MATHS TUTORIAL-3 (Group Theory)

Q1.)

1. In the following determine whether the systems described are groups. If they are not, point out which of the group axioms fail to hold.
   (a) G = set of all integers, $a \cdot b \equiv a - b$.
   (b) G = set of all positive integers, $a \cdot b = ab$, the usual product of integers.
   (c) $G = a_0, a_1, \ldots, a_6$ where

$$a_i \cdot a_j = a_{i+j} \quad \text{if} \quad i + j < 7,$$

$$a_i \cdot a_j = a_{i+j-7} \quad \text{if} \quad i + j \geq 7$$

   (for instance, $a_5 \cdot a_4 = a_{5+4-7} = a_2$ since $5 + 4 = 9 > 7$).
   (d) G = set of all rational numbers with odd denominators, $a \cdot b \equiv a + b$, the usual addition of rational numbers.

Q2.)

2. Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(a \cdot b)^n = a^n \cdot b^n$.

Q3.)

3. If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, show that G must be abelian.

Q4.)

9. (a) If the group G has three elements, show it must be abelian.
   (b) Do part (a) if G has four elements.
   (c) Do part (a) if G has five elements.

Q5.)

11. If G is a group of even order, prove it has an element $a \neq e$ satisfying $a^2 = e$.

Q6.)

14. Suppose a *finite* set G is closed under an associative product and that both cancellation laws hold in G. Prove that G must be a group.

Q7.)

#20. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$ is a rational number. Prove that G forms a group under matrix multiplication.

Q8.)

#21. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$. Prove that G forms a group under matrix multiplication. Is G abelian?

Q9.)

#22. Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ where $a \neq 0$. Prove that G is an abelian group under matrix multiplication.

Q10.)

#24. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that $ad - bc \neq 0$. Using matrix multiplication as the operation in G, prove that G is a group of order 6.

Q11.)

#25. (a) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$ and a, b, c, d are integers modulo 3, relative to matrix multiplication. Show that $o(G) = 48$.

(b) If we modify the example of G in part (a) by insisting that $ad - bc = 1$, then what is $o(G)$?

Q12.)

#*26. (a) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(b) Let H be the subgroup of the G of part (a) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \mid ad - bc = 1 \right\}.$$

What is $o(H)$?