Question

Economic conditions cause fluctuations in the prices of raw commodities as well as in finished products. Let X denote the price paid for a barrel of crude oil by the initial carrier, and let Y denote the price paid by the refinery purchasing the product from the carrier. Assume that the joint density for (X, Y) is given by

$$f_{XY}(x, y) = c \qquad 20 < x < y < 40$$

“(a) Find the value of c that makes this a joint density for a two-dimensional random variable.

(b) Find the probability that the carrier will pay at least \$25 per barrel and the refinery will pay at most \$30 per barrel for the oil.

(c) Find the probability that the price paid by the refinery exceeds that of the carrier by at least \$10 per barrel.

(d) Find the marginal densities for X and Y.

(e) Find the probability that the price paid by the carrier is at least \$25.

(f) Find the probability that the price paid by the refinery is at most \$30.

(g) Are X and Y independent? Explain.”

Answer

Step 1 of 8

(a)

Find the value of c that makes a joint density for a two-dimensional random variable.

Here, the joint density function is,

$$f_{XY}(xy) = c \quad ;20 < x < y < 40$$

The integral value of the probability density function over the entire region is 1.

Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(xy)\,dxdy = 1 \quad ;0 \le x \le \infty$$

Therefore, the probability density function is given by,

$$\int_{x=20}^{40}\int_{y=x}^{40} c\,dxdy = 1$$

$$c\int_{x=20}^{40} [y]_x^{40}\,dx = 1$$

$$c\int_{x=20}^{40} [40-x]\,dx = 1$$

$$c\left[40x - \frac{x^2}{2}\right]_{20}^{40} = 1$$

$$c\left\{\left[40(40) - \frac{40^2}{2}\right] - \left[40(20) - \frac{20^2}{2}\right]\right\} = 1$$

$$c\{[1,600-800]-[800-200]\} = 1$$

$$c(200) = 1$$

$$c = \frac{1}{200}$$

Thus, the value of *c* is

$\boxed{\frac{1}{200}}$

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Step 2 of 8

(b)

Find the probability that the carrier will pay at least \$25 per barrel and the refinery will pay at most \$30 per barrel for the oil.

The required probability is,

$P(x \geq 25, y \leq 30)$

$$\begin{aligned} P(x \geq 25, y \leq 30) &= \int_{x=25}^{30} \int_{y=x}^{30} \frac{1}{200} dxdy \\ &= \frac{1}{200} \int_{x=25}^{30} [y]_x^{30} dx \\ &= \frac{1}{200} \int_{x=25}^{30} [30 - x] dx \\ &= \frac{1}{200} \left[30x - \frac{x^2}{2} \right]_{25}^{30} \end{aligned}$$

$$= \frac{1}{200}\left[\left(30 \times 30 - \frac{30^2}{2}\right) - \left(30 \times 25 - \frac{25^2}{2}\right)\right]$$

$$= \frac{1}{200}[(900 - 450) - (750 - 312.5)]$$

$$= \frac{1}{200}[450 - 437.5]$$

$= 0.0625$

Therefore, the probability that the carrier will pay at least $25 per barrel and the refinery will pay at most $30 per barrel for the oil is

$\boxed{0.0625}$

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Step 3 of 8

(c)

Find the probability that the price paid by the refinery exceeds that of carrier by at least $10 per barrel.

The required probability is,

$P(y \geq x+10)$

$$\begin{aligned}
P(y \geq x+10) &= \int_{x=20}^{30} \int_{y=x+10}^{40} \frac{1}{200}\,dx\,dy \\
&= \frac{1}{200} \int_{x=20}^{30} [y]_{x+10}^{40}\,dx \\
&= \frac{1}{200} \int_{x=20}^{30} [40-x-10]\,dx \\
&= \frac{1}{200} \int_{x=20}^{30} [30-x]\,dx
\end{aligned}$$

$$=\frac{1}{200}\left[30x-\frac{x^2}{2}\right]_{20}^{30}$$
$$=\frac{1}{200}\left[\left(30\times 30-\frac{30^2}{2}\right)-\left(30\times 20-\frac{20^2}{2}\right)\right]$$
$$=\frac{1}{200}\left[(900-450)-(600-200)\right]$$
$$=\frac{1}{200}\left[450-400\right]$$

$$=0.25$$

Therefore, the probability that the price paid by the refinery exceeds that of carrier by at least $10 per barrel is

$\boxed{0.25}$

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Step 4 of 8

(d)

Find the marginal densities of X and Y.

The formula for obtaining the marginal density of X is

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$$f_X(x)=\int_{-\infty} f_{XY}(x,y)\,dy$$

Now,

$$\begin{aligned}f_X(x)&=\int_x^{40}\frac{1}{200}\,dy\\&=\frac{1}{200}\int_x^{40}1\,dy\\&=\frac{1}{200}(y)_x^{40}\\&=\frac{1}{200}(40-x)\end{aligned}$$

$$\begin{aligned}&=\frac{40}{200}-\frac{x}{200}\\&=0.2-0.005x\end{aligned}$$

Therefore, the marginal density of X is

$$f_X(x)=0.2-0.005x\ ;\ 20<x<40$$

Step 5 of 8

The formula for obtaining the marginal density of Y is

$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

Now,

$$
\begin{aligned}
f_Y(y) &= \int_{20}^{y} \frac{1}{200}\, dx \\
&= \frac{1}{200}\int_{20}^{y} 1\, dx \\
&= \frac{1}{200}(x)_{20}^{y} \\
&= \frac{1}{200}(y-20)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{y}{200} - \frac{20}{200} \\
&= 0.005y - 0.1
\end{aligned}
$$

Therefore, the marginal density of Y is

$$f_Y(y) = 0.005y - 0.2 \; ; \; 20 < y < 40$$

Step 6 of 8

(e)

Find the probability that the price paid by the carrier is at least $25.

That is,

$$P(x \geq 25)$$

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\begin{aligned}
P(x \geq 25) &= \int_{25}^{40} f_X(x)\,dx \\
&= \int_{25}^{40} 0.2 - 0.005x\,dx \\
&= \left(0.2x - 0.005\frac{x^2}{2}\right)_{25}^{40} \\
&= \left(0.2(40) - 0.005\frac{40^2}{2}\right) - \left(0.2(25) - 0.005\frac{25^2}{2}\right)
\end{aligned}
$$

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\begin{aligned}
&= (8-4) - (5-1.5625) \\
&= 4 - 3.4375 \\
&= 0.5625
\end{aligned}
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Therefore, the probability that the price paid by the carrier is at least $25 is

$\boxed{0.5625}$

Step 7 of 8

(f)

Find the probability that the price paid by the refinery is at most $30.

That is,

$P(y \leq 30)$

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\begin{aligned}
P(y \le 30) &= \int_{20}^{30} f_Y(y)\,dy \\
&= \int_{25}^{40} 0.005y - 0.1\,dy \\
&= \left(0.005\frac{y^2}{2} - 0.1y\right)_{20}^{30} \\
&= \left(0.005\frac{30^2}{2} - 0.1(30)\right) - \left(0.005\frac{20^2}{2} - 0.1(20)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (2.25 - 3) - (1 - 2) \\
&= -0.75 + 1 \\
&= 0.25
\end{aligned}
$$

Therefore, the probability that the price paid by the refinery is at most \$30 is

$\boxed{0.25}$

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Step 8 of 8

(g)

Check whether X and Y are independent.

The condition for independence is

$$f_{XY}(x,y)=f_X(x)f_Y(y)$$

Now,

$$f_{XY}(x,y)=\frac{1}{200}$$
$$f_X(x)f_Y(y)=(0.2-0.005x)(0.005y-0.2)$$

Here,

$$f_{XY}(x,y)\neq f_X(x)f_Y(y)$$

Therefore, *X* and *Y* are not independent.