LATTICE HOMOMORPHISM

Definition

If $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus, *\}$ are two lattices, a mapping $f: L_1 \to L_2$ is called *a lattice homomorphism* from L_1 to L_2, if for any $a, b \in L_1$,

$$f(a \vee b) = f(a) \oplus f(b) \text{ and } f(a \wedge b) = f(a) * f(b).$$

If a homomorphism $f: L_1 \to L_2$ of two lattices $\{L_1, \vee, \wedge\}$ and $\{L_2, \oplus\ *\}$ is objective, i.e. one-to-one onto, then f is called an *isomorphism*. If there exists an isomorphism between two lattices, then the lattices are said to be *isomorphic*.

SOME SPECIAL LATTICES

(a) A lattice L is said to have a *lower bound* denoted by 0, if $0 \leq a$ for all $a \in L$. Similarly L is said to have an *upper bound* denoted by 1, if $a \leq 1$ for all $a \in L$. The lattice L is said to be *bounded*, if it has both a lower bound 0 and an upper bound 1.

The bounds 0 and 1 of a lattice $\{L, \vee, \wedge, 0, 1\}$ satisfy the following identities, which are seen to be true by the meanings of $\vee$ and $\wedge$.

For any $a \in L$, $a \vee 1 = 1$; $a \wedge 1 = a$ and $a \vee 0 = a$; $a \wedge 0 = 0$.

Since $a \vee 0 = a$ and $a \wedge 1 = a$, 0 is the identity of the operation $\vee$ and 1 is the identity of the operation $\wedge$.

Since $a \vee 1 = 1$ and $a \wedge 0 = 0$, 1 and 0 are the zeros of the operations $\vee$ and $\wedge$ respectively.

Note 1 If we treat 1 and 0 as duals of each other in a bounded lattice, the principle of duality can be extended to include the interchange of 0 and 1. Thus the identities $a \vee 1 = 1$ and $a \wedge 0 = 0$ are duals of each other; so also are $a \vee 0 = a$ and $a \wedge 1 = a$.

Note 2 If $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice, then $a_1 \vee a_2 \vee a_3 \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$ are upper and lower bounds of L respectively and hence we conclude that every finite lattice is bounded.

(ii) A lattice $\{L, \vee, \wedge\}$ is called a *distributive lattice*, if for any elements a, $b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In other words if the operations $\vee$ and $\wedge$ distribute over each other in a lattice, it is said to be distributive. Otherwise it is said to be *non distributive*.

(iii) If $\{L, \vee, \wedge, 0, 1\}$ is a bounded lattice and $a \in L$, then an element $b \in L$ is called a *complement* of a, if

$$a \vee b = 1 \text{ and } a \wedge b = 0$$

Since $0 \vee 1 = 1$ and $0 \wedge 1 = 0$, 0 and 1 are complements of each other. When $a \vee b = 1$, we know that $b \vee a = 1$ and when $a \wedge b = 0$, $b \wedge a = 0$. Hence when b is the complement of a, a is the complement of b.

An element $a \in L$ may have no complement. Similarly an element, other than 0 and 1, may have more than one complement in L as seen from Fig. 2.28.

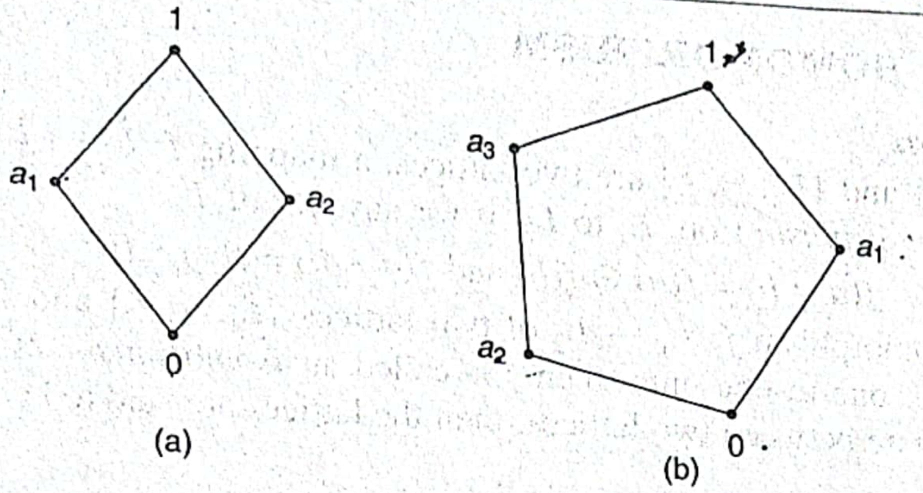

Fig. 2.28

In Fig. 2.28(a), complement of a_1 is a_2, whereas in (b), complement of a_1 is a_2 and a_3. It is to be noted that 1 is the only complement of 0. If possible, let $x \neq 1$ be another complement of 0, where $x \in L$.

Then $\quad 0 \vee x = 1 \quad$ and $\quad 0 \wedge x = 0$

But $\quad 0 \vee x = x \quad \therefore \quad x = 1$, which contradicts the assumption $x \neq 1$.

Similarly we can prove that 0 is the only complement of 1.

Now a lattice $\{L, \vee, \wedge, 0, 1\}$ is called a *complemented lattice* if every element of L has at least one complement.

The following property holds good for a distributive lattice.

Property

In a distributive lattice $\{L, \vee, \wedge\}$ if an element $a \in L$ has a complement, then it is unique.

Proof

If possible, let b and c be the complements of $a \in L$.

Then $\quad a \vee b = a \vee c = 1 \quad (1)$

and $\quad a \wedge b = a \wedge c = 0 \quad (2)$

Now
$$\begin{aligned} b &= b \vee 0 = b \vee (a \wedge c), \text{ by (2)} \\ &= (b \vee a) \wedge (b \vee c), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (b \vee c), \text{ by (1)} \\ &= b \vee c \qquad (3) \end{aligned}$$

Similarly,
$$\begin{aligned} c &= c \vee 0 = c \vee (a \wedge b), \text{ by (2)} \\ &= (c \vee a) \wedge (c \vee b), \text{ since } L \text{ is distributive} \\ &= 1 \wedge (c \vee b), \text{ by (1)} \\ &= c \vee b \qquad (4) \end{aligned}$$

From (3) and (4), since $b \vee c = c \vee b$, we get $b = c$.

Note From the definition of complemented lattice and the previous property, it follows that every element a of a complemented and distributive lattice has a unique complement denoted by a'.

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BOOLEAN ALGEBRA

Definition

A lattice which is complemented and distributive is called a Boolean Algebra, (which is named after the mathematician George Boole). Alternatively, Boolean Algebra can be defined as follows:

Definition

If B is a nonempty set with two binary operations + and •, two distinct elements 0 and 1 and a unary operation ′, then B is called a *Boolean Algebra* if the following basic properties hold for all a, b, c in B:

B1: $\left.\begin{array}{l} a + 0 = a \\ a \cdot 1 = a \end{array}\right\}$ Identity laws

B2: $\left.\begin{array}{l} a + b = b + a \\ a \cdot b = b \cdot a \end{array}\right\}$ Commutative laws

B3: $\left.\begin{array}{l} (a + b) + c = a + (b + c) \\ (a \cdot b) \cdot c = a \cdot (b \cdot c) \end{array}\right\}$ Associative laws

B4 $\left.\begin{array}{l} a + (b \cdot c) = (a + b) \cdot (a + c) \\ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \end{array}\right\}$ Distributive laws

B5: $\left.\begin{array}{l} a + a' = 1 \\ a \cdot a' = 0 \end{array}\right\}$ Complement laws.

Note
1. We have switched over to the symbols + and • instead of ∨ (join) and ∧ (meet) used in the study of lattices. The operations + and •, that will be used hereafter in Boolean algebra, are called *Boolean sum* and *Boolean product* respectively. We may even drop the symbol • and instead use juxtaposition. That is $a \bullet b$ may be written as ab.
2. If B is the set {0, 1} and the operations +, •, ′ are defined for the elements of B as follows:

$$0 + 0 = 0;\ 0 + 1 = 1 + 0 = 1 + 1 = 1$$
$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0;\ 1 \cdot 1 = 1$$
$$0' = 1 \text{ and } 1' = 0,$$

then the algebra $\{B, +, \bullet, ', 0, 1\}$ satisfies all the 5 properties given above and is the simplest Boolean algebra called a two-element Boolean algebra. It can be proved that two element Boolean algebra is the only Boolean algebra.
If a variable x takes on only the values 0 and 1, it is called a *Boolean variable*.
3. 0 and 1 are merely symbolic names and, in general, have nothing to do with the numbers 0 and 1. Similarly + and • are merely binary operators and, in general, have nothing to do with ordinary addition and multiplication.

ADDITIONAL PROPERTIES OF BOOLEAN ALGEBRA

If $\{B, +, \bullet, ', 0, 1\}$ is a Boolean algebra, the following properties hold good. They can be proved by using the basic properties of Boolean algebra listed in the definition.

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(i) Idempotent Laws

$$a + a = a \quad \text{and} \quad a \cdot a = a, \quad \text{for all } a \in B$$

Proof

$$\begin{aligned} a &= a + 0, \text{ by } B1 \\ &= a + a \cdot a', \text{ by } B5 \\ &= (a + a) \cdot (a + a'), \text{ by } B4 \\ &= (a + a) \cdot 1, \text{ by } B5 \\ &= a + a, \text{ by } B1 \end{aligned}$$

Now,

$$\begin{aligned} a &= a \cdot 1, \text{ by } B1 \\ &= a \cdot (a + a'), \text{ by } B5 \\ &= a \cdot a + a \cdot a', \text{ by } B4 \\ &= a \cdot a + 0, \text{ by } B5 \\ &= a \cdot a, \text{ by } B1. \end{aligned}$$

(ii) Dominance Laws

$$a + 1 = 1 \text{ and } a \cdot 0 = 0, \text{ for all } a \in B.$$

Proof

$$\begin{aligned} a + 1 &= (a + 1) \cdot 1, \text{ by } B1 \\ &= (a + 1) \cdot (a + a'), \text{ by } B5 \\ &= a + 1 \cdot a', \text{ by } B4 \\ &= a + a' \cdot 1, \text{ by } B2 \\ &= a + a', \text{ by } B1 \\ &= 1, \text{ by } B5. \end{aligned}$$

Now

$$\begin{aligned} a \cdot 0 &= a \cdot 0 + 0, \text{ by } B1 \\ &= a \cdot 0 + a \cdot a', \text{ by } B5 \\ &= a \cdot (0 + a'), \text{ by } B4 \\ &= a \cdot (a' + 0), \text{ by } B2 \\ &= a \cdot a', \text{ by } B1 \\ &= 0, \text{ by } B5 \end{aligned}$$

(iii) Absorption Laws

$$a \cdot (a + b) = a \text{ and } a + a \cdot b = a, \text{ for all } a, b \in B.$$

Proof

$$\begin{aligned} a \cdot (a + b) &= (a + 0) \cdot (a + b), \text{ by } B1 \\ &= a + 0 \cdot b, \text{ by } B4 \\ &= a + b \cdot 0, \text{ by } B2 \\ &= a + 0, \text{ by dominance law} \\ &= a, \text{ by } B1. \end{aligned}$$

Now

$$\begin{aligned} a + a \cdot b &= a \cdot 1 + a \cdot b, \text{ by } B1 \\ &= a \cdot (1 + b), \text{ by } B4 \\ &= a \cdot (b + 1), \text{ by } B2 \\ &= a \cdot 1 \text{ by dominance law} \\ &= a \text{ by } B1. \end{aligned}$$

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(iv) De Morgan's Laws

$$(a + b)' = a' \cdot b' \text{ and } (a \cdot b)' = a' + b', \text{ for all } a, b \in B.$$

Proof

Note If y is to be the complement of x, by definition, we must show that $x + y = 1$ and $x \cdot y = 0$.

$$\begin{aligned}
(a + b) + a'b' &= \{(a + b) + a'\} \cdot \{(a + b) + b'\}, \text{ by } B4 \\
&= \{(b + a) + a'\} \cdot \{(a + b) + b'\}, \text{ by } B2 \\
&= \{b + (a + a')\} \cdot \{a + (b + b')\}, \text{ by } B3 \\
&= (b + 1) \cdot (a + 1), \text{ by } B5 \\
&= 1 \cdot 1, \text{ by dominance law} \\
&= 1, \text{ by } B1. \qquad (1)
\end{aligned}$$

Now

$$\begin{aligned}
(a + b) \cdot a'b' &= a'b' \cdot (a + b), \text{ by } B2 \\
&= a'b' \cdot a + a'b' \cdot b, \text{ by } B4 \\
&= a \cdot (a'b') + a' \cdot b'b, \text{ by } B3 \\
&= (a \cdot a') \cdot b' + a' \cdot (bb'), \text{ by } B_3 \text{ and } B_2 \\
&= 0 \cdot b' + a' \cdot 0, \text{ by } B5 \\
&= b' \cdot 0 + a' \cdot 0, \text{ by } B2 \\
&= 0 + 0, \text{ by dominance law} \\
&= 0, \text{ by } B1. \qquad (2)
\end{aligned}$$

From (1) and (2), we get $a'b'$ is the complement of $(a + b)$. i.e. $(a + b)' = a'b'$.

[$\because$ the complement is unique]

Note The students are advised to give the proof for the other part in a similar manner.

(v) Double Complement or Involution Law

$$(a')' = a, \text{ for all } a \in B.$$

Proof

$$a + a' = 1 \text{ and } a \cdot a' = 0, \text{ by } B5$$

i.e. $a' + a = 1$ and $a' \cdot a = 0$, by $B2$

$\therefore$ a is the complement of a'

i.e. $(a')' = a$, by the uniqueness of the complement of a'. [See example (14)]

(vi) Zero and One Law

$$0' = 1 \text{ and } 1' = 0$$

Proof

$$\begin{aligned}
0' &= (aa')', \text{ by } B5 \\
&= a' + (a')', \text{ by De Morgan's law} \\
&= a' + a, \text{ by involution law} \\
&= a + a', \text{ by } B2 \\
&= 1, \text{ by } B5
\end{aligned}$$

Now $(0')' = 1'$

i.e. $0 = 1'$ or $1' = 0$.

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DUAL AND PRINCIPLE OF DUALITY

Definition

The *dual* of any statement in a Boolean algebra B is the statement obtained by interchanging the operations $+$ and $\bullet$ and interchanging the elements 0 and 1 in the original statement.

For example, the dual of $a + a(b + 1) = a$ is $a \bullet (a + b \bullet 0) = a$.

PRINCIPLE OF DUALITY

The dual of a theorem in a Boolean algebra is also theorem.

For example, $(a \cdot b)' = a' + b'$ is a valid result, since it is the dual of the valid statement $(a + b)' = a' \cdot b'$ [De Morgan's laws]. If a theorem in Boolean algebra is proved by using the axioms of Boolean algebra, the dual theorem can be proved by using the dual of each step of the proof of the original theorem. This is obvious from the proofs of additional properties of Boolean algebra.

SUBALGEBRA

If C is a nonempty subset of a Boolean algebra such that C itself is a Boolean algebra with respect to the operations of B, then C is called a *subalgebra* of B.

It is obvious that C is a subalgebra of B if and only if C is closed under the three operations of B, namely, $+$, $\bullet$ and $'$ and contains the element 0 and 1.

BOOLEAN HOMOMORPHISM

If $\{B +, \bullet, ', 0, 1\}$ and $\{C, \cup, \cap, -, \alpha, \beta\}$ are two Boolean algebras, then a mapping $f : B \to C$ is called a *Boolean homomorphism*, if all the operations of Boolean algebra are preserved. viz., for any $a, b \in B$,

$$f(a + b) = f(a) \cup f(b), f(a \cdot b) = f(a) \cap f(b),$$

$$f(a') = \overline{f(a)}, f(0) = \alpha \text{ and } f(1) = \beta,$$

where α and β are the zero and unit elements of C.

ISOMORPHIC BOOLEAN ALGEBRAS

Two Boolean algebras B and B' are said to be *isomorphic* if there is one-to-one correspondence between B and B' with respect to the three operations, viz. there exists a mapping $f: B \to B'$ such that $f(a + b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(a') = \{f(a)\}'$.

BOOLEAN EXPRESSIONS AND BOOLEAN FUNCTIONS

Definitions

A *Boolean expression* in n Boolean variables $x_1, x_2, \ldots x_n$ is a finite string of symbols formed recursively as follows:

1, 0, 1, x_1, x_2, ... x_n are Boolean expressions.

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2. If E_1 and E_2 are Boolean expressions, then $E_1 \cdot E_2$ and $E_1 + E_2$ are also Boolean expressions.
3. If E is a Boolean expression, E' is also a Boolean expression.

Note A Boolean exprssion in n variables may or may not contain all the n *literals*, viz., variables or their complements.

If $x_1, x_2, \ldots, x_n$ are Boolean variables, a function from $B^n = \{(x_1, x_2, \ldots, x_n\}$ to $B = \{0, 1\}$ is called a *Boolean function of degree n*. Each Boolean expression represents a Boolean function, which is evaluated by substituting the value 0 or 1 for each variable. The values of a Boolean function for all possible combinations of values of the variables in the function are often displayed in truth tables.

For example, the values of the Boolean function $f(a, b, c) = ab + c'$ are displayed in the following truth table:

$(x_1, x_2, \ldots x_n)$ $B \to \{0, 1\}$

a	b	c	ab	c'	$ab + c'$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Note Although the order of the variable values may be random, a symmetric way of writing them in a cyclic manner which will be advantageous is as follows:

If there be n variables in the Boolean function, there will obviously be 2^n rows in the truth table corresponding to all possible combinations of the values 0 and 1 of the variables.

We write $\frac{1}{2} \times 2^n$ ones followed by $\frac{1}{2} \times 2^n$ zeros in the first column representing the values of the first variable.

Then in the second column, we write $\frac{1}{4} \times 2^n$ ones and $\frac{1}{4} \times 2^n$ zeros alternately, representing the values of the second variable. Next in the third column, we write $\frac{1}{8} \times 2^n$ ones and $\frac{1}{8} \times 2^n$ zeros alternately, representing the values of the third variable. We continue this procedure and in the final column, we write $\frac{1}{2^n} \times 2^n$ (=1) one and 1 zero alternately, representing the values of the n^{th} variable.]

Definitions of

1. A *minterm of* n Boolean variables is a Boolean product of the n literals (variables of complements) in which each literal appears exactly once.

For example, ab, $a'b$, ab' and $a'b'$ form the complete set of minterms of two variables a and b, abc, abc', $ab'c$, $a'bc$, $ab'c'$, $a'bc'$, $a'b'c$ and $a'b'c'$ form the complete set of minterms of three variables a, b, c.

2. A *maxterm* of n Boolean variables is a Boolean sum of the n literals in which each literal appears exactlty once.
   For example, $a + b$, $a' + b$, $a + b'$ and $a' + b'$ form the complete set of maxterms in two variables a and b.
3. When a Boolean function is expressed as a sum of minterms, it is called its *sum of products expansion* or it is said to be in the *disjunctive normal form* (DNF).
4. When a Boolean function is expressed as a product of maxterms, it is called its *product of sums expansion* or it is said to be in the *conjunctive normal form* (CNF).
5. Boolean function expressed in the DNF or CNF are said to be in *canonical form.*
6. If a Boolean function in n variables is expressed as the sum (product) of all the 2^n minterms (maxterms), it is said to be in *complete DNF* (*complete CNF*).
7. Boolean functions expressed in complete DNF or complete CNF are said to be *complete canonical form.*

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⇒ simplify a Boolean Expression,
$a'b'c + ab'c + a'b'c'$.

⇒ $a'b'c + ab'c + a'b'c'$

$a'b'(c + c') + ab'c$

$a'b' + ab'c$

$b'(a' + ac)$ $\qquad a + a'b = a + b$

$b'((a' + a) \cdot (a' + c))$

$b'(1 \cdot (a' + c))$

$\boxed{= b'c + b'a'}$

Ex Show that $ab + abc + a'b + ab'c = b + ac$

⇒ $ab + abc + a'b + ab'c$

$= ab + a'b + ab'c$

$= b + ab'c$

$= b + b'(ac)$

$\boxed{= b + ac}$

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to be isomorphic if there is one to one correspondance between B & C with respect to the operation

Min terms & Max terms in B.A. & K-Map

Min term: Each individual product term in SSOP (Standard sum of Product) form is called as Min term

Max term: Each individual sum term in SPOS (standard Product of sum) form is called as Max-term

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eg: For two variable function

Variable A	Variable B	Min term SSOP	Max term SPOS
0	0	$\bar{A}\bar{B} \to m_0$	$A+B \to M_0$
0	1	$\bar{A}B \to m_1$	$A+\bar{B} \to M_1$
1	0	$A\bar{B} \to m_2$	$\bar{A}+B \to M_2$
1	1	$AB \to m_3$	$\bar{A}+B \to M_3$

(×) Min terms → product term
(m)
if 0 is there → of Actual
if 1 is there → will take a[illegible]
Variable

(+) Max term → sum term
(M)
if 0 is there → taking Actual Va[illegible]
if 1 is there → complement of V[illegible]

[illegible] terms & Max