18. For each of the following statements in which A, B, C and D are arbitrary sets, either prove that it is true or give a counter example to show that it is false.
   (a) $A \cap C = B \cap C \rightarrow A = B$
   (b) $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C \rightarrow B = C$
   (c) $(A - C) = (B - C) \rightarrow A = B$
   (d) $A \cap C = B \cap C$ and $A - C = B - C \rightarrow A = B$
   (e) $A \cup C = B \cup C$ and $A - C = B - C \rightarrow A = B$
   (f) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (g) $A \cap (B \times C) = (A \cap B) \times (A \cap C)$
   (h) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
   (i) $(A - B) \times C = (A \times C) - (B \times C)$
   (j) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
   (k) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$
   (l) $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
19. Simplify the following set expressions, using set identities:
   (a) $\overline{(A \cup B)} \cap (\bar{A} \cup \bar{C}) \cap (\bar{\bar{B}} \cup C)$
   (b) $(A \cap B) \cup (A \cap B \cap \bar{C} \cap D) \cup (\bar{A} \cap B)$
   (c) $(A - B) \cup (A \cap B)$
20. Write the dual of each of the following statements:
   (a) $(A \cup B) \cap (A \cup \phi) = A$
   (b) $A \cup B = (A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$
   (c) $\overline{(A \cap B \cap C)} = \overline{(A \cap C)} \cup \overline{(A \cap B)}$.

RELATIONS

Introduction

A *relation* can be thought of as a structure (for example, a table) that represents the relationship of elements of a set to the elements of another set. We come across many situations where relationships between elements of sets, such as those between roll numbers of students in a class and their names, industries and their telephone numbers, employees in an organization and their salaries occur. Relations can be used to solve problems such as producing a useful way to store information in computer databases.

The simplest way to express a relationship between elements of two sets is to use ordered pairs consisting of two related elements. Due to this reason, sets of ordered pairs are called *binary relations*. In this section, we introduce the basic terminology used to describe binary relations, discuss the mathematics of relations defined on sets and explore the various properties of relations.

Definition

When A and B are sets, a subset R of the Cartesian product $A \times B$ is called a *binary relation* from A to B. viz., If R is a binary relation from A to B, R is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$. When $(a, b) \in R$, we use the notation $a \,R\, b$ and read it as "a is related to b by R". If $(a, b) \notin R$, it is denoted as $a \not R\, b$.

Note Mostly we will deal with relationships between the elements of two sets. Hence the word 'binary' will be omitted hereafter.

Relation

If R is a relation from a set A to itself, viz., if R is a subset of $A \times A$, then R is called *a relation on the set A*.

domain

The set $\{a \in A | a \,R\, b$, for some $b \in B\}$ is called *the domain* of R and denoted by $D(R)$.

range

The set $\{b \in B | a \,R\, b$, for some $a \in A\}$ is called *the range* of R and denoted by $R(R)$.

Examples

1. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3\}$ and $a \,R\, b$ if and only if $a + b = 4$.
   Then $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$
   The domain of $R = \{1, 2, 3, 4\}$ and the image of $R = \{0, 1, 2, 3\}$
2. Let R be the relation on $A = \{1, 2, 3, 4\}$, defined by $a \,R\, b$ if $a \le b$; $a, b \in A$.
   Then $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
   The domain and range of R are both equal to A.

TYPES OF RELATIONS

A relation R on a set A is called a *universal* relation, if $R = A \times A$.
For example if $A = \{1, 2, 3\}$, then $R = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is the universal relation on A.

A relation R on a set A is called a *void relation* if R is the null set ϕ. For example if $A = \{3, 4, 5\}$ and R is defined as $a \,R\, b$ if and only if $a + b > 10$, then R is a null set, since no element in $A \times A$ satisfies the given condition.

Note The entire Cartesian product $A \times A$ and the empty set are subsets of $A \times A$.

A relation R on a set A is called an *identity relation*, if $R = \{(a, a) | a \in A\}$ and is denoted by I_A.

For example, if $A = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

When R is any relation from a set A to a set B, *the inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs got by interchanging the elements of the ordered pairs in R.

viz., $R^{-1} = \{(b, a) | (a, b) \in R\}$
viz., if $a \,R\, b$, then $b \,R^{-1}\, a$.

For example, if $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ and $a \,R\, b$ if and only if $a \in A$ divides $b \in B$, then $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (5, 10)\}$.

Now $R^{-1} = \{(6, 2), (8, 2), (10, 2), (6, 3), (10, 5)\}$

We note that $b \,R^{-1}\, a$, if and only if $b \in B$ is a multiple of $a \in A$. Also we note that

$$D(R) = R(R^{-1}) = \{2, 3, 5\} \text{ and}$$

$R(R) = D(R^{-1}) = \{6, 8, 10\}$

$\vee \to$ disjunction

 $\wedge \to$ conjunction

SOME OPERATIONS ON RELATIONS

As binary relations are sets of ordered pairs, all set operations can be done on relations. The resulting sets are ordered pairs and hence are relations.

If R and S denote two relations, *the intersection of R and S* denoted by $R \cap S$, is defined by conjunction $\boxed{a\,(R \cap S)b = a \text{ R } b \wedge a \text{ S } b}$

and *the union of R and S*, denoted by $R \cup S$, is defined by $a\,(R \cup S)\,b = a \text{ R } b \vee a \text{ S } b$.

disjunction *The difference of R and S*, denoted by $R - S$, is defined by $a\,(R - S)\,b = a \text{ R } b \wedge a \not\text{S}\, b$.

The *complement of R*, denoted by R' or $\sim R$ is defined by $a(R')b = a \not\text{R}\, b$.

For example, let $A = \{x, y, z\}$, $B = \{1, 2, 3\}$, $C = \{x, y\}$ and $D = \{2, 3\}$. Let R be a relation from A to B defined by $R = \{(x, 1), (x, 2), (y, 3)\}$ and let S be a relation from C to D defined by $S = \{(x, 2), (y, 3)\}$.

Then $R \cap S = \{(x, 2), (y, 3)\}$ and $R \cup S = R$.

$$R - S = \{(x, 1)\}$$
$$R' = \{(x, 3), (y, 1), (y, 2), (z, 1), (z, 2), (z, 3)\}$$

COMPOSITION OF RELATIONS

If R is a relation from set A to set B and S is a relation from set B to set C, viz., R is a subset of $A \times B$ and S is a subset of $B \times C$, then *the composition of R and S*, denoted by $R \bullet S$, [some authors use the notation $S \bullet R$ instead of $R \bullet S$] is defined by

$a(R \bullet S)\, c$, if for some $b \in B$, we have $a \text{ R } b$ and $b \text{ R } c$.

viz., $R \bullet S = \{(a, c) |$ there exists some $b \in B$ for which $(a, b) \in R$ and $(b, c) \in S\}$

Note
1. For the relation $R \bullet S$, the domain is a subset of A and the range is a subset of C.
2. $R \bullet S$ is empty, if the intersection of the range of R and the domain of S is empty.
3. If R is a relation on a set A, then $R \bullet R$, the composition of R with itself is always defined and sometimes denoted as R^2.

For example, let $R = \{(1, 1), (1, 3), (3, 2), (3, 4), (4, 2)\}$ and $S = \{(2, 1), (3, 3), (3, 4), (4, 1)\}$.

Any member (ordered pair) of $R \bullet S$ can be obtained only if the second element in the ordered pair of R agrees with the first element in the ordered pair of S.

Thus (1, 1) cannot combine with any member of S.

(1, 3) of R can combine with (3, 3) and (3, 4) of S producing the members (1, 3) and (1, 4) respectively of $R \bullet S$. Similarly the other members of $R \bullet S$ are obtained.

Thus $R \bullet S = \{(1, 3), (1, 4), (3, 1), (4, 1)\}$

$\ldots (4, 3)\}$

$$(R \bullet S) \bullet R = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R \bullet (S \bullet R) = \{(1, 2), (1, 4), (3, 1), (3, 3), (4, 1), (4, 3)\}$$
$$R^3 = R \bullet R \bullet R = (R \bullet R) \bullet R = R \bullet (R \bullet R)$$
$$= \{(1, 1), (1, 3), (1, 2), (1, 4)\}$$

PROPERTIES OF RELATIONS

(i) A relation R on a set A is said to be *reflexive*, if $a \text{ R } a$ for every $a \in A$, viz., if $(a, a) \in R$ for every $a \in A$.

For example, if R is the relation on $A = \{1, 2, 3\}$ defined by $(a, b) \in R$ if $a \le b$, where $a, b \in A$, then $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$.

Now R is reflexive, since each of the elements of A is related to itself, as (1, 1), (2, 2) and (3, 3) are members in R.

Note A relation R on a set A is irreflexive, if, for every $a \in A$, $(a, a) \notin R$. viz., if there is no $a \in A$ such that $a \text{ R } a$.

For example, R, $\{(1, 2), (2, 3), (1, 3)\}$ in the above example is irreflexive.

(ii) A relation R on a set A is said to be *symmetric*, if whenever $a \text{ R } b$ then $b \text{ R } a$, viz., if whenever $(a, b) \in R$ then (b, a) also $\in R$.

Thus a relation R on A is not symmetric if there exist $a, b \in A$ such that $(a, b) \in R$, but $(b, a) \notin R$.

(iii) A relation R on a set A is said to be *antisymmetric*, whenever (a, b) and $(b, a) \in R$ then $a = b$. If there exist $a, b \in A$ such that (a, b) and $(b, a) \in R$, but $a \ne b$, then R is not antisymmetric.

For example, the relation of perpendicularity on a set of lines in the plane is symmetric, since if a line a is perpendicular to the line b, then b is perpendicular to a.

The relation $\le$ on the set Z of integers is not symmetric, since, for example, $4 \le 5$, but $5 \not\le 4$.

The relation of divisibility on N is antisymmetric, since whenever m is divisible by n and n is divisible by m then $m = n$.

Note Symmetry and antisymmetry are not negative of each other. For example, the relation $R = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric, whereas the relation $S = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

(iv) A relation R on a set A is said to be *transitive*, if whenever $a \text{ R } b$ and $b \text{ R } c$ then $a \text{ R } c$. viz., if whenever (a, b) and $(b, c) \in R$ then $(a, c) \in R$.

Thus if there exist $a, b, c \in A$ such that (a, b) and $(b, c) \in R$ but $(a, c) \notin R$, then R is not transitive.

For example, the relation of set inclusion on a collection of sets is transitive, since if $A \subseteq B$ and $B \subseteq C$, $A \subseteq C$.

(v) A relation R on a set A is called an *equivalence relation*, if R is reflexive, symmetric and transitive.

viz. R is an equivalence relation on a set A, if it has the following three properties: 1. $a R a$, for $\forall a \in A$,

1. a R a, for every a ∈ A.
2. If a R b, then b R a
3. If a R b and b R c, then a R c

For example, the relation of similarity with respect to a set of triangles T is an equivalence relation, since if T_1, T_2, T_3 are elements of the set T, then

$T_1 ||| T_1$, i.e., T_1 R T_1 for every $T_1 \in T$,
$T_1 ||| T_2$ implies $T_2 ||| T_1$ and
$T_1 ||| T_2$ and $T_2 ||| T_3$ simplify $T_1 ||| T_3$

viz., the relation of similarity of triangles is reflexive, symmetric and transitive.

(vi) A relation R on a set A is called a *partial ordering* or *partial order relation*, if R is reflexive, antisymmetric and transitive.

viz., R is a partial order relation on A if it has the following three properties:

(a) a R a, for every $a \in A$
(b) a R b and b R $a \Rightarrow a = b$
(c) a R b and b R $c \Rightarrow a$ R c

A set A together with a partial order relation R is called *a partially ordered set* or *poset*. For example, the greater than or equal to (≥) relation is a partial ordering on the set of integers Z, since

(a) $a \geq a$ for every integer a, i.e. ≥ is reflexive
(b) $a \geq b$ and $b \geq a \Rightarrow a = b$, i.e. ≥ is antisymmetric
(c) $a \geq b$ and $b \geq c \Rightarrow a \geq c$, i.e. ≥ is transitive

Thus $(Z, \geq)$ is a poset.

EQUIVALENCE CLASSES

Definition

If R is an equivalence relation on a set A, the set of all elements of A that are related to an element a of A is called the *equivalence class of a* and denoted by $[a]_R$.

When there is no ambiguity regarding the relation, viz., when we deal with only one relation, the equivalence class of a is denoted by just $[a]$.

In other words, the equivalence class of a under the relation R is defined as

$$[a] = \{x|(a, x) \in R\}$$

Any element $b \in [a]$ is called *a representative* of the equivalence class $[a]$.

The collection of all equivalence classes of elements of A under an equivalence relation R is denoted by A/R and is called the *quotient set* of A by R.

viz. $$A/R = \{[a]|a \in A\}$$

For example, the relation R on the set $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ is an equivalence relation, since R is reflexive symmetric and transitive.

Now $[1] = \{1, 2\}$, $[2] = \{1, 2\}$ and $[3] = \{3\}$

Thus [1], [2] and [3] are the equivalence classes of A ~~under~~ R and hence form A/R.

Theorem

If R is an equivalence relation on non-empty set A and if a and $b \in A$ a[re] arbitrary, then

(i) $a \in [a]$, for every $a \in A$
(ii) $[a] = [b]$, if and only if $(a, b) \in R$
(iii) If $[a] \cap [b] \neq \phi$, then $[a] = [b]$

Proof:

(i) Since R is reflexive, $(a, a) \in R$ for every $a \in A$.
Hence $a \in [a]$,

(ii) Let us assume that $(a, b) \in R$ or a R b (
Let $x \in [b]$. Then $(b, x) \in R$ or b R x (
From (1) and (2), it follows that a R x or $(a, x) \in R$ (∵ R is transitiv[e])
$\therefore\ x \in [a]$
Thus $x \in [b] \Rightarrow x \in [a]$ ∴ $[b] \subseteq [a]$ (
Let $y \in [a]$. Then a R y (
From (1), we have b R a, since R is symmetric. (
From (5) and (4), we get b R y, since R is transitive.
$\therefore\ y \in [b]$
Thus $y \in [a] \Rightarrow y \in [b]$ ∴ $[a] \subseteq [b]$ (
From (3) and (6), we get $[a] = [b]$
Conversely, let $[a] = [b]$
Now $b \in [b]$ by (i)
i.e., $b \in [a]$ ∴ $(a, b) \in R$

(iii) Since $[a] \cap [b] \neq \phi$, there exists an element $x \in A$ such that $x \in [a] \cap [$
Hence $x \in [a]$ and $x \in [b]$
i.e., x R a and x R b
or a R x and x R b
∴ a R b, since R is transitive
Hence, by (ii), $[a] = [b]$
Equivalently, if $[a] \neq [b]$, then $[a] \cap [b] = \phi$.

Note From (ii) and (iii) of the above theorem, it follows that the equivalen[ce] classes of two arbitrary elements under R are identical or disjoint.

PARTITION OF A SET

Definition

If S is a non empty set, a collection of disjoint non empty subsets of S who[se] union is S is called *a partition* of S. In other words, the collection of subsets is a partition of S if and only if

(i) $A_i \neq \phi$, for each i
(ii) $A_i \cap A_j = \phi$, for $i \neq j$ and
(iii) $\bigcup_i A_i = S$, where $\bigcup_i A_i$ represents the union of the subsets A_i for all i.

Note The subsets in a partition are also called *blocks* of the partition.

For example, if $S = \{1, 2, 3, 4, 5, 6\}$

(i) $\{\{1, 3, 5\}, \{2, 4\}\}$ is not a partition since the union of the subsets is n[ot] S, as the element 6 is m[issing]