TUTORIAL - 1

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AUTOMATA AND FORMAL LANGUAGES
MATHEMATICAL INDUCTION

U19CS012 [BHAGYA VINOD RANA]

Q.1)

1.) $S(n) = 1 + 2 + 3 + \dots + n$, $n \in N$ [Natural Number Set]

Prove by mathematical induction the statement

$$P(n) : S(n) = \frac{n * (n+1)}{2}, \quad \forall \text{ integers } n \geq 1$$

For any integer $n \geq 1$, let $P(n)$ be the statement

$$P(n) : \quad 1 + 2 + 3 + \dots + n = \frac{n*(n+1)}{2}$$

(A) Base Case : The statement $P(1)$ says that

$LHS = 1$

$RHS = \frac{n*(n+1)}{2} = \frac{1*2}{2} = 1$

$LHS = RHS$, Hence $P(1)$ is true

(B) Inductive Step : Fix $k \geq 1$, and suppose $P(k)$ holds true,

$$\boxed{1 + 2 + 3 + \dots + k = \frac{k*(k+1)}{2}} \quad - (1)$$

To show: $P(k+1)$ is also true, i.e

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$LHS = \{1 + 2 + 3 + \dots + k\} + (k+1)$ $\qquad$ $RHS = \frac{(k+1)(k+2)}{2}$

$= \left\{\frac{k \times (k+1)}{2}\right\} + (k+1)$

$= (k+1)\left(\frac{k}{2} + 1\right)$ $\qquad$ $\therefore$ $LHS = RHS$

$P(k+1)$ hold's true

$= \frac{(k+1)(k+2)}{2}$

Thus, by the Principle of mathematical induction, for all $n \geq 1$, $P(n)$ holds true

Q.1.) 2.) Let $S(n) = 1+3+5+ \dots + (2n-1)$. Prove by mathematical induction, the statement $P(n)$: $S(n) = n^2$, for all integers $n \geq 1$

For any integer $n \geq 1$, let $P(n)$ be the statement

$P(n)$: $1+3+5+ \dots + (2n-1) = n^2$

(A) Base Case : The statment $P(1)$ says that

$LHS = 1$ $RHS = n^2 = (1)^2 = 1$

$LHS = RHS$,

Hence $P(1)$ is true.

(B) Inductive Step : Fix $K \geq 1$, and suppose $P(K)$ holds,

$$\boxed{1+3+5+ \dots + (2k-1) = k^2} \quad \text{--- (1)}$$

It remains To show: $P(K+1)$ is also true,

i.e.

$$\{1+3+5+ \dots + (2K-1)\} + (2(K+1)-1) = (K+1)^2$$

$LHS = \{1+3+5+ \dots + 2K-1\} + (2K+1)$

$= K^2 + (2K+1)$ $(a \in N)$

$= (K+1)^2$ $[\because (a+1)^2 = a^2 + 2a + 1]$

$= RHS$

$\therefore LHS = RHS$

Hence, $P(K+1)$ hold's true.

Thus, by the principle of mathematical induction, for all $n \geq 1$, $P(n)$ holds true.

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Q.1) 3) Prove by mathematical induction, that

$(n+1)^2 + (n+2)^2 + (n+3)^2 + \dots + (2n)^2 = \frac{n*(2n+1)*(7n+1)}{6}$ is true

for all natural number n ie
[$\forall$ integer $n \geq 1$]

For any integer $n \geq 1$, let P(n) be the statement

$$\boxed{P(n) : (n+1)^2 + (n+2)^2 + \dots + (2n)^2 = \frac{n*(2n+1)(7n+1)}{6} \quad \forall n \in \mathbb{N}}$$

(A)

Base Case : The statement P(1) says that

$LHS = (1+1)^2 = (2(1))^2 = 4$

$RHS = \frac{n*(2n+1)*(7n+1)}{6} = \frac{1(3)(8)}{6} = \frac{24}{6} = 4$

LHS = RHS, Hence P(1) is true

(B)

Inductive Step : Fix $k \geq 1$, and suppose that P(k) holds

$$\boxed{(k+1)^2 + (k+2)^2 + (k+3)^2 + \dots + (2k)^2 = \frac{k*(2k+1)*(7k+1)}{6}} \quad -(1)$$

To show: P(k+1) is also true,

ie. $((k+1)+1)^2 + (k+3)^2 + \dots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(7(k+1)+1)}{6}$

$LHS = \{(k+2)^2 + (k+3)^2 + \dots + (2k)^2\} + (2k+1)^2 + (2k+2)^2$

$= \left\{\frac{k*(2k+1)*(7k+1)}{6} - (k+1)^2\right\} + (2k+1)^2 + (2k+2)^2$ From (1)

$= \frac{k(2k+1)(7k+1)}{6} - [k^2+2k+1] + [4k^2+4k+1] + [4k^2+8k+4]$

$= \frac{k(2k+1)(7k+1)}{6} + 7k^2 + 10k + 4$

$= \frac{k(14k^2+9k+1) + 6(7k^2+10k+4)}{6}$

$= \left(\frac{14k^3 + 9k^2 + k + 42k^2 + 60k + 24}{6}\right) = \frac{(14k^3 + 51k^2 + 61k + 24)}{6}$

continued...

$$\boxed{LHS = \frac{14k^3 + 51k^2 + 61k + 24}{6}}$$

$$RHS = \frac{(k+1)(2k+3)(7k+8)}{6} = \frac{(k+1)(14k^2 + 16k + 21k + 24)}{6}$$

$$= \frac{(k+1)(14k^2 + 37k + 24)}{6}$$

$$= \frac{k(14k^2 + 37k + 24) + (14k^2 + 37k + 24)}{6}$$

$$= \frac{14k^3 + 37k^2 + 24k + 14k^2 + 37k + 24}{6}$$

$$\boxed{= \frac{(14k^3 + 51k^2 + 61k + 24)}{6} = LHS}$$

$\therefore$ LHS = RHS

Thus, P(k+1) hold's true.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P(n) holds.

Q.1.> 4.> Prove by mathematical induction, that

$(1.n) + (2.(n-1)) + (3.(n-2)) + \dots + ((n-1)2) + (n.1) = \frac{n(n+1)(n+2)}{6}$

is true for all natural numbers n.

For any integer $n \geq 1$, let P(n) be the statement

P(n): $\boxed{(1.n) + (2.(n-1)) + \dots + ((n-1)2) + (n.1) = \frac{n*(n+1)(n+2)}{6}}$

(A) Base case: The statement P(1) says that

LHS = 1.1 = 1

$RHS = \frac{1}{6}(n)(n+1)(n+2) = \frac{(1)(2)(3)}{6} = 1$

LHS = RHS, Hence P(1) is true.

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Q.1.> 4.> continued..

(B) Inductive Step : Fix $K \geq 1$, and suppose $p(k)$ holds true

$$P(K) : \boxed{1.K + 2.(K-1) + 3.(K-2) + \dots + (K-1).2 + K.1 = \frac{K(K+1)(K+2)}{6}} \quad -(1)$$

To show: $P(K+1)$ is also true, i.e.

$$(1.(K+1)) + (2.(K+1-1)) + \dots + ((K+1-1).2) + ((K+1).1) = \frac{1(K+1)((K+1)+1)((K+1)+2)}{6}$$

$$LHS = [1.(K+1)] + [2.K] + \dots + [K.2] + [(K+1).1]$$

(add $+1-1 \Rightarrow 0$)

$$= [1.(K+1+1-1)] + [2.(K+1-1)] + \dots [K.(2+1-1)] + [(K+1).(1+1-1)]$$

$$= [1.(K+1-1) + 1.(1)] + [2(K-1) + 2.1] + \dots [K.(2-1) + K] + [(K+1)(1-1) + (K+1)]$$

$$= [(1.K) + 1] + [2.(K-1) + 2] + \dots [K(1) + K] + [(K+1)(0) + (K+1)]$$

$$= \{1.K + 2.(K-1) + \dots + K(1)\} + \{1 + 2 + \dots + K + (K+1)\}$$

(from (1))

From (1)

Sum of first n no.'s $= \frac{n(n+1)}{2}$

$$= \left\{\frac{K(K+1)(K+2)}{6}\right\} + \frac{(K+1)(K+2)}{2}$$

$$= \left(\frac{K}{3} + 1\right) \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)(K+2)(K+3)}{6}$$

$$= \frac{(K+1)((K+1)+1)((K+1)+2)}{6} = RHS$$

LHS = RHS, $P(K+1)$ hold's true

Thus, by the principle of mathematical induction, for all $n \geq 1$, $P(n)$ holds true.

Q.1) 5) Prove, by Mathematical Induction, that $n(n+1)(n+2)(n+3)$ is divisible by 24, $\forall\ n \in N$.
[Mathematical induction cannot be applied directly. Here we break the proposition into three parts.]

Let $P(n)$ be the proposition:
1.) $n(n+1)$ is divisible by $2! = 2$
2.) $n(n+1)(n+2)$ is divisible by $3! = 6$
3.) $n(n+1)(n+2)(n+3)$ is divisible by $4! = 24$

For $P(1)$, [Base Case]
1.) $1 \times 2 = 2$ is divisible by 2
2.) $1 \times 2 \times 3 = 6$ is divisible by 3
3.) $1 \times 2 \times 3 \times 4 = 24$ is divisible by 24
$\therefore$ $P(1)$ is true

Assume that $P(K)$ is true for some natural number K, that is
1.) $K(K+1)$ is divisible by 2, that is, $\boxed{K(K+1) = 2a}$ – (1)
2.) $K(K+1)(K+2)$ is divisible by 6, ie $\boxed{K(K+1)(K+2) = 6b}$ – (2)
3.) $K(K+1)(K+2)(K+3)$ is divisible by 24,
ie $K(K+1)(K+2)(K+3) = 24C$ – (3) $(a,b,c) \in N$

For $P(K+1)$,
1.) $(K+1)(K+2) = K(K+1) + 2(K+1) = 2a + 2(K+1)$, by (1)
$= \boxed{2[a+K+1]}$, which is divisible by 2 – (4)

2.) $(K+1)(K+2)(K+3) = K(K+1)(K+2) + 3(K+1)(K+2)$
$= 6b + 3 \times 2[a+K+1]$ (from (2) & (4))
$= \boxed{6[b+a+K+1]}$, divisible by 6 – (5)

3.) $(K+1)(K+2)(K+3)(K+4) = K((K+1)(K+2)(K+3)) + 4((K+1)(K+2)(K+3))$
$= 24C + 4(6[b+a+K+1])$
$= \boxed{24(C+b+a+K+1)}$, divisible by 24 – (6)

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Q1.> 5.> From 4, 5, 6

cont. $\therefore$ $P(k+1)$ is true

By the principle of mathematical Induction, $P(n)$ is true for all natural numbers, n.

Q.2.> Prove the following by contradiction

1> The square root of 2 is irrational

Lets assume $\sqrt{2}$ is rational number

$\sqrt{2} = \frac{a}{b}$, where a and b are whole numbers and it is the simplest form, so a and b can't be even at same time.

They both can be odd or one of them can be odd.

$\sqrt{2} = \frac{a}{b}$ $\Rightarrow$ Square both sides $\left[2 = \frac{a^2}{b^2}\right]$ - Eq^n 1

$[a^2 = 2b^2]$

a^2 is even, so a will also be even

Let $a = 2k$, Substituting in eq^n 1

$$2 = \frac{(2k)^2}{b^2}$$

$$2 = \frac{4k^2}{b^2}$$

$$\boxed{b^2 = 2k^2}$$

b^2 is even, so b is also even

But a and b can't be even at the same time

$\therefore$ It is a contradiction so square root of 2 is irrational number.

2) The cube root of 2 is irrational.

Let assume $\sqrt[3]{2}$ is rational number

$\sqrt[3]{2} = \frac{a}{b}$, where a and b are whole numbers and it is the simplest form

so a and b can't be even at same time.

they both can be odd or one of them can be odd.

$$\sqrt[3]{2} = \frac{a}{b} \xrightarrow[\text{sides}]{\text{cubing both}} \boxed{2 = \frac{a^3}{b^3}} \quad -(1)$$

$$a^3 = 2b^3$$

a^3 is even so a will also be even.

Let $a = 2k$

putting in Eq^n (1), $2 = \frac{(2k)^3}{b^3}$

$$b^3 = 4k^3$$

b^3 is also even, so b also will be even.

but a and b can't be even at same time.

∴ It is contradiction, cube root of 2 is irrational

3.) For every n, if $n > 2$ and n is prime, the n is odd.

If $n > 2$ and n is prime, Prove: n is odd

Let's assume n is even,

$$\boxed{n = 2k} \quad -(1)$$

From the defination of prime,

Prime number, it has only two factors one of them is 1 and other is number itself.

But from Eq^n (1)

n is divisible by 1, 2, and n

∴ n has 3 or more than 3 factors, which contradicts the Prime Number defination.

∴ It is contradiction and every prime number (> 2) greater than 2 is odd.

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Q 2 >

4.> If n is perfect square, the $n+2$ is not

Assume for n, a perfect square

$\{ n = a^2 \}$

So, $(n+2)$ is also a perfect square $\{a, b \in N\}$

$(n+2) = b^2$

$a^2 + 2 = b^2$

$2 = b^2 - a^2$

$2 = (b-a)(b+a)$ — 1

(n) and $(n+2)$ is perfect square, so a and b will be integer

From eq 1,

$2 = (b-a)(b+a)$

$(b+a)$ and $(b-a)$ will be 2 and 1 respectively

$\left.\begin{array}{l} b+a=2 \\ b-a=1 \end{array}\right\}$ adding → $\left[b = \frac{3}{2} \right]$

By contradiction, But, b is an integer, so our assumption is false

∴ If n is perfect square, then $(n+2)$ is not.

5.> If $m.n$ is even, where m and n are integers, then either m is even or n is even.

let assume m, n are odd $m = 2k+1$

$m.n = (2k+1)(2l+1)$ $n = 2l+1$ $[k, l \in z]$

$= 4kl + 2k + 2l + 1$

$= 2(2kl + k + l) + 1 = 2(C)+1$ $C = \text{constent} = (2kl + k + l)$

$m.n =$ odd, which is false,

So our Assumption is false,

Either m or n or both can be even, so that $m.n$

$\left.\begin{array}{l} m = 2k+1 \\ n = 2l \end{array}\right\}$ $m.n = 2\underbrace{(2k+1)l}$ $= (2P)$ $\left.\begin{array}{l} m = 2k \\ n = 2l \end{array}\right\}$ $m.n = 4kl$ is even $= 2(2kl) = 2(p)$

∴ By contradiction, we have proved, if $m.n$ is even, either of m or n needs to be even.

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