Equivalently, if $[a] \neq [b]$, then $[a] \cap [b] = \phi$.

Note: From (ii) and (iii) of the above theorem, it follows that the equivalence classes of two arbitrary elements under R are identical or disjoint.)

PARTITION OF A SET

Definition

If S is a non empty set, a collection of disjoint non empty subsets of S whose union is S is called *a partition* of S. In other words, the collection of subsets A_i is a partition of S if and only if

(i) $A_i \neq \phi$, for each i

(ii) $A_i \cap A_j = \phi$, for $i \neq j$ and

(iii) $\bigcup_i A_i = S$, where $\bigcup_i A_i$ represents the union of the subsets A_i for all i.

Note: The subsets in a partition are also called *blocks* of the partition.

For example, if $S = \{1, 2, 3, 4, 5, 6\}$

(i) $\{\{1, 3, 5\}, \{2, 4\}\}$ is not a partition since the union of the subsets is not S, as the element 6 is missing.

(ii) $[\{1, 3\}, \{3, 5\}, \{2, 4, 6\}]$ is not a partition, since $\{1, 3\}$ and $\{3, 5\}$ are not disjoint.

(iii) $[\{1, 2, 3\}, \{4, 5\}, \{6\}]$ is a partition.

PARTITIONING OF A SET INDUCED BY AN EQUIVALENCE RELATION

Let R be an equivalence relation of a non-empty set A.

Let $A_1, A_2, \ldots, A_k$ be the distinct equivalence classes of A under R. For every $a \in A_i$, $a \in [a]_R$, by the above theorem.

$$\therefore \quad A_i = [a]_R$$

$$\therefore \quad \bigcup_{a \in A_i} [a]_R = \bigcup_i A_i = A$$

Also by the above theorem, when $[a]_R \neq [b]_R$, then

$$[a]_R \cap [b]_R = \phi. \text{ viz., } A_i \cap A_j = \phi, \text{ if } [a]_R = A_i \text{ and } [b]_R = A_j$$

$\therefore$ The equivalence classes of A form a partition of A.

In other words, the quotient set A/R is a partition of A.

For example, let $A \equiv$ {blue, brown, green, orange, pink, red, white, yellow} and R be the equivalence relation of A defined by "has the same number of letters", then

A/R = [{red}, {blue, pink}, {brown, green, white}, {orange, yellow}]

The equivalence classes contained in A/R form a partition of A.

MATRIX REPRESENTATION OF A RELATION

If R is a relation from the set $A = \{a_1, a_2, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$, where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix

$$M_R = [m_{ij}], \text{ where}$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero-one matrix M_R has a 1 in its $(i - j)$th position when a_i is related to b_j and a 0 in this position when a_i is not related by b_j.

For example, if $A = [a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_4)\}$, then the matrix of R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Conversely, if R is the relation on $A = \{1, 3, 4\}$ represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$, since $m_{ij} = 1$ means that the ith element of A is related to the jth element of A.

1. If R and S are relations on a set A, represented by M_R and M_S respectively, then the matrix representing $R \cup S$ is the *join* of M_R and M_S obtained by putting 1 in the positions where either M_R or M_S has a 1 and denoted by $M_R \vee M_S$ i.e., $M_{R \cup S} = M_R \vee M_S$.
2. The matrix representing $R \cap S$ is the *meet* of M_R and M_S obtained by putting 1 in the positions where both M_R and M_S have a 1 and denoted by $M_R \wedge M_S$ i.e., $M_{R \cap S} = M_R \wedge M_S$.

Note The operations 'join' and 'meet', denoted by $\vee$ and $\wedge$ respectively are Boolean operations which will be discussed later in the topic on Boolean Algebra.

For example, if R and S are relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ respectively,}$$

then
$$M_{R \cup S} = M_R \vee M_S$$
$$= \begin{bmatrix} 1\vee1 & 0\vee0 & 1\vee1 \\ 0\vee1 & 1\vee0 & 1\vee0 \\ 1\vee0 & 0\vee1 & 0\vee0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and
$$M_{R \cap S} = M_R \wedge M_S$$
$$= \begin{bmatrix} 1\wedge1 & 0\wedge0 & 1\wedge1 \\ 0\wedge1 & 1\wedge0 & 1\wedge0 \\ 1\wedge0 & 0\wedge1 & 0\wedge0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. If R is a relation from a set A to a set B represented by M_R, then the matrix representing R^{-1} (the inverse of R) is M_R^T, the transpose of M_R. For example, if $A = \{2, 4, 6, 8\}$ and $B = \{3, 5, 7\}$ and if R is defined by $\{(2, 3), (2, 5), (4, 5), (4, 7), (6, 3), (6, 7), (8, 7)\}$, then

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R^{-1} is defined by $\{(3, 2), (5, 2), (5, 4), (7, 4), (3, 6), (7, 6), (7, 8)$

Now
$$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T.$$

4. If R is a relation from A to B and S is a relation from B to C, then the composition of the relations R and S (if defined), viz., $R \bullet S$ is represented by the Boolean product of the matrices M_R and M_S, denoted by $M_R \bullet M_S$.

Note The Boolean product of two matrices is obtained in a way similar to the ordinary product, but with multiplication replaced by the Boolean operation $\wedge$ and with addition replaced by the Boolean operation $\vee$.

For example, the matrix representing $R \bullet S$

$$\text{where} \quad M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R \bullet S} = M_R \odot M_S = \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Since the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$, $m_{ii} = 1$ for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements in the principal diagonal of M_R are equal to 1.

6. Since the relation R on the set $A = \{a_2, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$, we will have $m_{ji} = 1$ whenever $m_{ij} = 1$ (or equivalently $m_{ji} = 0$ whenever $m_{ij} = 0$). In other words, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j ($i, j = 1, 2, \ldots, n$). This means that R is symmetric, if $M_R = (M_R)^T$, viz., M_R is a symmetric matrix.

Note The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ $(i \neq j)$, then $m_{ji} = 0$.

7. There is no simple way to test whether a relation R on a set A is transitive by examining the matrix M_R. However, we can easily verify that a relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

REPRESENTATION OF RELATIONS BY GRAPHS

Let R be a relation on a set A. To represent R graphically, each element of A is represented by a point. These points are called *nodes* or *vertices*. Whenever the element a is related to the element b, an arc is drawn from the point 'a' to the point 'b'. These arcs are called *arcs* or *edges*. The arcs start from the first element of the related pair and go to the second element. The direction is indicated by an arrow. The resulting diagram is called the *directed graph* or

The edge of the form (a, a), represented by using an arc from the vertex a back to itself, is called a *loop*.

For example, if $A = \{2, 3, 4, 6\}$ and R is defined by a R b if a divides b, then

$R = (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

The digraph representing the relation R is given in Fig. 2.14.

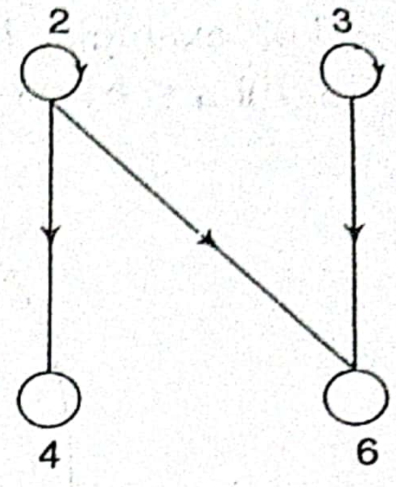

Fig. 2.14

Note The digraph of R^{-1}, the inverse of R, has exactly the same edges of the digraph of R, but the directions of the edges are reversed.

The digraph representing a relation can be used to determine whether the relation has the standard properties explained as follows:

(i) A relation R is reflexive if and only if there is a loop at every vertex of the digraph of the relation R, so that every ordered pair of the form (a, a) occurs in R. If no vertex has a loop, then R is irreflexive.

(ii) A relation R is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (b, a) is in R whenever (a, b) is in R.

(iii) A relation R is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices.

(iv) A relation R is transitive if and only if whenever there is an edge from a vertex a to a vertex b and from the vertex b to a vertex c, there is an edge from a to c.

Example 2.14 If R is the relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$, if and only if $a + b$ = even, find the relational matrix M_R. Find also the relational matrices R^{-1}, $\bar{R}$ and R^2.

$$R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$$

$$\therefore \qquad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\bar{R}$ is the complement R that consists of elements of $A \times A$ that are not in R.

Thus $\bar{R} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$

$\therefore \quad M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which is the same as the matrix obtained from M_R by changing 0's to 1's and 1's to 0's.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It can be found that $R^2 = R \bullet R = R$. Hence $M_{R^2} = M_R$

Example 2.15 If R and S be relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

viz. $R^2 \subseteq R$

$\therefore$ R is a transitive relation.

Hence R is an equivalence relation.

Example 2.17 List the ordered pairs in the relation on $\{1, 2, 3, 4\}$ corresponding to the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also draw the directed graph representing this relation. Use the graph to find if the relation is reflexive, symmetric and/or transitive.

The ordered pairs in the given relation are $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$. The directed graph representing the relation is given in Fig. 2.18.

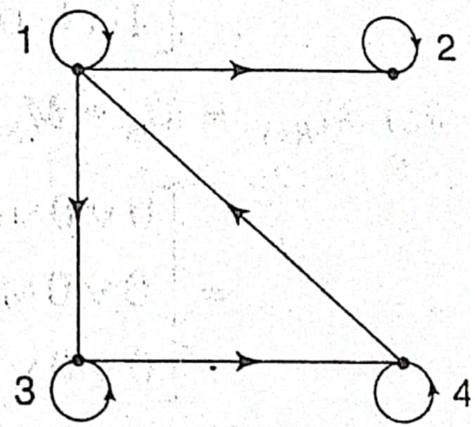

Fig. 2.18

Since there is a loop at every vertex of the digraph, the relation is reflexive. The relation is not symmetric. For example, there is an edge from 1 to 2, but there is no edge in the opposit direction, i.e. from 2 to 1. The relation is not transitive. For example, thoug there are edges from 1 to 3 and 3 to 4, there is no edge from 1 to 4.