Discrete Mathematics

Chapter 1

Sets Theory

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Course objectives

We will focus on two major goals:

- Basic tools and techniques in discrete mathematics
  - Propositional logic
  - Set Theory
  - Simple algorithms
  - Functions, sequences, Relations
  - Counting methods
  - Introduction to number theory
  - Graph theory
  - Trees
  - Network models
- Practice on Boolean algebra and Combinatorial Circuits.

To do well you should

- Study with pen and paper
- Ask for help immediately
- Practice, practice, practice…
- Follow along in class rather than take notes
- Ask questions in class
- Keep up with the class
- Read the book, not just the slides

What *Is* Discrete Mathematics?

- Discrete: consisting of distinct or unconnected elements.

- **Definition** Discrete Mathematics
  - *Discrete Mathematics* is a collection of mathematical topics that examine and use finite or countably infinite mathematical objects.

Discrete vs. Continuous Mathematics

Continuous Mathematics

It considers objects that vary continuously;

Example: analog wristwatch (separate hour, minute, and second hands).

From an analog watch perspective, between 1 :25 p.m. and 1 :26 p.m.

there are infinitely many possible different times as the second hand moves around the watch face.

Real-number system --- core of continuous mathematics;

Continuous mathematics --- models and tools for analyzing real-world phenomena that change smoothly over time. (Differential equations etc.)

Discrete vs. Continuous Mathematics

Discrete Mathematics

It considers objects that vary in a discrete way.

Example: digital wristwatch.

On a digital watch, there are only finitely many possible different times

between 1 :25 P.M. and 1:27 P.M. A digital watch does not show split seconds: - no time between 1 :25:03 and 1 :25:04. The watch moves from one time to the next.

Integers --- *core of discrete mathematics*

Discrete mathematics --- models and tools for analyzing real-world phenomena that change discretely over time and therefore ideal for studying computer science.

Sets

- Definition: Well-defined collection of distinct objects
- Members or Elements: part of the collection
- Roster Method: Description of a set by listing the elements, enclosed with braces
  - Examples:
    - Vowels = {a,e,i,o,u}
    - Primary colors = {red, blue, yellow}
- Membership examples
  - "a belongs to the set of Vowels" is written as:
    $a \in$ Vowels
  - "j does not belong to the set of Vowels:
    $j \notin$ Vowels

Sets

- Set-builder method
  - $A = \{ x \mid x \in S, P(x) \}$ or $A = \{ x \in S \mid P(x) \}$
    - A is the set of all elements x of S, such that x satisfies the property P
    - Example:
      - If $X = \{2,4,6,8,10\}$, then in set-builder notation, X can be described as

$$X = \{n \in \mathbb{Z} \mid n \text{ is even and } 2 \leq n \leq 10\}$$

Sets

- Standard Symbols which denote sets of numbers

- $\mathbb{N}$: The set of all natural numbers (i.e.,all positive integers)
- $\mathbb{Z}$: The set of all integers
- $\mathbb{Z}^{\not\supset}$: The set of all positive integers
- $\mathbb{Z}\not\subset$: The set of all nonzero integers
- $\mathbb{E}$: The set of all even integers
- $\mathbb{Q}$: The set of all rational numbers
- $\mathbb{Q}\not\subset$: The set of all nonzero rational numbers
- $\mathbb{Q}^{\not\supset}$: The set of all positive rational numbers
- $\mathbb{R}$: The set of all real numbers
- $\mathbb{R}\not\subset$: The set of all nonzero real numbers
- $\mathbb{R}^{\not\supset}$: The set of all positive real numbers
- $\mathbb{C}$: The set of all complex numbers
- $\mathbb{C}\not\subset$: The set of all nonzero complex numbers

Sets

- Subsets
  - "X is a subset of Y" is written as $X \subseteq Y$
  - "X is not a subset of Y" is written as $X \nsubseteq Y$
  - Example:
    - $X = \{a,e,i,o,u\}$, $Y = \{a, i, u\}$ and

      $Z= \{b,c,d,f,g\}$
      - $Y \subseteq X$, since every element of Y is an element of X
      - $Y \nsubseteq Z$, since $a \in Y$, but $a \notin Z$

Sets

- Superset
  - X and Y are sets. If $X \subseteq Y$, then "X is contained in Y" or "Y contains X" or Y is a superset of X, written $Y \supseteq X$
- Proper Subset
- X and Y are sets. If X is a subset of Y and X does not equal Y, we say that X is a proper subset of Y and write $X \subset Y$.
  - Example:
    - $X = \{a,e,i,o,u\}$, $Y = \{a,e,i,o,u,y\}$
      - $X \subset Y$, since $y \in Y$, but $y \notin X$

Sets

- Set Equality
  - X and Y are sets. They are said to be equal if every element of X is an element of Y and every element of Y is an element of X, i.e. $X \subseteq Y$ and $Y \subseteq X$
  - Examples:
    - $\{1,2,3\} = \{2,3,1\}$
    - X = {red, blue, yellow} and Y = {c | c is a primary color} Therefore, X=Y
- Empty (Null) Set
  - A Set is Empty (Null) if it contains no elements.
  - The Empty Set is written as $\varnothing$
  - The Empty Set is a subset of every set

Sets

- Finite and Infinite Sets
  - X is a set. If there exists a nonnegative integer n such that X has n elements, then X is called a finite set with n elements.
  - If a set is not finite, then it is an infinite set.
  - Examples:
    - Y = {1,2,3} is a finite set
    - P = {red, blue, yellow} is a finite set
    - $\mathbb{E} \nparallel$ the set of all even integers, is an infinite set
    - $\emptyset$, the Empty Set, is a finite set with 0 elements

Sets

- Cardinality of Sets
  - Let S be a finite set with n distinct elements, where $n \geq 0$. Then $|S| = n$, where the cardinality (number of elements) of S is n
  - Example:
    - If P = {red, blue, yellow}, then $|P| = 3$
  - Singleton
    - A set with only one element is a singleton
    - Example:
      - $H = \{ 4 \}$, $|H| = 1$, H is a singleton

Sets

- Power Set
  - For any set X ,the power set of X ,written $\mathcal{P}$(X),is the set of all subsets of X
  - Example:
    - If X = {red, blue, yellow}, then $\mathcal{P}$(X) = { ∅ , {red}, {blue}, {yellow}, {red,blue}, {red, yellow}, {blue, yellow}, {red, blue, yellow} }
- Universal Set
  - An arbitrarily chosen, but fixed set

Sets

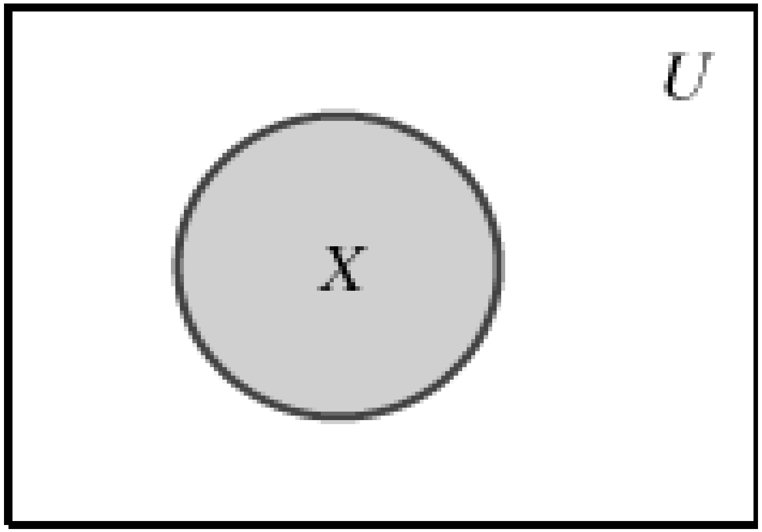

FIGURE 1.1 Set X

- Venn Diagrams
  - Abstract visualization of a Universal set, U as a rectangle, with all subsets of U shown as circles.
  - Shaded portion represents the corresponding set
  - Example:
    - In Figure 1, Set X, shaded, is a subset of the Universal set, U

Sets

- Intersection of Sets

The **intersection** of two sets X and Y, denoted by $X \cap Y$, is defined to be the set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}.$$

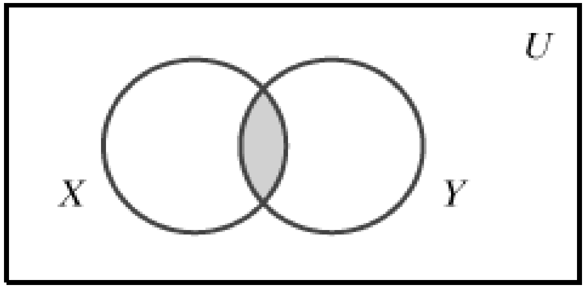

FIGURE 1.3 Venn diagram of $X \cap Y$

Example: If X = {1,2,3,4,5} and Y = {5,6,7,8,9}, then X ∩ Y = {5}

Sets

- Disjoint Sets

Two sets X and Y are said to be **disjoint** if $X \cap Y = \emptyset$.

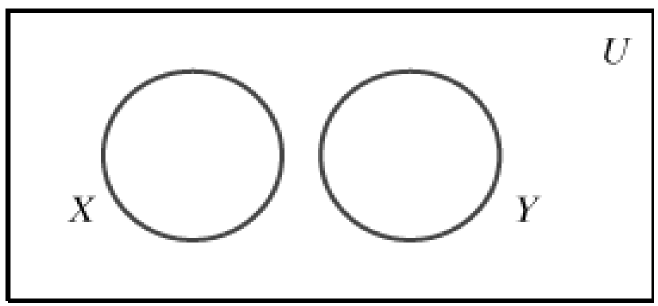

FIGURE 1.4 $X \cap Y = \emptyset$

Example: If X = {1,2,3,4,} and Y = {6,7,8,9}, then $X \cap Y = \varnothing$

Sets

- Difference

Let X and Y be sets. The **difference** of X and Y (or the **relative complement** of Y in X), written $X - Y$, is the set

$$X - Y = \{x \mid x \in X \text{ but } x \notin Y\}.$$

Example: If X = {a,b,c,d} and Y = {c,d,e,f}, then X – Y = {a,b} and Y – X = {e,f}

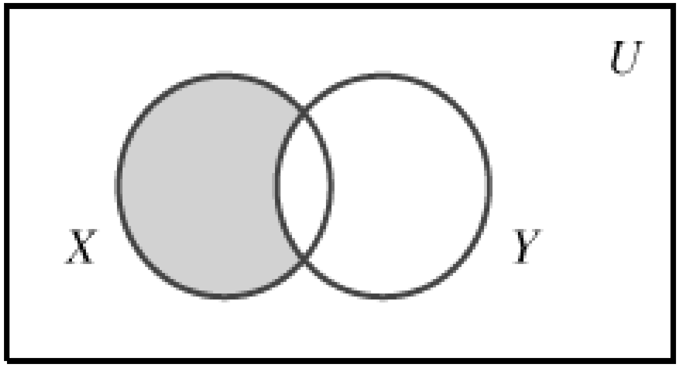

FIGURE 1.6 Venn diagram of $X - Y$

Sets

- Complement

The complement of a set X with respect to a universal set U, denoted by $\overline{X}$, is defined to be $\overline{X}$ = {x |x $\in$ U, but x $\notin$ X}

Example: If U = {a,b,c,d,e,f} and X = {c,d,e,f}, then $\overline{X}$ = {a,b}

Sets

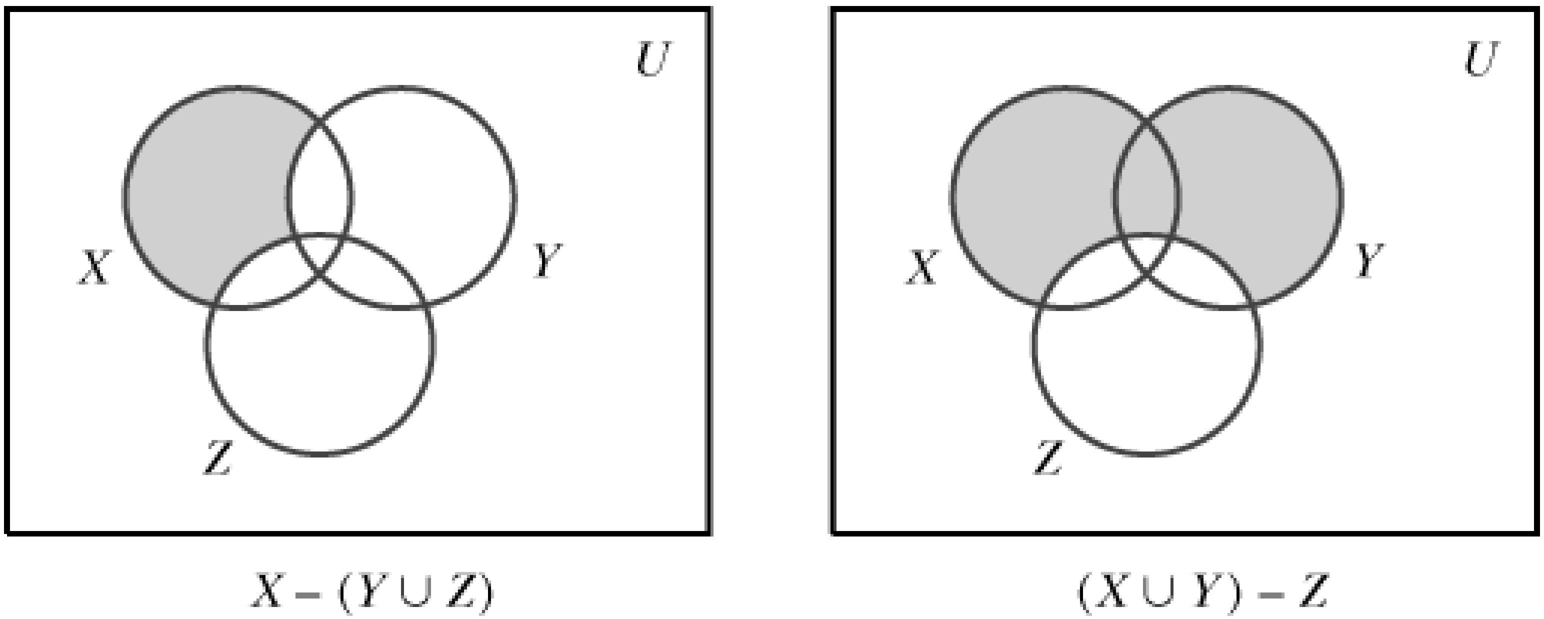

FIGURE 1.8 Venn diagrams of the sets $X - (Y \cup Z)$ and $(X \cup Y) - Z$

Sets

- Ordered Pair
  - X and Y are sets. If x ∈ X and y ∈ Y, then an ordered pair is written (x,y)
  - Order of elements is important. (x,y) is not necessarily equal to (y,x)
- Cartesian Product
  - The **Cartesian product** of two sets X and Y ,written $X \times Y$,is the set
  - $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
    - For any set X, $X \times \varnothing = \varnothing = \varnothing \times X$
  - Example:
    - X = {a,b}, Y = {c,d}
      - $X \times Y = \{(a,c), (a,d), (b,c), (b,d)\}$
      - $Y \times X = \{(c,a), (d,a), (c,b), (d,b)\}$

Fundamental Set Properties

Idempotence

$A \cup A = A$

$A \cap A = A$

Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Distributivity ($\cap$ over $\cup$)

$[A \cap (B \cup C)] = [(A \cap B) \cup (A \cap C)]$

$[(A \cup B) \cap C] = [(A \cap C) \cup (B \cap C)]$

Complement

$A \cup \overline{A} = U$

$A \cap \overline{A} = \varnothing$

Involution

$\overline{(\overline{A})} = A$

Domination

$A \cup U = U$

$A \cap \varnothing = \varnothing$

Identity

$A \cup \varnothing = A$

$A \cap U = A$

De Morgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Distributivity ($\cup$ over $\cap$)

$[A \cup (B \cap C)] = [(A \cup B) \cap (A \cup C)]$

$[(A \cap B) \cup C] = [(A \cup C) \cap (B \cup C)]$

Complement (continued)

$\overline{\varnothing} = U$

$\overline{U} = \varnothing$

Computer Representation of Sets

- A Set may be stored in a computer in an array as an unordered list
  - Problem: Difficult to perform operations on the set.
- Linked List
- Solution: use Bit Strings (Bit Map)
  - A Bit String is a sequence of 0s and 1s
  - Length of a Bit String is the number of digits in the string
  - Elements appear in order in the bit string
    - A 0 indicates an element is absent, a 1 indicates that the element is present
- A set may be implemented as a file