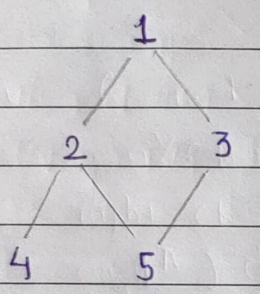
TUTORIAL - 4

[U19CS012]

Lattice and Boolean Algebra

Q1.> Let $A = \{1, 2, 3, 4, 5\}$ be ordered by the following Hasse diagram. Insert the correct symbol $<$, $>$ or $\|$ (not comparable) between each pair of elements:



1) 1 $<$ 5
2) 2 $\|$ 3
3) 4 $<$ 1
4) 3 $>$ 4

Q2.> Consider the ordered set A in the previous Hasse diagram.

(1) Find all minimal and maximal elements of A.
(2) Does A have a lower bound and an upper bound? Also discuss glb and lub for the set A.

A2.>

(1) Maximal Elements = 1
Minimal Element = 4, 5

(2)

lower bound of A = ϕ
4 does not relate to 5
upper bound of A = 1

least upperbound = 1
greatest lower bound = ϕ

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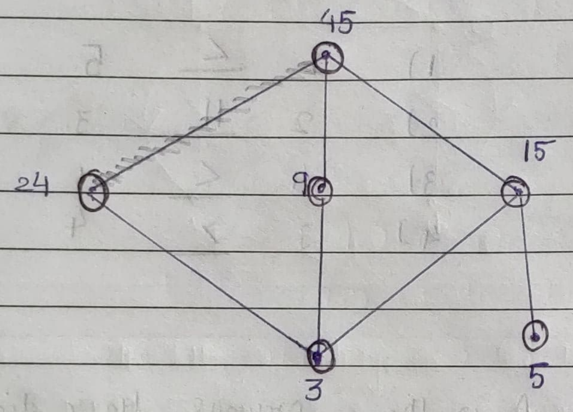
Q3.> For the poset ({3, 5, 9, 15, 24, 45}) divisor of I Find

1.) The maximal and minimal elements.
2.) The greatest and the least elements.
3.) the upper bounds and the lub at {3, 5}
4.) The lower bounds and glb at {15, 45}



1) maximal Elements = 24, 45
minimal elements = 3, 5

2) Greatest Element = Does not exist
Least Element = Does not Exist

3) Upper Bound = 15, 45 , LUB = 15

4) lower Bound = 3, 5, 15 , GLB = 15

Q4.> If R and S are relations on A = {1, 2, 3} represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the matrices that represents

1) $R \cup S$ 2) $R \cap S$ 3) $R.S$ 4) $S.R$ 5) $R \oplus S$

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1) $M_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $M_{R.S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2) $M_{R \cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4) $M_{S.R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

5) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S}$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

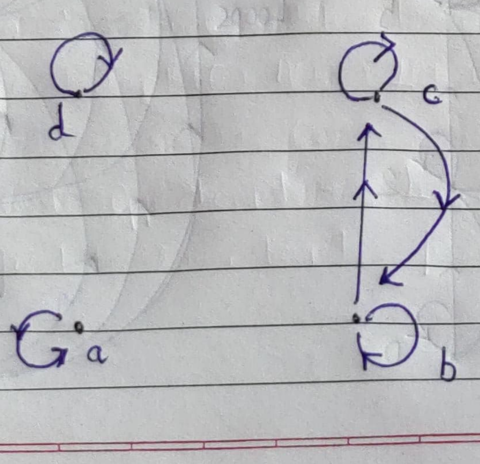
Q5.) List the ordered pairs in the relations R and S whose matrix representation are given as follows:

1) $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2) $M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

Also draw the directed graphs representing R and S. Use the graphs to find if R and S are equivalence relⁿ.

$R = \{ (a,a), (b,b), (b,c), (c,b), (c,c), (d,d) \}$



✓ Self Loop ⇒ Reflexive
✓ Symmetric
✓ Transitive
⇒ Equivalence

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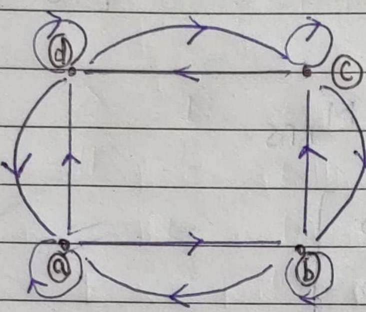
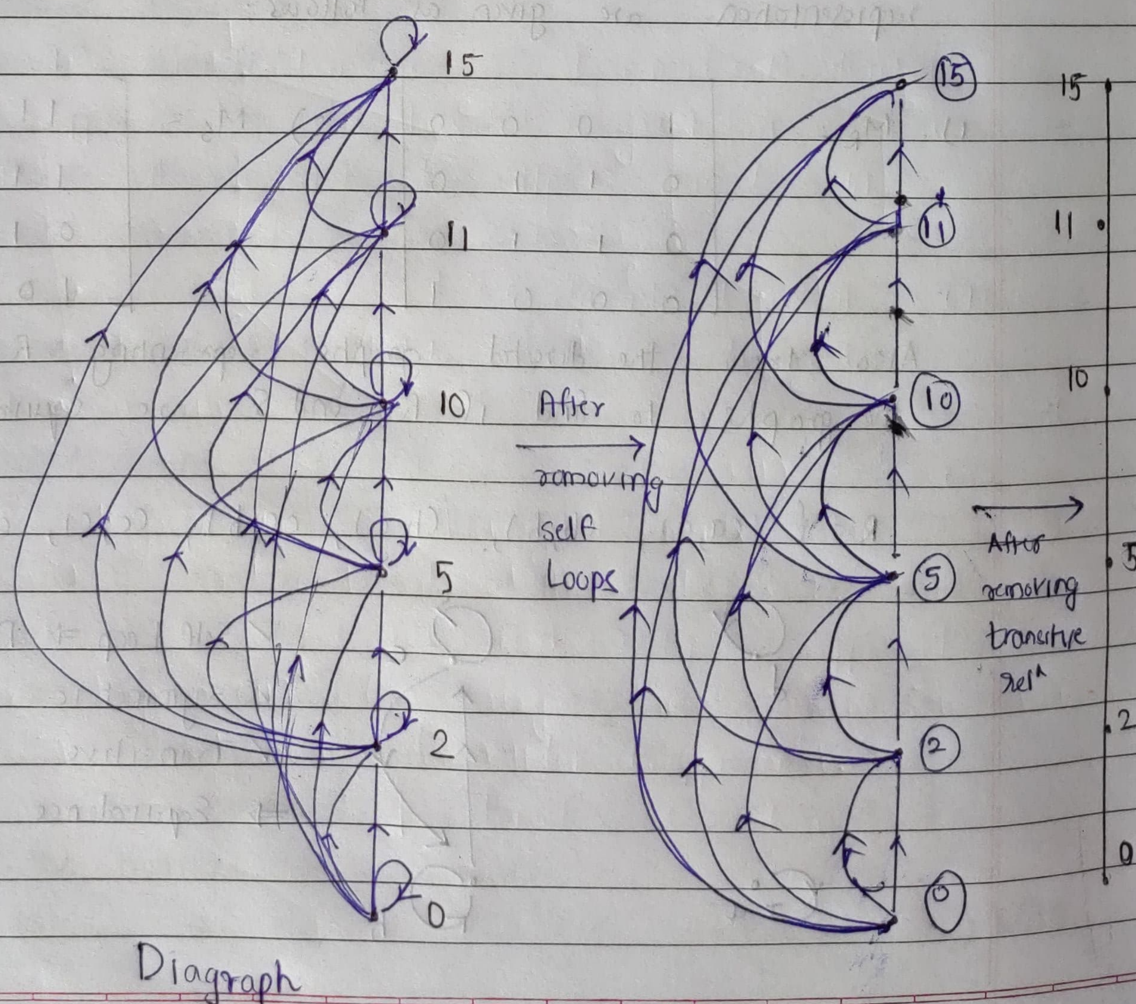
S = { (a, a), (a, b), (a, d),
(b, a), (b, b), (b, c),
(c, b), (c, c), (c, d),
(d, a), (d, c), (d, d) }



✓ Self Loop ⇒ Reflexive
✓ Symmetric
✓ Not transitive
⇒ Not equivalence

Q6.) Draw the Hasse diagram for the "Less than or equal to" relation on { 0, 2, 5, 10, 11, 15 } starting from the diagraph.



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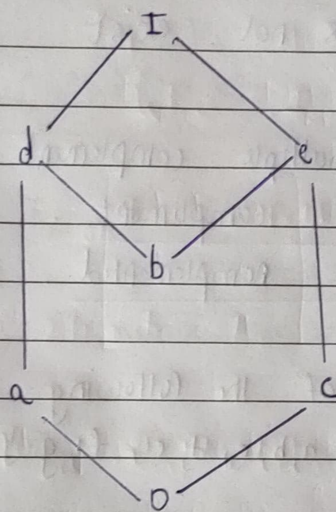
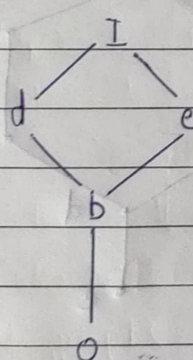
Q7.) Consider the lattice L in the following figure

(1.) Find all sub-lattice with five elements
(2) Find complements of a and b, if they exist
(3) Is L distributive? complimented?

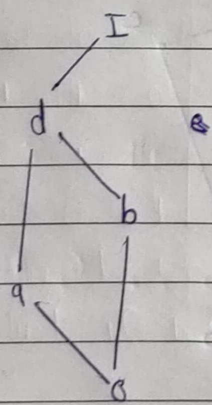


(1)

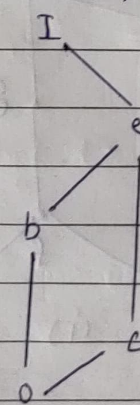
$\{I, d, e, b, 0\}$



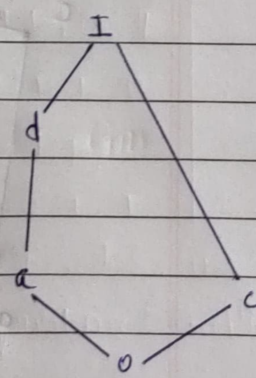
$\{I, d, b, a, 0\}$

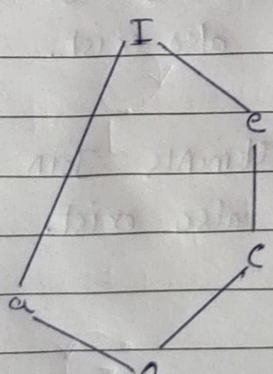


$\{I, e, b, c, 0\}$



$\{I, d, a, 0, c\}$





$\{I, e, c, 0, a\}$

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2) $a \vee c = I$ $\quad a \wedge c = 0$ $\quad a \vee e = I$ $\quad a \wedge e = 0$

$\Rightarrow a^c = \{c, e\}$

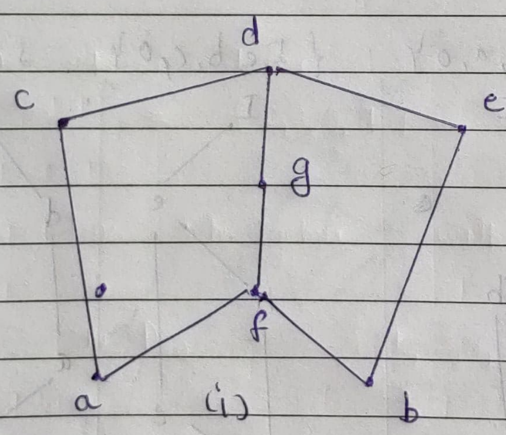
$\forall$ elements in L, no element satisfy complement of 'b'

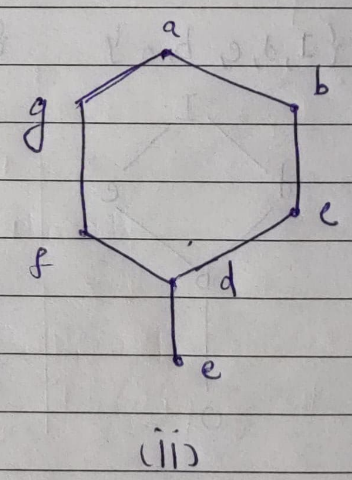
$\Rightarrow b^c$ does not exist

3) Since a has multiple complement so L is not distributive
And b has no compliment.
So L is not complemented.

$\left[\begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array}\right.$
[Distributive]

Q8.) Decide ~~whether~~ which of the following Hasse diagram define a lattice on {a, b, c, d, e, f, g}





For diagram (i) for every elements Join semi-lattice exist and meet semi-lattice also exist.

For diagram (ii) for every elements Join Semilattice exist and meet semi lattice also exist.

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Q9.> Write the duals of each Boolean eqⁿ

(1) $(a \ast 1) \ast (a + a')$

(2) $a + a'b = a + b$

$0 \longrightarrow 1$

$1 \longrightarrow 0$

$+ \longrightarrow \ast$

$\ast \longrightarrow +$

(1) Dual = $(a + 0) + (a \ast a')$

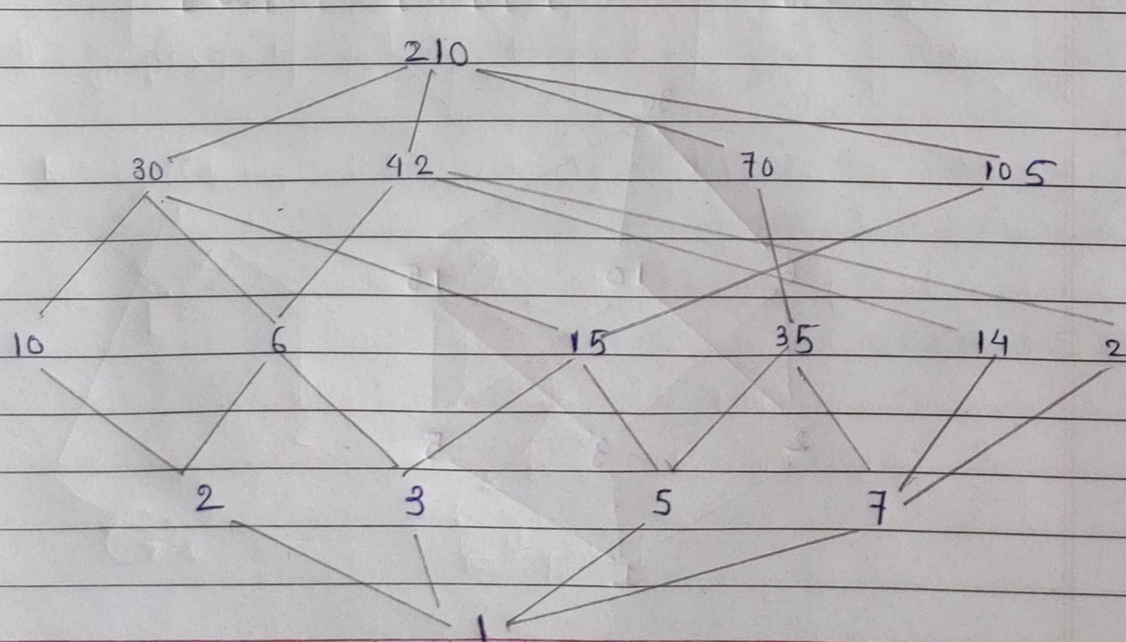
(2) $a \ast (a' + b) = a \ast b$

Q10.> Given the set D_m of divisors of m is bounded, distributive lattice with $a + b = a \vee b = \text{lcm}(a, b)$
and $a \ast b = a \wedge b = \gcd(a, b)$

(1) S.T. D_m is a B.A. If m is a square free i.e if m is a product of distinct prime.

(2) Find the atoms of D_m.

(1) Let $m = 210$
Factors of $m = 2, 3, 5, 7$



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Hence, the, it is distributive and complimented and obeys laws of boolean Algebra since complemented also exist.
Hence is Boolean Algebra.

(ii) Find atoms of D_m

All the prime factors of 'm'

eg $210 = \{$ ~~[illegible]~~ $2, 3, 5, 7 \}$

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(11) Consider the boolean Algebra D_{210}

(i) List all elements and draw its diagram

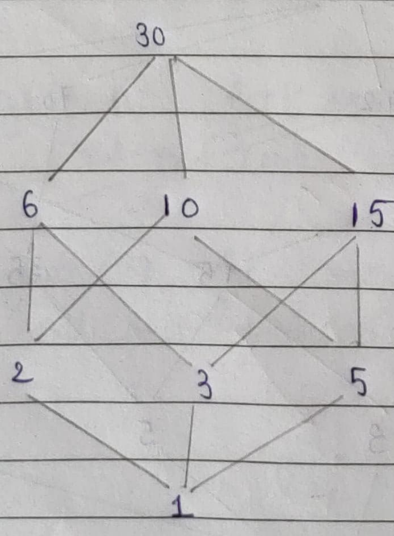
$D_{210} = \{1, 2, 3, 5, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$

(ii) set of atoms

$= \{2, 3, 5, 7\}$

(iii) two sub algebra with 8 elements

(i) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



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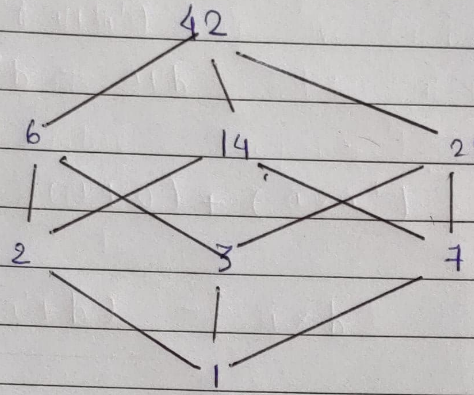
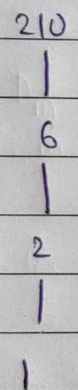
(ii) $D_{42} = \{1, 2, 3, 7, 6, 14, 21, 42\}$



(iii) Is $X = \{1, 2, 6, 210\}$ a sub lattice of D_{210}? is it a sub algebra?

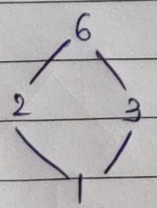


Yes, it is a sublattice of D_{210} as every pair of element has lub and glb.

X → is distributive but not complemented
X is not subalgebra.

(v) Is $Y = \{1, 2, 3, 6\}$ a subtlattice of D_{210}? a sub-algebra.

Ans:



Y is sublattice of D_{210} as every pair of element has lub and glb.

Y is distributive and complimented thus y is sub-algebra