21. Linear mappings.

Let V and W be vector spaces over the same field F. A mapping $T: V \to W$ is said to be a *linear mapping* (or a *linear transformation*) if it satisfies the following conditions –

1. $T(\alpha + \beta) = T(\alpha) + T(\beta)$ for all α, β in V
2. $T(c\alpha) = cT(\alpha)$ for all c in F and all α in V.

These two conditions can be replaced by the single condition–

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \text{ for all } a, b \text{ in } F \text{ and all } \alpha, \beta \text{ in } V.$$

Note 1. A linear mapping $T : V \to W$ is also a *homomorphism* of V to W.

2. Generally, a linear mapping T is a transformation from one vector space V to another vector space W, both over the same field of scalars. But the co-domain space W may be the space V itself. In this case T is said to be a *linear mapping on* V.

There is another important case when the co-domain space is F, regarded as a vector space over itself. In this case $T : V \to F$ is said to be a *linear functional.*

Examples.

1. **The identity mapping.** The mapping $T : V \to V$ defined by $T(\alpha) = \alpha$ for all α in V, is a linear mapping. This is called the *identity mapping* on V and is denoted by I_V.

2. Th

This shows that T is not a linear mapping.

Theorem 4.21.1. Let V and W be vector spaces over a field F a $T : V \to W$ be a linear mapping. Then

(i) $T(\theta) = \theta'$, where θ, θ' are null elements in V and W respectivel

(ii) $T(-\alpha) = -T(\alpha)$ for all $\alpha \in V$.

Proof. (i) In $V, \theta + \theta = \theta$.

Since T is linear , $T(\theta) + T(\theta) = T(\theta)$ in W.

This implies $-T(\theta) + [T(\theta) + T(\theta)] = -T(\theta) + T(\theta)$

$\Rightarrow [-T(\theta) + T(\theta)] + T(\theta) = \theta'$, since θ' is the null vector in W

$\Rightarrow \theta' + T(\theta) = \theta'$

$\Rightarrow T(\theta) = \theta'$.

(ii) Proof left as an exercise.

Kernel of a linear mapping.

Let V and W be vector spaces over a field F. Let $T : V \to W$ a linear mapping. The set of all vectors $\alpha \in V$ such that $T(\alpha) = \theta'$ being the null vector in W, is said to be the *kernel* of T and is deno by $Ker\,T$.

$$Ker\,T = \{\alpha \in V : T(\alpha) = \theta'\}.$$

Theorem 4.21.2. Let V and W be vector spaces over a field F. $T : V \to W$ be a linear mapping. Then $Ker\,T$ is a subspace of V.

Proof. $Ker\,T = \{\alpha \in V : T(\alpha) = \theta'\}$.

Since $T(\theta) = \theta', \theta \in Ker\,T$. Therefore $Ker\,T$ is non-empty.

Case 1. $Ker\,T = \{\theta\}$. Then $Ker\,T$ is a subspace of V.

Case 2. $Ker\,T \neq \{\theta\}$. Let $\alpha \in Ker\,T$. Then $T(\alpha) = \theta'$.

Let $c \in F$. Then $T(c\alpha) = cT(\alpha)$, since T is linear
$= c\theta' = \theta'$.

This implies $c\alpha \in Ker\ T$.

Let $\alpha, \beta \in Ker\ T$. Then $T(\alpha) = \theta'$, $T(\eta) = \theta'$.
$T(\alpha + \beta) = T(\alpha) + T(\beta)$, since T is linear
$= \theta' + \theta' = \theta'$.

This implies $\alpha + \beta \in Ker\ T$.

Thus $\alpha, \beta \in Ker\ T \Rightarrow \alpha + \beta \in Ker\ T$ and $\alpha \in Ker\ T \Rightarrow c\alpha \in Ker\ T$ for all $c \in F$. This proves that $Ker\ T$ is a subspace of V. This completes the proof.

Note. $Ker\ T$ is also called the *null space* of T and is denoted by $N(T)$.

Theorem 4.21.3. Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear mapping. Then T is injective if and only if $Ker\ T = \{\theta\}$.

Proof. Let T be injective. Since $T(\theta) = \theta'$ in W, θ is a pre-image of θ' and since T is injective, θ is the only pre-image of θ'. So $Ker\ T = \{\theta\}$.

Conversely, let $Ker\ T = \{\theta\}$ and α, β be two elements of V such that $T(\alpha) = T(\beta)$ in W.

$$\theta' = T(\alpha) - T(\beta)$$
$$= T(\alpha - \beta), \text{ since } T \text{ is linear.}$$

This implies $\alpha - \beta \in Ker\ T$ and since $Ker\ T = \{\theta\}, \alpha = \beta$.
Thus $T(\alpha) = T(\beta) \Rightarrow \alpha = \beta$ and therefore T is injective.
This completes the proof.

This proves

Image of a linear mapping.

Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear mapping. The images of the elements of V under the mapping T form a subset of W. This subset is said to be the *image* of T and is denoted by $Im\ T$.

$$Im\ T = \{T(\alpha) : \alpha \in V\}.$$

over a field F. Let