Discrete Models

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1 Introduction

A *dynamical system* is simply a system that changes over time. When time is measured in discrete increments, the system is called a *discrete dynamical system*. In mathematical terms, a discrete dynamical system is simply a sequence of numbers. Consider a savings account that is compounded yearly and the interest is added at the end of each year. If a_n is the amount in the account at the end of year $n (n = 0, 1, ...)$ and r is the interest rate, we have the sequence

$$a_1 = a_0 + ra_0 = (1 + r)a_0$$

$$a_2 = a_1 + ra_1 = (1 + r)a_1$$

Because this dynamical system changes over discrete time intervals, the system is called a discrete dynamical system

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$a_{n+1} = a_n + ra_n = (1 + r)a_n$$

difference equation

Based on the last equation, we can define the formal definition of linear dynamical system.

Definition 5.1: A linear discrete dynamical system is a sequence of numbers $an|n = 0, 1, ...$ defined by a relation of the form

$$a_{n+1} = ba_n$$

for some constant $b \neq 0$.

b = (1+r)

Any dynamical system that does not have this linear form is generically referred to as *non-linear*. A solution of a discrete dynamical system is an explicit description of a_n in terms of n and the initial state a_0.

Theorem 5.1: The solution of a linear dynamical system $a_{n+1} = ba_n$ for $b \neq 0$ is

$$a_n = b^n a_0$$

where a_0 is the initial state.

analytical solution of the difference equation

Long-Term Behavior

Theorem 5.1 gives us an easy-to-use formula for finding the exact value of a_n. However, we are usually more interested in describing the long-term behavior of the system than in finding exact values at points in time. That is, we want to know what happens to a_n for large values of n.

Activity 01

Graphically examine the long-term behavior of a linear dynamical system $a_{n+1} = ba_n$ for various values of b by considering, $a_0 = 0.1, a_0 = 1$ and $a_0 = 1.4$.

1. When $b < -1$
2. When $b = -1$
3. When $-1 < b < 0$
4. When $b = 0$
5. When $0 < b < 1$
6. When $b = 1$
7. When $b > 1$

Then we can come summarize above observations as below.

refer activity 1 word document

Value of b	**Behavior of a_n**
$b < -1$	Oscillates between pos. and neg., $\lvert a_n \rvert$ grows without bound
$b = -1$	Oscillates between $-a_0$ and $+a_0$
$-1 < b < 0$	Oscillates between pos. and neg., $\lvert a_n \rvert$ approaches 0
$b = 0$	$a_n = 0$ for $n > 0$
$0 < b < 1$	a_n approaches 0
$b = 1$	$a_n = a_0$ for all n
$b > 1$	a_n grows without bound

Activity 02

Now consider a savings account that pays 5% interest compounded yearly. If the interest rate is r (annual) and a_n is the total money at the end of the year, can you derive a model to describe a_n?

Activity 03

Consider the Saving account described in Activity 02. Now suppose we want to withdraw \$ 4000 at the end of each year to supplement our income. Can you calculate the minimum amount of money to deposit so that we never run out of money?

Activity 04

1. Using a single Octave script, find money in the account for the first 20 years when the initial deposit is
   (a) $a_0 = \$80000$.
   (b) $a_0 = \$100000$.
   (c) $a_0 = \$60000$.
2. Show the above results graphically in one figure.
3. What is your observation?

Activity 05

An infant is given the antibiotic co-amoxiclav to treat a respiratory infection. When taking an antibiotic, it is important to keep the amount of the drug in the bloodstream fairly constant. If the amount gets too low, the bacteria can begin to regrow. If the amount gets too high, it could cause other complications. Suppose the half-life of the drug is 1 day (meaning that half the co-amoxiclav remains in the blood after each 1-day period) and a dosage of 0.1 mg is given at the end of each day. Create a graph to represent the amount of antibiotics in the blood at the end of the day for the next 15 days. If the daily dosage increases up to 0.15 mg how many days will it take to achieve the constant antibiotic level in the bloodstream?