Discrete models II- Discrete Logistic models

In many situations involving populations, the larger the population, the faster it grows. This suggests a proportionality relationship between the population and the rate of growth. Table 2.3 shows the population (in thousands) of bacteria in a Petri dish at different points in time. The third column contains the change in population between time periods.

TABLE 2.3

Day n	Actual Population p_n	Change in Population $\Delta p_n = p_{n+1} - p_n$
0	10.3	6.9
1	17.2	9.8
2	27	18.3
3	45.3	34.9
4	80.2	45.1
5	125.3	50.9
6	176.2	79.4
7	255.6	

Observe that as n increases, p_n increases, and so does Δp_n. This suggests that p_n is proportional to Δp_n. A graph of Δp_n vs. p_n is shown in Figure 2.13.

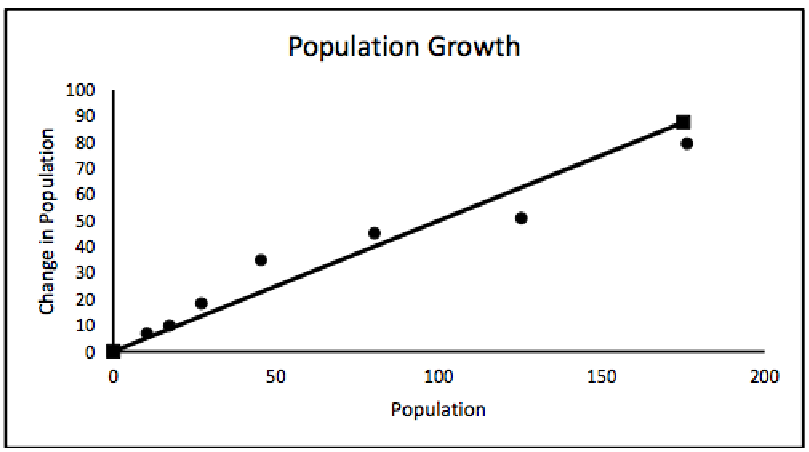

FIGURE 2.13

We see that the points do fall near a straight line through the origin, suggesting that proportionality is a reasonable assumption. The slope of this line is approximately 0.5. This gives a model that relates the population at one day, p_n, to the population at the next, p_{n+1}:

$$\Delta p_n = p_{n+1} - p_n = 0.5 p_n \quad \Rightarrow \quad p_{n+1} = 1.5 p_n$$

This model predicts that the population grows by about 50% each time period, which means the population will grow without bound. In other words, this model predicts a population that increases forever, which is questionable. Since this seems unreasonable, so the model needs to be refined.

1 Model Refinement: Modeling Births, Deaths, and Resources

If both births and deaths during a period are proportional to the population, then the change in population should be proportional to the population, as was illustrated in the above example. However, certain resources (e.g., food) can support only a maximum population (**carrying capacity**) level rather than one that increases indefinitely. As these maximum levels are approached, growth should slow.

Definition 1.1 *The **carrying capacity** of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.*

We'll study how to refine a model using the following example which in the textbook "Mathematical Modeling with Excel, Second edition"

Example: Bacteria in a Petri dish

Table 4.2 gives the number of bacteria in a Petri dish, a_n, at the end of each hour n. This data is graphed in Figure 4.10. We want to model a_n in terms of n.

When modeling a dynamical system, it is often convenient to think about the way the variable(s) change between time periods. Specifically, we consider the change between time periods $\Delta a_n = a_{n+1} - a_n$. The values of Δa_n for the first 7 values of n are given in Table 4.3.

Notice that as a_n increases, Δa_n also increases. This suggests that Δa_n is proportional to a_n, which leads to the equation

$$\Delta a_n = a_{n+1} - a_n = r\, a_n \tag{4.3}$$

TABLE 4.2

n	0	1	2	3	4	5	6	7	8	9
a_n	10.3	17.2	27	45.3	80.2	125.3	176.2	255.6	330.8	390.4
n	10	11	12	13	14	15	16	17	18	19
a_n	440	520.4	560.4	600.5	610.8	614.5	618.3	619.5	620	621

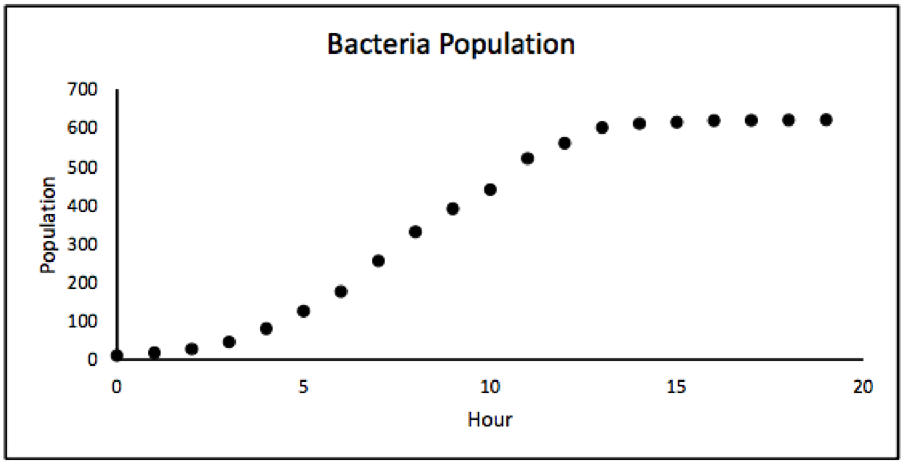

FIGURE 4.10

TABLE 4.3

n	0	1	2	3	4	5	6
a_n	10.3	17.2	27	45.3	80.2	125.3	176.2
Δa_n	6.9	9.8	18.3	34.9	45.1	50.9	79.4

where r is some positive constant. An equation describing the difference in populations between time periods, such as (4.3), is called a *difference equation.* Forming a difference equation is often the first step in modeling a discrete dynamical system. Solving this equation for a_{n+1} yields the model

$$a_{n+1} = (1 + r)\, a_n$$

The parameter r can be interpreted as the constant hourly growth rate. However, the graph of population vs. hour shows that the population does not grow at a constant rate. Also note that this constant hourly growth rate would predict a population that grows without bound, which the data do not support either.

To refine the model, note that the graph shows that the rate of growth decreases as the population nears 621. This number is called the *carrying capacity* of the system. So instead of assuming a constant growth rate r, we assume a growth rate that approaches 0 as the population approaches 621. An equation implementing this assumption is

$$\Delta a_n = a_{n+1} - a_n = b(621 - a_n)\, a_n \tag{4.4}$$

where $b > 0$ is a constant. Solving for a_{n+1} yields the model

$$a_{n+1} = a_n + b(621 - a_n)\, a_n \tag{4.5}$$

621 = carrying capacity
b = growth rate that balances out other elements (the limitations)

Equation (4.5) is an example of a *discrete logistic equation.*

This is called a logistic model

Definition 1.2 ***(Discrete Logistic Equation)****. A discrete logistic equation (also called a logistic map or a constrained growth model) is an equation of the form*

$$a_{n+1} = a_n + b(c - a_n)a_n,$$

where b and c are constants. This type of equation is often used to model population growth where an is the population at time n. The constant b is called the intrinsic growth rate and c is called the carrying capacity.

Activity: Testing the logistic growth model

1. Plot the observed data in the table 4.2.
2. We have already noticed that the model $a_{n+1} = (1+r)a_n$ is not described the observed data. We'll now test the refined model. Then we need to find the suitable value for b. Propose a method to find b.
   Notice that $a_{n+1} - a_n = b(c - a_n)a_n$.
   What does this tell you? Is $a_{n+1} - a_n \propto (c - a_n)a_n$? Recalling your previous knowledge, find b and write the logistic equation.
3. Use the refine model to predict the population for the first 19 hours.
4. Plot the predicted values and observed values in the same figure and validate the model.