Example 2.1 List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if (i) $a = b$, (ii) $a + b = 4$, (iii) $a > b$, (iv) $a|b$ (viz., a divides b), (v) $gcd(a, b) = 1$ and (vi) $lcm\ (a, b) = 2$.

(i) Since $a \in A$ and $b \in B$ and a R b when $a = b$, $R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$.

(ii) Since a R b if and only if $a + b = 4$, $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$.

(iii) Since a R b, if and only if $a > b$, $R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1),$

greatest element $(3,2), (4,0), (4,1), (4,2), (4,3)\}$.

(iv) Since a R b, if and only if $a|b$, $R = \{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$.

Note $\frac{0}{0}$ is indeterminate and so 0 does not divide 0.

(v) Since a R b, if and only if $gcd\ (a, b) = 1$, $R = \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$.

(vi) Since a R b, if and only if $lcm\ (a, b) = 2$, $R = \{(1, 2), (2, 1), (2, 2)\}$.

Example 2.2 The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is defined by the rule $(a, b) \in R$, if 3 divides $a - b$.

(i) List the elements of R and R^{-1},
(ii) Find the domain and range of R.
(iii) Find the domain and range of R^{-1}.
(iv) List the elements of the complement of R.

The Cartesian product $A \times A$ consists of $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), \ldots, (2, 5), (3, 1), (3, 2) \ldots, (3, 5), (4, 1), (4, 2), \ldots, (4, 5), (5, 1), (5, 2), \ldots, (5, 5)\}$

(i) Since $(a, b) \in R$, if 3 divides $(a - b)$, $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

R^{-1} (the inverse of R) $= \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

We note that $R^{-1} = R$

(ii) Domain of R = Range of $R = \{1, 2, 3, 4, 5\}$

(iii) Domain of R^{-1} = Range of $R^{-1} = \{1, 2, 3, 4, 5\}$

(iv) R' (the complement of R) = the elements of $A \times A$, that are not in $R = \{(1, 2), (1, 3), (1, 5), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 1), (5, 3), (5, 4)\}$

Example 2.3 If $R = \{(1, 2), (2, 4), (3, 3)\}$ and $S = \{(1, 3), (2, 4), (4, 2)\}$, find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$. Also verify that dom $(R \cup S)$ = dom$(R) \cup$ dom (S) and range $(R \cap S) \subseteq$ range $(R) \cap$ range (S).

(i) $R \cup S = \{(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)\}$
(ii) $R \cap S = \{(2, 4)\}$
(iii) $R - S = \{(1, 2), (3, 3)\}$
(iv) $S - R = \{(1, 3), (4, 2)\}$
(v) $R \oplus S = (R \cup S) - (R \cap S)$

$$= \{(1, 2), (1, 3), (3, 3), (4, 2)\}$$

dom $(R) = \{1, 2, 3\}$; dom $(S) = \{1, 2, 4\}$

Now dom $(R) \cup$ dom $(S) = \{1, 2, 3, 4\}$

$=$ domain $(R \cup S)$

Range $(R) = \{2, 3, 4\}$; Range $(S) = \{2, 3, 4\}$

Range $(R \cap S) = \{4\}$

relations respectively on the set of integers. That is
$R = \{(a, b) \mid a \equiv b \pmod 3\}$ and $S = \{(a, b) \mid a \equiv b \pmod 4\}$

Find (i) $R \cup S$, (ii) $R \cap S$, (iii) $R - S$, (iv) $S - R$, (v) $R \oplus S$.

$R = \{(a, b)$, where $(a - b)$ is a multiple of 3

i.e. $a - b = \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots$

i.e. $a - b = \{\ldots, -9, 3, 15, 27, 39, \ldots\}, \{\ldots, -6, 6, 18, 30, \ldots\}, \{\ldots, -3, 9, 21, 33, \ldots\}, \{\ldots, 0, 12, 24, 36, \ldots\}$

i.e. $a - b = 3 \pmod{12}$ or $6 \pmod{12}$ or $9 \pmod{12}$ or $0 \pmod{12}$ (1)

$S = \{(a, b)\}$, where $(a - b)$ is a multiple of 4

i.e. $a - b = \ldots, -12, -8, -4, 0, 4, 8, 12, \ldots$

i.e. $a - b = \{\ldots, -8, 4, 16, 28, \ldots\}, \{\ldots, -16, -4, 8, 20, \ldots\}, \{\ldots, -24, -12, 0, 12, 24, \ldots\}$

i.e. $a - b = 4 \pmod{12}$ or $8 \pmod{12}$ or $0 \pmod{12}$ (2)

$\therefore$ $R \cup S = \{(a, b) \mid a - b = 0 \pmod{12}, 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$

$R \cap S = \{(a, b) \mid a - b = 0 \pmod{12}$, from (1) and (2)

$R - S = \{(a, b) \mid a - b = 3 \pmod{12}, 6 \pmod{12}$ or $9 \pmod{12}\}$

$S - R = \{(a, b) \mid a - b = 4 \pmod{12}$ or $8 \pmod{12}\}$

$R \oplus S = \{(a, b) \mid a - b = 3 \pmod{12}, 4 \pmod{12}, 6 \pmod{12}, 8 \pmod{12}$ or $9 \pmod{12}\}$.

Example 2.5 If the relations $R_1, R_2, \ldots, R_6$ are defined on the set of real numbers as given below,

$$R_1 = \{(a, b) \mid a > b\}, \quad R_2 = \{(a, b) \mid a \geq b\},$$
$$R_3 = \{(a, b) \mid a < b\}, \quad R_4 = \{(a, b) \mid a \leq b\},$$
$$R_5 = \{(a, b) \mid a = b\}, \quad R_6 = \{(a, b) \mid a \neq b\},$$

find the following composite relations:

$R_1 \bullet R_2, R_2 \bullet R_2, R_1 \bullet R_4, R_3 \bullet R_5, R_5 \bullet R_3, R_6 \bullet R_3, R_6 \bullet R_4$ and $R_6 \bullet R_6$

(i) $R_1 \bullet R_2 = R_1$. For example, let $(5, 3) \in R_1$ and let $(3, 1), (3, 2), (3, 3) \in R_2$
Then $R_1 \bullet R_2$ consists of $(5, 1), (5, 2), (5, 3)$ which belong to R_1

(ii) $R_2 \bullet R_2 = R_2$. For example, let $(5, 5), (5, 3), (5, 2) \in R_2$
Then $R_2 \bullet R_2 = \{(5, 5), (5, 3), (5, 2)\} = R_2$

(iii) $R_1 \bullet R_4 = R^2$ (the entire 2 dimensional vector space). For example, let $R_1 = \{(5, 4), (5, 3)\}$ and $R_4 = \{(4, 4), (4, 6), (3,3), (3, 5)\}$
Then $R_1 \bullet R_4 = \{(5, 4), (5, 6), (5, 3), (5, 5)\}$
Thus $R_1 \bullet R_4 = \{(a, b) \mid a > b, a = b$ and $a < b\}$

(iv) $R_3 \bullet R_5 = R_3$. For example, let $R_3 = \{(3, 4), (2, 4), (2, 5)\}$ and $R_5 = \{(3, 3), (4, 4), (5, 5)\}$
Then $R_3 \bullet R_5 = \{(3, 4), (2, 4), (2, 5)\} = R_3$

(v) $R_5 \bullet R_3 = R_3$. For example, let $R_5 = \{(3, 3), (4, 4), (5, 5)\}$ and $R_3 = \{(3, 4), (4, 6), (5, 7)\}$
Then $R_5 \bullet R_3 = \{(3, 4), (4, 6), (5, 7)\} = R_3$

(vi) $R_6 \bullet R_3 = R^2$. For exasmple, let $R_6 = \{(1, 2), (4, 3), (5, 2)\}$ and $R_3 = \{(2, 5), (3, 4), (2, 3)\}$
Then $R_6 \bullet R_3 = \{(1, 5), (1, 3), (4, 4), \ldots$

(vii) $R_6 \circ R_4 = R^2$. For example, let $R_6 = \{(1, 2), (4, 3), (5, 2)\}$ and $R_4 = \{(2, 3), (2, 5), (3, 3)\}$

Then $R_6 \circ R_4 = \{(1, 3), (1, 5), (4, 3), (5, 3), (5, 5)\} \rightarrow R^2$

(viii) $R_6 \circ R_6 = R^2$. For example, let $R_6 = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4)\}$

Then $R_6 \circ R_6 = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3)\} \rightarrow R^2$

Example 2.6 Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric and/or transitive, where $a\, R\, b$ if and only if (i) $a \neq b$, (ii) $ab \geq 0$, (iii) $ab \geq 1$, (iv) a is a multiple of b, (v) $a \equiv b$ (mod 7), (vi) $|a - b| = 1$, (vii) $a = b^2$, (viii) $a \geq b^2$.

(i) '$a \neq a$' is not true. Hence R is not reflexive

$a \neq b \Rightarrow b \neq a$. $\therefore$ R is symmetric

$a \neq b$ and $b \neq c$ does not necessarily imply that $a \neq c$. $\therefore$ R is not transitive

Hence R is symmetric only.

(ii) $a^2 \geq 0$. $\therefore$ R is reflexive.

$ab \geq 0 \Rightarrow ba \geq 0$. $\therefore$ R is symmetric.

Consider (2, 0) and (0, –3), that belong to R. But $(2, -3) \notin R$, as $2(-3) < 0$. $\therefore$ R is not transitive.

$\therefore$ R is reflexive, symmetric and not transitive.

(iii) '$a^2 \geq 1$' need not be true, since a may be zero. $\therefore$ R is not reflexive.

$ab \geq 1 \Rightarrow ba \geq 1$ $\therefore$ R is symmetric.

$ab \geq 1$ and $bc \geq 1 \Rightarrow$ all of $a, b, c > 0$ or < 0

If all of $a, b, c > 0$, least a = least b = least $c = 1$

$\therefore$ $ac \geq 1$

If all of $a, b, c < 0$, greatest a = greatest b = greatest $c = -1$

$\therefore$ $ac \geq 1$. Hence R is transitive.

$\therefore$ R is symmetric and transitive.

(iv) a is a multiple of a. $\therefore$ R is reflexive. If a is a multiple of b, b is not a multiple of a in general. But if a is a multiple of b and b is a multiple of a, then $a = b$.

$\therefore$ R is antisymmetric.

When a is a multiple of b and b is a multiple of c, then a is a multiple of c.

$\therefore$ R is transitive.

Thus R is reflexive, antisymmetric and transitive.

(v) $(a - a)$ is a multiple of 7 $\therefore$ R is reflexive. When $(a - b)$ is a multiple of 7, $(b - a)$ is also a multiple of 7. $\therefore$ R is symmetric.

When $(a - b)$ and $(b - c)$ are multiples of 7, $(a - b) + (b - c) = (a - c)$ is also a multiple of 7.

$\therefore$ R is transitive.

Hence R is reflexive, symmetric and transitive.

(vi) $|a - a| \neq 1$. $\therefore$ R is not reflexive

$|a - b| = 1 \Rightarrow |b - a| = 1$. $\therefore$ R is symmetric.

$|a - b| = 1 \Rightarrow a - b = 1$ or -1 (1)

$|b - c| = 1 \Rightarrow b - c = 1$ or -1

(1) + (2) gives $a - c = \pm 2$ or 0

i.e. $|a - c| = 2$ or 0

i.e. $|a - c| \neq 1$

Hence R is symmetric only.

(vii) '$a = a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a = b^2$ and $b = a^2$, for $a = b = 0$ or 1

$\therefore$ R is antisymmetric.

$a = b^2$ and $b = c^2$ does not simply $a = c^2$

$\therefore$ R is not transitive

Hence R is antisymmetric only.

(viii) '$a \geq a^2$' is not true for all integers.

$\therefore$ R is not reflexive.

$a \geq b^2$ and $b \geq a^2$ imply that $a = b$

$\therefore$ R is antisymmetric

When $a \geq b^2$ and $b \geq c^2$, $a \geq c^2$

$\therefore$ R is transitive

Hence R is antisymmetric and transitive.

Example 2.7 Which of the following relations on $\{0, 1, 2, 3\}$ are equivalence relations? Find the properties of an equivalence relation that the others lack.

(a) $R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

(b) $R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)$

(c) $R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

(d) $R_4 = \{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(e) $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

(a) R_1 is reflexive, symmetric and transitive.

$\therefore$ R_1 is an equivalence relation.

(b) R_2 is reflexive

R_2 is symmetric, but not transitive, since (3, 2) and (2, 0) $\in R_2$, but (3, 0) $\notin R_2$

$\therefore$ R_2 is not an equivalence relation.

(c) R_3 is reflexive, symmetric and transitive. $\therefore$ R_3 is an equivalence relation.

(d) R_4 is reflexive and symmetric, but not transitive, since (1, 3) and (3, 2) $\in R_4$, but (1, 2) $\notin R_4$. $\therefore$ R_4 is not an equivalence relation.

(e) R_5 is reflexive, but not symmetric since (1, 2) $\in R$, but (2, 1) $\notin R$. Also R_5 is not transitive, since (2, 0) and (0, 1) $\in R$, but (2, 1) $\notin R$.

$\therefore$ R_5 is not an equivalence relation.

Example 2.8 Show that the following relations are equivalence relations:

(i) R_1 is the relation on the set of integers such that aR_1b if and only if $a = b$ or $a = -b$.

(ii) R_2 is the relation on the set of integers such that aR_2b if and only if $a \equiv b$ (mod m), where m is a positive integer > 1.

(iii) R_3 is the relation on the set of real numbers such that aR_3b if and only if $(a-b)$ is an integer