(ii) [{1, 3}, {3, 5}, {2, 4, 6}] is not a partition, since {1, 3} and {3, 5} are not disjoint.

(iii) [{1, 2, 3}, {4, 5}, {6}] is a partition.

PARTITIONING OF A SET INDUCED BY AN EQUIVALENCE RELATION

Let R be an equivalence relation of a non-empty set A.

Let $A_1, A_2, \ldots, A_k$ be the distinct equivalence classes of A under R. For every $a \in A_i$, $a \in [a]_R$, by the above theorem.

$$\therefore \qquad A_i = [a]_R$$

$$\therefore \qquad \bigcup_{a \in A_i} [a]_R = \bigcup_i A_i = A$$

Also by the above theorem, when $[a]_R \neq [b]_R$, then

$$[a]_R \cap [b]_R = \phi. \text{ viz., } A_i \cap A_j = \phi, \text{ if } [a]_R = A_i \text{ and } [b]_R = A_j$$

$\therefore$ The equivalence classes of A form a partition of A.

In other words, the quotient set A/R is a partition of A.

For example, let $A \equiv$ {blue, brown, green, orange, pink, red, white, yellow} and R be the equivalence relation of A defined by "has the same number of letters", then

$$A/R = [\{\text{red}\}, \{\text{blue, pink}\}, \{\text{brown, green, white}\}, \{\text{orange, yellow}\}]$$

The equivalence classes contained in A/R form a partition of A.

MATRIX REPRESENTATION OF A RELATION

If R is a relation from the set $A = \{a_1, a_2, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$, where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix

$$M_R = [m_{ij}], \text{ where}$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero-one matrix M_R has a 1 in its $(i-j)$th position when a_i is related to b_j and a 0 in this position when a_i is not related by b_j.

For example, if $A = [a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_4)\}$, then the matrix of R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Conversely, if R is the relation on $A = \{1, 3, 4\}$ represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$, since $m_{ij} = 1$ means that the ith element of A is related to the jth element of A.

1. If R and S are relations on a set A, represented by M_R and M_S respectively, then the matrix representing $R \cup S$ is the join of M_R and M_S obtained by putting 1 in the positions where either M_R or M_S has a 1 and denoted by $M_R \vee M_S$ i.e., $M_{R \cup S} = M_R \vee M_S$.
2. The matrix representing $R \cap S$ is the meet of M_R and M_S obtained by putting 1 in the positions where both M_R and M_S have a 1 and denoted by $M_R \wedge M_S$ i.e., $M_{R \cap S} = M_R \wedge M_S$.

Note: The operations 'join' and 'meet', denoted by $\vee$ and $\wedge$ respectively are Boolean operations which will be discussed later in the topic on Boolean Algebra.

For example, if R and S are relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ respectively,}$$

then $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1\vee 1 & 0\vee 0 & 1\vee 1 \\ 0\vee 1 & 1\vee 0 & 1\vee 0 \\ 1\vee 0 & 0\vee 1 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 1\wedge 1 & 0\wedge 0 & 1\wedge 1 \\ 0\wedge 1 & 1\wedge 0 & 1\wedge 0 \\ 1\wedge 0 & 0\wedge 1 & 0\wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. If R is a relation from a set A to a set B represented by M_R, then the matrix representing R^{-1} (the inverse of R) is M_R^T, the transpose of M_R.
For example, if $A = \{2, 4, 6, 8\}$ and $B = \{3, 5, 7\}$ and if R is defined by $\{(2, 3), (2, 5), (4, 5), (4, 7), (6, 3), (6, 7), (8, 7)\}$, then

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R^{-1} is defined by $\{(3, 2), (5, 2), (5, 4), (7, 4), (3, 6), (7, 6), (7, 8)$

Now $$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T.$$

Example 2.11

(i) Prove that the relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$ if and only if $b = a^m$ for some positive integer m, show that R is a partial ordering.

(i) $(A, B) \in R$, if and only if $A \subseteq B$, where A and B are any two sets.
Now $A \subseteq A \quad \therefore (A, A) \in R$. i.e. R is reflexive.
If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
i.e. R is antisymmetric.
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
i.e. $(A, B) \in R$ and $(B, C) \in R \Rightarrow (A, C) \in R$
$\therefore$ R is transitive
Hence R is a partial ordering.

(ii) $a = a^1 \therefore (a, a) \in R$.
Let $(a, b) \in R$ and $(b, a) \in R$
i.e. $b = a^m$ and $a = b^n$ (1)
where m and n are positive integers.
$\therefore \quad a = (a^m)^n = a^{mn}$.
This means that $mn = 1$ or $a = 1$ or $a = -1$
Case (1): If $mn = 1$, then $m = 1$ and $n = 1$
$\therefore a = b$ [from (1)]
Case (2): If $a = 1$, then, from (1), $b = 1^m = 1 = a$
If $b = 1$, then, from (1), $a = 1^n = 1 = b$
Either way, $a = b$.
Case (3): If $a = -1$, then $b = -1$
Thus in all the three cases, $a = b$.
$\therefore$ R is antisymmetric.
Let $(a, b) \in R$ and $(b, c) \in R$
i.e. $b = a^m$ and $c = b^n$
$\therefore \quad c = (a^m)^n = a^{mn}$
$\therefore \quad (a, c) \in R$. i.e. R is transitive.
$\therefore$ R is a partial ordering.

Example 2.12

(i) If R is the equivalence relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ given below, find the partition of A induced by R:
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$

(ii) If R is the equivalence relation on the set $A = \{(-4, -20), (-3, -9), (-2, -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12)\}$, where $(a, b)\, R\, (c, d)$ if $ad = bc$, find the equivalent classes of R.

(i) The elements related to 1 are 1 and 2.
$\therefore \quad [1]_R = \{1, 2\}$
Also $[2]_R = \{1, 2\}$
The element related to 3 is 3 only
i.e. $[3]_R = \{3\}$ The elements related to 4 are $\{4, 5\}$

i.e. $[4]_R = \{4, 5\} = [5]_R$

The element related to 6 is 6 only

i.e. $[6]_R = \{6\}$

$\therefore$ $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ is the partition induced by R.

(ii) The elements related to $(-4, -20)$ are $(1, 5)$ and $(2, 10)$

i.e. $$[(-4, -20)] = \{(-4, -20), (1, 5), (2, 10)\}$$

The elements related to $(-3, -9)$ are $(-1, -3)$ and $(4, 12)$

i.e. $$[(-3, -9)] = \{(-3, -9), (-1, -3), (4, 12)\}$$

The elements related to $(-2, -4)$ are $(-2, -4), (1, 2), (3, 6)$ and $(4, 8)$

i.e. $$[(-2, -4)] = \{(-2, -4), (1, 2), (3, 6), (4, 8)\}.$$

The element related to $(-1, -11)$ is itself only.

The element related to $(2, 14)$ is itself only.

$\therefore$ The partition induced by R consists of the cells $[(-4, -20)], [(-3, -9)], [(-2, -4)], [(-1, -11]$ and $[(2, 14)]$.

Example 2.13

(i) If $A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and the relation R is defined on A by $(a, b)\ R\ (c, d)$ if $a + b = c + d$, verify that A is an equivalence relation on A and also find the quotient set of A by R.

(ii) If the relation R on the set of integers Z is defined by $a\ R\ b$ if $a \equiv b$ (mod 4), find the partition induced by R.

(i) $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

If we take $R \equiv A$, it can be verified that R is an equivalence relation.

The quotient set A/R is the collection of equivalence classes of R.

It is easily seen that

$$[(1, 1)] = \{(1, 1)\}$$
$$[(1, 2)] = \{(1, 2), (2, 1)\}$$
$$[(1, 3)] = \{(1, 3), (2, 2), (3, 1)\}$$
$$[(1, 4)] = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$
$$[(2, 4)] = \{(2, 4), (3, 3), (4, 2)\}$$
$$[(3, 4)] = \{(3, 4), (4, 3)\}$$
$$[(4, 4)] = \{(4, 4)\}$$

Thus $[(1, 1)], [(1, 2)], [(1, 3)], [(1, 4)], [(2, 4)], [(3, 4)], [(4, 4)]$ form the quotient set A/R.

(ii) The equivalence classes of R are the following:

$$[0]_R = \{\cdots, -8, -4, 0, 4, 8, 12, \cdots\}$$
$$[1]_R = \{\cdots, -7, -3, 1, 5, 9, 13, \cdots\}$$
$$[2]_R = \{\cdots, -6, -2, 2, 6, 10, 14, \cdots\}$$
$$[3]_R = \{\cdots, -5, -1, 3, 7, 11, 15, \cdots\}$$

Thus $[0]_R, [1]_R, [2]_R$ and $[3]_R$ form the partition of R.

Note These equivalence classes are also called *the congruence classes* modulo 4 and also denoted $[0]_4, [1]_4, [2]_4$ and $[3]_4$.

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Example 2.14 If R is the relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$, if and only if $a + b =$ even, find the relational matrix M_R. Find also the relational matrices R^{-1}, $\bar{R}$ and R^2.

$$R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$$

$$\therefore \qquad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\bar{R}$ is the complement R that consists of elements of $A \times A$ that are not in R.

Thus $\bar{R} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$

$\therefore \quad M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which is the same as the matrix obtained from M_R by changing 0's to 1's and 1's to 0's.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It can be found that $R^2 = R \bullet R = R$. Hence $M_{R^2} = M_R$

Example 2.15 If R and S be relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

find the matrices that represent

(a) $R \cup S$ (b) $R \cap S$ (c) $R \bullet S$ (d) $S \bullet R$ (e) $R \oplus S$

(a) $M_{R \cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 0 \vee 0 & 1 \vee 1 & 0 \vee 0 \\ 1 \vee 0 & 1 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) $M_{R \cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 0 \wedge 0 & 1 \wedge 1 & 0 \wedge 0 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) $M_{R \bullet S} = M_R \bullet M_S$

$$= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 1 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \\ 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d) $M_{S \bullet R} = M_S \bullet M_R$

$$= \begin{bmatrix} 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 1 \vee 1 & 1 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(e) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example 2.16 Examine if the relation R represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of M_R.

Since all the elements in the main diagonal of M_R and equals to 1 each, R is a reflexive relation.

Since M_R is a symmetric matrix, R is a symmetric relation.

$$M_{R^2} = M_R \bullet M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$$

viz. $R^2 \subseteq R$

$\therefore$ R is a transitive relation.

Hence R is an equivalence relation.

Example 2.17 List the ordered pairs in the relation on $\{1, 2, 3, 4\}$ corresponding to the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Also draw the directed graph representing this relation. Use the graph to find if the relation is reflexive, symmetric and/or transitive.

The ordered pairs in the given relation are $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$. The directed graph representing the relation is given in Fig. 2.18.

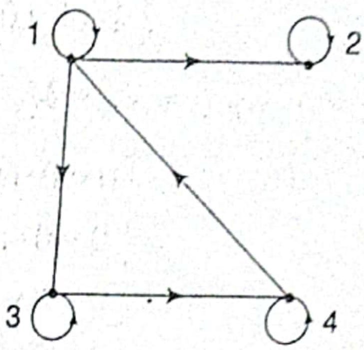

Fig. 2.18

Since there is a loop at every vertex of the digraph, the relation is reflexive. The relation is not symmetric. For example, there is an edge from 1 to 2, but there is no edge in the opposite direction, i.e. from 2 to 1. The relation is not transitive. For example, though there are edges from 1 to 3 and 3 to 4, there is no edge from 1 to 4.

Example 2.18 List the ordered pairs in the relation represented by the digraph given in Fig. 2.19. Also use the graph to prove that the relation is a partial ordering. Also draw the directed graphs representing R^{-1} and $\bar{R}$.

The ordered pairs in the relation are $\{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)\}$.

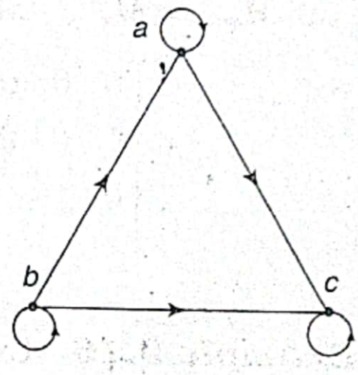

Fig. 2.19

Since there is a loop at every vertex, the relation is reflexive.

Though there are edges $b - a$, $a - c$ and $b - c$, the edges $a - b$, $c - a$ and $c - b$ are not present in the digraph. Hence the relation is antisymmetric.

When edges $b - a$ and $a - c$ are present in the digraph, the edge $b - c$ is also present (for example). Hence the relation is transitive.

Hence the relation is a partially ordering. The digraph of R^{-1} is got by reversing the directions of the edges (Fig. 2.20). The digraph of $\bar{R}$ contains the edges (a, b), (c, a), and (c, b) as shown in Fig. 2.21.

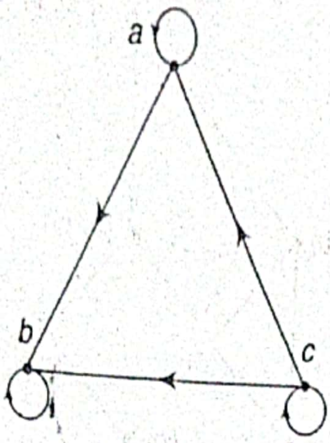

Fig. 2.20

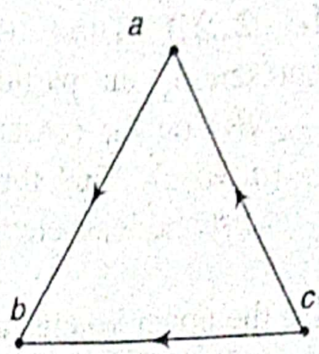

Fig. 2.21

Example 2.19 Draw the digraph representing the partial ordering $\{(a, b)\} \mid a$ divides $b\}$ on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Reduce it to the Hasse diagram representing the given partial ordering.

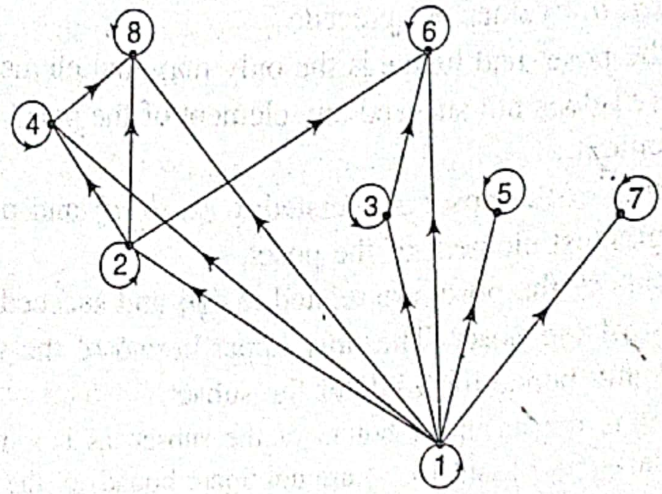

Fig. 2.22

Deleting all the loops at the vertices, deleting all the edges occurring due to transitivity, arranging all the edges to point upward and deleting all arrows, we get the corresponding Hasse diagram as given in Fig. 2.23.

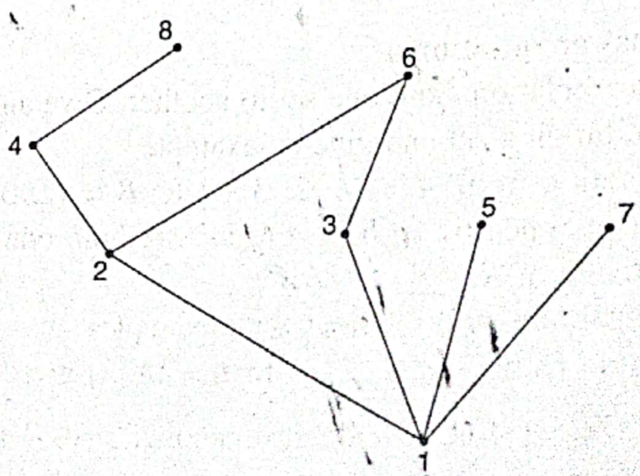

Fig. 2.23