TUTORIAL - 5

BOOLEAN ALGEBRA

[U19CS012]

⊛ Theory

(A) Distributive Lattice - A Lattice (L) is said to be distributive if $\forall (a,b,c) \in L$ (L = Lattice)

a) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\vee$ = Join

b) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\wedge$ = Meet

OR

If ~~the~~ every element of Lattice has atmost 1 complement

$\boxed{\forall\ e \in L,\ \text{no of elements } (e^c) \leq 1}$

ATMOST 1 COMPLEMENT

(B) Complemented Lattice - A Lattice L is said to be complemented if every element $\forall a \in L$, must have atleast one complement.

OR

$\forall\ a \in L,$ $\boxed{\text{no. of element } (a^c) \geq 1}$

ATLEAST 1 COMPLEMENT

(C) Boolean Algebra - A Lattice 'L' is said to be Boolean Algebra, if it is complemented and Distributive Lattice.

OR

$\forall a \in L$ $((\text{no. of element } (a^c) \leq 1)$ && $(\text{no. of element } (a^c) \geq 1))$

$\Downarrow$

$\boxed{\text{Exactly one complement} \quad \text{no. of element } (a^c) = 1}$

EXACTLY 1 COMPLEMENT

(D) Complement of Lattice : In a bounded Lattice 'L', for any element 'a' $\in L$, there exist element 'b' $\in L$, such that

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$a \vee b = I$ — ①

(Join) (Upper bound of Lattice) [maximum]

$a \wedge b = 0$ — ②

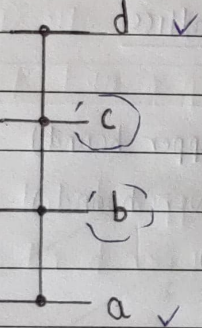
(meet) (lower bound of Lattice) [minimum]

then

$b =$ complement of $a = a^c = b$

$b^c = a$ (Commutative Law)

①



$I = d$ (upper bound of Lattice)

$0 = a$ (lower bound of Lattice)

$a^c = d$	1
$d^c = a$	1
$b^c = (?)$ Does not exist	0
$c^c = (?)$ Does not exist	0

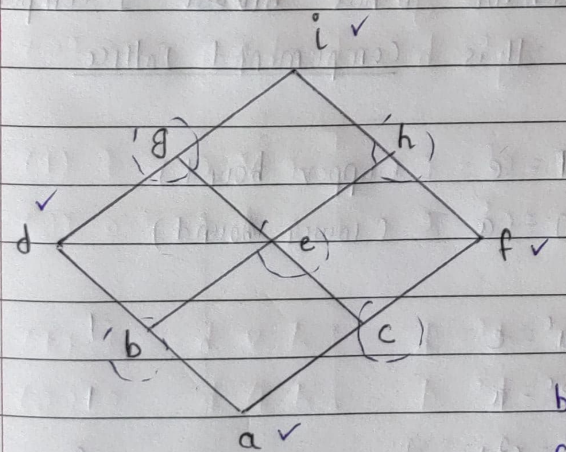
$\therefore$ Almost 1 complement (Every Element)

$\therefore$ This is Distributive Lattice

②

$I = i$ (upper bound of Lattice)

$0 = a$ (lower bound of Lattice)



$a^c = i$

$i^c = a$

$d^c = f$ ($\because$ $d \vee f = I = i$, $d \wedge f = a = 0$)

$f^c = d$

$b^c = (?)$ $g^c = (?)$ $e^c = (?)$

$c^c = (?)$ $h^c = (?)$

Does Not Exist $= (?)$

$\because$ all elements in Given Lattice have almost 1 complement,

$\therefore$ This is Distributive Lattice

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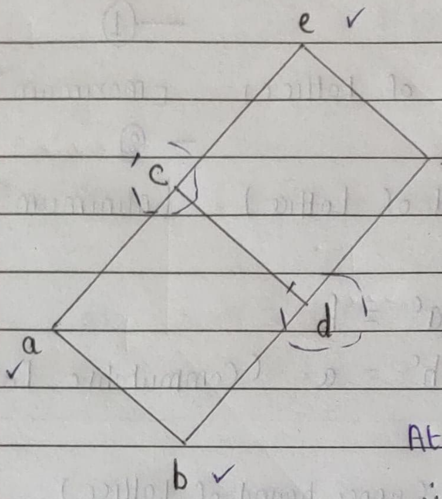
(3)



$I = e$ (upper bound)

$0 = b$ (lower bound)

$b^c = e$

$e^c = b$

$a^c = f$ } ($\because$ $a \vee f = e = I$

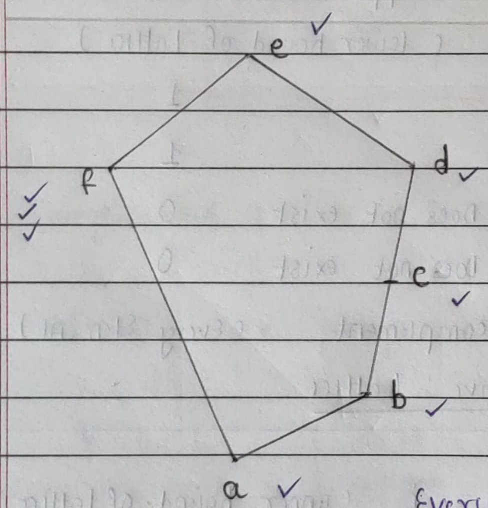
$f^c = a$ } $a \wedge f = b = 0$)

$c^c = (?)$ $d^c = (?)$ $(?)$ = Does not exist

Atmost 1 complement

$\therefore$ Distributed Lattice

(4)



$I = e$ (upper bound)

$0 = a$ (lower bound)

$a^c = e$ (1)

$e^c = a$ (1)

$d^c = f$ (1) $(>= 1)$

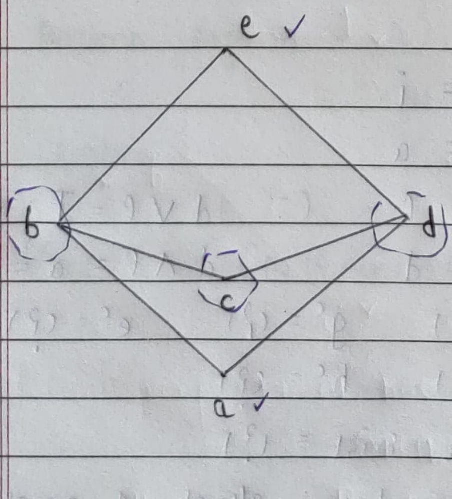
$c^c = f$ (1) $f^c = d, c, b$ (3)

$b^c = f$ (1)

Every element has atleast 1 complement

$\therefore$ It is Complemented Lattice

(5)



$I = e$ (upper bound)

$0 = a$ (lower bound)

$a^c = e$ 1

$e^c = a$ 1

$b^c = (?)$ } 0

$d^c = (?)$ } Does Not Exist 0

$c^c = (?)$ } 0

$\because$ Every element has atmost 1 complement,

$\therefore$ It is Distributive Lattice.

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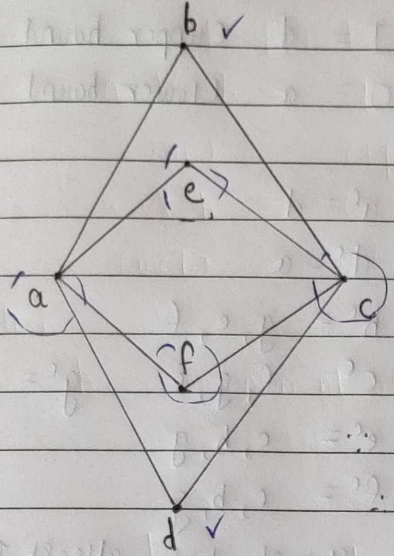
(6)



I = b (upper bound) 0 = d (lower bound)

$b^c = d$ 1

$d^c = b$ 1

$a^c = (?)$
$c^c = (?)$
$e^c = (?)$
$f^c = (?)$
} Does not exist (0)

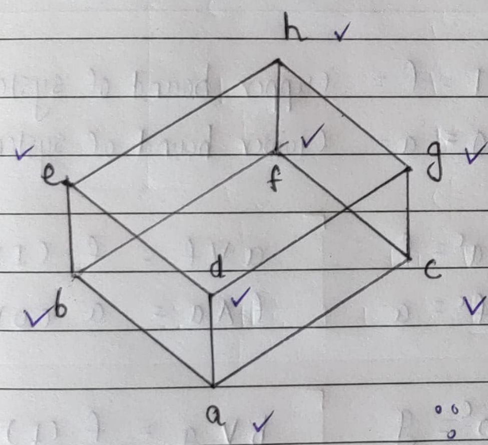
∴ Every Element has atmost 1 complement

∴ It is Distributive Lattice

(7)

I = h (upper bound)

0 = a (lower bound)



$a^c = h$ $b^c = g$ $e^c = c$

$h^c = a$ $g^c = b$ $c^c = e$

$d^c = f$

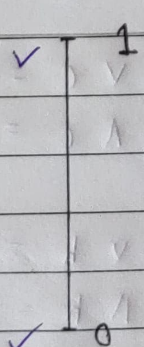
$f^c = d$

∴ Every Element has only one complement

∴ It is Boolean Algebra

(Both Distributive and Complement Lattice as well)

(8)



I = 1 (upper bound)

0 = 0 (lower bound)

$1^c = 0$ (only 1 complement)

$0^c = 1$

∴ Every Element has only one complement

∴ It is Boolean Algebra

(Both Distributive and Complement lattice as well)

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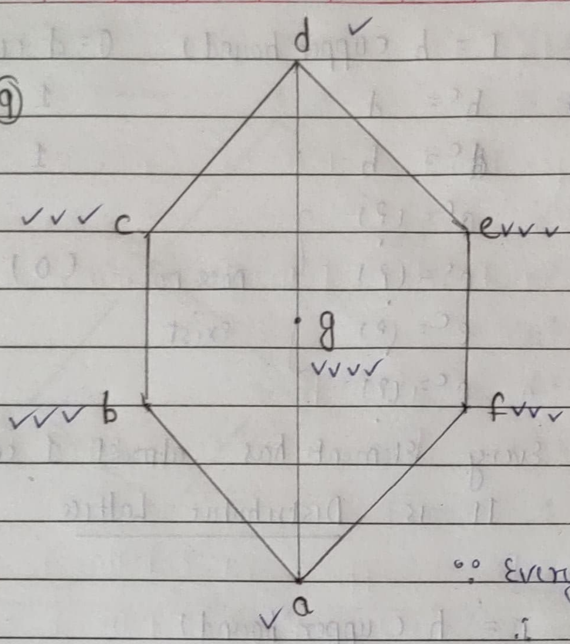
(9)

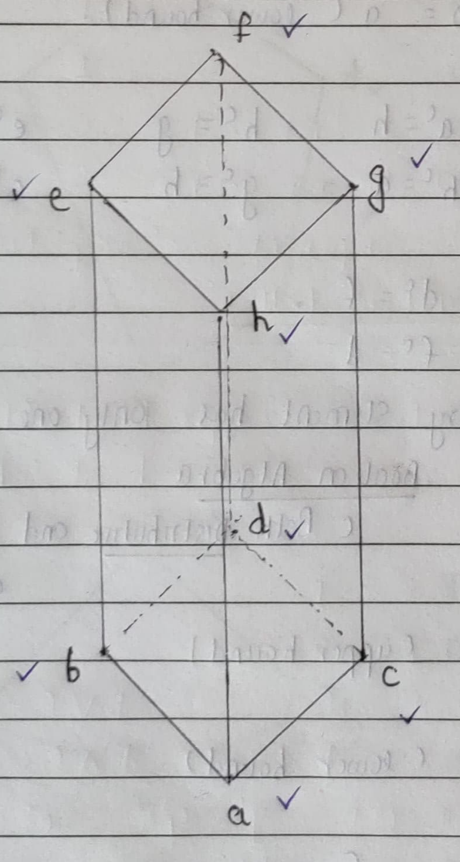


$I = d$ (upper bound of Lattice)
$O = a$ (lower bound of Lattice)

$a^c = d$
$d^c = a$
$b^c = g, e, f$
$c^c = f, g, e$ $\quad g^c = c, b, e, f$
$e^c = c, b, g$
$f^c = c, b, g$

$\because$ Every Element has atleast 1 complement
$\therefore$ It is Complemented Lattice

(10)



$I = f =$ (upper bound of system)
$O = a =$ (lower bound of system)

$a^c = f$
$f^c = a$
$\left[\begin{array}{l} a \vee f = f \ (I) \\ f \wedge a = a \ (O) \end{array} \right]$

$b^c = g$
$g^c = b$
$\left[\begin{array}{l} b \vee g = f \ (I) \\ b \wedge g = a \ (O) \end{array} \right]$

$e^c = c$
$c^c = e$
$\left[\begin{array}{l} e \vee c = f \ (I) \\ e \wedge c = a \ (O) \end{array} \right]$

$d^c = h$
$h^c = d$
$\left[\begin{array}{l} d \vee h = f \ (I) \\ d \wedge h = a \ (O) \end{array} \right]$

$\because$ Every element has EXACTLY 1 complement
$\therefore$ This is BOOLEAN ALGEBRA (Both Distributive and Complemented Lattice as well)