50. $\begin{bmatrix} 1 & 2+4i & 1-i \\ -2+4i & -5 & 3-5i \\ -1-i & -3-5i & 6 \end{bmatrix}$.

51. $\begin{bmatrix} 0 & 2+4i & 1-i \\ -2+4i & 0 & 3-5i \\ -1-i & -3-5i & 0 \end{bmatrix}$.

52. $\begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$.

53. $\begin{bmatrix} 0 & -i & 1+i \\ -i & -2i & 0 \\ -1+i & 0 & i \end{bmatrix}$.

54. $\begin{bmatrix} 1 & -1 & i \\ -1 & 0 & 1-i \\ -i & 1+i & 2 \end{bmatrix}$.

55. $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$.

3.3 Vector Spaces

Let V be a non-empty set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a vector. The elements of V are denoted by **a**, **b**, **c**, **u**, **v**, etc. Assume that the two algebraic operations

(i) vector addition and (ii) scalar multiplication

are defined on elements of V.

If the vector addition is defined as the usual addition of vectors, then

$$\mathbf{a} + \mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

If the scalar multiplication is defined as the usual scalar multiplication of a vector by the scaler α, then

$$\alpha\mathbf{a} = \alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

The set V defines a vector space if for any elements **a**, **b**, **c** in V and any scalars α, β the following properties (axioms) are satisfied.

Properties (axioms) with respect to vector addition

1. $\mathbf{a} + \mathbf{b}$ is in V.
2. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (commutative law)
3. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. (associative law)
4. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$. (existence of a unique zero element in V)
5. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. (existence of additive inverse or negative vector in V)

Properties (axioms) with respect to scalar multiplication

6. $\alpha\mathbf{a}$ is in V.
7. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$. (left distributive law)
8. $(\alpha\beta)\,\mathbf{a} = \alpha(\beta\mathbf{a})$.
9. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$. (right distributive law)
10. $1\mathbf{a} = \mathbf{a}$. (existence of multiplicative inverse)

The properties defined in **1** and **6** are called the *closure* properties. When these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication. The vector addition and scalar multiplication defined above need not always be the usual addition and multiplication operators. Thus, *the vector space depends not only on the set V of vectors, but also on the definition of vector addition and scalar multiplication on V.*

If the elements of V are real, then it is called a *real vector space* when the scalars α, β are real numbers, whereas V is called a *complex vector space*, if the elements of V are complex and the scalars α, β may be real or complex numbers or if the elements of V are real and the scalars α, β are complex numbers.

Remark 7

(a) If even one of the above properties is not satisfied, then V is not a vector space. We usually check the closure properties first before checking the other properties.

(b) The concepts of length, dot product, vector product etc. are not part of the properties to be satisfied.

(c) The set of real numbers and complex numbers are called *fields* of scalars. We shall consider vector spaces only on the fields of scalars. In an advanced course on linear algebra, vector spaces over arbitrary fields are considered.

(d) The vector space $V = \{\mathbf{0}\}$ is called a trivial vector space.

The following are some examples of vector spaces under the usual operations of vector addition and scalar multiplication.

1. The set V of real or complex numbers.
2. The set of real valued continuous functions f on any closed interval $[a, b]$. The $\mathbf{0}$ vector defined in property **4** is the zero function.
3. The set of polynomials P_n of degree less than or equal to n.
4. The set V of n-tuples in $\mathbb{R}^n$ or $\mathbb{C}^n$.
5. The set V of all ... matrix of order

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Example 3.12 Let V be the set of all polynomials, with real coefficients, of degree n, where addition is defined by $\mathbf{a} + \mathbf{b} = \mathbf{ab}$ and under usual scalar multiplication. Show that V is not a vector space.

Solution Let P_n and Q_n be two elements in V. Now, $P_n + Q_n = (P_n)(Q_n)$ is a polynomial of degree $2n$, which is not in V. Therefore, V does not define a vector space.