Mathematical Logi

INTRODUCTION

Logic is the discipline that deals with the methods of reasoning. One of t aims of logic is to provide rules by which we can determine whether a particu reasoning or argument is valid. Logical reasoning is used in many disciplin to establish valid results. Rules of logic are used to provide proofs of theore in mathematics, to verify the correctness of computer programs and to dr conclusions from scientific experiments. In this chapter, we shall introdu certain logical symbols using which we shall state and apply rules of va inference and hence understand how to construct correct mathemati arguments.

PROPOSITIONS

A declarative sentence (or assertion) which is true or false, but not both, called a *proposition* (or *statement*). Sentences which are exclamato interrogative or imperative in nature are not propositions. Lower case lett such as p, q, r ... are used to denote propositions. For example, we consider t following sentences:

1. New Delhi is the capital city of India.
2. How beautiful is Rose?
3. $2 + 2 = 3$
4. What time is it?
5. $x + y = z$
6. Take a cup of coffee.

In the given statements, (2), (4) and (6) are obviously not propositions ince (1) is true, (3) is false and (5) is neither true nor false as the values of x, y ınd z are not assigned.

If a proposition is true, we say that the *truth value* of that proposition is true, lenoted by T or 1. If a proposition is false, the truth value is said to be false, lenoted by F or 0.

Propositions which do not contain any of the logical operators or connectives to be introduced in the next section) are called *atomic* (*primary* or *primitive*) *ɔropositions*. Many mathematical statements which can be constructed by :ombining one or more atomic statements using connectives are called molecular ɔr *compound propositions*.

The truth value of a compound proposition depends on those of sub-ɔropositions and the way in which they are combined using connectives.

The area of logic that deals with propositions is called *propositional logic* or *ɔropositional calculus*.

CONNECTIVES

Definition

When p and q are any two propositions, the proposition "p and q" denoted by $p \wedge q$ and called the *conjunction* of p and q is defined as the compound proposition that is true when both p and q are true and is false otherwise. ($\wedge$ is the connective used) A *truth table* is a table that displays the relationships between the truth values of sub-propositions and that of compound proposition constructed from them.

Table 1.1 is the truth table for the conjunction of two propositions p and q viz., "p and q".

Table 1.1

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition

When p and q are any two propositions, the propositions "p or q" denoted by $p \vee q$ and called the *disjunction* of p and q is defined as the compound proposition that is false when both p and q are false and is true otherwise. ($\vee$ is the connective used)

Table 1.2 is the truth table for the disjunction of two propositions p and q, viz., "$p \vee q$".

Table 1.2

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition

Given any proposition p, another proposition formed by writing "It is not the case that" or "It is false that" before p or by inserting the word 'not' suitably in p is called the *negation of p* and denoted by $\rceil p$ (read as 'not p'). $\rceil p$ is also denoted as p', $\bar{p}$ and $\sim p$. It p is true, then $\rceil p$ is false and if p is false, then $\rceil p$ is true.

Table 1.3 is the truth table for the negation of p. For example, if p is the statement "New Delhi is in India", the $\rceil p$ is any one of the following statements.

Table 1.3

p	$\rceil p$
T	F
F	T

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(a) It is not the case ...
(b) It is false that New Delhi is in India
(c) New Delhi is not in India

The truth value of p is T and that of $\neg p$ is F.

ORDER OF PRECEDENCE FOR LOGICAL CONNECTIVES

We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For example, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However to avoid the use of an excessive number of parentheses, we adopt an order of precedence for the logical operators, given as follows:

(i) The negation operator has precedence over all other logical operators. Thus $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$.

(ii) The conjunction operator has precedence over the disjunction operator. Thus $p \wedge q \vee r$ means $(p \wedge q) \vee r$, but not $p \wedge (q \vee r)$.

(iii) The conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ (to be introduced subsequently) have lower precedence than other operators. Among them, $\rightarrow$ has precedence over $\leftrightarrow$.

CONDITIONAL AND BICONDITIONAL PROPOSITIONS

Definition

If p and q are propositions, the compound proposition "if p, then q", that is denoted by $p \rightarrow q$ is called *a conditional proposition*, which is false when p is true and q is false and true otherwise.

In this conditional proposition, p is called the *hypothesis* or *premise* and q is called the *conclusion* or *consequence*.

Note Some authors call $p \rightarrow q$ as an implication.

For example, let us consider the statement.

"If I get up at 5 A.M., I will go for a walk", which may be represented as $p \rightarrow q$ and considered as a contract.

If p is true and q is also true, the contract is not violated and so '$p \rightarrow q$' is true.

If p is true and q is false (viz., I get up at 5 A.M., but I do not go for a walk), the contract is violated and so '$p \rightarrow q$' is false.

If p is false and whether q is true or false (viz., when I have not got up at 5 A.M; I may or may not go for a walk), the contract is not violated and so '$p \rightarrow q$' is true.

Accordingly, the truth table for the conditional proposition $p \rightarrow q$ will be as given in Table 1.4.

Table 1.4

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The alternative terminologies used to express $p \rightarrow q$ (if p, then q) are the following:

T is not q true

T has no value

(i) p implies q, (ii) p only if q ["If p, then q" formulation emphasizes the hypothesis, whereas "p only if q" formulation emphasizes the conclusion; the difference is only stylistic], (iii) q if p or q when p, (iv) q follows from p, (v) p is sufficient for q or a sufficient condition for q is p and (vi) q is necessary for p or a necessary conditions for p is q.

Definition

If p and q are propositions, the compound proposition "p if and only if q", that is denoted by $p \leftrightarrow q$, is called a *biconditional proposition*, which is true when p and q have the same truth values and is false otherwise.

It is easily verified that '$p \leftrightarrow q$' is true when both the conditionals $p \to q$ and $q \to p$ are true. This is the reason for the symbol $\leftrightarrow$ which is a combination of $\to$ and $\leftarrow$.

Alternatively, '$p \leftrightarrow q$' is also expressed as 'p iff q' and 'p is necessary and sufficient for q'.

The truth table for '$p \leftrightarrow q$' is given in Table 1.5.

Note The notation $p \rightleftarrows q$ is also used instead of $p \leftrightarrow q$.

Table 1.5

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TAUTOLOGY AND CONTRADICTION

A compound proposition $P = P(p_1, p_2, \ldots, p_n)$, where $p_1, p_2, \ldots, p_n$ are variables (elemental propositions), is called *a tautology*, if it is true for every truth assignment for $p_1, p_2, \ldots, p_n$.

P is called *a contradiction*, if it is false for every truth assignment for $p_1, p_2, \ldots, p_n$.

For example, $p \vee \neg p$ is a tautology, whereas $p \wedge \neg p$ is a contradiction, as seen from the Table 1.6 given below.

Table 1.6

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Note
1. The negation of a tautology is a contradiction and the negation of a contradiction is a tautology.
2. If $P(p_1, p_2, \ldots, p_n)$ is a tautology, then $P(q_1, q_2, \ldots, q_n)$ is also a tautology, where $q_1, q_2, \ldots, q_n$ are any set of propositions. This is known as the *principle of substitution*.

   For example, since $p \vee \neg p$ is a tautology, $((p \vee q) \wedge r) \vee \neg ((p \vee q) \wedge r)$ is also a tautology.
3. If a proposition is neither a tautology nor a contradiction, it is called a *contingency*.

EQUIVALENCE OF PROPOSITIONS

Two compound propositions $A(p_1, p_2, \ldots, p_n)$ and $B(p_1, p_2, \ldots, p_n)$ are said to be *logically equivalent* or simply *equivalent*, if they have identical truth tables, viz. if the truth value of A is equal to the truth value of B for every one of the 2^n

The equivalence of two propositions A and B is denoted as $A \Leftrightarrow B$ or $A \equiv B$ (which is read as 'A is equivalent to B'). $\Leftrightarrow$ or $\equiv$ is not a connective.
For example, let us consider the truth tables of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ (see Table 1.7). The final columns in the truth tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are identical. Hence $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Table 1.7

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Note We have already noted that the biconditional proposition $A \leftrightarrow B$ is true whenever both A and B have the same truth value, viz. $A \leftrightarrow B$ is a tautology, when A and B are equivalent.
Conversely, $A \equiv B$, when $A \leftrightarrow B$ is a tautology. For example, $(p \to q) \equiv (\neg p \vee q)$, since $(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology, as seen from the truth Table 1.8 given below:

Table 1.8

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow \neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

DUALITY LAW

The *dual* of a compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are special variables representing compound propositions that are tautologies and contradictions respectively. The dual of a proposition A is denoted by A^*.

DUALITY THEOREM

If $A(p_1, p_2, \ldots, p_n) \equiv B(p_1, p_2, \ldots, p_n)$, where A and B are compound propositions, then $A^*(p_1, p_2, \ldots, p_n) \equiv B^*(p_1, p_2, \ldots, p_n)$.

Proof

In Table (1.7), we have proved that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \text{ or } p \vee q \equiv \neg(\neg p \wedge \neg q) \quad (1)$$

Similarly we can prove that

$$p \wedge q \equiv \neg(\neg p \vee \neg q) \quad (2)$$

Note (1) and (2) are known as *De Morgan's laws*.

Using (1) and (2), we can show that

dual in which every variable (primary proposition) is replaced by its negation.

From Eq. (3), it follows that

$$A(p_1, p_2, \ldots, p_n) \equiv \neg A^*(\neg p_1, \neg p_2, \ldots, \neg p_n) \tag{4}$$

Now since $A(p_1, p_2, \ldots, p_n) \equiv B(p_1, p_2, \ldots, p_n)$, we have $A(p_1, p_2, \ldots, p_n) \leftrightarrow B(p_1, p_2, \ldots, p_n)$ is tautology

$\therefore \quad A(\neg p_1, \neg p_2, \ldots, \neg p_n) \leftrightarrow B(\neg p_1, \neg p_2, \ldots, \neg p_n)$ is also a tautology (5)

Using (4) in (5), we get

$\neg A^*(p_1, p_2, \ldots, p_n) \leftrightarrow \neg B^*(p_1, p_2, \ldots, p_n)$ is a tautology.

$\therefore A^* \leftrightarrow B^*$ is a tautology.

$\therefore A^* \equiv B^*$

ALGEBRA OF PROPOSITIONS

A proposition in a compound proposition can be replaced by one that is equivalent to it without changing the truth value of the compound proposition. By this way, we can construct new equivalences (or laws). For example, we have proved that $p \to q \equiv \neg p \vee q$ (Table 1.8). Using this equivalence, we get another equivalence $p \to (q \to r) \equiv p \to (\neg q \vee r)$. Some of the basic equivalences (laws) and their duals which will be of use later are given in Tables 1.9, 1.10 and 1.11. They can be easily established by using truth tables.

Table 1.9 Laws of Algebra of Propositions

Sl. No.	*Name of the law*	*Primal form*	*Dual form*
1.	Idempotent law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2.	Identity law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3.	Dominant law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4.	Complement law	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
5.	Commutative law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6.	Associative law	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7.	Distributive law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	De Morgan's law	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Table 1.10 Equivalences Involving Conditionals

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $p \wedge q \equiv \neg(p \to \neg q)$
5. $\neg(p \to q) \equiv p \wedge \neg q$
6. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Equivalences Involving Biconditionals

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TAUTOLOGICAL IMPLICATION

A compound proposition $A(p_1, p_2, \ldots, p_n)$ is said to *tautologically imply* or simply *imply* the compound proposition $B(p_1, p_2, \ldots, p_n)$, if B is true whenever A is true or equivalently if and only if $A \to B$ is a tautology. This is denoted by $A \Rightarrow B$, read as "A implies B".

Note $\Rightarrow$ is not a connective and $A \Rightarrow B$ is not a proposition).

For example, $p \Rightarrow p \vee q$, as seen from the following truth Table 1.12. We note that $p \vee q$ is true, whenever p is true and that $p \to (p \vee q)$ is a tautology.

Table 1.12

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Similarly we note that $(p \to q) \Rightarrow (\neg q \to \neg p)$ from the following truth Table 1.13.

Table 1.13

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$(p \to q) \to (\neg q \to \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Some important implications which can be proved by truth tables are given in Table 1.14.

Table 1.14 Implications

1. $p \wedge q \Rightarrow p$
2. $p \wedge q \Rightarrow q$
3. $p \Rightarrow p \vee q$
4. $\neg p \Rightarrow p \to q$
5. $q \Rightarrow p \to q$
6. $\neg(p \to q) \Rightarrow p$
7. $\neg(p \to q) \Rightarrow \neg q$
8. $p \wedge (p \to q) \Rightarrow q$
9. $\neg q \wedge (p \to q) \Rightarrow \neg p$
10. $\neg p \wedge (p \vee q) \Rightarrow q$
11. $(p \to q) \wedge (q \to r) \Rightarrow p \to r$
12. $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$

Note We can easily verify that if $A \Rightarrow B$ and $B \Rightarrow A$, then $A \equiv B$. Hence to prove the equivalence of two propositions, it is enough to prove that each implies the other.

NORMAL FORMS

To determine whether a given compound proposition $A(p_1, p_2, \ldots, p_n)$ is a tautology or a contradictor or at least *satisfiable* and whether two given compound propositions $A(p_1, p_2, \ldots, p_n)$ and $B(p_1, p_2, \ldots, p_n)$ are equivalent, we have to construct the truth tables and compare them.

Note $A(p_1, p_2, \ldots, p_n)$ is said to be satisfiable, if it has the truth value T for at least one combination of the truth values of $p_1, p_2, \ldots, p_n$.

But the construction of truth tables may not be practical, when the number of primary propositions (variables) $p_1, p_2, \ldots, p_n$ increases. A better method is to reduce A and B to some standard forms, called *normal forms* and use them for deciding the nature of A or B and for comparing A and B. There are two types of normal form—disjunctive normal form and conjunctive normal form. We shall use the word 'product' in place of 'conjunction' and 'sum' in place 'disjunction' hereafter in this section for convenience.

DISJUNCTIVE AND CONJUNCTIVE NORMAL FORMS

A product of the variables and their negations (a conjunction of primary statements and their negations) is called an *elementary product*.

Similarly, a sum of the variables and their negations is called an *elementary sum*. For example, p, $\neg p$, $p \wedge \neg p$, $\neg p \wedge q$, $p \wedge \neg q$ and $\neg p \wedge \neg q$ are some elementary products in 2 variables q, $\neg q$, $p \vee q$, $p \vee \neg q$ and $\neg p \vee \neg q$ are some elementary sums is 2 variables. A compound proposition (or a formula) which consists of a sum of elementary products and which is equivalent to a given proposition is called a *disjunctive normal form* (DNF) of the given proposition.

A formula which consists of a product of elementary sums and which is equivalent to a given formula is called a *conjunctive normal form* (CNF) of the given formula.

Procedure to Obtain the DNF or CNF of a Given Formula

Step 1

If the connectives $\rightarrow$ and $\leftrightarrow$ are present in the given formula they are replaced by $\wedge$, $\vee$ and $\neg$ viz. $p \rightarrow q$ is replaced by $7p \vee q$ and $p \leftrightarrow q$ is replaced by either $(p \wedge q) \vee (\neg p \wedge \neg q)$ or $(\neg p \vee q) \wedge (\neg q \vee p)$.

Step 2

If the negation is present before the given formula or a part of the given formula (not a variable), De Morgan's laws are applied so that the negation is brought before the variables only.

Step 3

If necessary, the distributive law and the idempotent law are applied.

Step 4

If there is an elementary product which is equivalent to the truth value F in the DNF, it is omitted. Similarly if there is an elementary sum which is equivalent to the truth value T in the CNF, it is omitted.

For example, the DNF of $q \to (q \to p)$ is given by

$$\begin{aligned} q \to (q \to p) &\equiv \neg q \vee (q \to p) \\ &\equiv \neg q \vee (\neg q \vee p) \\ &\equiv (\neg q \vee \neg q) \vee p, \text{ by associative law} \\ &\equiv \neg q \vee p, \text{ by idempotent law.} \end{aligned}$$

The CNF of $\neg(p \vee q) \leftrightarrow (p \wedge q)$ is given by

$$\begin{aligned} \neg(p \vee q) \leftrightarrow (p \wedge q) &\equiv (\neg(p \vee q) \wedge (p \wedge q)) \vee (\neg(\neg(p \vee q)) \wedge \neg(p \wedge q)) \\ &\equiv (\neg p \wedge \neg q) \wedge (p \wedge q) \vee (p \vee q) \wedge (\neg p \vee \neg q) \\ &\equiv (p \wedge \neg p) \wedge (q \wedge \neg q) \vee (p \vee q) \wedge (\neg p \vee \neg q) \\ &\equiv F \wedge F \vee (p \vee q) \wedge (\neg p \vee \neg q) \\ &\equiv (p \vee q) \wedge (\neg p \vee \neg q) \end{aligned}$$

PRINCIPAL DISJUNCTIVE AND PRINCIPAL CONJUNCTIVE NORMAL FORMS

Given a number of variables, the products (or conjunctions) in which each variable or its negation, but not both, occurs only once are called the *minterms*. For two variable p and q, the possible minterms are $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$ and $\neg p \wedge \neg q$.

For three variables p, q and r, the possible minterms are $p \wedge q \wedge r$, $\neg p \wedge q \wedge r$, $p \wedge \neg q \wedge r$, $p \wedge q \wedge \neg r$, $\neg p \wedge \neg q \wedge r$, $p \wedge \neg q \wedge \neg r$, $\neg p \wedge q \wedge \neg r$ and $\neg p \wedge \neg q \wedge \neg r$.

We note that there are 2^n minterms for n variables.

Given a number of variables, the sums (or disjunctions) in which each variable or its negation, but not both, occurs only once are called the *maxterms*.

For the two variables p and q, the possible maxterms are $p \vee q$, $p \vee \neg q$, $\neg p \vee q$ and $\neg p \vee \neg q$. The maxterms are simply the duals of minterms.

A formula (compound proposition) consisting of disjunctions of minterms in the variables only and equivalent to a given formula is known as its *principal disjunctive normal form* (PDNF) or its *sum of products canonical form* of the given formula. Similarly, a formula consisting of conjunctions of maxterms in the variables only and equivalent to given formula is known as its principal conjunctive normal form (PCNF) or its *product of sums canonical form.*

In order to obtain the PDNF of a formula, we first obtain a DNF of the formula by using the procedure given above. To get the minterms in the disjunctions, the missing factors are introduced through the complement law (viz. $P \vee \neg P = T$) and then applying the distributive law. Identical minterms appearing in the disjunctions are then deleted, as $P \vee P \equiv P$. A similar procedure with necessary modifications is adopted to get the PCNF of a formula.

In order to verify whether two given formulas are equivalent, we may obtain either PDNF or PCNF of both the formulas and compare them.

Note If the PDNF of a formula A is known, the PDNF of $\neg A$ will consist of the disjunctions of the remaining minterms which are not included in the PDNF of A.

To obtain the PCNF of A, we use the fact that $A = \neg(\neg A)$ and apply De Morgan's laws to the PDNF of $\neg A$ repeatedly.

Examples

(a) The PDNF of $(p \vee \neg q)$ is given by

$$p \vee \neg q \equiv p \wedge (q \vee \neg q) \vee \neg q \wedge (p \vee \neg p), \text{ by complement law}$$

$$\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg q \wedge p) \vee (\neg q \wedge \neg p), \text{ by distributive law}$$

$$\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q), \text{ by commutative and idempotent laws.}$$

(b) To get the PCNF of $p \leftrightarrow q$, we proceed as follows:

The PDNF of $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ [assumed from Table (1.11)]

$\therefore$ PDNF of $\neg(p \leftrightarrow q) \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$ (remaining minterms) (1)

$$\therefore \quad (p \leftrightarrow q) \equiv \neg\neg(p \leftrightarrow q)$$

$$\equiv \neg((\neg p \wedge q) \vee (p \wedge \neg q)), \text{ form (1)}$$

$$\equiv \neg(\neg p \wedge q) \wedge \neg(p \wedge \neg q), \text{ by De Morgan's law}$$

$$\equiv (p \vee \neg q) \wedge (\neg p \vee q), \text{ by De Morgan's law,}$$

which is the same as

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

WORKED EXAMPLES 1(A)

Example 1.1 Construct a truth table for each of the following compound propositions:

(a) $(p \vee q) \to (p \wedge q)$; (b) $(p \to q) \to (q \to p)$;

(c) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$; (d) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$;

(e) $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$.

(a) **Table 1.15** Truth Table for $(p \vee q) \to (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(c) **Table 1.17** Truth Table for $(q \to \daleth p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\daleth p$	$q \to \daleth q$	$p \leftrightarrow q$	$(q \to \daleth p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

(d) **Table 1.18** Truth Table for $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\daleth p \wedge \daleth q))$

p	q	$\daleth p$	$\daleth q$	$p \leftrightarrow q$	$p \wedge q$	$\daleth p \wedge \daleth q$	$(p \wedge q) \vee (\daleth p \wedge \daleth q)$	*given formula*
T	T	F	F	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	F	T	T	T	F	T	T	T

(e) **Table 1.19** Truth Table for $(\daleth p \vee \daleth q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\daleth p$	$\daleth q$	$(\daleth p \leftrightarrow \daleth q)$	$(p \leftrightarrow q)$	$(\daleth p \leftrightarrow \daleth q) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Note Formulas given in (d) and (e) are tautologies.

Example 1.2 Construct the truth table for each of the compound propositions given as follows:

(a) $((p \to (q \to r)) \to ((p \to q) \to (p \to r))$
(b) $\daleth(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \to r))$
(c) $(\daleth p \leftrightarrow \daleth q) \leftrightarrow (q \leftrightarrow r)$
(d) $(p \to (q \to s)) \wedge (\daleth r \vee p) \wedge q$
(e) $((p \to q) \to r) \to s$

Note If there are n distinct components (sub-propositions) in a statement (compound proposition), the corresponding truth table will consist of 2^n rows corresponding to 2^n possible combinations. In order not to miss any of the combinations, we adopt the following procedure: In the first column of the truth table corresponding to the first component, we will write $\frac{1}{2} \times 2^n$ entries each equal to T, followed by $\frac{1}{2} \times 2^n$ entries each equal to F. In the second column, $\left(\frac{1}{4} \times 2^n\right)$ T's will be first written, then $\left(\frac{1}{4} \times 2^n\right)$ F's will be written followed again by $\left(\frac{1}{4} \times 2^n\right)$ T's. Finally $\left(\frac{1}{4} \times 2^n\right)$ F's will be written. In the third column $\left(\frac{1}{8} \times 2^n\right)$ T's and $\left(\frac{1}{8} \times 2^n\right)$ F's will be alternately written starting with T's and so on.

(a) **Table 1.20** Truth Table for $(p \to (q \to r)) \to ((p \to q) \to (p - r))$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$ $\equiv a$	$(p \to q) \to (p \to r)$ $\equiv b$	$a \to b$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

The given compound proposition is a tautology.

(b) **Table 1.21** Truth Table for $\neg(p \vee (q \wedge r) \leftrightarrow ((p \vee q) \wedge (p \to r))$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$ $\equiv a$	$\neg a$	$p \vee q$	$p \to r$	$(p \vee q) \wedge (p \to r)$ $\equiv b$	$\neg a \leftrightarrow b$
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	F	F	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	T	T	T
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	T	F	F

(c) **Table 1.22** Truth Table for $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

p	q	r	$\neg p$	$\neg q$	$(\neg p \leftrightarrow \neg q) \equiv a$	$q \leftrightarrow r \equiv b$	$a \leftrightarrow b$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

(d) **Table 1.23** Truth Table for $(p \to (q \to s)) \wedge (\urcorner r \vee p) \wedge q$

p	q	r	s	$q \to s \equiv a$	$p \to a \equiv b$	$\urcorner r$	$(\urcorner r \vee p) \equiv c$	$b \wedge c$	$b \wedge c \wedge q$
T	T	T	T	T	T	F	T	T	T
T	T	T	F	F	F	F	T	T	T
T	T	F	T	T	T	T	T	F	F
T	T	F	F	F	F	T	T	T	T
T	F	T	T	T	T	T	T	F	F
T	F	T	F	T	T	F	T	T	F
T	F	F	T	T	T	F	T	T	F
T	F	F	F	T	T	T	T	T	F
F	T	T	T	T	T	T	T	T	F
F	T	T	F	F	T	F	F	F	F
F	T	F	T	T	T	F	F	F	F
F	T	F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	T	F	F	F	F
F	F	F	T	T	T	F	F	F	F
F	F	F	F	T	T	T	T	T	F

(e) **Table 1.24** Truth Table for $((p \to q) \to r) \to s$

p	q	r	s	$p \to q$	$(p \to q) \to r$	$((p \to q) \to r) \to s$
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

Example 1.3 Determine which of the following compound propositions are tautologies and which of them are contradictions, using truth tables:

(a) $(\urcorner q \wedge (p \to q) \to \urcorner p$

(b) $((p \to q) \wedge (q \to r)) \to (p \to r)$

(c) $\urcorner(q \to r) \wedge r \wedge (p \to q)$

(d) $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r.$

(a) **Table 1.25** Truth Table for $(\neg q \wedge (p \to q)) \to \neg p$

p	q	$\neg p$	$\neg q$	$(p \to q)$	$\neg q \wedge (p \to q)$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the truth value of the given compound proposition is T for all combinations of p and q, it is a tautology.

(b) **Table 1.26** Truth Table for $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$p \to r$	$q \to r$	$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q - r)) \to (p - r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the truth value of the given statement is T for all combinations of truth values of p, q and r, it is a tautology.

(c) **Table 1.27** Truth Tables for $\neg(q \to r) \wedge r \wedge (p \to q)$

p	q	r	$p \to q$	$q \to r$	$\neg(q \to r)$	$\neg(q \to r) \wedge r$	$\neg(q \to r) \wedge r \wedge (p \to q)$
T	T	T	T	T	F	F	F
T	T	F	T	F	T	F	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	T	F	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	F

The last column contains only F as the truth values of the given statement. Hence it is a contradiction.

(d) **Table 1.28** Truth Table for $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r$

p	q	r	$p \vee q \equiv a$	$p \to r \equiv b$	$a \wedge b$	$q \to r \equiv c$	$a \wedge b \wedge c$	$(a \wedge b \wedge c) \to r$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F	T

(Contd.)

(Contd.)

F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	F	T	F	T	F	T
F	F	F	F	T	F	T	F	T

Since all the entries in the last column are T's, the given statement is a tautology

Example 1.4 Without using truth tables, prove the following:

(i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q) \equiv p \wedge q$

(ii) $p \to (q \to p) \equiv \neg p \to (p \to q)$

(iii) $\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \vee q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

(i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv (\neg p \vee q) \wedge (p \wedge p) \wedge q$, by associative law

$\equiv (\neg p \vee q) \wedge (p \wedge q)$, by idempotent law

$\equiv (p \wedge q) \wedge (\neg p \vee q)$, by commutative law

$\equiv ((p \wedge q) \wedge \neg p) \vee ((p \wedge q) \wedge q)$, by distributive law

$\equiv (\neg p \wedge (p \wedge q)) \vee ((p \wedge q) \wedge q)$, by commutative law

$\equiv ((\neg p \wedge p) \wedge q) \vee (p \wedge (q \wedge q)$, by associative law

$\equiv (F \vee q) \vee (p \wedge q)$, by complement and idempotent law

$\equiv F \vee (p \wedge q)$, by dominant law

$\equiv p \wedge q$, by dominant law.

(ii) $p \to (q \to p) \equiv \neg p \vee (q \to p)$ [Refer to Table 1.10]

$\equiv \neg p \vee (\neg q \vee p)$ [Refer to Table 1.10]

$\equiv \neg q \vee (p \vee \neg p)$, by commutative and associative laws

$\equiv \neg p \vee$ T, by complement law

$\equiv$ T, by dominant law (1)

$\neg p \to (p \to q) \equiv p \vee (p \to q)$, by (1) of Table 1.10

$\equiv p \vee (\neg p \vee q)$, by (1) of Table 1.10

$\equiv (p \vee \neg p) \vee q$, by associative law

$\equiv T \vee q$, by complement law

$\equiv T$, by dominant law, (2)

From (1) and (2), the result follows.

(iii) $\neg(p \leftrightarrow q) \equiv \neg((p \to q) \wedge (q \to p))$, from Table 1.11

$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee q))$, from Table 1.10

$\equiv \neg[((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p]$, by distributive law

$\equiv \neg[((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \wedge p)) \vee (q \wedge p))]$, by distributive law

$\equiv \neg[((\neg p \wedge \neg q) \vee F) \vee ((F \vee (q \wedge p))]$, by complement law

$\equiv \neg[(\neg p \wedge \neg q) \vee (q \wedge p)]$, by identity law

$\equiv \neg[\neg(p \vee q) \vee (q \wedge p)]$, by De Morgan's law (1)

$\equiv (p \vee q) \wedge \neg(q \wedge p)$, by De Morgan's law

$\equiv (p \vee q) \wedge (\neg q \vee \neg p)$, by De Morgan's law

$\equiv ((p \vee q) \wedge \neg q)) \vee ((p \vee q) \wedge \neg p))$, by distributive law

$\equiv ((p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((p \wedge \neg p) \vee (q \wedge \neg p))$, by distributive law

$\equiv ((p \wedge \neg q) \vee F) \vee ((F \vee (q \wedge \neg p))$, by complement law
$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$, by identity law
$\equiv (p \wedge \neg q) \vee (\neg p \wedge q)$, by commutative law (2)

From (1) and (2), the result follows.

Example 1.5 Without constructing the truth tables, prove the following:

(i) $\neg p \to (q \to r) \equiv q \to (p \vee r)$
(ii) $p \to (q \to r) \equiv p \to (\neg q \vee r) \equiv (p \wedge q) \to r$
(iii) $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.

(i) $\neg p \to (q \to r) \equiv p \vee (q \to r)$, from Table 1.10
$\equiv p \vee (\neg q \vee r)$, from Table 1.10
$\equiv (p \vee \neg q) \vee r$, by associative law
$\equiv (\neg q \vee p) \vee r$, by commutative law
$\equiv \neg q \vee (p \vee r)$, by associative law
$\equiv q \to (p \vee r)$, from Table 1.10.

(ii) $p \to (q \to r) \equiv p \to (\neg q \vee r)$, from Table 1.10 (1)
Now $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$, from Table 1.10
$\equiv (\neg p \vee \neg q) \vee r$, by associative law
$\equiv \neg(p \wedge q) \vee r$, by De Morgan's law
$\equiv (p \wedge q) \to r$ (2)

(iii) $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$
$\equiv ((p \vee q) \wedge \neg(\neg p \wedge \neg(q \wedge r))) \vee \neg(p \vee q) \vee \neg(p \vee r)$,
by De Morgan's law
$\equiv ((p \vee q) \wedge (p \vee (q \wedge r))) \vee \neg(p \vee q) \vee \neg(p \vee r)$,
by De Morgan's law
$\equiv ((p \vee q) \wedge [(p \vee q) \wedge (p \vee r)]) \vee [\neg(p \vee q) \vee \neg(p \vee r)]$,
by distributive law
$\equiv [(p \vee q) \wedge (p \vee r)] \vee \neg[(p \vee q) \wedge (p \vee r)]$,
by idempotent and De Morgan's laws

The final statement is in the form of $p \vee \neg p$.

$\therefore$ L.H.S. $\equiv$ T

Hence the given statement is tautology.

Example 1.6 Prove the following equivalences by proving the equivalences of the duals:

(i) $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$
(ii) $(p \vee q) \to r \equiv (p \to r) \wedge (q \wedge r)$
(iii) $(p \wedge (p \leftrightarrow q)) \to q \equiv T$

(i) The dual of the given equivalence is

$$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p$$

Let us now prove the dual equivalence.

L.H.S. $\equiv \neg(\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$, by distribution law
$\equiv \neg(\neg p \vee F) \wedge (p \vee q)$, by complement law
$\equiv \neg(\neg p) \wedge (p \vee q)$, by identity law
$\equiv p \wedge (p \vee q)$

(ii) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

i.e., $\neg(p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$

Dual of this equivalence is

$$\neg(p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r)$$

L.H.S $\equiv (\neg p \vee \neg q) \wedge r$, by De Morgan's law

$\equiv (\neg p \wedge r) \vee (\neg q \wedge r)$, by distributive law

$\equiv$ R.H.S.

(iii) $(p \wedge (p \leftrightarrow q)) \rightarrow q \equiv T$

i.e. $p \wedge ((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow q \equiv T$, from Table 1.11

i.e. $p \wedge ((\neg p \vee q) \wedge (\neg q \vee p)) \rightarrow q \equiv T$

i.e. $\neg(p \wedge ((\neg p \vee q) \wedge (\neg q \vee p))) \vee q \equiv T$

Dual of this equivalence is

$$\neg(p \vee ((\neg p \wedge q) \vee (\neg q \wedge p))) \wedge q \equiv F$$

L.H.S. $\equiv \neg[(p \vee (\neg p \wedge q)) \vee (\neg q \wedge p)] \wedge q$, by associative law

$\equiv \neg[(T \wedge (p \vee q)) \vee (\neg q \wedge p)] \wedge q$, by distributive and complement laws

$\equiv \neg[(p \vee q) \vee (\neg q \wedge p)] \wedge q$, by identity law

$\equiv \neg[((p \vee q) \vee \neg q) \wedge ((p \vee q) \vee p)] \wedge q$, by distributive law

$\equiv \neg[(p \vee T) \wedge (p \vee q)] \wedge q$, by idempotent and complement laws.

$\equiv \neg[T \wedge (p \vee q)] \wedge q$, by dominant law

$\equiv \neg[p \vee q] \wedge q$, by identity law

$\equiv (\neg p \wedge \neg q) \wedge q$, by De Morgan's law

$\equiv (\neg p \wedge F)$, by complement law

$\equiv F$, by dominant law.

Example 1.7 Prove the following implications by using truth tables:

(i) $p \rightarrow ((p \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$

(ii) $(p \rightarrow (q \rightarrow s)) \wedge (7r \vee p) \wedge q \Rightarrow r \rightarrow s$

(i) We have defined that $A \Rightarrow B$, if and only if $A \rightarrow B$ is a tautology

(i) **Table 1.29**

p	q	r	$p \rightarrow q$ (a)	$q \rightarrow r$ (b)	$p \rightarrow r$ (c)	$p \rightarrow b$ (d)	$a \rightarrow c$ (e)	$d \rightarrow e$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Since $d \rightarrow e$, viz., $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology, the required implication follows

(ii)

p	q	r	s	$q \to s$ (a)	$p \to a$ (b)	$\urcorner r$	$(\urcorner r \vee p)$ (c)	$b \wedge c$ (d)	$d \wedge q$ (e)	$r \to s$ (f)	$e \to f$
T	T	T	T	T	T	F	T	T	T	T	T
T	T	T	F	F	F	F	T	F	F	F	T
T	T	F	T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	T	F	F	T	T
T	F	T	T	T	T	F	T	T	F	T	T
T	F	T	F	T	T	F	T	T	F	F	T
T	F	F	T	T	T	T	T	T	F	T	T
T	F	F	F	T	T	T	T	T	F	T	T
F	T	T	T	T	T	F	F	F	F	T	T
F	T	T	F	F	T	F	F	F	F	F	T
F	T	F	T	T	T	T	T	T	T	T	T
F	T	F	F	F	T	T	T	T	T	T	T
F	F	T	T	T	T	F	F	F	F	T	T
F	F	T	F	T	T	F	F	F	F	F	T
F	F	F	T	T	T	T	T	T	F	T	T
F	F	F	F	T	T	T	T	T	F	T	T

Since $e \to f$ is a tautology, $e \Rightarrow f$.

Example 1.8 Prove the following implications without using truth tables:

(i) $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$

(ii) $((p \vee \urcorner p) \to q) \to ((p \vee \urcorner p) \to r) \Rightarrow q \to r$

(i) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

$\equiv (p \vee q) \wedge ((p \vee q) \to r) \to r$, from Table 1.10

$\equiv (p \vee q) \wedge (\urcorner(p \vee q) \vee r) \to r$

$\equiv (F \vee (p \vee q) \wedge r) \to r$

$\equiv ((p \vee q) \wedge r) \to r$

$\equiv \urcorner((p \vee q) \wedge r) \vee r$

$\equiv \urcorner((p \wedge r) \vee (q \wedge r)) \vee r$

$\equiv (\urcorner(p \wedge r) \wedge \urcorner(q \wedge r)) \vee r$

$\equiv (\urcorner(p \wedge r) \vee r) \wedge (\urcorner(q \wedge r) \vee r)$

$\equiv (\urcorner p \vee \urcorner r \vee r) \wedge (\urcorner q \vee \urcorner r \vee r)$

$\equiv (\urcorner p \wedge T) \wedge (\urcorner q \wedge T)$

$\equiv T \wedge T$

$\equiv T$

(ii) $[((p \vee \urcorner p) \to q) \to ((p \vee \urcorner p) \to r)] \to (q \to r)$

$\equiv [(T \to q) \to (T \to r)] \to (q \to r)$

$\equiv [(F \vee q) \to (F \vee r)] \to (q \to r)$

$\equiv (q \to r) \to (q \to r)$

$\equiv T$

Example 1.9 F[illegible] the disjunctive normal forms of the following statements:

(i) $\neg(\neg(p \leftrightarrow q) \wedge r)$

(ii) $p \vee (\neg p \to (q \vee (q \to \neg r)))$

(iii) $p \wedge \neg(q \wedge r) \vee (p \to q)$

(iv) $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$

(i) $\neg(\neg(p \leftrightarrow q) \wedge r) \equiv \neg(\neg((p \wedge q) \vee (\neg p \wedge \neg q)) \wedge r)$

$\equiv \neg[(\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)) \wedge r]$

$\equiv \neg[((\neg p \vee \neg q) \wedge (p \vee q)) \wedge r]$

$\equiv \neg[((\neg p \wedge p) \vee (\neg p \wedge q) \vee (\neg q \wedge p) \vee (\neg q \wedge q)) \wedge r]$, by extended distributive law

$\equiv \neg[((\neg p \wedge q) \vee (\neg q \wedge p)) \wedge r)$

$\equiv \neg[((\neg p \vee \neg q) \wedge (\neg p \vee p) \wedge (q \vee \neg q) \wedge (q \vee p)) \wedge r]$

$\equiv \neg[((p \vee q) \wedge (\neg p \vee \neg q)) \wedge r]$

$\equiv \neg(p \vee q) \vee \neg(\neg p \vee \neg q) \vee \neg r$

$\equiv (\neg p \wedge \neg q) \vee (p \wedge q) \vee \neg r$

(ii) $p \vee (\neg p \to (q \vee (q \to \neg r)))$

$\equiv p \vee (\neg p \to (q \vee (\neg q \vee \neg r)))$

$\equiv p \vee (p \vee (q \vee (\neg q \vee \neg r)))$

$\equiv p \vee p \vee q \vee \neg q \vee \neg r$

$\equiv p \vee q \vee \neg q \vee \neg r$

Note The given statement is a tautology, as $p \vee (q \vee \neg q) \vee \neg r \equiv P \vee T \vee \neg r \equiv T$

(iii) $p \wedge \neg(q \wedge r) \vee (p \to q)$

$\equiv p \wedge \neg(q \wedge r) \vee (\neg p \vee q)$

$\equiv (p \wedge (\neg q \vee \neg r)) \vee (\neg p \vee q)$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg p \vee q)$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee \neg p \vee q$

(iv) $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$

$\equiv (p \wedge (\neg q \wedge \neg r)) \vee ((p \wedge q) \wedge q) \vee (\neg r \wedge p)$

$\equiv (p \wedge \neg q \wedge \neg r) \vee (p \wedge q) \vee (p \vee \neg r)$

Example 1.10 Find the conjunction normal forms of the following statements:

(i) $(p \wedge \neg(q \wedge r)) \vee (p \to q)$

(ii) $(q \vee (p \wedge q)) \wedge \neg((p \vee r) \wedge q)$

(iii) $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \vee p$

(i) $(p \wedge \neg(q \wedge r)) \vee (p \to q)$

$\equiv (p \wedge (\neg q \vee \neg r)) \vee (\neg p \vee q)$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg p \vee q)$

$\equiv (p \vee p) \wedge (p \vee \neg r) \wedge (\neg q \vee p) \wedge (\neg q \vee \neg r) \vee (\neg p \vee q)$

$\equiv (p \vee p) \wedge (p \vee \neg r) \wedge (p \vee \neg q) \wedge (\neg p \vee q \vee \neg q \vee \neg r)$

$\equiv (p \vee p) \wedge (p \vee \neg r) \wedge (p \vee \neg q) \wedge (\neg p \vee T \vee \neg r)$

$\equiv (p \wedge (p \vee \neg r) \wedge (p \vee \neg q)$

(ii) $[q \vee (p \wedge q)] \wedge \neg[(p \vee r) \wedge q]$

$\equiv q \wedge \neg[(p \vee r) \wedge q]$, by absorption law

$\equiv q \wedge [\neg(p \vee r) \vee \neg q]$

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$\equiv q \wedge [(\neg p \wedge \neg r) \vee \neg q]$

$\equiv q \wedge (\neg p \vee \neg q) \wedge (\neg q \vee \neg r)$

(iii) $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$

$\equiv (p \wedge (\neg q \wedge \neg r)) \vee ((p \vee \neg r) \wedge (q \vee \neg r) \wedge p)$

$\equiv (p \wedge \neg q \wedge \neg r) \vee (p \wedge (p \vee \neg r) \wedge q \vee \neg r))$

$\equiv (p \wedge \neg q \wedge \neg r) \vee (p \wedge (q \vee \neg r))$, by absorption law

$\equiv [(p \wedge (\neg q \wedge \neg r)) \vee p] \wedge [(p \wedge \neg q \wedge \neg r) \vee (q \vee \neg r)]$

$\equiv p \wedge [((p \wedge \neg q \wedge \neg r) \vee \neg r) \vee q]$, by absorption law

$\equiv p \wedge (\neg r \vee q)$, by absorption law

$\equiv p \wedge (q \vee \neg r)$

Example 1.11 Obtain the principal disjunctive normal forms and the principal conjunctive normal forms of the following statements using truth tables:

(i) $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$

(ii) $p \vee (\neg p \rightarrow (q \vee (\neg q \rightarrow r)))$

(iii) $(p \rightarrow (q \wedge r)) \wedge (\neg p \rightarrow (\neg q \wedge \neg r))$

Procedure If the given statement is not a contradiction, then the disjunction (sum) of the minterms corresponding to the rows of the truth table having truth value T is the required PDNF, as it is equivalent to the given statement.

For example, if the truth value T of the statement corresponds to the truth values T, T and F for the variables p, q and r respectively, then the corresponding minterm is taken as $(p \wedge q \wedge \neg r)$.

If the given statement A is not a tautology, we can find the equivalent PCNF as follows:

We write down the PDNF of $\neg$A, which is the disjunction of the minterms corresponding to the rows of the truth table having the truth value F. Then if we find $\neg\neg$A(=A), we will get the required PCNF of A. Equivalently the PCNF is the conjunction of maxterms corresponding to the F values of A. But the maxterm corresponding to T, T, F value of p, q, r is $[(\neg p \vee \neg q \vee r)]$

(i) **Table 1.31**

p	q	$\neg p$	$\neg q$	$(\neg p \vee \neg q) \equiv a$	$p \leftrightarrow \neg q \equiv b$	$a \rightarrow b$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	F

PDNF of $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q) \equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$, since the minterms corresponding to the 3 T values of the last column are $p \wedge q, p \wedge \neg q, \neg p \wedge q$.

Now PDNF of $\neg(a \rightarrow b) \equiv \neg p \wedge \neg q$

$\therefore$ PCNF of $(a \rightarrow b) \equiv \neg(\neg p \wedge \neg q) = p \vee q$

(ii)

Table 1.32

p	q	r	$\neg p$	$\neg q$	$\neg q \to r \equiv a$	$q \vee a \equiv b$	$\neg p \to b \equiv c$	$p \vee c$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	T	T
F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	F	F

PDNF of the given statement

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$
$\vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$.

Now PCNF of the given statement $= \neg(\neg p \wedge \neg q \wedge \neg r)$
$= p \vee q \vee r$

(iii)

Table 1.33

p	q	r	$\neg p$	$\neg q$	$\neg r$	$q \wedge r \equiv a$	$p \to a \equiv b$	$\neg q \wedge \neg r \equiv c$	$\neg p \to c \equiv d$	$b \wedge d$
T	T	T	F	F	F	T	T	F	T	T
T	T	F	F	F	T	F	F	F	T	F
T	F	T	F	T	F	F	F	F	T	F
T	F	F	F	T	T	F	F	T	T	F
F	T	T	T	F	F	T	T	F	F	F
F	T	F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	T	F	F	F
F	F	F	T	T	T	F	T	T	T	T

PDNF of the given statement $\equiv (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

PDNF of $\neg(b \wedge d) \equiv (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$
$\vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

$\therefore$ PCNF of $(b \wedge d) \equiv (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r)$
$\wedge (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$

Example 1.12 Without constructing the truth tables, find the principal disjunctive normal forms of the following statements:

(i) $(\neg p \to q) \wedge (q \leftrightarrow p)$
(ii) $(p \wedge q) \vee (\neg p \wedge q) \vee (q \wedge r)$
(iii) $p \wedge \neg(q \wedge r) \vee (p \to q)$
(iv) $(q \vee (p \wedge r)) \wedge \neg((p \vee r) \wedge q)$

(i) $(\neg p \to q) \wedge (q \leftrightarrow p) \equiv (p \vee q) \wedge ((q \wedge p) \vee (\neg q \wedge \neg p))$
$\equiv (p \vee q) \wedge ((p \wedge q) \vee \neg(p \vee q))$
$\equiv ((p \vee q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge \neg(p \vee q))$
$\equiv ((p \vee q) \wedge (p \wedge q)) \vee \mathrm{F}$
$\equiv (p \wedge (p \wedge q)) \vee ((q \wedge (p \wedge q))$

$\equiv p \wedge q$

(ii) $(p \wedge q) \vee (\neg p \wedge q) \vee (q \wedge r)$

$\equiv ((p \wedge q) \wedge (r \vee \neg r)) \vee ((\neg p \wedge q) \wedge (r \vee \neg r)) \vee ((q \wedge r) \wedge (p \vee \neg p))$

($\because$ Already the given statement is in the DNF, but not in PDNF)

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$
$\vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r)$

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$

(Deleting repetition of identical minterms)

(iii) $p \wedge \neg(q \wedge r) \vee (p \to q)$

$\equiv (p \wedge (\neg q \vee \neg r)) \vee (\neg p \vee q)$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee \neg p \vee q$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge q)$

$\equiv (p \wedge \neg q) \vee (p \wedge \neg r) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge q)$

[Omitting the repletion of $(\neg p \wedge q)$]

$\equiv ((p \wedge \neg q) \wedge (r \vee \neg r)) \vee ((p \wedge \neg r) \wedge (q \vee \neg q)) \vee ((\neg p \wedge q)$
$\wedge (r \vee \neg r)) \vee ((\neg p \wedge \neg q) \wedge (r \vee \neg r)) \vee ((p \wedge q) \wedge (r \vee \neg r))$

$\equiv (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$
$\vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q$
$\wedge \neg r) \vee (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r)$

$\equiv (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$
$\vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$

(Omitting repetitions)

Note: Since all possible minterms are present in the PDNF, we infer that the given statement is a tautology.

(iv) $(q \vee (p \wedge r)) \wedge \neg((p \vee r) \wedge q)$

$\equiv (q \vee (p \wedge r)) \wedge (\neg(p \vee r) \vee \neg q)$

$\equiv (q \vee (p \wedge r)) \wedge ((\neg p \wedge \neg r) \vee \neg q)$

$\equiv (q \wedge \neg p \wedge \neg r) \vee (q \wedge \neg q) \vee (p \wedge r \wedge \neg p \wedge \neg r) \vee (p \wedge r \wedge \neg q)$

(By extended distribution law)

$\equiv (\neg p \wedge q \wedge \neg r) \vee F \vee F \vee (p \wedge \neg q \wedge r)$

$\equiv (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$, deleting F's

Example 1.13 Without constructing the truth tables, find the principal conjunctive normal forms of the following statements:

(i) $(p \wedge q) \vee (\neg p \wedge q \wedge r)$

(ii) $(p \vee q) \wedge (r \vee \neg p) \wedge (q \vee \neg r)$

(iii) $(p \vee \neg(q \vee r)) \vee (((p \wedge q) \wedge \neg r) \wedge p)$

(iv) $(p \to (q \wedge r)) \wedge (\neg p \to (\neg q \wedge \neg r)$

(i) $(p \wedge q) \vee (\neg p \wedge q \wedge r) \equiv ((p \wedge q) \vee \neg p) \wedge ((p \wedge q) \vee q) \wedge ((p \wedge q) \vee r)$

$\equiv (p \vee \neg p) \wedge (q \vee \neg p) \wedge (p \vee q) \wedge (q \vee q) \wedge (p \vee r) \wedge (q \vee r)$

$\equiv T \wedge (\neg p \vee q) \wedge (p \vee q) \wedge q \wedge (p \vee r) \wedge (q \vee r)$

$\equiv ((\neg p \vee q) \vee (r \wedge \neg r)) \wedge ((p \vee q) \vee (r \wedge \neg r)) \wedge q \vee (p \wedge \neg p)$
$\wedge (p \vee r) \vee (q \wedge \neg q) \wedge (q \vee r) \vee (p \wedge \neg p)$

($\because$ A $\vee$ F = A)

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$\wedge (q \vee p) \wedge (q \vee \neg p) \wedge (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (q \vee r \vee p)$
$\wedge (q \vee r \vee \neg p)$

$\equiv (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$
$\wedge (p \vee \neg q \vee r) \wedge ((q \vee p) \vee (r \wedge \neg r)) \wedge ((q \vee \neg p) \vee (r \wedge \neg r))$

(Omitting repetitions)

$\equiv (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee q \vee r) \wedge (p \vee q \vee \neg r)$
$\wedge (p \vee \neg q \vee r)$ (1)

(Deleting repetitions)

Note In this process, we have directly found out the PCNF of the given statement S. Alternatively we can first find the PDNF of S, write down the PDNF of $\neg S$ and hence get the PCNF of S given as follows:

Aliter

$S \equiv (p \wedge q) \wedge (r \vee \neg r) \vee (\neg p \wedge q \wedge r)$
$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$

$\therefore \quad \neg S \equiv (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$
$\vee (\neg p \wedge \neg q \wedge \neg r)$

$\therefore \quad S = \neg\neg S \equiv \neg(p \wedge \neg q \wedge r) \wedge \neg(\neg p \wedge \neg q \wedge r) \wedge \neg(\neg p \wedge q \wedge \neg r)$
$\wedge \neg(p \wedge \neg q \wedge \neg r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r)$
$\equiv (\neg p \vee q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$ (2)
$\wedge (p \vee q \vee r)$

We see that PCNF's of S in (1) and (2) are one and the same.

(ii) Let $S \equiv (p \vee q) \wedge (r \vee \neg p) \wedge (q \vee \neg r)$

Already S is in the CNF. Hence we can get the PCNF directly quickly.

$S \equiv ((p \vee q) \vee (r \wedge \neg r) \wedge ((\neg p \vee r) \vee (q \wedge \neg q)) \wedge ((q \vee \neg r) \vee (p \wedge \neg p))$
$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r)$
$\wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$
$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r)$
$\wedge (\neg p \vee q \vee \neg r)$

(iii) Let $S \equiv (p \vee \neg(q \vee r)) \vee ((p \wedge q) \wedge \neg r) \wedge p$
$\equiv (p \vee (\neg q \vee \neg r)) \vee (p \wedge q \wedge \neg r \wedge p)$
$\equiv (p \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$
$\equiv p \wedge (q \vee \neg q) \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$
$\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$
$\equiv ((p \wedge q) \wedge (r \vee \neg r)) \vee ((p \wedge \neg q) \wedge (r \vee \neg r)) \vee ((\neg q \wedge \neg r)$
$\wedge (p \vee \neg p)) \vee (p \wedge q \wedge \neg r)$
$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$
$\vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$
$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$ (1)
$\vee (\neg p \wedge \neg q \wedge \neg r)$

In (1), we have got the PDNF of S.

Now $\neg S \equiv (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

$\therefore \quad S \equiv \neg\neg S \equiv (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$ (2)

(2) is the required PCNF of S.

(iv) Let $S \equiv (p \rightarrow (q \wedge r)) \wedge (\neg p \rightarrow (\neg q \wedge \neg r))$
$\equiv (\neg p \vee (q \wedge r)) \wedge (p \vee (\neg q \wedge \neg r))$
$\equiv (\neg p \vee q) \wedge (\neg p \vee r)) \wedge (p \vee \neg q) \wedge (p \vee \neg r)$

$$\equiv ((\neg p \vee q) \vee (r \wedge \neg r)) \wedge ((\neg p \vee r) \vee (q \wedge \neg q)) \wedge ((p \vee \neg q) \vee (r \wedge \neg r)) \wedge ((p \vee \neg r) \vee (q \wedge \neg q))$$

$$\equiv (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$$

$$\equiv (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$$