CONNECTIVITY OF GRAPHS

ASWATHY.A

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CONTENTS

PREFACE

In the present times Mathematics occupies an important place in curriculum. The “Connectivity of graphs” is one of the important subject at

MSc level. In this book an attempt has been made to cover up most of topics included in the MSc syllabus by Indian Universities. This book covers the syllabi of Regular and Correspondence courses of Indian Universities.

The book has been written in a simple and lucid manner and is up-to-date in its contents. To illustrate theory some examples have been given.

It is hoped that the book will be appreciated by teachers and students alike. While preparing the book, material has been drawn from works of different authors, periodicals and journals and the author is indebted to all such persons and their publishers.

All suggestions for improvement of the book shall be thankfully accepted.

Author

INTRODUCTION

Graph Theory is regarded as one of the areas of Applied Mathematics. Graph Theory has been independently discovered many times. Leonhard Euler is known as “Father of Graph Theory”. Subsequent rediscoveries of Graph Theory has been made by Gustav Kirchhoff and Arthur Cayley. Another approach was of Hamilton’s. The origin of graph theory can be traced back to Euler's work on the Konigsberg bridges problem (1735), which subsequently led to the concept of an Eulerian graph. The study of cycles on polyhedra by the Thomas P. Kirkman (1806 - 95) and William R. Hamilton (1805-65) led to the concept of a Hamiltonian graph. The concept of a tree, a connected graph without cycles, appeared implicitly in the work of Gustav Kirchhoff (1824-87), who employed graph-theoretical ideas in the calculation of currents in electrical networks or circuits. Later, Arthur Cayley (1821-95), James J. Sylvester (1806-97), George Polya (1887-1985), and others use 'tree' to enumerate chemical molecules. The study of planar graphs originated in two recreational problems involving the complete graph K_5 and the complete bipartite graph $K_{3,3}$. These graphs proved to be planarity, as was subsequently demonstrated by Kuratowski. First problem was presented by A. F. Mobius around the year 1840.

The development of graph theory is very similar the development of probability theory, where much of the original work was motivated by efforts to understand games of chance. The large portions of graph theory have been motivated by the study of games and recreational mathematics. Generally speaking, we use graphs in two situations. Firstly, since a graph is a very convenient and natural way of representing the relationships between objects we represent objects by vertices and the relationship between them by lines. In many situations (problems) such a pictorial representation may be all that is needed. Secondly, we take the graph as mathematical model, solve the appropriate graph-theoretic problem, and then interpret the solution in terms of the original problem.

In the past few years, Graph Theory has been established as an important mathematical tool in wide variety of subjects. Graph Theory has application to some areas of Physics, Chemistry, Communicative Science, Computer Technology, Electrical and Civil Engineering, Architecture, Operation Research, Genetics, Psychology, Sociology, Economics, Anthropology and Linguistics. The theory had also emerged as a worthwhile mathematical discipline in its own right. Graph Theory is intimately related to many branches of Mathematics including Graph Theory, Matrix Theory, Numerical Analysis, Probability and Topology. In fact Graph Theory serves as a mathematical model for any system involving a binary relation.

In this text book, discuss the connectivity of graphs. This textbook is divided into two chapters. In the first chapter, discussed about definitions and theorems on connectivity. The second chapter deals with some applications of connectivity.

CHAPTER –1

CONNECTIVITY OF GRAPHS

Definition (2.1)

An edge of a graph G is called a *bridge* or a *cut edge* if the subgraph $G - e$ has more connected components than G has.

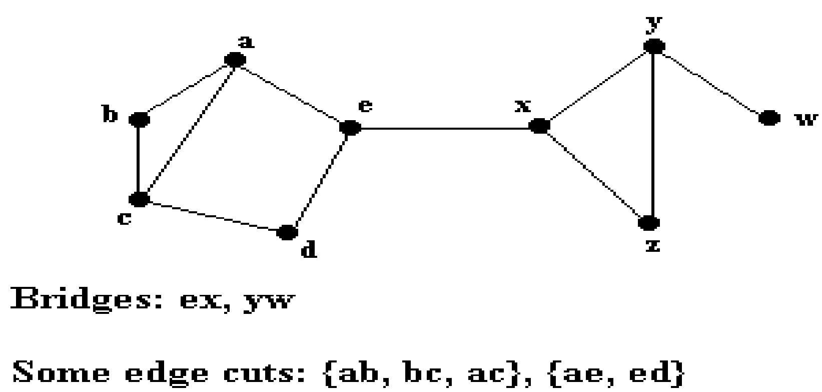


Figure (2.1)

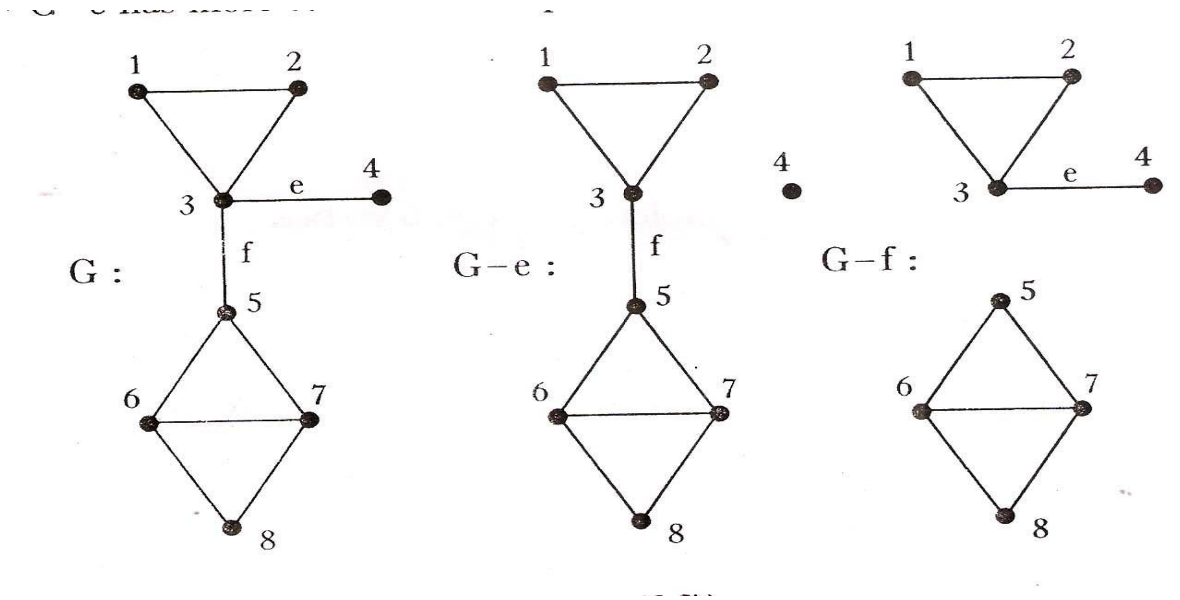


Figure (2.2)

A graph and two bridge deletions

Theorem (2.1)

An edge e of a graph G is a bridge if and only if e is not part of any cycle in G.

Proof:

Let e have end vertices u and v. If e is not a bridge then, it is either a loop or there is a path $P = uu_1u_2 . . . u_nv$, from u to v, different from the edge e. If it is a loop then it forms a cycle. If there is such a path P then $C = uu_1u_2 \ldots u_nvu$, the contradiction of P with , is a cycle in G.This shows that if e is not a bridge then it is part of a cycle. This is equivalent to saying that if e is not part of any cycle then e must be a bridge.

Conversely, suppose that e is part of some cycle $C = u_0u_1u_2...u_m$ in G. Let $e = u_iu_{i+1}$. In the case where $m = 1, C = u_0u_1$, and so C is just the edge e and e is a loop. On the other hand , $if\ m > 1$ then $P = u_iu_{i-1}\ldots u_0u_{m-1}\ldots u_{i+1}$ is a path from u to v different from e. Thus e is not a bridge .This shows that if e is a bridge then it is not part of any cycle in G , completing the proof.

Theorem (2.2)

Let G be a connected graph .Then G is a tree if and only if every edge of G is a bridge i.e., if and only if for every edge e of G the subgraph $G - e$ has two components.

Proof:-

Suppose that G is a tree. Then G is acyclic i.e., it has no cycle, and so no edge of G belongs to a cycle. In other words, if e is any edge of G then it is a bridge.

Conversely, suppose that G is connected and that every edge e of G is a bridge. Then G can have no cycles since any edge belonging to a cycle is not a bridge. Hence G is acyclic and so is a tree.

Problem (2.1)

Find all bridges in the graph.

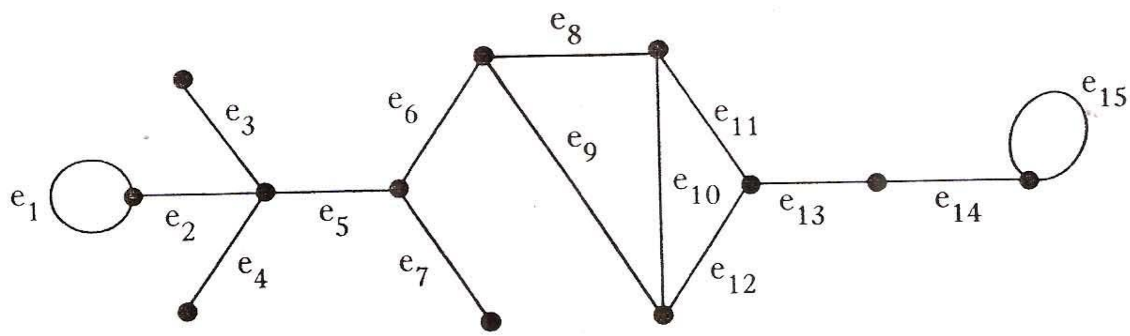


Figure (2.3)

Solution :-

Bridges in the graph are e_2,e_3, e_4, e_5,e_6,e_7,$e_{13}$and e_{14}.

Definition (2.2)

A vertex v of a graph G is called a *cut vertex* of G if $\omega(G-v) > \omega(G)$.In other words , a vertex v is a cut vertex of G if its deletion disconnects some connected components of G , there by producing a subgraph having more connected components than G has.

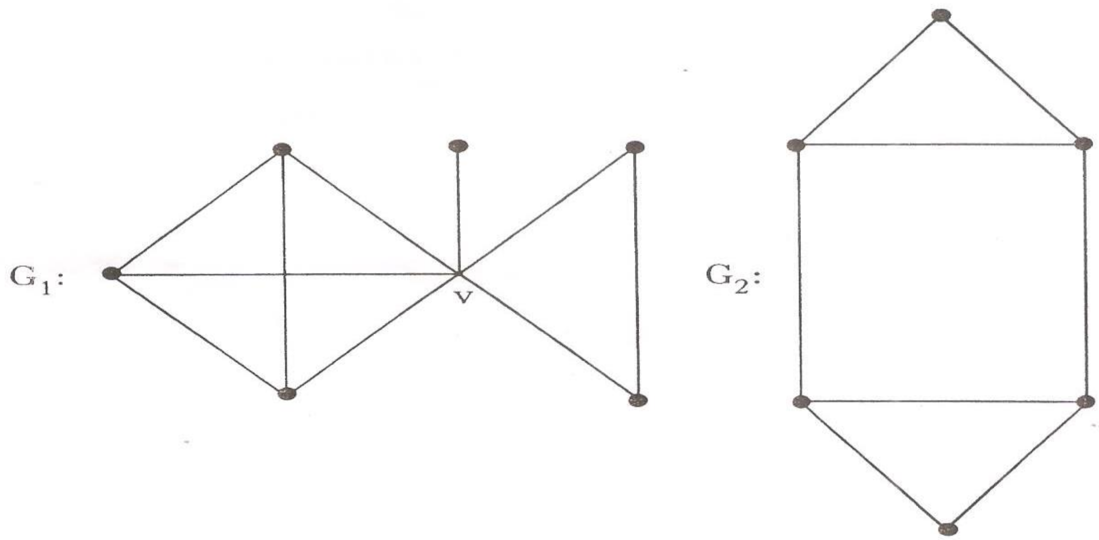


Figure (2.4)

V is a cut vertex of G_1,since $\omega(G_1) = 1$ while $\omega(G_1 - v) = 3$.The graph G_2 has no cut vertices.

Theorem (2.3)

Let v be a vertex of the connected graph G. Then v is a cut vertex of G if and only if there are two vertices u and w of G, both different from v, such that v is on every $u - w$ path in G.

Proof:-

First let v be a cut vertex of G .Then $G - v$ is disconnected and so there are vertices u and w of G which lie in different components of $G - v$ .Thus, although there is a path in G from u and w, there is no such path in $G - v$.This implies that every path in G from u to w contains the vertex v.

Conversely , suppose that u and w are two vertices of G , different from v, such that every path in G from u to w contains v.Then there can be no path from u

to w in $G - v$.Thus $G - v$ disconnected (with u and w lying in different components).hence v is a cut vertex.

Note:

No vertex of a complete graph is a cut vertex.

Theorem (2.4)

Let G be a graph with n vertices, where $n \geq 2$.Then G has at least two vertices which are not cut vertices.

Proof:-

We may suppose that G is a connected graph. We proceed by assuming the result is false for our G and so the proof will be complete if we derive a contradiction from this.

Thus we are assuming that there is at most one vertex in G which is not cut vertex. Now let u , v be vertices in G such that the distance $d\ (u , v)$between them is the greatest of distances between pairs of vertices in G. i.e., $d(u , v) = diam(G)$.since G is connected and has at least two vertices, $u \neq v$. Thus by our assumption, one of these vertices must be a cut vertex, say v. Then $G - v$ is disconnected and so there is a vertex w in G which does not belong to the same component as u does in $G - v$. This implies that every uw path in G contains the vertex v. It follows from this that the shortest path in G from u to w contains the shortest path from u to v and this contradiction completes our proof.

Definition (2.3)

A *vertex cut* of G is a subset V' of V such that $G - V'$ is disconnected.

A k-vertex cut is a vertex cut of k elements.

Definition (2.4)

Let G be a simple graph .The *connectivity (vertex connectivity)* of G, denoted by $k(G)$,is the minimum number of vertices in G whose deletion from G leaves either a disconnected graph or trivial graph.

Note:-

For $n \geq 2$, the deletion of any vertex from K_n,results in K_{n-1} and in general the deletion of t vertices $(t < n)$results in $Kn - t$.

This shows that $k(Kn) = n - 1$.

A connected graph G has $k(G) = 1$ if and only if either $G = K_2$ or G has a cut vertex.$k(G) = 0$ if and only if either G=K_1 or G is disconnected.

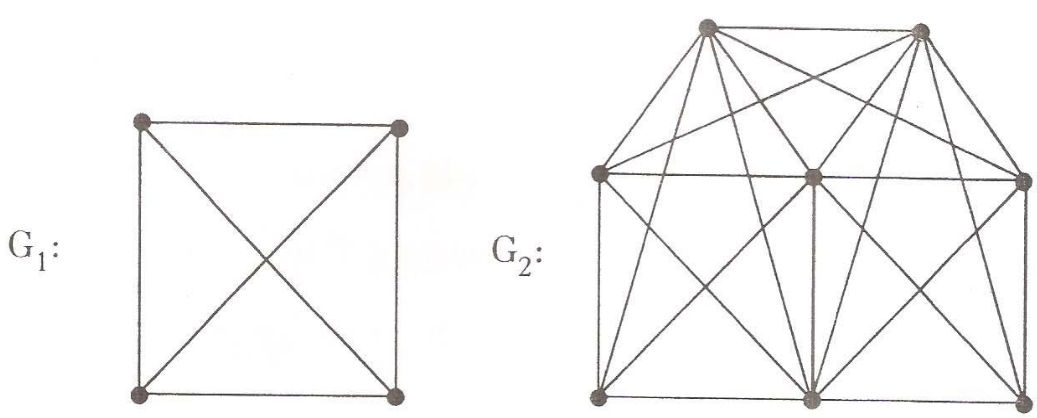


Figure (2.5)

$k(G_1) = 3$ and $k(G_2) = 4$.

Theorem (2.5)

The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof:-

Let α denote the edge connectivity of G. Therefore, there exists a cut set S in G with α edges. Let S partition the vertices of G into subsets V_1 and V_2. By removing at most α vertices from V_1(or V_2) on which the edges in S are incident, we can effect the removal of S (together with all other edges incident on these vertices) from G. Hence the theorem.

Theorem (2.6)

A graph of order n is k-connected (where $1 \le k \le n-1$) if the degree of each vertex is at least $(n+k-2)/2$.

Proof:-

If the graph is complete, it is k-connected for $k \le (n-1)$. Assume that the graph is not complete and not k-connected. So there is a disconnecting set S of s vertices such that G-S is a disconnected graph. Let H be a component of G-S with as few vertices as possible. If the order of H is r, $r \le (n-s-r)$, which gives the upper bound $(n-s)/2$ for the order r. If v is any vertex of H, the degree of v in the graph

G cannot exceed $(r-1)-s$. Thus $\deg v \le (n-s)/2-1+s=(n+s-2)/2<(n+k-2)/2$, violating the given inequality.

Theorem (2.7)

A graph is n-edge connected if and only if the number of edges in any cut is at least k.

Theorem (2.8)

A vertex of a tree is a cut vertex if and only if it is not a pendant vertex.

Proof:-

Let v be a pendant vertex of a tree T. The removal of v results in the removal of v and the only edge incident on it so that T-v is still connected. Therefore v is not a cut vertex of T.

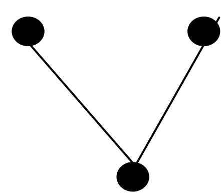


Figure (2.6)

Let u be a non pendant vertex of a tree T so that the degree of u is greater than 1. Therefore there exists at least two edges uv and uw incident on u. Since T is a tree vuw is the unique path connecting v and w. The removal of u deletes the edges uv and uw. Therefore there is no path connecting the vertices v and w in T-u and so T-u is disconnected. Hence u is a cut vertex of the tree T.

Definition (2.5)

A simple graph G is called n-connected if $k(G) \geq n$.

Note:-

G is 1-connected if and only if G is connected and has atleast two vertices.

G is 2- connected if and only if G is connected with atleast 3 vertices but no cut vertices.

In figure (2.4), G_1 is 3-connected and G_2 is 4-connected.

Definition (2.6)

An *edge cut* of G is a subset of e of the form $[S , \overline{S}]$ where S is a non-empty proper subset of V.

An k $-$edge cut is an edge cut of k elements.

If G is non-trivial and E is an edge cut of G, then $G - E$ is disconnected.

Definition (2.7)

Let G be a simple graph. The *edge connectivity* of G, denoted by k'(G), is the smallest number of edges in G whose deletion from G either leaves a disconnected graph or an empty graph.

Any non empty simple graph G with a bridge has k'(G) =1.

k'(G)=0 if and only if either G is disconnected or an empty graph.

All non-trivial connected graphs are 1-edge connected.

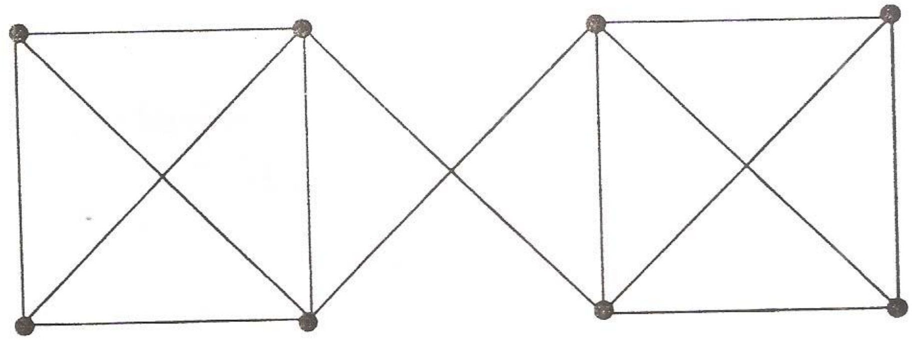


Figure (2.7)

Here $k'(G)$ =2.

Theorem (2.9)

The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof:-

Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem.

Definition (2.8)

A simple graph G is called *n-edge connected* if $k'(G) \geq n$.

Theorem (2.10)

A simple graph G is n-edge connected if and only if given any pair of distinct vertices u and v of G, there are atleast n edge disjoint paths from u to v.

Proof :-

Suppose that G is n-edge connected and u and v are two distinct vertices of G. Then any $u - v$ separating set of edges must be atleast n members. Thus, there must be atleast n-edge disjoint $u - v$ paths, as required.

Conversely, suppose that given any pair of distinct vertices u and v of G , there are atleast n −edge disjoint paths from u to v. Then, for each pair of vertices u and v every $u - v$ separating set of edges must have atleast n members. Thus it requires the deletion of atleast n edges from G inorder to produce a disconnected graph or an empty graph. In other words, G is n-edge connected, as required .

Theorem (2.11) (Whitney's Theorem)

For any graph , $k \leq k' \leq \delta$.

Proof :-

Let G be a connected graph.

If G has no edges, then k'=0. Otherwise a disconnected graph results when all the edges incident with a vertex of minimum degree are removed. In either case, $k' \leq \delta$.

To show $k \le k'$.

If G is disconnected or trivial, then $k = k' = 0$. If G is connected and has a bridge x, then k'=1. In this case, $k = 1$ since either G has a cut vertex incident with x or G is K_2.

Finally, suppose G has $k' \ge 2$ edges whose removal disconnects it. Clearly, the removal of $k' - 1$ of these lines produces a graph with a bridge $x = uv$. For each of these $k' - 1$ edges, select an incident vertex different from u or v. The removal of these vertices also removes the $k' - 1$ edges and quite possibly more. If the resulting graph is disconnected, then $k < k'$, if not x is a bridge, and hence the removal of u or v will result in either a disconnected or a trivial graph, so $k \le k'$ in every case.

Thus for any graph G, $k \le k' \le \delta$.

Theorem (2.12)

For any three integers r, s, t such that $0 \le r \le s \le t$ there is a graph G with $k = r, k' = s$ and $\delta = t$.

Proof :-

Take two disjoint copies of K_{t+1}. Let A be a set of r vertices in one of them and B be a set of s vertices in the other. Join the vertices of A and B by s edges utilizing all the vertices of B and all the vertices of A. Since A is a vertex cut and the set of these s edges is an edge cut of the resulting graph G, it is clear that

$k(G) = r$ and $k'(G) = s$. Also there is atleast one vertex which is not in $A \cup B$ and it has degree t, so that $\delta(G) = t$.

Theorem (2.13)

$\delta \geq \frac{n}{2}$ ensures $k' = \delta$ in a graph.

Proof :-

Since it is enough to rule out $k' < \delta$, suppose G is a graph with $\delta \geq \frac{n}{2}$ and $k' < \delta$. Let F be a set of k' edges disconnecting G, and C_1 and C_2 be the components of $G - F$ and A_1 and A_2 the end vertices of the edges of F in C_1 and C_2 respectively. Let $|A_1| = r$ and $|A_2| = s$ and suppose $V(C_1) = A_1$. Then each vertex of C_1 is adjacent with atleast one edge of F. So the number m_1 of edges in C_1 satisfies the following inequality.

$$m_1 \geq \frac{1}{2}(r\delta - k')$$

$$> \frac{1}{2}(r\delta - \delta) \quad \text{since } k' < \delta .$$

$$= \frac{1}{2}(r-1)\delta$$

$$> \frac{1}{2}(r-1)r \quad \text{since } r \leq |F| = k' < \delta .$$

But a graph on r vertices cannot have more than $\frac{1}{2}(r-1)r$ edges. Thus, $|V(C_1)| > |A_1|$ and similarly, $|V(C_2)| > |A_2|$.

Thus each of C_1 and C_2 contains atleast $\delta + 1$ vertices.

$$\begin{aligned}\therefore n &= |V(G)| \\ &\geq 2(\delta + 1) \\ &\geq 2\left(\frac{n}{2} + 1\right) \\ &= n + 2\end{aligned}$$

This contradiction establishes the theorem.

Problem (2.2)

Show that if a and b are the only two odd degree vertices of a graph G, then a and b are connected in G.

Solution:

If G is connected, nothing to prove.

Let G be disconnected.

If possible assume that a and b are not connected.

Then a and b lie in the different components of G. Hence the component of G containing a (similarly containing b) contains only one odd degree vertex a, which is not possible as each component of G is itself a connected graph and in a graph number of odd degree vertices should be even.

Therefore a and b lie in the same component of G.

Hence they are connected.

Problem (2.3)

Prove that a connected graph G remains connected after removing an edge e from G if and only if e lie in some circuit in G.

Solution:

If an edge e lies in a circuit C of the graph G then between the end vertices of e, there exist at least two paths in G.

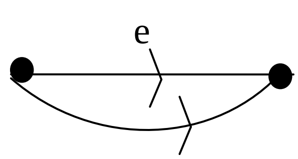


Figure (2.8)

Hence removal of such an edge e from the connected graph G will not effect the connectivity of G. Conversely, if e does not lies in any circuit of G then removal of e disconnects the end vertices of e.

Hence G is disconnected.

Definition (2.9)

A connected graph that has no cut vertices is called a *block*.

Every block with atleast 3 vertices is 2-connected.

Definition (2.10)

A *block of a graph* G is a maximal subgraph H of G such that H is a block.

Every graph is the union of its blocks.

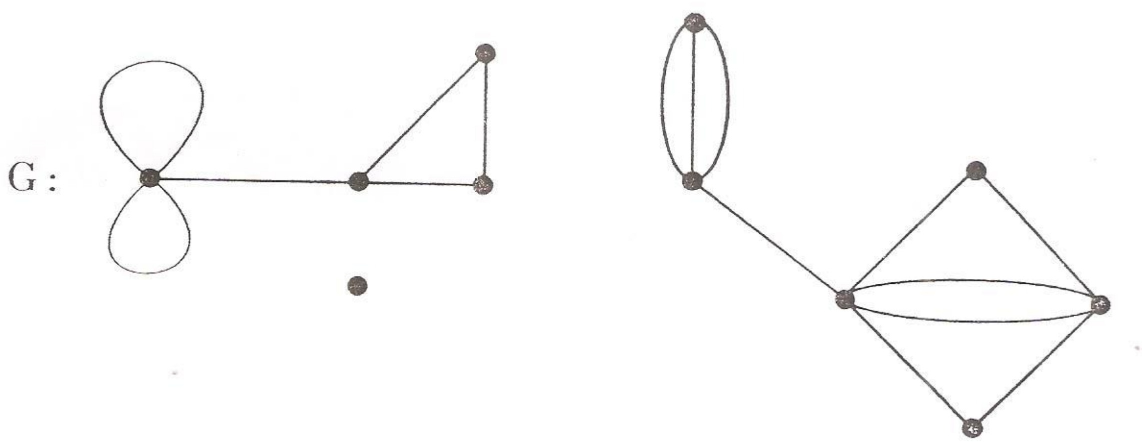
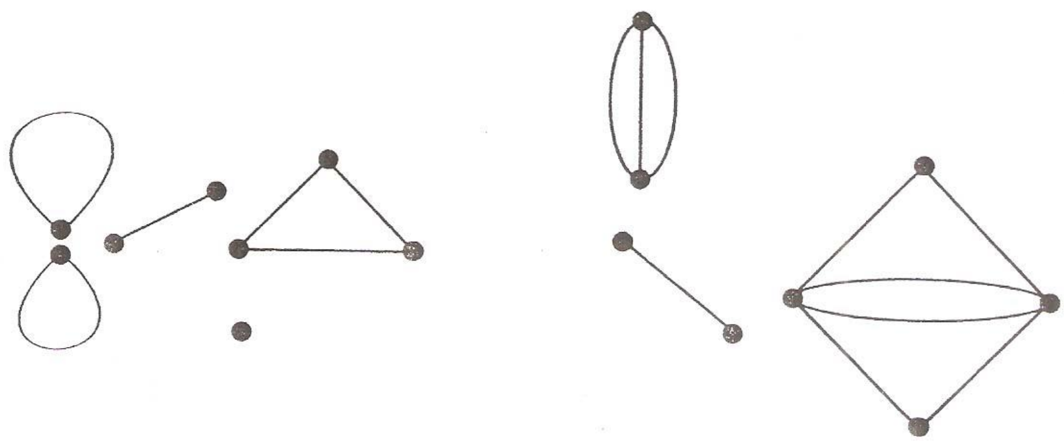


Figure (2.9)

The blocks of G are



Definition (2.11)

Let u and v be two vertices of a graph G. A collection $\{P(1), P(2), \ldots, P(n)\}$ of $u - v$ paths is said to be *internally disjoint* if , given any distinct pair $P(i)$ and $P(j)$ in the collection, u and v are the only vertices $P(i)$ and $P(j)$in common.

Theorem (2.14)

Let G be a simple graph with atleast 3 vertices. Then G is 2-connected if and only if for each pair of distinct vertices u and v of G there are two internally disjoint $u - v$ paths in G.

Proof :-

Suppose that any pair of distinct vertices is connected by a pair of internally disjoint paths. Then clearly G is connected so it remains to prove that G has no cut vertices. Assume to the contrary, that v is a cut vertex of G. Then there are two vertices u and w of G, both different from v, such that v is on every $u - w$ path in G. However, by the hypothesis there are two internally disjoint $u - w$ paths in G and atmost one of these can then pass through v. Thus v is not on every $u - w$ path, a contradiction. Hence G has no cut vertices.

Conversly, suppose that G is 2-connected. Let u and v be a pair of distinct vertices of G. We use induction on $d(u, v)$, the distance between u and v, to show that there is a pair of internally disjoint $u - v$ paths. First, if $d(u, v) = 1$ then u and v are joined by an edge, say e. It follows that e is not a bridge, since G has no cut vertices. Hence , there is a $u - v$ path P different and so internally disjoint from the $u - v$ path Q given by the single edge e.

We now assume that $d(u,v) = k \geq 2$ and that if x and y are any pair of vertices with $d(x, y) < k$ then there are two internally disjoint $x - y$ paths. Let P be a path of length k from u to v and let w be the second last vertex of P. Then $d(u, w) = k - 1$ and there are two internally disjoint $u - w$ paths say Q_1 and Q_2 .

Since G is 2-connected, w is not a cut vertex, i.e., $G - w$ is connected and so there is a $u - v$ path P' which does not pass through w. Let x be the last vertex of p' which is also a vertex of either Q_1 or Q_2 .

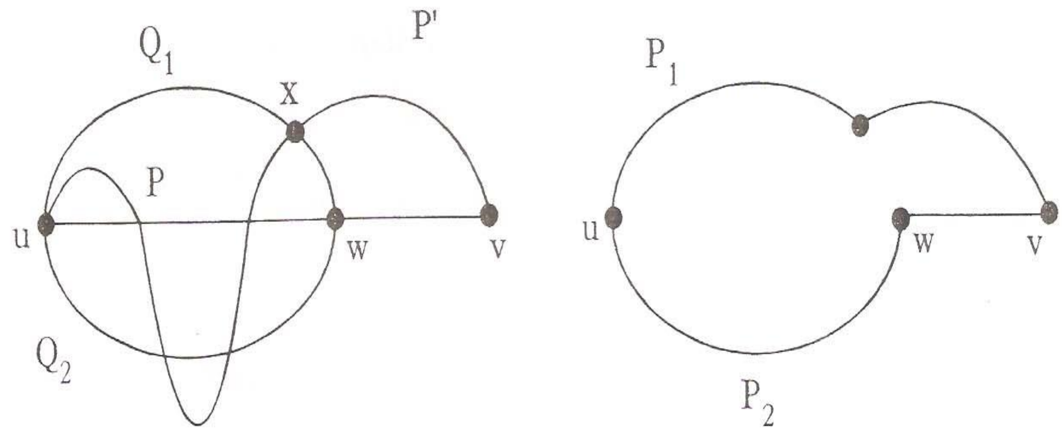


Figure (2.10)

Suppose that $x \in Q_1$. Let P_1 be the $u - v$ path given by the $u - x$ section of Q_1 followed by the $x - v$ section of P'. Let P_2 be the $u - v$ path given by the path Q_2 followed by the edge wv. Then P_1 and P_2 are internally disjoint by the definition of Q_1, Q_2, x and P'. The proof now follows by induction and is complete.

Corollary (2.15)

If G is 2-connected, then any two vertices of G lie on a common cycle.

Definition (2.12)

An edge e is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex.

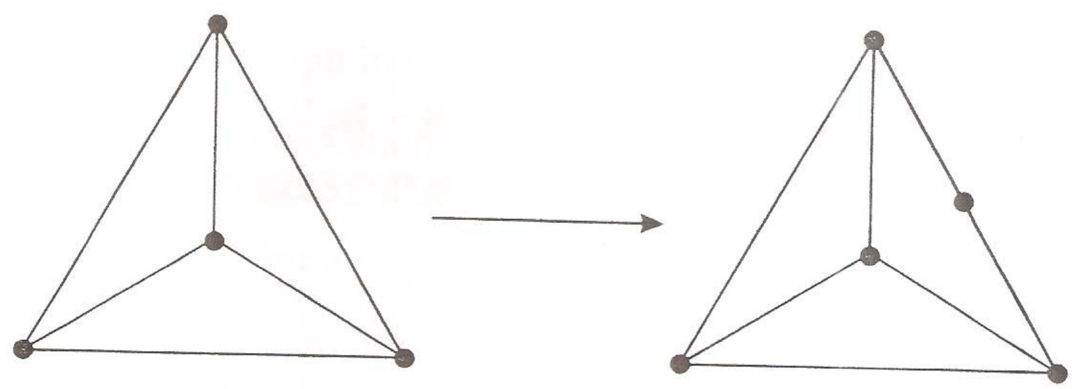


Figure (2.11)

Subdivision of an edge

Note :-

The class of blocks with atleast 3 vertices is closed under the operation of subdivision.

Corollary (2.16)

If G is a block with $n \geq 3$, then any two edges of G lie on a common cycle.

Proof :-

Let G be a block with $n \geq 3$, and let e_1 and e_2 be two edges of G. Form a new graph G' by subdividing e_1 and e_2 and denote the new vertices by v_1 and v_2. Clearly G' is a block with atleast five vertices, and hence is 2-connected. It follows that v_1 and v_2 lie on a common cycle of G'. Thus e_1 and e_2 lie on a common cycle of G.

CHAPTER-2

APPLICATIONS

Application (3.1)

If we think of graph as representing a communication network, the connectivity (or edge connectivity) becomes the smallest number of communication stations (or communication links) whose break down would jeopardize communication in the system. The higher the connectivity and edge connectivity, the more reliable the network. From this point of view, a tree network, such as the one obtained by Kruskal's Algorithm, is not very reliable, and one is led to consider the following generalization of the connector problem.

Let k be a given positive integer and let G be a weighted graph. Determine a minimum-weight k-connected spanning subgraph of G.

For k=1, this problem reduces to the connector problem, which can be solved by Kruskal's Algorithm. For values of k greater than one, the problem is unsolved and is known to be difficult. However if G is a complete graph in which each edge is assigned unit weight, then the problem has a simple solution which we now present.

Observe that, for a weighted complete graph on n-vertices in which edge is assigned unit weight, a minimum-weighted m-connected spanning subgraph is simply an m-connected graph on n vertices with as few edges as possible. We shall denote by $f(m,n)$ the least number of edges that an m-connected graph on n vertices can have (It is, of course, assumed that $m < n$).

$$f(m,n) \geq \left[\frac{mn}{2}\right] \longrightarrow (1)$$

We shall show that equality holds in (1) by constructing an m-connected graph $H_{m,n}$ on n vertices that has exactly $\left\lceil\frac{mn}{2}\right\rceil$ edges. The structure of $H_{m,n}$ depends on the pattern of m and n, there are 3 cases.

<u>Case 1</u> : m even. Let $m = 2r$. Then $H_{2r,m}$ is constructed as follows. It has vertices $0,1,\dots.,n-1$ and two vertices i and j are joined if $i-r \leq j \leq i+r$. $H_{8,n}$ is shown in figure 3.1 (a).

<u>Case 2</u> : m odd, n even. Let $m = 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i+\frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$. $H_{9,8}$ is shown in figure 3.1(b).

Case 3 : m odd, n odd. Let $= 2r+1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and vertex i to vertex $i+\frac{n+1}{2}$ for $1 \leq i < \frac{n-1}{2}$. $H_{9,9}$ is shown in figure 3.1 (c).

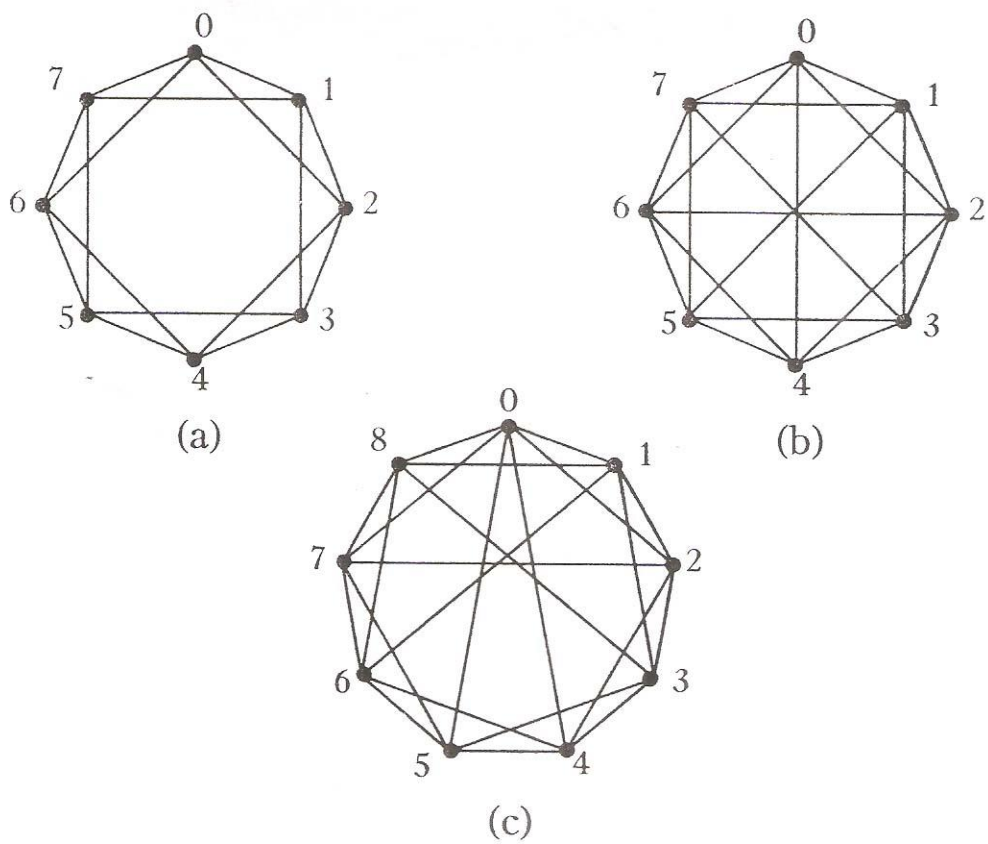


Figure (3.1)

Theorem (3.1) (Harary Theorem)

The graph $H_{m,n}$ is m-connected.

Proof :-

Consider the case $m - 2r$. We shall show that $H_{m,n}$ has no vertex cut of fewer than $2r$ vertices. If possible, let V' be a vertex cut with $|V'| < 2r$. Let i and j be vertices belonging to different components of $H_{2r,n} - V'$. Consider the two sets of vertices $S = \{i, i+1, \dots, j-1, j\}$ and $T = \{j, j+1, \dots, i-1, i\}$ where addition is taken modulo n. Since $|V'| < 2r$, we may assume, without loss of generality, that $|V' \cap s| < r$. Then there is clearly a sequence of distinct vertices in $S|\ V'$ which starts with i, ends with j, and is such that the difference between any two consecutive terms is atmost i. But such a sequence is an (i, j) path in $H_{2r,n} - V'$, a contradiction. Hence $H_{2r,n}$ is $2r$-connected. The case $m = 2r + 1$ is also $2r + 1$ -connected.

It is easy to see that $f(m,n) \le \left[\frac{mn}{2}\right] \longrightarrow$ (2)

It now follows from (1) and (2) that $f(m,n) = \left[\frac{mn}{2}\right]$ and that $H_{m,n}$ is an m-connected graph on n-vertices with a few edges as possible.

Application (3.2)

Suppose we are given n stations that are to be connected by means of e lines (telephone lines, bridges, rail roads, tunnels, or highways) where $e \ge n-1$. What is the best way of connecting? By “best” we mean that the network should be as invulnerable to destruction of individual stations and individual lines as possible. In other words, construct a graph with n vertices and e edges that has the maximum possible edge connectivity and vertex connectivity.

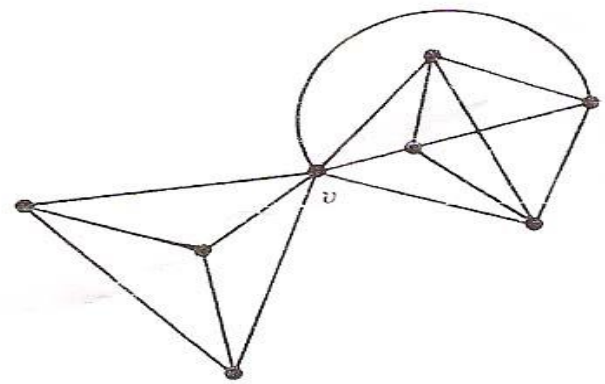


Figure (3.2)

For example, the graph in figure (3.2) has n=8, e=16, and has vertex connectivity of one and edge connectivity of three. Another graph with the same number of vertices and edges (8 and 16, respectively) can be drawn as shown in figure (3.3).

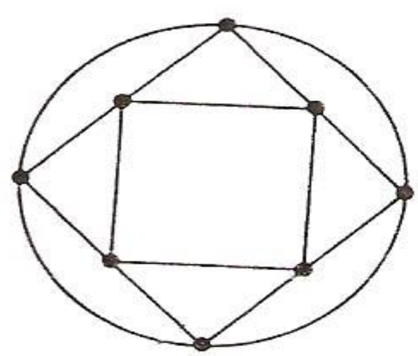


Figure (3.3)

Graph with 8 vertices and 16 edges

It can easily be seen that the edge connectivity as well as the vertex connectivity of this graph is four. Consequently, even after any three stations are bombed, or any three lines destroyed, the remaining stations can still continue to "communicate" with each other. Thus the network of figure (3.3) is better connected than that of figure (3.2) (although both consist of the same number of lines-16).

Theorem (3.2)

The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges $(e \geq n-1)$ is the integral part of the number $2e/n$

Proof :-

Every edge in G contributes two degrees. The total (2e degrees) is divided among n vertices. Therefore, there must be at least one vertex in G whose degree is equal to or less than the number $2e/n$. The vertex connectivity of G cannot exceed this number.

To show that this value can actually be achieved, one can first construct an n-vertex regular graph of degree equal to the integral part of the number $2e/n$ and then add the remaining $e-(n/2).(\text{integral part of the number } 2e/n)$ edges arbitrarily.

Thus we can summarize as follows:

$$\text{vertex connectivity} \leq \text{edge connectivity} \leq \frac{2e}{n},$$

and

maximum vertex connectivity possible = integral part of the number $2e/n$.

Thus for a graph with 8 vertices and 16 edges, we can achieve a vertex connectivity(and therefore edge connectivity) s high as four ($= 2 \bullet 16/8$).

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