U19CS012
[BHAGYA RANA]

TUTORIAL 2 : NUMBER THEORY

DIVISIBILITY THEORY IN INTEGERS

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1. Find gcd (306, 657) and gcd (272, 1479)

1. (i) gcd (306, 657)

$$657 = (2 \times 306) + 45$$
$$306 = (45 \times 6) + 36$$
$$45 = (36 \times 1) + 9$$
$$36 = (9 \times 4) + \boxed{0}$$

So, by Division Algorithm, gcd (306, 657) = $\boxed{9}$

(ii) gcd (272, 1479)

$$1479 = (272 \times 5) + 119$$
$$272 = (119 \times 2) + 34$$
$$119 = (34 \times 3) + 17$$
$$34 = 17 \times 2 + \boxed{0}$$

So, by Division Algorithm, gcd(272, 1479) = $\boxed{17}$

2. Use Euclidean Algorithm to obtain integers x and y satisfying the following :

(a) gcd (56, 72) = $56x + 72y$

(b) gcd (1769, 2378) = $1769x + 2378y$

2. (a) gcd (56, 72) = $56x + 72y$

$$72 = (1 \times 56) + 16$$
$$56 = (3 \times 16) + 8$$
$$16 = (2 \times 8) + 0$$

So, By Division Algorithm, gcd (56, 72) = 8

Now,

$$8 = 56 - 3 \times 16$$
$$= 56 - 3 \times (72 - 56)$$
$$= (4) \times 56 - (3) \times 72$$

So, $\boxed{x = 4 \quad y = -3}$

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(b) gcd (1769, 2376) = 1769 x + 2376 y

$$2376 = (1)\times(1769) + 609$$
$$1769 = (2)\times(609) + 551$$
$$1769 = (3)\times(609) - 58$$
$$609 = 10\times(58) + 29$$
$$58 = 2\times(29) + 0$$

So, by Division Algorithm, gcd (1769, 2376) = 29

$$\therefore\ 29 = 609 - 10\times 58$$
$$= 609 - 10\,(3\times 609 - 1769)$$
$$= (-29)(609) + (10).\,1769$$
$$= (-29)(2378 - 1769) + (10).(1769)$$
$$= (39)(1769) - (29)\,2328$$

ANS : $\boxed{x = 39 \ \& \ y = -29}$

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(3) Prove that if d is a common divisor of a and b, then $d = \gcd(a,b)$ if and only if $\gcd(a/d, b/d) = 1$.

(3) Proof : Let m, n be such that $d\times m = a$

$d \times n = b$

(a) if $d = \gcd(a,b)$, Then by theorem 2.7 ($\because d > 0$)

(If $k>0$, then $\gcd(ka, kb) = k\gcd(a,b)$)

$$d = \gcd(dm, dn) = d\gcd(m,n)$$
$$= d\gcd\left(\frac{a}{d}, \frac{b}{d}\right)$$

$$\therefore\ 1 = \gcd(a/d, b/d)$$

(b) if $\gcd(a/d, b/d) = 1$, Then by Theorem 2.7

$$\gcd(a,b) = \gcd\left(d.\frac{a}{d}, d.\frac{b}{d}\right) = |d|\gcd\left(\frac{a}{d}, \frac{b}{b}\right) = |d|\times 1 = |d|$$

, Hence Proved,

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4) Assuming $gcd(a,b) = 1$ prove

(a) $gcd(a+b, a-b) = 1$ or 2

(b) $gcd(a+b, a^2-ab+b^2) = 1$ or 3

4) (a) It is Given that $gcd(a,b) = 1$

Let $gcd(a-b, a+b) = d$

$\Rightarrow$ d divides $a-b$ and $a+b$

there exist integers m and n such that

$$a+b = m \times d \quad \text{---} \ (1)$$

$$a-b = n \times d \quad \text{---} \ (2)$$

Adding and subtracting eq^n (1) and (2)

$$2a = (m+n) \times d \quad \text{---} \ (3)$$

$$2b = (m-n) \times d \quad \text{---} \ (4)$$

Since $GCD(a,b) = 1$, (given)

$\therefore$ $2 \times \overbrace{GCD(a,b)}^{1} = 2$

$\therefore$ $GCD(2a, 2b) = 2$ [$\because GCD(ka, kb) = k \, GCD(a,b)$]

From eq^n (3) & (4)

$\therefore$ $GCD((m+n) \times d, (m-n) \times d) = 2$

$\therefore$ $d \times \underbrace{GCD(m+n, m-n)} = 2$

$\therefore$ $d \times$ (some integer) $= 2$

d divides 2

$d \le 2$ if x divides y then $|x| \le |y|$

$\therefore d = 1$ or 2 $\because$ gcd is alway's positive integer.

Hence, $gcd(a+b, a-b) = 1$ or 2.

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(b) It is given that $GCD(a,b) = 1$ — Eq^n (0)

let $GCD(a+b, a^2-ab+b^2) = d$

So $d \mid a+b$ — (1)

and $d \mid a^2-ab+b^2$ — (2)

Using Basic algebra, a^2-ab+b^2 can be written as $\underline{(a+b)^2 - 3ab}$

$\therefore d \mid (a+b)^2 - 3ab$ — (3)

By using Eq^n 1 $\underline{[d \mid (a+b)^2]}$ and d should also divide $\underline{-3ab}$

$$d \mid (-3ab)$$

Using Eq^n (0) $\underline{gcd(a,b) = 1}$ implies $d \nmid ab$

thus $d \mid 3$ [Euclid's Lemma]

So, the

$GCD(a+b, a^2-ab+b^2) = 1$ or 3.

$\therefore \underline{d \le 3}$ Since, $gcd(2,3) = 1$, Then if $d = 2$, then

$2 \mid 3ab \longrightarrow 2 \mid ab$

$\therefore 2 \mid a$ or $2 \mid b$, either of

which contradicts $\underline{gcd(a,d) = gcd(b,d) = 1 \quad \therefore [d \neq 2]}$

$\therefore \underline{d = 1 \text{ or } 3}$

$GCD(a+b, a^2-ab+b^2) = 1$ or 3,

Hence proved

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5.) Prove that if $\gcd(a,b) = 1$, then $\gcd(a+b, ab) = 1$

5) Given $GCD(a,b) = 1$ —— (A)

Let $GCD(a+b, ab) = d$

then, $d \mid ab$ —— (1)

& $d \mid a+b$ —— (2)

From eqⁿ (2), it shows that $d \mid a$ and $d \mid b$ at the same time

but from eqⁿ (A) $GCD(a,b) = 1$ → So, $d \mid a$ & $d \mid b$

So, it is a contradiction, so $d = 1$ $\therefore d \le \gcd(a,b)$

$[d \le 1]$

$\therefore GCD(a+b, ab) = 1$ $[\therefore d = 1]$

6) Find $LCM(143, 227)$, $LCM(306, 657)$

6) (I) Let $LCM(143, 227) = t$

then $t = \dfrac{143 \times 227}{GCD(143, 227)}$ —— (1) [$\underset{\text{of } a,b}{HCF} \times \underset{\text{of } a,b}{LCM} = a \times b$]

$\therefore GCD(143, 227)$

Euclidean method:

$227 = 143 * 1 + 84$

$143 = 84 \times 1 + 59$

$84 = 59 \times 1 + 25$

$59 = 25 \times 2 + 9$

$25 = 9 \times 2 + 7$

$9 = 7 \times 1 + 2$

$7 = 2 \times 3 + 1$

$2 = 1 \times 2 + 0$

$\therefore GCD(143, 227) = 1$

[using Eq (1)]

$LCM = \dfrac{143 \times 227}{(1)} = [32461]$

$\therefore [LCM(143, 227) = 32461]$

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(II) Let LCM (306, 657) = t (of a, b) (a, b ∈ I)

then $t = \dfrac{306 \times 657}{GCD(306, 657)}$ – Eq ① [HCF × LCM = a × b]

GCD (306, 657)

Euclidian method

$657 = 306 \times 2 + 45$

$306 = 45 \times 6 + 36$

$45 = 36 \times 1 + 9$

$36 = 9 \times 4 + 0$

GCD (306, 657) = 9

using [Eqⁿ ①] $LCM(306, 657) = \dfrac{\cancel{306}^{34} \times 657}{\cancel{9}}$ = [22338]

∴ LCM (306, 657) = 22338

Q7.7 Which of the following diophantine eqⁿ cannot be solved?

(a) $6x + 51y = 22$

{ $ax + by = c$ Diophantine eqⁿ has solution when $d \mid c$ where $d = GCD(a, b)$ }

GCD (6, 51) :-

$51 = 6 \times 8 + 3$

$6 = 3 \times 2 + 0$

∴ GCD (6, 51) = 3

∵ But 3 does not divide 22 & $3 \nmid 22$,

∴ Can't be solved.

b) $33x + 14y = 15$

gcd (33, 14) :- = 1

$33 = 14 \times 2 + 5$

$14 = 5 \times 2 + 4$

$5 = 4 \times 1 + 1$

$4 = 1 \times 4 + 0$ (1 divides 15)

GCD (33, 14) = 1 & $1 \mid 15$ ∴ It can be solved

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(c) $14x + 35y = 93$ { Diophantine eq^n $ax + by = c$ has sol^n when $d \mid c$ where $d = \gcd(a,b)$ }

$d = GCD(14, 35)$

$GCD(14, 35)$:-

$35 = 14 \times 2 + 7$

$14 = 7 \times 2 + 0$

$GCD(14, 35) = 7$

But $7 \nmid 93$ (7 doesn't divides 93) $\therefore$ Can't be solved.

(8) Determine all solutions in the integers of the following Diophantine eqn

(a) $56x + 72y = 40$

SOLVABLE : $GCD(56, 72)$:-

$72 = 56 \times 1 + 16$

$56 = 16 \times 3 + 8$

$16 = 8 \times 2 + 0$

$\because GCD(56, 72) = 8$ & $8 \mid 40$ $\therefore$ It can be solved

$GCD(56, 72) \Rightarrow$

$$8 = 56(1) - 16 \times 3$$
$$= 56 - 3 \times (72 - 56)$$
$$= 56 - (3 \times 72) + (3 \times 56)$$
$$= 4 \times 56 - 3 \times 72$$

Multiply by 5, both sides

$$40 = (5 \times 4 \times 56 - 5 \times 3 \times 72)$$
$$= (20 \times 56 - 15 \times 72)$$

$x = x_0 + \left(\frac{b}{d}\right) t$ $\quad$ $y = y_0 - \left(\frac{a}{d}\right) t$

$= 20 + \left(\frac{72}{8}\right) t$ $\quad$ $y = -15 - \left(\frac{56}{8}\right) t$

$[x = 20 + 9t \quad y = -15 - 7t]$ $[t \in \mathbb{Z}]$

Ans: $x = 20 + 9t$

$y = -15 - 7t$, $\forall t \in \mathbb{Z}$ (integer).

(b) $24x + 138y = 18$

GCD : $138 = 24 \times 5 + 18$

$24 = 18 \times 1 + 6$

$18 = 6 \times 3 + 0$

$\therefore \gcd(24, 138) = 6$ — ①

& $6 \mid 18$ $\longrightarrow$ [SOLVABLE]

$6 = 24 - 18$

$= 24 - (138 - 5.24)$

$6 = 6.24 - 138$ — ②

$\Rightarrow$ Multiplying Eqⁿ ② by ③

$18 = 3(6).(24) - (3)(138)$

$18 = (18)\,24 - (3)\,138$

$\therefore$ $(18, -3)$ is a solution

$\therefore$ $x = x_0 + \left(\frac{b}{d}\right)t$ $\quad$ $y = y_0 + \left(\frac{a}{d}\right)t$

$= 18 + \left(\frac{138}{6}\right)t$ $\quad$ $= -3 - \left(\frac{24}{6}\right)t$

ANS. $[x = 18 + 23t \quad y = -3 - 4t]$ $\forall t \in Z$

(C) $221x + 35y = 11$

$221 = 35 \times 6 + 11$

$35 = 11 \times 3 + 2$

$11 = 2 \times 5 + 1$

$2 = 1 \times 2 + 0$

$\therefore GCD(221, 35) = 1$ &

$1 \mid 11$

Eqⁿ is solvable.

$\therefore 1 = 11 - 5 \times 2$

$= 11 - 5(35 - 3.11)$

$= 16.11 - 5.35$

$= 16.(221 - 35 \times 6) - 5.35$

$1 = [16.221 - 101.35]$ — Eqⁿ ②

· Multiplying 11 both sides

$11 = 176 \times (221) - (1111) \times 35$

$[x = x_0 + (b/d)t$

$= 176 + (35/1)\,t$

$x = 176 + 35t$

$y = y_0 + (a/d)t$

$= -1111 - (221/1)\,t$

$= -1111 - 221t]$

[$\therefore$ $x = 176 + 35t$ & $y = -1111 - 221t$ $\forall t \in Z$]

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9.) Determine all solution's in the positive integers of following diophantine eqn's

(a) $18x + 5y = 48$

GCD(18, 5):

$18 = 5 \times 3 + 3$

$5 = 3 \times 1 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

GCD(18, 5) = 1 & 1 | 48 } SOLVABLE

$$1 = 3 - 2 \times 1$$
$$= 3 - (5 - 3)$$
$$= 2.3 - 5$$
$$= 2.(18 - 3.5) - 5$$
$$1 = 2.18 - 7.5$$

multiply eqn by (48) $\therefore 48 = (96).18 - (48.7).5$

$\therefore$ (96, −336) is a solution

$\therefore x = x_0 + (b/d)t = 96 + (5/1)t = 96 + 5t$

$y = y_0 - (a/d)t = 336 - (18/1)t = -336 - 18t$

$x = 96 + 5t$ & $y = -336 - 18t$

Since, $x, y \geq 0$, $96 + 5t > 0 \Rightarrow t > -19.2$

$-336 - 18t > 0 \Rightarrow t < -18.7$

$\therefore [t = -19]$ (only integer)

$\therefore$

ANS:	
	$x = 96 + 5(-19) = 1$
	$y = -336 - 18(-19) = 6$

(b) $54x + 21y = 906$ [EXTRA]

GCD(54, 21):

$54 = 21 \times 2 + 12$

$21 = 12 \times 1 + 9$

$12 = 9 \times 1 + 3$

$9 = 3 \times 3 + 0$

GCD(54, 21) = 3 & 3 | 906 } $\therefore$ SOLVABLE

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$\therefore\ 3 = 12 - 9$

$= 12 - (21 - 12)$

$= 2.12 - 21$

$= 2.(54 - 2.21) - 21$

$3 = 2.54 - 5.21$

$\therefore$ Multiply whole eq^n by 302

$\therefore 906 = (302.2)\ (54) - (302 \times 5)\ (21)$

$\therefore \Rightarrow (604, -1510)$ is a solution

$\therefore\ x = 604 + 7t > 0 \Rightarrow t > -86.3$

$y = -1510 - 18t > 0 \Rightarrow t < -83.9$

$\therefore\ t = -84, -85, -86$

$\therefore (x, y) = \{ (604 + 7(-84), -1510 - 18(-84)),$
$(604 + 7(-85), -1510 - 18(-85)),$
$(604 + 7(-86), -1510 - 18(-86)) \}$

ANS :
$(x, y) = \{(16, 2), (9, 20), (2, 38)\}$

(b) $123x + 360y = 99$

GCD $(123, 360)$:- $360 = 123 \times 2 + 114$

$123 = 114 \times 1 + 9$

$114 = 9 \times 12 + 6$

$9 = 6 \times 1 + 3$

$6 = 3 \times 2 + 0$

GCD $(123, 360) = 3$
&
$3 \mid 99$
SOLVABLE ✓

$3 = 9 - 6 = 9 - (114 - 9 \times 12)$

$= 13 \times 9 - 114$

$= 13 \times (123 - 114) - 114$

$= 13 \times (123) - 14.114$

$= 13 \times 123 - 14(360 - 123 \times 2)$

$3 = 41 \times 123 - 14 \times 360$

Multiplying 33 both sides $[\therefore 99 = 1353 \times 123 - 462 \times 360]$

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$x = x_0 + (b/d)t$ $\qquad$ $y = y_0 - (a/d)t$

$= 1353 + (360/3)t$ $\qquad$ $= -462 - (123/3)t$

$= 1353 + 120t$ $\qquad$ $= -462 - 41t$

For Positive integer's $x, y \geq 0$

$x > 0$ $\qquad$ $y > 0$

$1353 + 120t > 0$ $\qquad$ $-462 - 41t > 0$ $\qquad$ $t > -11.275$

$t > \frac{-1353}{120}$ $\qquad$ $t < \frac{-462}{41}$ $\qquad$ $t < -11.3$

ANS: $\therefore$ No t exist [integer], So no Positive solution

(c) $158x - 57y = 7$

GCD(158, 57) :- $\quad 158 = 57 \times 3 - 13$

$57 = 4 \times 13 + 5$

$13 = 2 \times 5 + 3$

$5 = 3 \times 1 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

$\therefore$ GCD(158, 57) = 1 & $1 \mid 7$ } SOLVABLE

$1 = 3 - 2 = 3 - (5 - 3)$

$= 2.3 - 5 = 2.(13 - 2.5) - 5$

$= 2.13 - 5.5$

$= 2.13 - 5(57 - 4.13)$

$= 22.13 - 5.57$

$= 22.(3.57 - 158) - 5.57$

$= 61.(57) - 22.(158)$

$\therefore [7 = 427.57 - 154.158]$

$(-154, -427)$ is a solution

$\therefore$ $x = -154 - 57t > 0 \Rightarrow t < -2.7$

$y = -427 - 158t > 0 \Rightarrow t < -2.7$ } $(t \leq -3)$

ANS: $\therefore$ $x = -154 - 57t$, $y = -427 - 158t$ for $t \in \{\ldots -8, -7, -6, -5, -4, -3\}$

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10.) The neighborhood theatre charges $1.80 for adult admissions $ 0.75 for children. On a particular evening the total receipts were $90. Assuming that more adults than children were present, how many people attended?

10.) Let x : Number of adults y : Number of childrens

$$x(1.80) + y(0.75) = 90$$

$$180x + 75y = 9000 \quad \text{— (1)}$$

GCD (180, 75) :- $180 = 75 \times 2 + 30$

$$75 = 30 \times 2 + 15$$

$$30 = 15 \times 2 + 0$$

$\therefore$ GCD (180, 75) = 15 & $15 \mid 90$ ∴ SOLVABLE

$$15 = 75 - 30 \times 2$$

$$= 75 - 2(180 - 75 \times 2)$$

$$15 = 75(5) - 2(180)$$

Multiplying both sides by 600,

$$9000 = 3000 \times 75 - 1200 \times 180$$

$x = x_0 + (b/d)t$ $y = y_0 \mp (a/d)t$

$x = -1200 + 5t$ $y = 3000 - 12t$

Childrens & adult count needs to be positive (≥ 0)

$x \geq 0$ $y \geq 0$

$-1200 + 5t \geq 0$ $3000 - 12t \geq 0$

$t \geq 240$ $250 \geq t$

$\therefore$ $[240 \leq t \leq 250]$ — (2)

$\because$ more adults than children $x > y$

$$-1200 + 5t > 3000 - 12t$$

$$t > 4200/17$$

$$[t > 247.06] \text{ — (3)}$$

$\therefore$ $t = 248, 249, 250$

$x = -1200 + 5t$ $y = 3000 - 12t$

	ADULTS	CHILD	
Ans:	$x = 40$,	$y = 24$	$t = 248$
	$x = 45$,	$y = 12$	$t = 249$
	$x = 50$,	$y = 0$	$t = 250$.

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11> A certain number of sixes and nines is added to give a sum of 126, if no. of sixes and no. of nines are interchanged, the new sum is 114. How many of each were there originally.

11> Let x : no. of nines

y : no. of sixes

Acc. to question, $9x+6y = 126$ — ①

[After interchanging] $9y+6x = 114$ — ②

[Both eqⁿ's must be true simultaneously, observing both eqⁿ → only 1 solⁿ exist

Eqⁿ ① × 6 & Eqⁿ ② × 9

$$54x + 36y = 756$$

$$-(54x + 81y = 1026)$$

$$-45y = -270$$

$$[y = 6]$$

$$\left[x = \frac{(756 - 36(6))}{54} = \frac{540}{54} = 10\right]$$

Originally, there were 6 sixes and 10 nines.

12> Alcuin of York, 775. One hundred bushels of grain are distributed among 100 persons in such a way that each man ~~receea~~ receives 3 bushels, each women received 2 bushels and each child $\frac{1}{2}$ bushels. How many men, women and children are there?

12> Let x : Number of men

y : Number of women

z : Number of children

$$x + y + z = 100$$

$$3x + 2y + 0.5z = 100 \quad — ①$$

From $x+y+z = 100$, we can substitute $z = 100 - x - y$ in eqⁿ ①, & multiplying by 2,

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$$3x + 2y + \frac{1}{2}(100 - x - y) = 1001$$

$$6x + 4y + (100 - x - y) = 200$$

$$[5x + 3y = 100] \quad — ②$$

{Diaphontine eqⁿ}

GCD(3, 5)

$$5 = 3 \times 1 + 2$$

$$3 = 2 \times 1 + 1$$

$$2 = 1 \times 2 + 0$$

$$1 = 3 - 2 \times 1$$

$$= 3 - (5 - 3)$$

$$1 = 2 \times 3 - 5$$

multipling by 100 both sides,

$$100 = 200 \times 3 - 100 \times 5$$

$$x = x_0 + (b/d)t \qquad y = y_0 - (a/d)t$$

$$[x = -100 + 3t] \qquad [y = 200 - 5t]$$

$$x \geq 0 \qquad y \geq 0$$

$$-100 + 3t \geq 0 \qquad 200 - 5t \geq 0$$

$$[t \geq 33.33] \qquad t \leq 40$$

$$t = \{34, 35, 36, 37, 38, 39, 40\}$$

$$x = -100 + 3t \qquad y = 200 - 5t \qquad z = 100 - x - y$$

ANS:

	t	34	35	36	37	38	39	40
MEN	x	2	5	8	11	14	17	20
WOMEN	y	30	25	20	15	10	5	0
CHILDREN	z	68	70	72	74	76	78	80

—x—