Conclusion: The checkbook was on the breakfast table.

Proof Techniques:

Any theorem that a problem is a statement that can be shown to be true.

eg. If x is an integer, and x is odd, then x^2 is odd

or equivalently,

For all integers, x, if x is odd, then x^2 is odd.

This statement can be shown to be true.

Now.

As facts.

eg. 6 is an even integer (or)

As implication. $x^2+1=0$ has no sol^{n}s in real no.'s.

eg. for eg, for all integers x, if x is even, then $x+1$ is odd.

As biimplication.

eg. for all integers x, x is even if and only if x is divisible by 2.

⇒ Here, A proof may consist of previously known facts, proved results (or) previous statements of the proof.

⇒ There are several known techniques for constructing a proof.

1) Direct Proof.
2) Indirect Proof
3) Proof by contradiction
4) Proving Biimplication
5) Proving Equivalent statement.
6) Errors in the proof.

→ 1) Direct Proof:-

It includes proof of those theorems that can be expressed in the form,

$\forall x\,(\underbrace{P(x)}_{\text{Premises}} \rightarrow \underbrace{Q(x)}_{\text{conclusion}})$, D is the domain of discourse eg. For all Integers x, if x is odd, then x^2 is odd.

→ let us verify that the theorem is true for certain values of x,

If $x = 3 \Rightarrow x^2 = 9$ so x^2 is odd.
$x = -5 \Rightarrow x^2 = 25$ " " " "

It means theorem is true.

→ but we must prove that the theorem is true for an arbitrary value of the domain of discourse.

let $P(x)$ = " x is an odd integer "
$Q(x)$ = " x^2 is an odd integer "

Then symbolically,

$\forall x (P(x) \rightarrow Q(x))$, the domain of discourse is the set of all integers $\mathbb{Z}$

here we will start the proof by assuming 'a' is a particular but arbitrarily chosen element of $\mathbb{Z}$.

for 'a', we assume that $P(a)$ is true, then we show that $Q(a)$ is true.

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Main Proof

$\Rightarrow$ let 'a' be an integer, such that 'a' is odd.

so we can write, ~~a = 2n + n f~~
$a = 2n+1$, for some integer 'n'.

$\Rightarrow a^2 = (2n+1)^2$
$= 4n^2 + 4n + 1$
$= 2(2n^2 + 2n) + 1$

let $m = 2n^2 + 2n$ because n = integer
$\Rightarrow m (= 2n^2 + 2n)$ is also integer

$\therefore$ we can write,

$a^2 = 2m + 1$, for some integer m
$\Rightarrow a^2$ = odd

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Proof: let 'a' be an odd integer then

a is an odd integer.
$\Rightarrow a = 2n+1$, for some integer n
$\Rightarrow a^2 = (2n+1)^2$
$\Rightarrow a^2 = 4n^2 + 4n + 1$
$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$
$\Rightarrow a^2 = 2m + 1$, $m = 2n^2 + 2n$ is an integer
$\Rightarrow a^2$ is an odd integer.

$\Rightarrow$ for all integer x, if x is odd, then x^2 is odd.

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2) <u>Indirect</u> <u>proof</u>:-

→ Consider the implication $p \to q$. This implication is equivalent to the implication $\sim q \to \sim p$. This means that in order to show that $p \to q$ is true, we can also show that the implication $\sim q \to \sim p$ is true.

→ to show $\sim q \to \sim p$ is true, we have to show $\sim q$ is true. & prove that $\sim p$ is true.

→ This type of proof is called <u>Indirect</u> proof.

<u>Ex</u> Prove that n is an integer, n^2+3 is odd then n is even

→ let $P(n) = n^2+3$ is an odd integer
$Q(n) = n$ is an even integer

$\forall n \, (P(n) \to Q(n))$ the domain of discourse is the set Z of all integers.

1 a/n proof → assume that n is a particular but arbitarily chosen element of Z.

for this n, will show that $P(n) \to Q(n)$ is true.

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→ it is logically equivalent to $\sim Q(n) \to \sim P(n)$ so we have to show that it is true.

→ suppose $\sim Q(n)$ is true
we have to show $\sim P(n)$ is true.

→ Because $\sim Q(n)$ is true, 'n' is not even
⇒ n is odd
So $n = 2k+1$ for some integer 'k'. Thus

$$n^2+3 = (2k+1)^2+3 \quad , \text{put } n=2k+1$$
$$= 4k^2+4k+1+3$$
$$= 4k^2+4k+4$$
$$= 2(2k^2+2k+2)$$

let us write $t = 2k^2+2k+2$
⇒ $n^2+3 = 2t$, for some int. t
→ as it's a multiple of 2 ⇒ even

⇒ n^2+3 is an even integer
⇒ n^2+3 is not an odd ".

⇒ $\sim P(n)$ is true

→ $\sim Q(n) \to \sim P(n)$ is true.
⇒ $P(n) \to Q(n)$

∴ $\forall n(P(n) \to Q(n))$, in the domain Z of all integers

⇒ by the rule universal generalization
if n^2+3 is odd
⇒ n is even

3) **Proof by contradiction:**

→ In this, we assume that the conclusion is not true and then arrive at a contradiction.

eg:- Show that $\sqrt{2}$ is an irration number.

→ let us assume that $\sqrt{2}$ is not an Irration numbers.

⇒ $\sqrt{2}$ is rational numbers,

$$\Rightarrow \sqrt{2} = \frac{a}{b}, \Rightarrow a \& b = \text{integers}, \; b \neq 0.$$

here we assume that $\frac{a}{b}$ = lowest term

⇒ a & b have no common factor other than one (if it have then, cancel it)

$$\therefore \sqrt{2} = \frac{a}{b}$$

$$\Rightarrow (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow a^2 = 2b^2$$

⇒ a^2 is an integer

⇒ a is an integer (by 1^{st} eg.)

We can write $a = 2n$, for some integer 'n'

$\Rightarrow a^2 = 4n^2$

Now substitute the value of a^2 into $a^2 = 2b^2$ to obtain

$$2b^2 = a^2 = 4n^2$$

implies $\Rightarrow b^2 = 2n^2$

$\Rightarrow b^2 = \text{even}$

$\Rightarrow b = \text{even}$

$\Rightarrow$ a & b both are even.

$\Rightarrow$ both have 2 as a common factor

This contradicts our assumption that a & b have no common factors other than 1.

→ We now arrived at a contradiction

$\Rightarrow \sqrt{2}$ is an irrational number.

4) ⇒ Proving Biimplications: $\forall x (P(x) \leftrightarrow Q(x))$ in the domain $\mathbb{Z}$.

consider the following theorem

ex: An integer x is even, if and only if $x+1$ is odd.

→ $P(x)$: x is even

$Q(x)$: $x+1$ is odd.

So for all integers x, $P(x)$ iff $Q(x)$.

OR

$$\forall x (P(x) \leftrightarrow Q(x)); \text{ in the domain } \mathbb{Z}.$$

Proof: Assume that 'x' is a particular but arbitrary integer such that x is even and show that $x+1$ is odd.

→ Then assume that $x+1$ is odd and prove that x is even.

→ 1st assume that x is even

$\Rightarrow x = 2n$, for some integer n.

$\Rightarrow x+1 = 2n+1$

$\Rightarrow x+1$ is an odd integer.

→ Let us now suppose that $x+1$ is odd

$\Rightarrow x+1 = 2m+1$ for some integer m

$\Rightarrow x = 2m$

$\Rightarrow x$ is an even integer

⇒ we conclude that an integer x is even iff $x+1$ is od.

5) Proving Equivalent statements:

→ consider the following statements:

let x be an integer.

Ex

p: x is divisible by 6.

q: x is divisible by 2 & 3.

r: x is an even number & x is divisible by 3.

we can prove that, $p \rightarrow q$, $p \Rightarrow r$, $q \Rightarrow r$ } In words p, q & r are equivalent statement

We have cycle $p \to q \to r \to p$

let

i) p : x is divisible by 6.
ii) q : x is divisible by 2 & 3.
iii) r : x is an even number & x is divisible by 3.

Proof. To prove that these statements are equivalent, we show that,

(i) → (ii), (ii) → (iii) & (iii) → (i)
(or)
(i) → (ii) → (iii) → (i)

1) (i) → (ii)
Suppose x is divisible by 6
Then $x = 6n$, for some integer n.
Then $x = 6n = 2 \cdot 3n = 3 \cdot 2n$
it follows x is divisible by 2 & 3.

2) (ii) → (iii)
Suppose x is divisible by 2 & 3.
because x is divisible by 2
→ x is an even integer
Hence, x is an even integer.
and x is divisible by 3.

3) (iii) → (i)
suppose that x is an even no. & x is divisible by 3
because x is an even integer.
$x = 2n$, for some integer n.
→ 3 is divides $2n$
thus $2n = 3t$ for some integer t.

Now,

$$n = 3n - 2n$$

This implies that,

$$n = 3n - 3t$$

that is,

$$n = 3(n - t)$$

let $n - t = s \Rightarrow n = 3s$, s is an some integer

It follows that,

$$x = 2n = 2 \cdot 3s = 6s$$, for some integer s.

$\Rightarrow x$ is divisible by 6.

5) Errors (Fallacies) in the proofs:

- As we know, proof may consist of previously known facts, proves results or previous statements of the proof.
- However, if we are not careful, errors can occurs in the proofs.

eg.
$$1 = \sqrt{1}$$
$$= \sqrt{(-1) \cdot (-1)}$$
$$= \sqrt{(-1)} \cdot \sqrt{(-1)}$$
$$= (\sqrt{-1})^2$$
$$= -1$$

but we know that $1 = -1$, is not true.
so here, there is some error. & error is in the equality $\sqrt{(-1) \cdot (-1)} = \sqrt{-1} \cdot \sqrt{-1}$.

→ here we are using $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for all real no's.
however it's true when a & b are non negative real numbers.