Boolean Algebra

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The "WHY" Boolean Algebra

- Algebra

When we learned numbers like 1, 2, 3, we also then learned how to add, multiply, etc. with them. Boolean Algebra covers operations that we can do with 0's and 1's. Computers do these operations ALL THE TIME and they are basic building blocks of computation inside your computer program.

Axioms, laws, theorems

We need to know some rules about how those 0's and 1's can be operated on together. There are similar axioms to decimal number algebra, and there are some laws and theorems that are good for you to use to simplify your operation.

How does Boolean Algebra fit into the big picture?

- It is part of the Combinational Logic topics (memoryless)
  - Different from the Sequential logic topics (can store information)

- Learning Axioms and theorems of Boolean algebra
  - Allows you to design logic functions
  - Allows you to know how to combine different logic gates
  - Allows you to simplify or optimize on the complex operations

Boolean algebra

- A Boolean algebra comprises...
  - A set of elements B
  - Binary operators {+ , •} Boolean sum and product
  - A unary operation { ' } (or { ‾ }) example: A’ or $\bar{A}$
- ...and the following axioms
  - 1. The set B contains at least two elements {a b} with a $\neq$ b
  - 2. Closure: a+b is in B a•b is in B
  - 3. Commutative: a+b = b+a a•b = b•a
  - 4. Associative: a+(b+c) = (a+b)+c a•(b•c) = (a•b)•c
  - 5. Identity: a+0 = a a•1 = a
  - 6. Distributive: a+(b•c)=(a+b)•(a+c) a•(b+c)=(a•b)+(a•c)
  - 7. Complementarity: a+a' = 1 a•a' = 0

Digital (binary) logic is a Boolean algebra

Substitute

- {0, 1} for B
- AND for • Boolean Product.
- OR for + Boolean Sum.
- NOT for ‘ Complement.

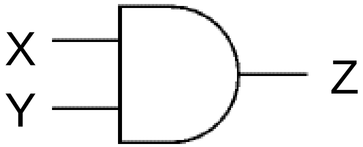
All the axioms hold for binary logic

Definitions

- Boolean function
  - Maps inputs from the set {0,1} to the set {0,1}
- Boolean expression
  - An algebraic statement of Boolean variables and operators

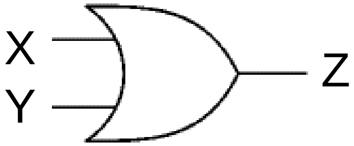
Logic Gates (AND, OR, Not) & Truth Table

AND $X \bullet Y$ XY



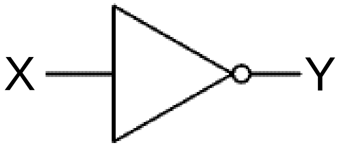
X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

OR $X+Y$



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

NOT $\overline{X}$ X'



X	Y
0	1
1	0

Logic functions and Boolean algebra

- *Any* logic function that is expressible as a truth table can be written in Boolean algebra using +, •, and '

X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

Z=X•Y

X	Y	X'	Z
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	0

Z=X'•Y

X	Y	X'	Y'	X • Y	X' •Y'	Z
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

Z=(X•Y)+(X' •Y')

Two key concepts

- **Duality (a meta-theorem—** *a theorem about theorems*)
  - All Boolean expressions have logical duals
  - Any theorem that can be proved is also proved for its dual
  - Replace: • with +, + with •, 0 with 1, and 1 with 0
  - Leave the variables unchanged

- **De Morgan's Theorem**
  - Procedure for complementing Boolean functions
  - Replace: • with +, + with •, 0 with 1, and 1 with 0
  - Replace all variables with their complements

Universal Gate Implementation

NAND Gate as Universal Gate

OR Gate using NAND Gate

NOR Gate using NAND Gate

EXOR using NAND Gate

NOR Gate as Universal Gate: TO DO

EX-OR Gate Using NOR gate

Boolean Functions

- Boolean function is described by an a algebraic expression called Boolean expressions which consists of binary variables and logic operation symbols.
- Mathematical functions can be expressed in two ways:

An expression is finite but not unique

$$
\begin{aligned}
f(x,y) &= 2x + y \\
&= x + x + y \\
&= 2(x + y/2) \\
&= \ldots
\end{aligned}
$$

A function table is unique

x	y	f(x,y)
0	0	0
…	…	…
2	2	6
…	…	…
23	41	87
…	…	…

We can represent logical functions in two analogous ways too:

- A finite, but non-unique Boolean expression.
- A truth table, which will turn out to be unique.

Boolean expressions

- We can use these basic operations to form more complex expressions:

$$f(x,y,z) = (x + y')z + x'$$

  - f is the name of the function.
  - (x, y, z) are the input variables, each representing 1 or 0. Listing the inputs is optional, but sometimes helpful.
  - A literal is any occurrence of an input variable or its complement. The function above has four literals: x, y', z, and x'.
- Precedence is important, but not too difficult.
  - NOT has the highest precedence, followed by AND, and then OR.
  - Fully parenthesized, the function above would be kind of messy:

$$f(x,y,z) = (((x +(y'))z) + x')$$

Useful laws and theorems

Identity:	$X + 0 = X$	Dual:	$X \bullet 1 = X$
Null:	$X + 1 = 1$	Dual:	$X \bullet 0 = 0$
Idempotent:	$X + X = X$	Dual:	$X \bullet X = X$
Involution:	$(X')' = X$		
Complementarity:	$X + X' = 1$	Dual:	$X \bullet X' = 0$
Commutative:	$X + Y = Y + X$	Dual:	$X \bullet Y = Y \bullet X$
Associative:	$(X+Y)+Z=X+(Y+Z)$	Dual:	$(X\bullet Y)\bullet Z=X\bullet(Y\bullet Z)$
Distributive:	$X\bullet(Y+Z)=(X\bullet Y)+(X\bullet Z)$	Dual:	$X+(Y\bullet Z)=(X+Y)\bullet(X+Z)$
Uniting:	$X\bullet Y+X\bullet Y'=X$	Dual:	$(X+Y)\bullet(X+Y')=X$

Proof of law with Truth table

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

$$(A + B) + C = A + (B + C)$$

Useful laws and theorems (con't)

Absorption: X+X•Y=X Dual: X•(X+Y)=X

Absorption (#2): (X+Y')•Y=X•Y Dual: (X•Y')+Y=X+Y

$$x + \bar{x}y = x + y$$

$$(x + y)(x + z) = x + yz$$

de Morgan's: (X+Y+...)'=X'•Y'•... Dual: (X•Y•...)'=X'+Y'+...

Duality: $(X+Y+...)^D$=X•Y•... Dual: $(X•Y•...)^D$=X+Y+...

Multiplying & factoring: (X+Y)•(X'+Z)=X•Z+X'•Y

Dual: X•Y+X'•Z=(X+Z)•(X'+Y)

Consensus: (X•Y)+(Y•Z)+(X'•Z)= X•Y+X'•Z

Dual: (X+Y)•(Y+Z)•(X'+Z)=(X+Y)•(X'+Z)

Proofing the theorems using axioms

- Idempotency: $x + x = x$

  Proof:

$x + x = (x + x) \bullet 1$	by identity
$= (x + x) \bullet (x + x')$	by complement
$= x + x \bullet x'$	by distributivity
$= x + 0$	by complement
$= x$	by identity

- Idempotency: $x \bullet x = x$

  Proof:

$x \bullet x = (x \bullet x) + 0$	by identity
$= (x \bullet x) + (x \bullet x')$	by complement
$= x \bullet (x + x')$	by distributivity
$= x \bullet 1$	by complement
$= x$	by identity

Theorems

Prove the uniting theorem-- $X \bullet Y + X \bullet Y' = X$

Distributive	$X \bullet Y + X \bullet Y' = X \bullet (Y + Y')$
Complementarity	$= X \bullet (1)$
Identity	$= X$

Prove the absorption theorem-- $X + X \bullet Y = X$

Identity	$X + X \bullet Y = (X \bullet 1) + (X \bullet Y)$
Distributive	$= X \bullet (1 + Y)$
Null	$= X \bullet (1)$
Identity	$= X$

To Prove

$$A + \bar{A}B = A + B$$

$$(\boldsymbol{A} + \boldsymbol{B})(\boldsymbol{A} + \boldsymbol{C}) = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}$$

More proves

$$(A+\bar{B}+AB)(A+\bar{B})(\bar{A}B)=0$$

$$A+\bar{A}B+AB=A+B$$

Simplify the following expression

$$Y=\overline{(\overline{AB}+\bar{A}+AB)}$$

Theorems

Prove the consensus theorem--

$$(XY)+(YZ)+(X'Z)= XY+X'Z$$

Complementarity $XY+YZ+X'Z = XY+(X+X')YZ + X'Z$

Distributive $= XYZ+XY+X'YZ+X'Z$

- *Use absorption {AB+A=A} with A=XY and B=Z*

$= XY+X'YZ+X'Z$

Rearrange terms $= XY+X'ZY+X'Z$

- *Use absorption {AB+A=A} with A=X'Z and B=Y*

$$XY+YZ+X'Z = XY+X'Z$$

Logic simplification

Example:

$$Z = A'BC + AB'C' + AB'C + ABC' + ABC$$

$= A'BC + AB'(C' + C) + AB(C' + C)$ distributive

$= A'BC + AB' + AB$ complementary

$= A'BC + A(B' + B)$ distributive

$= A'BC + A$ complementary

$= BC + A$ absorption #2 Duality

$(X \bullet Y')+Y=X+Y$ with $X=BC$ and $Y=A$

Algebraic manipulation

$x'y' + xyz + x'y$

$= x'(y' + y) + xyz$ [Distributive; $x'y' + x'y = x'(y' + y)$]

$= x' \bullet 1 + xyz$ [Axiom 5; $y' + y = 1$]

$= x' + xyz$ [Axiom 2; $x' \bullet 1 = x'$]

$= (x' + x)(x' + yz)$ [Distributive]

$= 1 \bullet (x' + yz)$ [Axiom 5; $x' + x = 1$]

$= x' + yz$ [Axiom 2 ; $x' \bullet 1 = x'$]

More Problems :

$$A\bar{B} + \bar{A}B + \bar{A}\bar{B} + AB$$

$$(AB + C)(AB + D)$$

$$AB + ABC + A\bar{B} = A \qquad \text{(Prove)}$$

De Morgan’s Theorem

Use de Morgan’s Theorem to find complements

Example: $F=(A+B)\bullet(A'+C)$, so $F'=(A'\bullet B')+(A\bullet C')$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A	B	C	F′
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Complement of a function

- The complement of a function always outputs 0 where the original function outputted 1, and 1 where the original produced 0.
- In a truth table, we can just exchange 0s and 1s in the output column(s)

$f(x,y,z) = x(y'z' + yz)$

x	y	z	f(x,y,z)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	y	z	f'(x,y,z)
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Complementing a function algebraically

$$f(x,y,z) = x(y'z' + yz)$$

$$
\begin{aligned}
f'(x,y,z) &= (\ x(y'z' + yz)\ )' && [\text{ complement both sides }] \\
&= x' + (y'z' + yz)' && [\text{ because } (xy)' = x' + y'\ ] \\
&= x' + (y'z')'\ (yz)' && [\text{ because } (x + y)' = x'\ y'\ ] \\
&= x' + (y + z)(y' + z') && [\text{ because } (xy)' = x' + y',\text{ twice}]
\end{aligned}
$$

- You can use DeMorgan's law to keep "pushing" the complements inwards
- You can also take the dual of the function, and then complement each literal
  - If $f(x,y,z) = x(y'z' + yz)$…
  - …the dual of f is $x + (y' + z')(y + z)$…
  - …then complementing each literal gives $x' + (y + z)(y' + z')$…
  - …so $f'(x,y,z) = x' + (y + z)(y' + z')$

Canonical Forms

- Any boolean function that is expressed as a sum of minterms or as a product of maxterms is said to be in its canonical form.
- A minterm is a special product of literals, in which each input variable appears exactly once.
- A function with n variables has 2^n minterms (since each variable can appear complemented or not) A three-variable function, such as f(x,y,z), has $2^3 = 8$ minterms:

$$x'y'z' \quad x'y'z \quad x'yz' \quad x'yz$$
$$xy'z' \quad xy'z \quad xyz' \quad xyz$$

Minterms

- Each minterm is true for exactly one combination of inputs:

Minterm	Is true when…	Shorthand
$x'y'z'$	$x=0, y=0, z=0$	m_0
$x'y'z$	$x=0, y=0, z=1$	m_1
$x'yz'$	$x=0, y=1, z=0$	m_2
$x'yz$	$x=0, y=1, z=1$	m_3
$xy'z'$	$x=1, y=0, z=0$	m_4
$xy'z$	$x=1, y=0, z=1$	m_5
xyz'	$x=1, y=1, z=0$	m_6
xyz	$x=1, y=1, z=1$	m_7

Sum of minterms form

- Every function can be written as a sum of minterms, which is a special kind of sum of products form
- The sum of minterms form for any function is *unique*
- If you have a truth table for a function, you can write a sum of minterms expression just by picking out the rows of the table where the function output is 1 (1-minterm).

x	y	z	f(x,y,z)	f'(x,y,z)
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

$$\begin{aligned} f &= x'y'z' + x'y'z + x'yz' + x'yz + xyz' \\ &= m_0 + m_1 + m_2 + m_3 + m_6 \\ &= \Sigma(0,1,2,3,6) \end{aligned}$$

$$\begin{aligned} f' &= xy'z' + xy'z + xyz \\ &= m_4 + m_5 + m_7 \\ &= \Sigma(4,5,7) \end{aligned}$$

f' contains all the minterms not in f

Sum of minterms: practise

F = x + yz, how to express this in the sum of minterms?

$$= x(y + y')(z + z') + (x + x')yz$$
$$= xyz + xyz' + xy'z + xy'z' + xyz + x'yz$$
$$= x'yz + xy'z' + xy'z + xyz' + xyz$$
$$= \Sigma(3,4,5,6,7)$$

or, convert the expression into truth-table and then read the minterms from the table