## Introduction

Inference theory is concerned with the inferring of a *conclusion* from certain hypotheses or basic assumptions, called *premises*, by applying certain principles of reasoning, called *rules of inference*. When a conclusion derived from a set of premises by using rules of inference, the process of such derivation is called a *formal proof.* The rules of inference are only means used to draw a conclusion from a set of premises in a finite sequence of steps, called *argument.* These rules will be given in terms of statement formulas rather than in terms of any specific statements or their truth values. In this section we deal with the rules of inference by which conclusions are derived from premises. Any conclusion which is arrived at by following these rules is called a *valid conclusion* and the argument is called a *valid argument*. The actual truth values of the premises and that of the conclusion do not play any part in the determination of the validity of the argument. However, if the premises are believed to be true and if proper rules of inference are used, then the conclusion may be expected to be true.

## TRUTH TABLE TECHNIQUE

When $A$ and $B$ are two statement formulas, then $B$ is said to (logically) follow $A$ or $B$ is a valid conclusion of the premise $A$, if $A \to B$ is a tautology, viz., $A \Rightarrow B$. Extending, a conclusion $C$ is said to follow from a set of premises $H_1$, $H_2$, … $H_n$, if $(H_1 \wedge H_2 \wedge \ldots \wedge H_n) \Rightarrow C$. If a set of premises and a conclusion are given, it is possible to determine whether the conclusion follows from the premises by constructing relevant truth tables, as explained in the following example. This method is known as truth table technique.

For example, let us consider

(i) $H_1$: $\neg p$, $H_2$: $p \vee q$, $C : q$

(ii) $H_1$: $p \to q$, $H_2$: $q$, $C : p$

**Table 1.34**

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | T |

(i) $H_1$ and $H_2$ are true only in the third row, in which case $C$ is also true. Hence (i) is valid

(ii) $H_1$ and $H_2$ are true in the first and third rows, but $C$ is not true in the third row. Hence (ii) is not a valid conclusion.

**Note** The truth table technique becomes tedious, if the premises contain a large number of variables.

## RULES OF INFERENCE

Before we give the frequently used rules of inference in the form of tautologies in a table, we state two basic rules of inference called rules P and T.

*Rule P* A premise may be introduced at any step in the derivation.

*Rule T* A formula $S$ may be introduced in the derivation, if $S$ is tautologically implied by one or more preceding formulas in the derivation.

**Table 1.35** Rules of Inference

| *Rule in tautological form* | *Name of the rule* |
|---|---|
| $(p \wedge q) \rightarrow p$ (viz., $p \wedge q \Rightarrow p$)<br>$(p \wedge q) \rightarrow q$ (viz., $p \wedge q \Rightarrow q$) | Simplification |
| $p \rightarrow (p \vee q)$<br>$q \rightarrow (p \vee q)$ | Addition |
| $((p) \wedge (q)) \rightarrow (p \wedge q)$ | Conjunction |
| $[p \wedge (p \rightarrow q)] \rightarrow q$ | Modus ponens |
| $[\urcorner q \wedge (p \rightarrow q)] \rightarrow \urcorner p$ | Modus tollens |
| $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ | Hypothetical syllogism |
| $[(p \vee q) \wedge \urcorner p] \rightarrow q$ | Disjunctive syllogism |
| $[(p \vee q) \wedge (\urcorner p \vee r)] \rightarrow (q \vee r)$ | Resolution |
| $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ | Dilemma |

## FORM OF ARGUMENT

When a set of given statements constitute a valid argument, the argument form will be presented as in the following example: "If it rains heavily, then travelling will be difficult. If students arrive on time, then travelling was not difficult. They arrived on time. Therefore, it did not rain heavily."

Let the statements be defined as follows:

$p$: It rains heavily

$q$: Travelling is difficult

$r$: Students arrived on time

Now we have to show that the premises $p \rightarrow q$, $r \rightarrow \urcorner q$ and $r$ lead to the conclusion $\urcorner p$. The form of argument given as follows shows that the premises lead to the conclusion.

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $p \rightarrow q$ | Rule $P$ |
| 2. | $\urcorner q \rightarrow \urcorner p$ | $T$, Contrapositive of 1 |
| 3. | $r \rightarrow \urcorner q$ | Rule $P$ |
| 4. | $r \rightarrow \urcorner p$ | $T$, Steps 2, 3 and hypothetical syllogism |
| 5. | $r$ | Rule $P$ |
| 6. | $\urcorner p$ | $T$, steps 4, 5 and Modus ponen |

## RULE CP OR RULE OF CONDITIONAL PROOF

In addition to the two basic rules of inference $P$ and $T$, we have one more basic rule called Rule CP, which is stated below:

If a formula $s$ can be derived from another formula $r$ and a set of premises, the statement $(r \rightarrow s)$ can be derived from the set of premises alone.

The rule CP follows from the equivalence

$$(p \wedge r) \rightarrow s \equiv p \rightarrow (r \rightarrow s)$$

**Note:** If the conclusion is of the form $r \rightarrow s$, we will take $r$ as an additi premise and derive $s$ using the given premises and $r$.

## INCONSISTENT PREMISES

A set of premises (formulas) $H_1, H_2, \ldots H_n$ is said to be inconsistent, if t conjunction implies a contradiction.

viz. if $H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow R \wedge \neg R$, for some formula $R$.

A set of premises is said to be consistent, if it is not inconsistent.

## INDIRECT METHOD OF PROOF

The notion of inconsistency is used to derive a proof at times. This proce is called the indirect method of proof or proof by contradiction or reduc and absurdum.

In order to show that a conclusion $C$ follows from the premises $H_1, H_2$ $H_n$ by this method, we assume that $C$ is false and include $\neg C$ as an additi premise. If the new set of premises is inconsistent leading to a contradic then the assumption that $\neg C$ is true does not hold good. Hence $C$ is whenever $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is true. Thus $C$ follows from $H_1, H_2, \ldots H_n$.

For example, we prove that the premises $\neg q, p \rightarrow q$ result in the conclu $\neg p$ by the indirect method of proof.

We now include $\neg\neg p$ as an additional premise. The argument form is g below:

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\neg\neg p$ | $C$ |
| 2. | $p$ | $T$, double negation, 1 |
| 3. | $p \rightarrow q$ | $C$ |
| 4. | $\neg q \rightarrow \neg p$ | $T$, Contrapositive, 3 |
| 5. | $\neg q$ | $C$ |
| 6. | $\neg p$ | $T$, Modus ponens, 4, |
| 7. | $p \wedge \neg p$ | $T$, Conjunction, 2, 6 |

Thus the inclusion of $\neg C$ leads to a contradiction. Hence $\neg q, p \rightarrow q \Rightarrow \neg$

## PREDICATE CALCULUS OR PREDICATE LOGIC

### Introduction

In mathematics and computer programs, we encounter statements invol variables such as "$x > 10$", "$x = y + 5$" and "$x + y = z$". These statements neither true nor false, when the values of the variables are not specified.

The statement "$x$ is greater than 10" has 2 parts. The first part, the vari $x$, is the subject of the statement. The second part "is greater than 10", wl refers to a property that the subject can have, is called the *predicate*. We can denote the statement "$x$ is greater than 10" by the notation $P(x)$, where $P$ denotes the predicate "is greater than 10" and $x$ is the variable. $P(x)$ is called the *propositional function* at $x$. Once a value has been assigned to the variable $x$, the statement $P(x)$ becomes a proposition and has a truth value. For example, the truth values of $P(15)\{\equiv 15 > 10\}$ and $P(5)\{\equiv 5 > 10\}$ are T and F respectively. The statements "$x = y + 5$" and "$x + y = z$" will be denoted by $P(x, y)$ and $P(x, y, z)$ respectively. The logic based on the analysis of predicates in any statement is called *predicate logic* or *predicate calculus.*

## QUANTIFIERS

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the *universe of discourse*. Such a statement is expressed using a universal quantification. The universal quantification of $P(x)$ is the statement.

"$P(x)$ is true for all values of $x$ in the universe of discourse" and is denoted by the notation $(x)P(x)$ or $\forall\, xP(x)$. The proposition $(x)P(x)$ or $\forall\, xP(x)$ is read as "for all $x$, $P(x)$" or "for every $x$, $P(x)$". The symbol $\forall$ is called the *universal quantifier*.

[**Note** Let us consider $\forall x\, P(x) \equiv \forall x, (x^2 - 1) = (x - 1)(x + 1)$ (1)

(1) is a proposition and not a propositional function, even though a variable $x$ appears in it. We need not replace $x$ by a number to obtain a statement. The truth value of $\forall x\, P(x)$ is T.]

### Examples

1. If $P(x) \equiv \{(-x)^2 = x\}$, where the universe consists of all integers, then the truth value of $\forall x((-x)^2 = x^2)$ is T.
2. If $Q(x) \equiv$ "$2x > x$", where the universe consists of all real numbers, then the truth value of $\forall x\, Q(x)$ is F.
3. If $P(x) \equiv$ "$x^2 < 10$", where the universe consists of the positive integers 1, 2, 3 and 4, then $\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ and so the truth value of $\forall x\, P(x) = \text{T} \wedge \text{T} \wedge \text{T} \wedge \text{F} = \text{F}$.

   [**Note** We have so far applied universal quantification to propositional functions of a single variable only. Universal quantification (and also existential quantification, that is discussed below) can be applied to compound propositional functions such as $P(x) \wedge Q(x)$, $P(x) \to Q(x)$, $\neg P(x)$, $P(x) \vee \neg Q(x)$ etc. and to propositional functions of many variables, as given in the following examples.]
4. Let $P(x) \equiv x$ is an integer and $Q(x) \equiv x$ is either positive or negative.
   Then $P(x) \to Q(x)$ is a compound propositional function.
   Obviously $\forall x(P(x) \to Q(x))$, where the universe of discourse consists of integers.
5. Let $P(x, y)$: $x$ is taller than $y$.
   If $x$ is taller than $y$, then $y$ is not taller than $x$.
   viz. $P(x, y) \to \neg P(y, x)$

as

$$\forall x \forall y\, (P(x, y) \rightarrow \neg P(y, x))$$

## EXISTENTIAL QUANTIFIER

The existential quantification of $P(x)$ is the proposition.
"There exists at least one $x$ (or an $x$) such that $P(x)$ is true" and is denoted by the notation $\exists x P(x)$. The symbol $\exists$ is called the *existential quantifier.*

The proposition $\exists x P(x)$ is read as "For some $x$, $P(x)$".

### Examples

1. When $P(x)$ denotes the propositional function "$x > 3$", the truth value of $\exists x P(x)$ is T, where the universe of discourse consists of all real numbers, since "$x > 3$" is true for $x = 4$.

   **Note:** When the elements of the universe of discourse is finitely many, viz., consists of $x_1, x_2, \ldots x_n$, then $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$, since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

2. When $P(x)$ denotes "$x^2 > 10$", where the universe of discourse consists of the positive integers not exceeding 4, then the truth value of $\exists x P(x)$ is T, since $P(1) \vee P(2) \vee P(3) \vee P(4)$ is true as $P(4)$ [viz., $4^2 > 10$] is true.

## NEGATION OF A QUANTIFIED EXPRESSION

If $P(x)$ is the statement "$x$ has studied computer programming", then $\forall x P(x)$ means that "every student (in the class) has studied computer programming". The negation of this statement is "It is not the case that every student in the class has studied computer programming" or equivalently "There is a student in the class who has not studied computer programming" which is denoted by $\exists x \neg P(x)$. Thus we see that $\neg \forall x P(x) \equiv \exists x \neg P(x)$.

Similarly, $\exists x P(x)$ means that "there is a student in the class who has studied computer programming "The negation of this statement is "Every student in this class has not studied computer programming", which is denoted by $\forall x \neg P(x)$. Thus we get

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Further we note that $\neg \forall x P(x)$ is true, when there is an $x$ for which $P(x)$ is false and false when $P(x)$ is true for every $x$, since

$$\begin{aligned} \neg \forall x P(x) &\equiv \exists x \neg P(x) \\ &\equiv \neg P(x_1) \vee \neg P(x_2) \ldots \vee \neg P(x_n) \end{aligned}$$

$\neg \exists x P(x)$ is true, when $P(x)$ is false for every $x$ and false when there is an $x$ for which $P(x)$ is true,

since $$\begin{aligned} \neg \exists x P(x) &\equiv \forall x \neg P(x) \\ &\equiv \neg P(x_1) \wedge \neg P(x_2) \ldots \wedge \neg P(x_n) \end{aligned}$$

## NESTED (MORE THAN ONE) QUANTIFIERS

There are situations when quantifiers occur in combinations in respect of 1-place or $n$-place predicate formulas (i.e., propositional functions containing 1 or $n$ variables). For example let us consider a 2-place predicate formula $P(x, y)$.

Now $\quad \forall x \forall y\, P(x, y) \equiv \forall x[\forall y\, P(x, y)]$

$$\equiv \forall y[\forall x\, P(x, y)] \qquad (1)$$

and $\quad \exists x \exists y\, P(x, y) \equiv \exists x[\exists y\, P(x, y)] \equiv \exists y[\exists x\, P(x, y)] \qquad (2)$

From the meaning of quantifiers and by (1) and (2) the following simplifications hold good:

$$\forall x \forall y\, P(x, y) \Rightarrow (\exists y) \forall x\, P(x, y) \Rightarrow \forall x \exists y\, P(x, y)$$

$$\forall y \forall x\, P(x, y) \Rightarrow (\exists x) \forall y\, P(x, y) \Rightarrow \forall y \exists x\, P(x, y)$$

**Note** The negation of multiply quantified predicate formulas may be obtained by applying the rules for negation (given earlier) from left to right.

Thus $\quad \neg[\forall x \exists y\, P(x, y)] \equiv \exists x[\neg \exists y\, P(x, y)]$

$$\equiv \exists x\, \forall y[\neg P(x, y)]$$

## FREE AND BOUND VARIABLES

When a quantifier is used on a variable $x$ or when we have to assign a value to this variable to get a proposition, the occurrence of the variable is said to be *bound* or the variable is said to be a bound variable. An occurrence of a variable that is not bound by a quantifier or that is set equal to a particular value is said to be *free*.

The part of the logical expression or predicate formula to which a quantifier is applied is called the *scope* of the quantifier.

### Examples

**Table 1.36**

| *Predicate formula* | *Bound variable and scope* | *Free variable* |
|---|---|---|
| 1. $\forall x\, P(x, y)$ | $x$; $P(x, y)$ | $y$ |
| 2. $\forall x\, (P(x) \to Q(x))$ | $x$; $P(x) \to Q(x)$ | — |
| 3. $\forall x\, (P(x) \to E(y)Q(x, y))$ | $x$; $P(x) \to E(y)Q(x, y)$<br>$y$; $Q(x, y)$ | — |
| 4. $\forall x\, (P(x) \wedge Q(x)) \vee \forall y\, R(y)$ | $x$; $P(x) \wedge Q(x)$<br>$y$; $R(y)$ | — |
| 5. $\exists x\, P(x) \wedge Q(x)$ | First $x$; $P(x)$ | Second $x$ |

## VALID FORMULAS AND EQUIVALENCES

Let $A$ and $B$ be any two predicate formulas defined over a common universe of discourse $E$. When each of the variables appearing in $A$ and $B$ is replaced by any element (object name) of the universe $E$, if the resulting statements have the same truth values, then $A$ and $B$ are said to be *equivalent* to each other over

$E$ and denoted as $A \equiv B$ or $A \Leftrightarrow B$ over $E$. If $E$ is arbitrary, we simply say tha $A$ and $B$ are equivalent and denote it as $A \equiv B$ or $A \Leftrightarrow B$.

Generally, logically valid formulas in predicate calculus can be obtaine from tautologies of propositional calculus by replacing primary proposition (elementary statements) such as $p, q, r$ by propositional functions.

For example, $p \vee \neg p \equiv T$ and $(p \to q) \leftrightarrow (\neg p \vee q) \equiv T$ are tautologies i statement calculus.

If we replace $p$ by $\forall R(x)$ and $q$ by $\exists x\, S(x)$ in the above, we get the followin valid formulas in predicate calculus.

$$(\forall x\, R(x)) \vee (\neg \forall x\, R(x)) \equiv T$$

$$(\forall\, x\, R(x) \to \exists x\, S(x)) \leftrightarrow ((\neg \forall\, x\, R(x)) \vee \exists x\, S(x)) \equiv T$$

More generally, all the implications and equivalences of the statement calculu can also be considered as implications and equivalences of the predicate calculu if we replace elementary statements by primary predicate formulas. For example

from $\neg\neg p \Leftrightarrow p$, we get $\neg\neg P(x) \equiv P(x)$ (1

from $p \wedge q \equiv q \wedge p$, we get $P(x) \wedge Q(x, y) \equiv Q(x, y) \wedge P(x)$ (2

from $p \to q \equiv \neg p \vee q$, we get $P(x) \to Q(x) \equiv \neg P(x) \vee Q(x)$ (3

(1), (2) and (3) are some examples for valid formulas in predicate calculus.

Apart from the types of valid formulas given above, there are other vali formulas also which involve quantifiers. Such valid formulas are obtained b using the inference theory of predicate logic, discussed below:

## INFERENCE THEORY OF PREDICATE CALCULUS

Derivations of formal proof in predicate calculus are done mostly in the sam way as in statement calculus, using implications and equivalences, provide that the statement formulas are replaced by predicate formulas. Also the thre basic rules P, T and CP of Inference theory used in statement calculus can als be used in predicate calculus. Moreover, the indirect method of proof can als be used in predicate calculus.

Apart from the above rules of inference, we require certain additional rules to deal with predicate formulas involving quantifiers. If it becomes necessary to eliminate quantifiers during the course of derivation, we require two *rules o specification*, called US and ES rules. Once the quantifiers are eliminated, the derivation is similar to that in statement calculus. If it becomes necessary to quantify the desired conclusion, we require two *rules of generalisation*, called UG and EG rules.

**Rule US** *Universal Specification* is the rule of inference which states that one can conclude that $P(c)$ is true, if $\forall\, x\, P(x)$ is true, where $c$ is an arbitrary member of the universe of discourse. This rule is also called the *universal instantiation*.

**Rule ES** *Existential Specification* is the rule which allows us to conclude that $P(c)$ is true, if $\exists x\, P(x)$ is true, where $c$ is not an arbitrary member of the universe, but one for which $P(c)$ is true. Usually we will not know what ... but know that it exists. Since it exists, we may call it $c$. This rule is also called the existential instantiation.

**Rule UG** *Universal Generalisation* is the rule which states that $\forall x P(x)$ is true, if $P(c)$ is true, where $c$ is an arbitrary member (not a specific member) of the universe of discourse.

**Rule EG** *Existential Generalisation* is the rule that is used to conclude that $\exists x P(x)$ is true when $P(c)$ is true, where $c$ is a particular member of the universe of discourse.

## Examples

1. Let us consider the following "Famous Socrates argument" which is given by:

   All men are mortal.
   Socrates is a man.
   Therefore Socrates is a mortal.

   Let us use the notations H($x$): $x$ is a man
   M($x$): $x$ is a mortal
   $s$: Socrates

   With these symbolic notations, the problem becomes

   $$\forall x(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$$

   The derivation of the proof is as follows:

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\forall x (H(x) \rightarrow M(x))$ | P |
| 2. | $H(s) \rightarrow M(s)$ | US, 2 |
| 3. | $H(s)$ | P |
| 4. | $M(s)$ | T, 2, 3, Modus ponens |

2. Application of any of US, ES, UG and EG rules wrongly may lead to a false conclusion from a true premise as in the following example

   Let D($u$, $v$): $u$ is divisible by $v$, where the universe of discourse is (5, 6, 10, 11).

   Then $\exists u\, D(u, 5)$ is true, since D(5, 5) and D(10, 5) are true.

   But $\forall u\, D(u, 5)$ is false, since D(6, 5) and D(11, 5) are false.

   We now give the following derivation:

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\exists u\, D(u, 5)$ | P |
| 2. | $D(c, 5)$ | ES, 1 |
| 3. | $\forall x\, D(x, 5)$ | UG, 2 |

**Note** In step (3), UG has been applied wrongly, since $c$ is not an arbitrary member in step (2), as $c$(= 5 or 10) is only a specific member of the given universe of discourse.

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**Example 1.1** Find whether the conclusion $C$ follows from the premises $H_1$, $H_2$, $H_3$ in the following cases, using truth table technique:

(i) $H_1$: $\neg p$, $H_2$: $p \vee q$, $C$: $p \wedge q$

(ii) $H_1$: $p \vee q$, $H_2$: $p \rightarrow r$, $H_3$: $q \rightarrow r$, $C$: $r$

(i) **Table 1.37**

| $p$ | $q$ | $H_1 \equiv \neg p$ | $H_2 \equiv p \vee q$ | $H_1 \wedge H_2$ | $C \equiv p \wedge q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | F | T | F | F |
| F | T | T | T | T | F |
| F | F | T | F | F | F |

$H_1$ and $H_2$ and hence $H_1 \wedge H_2$ are true in the third row, in which $C$ is false.
Hence $C$ does no follow from $H_1$ and $H_2$.

(ii) **Table 1.38**

| $p$ | $q$ | $r$ | $H_1(p \vee q)$ | $H_2(p \rightarrow r)$ | $H_3(q \rightarrow r)$ | $H_1 \wedge H_2 \wedge H_3$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | F | F |
| T | F | T | T | T | T | T |
| T | F | F | T | F | T | F |
| F | T | T | T | T | T | T |
| F | T | F | T | T | F | F |
| F | F | T | F | T | T | F |
| F | F | F | F | T | T | F |

$H_1$, $H_2$, $H_3$ and hence $H_1 \wedge H_2 \wedge H_3$ are true in the first, third and fifth rows in which $r$ is also true.
Hence $C$ follows from $H_1$, $H_2$ and $H_3$.

**Example 1.2** Show that $(t \wedge s)$ can be derived from the premises $p \rightarrow q$, $q \rightarrow \neg r$, $r$, $p \vee (t \wedge s)$.

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $p \rightarrow q$ | $P$ |
| 2. | $q \rightarrow \neg r$ | $P$ |
| 3. | $p \rightarrow \neg r$ | $T$, 1, 2 and Hypothetical syllogism |
| 4. | $r \rightarrow \neg p$ | $T$, 3 and $p \rightarrow q \equiv \neg q \rightarrow \neg p$ |
| 5. | $r$ | $P$ |
| 6. | $\neg p$ | $T$, 4, 5 and Modus ponens |
| 7. | $p \vee (t \wedge s)$ | $P$ |
| 8. | $t \wedge s$ | $T$, 6, 7 and Disjunctive syllogism |

**Example 1.3** Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \rightarrow \urcorner r$, $\urcorner r \rightarrow (s \wedge \urcorner t)$ and $(s \wedge \urcorner t) \rightarrow (a \vee b)$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $(p \vee q) \rightarrow \urcorner r$ | $P$ |
| 2. | $\urcorner r \rightarrow (s \wedge \urcorner t)$ | $P$ |
| 3. | $(p \vee q) \rightarrow (s \wedge \urcorner t)$ | $T$, 1, 2 and hypothetical syllogism |
| 4. | $p \vee q$ | $P$ |
| 5. | $s \wedge \urcorner t$ | $T$, 3, 4 and modus ponens |
| 6. | $(s \wedge \urcorner t) \rightarrow (a \vee b)$ | $P$ |
| 7. | $a \vee b$ | $T$, 5, 6 and Modus ponens |

**Example 1.4** Show that $(p \rightarrow q) \wedge (r \rightarrow s)$, $(q \rightarrow t) \wedge (s \rightarrow u)$, $\urcorner(t \wedge u)$ and $(p \rightarrow r) \Rightarrow \urcorner p$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $(p \rightarrow q) \wedge (r \rightarrow s)$ | $P$ |
| 2. | $p \rightarrow q$ | $T$, 1 and simplification |
| 3. | $r \rightarrow s$ | $T$, 1 and simplification |
| 4. | $(q \rightarrow t) \wedge (s \rightarrow u)$ | $P$ |
| 5. | $q \rightarrow t$ | $T$, 4 and simplification |
| 6. | $s \rightarrow u$ | $T$, 4 and simplification |
| 7. | $p \rightarrow t$ | $T$, 2, 5 and hypothetical syllogism |
| 8. | $r \rightarrow u$ | $T$, 3, 6 and hypothetical syllogism |
| 9. | $p \rightarrow r$ | $P$ |
| 10. | $p \rightarrow u$ | $T$, 8, 9 and hypothetical syllogism |
| 11. | $\urcorner t \rightarrow \urcorner p$ | $T$ and 7 |
| 12. | $\urcorner u \rightarrow \urcorner p$ | $T$ and 10 |
| 13. | $(\urcorner t \vee \urcorner u) \rightarrow \urcorner p$ | $T$, 11, 12, and $(a \rightarrow b), (c \rightarrow b) \Rightarrow (a \vee c) \rightarrow b$ |
| 14. | $\urcorner(t \wedge u) \rightarrow \urcorner p$ | $T$, 13 and De Morgan's law |
| 15. | $\urcorner(t \wedge u)$ | $P$ |
| 16. | $\urcorner p$ | $T$, 14, 15 and modus ponens. |

**Example 1.5** Show that $(a \rightarrow b) \wedge (a \rightarrow c)$, $\urcorner(b \wedge c)$, $(d \vee a) \Rightarrow d$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $(a \rightarrow b) \wedge (a \rightarrow c)$ | $P$ |
| 2. | $a \rightarrow b$ | $T$, 1 and simplification |
| 3. | $a \rightarrow c$ | $T$, 1 and simplification |
| 4. | $\urcorner b \rightarrow \urcorner a$ | $T$, 2 and contrapositive |
| 5. | $\urcorner c \rightarrow \urcorner a$ | $T$, 3 and contrapositive |
| 6. | $(\urcorner b \vee \urcorner c) \rightarrow \urcorner a$ | $T$, 4 and 5 |
| 7. | $\urcorner(b \wedge c) \rightarrow \urcorner a$ | $T$ and De Morgan's law |
| 8. | $\urcorner(b \wedge c)$ | $P$ |
| 9. | $\urcorner a$ | $T$, 7, 8 and modus ponens |
| 10. | $d \vee a$ | $P$ |

| | | |
|---|---|---|
| 11. | $(d \vee a) \wedge \neg a$ | $T$, 9, 10 and conjunction |
| 12. | $(d \wedge \neg a) \vee (a \wedge \neg a)$ | $T$, 11 and distributive law |
| 13. | $(d \wedge \neg a) \vee F$ | $T$, 12 and negation law |
| 14. | $d \wedge \neg a$ | $T$, 13 and identity law |
| 15. | $d$ | $T$, 14 and simplification |

**Example 1.6** Give a direct proof for the implication $p \to (q \to s)$, ( $p)$, $q \Rightarrow (r \to s)$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $\neg r \vee p$ | $P$ |
| 2. | $r \to p$ | $T$, 1 and equivalence of (1) |
| 3. | $p \to (q \to s)$ | $P$ |
| 4. | $r \to (q \to s)$ | $T$, 2, 3 and hypothetical syllogism. |
| 5. | $\neg r \vee (\neg q \vee s)$ | $T$, 4 and equivalence of (4) |
| 6. | $q$ | $P$ |
| 7. | $q \wedge (\neg r \vee \neg q \vee s)$ | $T$, 5, 6 and conjunction |
| 8. | $q \wedge (\neg r \vee s)$ | $T$, 7, 8 and negation and domination |
| 9. | $\neg r \vee s$ | $T$, 8 and simplification |
| 10. | $r \to s$ | $T$, 9 and equivalence of (9) |

**Example 1.7** Derive $p \to (q \to s)$ using the CP-rule (if necessary) f the premises $p \to (q \to r)$ and $q \to (r \to s)$.

We shall assume $p$ as an additional premise. Using $p$ and the two g premises, we will derive $(q \to s)$. Then, by CP-rule, $p \to (q \to s)$ is deeme have been derived from the two given premises.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $p$ | $P$(additional) |
| 2. | $p \to (q \to r)$ | $P$ |
| 3. | $q \to r$ | $T$, 1, 2 and modus ponens |
| 4. | $\neg q \vee r$ | $T$, 3 and equivalence of (3) |
| 5. | $q \to (r \to s)$ | $P$ |
| 6. | $\neg q \vee (r \to s)$ | $T$, 5 and equivalence of (5) |
| 7. | $\neg q \vee (r \wedge (r \to s))$ | $T$, 4, 6 and distributive law |
| 8. | $\neg q \vee s$ | $T$, 7 and modus ponens |
| 9. | $q \to s$ | $T$, 8 and equivalence of (8) |
| 10. | $p \to (q \to s)$ | $T$, 9 and CP-rule |

**Example 1.8** Use the indirect method to show that $r \to \neg q$, $r \vee s$, $s$ $\neg q$, $p \to q \Rightarrow \neg p$.

To use the indirect method, we will include $\neg\neg p \equiv p$ as an additio premise and prove a contradiction.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $p$ | $P$ (additional) |
| 2. | $p \to q$ | $P$ |

| | | |
|---|---|---|
| 3. | $q$ | $T$, 1, 2 and modus ponens |
| 4. | $r \to \neg q$ | $P$ |
| 5. | $s \to \neg q$ | $P$ |
| 6. | $(r \vee s) \to \neg q$ | $T$, 4, 5 and equivalence |
| 7. | $r \vee s$ | $P$ |
| 8. | $\neg q$ | $T$, 6, 7 and modus ponens |
| 9. | $q \wedge \neg q$ | $T$, 3, 8 and conjunction |
| 10. | $F$ | $T$, 9 and negation law |

**Example 1.9** Show that $b$ can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, $d$, by the indirect method.

Let us include $\neg b$ as an additional premise and prove a contradiction.

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $a \to b$ | $P$ |
| 2. | $c \to b$ | $P$ |
| 3. | $(a \vee c) \to b$ | $T$, 1, 2 and equivalence |
| 4. | $d \to (a \vee c)$ | $P$ |
| 5. | $d \to b$ | $T$, 3, 4 and hypothetical syllogism |
| 6. | $d$ | $P$ |
| 7. | $b$ | $T$, 5, 6 and modus ponens |
| 8. | $\neg b$ | $P$ (additional) |
| 9. | $b \wedge \neg b$ | $T$, 7, 8 and conjunction |
| 10. | $F$ | $T$, 9 and negation law. |

**Example 1.10** Using indirect method of proof, derive $p \to \neg s$ from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, $p$.

Let us include $\neg(p \to \neg s)$ as an additional premise and prove a contradiction.

Now $\neg(p \to \neg s) = \neg(\neg p \vee \neg s) = p \wedge s$

Hence the additional premise to be introduced may be taken as $p \wedge s$.

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $q \to (q \vee r)$ | $P$ |
| 2. | $p$ | $P$ |
| 3. | $q \vee r$ | $T$, 1, 2 and modus ponens |
| 4. | $p \wedge s$ | $P$(additional) |
| 5. | $s$ | $T$, 4 and simplification |
| 6. | $s \to \neg r$ | $P$ |
| 7. | $\neg r$ | $T$, 5, 6 and modus ponens |
| 8. | $q$ | $T$, 3, 7 and disjunctive syllogism |
| 9. | $q \to \neg p$ | $P$ |
| 10. | $\neg p$ | $T$, 8, 9 and modus ponens |
| 11. | $p \wedge \neg p$ | $T$, 2, 10 and conjunction |
| 12. | $F$ | $T$, 11 and negation law |

**Example 1.11** Prove that the premises $p \to q$, $q \to r$, $s \to \neg r$ and $p \wedge s$ are inconsistent.

If we derive a contradiction by using the given premises, it means that they are inconsistent.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $p \to q$ | $P$ |
| 2. | $q \to r$ | $P$ |
| 3. | $p \to r$ | $T$, 1, 2 and hypothetical syllogism |
| 4. | $s \to \neg r$ | $P$ |
| 5. | $r \to \neg s$ | $T$, 4 and contrapositive |
| 6. | $q \to \neg s$ | $T$, 2, 5 and hypothetical syllogism |
| 7. | $\neg q \vee \neg s$ | $T$, 6 and equivalence of (6) |
| 8. | $\neg(q \wedge s)$ | $T$, 7 and De Morgan's law |
| 9. | $q \wedge s$ | $P$ |
| 10. | $(q \wedge s) \wedge \neg(q \wedge s)$ | $T$, 8, 9 and conjunction |
| 11. | $F$ | $T$, 10 and negation law |

Hence the given premises are inconsistent

**Example 1.12** Prove that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $a \wedge d$ | $P$ |
| 2. | $a$ | $T$, 1 and simplification |
| 3. | $d$ | $T$, 1 and simplification |
| 4. | $a \to (b \to c)$ | $P$ |
| 5. | $b \to c$ | $T$, 2, 4 and modus ponens |
| 6. | $\neg b \vee c$ | $T$, 5 and equivalence of (5) |
| 7. | $d \to (b \wedge \neg c)$ | $P$ |
| 8. | $\neg(b \wedge \neg c) \to \neg d$ | $T$, 7 and contrapositive |
| 9. | $\neg b \vee c \to \neg d$ | $T$, 8 and equivalence |
| 10. | $\neg d$ | $T$, 6, 9 and modus ponens |
| 11. | $d \wedge \neg d$ | $T$, 3, 10 and conjunction |
| 12. | F | $T$, 11 and negation law |

Hence the given premises are inconsistent.

**Example 1.13** Construct an argument to show that the following premises imply the conclusion " it rained".

"If it does not rain or if there is no traffic dislocation, then the sports day will be held and the cultural programme will go on"; "If the sports day is held, the trophy will be awarded" and "the trophy was not awarded".

Let us symbolise the statement as follows:

$p$: It rains.

$q$: There is traffic dislocation.

$r$: Sports day will be held.
$s$: Cultural programme will go on.
$t$: The trophy will be awarded.
Then we have to prove that

$$\neg p \vee \neg q \rightarrow r \wedge s,\ r \rightarrow t,\ \neg t \Rightarrow p$$

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $\neg p \vee \neg q \rightarrow r \wedge s$ | $P$ |
| 2. | $(\neg p \rightarrow (r \wedge s)) \wedge (\neg q \rightarrow (r \wedge s))$ | $T$, 1 and the equivalence $(a \vee b) \rightarrow c \equiv (a \rightarrow c) \wedge (b \rightarrow c)$ |
| 3. | $\neg(r \wedge s) \rightarrow p$ | $T$, 2 and contrapositive of (2) |
| 4. | $r \rightarrow t$ | $P$ |
| 5. | $\neg t \rightarrow \neg r$ | $T$, 4 and contrapositive of (4) |
| 6. | $\neg t$ | $P$ |
| 7. | $\neg r$ | $T$, 5, 6 and modus ponens |
| 8. | $\neg r \vee \neg s$ | $T$, 7 and addition |
| 9. | $\neg(r \wedge s)$ | $T$, 8 and De Morgan's law |
| 10. | $p$ | $T$, 3, 9 and modus ponens |

**Example 1.14** Show that the following set of premises is inconsistent:
If Rama gets his degree, he will go for a job.
If he goes for a job, he will get married soon.
If he goes for higher study, he will not get married.
Rama gets his degree and goes for higher study.
Let the statements be symbolised as follows:

$p$: Rama gets his degree.
$q$: He will go for a job.
$r$: He will get married soon.
$s$: He goes for higher study.

Then we have to prove that

$$p \rightarrow q,\ q \rightarrow r,\ s \rightarrow \neg r,\ p \wedge s \text{ are inconsistent}$$

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $p \rightarrow q$ | $P$ |
| 2. | $q \rightarrow r$ | $P$ |
| 3. | $p \rightarrow r$ | $T$, 1, 2 and hypothetical syllogism |
| 4. | $p \rightarrow s$ | $P$ |
| 5. | $p$ | $T$, 4 and simplification |
| 6. | $s$ | $T$, 4 and simplification |
| 7. | $s \rightarrow \neg r$ | $P$ |
| 8. | $\neg r$ | $T$, 6, 7 and modus ponens |
| 9. | $r$ | $T$., 3, 5 and modus ponens |
| 10. | $r \wedge \neg r$ | $T$, 8, 9 and conjunction |
| 11. | $F$ | $T$, 10 and negation law |

Hence the set of given premises is inconsistent.

**Example 1.15** If $L(x, y)$ symbolises the statement "$x$ loves $y$"; where tl universe of discourse for both $x$ and $y$ consists of all people in the worl translate the following English sentences into logical expressions:

(a) Every body loves $z$.
(b) Every body loves somebody.
(c) There is somebody whom everybody loves.
(d) Nobody loves everybody.
(e) There is somebody whom no one loves.

(a) $L(x, z)$ for all $x$. Hence $\forall x\, L(x, z)$
(b) $L(x, y)$ is true for all $x$ and some $y$. Hence $\forall \exists y\, L(x, y)$
(c) Eventhough, (c) is the same as (b), the stress is on the existence ( somebody ($y$) whom all $x$ love.

$$\text{Hence } \exists y\, \forall x\, L(x, y)$$

(d) Nobody loves every body
i.e., There is not one who loves everybody
Hence $\neg \exists x\, \forall y\, L(x, y)$
$\equiv \forall x\, \neg \forall y\, L(x, y)$
$\equiv \forall x\, \exists y\, \neg L(x, y)$

(e) The sentence means that there is somebody whom every one does no love.
Hence $\neg \forall x\, \exists y\, L(x, y)$
$\equiv \exists x\, \neg \exists y\, L(x, y)$
$\equiv \exists x\, \forall y\, \neg L(x, y)$

**Example 1.16** Express each of the following statements using mathematica and logical operations, predicates and quantifiers, where the universe of discours consists of all computer science students/mathematics courses.

(a) Every computer science student needs a course in mathematics.
(b) There is a student in this class who owns a personal computer.
(c) Every student in this class has taken at least one mathematics course.
(d) There is a student in this class who has taken at least one mathematics course.

(a) Let $M(x) \equiv$ '$x$ needs a course in mathematics', where the universe of discourse consists of all computer science students.
Then $\forall x\, M(x)$.
(b) Let $P(x) \equiv$ '$x$ owns a personal computer', where the universe consists of all students in this class.
Then $\exists x\, P(x)$
(c) Let $Q(x, y) \equiv$ '$x$ has taken $y$', where the universe of $x$ consists of all students in this class and that of $y$ consists of all mathematics courses.
Then $\forall x\, \exists y\, Q(x, y)$
(d) Using the same assumptions as in (c), we have $\exists x\, \exists y\, Q(x, y)$.

**Example 1.17** Express the negations of the following statements using quantifiers and in English:

(a) If the teacher is absent, then some students do not keep quiet.
(b) All the students keep quiet and the teacher is present.
(c) Some of the students do not keep quiet or the teacher is absent.
(d) No one has done every problem in the exercise.

(a) Let T represent the presence of the teacher and $Q(x)$ represent "$x$ keeps quiet".

Then the given statement is:

$$\neg T \to \exists x\, Q(x) \equiv \neg T \to \neg \forall x\, Q(x)$$
$$\equiv T \vee \neg \forall x\, Q(x)$$

$\therefore$ Negation of the given statement is

$$\neg (T \vee \neg \forall x\, Q(x))$$
$$\equiv \neg T \wedge \forall x\, Q(x)$$

i.e., the teacher is absent and all the students keep quiet.

(b) The given statement is:

$$\forall x\, Q(x) \wedge T$$

$\therefore$ The negation of the given statement is

$$\neg (\forall x\, Q(x) \wedge T) \equiv \neg \forall x\, Q(x) \vee \neg T$$
$$\equiv \exists x \neg\, Q(x) \vee \neg T$$

i.e., some students do not keep quiet or the teacher is absent.

(c) The given statement is:

$$\exists x \neg Q(x) \vee \neg T \equiv \neg\, \forall x\, Q(x) \vee \neg T$$

$\therefore$ The negation of the given statement is

$$\neg (\neg \forall x\, Q(x) \vee \neg T)$$
$$\equiv \forall x\, Q(x) \wedge T$$

i.e., All the students keep quiet and the teacher is present.

(d) Let $D(x, y)$ represent "$x$ has done problem $y$".

The given statement is

$$(\neg \exists x)(\forall y\, D(x, y)) \qquad (1)$$

$\therefore$ The negation of the given statement is

$$(\neg \neg \exists (x))(\forall y\, D(x, y))$$
$$\equiv \exists x\, \forall y\, D(x, y) \qquad (2)$$

i.e., some one has done every problem in the exercise.

*Aliter:*

(1) can be re-written as

$$\forall x \neg \forall y\, D(x, y)$$
$$\equiv \forall x\, \exists y \neg D(x, y)$$

$\therefore$ The negative of this statement is

$$\neg \forall x\, \exists y \neg\, D(x, y) \equiv \exists x \neg \exists y \neg D(x, y)$$
$$\equiv \exists x\, \forall y\, D(x, y),$$

which is the same as (2).

**Example 1.18** Show that the premises "one student in this class knows how to write programs in JAVA" and "Everyone who knows how to write programs in JAVA can get a high-paying job" imply the conclusion "Someone in this class can get a high-paying job".

Let $C(x)$ represent "$x$ is in this class" $J(x)$ represent "$x$ knows JAVA programming" and $H(x)$ represent "$x$ can get a high paying job".

Then the given premises are $\exists x\ (C(x) \wedge J(x))$ and $\forall x\ (J(x) \rightarrow H(x))$. The conclusion is $\exists x(C(x) \wedge H(x))$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $\exists x(C\ (x) \wedge J(x))$ | $P$ |
| 2. | $C(a) \wedge J(a)$ | ES and 1 |
| 3. | $C(a)$ | $T$, 2 and simplification |
| 4. | $J(a)$ | $T$, 2 and simplification |
| 5. | $\forall x\ (J(x) \rightarrow H(x))$ | $P$ |
| 6. | $J(a) \rightarrow H(a)$ | US and 5 |
| 7. | $H(a)$ | T, 4, 6 and modus ponens |
| 8. | $C(a) \wedge H(a)$ | T, 3, 7 and conjunction |
| 9. | $\exists x\ (C(x) \wedge H(x))$ | EG and 8. |

**Example 1.19** Show, by indirect method of proof, that $\forall x\ (p(x) \vee q(x)) \Rightarrow (\forall x\ p(x)) \vee (\exists x\ q(x))$.

Let us assume that $\urcorner[(\forall x\ p(x)) \vee (\exists x\ q(x))]$ as an additional premise and prove a contradiction.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $\urcorner[(\forall x\ p(x)) \vee (\exists x\ q(x))]$ | $P$(additional) |
| 2. | $\urcorner(\forall x\ p(x)) \wedge \urcorner(\exists x\ q(x))$ | $T$, 1, De Morgan's law |
| 3. | $\urcorner(\forall x\ p(x))$ | $T$, 2, simplification |
| 4. | $\urcorner(\exists x\ q(x))$ | $T$, 2, simplification |
| 5. | $\exists x\ \urcorner p(x)$ | $T$, 3 and negation |
| 6. | $\forall x\ \urcorner q(x)$ | $T$, 4 and negation |
| 7. | $\urcorner p(a)$ | ES and 5 |
| 8. | $\urcorner q(a)$ | US and 6 |
| 9. | $\urcorner p(a) \wedge \urcorner q(a)$ | $T$, 7, 8 and conjunction |
| 10. | $\urcorner(p(a) \vee q(a))$ | $T$, 9 and De Morgan's law |
| 11. | $\forall x\ (p(x) \vee q(x))$ | $P$ |
| 12. | $p(a) \vee q(a)$ | US and 11 |
| 13. | $(p(a) \vee q(a)) \wedge \urcorner(p(a) \wedge q(a))$ | $T$, 10, 12 and conjunction |
| 14. | $F$ | $T$, 13 |

**Example 1.20** Prove that $\forall x\ (P(x) \rightarrow (Q(y) \wedge R(x))), \exists x\ P(x) \Rightarrow Q(y) \wedge \exists x\ (P(x) \wedge R(x))$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $\forall x\ (P(x) \rightarrow (Q(y) \wedge R(x)))$ | $P$ |
| 2. | $P(z) \rightarrow (Q(y) \wedge R(z))$ | US and 1 |

| | | |
|---|---|---|
| 3. | $\exists x\, P(x)$ | $P$ |
| 4. | $P(z)$ | ES and 3 |
| 5. | $Q(y) \wedge R(z)$ | $T$, 2, 4 and modus ponens |
| 6. | $Q(y)$ | $T$, 5 and simplification |
| 7. | $R(z)$ | $T$, 5 and simplification |
| 8. | $P(z) \wedge R(z)$ | $T$, 4, 7 and conjunction |
| 9. | $\exists x\, (P(x) \wedge R(x))$ | EG and 8 |
| 10. | $Q(y) \wedge \exists x\, (P(x) \wedge R(x)$ | $T$, 6, 10 and conjunction |

**Example 1.21** Show that the conclusion $\forall x(P(x) \rightarrow \urcorner Q(x))$ follows from the premises

$$\exists x\, (P(x) \wedge Q(x)) \rightarrow \forall y\, (R(y) \rightarrow S(y)) \text{ and } \exists y\, (R(y) \wedge \urcorner S(y)).$$

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\exists y\, (R(y) \wedge \urcorner S(y))$ | P |
| 2. | $R(a) \wedge \urcorner S(a)$ | ES and 1 |
| 3. | $\urcorner(R(a) \rightarrow S(a))$ | T, 2 and equivalence |
| 4. | $\exists y\, (\urcorner(R(y) \rightarrow S(y)))$ | EG and 3 |
| 5. | $7\forall y\, (R(y) \rightarrow S(y))$ | $T$, 4 and negation equivalence |
| 6. | $\exists x\, (P(x) \wedge Q(x)) \rightarrow \forall y$ $(R(y) \rightarrow S(y))$ | $P$ |
| 7. | $\urcorner\exists x\, (P(x) \wedge Q(x))$ | $T$, 5, 6 and modus tollens |
| 8. | $\forall x\, \urcorner(P(x) \wedge Q(x))$ | $T$, 7 and negative equivalence |
| 9. | $\urcorner(P(b) \wedge Q(b))$ | US and 8 |
| 10. | $\urcorner P(b) \vee \urcorner Q(b)$ | T, 9 and De Morgan's law |
| 11. | $P(b) \rightarrow \urcorner Q(b)$ | T, 10 and equivalence |
| 12. | $\forall x\, (P(x) \rightarrow \urcorner Q(x))$ | UG and 11. |

**Example 1.22** Prove the derivation

$$\exists x\, P(x) \rightarrow \forall x\, ((P(x) \vee Q(x)) \rightarrow R(x)),$$
$$\exists x\, P(x), \exists x\, Q(x) \Rightarrow \exists x \exists y\, (R(x) \wedge R(y))$$

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\exists x\, P(x) \rightarrow \forall x((P(x)$ $\vee Q(x)) \rightarrow R(x)$ | $P$ |
| 2. | $P(a) \rightarrow (P(b) \vee Q(b)$ $\rightarrow R(b))$ | ES, US and 1 |
| 3. | $\exists x\, P(x)$ | $P$ |
| 4. | $P(a)$ | ES and 3 |
| 5. | $(P(b) \vee Q(b)) \rightarrow R(b)$ | T, 2, 4 and modus ponens |
| 6. | $\exists x\, Q(x)$ | $P$ |
| 7. | $Q(b)$ | ES and 6 |
| 8. | $P(b) \vee Q(b)$ | T, 7 and addition |
| 9. | $R(b)$ | T, 5, 8 and modus ponens |
| 10. | $\exists x\, R(x)$ | EG and 9 |
| 11. | $R(a)$ | ES and 9 |

| | | |
|---|---|---|
| 12. | $R(a) \wedge R(b)$ | T, 9, 11 and conjunction |
| 13. | $\exists y\,(R(a) \wedge R(y))$ | EG and 12 |
| 14. | $\exists x\,\exists y\,(R(x) \wedge R(y))$ | EG and 13. |

**Example 1.23** Prove the implication

$$\forall x\,(P(x) \to Q(x)),\ \forall x\,(R(x) \to \neg Q(x)) \Rightarrow \forall x\,(R(x) \to \neg P(x)).$$

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\forall x\,(P(x) \to Q(x))$ | $P$ |
| 2. | $P(a) \to Q(a)$ | US and 1 |
| 3. | $\forall x\,(R(x) \to \neg Q(x))$ | $P$ |
| 4. | $R(a) \to \neg Q(a)$ | US and 2 |
| 5. | $Q(a) \to \neg R(a)$ | $T$, 4 and equivalence |
| 6. | $P(a) \to \neg R(a)$ | $T$, 2, 5 and hypothetical syllogism |
| 7. | $R(a) \to \neg P(a)$ | $T$, 6 and equivalence |
| 8. | $\forall x\, R(x) \to \neg P(x))$ | UG and 7 |

**Example 1.24** Use the indirect method to prove that the conclusion $\exists z\, Q(z)$ follows from the premises $\forall x\,(P(x) \to Q(x))$ and $\exists y\, P(y)$.

Let us assume the additional premise $\neg(\exists z\, Q(z))$ and prove a contradiction.

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\neg(\exists z\, Q(z))$ | $P$ (additional) |
| 2. | $\forall z\,(7Q(z))$ | $T$, 1 and negation equivalence |
| 3. | $\neg Q(a)$ | US and 2 |
| 4. | $\exists y\, P(y)$ | $P$ |
| 5. | $P(a)$ | ES and 4 |
| 6. | $P(a) \wedge \neg Q(a)$ | $T$, 3, 5 and conjunction |
| 7. | $\neg(\neg P(a) \vee Q(a)$ | $T$, 6 and equivalence |
| 8. | $\neg(P(a) \to Q(a))$ | $T$, 7 and equivalence |
| 9. | $\forall x\,(P(x) \to Q(x))$ | $P$ |
| 10. | $P(a) \to Q(a)$ | US and 9 |
| 11. | $(P(a) \to Q(a)) \wedge \neg(P(a) \to Q(a))$ | $T$, 8, 10 and conjunction |
| 12. | $F$ | $T$, 11 and negative law |

**Example 1.25** Show that $\forall x\,(P(x) \vee Q(x)) \Rightarrow \forall x\, P(x) \vee \exists x\, Q(x)$, using the indirect method

| *Step No.* | *Statement* | *Reason* |
|---|---|---|
| 1. | $\neg(\forall x\, P(x) \vee \exists x\, Q(x))$ | $P$ (additional) |
| 2. | $\neg(\forall x\, P(x) \wedge \neg(\exists x\, Q(x))$ | $T$, 1 and De Morgan's law |
| 3. | $\exists x\,(\neg P(x)) \wedge \forall x\,(\neg Q(x))$ | $T$, 2 and negation equivalence |
| 4. | $\exists x\,(\neg P(x))$ | $T$, 3 and simplification |
| 5. | $\forall x\,(\neg Q(x))$ | $T$, 3 and simplification |
| 6. | $\neg P(a)$ | ES and 4 |
| 7. | $\neg Q(a)$ | US and 5 |

## ⇒ Validity of Arguments:

- A finite sequence $A_1, A_2, \ldots, A_{n-1}, A_n$ of statements is called an argument.

- The final statement $A_n$ is called the conclu[sion] and the statements $A_1, A_2, \ldots, A_{n-1}$ are called the premises of the argument.

→ An argument $A_1, A_2, \ldots, A_{n-1}, A_n$ is called logically valid if the statement formula,

$$(A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_{n-1}) \rightarrow A_n$$

is a tautology.

→ Sometimes we write an argument in the following form,

$$\begin{array}{l} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{n-1} \\ \therefore A_n \end{array}$$

To test the logical validity of an argument written in a natural language, we first write each of premises & the conclusion with the help of statement letters and logical connectives.

Then we check whether the conjunction

$A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_{n-1}$

logically implies $A_n$. If it does ~~not~~ then the argument is logically valid, otherwise not.

Ex: consider the Argument:
If sheila solved seven problems correctly, then sheila obtained the grade A.

→ let,

p: sheila solved seven problems correctly.
q: sheila obtained the grade A.

so the argument takes the form

$$p \to q$$
$$p$$
$$\therefore q.$$

Now consider the truth table for $((p \to q) \wedge p) \to q$.

| $p$ | $q$ | $p \to q$ | $(p \to q) \wedge p$ | $((p \to q) \wedge p) \to q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

$((p \to q) \wedge p) \to q$ is a tautology, so it follows that the given argument is valid

⇒ Application:

Judicial court system:

In Judicial court work system you can't whatever the person say is True or not

Doctor says this.... $A_1$
police says this.... $A_2$
Some person's say this.... $A_{n-1}$
Using these premises, ↓
you can derive conclusion $A_n$

So this is how a judge deals with an argument whether doctor's argument, or a lawyer's arguments or other arguments are valid and judge take a decision that argument is valid or not!

So based on those premises, we are able to derive a conclusion 'c' then we can say the argument is valid otherwise it is not.

Note: we don't know whether all premises are T or F or conclusion is T or F but I'm saying using those premises, we can not derive a conclusion

If that is a case then the argument is Invalid otherwise it is valid.

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⇒ Some Valid argument forms:

1) Modus ponens:

consider the following argument form.

$p \rightarrow q$ If today is Tuesday then John will go to work
$p$ Today is Tuesday
$\therefore q$ ∴ John will go to work

eg. $((p \rightarrow q) \wedge p) \rightarrow q \Rightarrow$ tautology
Valid argument form
called Modus Ponens
Latin meaning is method of affirming.

2) Modus tollens:

consider the following argument form,

$p \rightarrow q$ If A is true then B is true
$\sim q$ B is not true
$\therefore \sim p$ ∴ A is not true

eg. $((p \rightarrow q) \wedge (\sim q)) \rightarrow (\sim p) \Rightarrow$ tautology
valid argument form
called Modus tollens
Latin meaning is method of denying

3) Disjuctive syllogisms

consider the following argument form,

a.) $p \vee q$     b.) $p \vee q$
$\sim p$ $[(p \vee q) \wedge \sim p] \rightarrow q$     $\sim q$
$\therefore q$     $\therefore p$

These two are valid argument forms.

It's not chocolate
∴ This cake is red velvet

4) hypothetical syllogism:

The following argument form is also valid argument form.

$S_1$: If a man is a bachelor he is unhappy
$S_2$: If a man is unhappy he is young
S: Bachelor die young

$$\left.\begin{array}{l} p \to q \\ q \to r \\ \therefore p \to r \end{array}\right\} \text{Valid argument form}$$

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

5) dilemma:

consider the following argument form.

$$\left.\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \therefore r \end{array}\right\} \text{Valid argument form.}$$

6) conjuctive simplifications:

consider the following two argument forms.

a) $p \wedge q$
$\therefore p$

b) $p \wedge q$
$\therefore q$

These two forms are valid argument form

7) disjunctive additions:

consider the following two argument forms.

a.) $p$
$\therefore p \vee q$

b.) $q$
$\therefore p \vee q$

These 2 are valid argument forms

8.) conjunctive addition:

The following argument form is also a valid argument form.

$$p$$
$$q$$
$$\therefore p \wedge q.$$

⇒ logically Valid Argument:

A statement formula 'A' is said to follow logically from the statement formulas $A_1, A_2, \ldots, A_{n-1}, A_n$ written as,

$$A_1, A_2, \cdots, A_{n-1}, A_n \models A$$

if there exists an argument $B_1, B_2, \cdots B_{m-1}, B_m$ satisfying the following conditions:

1.) $B_m$ is $A$.
2.) For $1 \leq i \leq m$, either

i) $B_i$ is one of $A_1, A_2, \ldots, A_{n-1}, A_n$ (we say $B_i$ is a hypothesis). or

ii) $B_i$ is a tautology or

iii) for $i \geq 2$, there exists $B_{i_1}, B_{i_2}, \ldots B_{i_t}$, where, $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, 3, \ldots, i-1, i\}$ such that $B_{i_1}, B_{i_2}, \ldots, B_{i_t}$ is a logically valid argument form and $B_{i_t}$ is $B_i$; i.e., $B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_{t-1}} \rightarrow B_{i_t}$ is a tautology.

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eg: Show that $p, q, p \to r, q \to s \models r \wedge s$.

→ $B_1 : p \to r$ — hypothesis

$B_2 : p$ — hypothesis

$B_3 : r$ — $B_1, B_2; B_3$ is a logically valid argument, by modus ponens. Sometimes we write $B_3$ follows from $B_2$ & $B_1$ by modus ponens.

$B_4 : q \to s$ — hypothesis

$B_5 : q$ — hypothesis

$B_6 : s$ — $B_4, B_5, B_6$ is a logically valid argument by modus ponens.

$B_7 : r \wedge s$ — $B_3, B_6, B_7$ is a logically valid argument by ~~addition~~ conjunction.

Thus, we find that there exists an argument $B_1, B_2, \ldots, B_7$ satisfying the conditions of the Definition.

Hence, $p, q, p \to r, q \to s \models r \wedge s$.

## Language Problem:

eg: consider the following statements.

i) If my checkbook is on my office table, then I paid my phone bill.

ii) I was looking at the phone bill for payment at breakfast (or) I was looking at the phone bill for payment in my office.

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iii) If I was looking at the phone bill at breakfast, then the checkbook is on breakfast table.

iv) I did not pay my phone bill.

v) If I was looking at the phone bill in my office, then the check-book is on my office table.

Ques: where was my checkbook?

→ let.

p: my checkbook is on my office table.
q: I paid my phone bill.
r: I was looking at the phone Bill for payment at breakfast.
s: I was looking at the phone bill for payment in my office.
t: the checkbook is on the breakfast table.
s: I was looking at the phone bill in my office.

Hence, in a symbolic notation, the given argument takes the form

$p \rightarrow q$

$r \vee s$

$r \rightarrow t$

$\sim q$

$s \rightarrow p$

We now consider the following argument.

$B_1 : s \to p$ hypothesis
$B_2 : p \to q$ hypothesis
$B_3 : s \to q$ from $B_1, B_2$ & by hypothetical syllogism
$B_4 : \sim q$ hypothesis
$B_5 : \sim s$ from $B_3, B_4$ & by modus tollens
$B_6 : r \vee s$ hypothesis
$B_7 : r$ from $B_5, B_6$ and by disjunctive syllogism
$B_8 : r \to t$
$B_9 : t$ ~~fro~~ from $B_7, B_8$ & by modus ponens

**Conclusion:** The checkbook was on the breakfast table.

## ⇒ Proof Techniques:

Any theorem that a problem is a statement that can be shown to be true.

eg. If $x$ is an integer, and $x$ is odd, then $x^2$ is odd

or equivalently,

For all integers $x$, if $x$ is odd, then $x^2$ is odd.

This statement can be shown to be true.

Now.