... and 3 to 4, there is no edge from 1 to 4.

Example 2.18 List the ordered pairs in the relation represented by the digraph given in Fig. 2.19. Also use the graph to prove that the relation is a partial ordering. Also draw the directed graphs representing R^{-1} and $\bar{R}$.

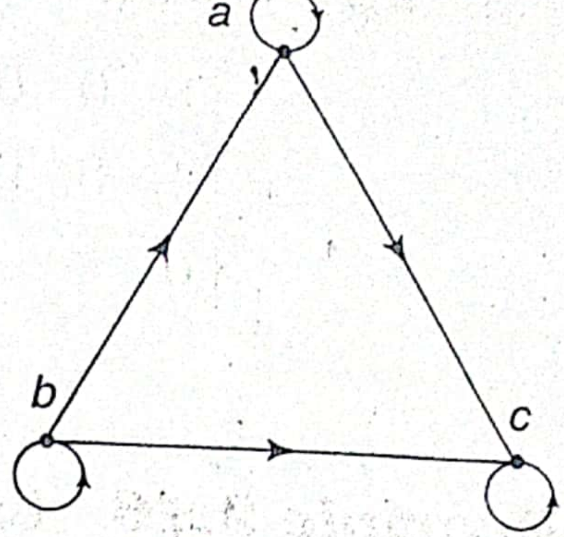


Fig. 2.19

The ordered pairs in the relation are $\{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)\}$.

Since there is a loop at every vertex, the relation is reflexive.

Though there are edges $b - a$, $a - c$ and $b - c$, the edges $a - b$, $c - a$ and $c - b$ are not present in the digraph. Hence the relation is antisymmetric.

When edges $b - a$ and $a - c$ are present in the digraph, the edge $b - c$ is also present (for example). Hence the relation is transitive.

Hence the relation is a partially ordering. The digraph of R^{-1} is got by reversing the directions of the edges (Fig. 2.20). The digraph of $\bar{R}$ contains the edges (a, b), (c, a), and (c, b) as shown in Fig. 2.21.

HASSE DIAGRAMS FOR PARTIAL ORDERINGS

The simplified form of the digraph of a partial ordering on a finite set that contains sufficient information about the partial ordering is called a *Hasse diagram*, named after the twentieth-century mathematician Helmut Haasse.

The simplification of the digraph as a Hasse diagram is achieved in three ways:

(i) Since the partial ordering is a reflexive relation, its digraph has loops at all vertices. We need not show these loops since they must be present.

(ii) Since the partial ordering is transitive, we need not show those edges that must be present due to transitivity. For example, if (1, 2) and (2, 3) are edges in the digraph of a partial ordering, (1, 3) will also be an edge due to transitivity. This edge (1, 3) need not be shown in the corresponding Hasse diagram.

(iii) If we assume that all edges are directed upward, we need not show the directions of the edges.

Thus the Hasse diagram representing a partial ordering can be obtained from its digraph, by removing all the loops, by removing all edges that are present due to transitivity and by drawing each edge without arrow so that its initial vertex is below its terminal vertex.

For example, let us construct the Hasse diagram for the partial ordering $\{(a, b) | a \le b\}$ on the set $\{1, 2, 3, 4\}$ starting from its digraph. (Fig. 2.15)

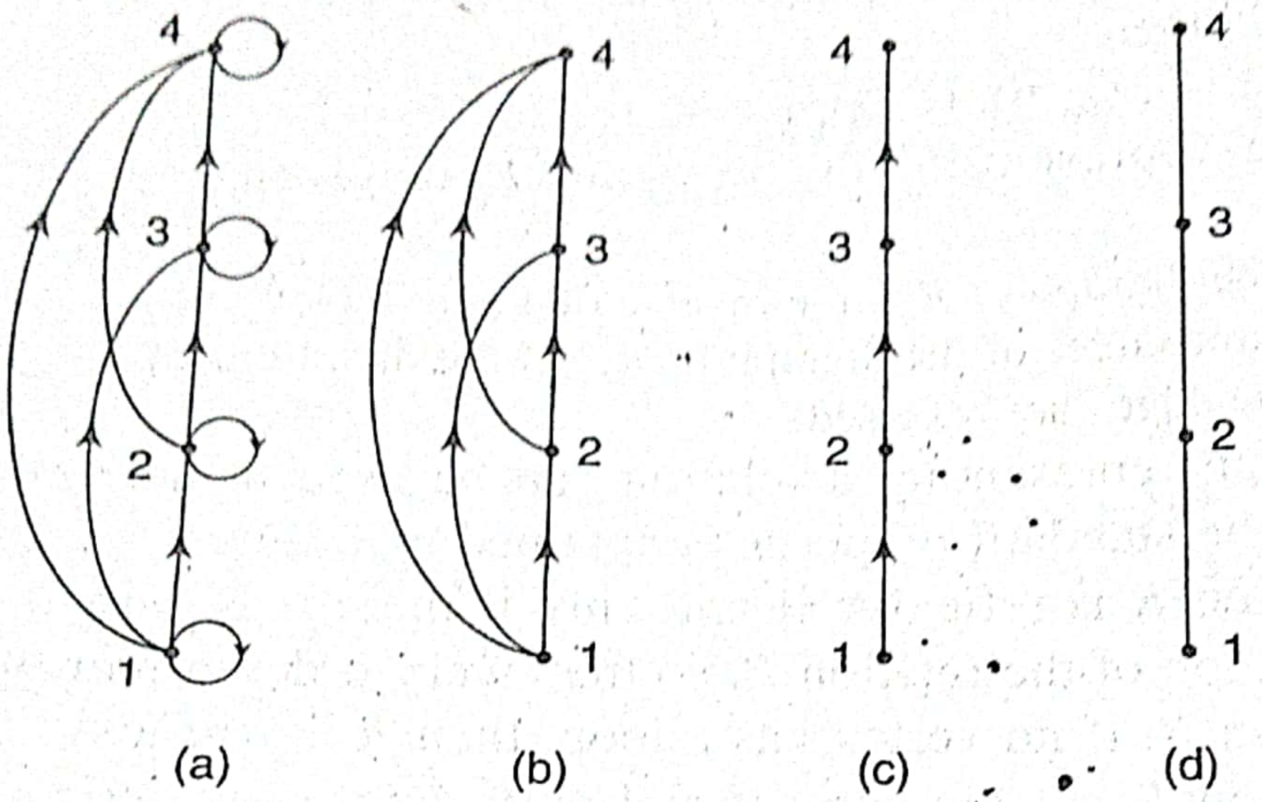


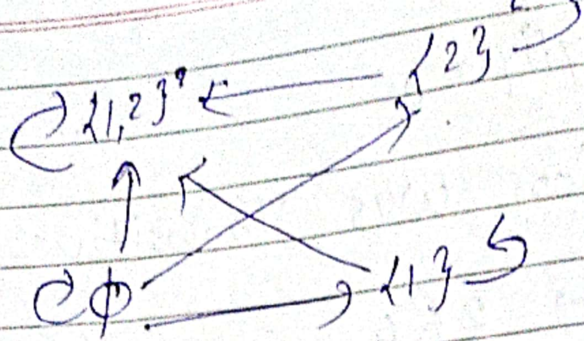
Fig. 2.15

Ex.

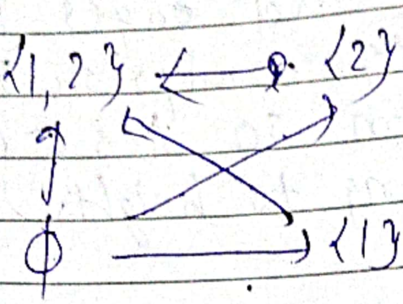
$A = \{1, 2\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$R = \{\{\phi, \{1\}\}, \{\phi, \{2\}\}, \{\phi, \{1,2\}\} \{\{1\}, \{1\}\}, \{\{1\}, \{1,2\}\}$
$\{\{2\}, \{2\}\}, \{\{2\}, \{1, 2\}\}, \{\{1,2\}, \{1,2\}\}. \}$

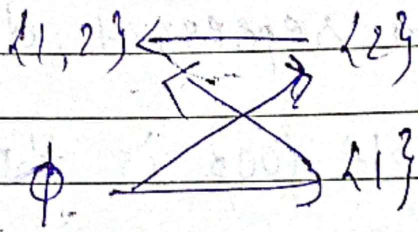


1) Remove self loop (Reflexive)

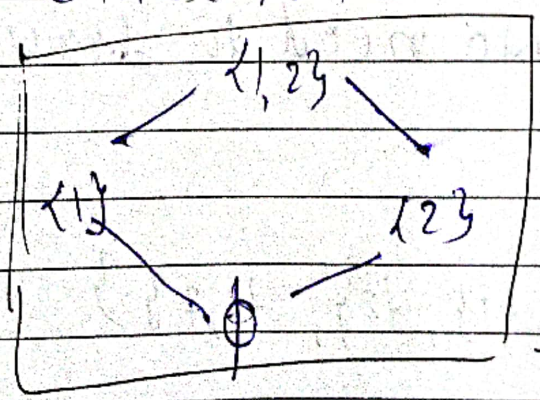


2) Remove Transitive edge.

$\phi R 1$ & $1 R \{1,2\} \rightarrow \phi R \{1,2\}$
$\phi R 2$ & $2 R \{1,2\} \rightarrow \phi R \{1,2\}$



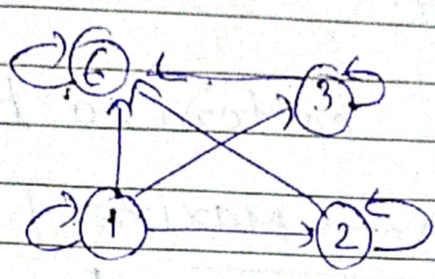
3) Remove direction



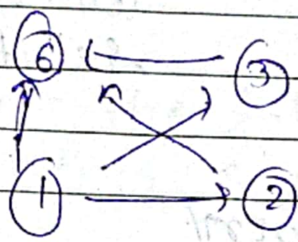
HASS Diagram

Ex. $A = \{1, 2, 3, 6\}$ and R = a divides b.

→ $R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$

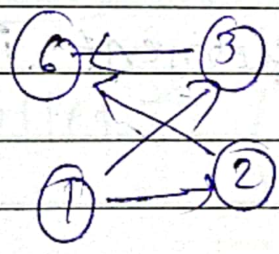


1) Remove self loop. (Reflexive)

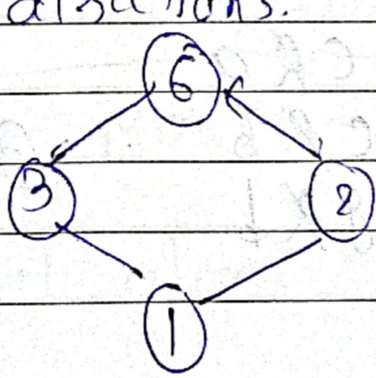


2) Remove Transitive edge.

$1R2,\ 2R6 \Rightarrow 1R6$
$1R3,\ 3R6 \Rightarrow 1R6$



3) Remove directions.



⟹ Hass diagram