MA - 212 TUTORIAL - 2

Consistency and Inconsistency of Matrix

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(A) **Concept Used**: Consistency and inconsistency of matrix

let
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

be set of Linear equations to solve,

① can be represented in Matrix form "$AX = B$"

Like

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{B}$$

A : Coefficient Matrix
X : Variables
B : Constant's

② Rank of matrix A = $\rho(A)$ = No. of Non-Zero rows (after transformation)

③ For checking consistency / Inconsistency, We will find $\rho(A:B)$ & $\rho(A)$

$$(A:B) = \left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

if we hide d_1, d_2, d_3, we get A matrix.

Ⓐ Consistent Solution $\boxed{\rho(A) = \rho(A:B)}$

① Unique solution $\boxed{\rho(A:B) = \rho(A) = n}$ (No. of Unknowns)

② Infinite Sol^n $\boxed{\rho(A:B) = \rho(A) < n}$ (No. of Unknowns)
(Assume $z = k$)

Ⓑ Inconsistent solution $\boxed{\rho(A) \neq \rho(A:B)}$

No solution exist

Q.1.> Test for consistency the following equations and solve them if consistent.

$$x - 2y + 3t = 2$$
$$2x + y + z + t = -4$$
$$4x - 3y + z + 7t = 8$$

A1.> Step 1 > Maxtrix form $AX = B$

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} \quad \text{— (i)}$$

Step 2.> Finding Rank of $(A:B)$

$$A:B = \left[\begin{array}{cccc:c} 1 & -2 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 & -4 \\ 4 & -3 & 1 & 7 & 8 \end{array}\right]$$

(a) $(R_2 \rightarrow R_2 - 2R_1)$ && $(R_3 \rightarrow R_3 - 4R_1)$

$$\left[\begin{array}{cccc:c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 5 & 1 & -5 & 0 \end{array}\right]$$

(b) $R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{cccc:c} 1 & -2 & 0 & 3 & 2 \\ 0 & 5 & 1 & -5 & -8 \\ 0 & 0 & 0 & 0 & +8 \end{array}\right]$$

$\rho(A:B) = 3 \qquad \rho(A) = 2$

Ans: $\boxed{\because \rho(A:B) \neq \rho(A)}$

Therefore, the given system of eqn's are Inconsistent and Hence No solution exist's

Q2> Solve the equations

(i) $x + 2y + 3z = 0$
$3x + 4y + 4z = 0$
$7x + 10y + 12z = 0$

$AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① $(A:B) = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{array}\right]$

(a) $R_2 \to R_2 - 3R_1$ AND $R_3 \to R_3 - 7R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{array}\right]$$

(b) $R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{— (iii)}$$

$\rho(A:B) = 3 \qquad \rho(A) = 3$

$\because \rho(A:B) = \rho(A) = 3$ (no. of variable)

$\therefore$ This set of eq^{n}s has UNIQUE solution.

from (iii) $z = 0$ — ①

$-2y - 5z = 0 \Rightarrow y = \frac{-5}{-2} z = 0$ — ②

$x + 2y + 3z = 0$

$[\, x = -2(0) + 3(0) = 0 \,]$ — ③

Ans: It has UNIQUE (trivial) solution

$$\boxed{x = y = z = 0}$$

(ii) $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + 0z + w = 0$

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① $\rho(A:B) = (?)$

$$(A:B) = \left[\begin{array}{cccc:c} 4 & 2 & 1 & 3 & 0 \\ 6 & 3 & 4 & 7 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

(a) $R_2 \to 2R_2 - 3R_1$ AND $R_3 \to 2R_3 - R_1$

$$\left[\begin{array}{cccc:c} 4 & 2 & 1 & 3 & 0 \\ 0 & 0 & 5 & 5 & 0 \\ 0 & 0 & -1 & -1 & -0 \end{array}\right]$$

(b) $R_3 \to 5 * R_3 + R_2$

$$\left[\begin{array}{cccc:c} 4 & 2 & 1 & 3 & 0 \\ 0 & 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A

A : B

$\rho(A) = 2$ $\rho(A:B) = 2$

$\therefore$ $\rho(A) = \rho(A:B) = 2 < 3$ (no. of unknown).

Given sys. of eq^n = <u>CONSISTENT</u>

$\therefore$ It has infinite solution

let $w = k$ — ①

$5z + 5w = 0$ $z = -k$ — ②

$6x + 3y = -4(-k) - 7(k)$ $\Rightarrow$ $6x + 3y = -3k$

$4x + 2y = -(-k) - 3(k)$ $\Rightarrow$ $4x + 2y = -2k$

$\boxed{2x + y = -k}$ $\forall \{k \in I\}$

ANS: Consistent, infinite many solution's where, $2x + y = -k$, $w = k$, $z = -k$

Q3.) For what values of k, the equation $x+y+z=1$, $2x+y+4z=k$, $4x+y+10z=k^2$ have a solution and solve them completely in each case.

A3.) $AX=B$ form
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

for solution, sys. of eq^n = CONSISTENT ie $\rho(A:B) = \rho(A)$.

$$(A:B) = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right]$$

(a) $R_2 \to R_2 - 2R_1$ AND $R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{array}\right]$$

(b) $R_3 \to R_3 - 3(R_1)$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & (k^2-4)-3(k-2) \end{array}\right]$$

$\because \rho(A) = A$ $\therefore$ $\rho(A:B)$ has to be '2'

$\therefore$ $k^2-4-3k+6 = 0$

$k^2-3k+2=0$

$k^2-k-2k+2=0$

$k(k-1)-2(k-1)=0$

$(k-1)(k-2)=0$

$\boxed{k = 1 \text{ OR } 2}$ for solution to exist !

for $k = 1$,

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & (k-1)(k-2) \end{array}\right] \quad \begin{array}{c} \\ \to (-1) \\ \to (0) \end{array}$$

$-y + z(2) = -1 \quad -(1)$ $\boxed{y = (2z+1)}$

$x + y + z = 1 \quad -(2)$

$x + (2z+1) + z = 1$

$\boxed{x = -3z}$

$\therefore$ for $k = 1$,

CONSISTENT,

Infinite Non-trivial Sol^n Exist.

for $k = 2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-y + 2z = 0 \quad -(1)$ $\boxed{y = 2z}$

$x + y + z = 1$

$x + 3z = 1$ $\boxed{x = 1 - 3z}$

$\therefore$ for $k = 2$

CONSISTENT,

Infinite Non-trivial Sol^n exist

ANS:

$\because \rho(A:B) = \rho(A) = 2 < 3$ (no. of variable)

$\therefore$ Infinite many sol^n Exist

few of them are: $k = 1$ $x = -3z$ $y = 2z+1$ $\forall z \in \mathbb{R}$

$k = 2$ $x = 1 - 3z$ $y = 2z$

Q4.> Test for consistency and solve

i) $2x-3y+7z=5$
$3x+y-3z=13$
$2x+19y-47z=32$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

① $(A:B) = \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$

(a) $R_2 \rightarrow 2R_2 - 3R_1$ AND $R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & -11 & 27 & 11 \\ 0 & 22 & -54 & 27 \end{array}\right]$$

(b) $R_3 \longrightarrow R_3 + 2(R_2)$

$$\left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & -11 & 27 & 11 \\ 0 & 0 & 0 & 49 \end{array}\right]$$

$\boxed{\rho(A) = 2}$ $\boxed{\rho(A:B) = 3}$

$\because \rho(A) \neq \rho(A:B)$

ANS: $\therefore$ system of eq^n = INCONSISTENT

No solution exist

(ii) $x+2y+z=3$

$2x+3y+2z=5$

$3x-5y+5z=2$

$3x+9y-z=4$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$$

① $(A:B) = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{array}\right]$

(a) $R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$ $R_4 \to R_4 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$$

(b) $R_3 \to R_3 - 11(R_2)$ $R_4 \to R_4 + 3R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

(c) $R_4 \to R_4 + 2(R_3)$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ — (iii)}$$

no. of variable = 3

$\rho(A) = 3$ $\rho(A:B) = 3$

$\because$ $\rho(A) = \rho(A:B) = 3$ = no. of unknowns

$\therefore$ system of eq^n = CONSISTENT & has UNIQUE sol^n

For (iii)

$2z = 4$ $\boxed{z = 2}$ — ①

$-y = -1$ $\boxed{y = 1}$ — ②

$x + 2y + z = 3$

$x + 2(1) + 2 = 3$

$[x = 3 - 4 = -1]$ — ③

ANS: Given Sys. of eq^n are consistent and has unique sol^n

$\boxed{x = -1,\ y = 1\ \&\ z = 2}$

Q5.) Find the values of a and b for which the equation

$x + ay + z = 3$

$x + 2y + 2z = b$

$x + 5y + 3z = 9$

$$\underbrace{\begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix}}_{B}$$

are consistent. When will these equations have a unique solution.

① $(A:B) = \begin{bmatrix} 1 & a & 1 & : & 3 \\ 1 & 2 & 2 & : & b \\ 1 & 5 & 3 & : & 9 \end{bmatrix}$

(a) $R_2 \to R_2 - R_1$ AND $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & a & 1 & : & 3 \\ 0 & 2-a & 1 & : & b-3 \\ 0 & 5-a & 2 & : & 6 \end{bmatrix}$$

For Consistency of Linear eq^n's $\rho(A) = \rho(A:B)$

det $\frac{2-a}{5-a} = \frac{1}{2} = \frac{b-3}{6}$

$\therefore \frac{b-3}{6} = \frac{1}{2}$ $\therefore 4-2a = 5-a$

$\boxed{b = 6}$ $\boxed{-1 = a}$

Therefore '3' Cases Arises

CASE I :> When $a = -1$ and $b = 6$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 6 & 2 & 6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \cancel{3}\ 2$

$\rho(A:B) = \cancel{3}\ 2$

no. of unknown $= 3$

$\boxed{\rho(A) = \rho(A;B) = 2 < 3}$ (no. of unknown)

$\therefore$ System of eq^n = CONSISTENT

and has infinite no. of solution

$z = k$ $3y = 3-z$ $x = 3 + y - z$

$y = (3-z)/3$ $= 3 + \frac{(3-z)}{3} - z$

$= \frac{(12-4z)}{3} = \frac{4(3-z)}{(3)}$

infinite sol^n $\left(\frac{4(3-k)}{3}, \frac{(3-k)}{3}, k \right) \forall (k \in R)$

CASE II.> When $a = -1$ and $b \neq 6$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 6 & 2 & 6 \end{array}\right]$$

$R_3 \to R_3 - (R_2 \times 2)$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 0 & 0 & 6-(2b-6) \end{array}\right]$$

$(12-2b)$

$b \neq 6 \Rightarrow (12-2b \neq 0)$

$\therefore R(A) = \cancel{3}\ 2$

$\because b \neq 6 \therefore 12-2b \neq 0$

$\therefore R(A:B) = 3$

$\therefore \rho(A) \neq \rho(A:B)$

system of eq^n = INCONSISTENT

and has no solution.

CASE III > When $a \neq -1$ and b be any value

$\therefore \rho(A) = 3$

$\therefore \rho(A:B) = 3$

No. of unknown = 3

$\therefore$ $\boxed{\rho(A) = \rho(A:B) = 3 = n}$ (No. of unknown)

$\therefore$ System of eq^n : consistent and have UNIQUE solution

Q6> Find the values of λ for which the equation

$$(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$$
$$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$$
$$2x + (3\lambda+1)y + 3(\lambda-1)z = 0$$

are consistent and find the ratios of $x:y:z$ when λ has the smallest of these values What happens when λ has greatest of value.

The given equation will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$$

(a) $R_2 \rightarrow R_2 - R_1$

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda & -\lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$$

$C_3 \rightarrow C_3 + C_2$

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$$

Expanding By R_2 $\quad (\lambda-3) \begin{vmatrix} (\lambda-1) & (5\lambda+1) \\ 2 & 2(3\lambda-1) \end{vmatrix} = 0$

$$= (\lambda-3)\left[2(\lambda-1)(3\lambda-1) - 2(5\lambda+1)\right]$$

$$= 2(\lambda-3)\left[3\lambda^2 - 4\lambda + 1 - \overset{5\lambda \;\; -1}{\cancel{10\lambda + 2}}\right]$$

$$= 2(\lambda-3)\left[3\lambda^2 \overset{-9\lambda}{\cancel{-9\lambda + 0}}\right]$$

$$= 6(\lambda-3)\left[\lambda^2 - 3\lambda\right]$$

$$\Rightarrow \quad 6\lambda(\lambda-3)^2 = 0$$

$$\therefore \quad \boxed{\lambda = 0 \;\; \cancel{\&} \;\; \lambda = 3}$$

or

① CASE I.) $\lambda = 0$

The Eq^n becomes

$$-x + y = 0 \quad -(i)$$

$$-x - 2y + 3z = 0 \quad -(ii)$$

$$2x + y - 3z = 0 \quad -(iii)$$

Solving (ii) & (iii), we get

$$\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4} \Rightarrow \boxed{x = y = z \neq 0}$$

$\therefore$ $x : y : z = 1 : 1 : 1$

② CASE II.) $\lambda = 3$

Equation's becomes identical $(2x + 10y + 6z = 0)$

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