https://www.geeksforgeeks.org/functional-dependency-and-attribute-closure/
https://www.gatevidyalay.com/closure-of-an-attribute-set/

Closure of an Attribute Set-

- The set of all those attributes which can be functionally determined from an attribute set is called as a closure of that attribute set.
- Closure of attribute set {X} is denoted as $\{X\}^+$.

Steps to Find Closure of an Attribute Set-

Following steps are followed to find the closure of an attribute set-

Step-01:

Add the attributes contained in the attribute set for which closure is being calculated to the result set.

Step-02:

Recursively add the attributes to the result set which can be functionally determined from the attributes already contained in the result set.

Example-

Consider a relation R (A , B , C , D , E , F , G) with the functional dependencies-

$$A \rightarrow BC$$

$$BC \rightarrow DE$$

$$D \rightarrow F$$

$$CF \rightarrow G$$

A+ = {A,B,C,D,E,F,G}

BC+ = {B,C,D,E,F,G}

D+= {D,F}

CF+ = {C,F,G}

ABC+ = { A,B,C,D,E,F,G}

Now, let us find the closure of some attributes and attribute sets-

Closure of attribute A-

A^+ = { A }

= { A , B , C } (Using $A \rightarrow BC$)

= { A , B , C , D , E } (Using $BC \rightarrow DE$)

= { A , B , C , D , E , F } (Using $D \rightarrow F$)

= { A , B , C , D , E , F , G } (Using $CF \rightarrow G$)

Thus, **A^+ = { A , B , C , D , E , F , G }**

Closure of attribute D-

$D^+ = \{ D \}$

$= \{ D , F \}$ (Using $D \rightarrow F$)

We can not determine any other attribute using attributes D and F contained in the result set.

Thus,

$$D^+ = \{ D , F \}$$

Closure of attribute set {B, C}-

$\{ B , C \}^+ = \{ B , C \}$

$= \{ B , C , D , E \}$ (Using $BC \rightarrow DE$)

$= \{ B , C , D , E , F \}$ (Using $D \rightarrow F$)

$= \{ B , C , D , E , F , G \}$ (Using $CF \rightarrow G$)

Thus,

$$\{ B , C \}^+ = \{ B , C , D , E , F , G \}$$

Finding the Keys Using Closure-

Super Key-

- If the closure result of an attribute set contains all the attributes of the relation, then that attribute set is called as a super key of that relation.
- Thus, we can say- **"The closure of a super key is the entire relation schema."**

Example-

In the above example,

- The closure of attribute A is the entire relation schema.
- Thus, attribute A is a super key for that relation.

Candidate Key-

- If there exists no subset of an attribute set whose closure contains all the attributes of the relation, then that attribute set is called as a candidate key of that relation.

Example-

In the above example,

- No subset of attribute A contains all the attributes of the relation.
- Thus, attribute A is also a candidate key for that relation.

Also Read-How To Find Candidate Keys?

PRACTICE PROBLEM BASED ON FINDING CLOSURE OF AN ATTRIBUTE SET-

Problem-

Consider the given functional dependencies-

$$AB \rightarrow CD$$
$$AF \rightarrow D$$
$$DE \rightarrow F$$
$$C \rightarrow G$$
$$F \rightarrow E$$
$$G \rightarrow A$$

AB+ = {A,B, C,D,G}

AF+ = {A,F,D,E}

DE+ = {D.E.F}

C+ = {C,G,A}

F+ = {F,E}

G + = {G,A}

Which of the following options is false?

(A) $\{ CF \}^+ = \{ A , C , D , E , F , G \}$

(B) $\{ BG \}^+ = \{ A , B , C , D , G \}$

(C) $\{ AF \}^+ = \{ A , C , D , E , F , G \}$

(D) $\{ AB \}^+ = \{ A , C , D , F ,G \}$

Solution-

Let us check each option one by one-

Option-(A):

$\{ CF \}^+ = \{ C , F \}$

$= \{ C , F , G \}$ (Using $C \rightarrow G$)

$= \{ C , E , F , G \}$ (Using $F \rightarrow E$)

$= \{ A , C , E , E , F \}$ (Using $G \rightarrow A$)

$= \{ A , C , D , E , F , G \}$ (Using $AF \rightarrow D$)

Since, our obtained result set is same as the given result set, so, it means it is correctly given.

Option-(B):

$\{ BG \}^{+} = \{ B , G \}$

$= \{ A , B , G \}$ (Using $G \to A$)

$= \{ A , B , C , D , G \}$ (Using $AB \to CD$)

Since, our obtained result set is same as the given result set, so, it means it is correctly given.

Option-(C):

$\{ AF \}^{+} = \{ A , F \}$

$= \{ A , D , F \}$ (Using $AF \to D$)

$= \{ A , D , E , F \}$ (Using $F \to E$)

Since, our obtained result set is different from the given result set, so,it means it is not correctly given.

Option-(D):

$\{ AB \}^{+} = \{ A , B \}$

$= \{ A , B , C , D \}$ (Using $AB \to CD$)

$= \{ A , B , C , D , G \}$ (Using $C \to G$)

Example

For the Given FD for Relation Stud(Stud_no, Stud_name, Stud_phone, Stud_state,Stud_country,stud_age) find the closureset of attributes

STUD_NO->STUD_NAME, STUD_NO->STUD_PHONE, STUD_NO->STUD_STATE, STUD_NO->STUD_COUNTRY,

STUD_NO -> STUD_AGE, STUD_STATE->STUD_COUNTRY

(STUD_NO)+ = {STUD_NO, STUD_NAME, STUD_PHONE, STUD_STATE, STUD_COUNTRY, STUD_AGE}

(STUD_STATE)+ = {STUD_STATE, STUD_COUNTRY}

How to find Candidate Keys and Super Keys using Attribute Closure?

- If attribute closure of an attribute set contains all attributes of relation, the attribute set will be super key of the relation.
- If no subset of this attribute set can functionally determine all attributes of the relation, the set will be candidate key as well.

For Example, using FD set of table:

(STUD_NO, STUD_NAME)+ = {STUD_NO, STUD_NAME, STUD_PHONE, STUD_STATE, STUD_COUNTRY, STUD_AGE}

(STUD_NO)+ = {STUD_NO, STUD_NAME, STUD_PHONE, STUD_STATE, STUD_COUNTRY, STUD_AGE}

(STUD_NO, STUD_NAME) will be super key but not candidate key because its subset (STUD_NO)+ is equal to all attributes of the relation. So, STUD_NO will be a candidate key.

Question: Consider the relation scheme R = {E, F, G, H, I, J, K, L, M, M} and the set of functional dependencies
{{E, F} -> {G},
{F} -> {I, J},
{E, H} -> {K, L},
K -> {M},
L -> {N}

on R. What is the key for R?

A. {E, F}
B. {E, F, H}
C. {E, F, H, K, L}
D. {E}

{EF}+ = {E,F,G,I,J}
{EFH}+ = {E,F,H,I,J,K,L,M,N}

Answer: Finding attribute closure of all given options, we get:
{E,F}+ = {EFGIJ}
{E,F,H}+ = {EFHGIJKLMN}
{E,F,H,K,L}+ = {{EFHGIJKLMN}
{E}+ = {E}
{EFH}+ and {EFHKL}+ results in set of all attributes, but EFH is minimal. So it will be candidate key. So correct option is (B).

How to check whether an FD can be derived from a given FD set?

To check whether an FD A->B can be derived from an FD set F,

1. Find (A)+ using FD set F.
2. If B is subset of (A)+, then A->B is true else not true.

Question: In a schema with attributes A, B, C, D and E following set of functional dependencies are given
{A -> B, A -> C, CD -> E, B -> D, E -> A}

Which of the following functional dependencies is NOT implied by the above set?
A. CD -> AC
B. BD -> CD
C. BC -> CD
D. AC -> BC

Answer: Using FD set given in question,
(CD)+ = {CDEAB} which means CD -> AC also holds true.
(BD)+ = {BD} which means BD -> CD can't hold true. So this FD is no implied in FD set. So (B) is the required option.
Others can be checked in the same way.

Prime and non-prime attributes

Attributes which are parts of any candidate key of relation are called as prime attribute, others are non-prime attributes. For Example, STUD_NO in STUDENT relation is prime attribute, others are non-prime attribute.

Question: Consider a relation scheme R = (A, B, C, D, E, H) on which the following functional dependencies hold: {A–>B, BC–> D, E–>C, D–>A}. What are the candidate keys of R?

(a) AE, BE
(b) AE, BE, DE
(c) AEH, BEH, BCH
(d) AEH, BEH, DEH

Answer:

(AE)+ = {ABECD} which is not set of all attributes. So AE is not a candidate key.
Hence option A and B are wrong.

(AEH)+ = {ABCDEH}
(BEH)+ = {BEHCDA}
(BCH)+ = {BCHDA} which is not set of all attributes. So BCH is not a candidate key. Hence option C is wrong.
So correct answer is D.

Example-1 : Consider the table student_details having (Roll_No, Name,Marks, Location) as the attributes and having two functional dependencies.

FD1 : Roll_No Name, Marks

FD2 : Name   Marks, Location

Now, We will calculate the closure of all the attributes present in the relation using the three steps mentioned below.

Step-1 : Add attributes present on the LHS of the first functional dependency to the closure.
{Roll_no}$^+$ = {Roll_No}

Step-2 : Add attributes present on the RHS of the original functional dependency to the closure.
{Roll_no}$^+$ = {Roll_No, Marks}

Step-3 : Add the other possible attributes which can be derived using attributes present on the RHS of the closure. So Roll_No attribute cannot functionally determine any attribute but Name attribute can determine other attributes such as Marks and Location using 2nd Functional Dependency(Name [icon name="long-arrow-right" class="" unprefixed_class=""] Marks, Location).
Therefore, complete closure of Roll_No will be :

{Roll_no}$^+$ = {Roll_No, Marks, Name, Location}

Similarly, we can calculate closure for other attributes too i.e “Name”.

Step-1 : Add attributes present on the LHS of the functional dependency to the closure.
{Name}$^+$ = {Name}

Step-2 : Add the attributes present on the RHS of the functional dependency to the closure.
{Name}$^+$ = {Name, Marks, Location}

Step-3 : Since, we don’t have any functional dependency where “Marks or Location” attribute is functionally determining any other attribute , we cannot add more attributes to the closure. Hence complete closure of Name would be :
{Name}$^+$ = {Name, Marks, Location}

NOTE : We don’t have any Functional dependency where marks and location can functionally determine any attribute. Hence, for those attributes we can only add the attributes themselves in their closures. Therefore,
{Marks}$^+$ = {Marks}   and

{Location}$^+$ = { Location}

Example-2 : Consider a relation R(A,B,C,D,E) having below mentioned functional dependencies.

FD1 : A ⟶ BC

FD2 : C ⟶ B

FD3 : D ⟶ E

FD4 : E ⟶ D

Now, we need to calculate the closure of attributes of the relation R. The closures will be:

$\{A\}^+ = \{A, B, C\}$

$\{B\}^+ = \{B\}$

$\{C\}^+ = \{B, C\}$

$\{D\}^+ = \{D, E\}$

$\{E\}^+ = \{E\}$

Closure Of Functional Dependency : Calculating Candidate Key

- "A Candidate Key of a relation is an attribute or set of attributes that can determine the whole relation or contains all the attributes in its closure."
- Let's try to understand how to calculate candidate keys.

Example-1 : Consider the relation R(A,B,C) with given functional dependencies :

FD1 : A ⟶ B

FD2 : B ⟶ C

Now, calculating the closure of the attributes as :

$\{A\}^+ = \{A, B, C\}$

$\{B\}^+ = \{B, C\}$

$\{C\}^+ = \{C\}$

Clearly, "A" is the candidate key as, its closure contains all the attributes present in the relation "R".

Example-2 : Consider another relation R(A, B, C, D, E) having the Functional dependencies :

FD1 : A → BC

FD2 : C → B

FD3 : D → E

FD4 : E → D

Now, calculating the closure of the attributes as :

$\{A\}^+ = \{A, B, C\}$

$\{B\}^+ = \{B\}$

$\{C\}^+ = \{C, B\}$

$\{D\}^+ = \{E, D\}$

$\{E\}^+ = \{E, D\}$

In this case, a single attribute is unable to determine all the attribute on its own like in previous example. Here, we need to combine two or more attributes to determine the candidate keys.

$\{A, D\}^+ = \{A, B, C, D, E\}$

$\{A, E\}^+ = \{A, B, C, D, E\}$

Hence, "AD" and "AE" are the two possible keys of the given relation "R". Any other combination other than these two would have acted as extraneous attributes.

NOTE : Any relation "R" can have either single or multiple candidate keys.

Closure Of Functional Dependency : Key Definitions

1. **Prime Attributes :** Attributes which are indispensable part of candidate keys. For example : "A, D, E" attributes are prime attributes in above example-2.
2. **Non-Prime Attributes :** Attributes other than prime attributes which does not take part in formation of candidate keys. For example.
3. **Extraneous Attributes :** Attributes which does not make any effect on removal from candidate key.

For example : Consider the relation R(A, B, C, D) with functional dependencies :

FD1 : A $\rightarrow$ BC

FD2 : B $\rightarrow$ C

FD3 : D $\rightarrow$ C

Here, Candidate key can be "AD" only. Hence,

Prime Attributes : A, D.

Non-Prime Attributes : B, C

Extraneous Attributes : B, C (As if we add any of the to the candidate key, it will remain unaffected). Those attributes, which if removed does not affect closure of that set.