Example 2.8 Show that the following relations are equivalence relations:

(i) R_1 is the relation on the set of integers such that aR_1b if and only if $a = b$ or $a = -b$.

(ii) R_2 is the relation on the set of integers such that aR_2b if and only if $a \equiv b \pmod{m}$, where m is a positive integer > 1.

(iii) R_3 is the relation on the set of real numbers such that aR_3b if and only if $(a-b)$ is an integer

(i) $a = a$ or $a = -a$, which is true for all integers.

$\therefore$ R_1 is reflexive.

When $a = b$ or $a = -b$, $b = a$ or $b = -a$.

$\therefore$ R_1 is symmetric

When $a, b, c \geq 0$, $a = b$ and $b = c$, if aR_1b and bR_1c

$\therefore$ $a = c$, i.e., aRc

Similarly when $a \geq 0$, $b \leq 0$, $c \leq 0$, we have $a = -b$ and $b = c$, if aR_1b and bR_1c.

$\therefore$ $a = -c$. i.e., aR_1c.

The result is true for all positive and negative value combinations of a, b, c.

$\therefore$ R_1 is transitive.

Hence R_1 is an equivalence relation.

(ii) $(a - a)$ is multiple of m

$\therefore$ $a \equiv a \pmod m$ i.e., R_2 is reflexive.

When $a - b$ is multiple of m, $b - a$ is also a multiple of m.

i.e. $a \equiv b \pmod m \Rightarrow b \equiv a \pmod m$

$\therefore$ R_2 is symmetric.

When $(a - b) = k_1m$ and $b - c = k_2m$, we get $a - c = (k_1 + k_2)m$ (by addition)

$\therefore$ When $a \equiv b \pmod m$ and $b \equiv c \pmod m$, $a \equiv c \pmod m$

$\therefore$ R_2 is transitive.

Hence R_2 is an equivalence relation.

(iii) $(a - a)$ is an integer. $\therefore$ R_3 is reflexive.

When $(a - b)$ is an integer, $(b - a)$ is an integer.

$\therefore$ R_3 is symmetric.

When $(a - b)$ and $(b - c)$ are integers, clearly $(a - c)$ is also an integer (by addition)

$\therefore$ R_3 is transitive.

Hence R_3 is an equivalence relation.

Example 2.9

(i) If R is the relation on the set of ordered pairs of positive integers such that $(a, b), (c, d) \in R$ whenever $ad = bc$, show that R is an equivalence relation.

(ii) if R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if ab is a perfect square, show that R is an equivalence relation.

(i) $(a, b)\ R\ (a, b)$, since $ab = ba$

$\therefore$ R is reflexive.

When $(a, b)\ R\ (c, d)$, $ad = bc$ i.e., $cb = da$

This means that $(c, d)\ R\ (a, b)$

$\therefore$ R is symmetric.

When $(a, b)\ R\ (c, d)$, $ad = bc$ (1)

When $(c, d)\ R\ (e, f)$, $cf = de$ (2)

(1) and (2) gives $af = be$ ($\because$ c and d are >

This means that $(a, b) R (e, f)$
$\therefore R$ is transitive.

Hence, R is an equivalence relation.

(ii) $(a, a) \in R_1$, since a^2 is a perfect square

$\therefore$ R is reflexive.

When ab is a perfect square, ba is also a perfect square.

i.e. $aRb \Rightarrow bRa$

$\therefore$ R is symmetric.

If, $a R b$, let $ab = x^2$ (1)

If $b R c$, let $bc = y^2$ (2)

(1) $\times$ (2) gives $ab^2c = x^2y^2$

$$\therefore \quad ac = \left(\frac{xy}{b}\right)^2 = \text{a present square.}$$

$\therefore$ aRc. i.e. R is transitive.

Hence R is an equivalence relation.

Example 2.10

(i) If R is the relation on the set of positive integers such that $(a, b) \in R$ if and only if $a^2 + b$ is even, prove that R is an equivalence relation.

(ii) If R is the relation on the set of integers such that $(a, b) \in R$, if and only if $3a + 4b = 7n$ for some integer n, prove that R is an equivalence relation.

(i) $a^2 + a = a(a + 1) =$ even, since a and $(a + 1)$ are consecutive positive integers.

$\therefore$ $(a, a) \in R$

Hence R is reflexive.

When $a^2 + b$ is even, a and b must be both even or both odd.

In either case, $b^2 + a$ is even

$\therefore$ $(a, b) \in R$ implies $(b, a) \in R$

Hence R is symmetric.

When a, b, c are even, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.

When a, b, c are odd, $a^2 + b$ and $b^2 + c$ are even. Also $a^2 + c$ is even.

Then $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ i.e., R is transitive.

$\therefore$ R is an equivalence relation.

(ii) $3a + 4a = 7a$, when a is an integer.

$\therefore$ $(a, a) \in R$. i.e., R is reflexive.

$$\begin{aligned} 3b + 4a &= 7a + 7b - (3a + 4b) \\ &= 7(a + b) - 7n \\ &= 7(a + b - n), \text{ where } a + b - n \text{ is an integer} \end{aligned}$$

$\therefore$ $(b, a) \in R$ when $(a, b) \in R$.

i.e. R is symmetric.

Let (a, b) and $(b, c) \in R$.

i.e. let $3a + 4b = 7m$ (1)

and $3b + 4c = 7n$ (2)

(1) and (2) gives, $3a + 4c = 7(m + n - b)$, where $m + n - b$ is an integer.

$\therefore$ $(a, c) \in R$

i.e. R is transitive $\Rightarrow \therefore R$ is an equivalence relation.