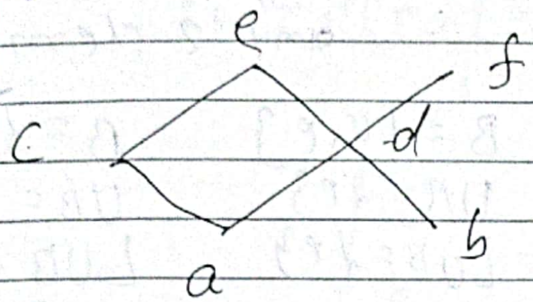
Lattice: (Join + Meet)

Join Semi Lattice and Meet semi Lattice.

① **Meet Semi Lattice:-** (single bottom) ($\wedge$)

In a poset, if GLB / Meet / Infimum / $\wedge$ exists for any pair of elements, then POSET is called "Meet - semi" lattice

ex-1



⇒ we will find where meet structure is not exist

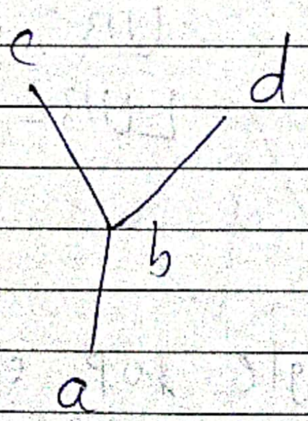
⇒ Not a Meet semi Lattice.

$B = \{e, f\}$ ← generalise for 2 elements so it will become binary operator

$LB = \{a, d, b\}$
$GLB = \{d\} \rightarrow$ Meet
$B = \{c, d\} \Rightarrow$ Meet $= a$
$a \wedge b$, meet $\Rightarrow \phi$

⇒ Not Meet Semi Lattice

ex-2



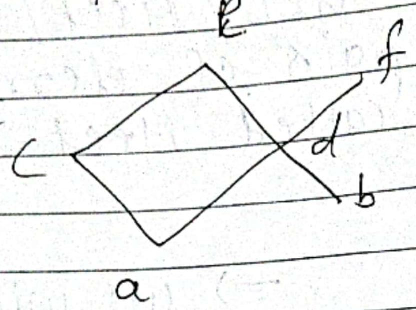
→ C & D
$LB = \{b, a\}$
$GLB =$ Meet $= b$
a & b meet = a
↓
single bottom

There exist Meet for all set of pairs

(2) Join-Semi Lattice: (single top) (V)

In a Poset if LUB / Join / supremum of element exists for any pair of element then poset is called Join-Semi Lattice

eg:



$x \vee y \rightarrow$ Binary

$\underbrace{x \vee y}_{\text{any 2 element}}$

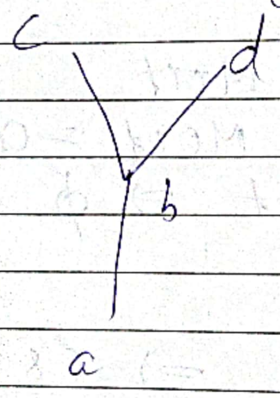
$B = \{a, b\}$
$UB = \{e, d, f\}$
$LUB = \{d\}$

$B = \{a, e\}$
$UB = \{e\}$
$LUB = \{e\}$

$B = \{e, f\}$
$UB = \phi$
$LUB = \phi$

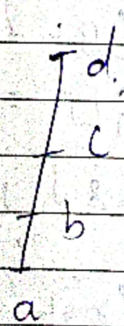
Not a Join Semi Lattice
(Not a single top.)

eg:



$B = \{c, d\}$
$UB = \phi$
$LUB = \phi$

eg.



single top exist
Join Semi Lattice.

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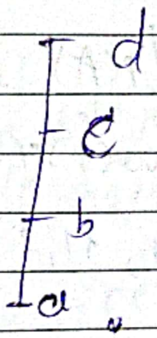
⇒ Lattice :-

A poset is called Lattice if it is both Meet Semi Lattice & Join Semi Lattice

Or

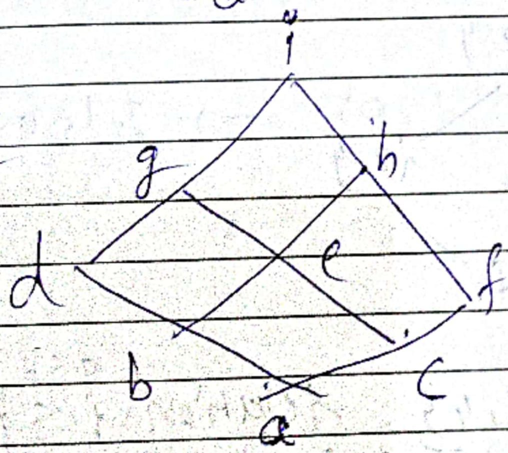
these exists single top & single bottom.

eg. for every pair of element these exist a Meet & Join then it is called Lattice



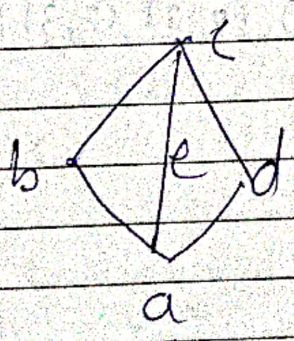
→ it's $\wedge$ - Lattice
$\vee$ - Lattice
↓
∴ it's a lattice

eg.



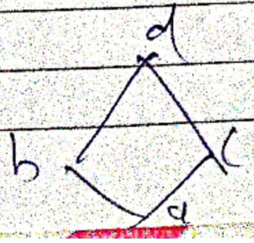
$\vee$ - Lattice
$\wedge$ - Lattice
⇓
it's a lattice

eg.



→ Lattice.

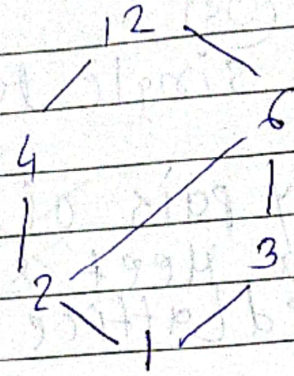
eg.



→ Lattice.

Ex $[D_{12}, /]$
↳ 12 ko divide kre.

$S = \{1, 2, 3, 4, 6, 12, /\}$

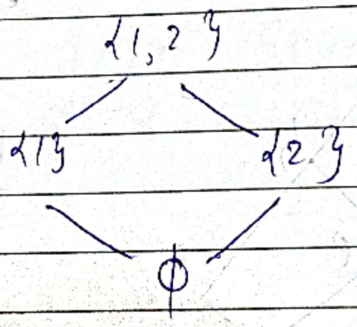


So it is Lattice.

eg: Set $A = \{1, 2\}$, $[P(A), \subseteq]$ → always Lattice

P(A) ↓ power set of A; $\subseteq$ ↳ subset

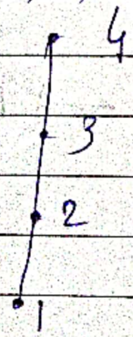
$P(A) = \{\{\phi\}, \{1\}, \{2\}, \{1, 2\}\}$



⟹ It's a Lattice.

eg: Set $A = \{1, 2, 3, 4\}$ relation $\{\leq\}$ ($a \leq b$)

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$



So it's lattice

LATTICES

Definitions

A partially ordered set $\{L, \leq\}$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *lattice*.

The LUB (supremum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \vee b$ [or $a \oplus b$ or $a + b$ or $a \cup b$] and is called the *join* or *sum* of a and b.

The GLB (infemum) of a subset $\{a, b\} \subseteq L$ is denoted by $a \wedge b$ [or $a * b$ or $a \bullet b$ or $a \cap b$] is called the *meet* or *product* of a and b.

Note Since the LUB and GLB of any subset of a poset are unique, both $\wedge$ and $\vee$ are binary operations on a lattice.

For example, let us consider the poset ($\{1, 2, 4, 8, 16\}$|), where | means 'divisor of'. The Hasse diagram of this poset is given in Fig. 2.26.

The LUB of any two elements of this poset is obviously the larger of them and the GLB of any two elements is the smaller of them. Hence this poset is a lattice.

Fig. 2.26

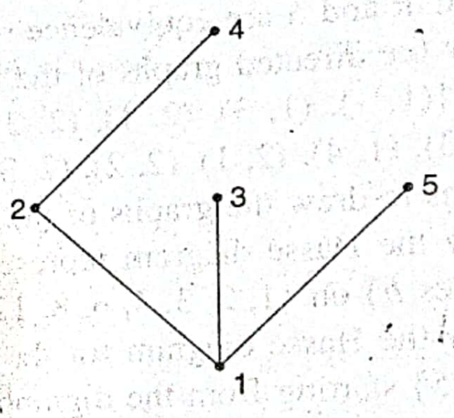


Fig. 2.27

Note All partially ordered sets are not lattices, as can be seen from the following example.

Let us consider the poset ($\{1, 2, 3, 4, 5\}$, |) whose Hasse diagram is given in Fig. 2.27.

The LUB's of the pairs (2, 3) and (3, 5) do not exist and hence they do not have LUB. Hence this poset is not a Lattice.

PRINCIPLE OF DUALITY

When $\leq$ is a partial ordering relation on a set S, the converse $\geq$ is also a partial ordering relation on S. For example if $\leq$ denotes 'divisor of', $\geq$ denotes 'multiple of'.

The Hasse diagram of $(S, \geq)$ can be obtained from that of $(S, \leq)$ by simply turning it upside down. For example the Hasse diagram of the poset ({1, 2, 4, 8, 16}) will be as given in Fig. 2.28.

multiple of, obtained from 2.26 will be as given in Fig 2.28.

From this example, it is obvious that LUB(A) with respect to $\leq$ is the same as GLB(A) with respect to $\geq$ and vice versa, where $A \subseteq S$. viz. LUB and GLB are interchanged, when $\leq$ and $\geq$ are interchanged.

In the case of lattices, if $\{L, \leq\}$ is a lattice, so also is $\{L, \geq\}$. Also the operations of join and meet on $\{L, \leq\}$ become the operations of meet and join respectively on $\{L, \geq\}$.

From the above observations, the following statement, known as *the principle of duality* follows:

Any statement in respect of lattices involving the operations $\vee$ and $\wedge$ and the relations $\leq$ and $\geq$ remains true, if $\vee$ is replaced by $\wedge$ and $\wedge$ is replaced by $\vee$, $\leq$ by $\geq$ and $\geq$ by $\leq$.

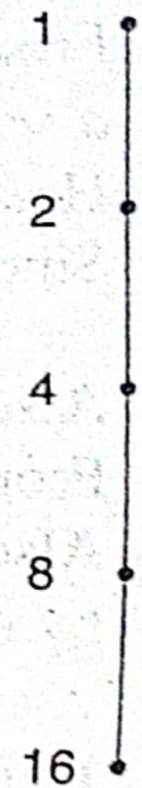


Fig. 2.28

The lattices $\{L, \leq\}$ and $\{L, \geq\}$ are called the *duals* of each other. Similarly the operations $\vee$ and $\wedge$ are duals of each other and the relations $\leq$ and $\geq$ are duals of each other.

PROPERTIES OF LATTICES