-y} y^{\alpha-1}\, dy \quad \text{(by letting } y = \lambda x\text{)} \end{aligned}$$

The integration by parts formula $\int u\, dv = uv - \int v\, du$ yields, with $u = y^{\alpha-1}$, $dv = e^{-y}dy$, $v = -e^{-y}$, that for $\alpha > 1$,

$$\begin{aligned} \int_0^\infty e^{-y} y^{\alpha-1}\, dy &= -e^{-y} y^{\alpha-1} \Big|_{y=0}^{y=\infty} + \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}\, dy \\ &= (\alpha-1)\int_0^\infty e^{-y} y^{\alpha-2}\, dy \end{aligned}$$

or

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \tag{5.7.1}$$

When α is an integer — say, $\alpha = n$ — we can iterate the foregoing to obtain that

$$\begin{aligned} \Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) && \text{by letting } \alpha = n-1 \text{ in Eq. 5.7.1} \\ &= (n-1)(n-2)(n-3)\Gamma(n-3) && \text{by letting } \alpha = n-2 \text{ in Eq. 5.7.1} \\ &\ \ \vdots \\ &= (n-1)!\,\Gamma(1) \end{aligned}$$

Because

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$$

* Optional section.

we see that

$$\Gamma(n) = (n-1)!$$

The function $\Gamma(\alpha)$ is called the *gamma* function.

It should be noted that when $\alpha = 1$, the gamma distribution reduces to the exponential with mean $1/\lambda$.

The moment generating function of a gamma random variable X with parameters (α, λ) is obtained as follows:

$$\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} e^{-\lambda x} x^{\alpha-1}\, dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda-t)x} x^{\alpha-1}\, dx \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^{\alpha-1}\, dy \quad [\text{by } y = (\lambda-t)x] \\
&= \left(\frac{\lambda}{\lambda-t}\right)^\alpha
\end{aligned} \tag{5.7.2}$$

Differentiation of Equation 5.7.2 yields

$$\begin{aligned}
\phi'(t) &= \frac{\alpha\lambda^\alpha}{(\lambda-t)^{\alpha+1}} \\
\phi''(t) &= \frac{\alpha(\alpha+1)\lambda^\alpha}{(\lambda-t)^{\alpha+2}}
\end{aligned}$$

Hence,

$$E[X] = \phi'(0) = \frac{\alpha}{\lambda} \tag{5.7.3}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\
&= \phi''(0) - \left(\frac{\alpha}{\lambda}\right)^2 \\
&= \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}
\end{aligned} \tag{5.7.4}$$

An important property of the gamma is that if X_1 and X_2 are independent gamma random variables having respective parameters (α_1, λ) and (α_2, λ), then $X_1 + X_2$ is a gamma random variable with parameters $(\alpha_1 + \alpha_2, \lambda)$. This result easily follows since

$$\begin{aligned}
\phi_{X_1+X_2}(t) &= E[e^{t(X_1+X_2)}] \\
&= \phi_{X_1}(t)\phi_{X_2}(t)
\end{aligned} \tag{5.7.5}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2} \qquad \text{from Equation 5.7.2}$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1+\alpha_2}$$

which is seen to be the moment generating function of a gamma $(\alpha_1 + \alpha_2, \lambda)$ random variable. Since a moment generating function uniquely characterizes a distribution, the result entails.

The foregoing result easily generalizes to yield the following proposition.

PROPOSITION 5.7.1 If $X_i, i = 1, \ldots, n$ are independent gamma random variables with respective parameters (α_i, λ), then $\sum_{i=1}^n X_i$ is gamma with parameters $\sum_{i=1}^n \alpha_i, \lambda$.

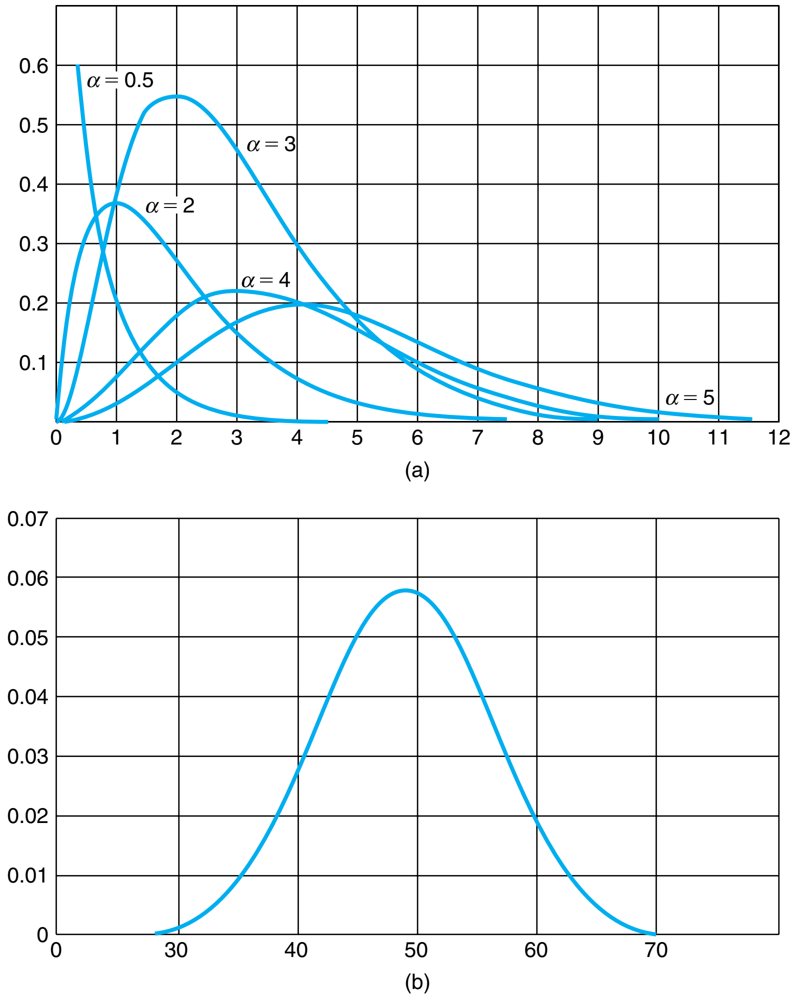

FIGURE 5.11 *Graphs of the gamma* $(\alpha, 1)$ *density for (a)* $\alpha = .5, 2, 3, 4, 5$ *and (b)* $\alpha = 50$.

Since the gamma distribution with parameters $(1, \lambda)$ reduces to the exponential with the rate λ, we have thus shown the following useful result.

Corollary 5.7.2

If $X_1, \ldots, X_n$ are independent exponential random variables, each having rate λ, then $\sum_{i=1}^{n} X_i$ is a gamma random variable with parameters (n, λ).

EXAMPLE 5.7a The lifetime of a battery is exponentially distributed with rate λ. If a stereo cassette requires one battery to operate, then the total playing time one can obtain from a total of n batteries is a gamma random variable with parameters (n, λ). ■

Figure 5.11 presents a graph of the gamma $(\alpha, 1)$ density for a variety of values of α. It should be noted that as α becomes large, the density starts to resemble the normal density. This is theoretically explained by the central limit theorem, which will be presented in the next chapter.

5.8 DISTRIBUTIONS ARISING FROM THE NORMAL

5.8.1 The Chi-Square Distribution

Definition

If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then X, defined by

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \tag{5.8.1}$$

is said to have a *chi-square distribution with n degrees of freedom*. We will use the notation

$$X \sim \chi_n^2$$

to signify that X has a chi-square distribution with n degrees of freedom.

The chi-square distribution has the additive property that if X_1 and X_2 are independent chi-square random variables with n_1 and n_2 degrees of freedom, respectively, then $X_1 + X_2$ is chi-square with $n_1 + n_2$ degrees of freedom. This can be formally shown either by the use of moment generating functions or, most easily, by noting that $X_1 + X_2$ is the sum of squares of $n_1 + n_2$ independent standard normals and thus has a chi-square distribution with $n_1 + n_2$ degrees of freedom.

If X is a chi-square random variable with n degrees of freedom, then for any $\alpha \in (0, 1)$, the quantity $\chi_{\alpha,n}^2$ is defined to be such that

$$P\{X \geq \chi_{\alpha,n}^2\} = \alpha$$

This is illustrated in Figure 5.12.

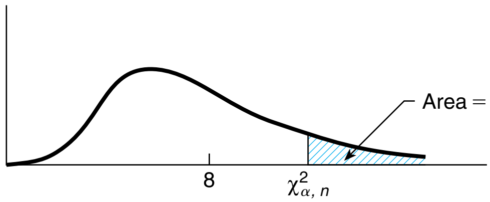

FIGURE 5.12 *The chi-square density function with 8 degrees of freedom.*

In Table A2 of the Appendix, we list $\chi^2_{\alpha,n}$ for a variety of values of α and n (including all those needed to solve problems and examples in this text). In addition, Programs 5.8.1a and 5.8.1b on the text disk can be used to obtain chi-square probabilities and the values of $\chi^2_{\alpha,n}$.

EXAMPLE 5.8a Determine $P\{\chi^2_{26} \leq 30\}$ when χ^2_{26} is a chi-square random variable with 26 degrees of freedom.

SOLUTION Using Program 5.8.1a gives the result

$$P\{\chi^2_{26} \leq 30\} = .7325 \quad ■$$

EXAMPLE 5.8b Find $\chi^2_{.05,15}$.

SOLUTION Use Program 5.8.1b to obtain:

$$\chi^2_{.05,15} = 24.996 \quad ■$$

EXAMPLE 5.8c Suppose that we are attempting to locate a target in three-dimensional space, and that the three coordinate errors (in meters) of the point chosen are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3 meters.

SOLUTION If D is the distance, then

$$D^2 = X_1^2 + X_2^2 + X_3^2$$

where X_i is the error in the ith coordinate. Since $Z_i = X_i/2, i = 1, 2, 3$, are all standard normal random variables, it follows that

$$\begin{aligned} P\{D^2 > 9\} &= P\{Z_1^2 + Z_2^2 + Z_3^2 > 9/4\} \\ &= P\{\chi^2_3 > 9/4\} \\ &= .5222 \end{aligned}$$

where the final equality was obtained from Program 5.8.1a. ■

*5.8.1.1 The Relation Between Chi-Square and Gamma Random Variables

Let us compute the moment generating function of a chi-square random variable with n degrees of freedom. To begin, we have, when $n = 1$, that

$$
\begin{aligned}
E[e^{tX}] &= E[e^{tZ^2}] \text{ where } Z \sim \mathcal{N}(0,1) \qquad (5.8.2)\\
&= \int_{-\infty}^{\infty} e^{tx^2} f_Z(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2(1-2t)/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\bar{\sigma}^2}\, dx \quad \text{where } \bar{\sigma}^2 = (1-2t)^{-1} \\
&= (1-2t)^{-1/2} \frac{1}{\sqrt{2\pi}\bar{\sigma}} \int_{-\infty}^{\infty} e^{-x^2/2\bar{\sigma}^2}\, dx \\
&= (1-2t)^{-1/2}
\end{aligned}
$$

where the last equality follows since the integral of the normal $(0, \bar{\sigma}^2)$ density equals 1. Hence, in the general case of n degrees of freedom

$$
\begin{aligned}
E[e^{tX}] &= E\left[e^{t\sum_{i=1}^{n} Z_i^2}\right] \\
&= E\left[\prod_{i=1}^{n} e^{tZ_i^2}\right] \\
&= \prod_{i=1}^{n} E[e^{tZ_i^2}] \quad \text{by independence of the } Z_i \\
&= (1-2t)^{-n/2} \quad \text{from Equation 5.8.2}
\end{aligned}
$$

However, we recognize $[1/(1-2t)]^{n/2}$ as being the moment generating function of a gamma random variable with parameters $(n/2, 1/2)$. Hence, by the uniqueness of moment generating functions, it follows that these two distributions — chi-square with n degrees of freedom and gamma with parameters $n/2$ and $1/2$ — are identical, and thus we can

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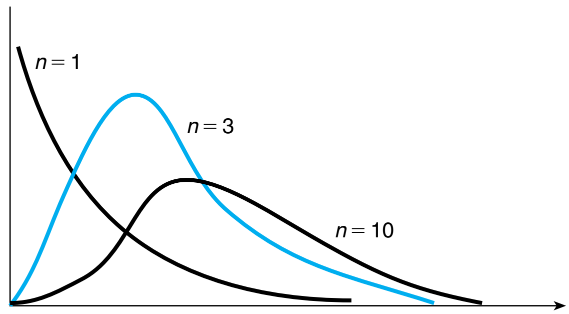

FIGURE 5.13 *The chi-square density function with n degrees of freedom.*

conclude that the density of X is given by

$$f(x) = \frac{\frac{1}{2}e^{-x/2}\left(\frac{x}{2}\right)^{(n/2)-1}}{\Gamma\left(\frac{n}{2}\right)}, \quad x > 0$$

The chi-square density functions having 1, 3, and 10 degrees of freedom, respectively, are plotted in Figure 5.13.

Let us reconsider Example 5.8c, this time supposing that the target is located in the two-dimensional plane.

EXAMPLE 5.8d When we attempt to locate a target in two-dimensional space, suppose that the coordinate errors are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3.

SOLUTION If D is the distance and $X_i, i = 1, 2$, are the coordinate errors, then

$$D^2 = X_1^2 + X_2^2$$

Since $Z_i = X_i/2, i = 1, 2$, are standard normal random variables, we obtain

$$P\{D^2 > 9\} = P\{Z_1^2 + Z_2^2 > 9/4\} = P\{\chi_2^2 > 9/4\} = e^{-9/8} \approx .3247$$

where the preceding calculation used the fact that the chi-square distribution with 2 degrees of freedom is the same as the exponential distribution with parameter 1/2. ■

Since the chi-square distribution with n degrees of freedom is identical to the gamma distribution with parameters $\alpha = n/2$ and $\lambda = 1/2$, it follows from Equations 5.7.3 and 5.7.4

that the mean and variance of a random variable X having this distribution is

$$E[X] = n, \qquad \mathrm{Var}(X) = 2n$$

5.8.2 The *t*-Distribution

If Z and χ_n^2 are independent random variables, with Z having a standard normal distribution and χ_n^2 having a chi-square distribution with n degrees of freedom, then the random variable T_n defined by

$$T_n = \frac{Z}{\sqrt{\chi_n^2/n}}$$

is said to have a *t-distribution with n degrees of freedom*. A graph of the density function of T_n is given in Figure 5.14 for $n = 1, 5$, and 10.

Like the standard normal density, the t-density is symmetric about zero. In addition, as n becomes larger, it becomes more and more like a standard normal density. To understand why, recall that χ_n^2 can be expressed as the sum of the squares of n standard normals, and so

$$\frac{\chi_n^2}{n} = \frac{Z_1^2 + \cdots + Z_n^2}{n}$$

where $Z_1, \ldots, Z_n$ are independent standard normal random variables. It now follows from the weak law of large numbers that, for large n, χ_n^2/n will, with probability close to 1, be approximately equal to $E[Z_i^2] = 1$. Hence, for n large, $T_n = Z/\sqrt{\chi_n^2/n}$ will have approximately the same distribution as Z.

Figure 5.15 shows a graph of the t-density function with 5 degrees of freedom compared with the standard normal density. Notice that the t-density has thicker "tails," indicating greater variability, than does the normal density.

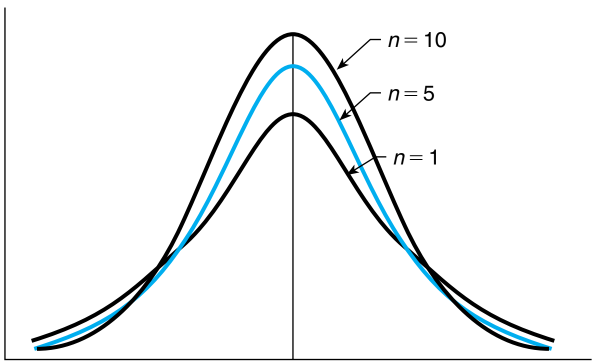

FIGURE 5.14 *Density function of T_n.*

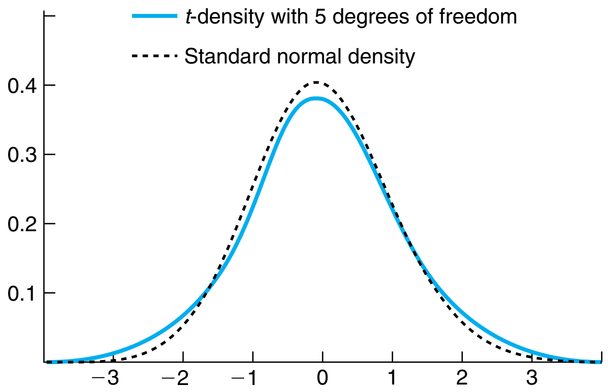

FIGURE 5.15 *Comparing standard normal density with the density of T_5.*

The mean and variance of T_n can be shown to equal

$$E[T_n] = 0, \qquad n > 1$$
$$\text{Var}(T_n) = \frac{n}{n-2}, \qquad n > 2$$

Thus the variance of T_n decreases to 1 — the variance of a standard normal random variable — as n increases to ∞. For α, $0 < \alpha < 1$, let $t_{\alpha,n}$ be such that

$$P\{T_n \geq t_{\alpha,n}\} = \alpha$$

It follows from the symmetry about zero of the t-density function that $-T_n$ has the same distribution as T_n, and so

$$\begin{aligned}\alpha &= P\{-T_n \geq t_{\alpha,n}\}\\ &= P\{T_n \leq -t_{\alpha,n}\}\\ &= 1 - P\{T_n > -t_{\alpha,n}\}\end{aligned}$$

Therefore,

$$P\{T_n \geq -t_{\alpha,n}\} = 1 - \alpha$$

leading to the conclusion that

$$-t_{\alpha,n} = t_{1-\alpha,n}$$

which is illustrated in Figure 5.16.

The values of $t_{\alpha,n}$ for a variety of values of n and α have been tabulated in Table A3 in the Appendix. In addition, Programs 5.8.2a and 5.8.2b on the text disk compute the t-distribution function and the values $t_{\alpha,n}$, respectively.

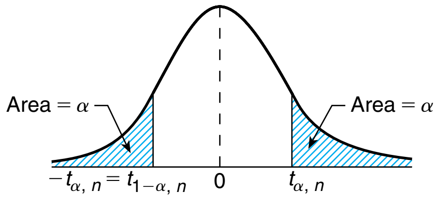

FIGURE 5.16 $t_{1-\alpha,n} = -t_{\alpha,n}$.

EXAMPLE 5.8e Find **(a)** $P\{T_{12} \leq 1.4\}$ and **(b)** $t_{.025,9}$.

SOLUTION Run Programs 5.8.2a and 5.8.2b to obtain the results.

(a) .9066 **(b)** 2.2625 ■