Course Name: Discrete Mathematics

Unit I - Set Theory and Logic

Department of Artificial Intelligence and Data Science Engineering

Course Name: Discrete Mathematics

Unit I – Set Theory & Logic

Course Contents		
Unit I	**Set Theory and Logic**	**(07 Hours)**
Introduction and significance of Discrete Mathematics, **Sets–** Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operations, Cardinality of set, Principle of inclusion and exclusion. **Types of Sets** – Bounded and Unbounded Sets, Diagonalization Argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set, **Propositional Logic-** logic, Propositional Equivalences, Application of Propositional Logic-Translating English Sentences, Proof by Mathematical Induction and Strong Mathematical Induction.		

Session 1

- Unit 1 Introduction
- Set Representation
- Subset
- Types of Set
- Examples

Why Discrete Mathematics ?

- Discrete mathematics : deals with discrete objects – which r separated from each other.

eg. Integers, automobiles, houses, people ,

distinct paths to travel from point A to point B on a map along a road network,

-- ways to pick a winning set of numbers in a lottery.

- A course in discrete mathematics provides the mathematical background needed for all subsequent courses in computer science.

- Computers run software and store files. The software and files are both stored as huge strings of 1s and 0s. Binary math is discrete mathematics.

- Encryption and decryption are part of cryptography, which is part of discrete mathematics. For example, secure internet shopping uses public-key cryptography.
- An analog clock has gears inside, and the sizes/teeth needed for correct timekeeping are determined using discrete math.
- Google Maps uses discrete mathematics to determine fastest driving routes and times.
- Railway planning uses discrete math: deciding how to expand train rail lines, train timetable scheduling, and scheduling crews and equipment for train trips use both graph theory and linear algebra.

- Cell phone communications: Making efficient use of the broadcast spectrum for mobile phones uses linear algebra and information theory. Assigning frequencies so that there is no interference with nearby phones can use graph theory or can use discrete optimization.
- Graph theory and linear algebra can be used in speeding up Facebook performance.
- Graph theory is used in DNA sequencing.
- **Computer graphics** (such as in video games) use linear algebra in order to transform (move, scale, change perspective) objects.

A program developer is to be able to choose the right algorithms and data structures for the program which is supposed to solve.

The importance of discrete mathematics used in the analysis of algorithms and in many common data structures – graphs, trees, sets and ordered sets etc.

Why ? :- The analysis of an algorithm requires one to carry out a proof of the correctness of the algorithm and a proof of its complexity.

Programmers use logic. While everyone has to think about the solution, a formal study of logical thinking helps you organize your thought process more efficiently.

Moral of the Story

- Discrete Math is needed to see mathematical structures in the object you work with, and understand their properties. This ability is important for software engineers, data scientists, business analyst ,security and financial analysts.

- it is not a coincidence that math puzzles are often used for interviews.

HAPPY LEARNING !!

What is discrete mathematics?

- Branch of mathematics
- Discrete or distinct objects

What is mean by discrete?

- Individual, separate, distinguishable or discontinuous

What is it’s scope?

- Whole numbers, but not floating numbers

Calculation of addition in DM:

$$\sum_{i=0}^{7} x_i = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

Distinct numbers are

{ 1, 2 , 3 , 4 …..9,…..23…. }

What are these numbers? **Natural**

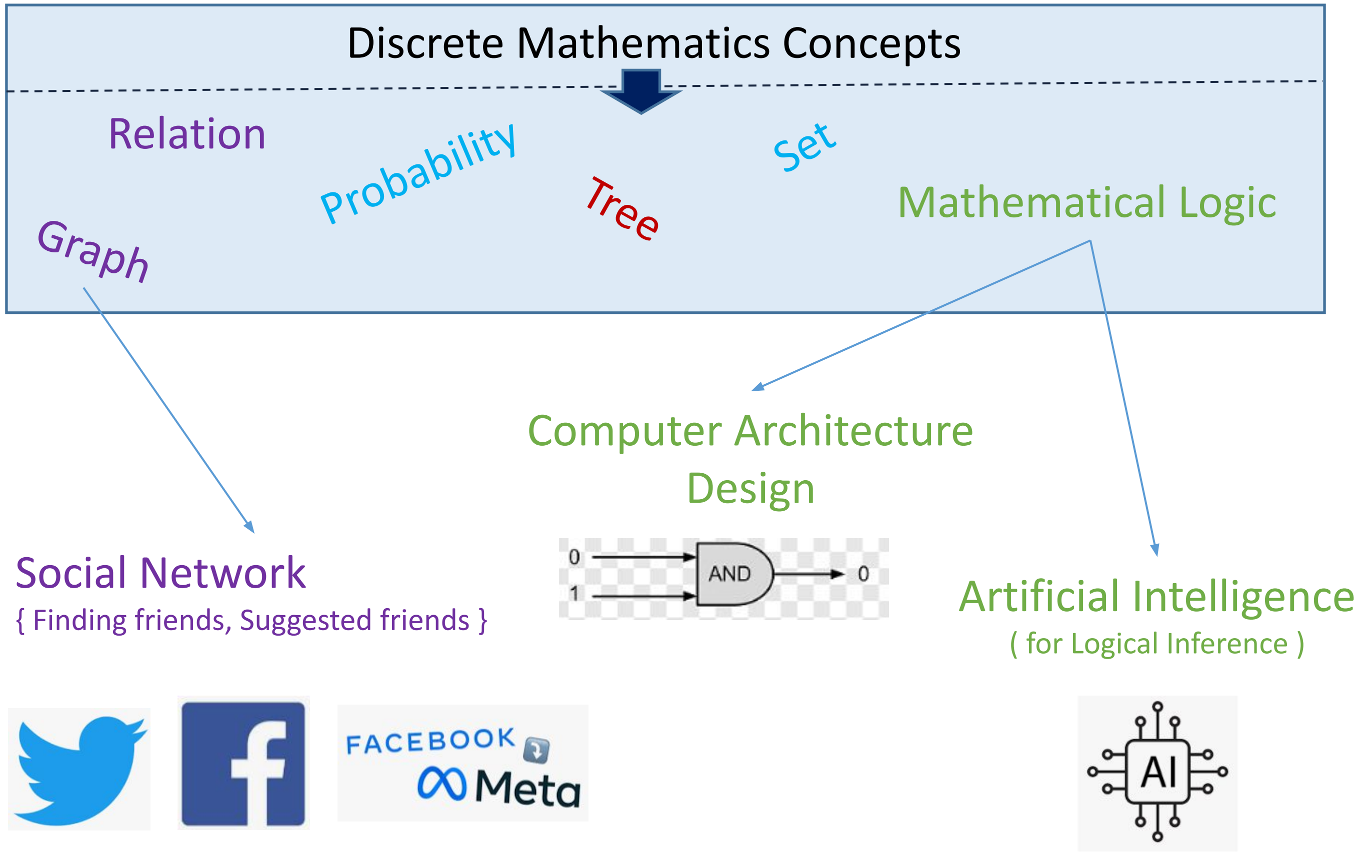
2.4, 6.98, 3½, ….

Significance of Discrete Mathematics in Computer Engineering

- Backbone of computer engineering
- A mathematical language of computer science

Usage: Describing objects & problems in computer engineering in

- Programming language
- Cryptography
- Software development
- Algorithms complexity
- & Many more….



- **Set** is collection of definite, distinguishable objects
- **Object** is called as **element** or **member** of the set

A = { a, e, i, o, u }

Jan = { 1, 2, 3 …., 31 }

X = { a1, a2, a3 }

Representation of Set

Roster Method

A = { 1, 3, 5, 7, 9 }

Statement Method

The set of all books in the library

Set Builder Method

A = { x | x is an odd positive integer less than 10 }

If every element of set A is also an element of set B

then

- A is **subset** of B
- A is **contained** in B

A = { **2, 4** }

B = { 1**, 2**, 3**, 4**, 5 }

<u>Subset</u>

Proper Subset

A = { **3, 6, 8** }

B = { 10, **3, 8**, 5, **6** }

if $A \subseteq B$ and $A \neq B$,

$A \subset B$ (A is a proper subset of B).

Improper Subset

A = { **1, 2, 3, 4** }

B = { **3, 4, 1, 2** }

All elements of A contains in set B

If every element in set 'A' is also an element of a set B, that is $a \in A$ and $a \in B$. Then set 'A' is a subset of set 'B' or set 'A' is contained in set 'B'.

It is denoted as $A \subseteq B$ or $B \supseteq A$

Here, $A \subseteq B$ means A is a subset of B.

$B \supseteq A$ mean B is superset of A.

$A = \{5, 6, 8\}$

$B = (1, 8, -3, 5, 12, 6)$

$C = \{15, -3, 5, 8\}$

$A \nsubseteq C$ $\quad$ $A \subseteq B$

Types of set

- Universal set A = { x | x is the student of class 10^{th} }, B = { x | x is the student of class 9^{th} }, **U = { x | x is the student of school }**
- Singleton set / Unit set

  A = { 1 } … Only one element
- Finite set

  A = { 10, 3, 8, 5, 6 } … Countable elements
- Infinite set

  A = { x | x is the natural number } … Uncountable elements
- Empty or Null set

  A = { x | x is the natural number less than 1 }

- Disjoint set :does not have any common element

  A = { 1, 2 } B = { 4, 5, 6 }

- Overlapping set: sets that have at least one element in common

  A = { 3, 6, 8 } B = { 10, 5, 3 }

- Equal set

  A = { 1, 2, 3 } B = { 3, 1, 2 }

- Equivalent setτ

  A = { a, d, f } B = { c, l, m }

- Power set

  Let A = { a, c } then P(A) = { { $\emptyset$ } , { a } , { c } , { a , c } } ……… 2^n Possible s

Ex. 1.6.1 : Which of the following sets are equal

$A = \{a, b, c\}$, $B = \{c, b, c, a\}$

$C = \{b, a, b, c\}$, $D = \{b, c, a, b\}$

Solution:

A = B = C = D

All are equal,

This is because order and repetition do not change a set

?

Let $A = \{a, b, \{a,b\}, \{\{a,b\}\}\}$

Identify true or False

1) $a \in A$
2) $\{a\} \in A$
3) $\{a, b\} \in A$
4) $\{\{a,b\}\} \in A$
5) $\{a, b\} \subseteq A$
6) $\{a, \{b\}\} \subseteq A$

Let $A = \{a, b, \{a, b\}, \{\{a, b\}\}\}$

Identify true or false

True 1) $a \in A$

False 2) $\{a\} \in A$ As { a } is subset of A, however its not an element of A

True 3) $\{a, b\} \in A$

True 4) $\{\{a, b\}\} \in A$

True 5) $\{a, b\} \subseteq A$

False 6) $\{a, \{b\}\} \subseteq A$ a is available in A, however { b } is not

?

Determine true or false

1) If $A \in B$ and $B \subseteq C$ then $A \in C$

2) If $A \in B$ and $B \subseteq C$ then $A \subseteq C$

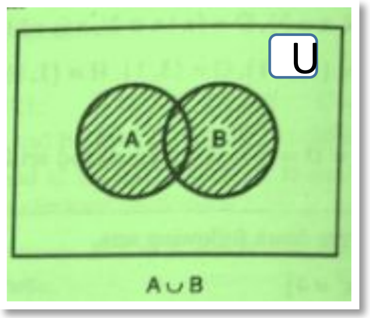
3) If $A \subseteq B$ and $B \in C$ then $A \in C$

4) If $A \subseteq B$ and $B \in C$ then $A \subseteq C$

Session 2

- Set Operations
- Set Operations – Examples
- Venn Diagram
- Venn Diagram – Examples
- Algebra of Set Operations
- Cardinality of Set

Union



Union A **U** B is set of all elements which belongs to set A or set B

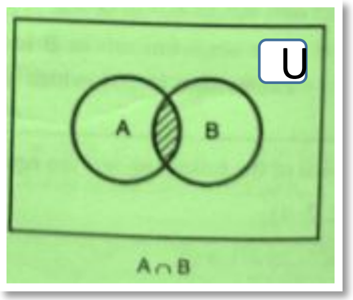
$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

Example

If A = { a, b, c, d }, B = { p, q, r, s }

Then A **U** B = { a, b, c, d, p, q, r, s }

Intersection



Intersection A ∩ B is set of all elements which belongs to set A and B

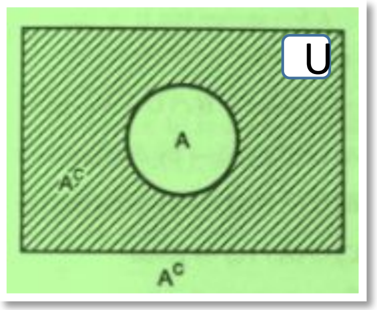
$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

Example

If A = { a, b, c, d }, B = { c, x, y, d }

Then A ∩ B = { c, d }

Complement



Complement of set A is set of elements which belongs to U but does not belong to A

$A^C = \{ x \mid x \in U, \ x \notin A \}$

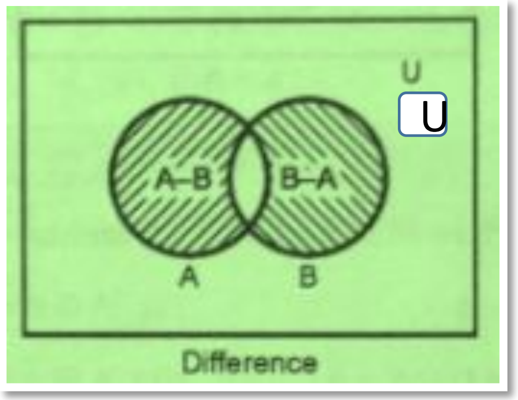
Example: If

A = { 1, 2, 3, 4 },

U = {1, 2, 3, 4, 5, ………, 10 }

Then A^c = { 5, 6, 7 …. , 10}

Difference



Difference of A ***and*** B is set elements which belongs to set A but not B

$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$

$B - A = \{ x \mid x \in B \text{ and } x \notin A \}$

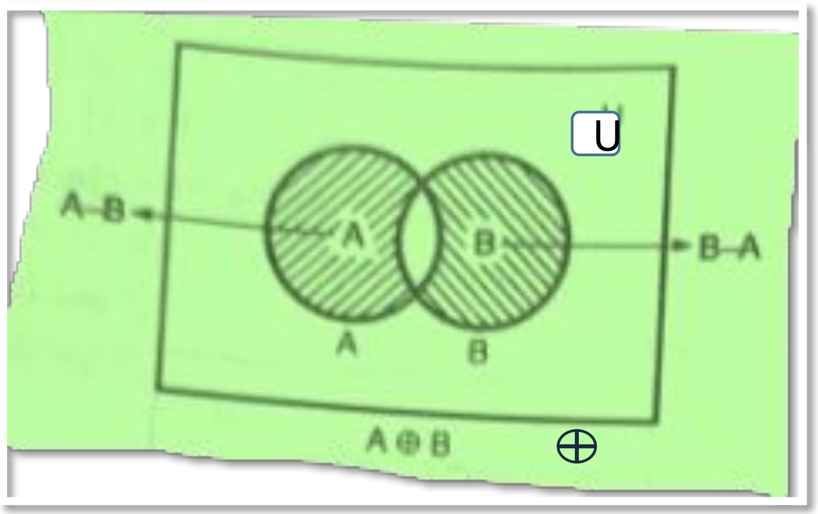
Example1

If A = { a, b, c, d, e }, B = { f, d, a, x, y }

Then A - B = { b, c, e }

Then B - A = { f, x, y }

Symmetric Difference



Symmetric difference A B is set of all elements which belongs to A or B but not belongs to both

$A \oplus B = \{ x \mid x \in A - B \text{ or } x \in B - A \}$

Example

If A = { a, b, c, d, e }, B = { c, d, e, f, g, h }

Then A - B = { a, b }, B – A = { g, h }

$A \oplus B = \{ a, b, g, h \}$

Complement and Difference Example

E2 If $U = \{n \mid n \in N,\ n \leq 15\}$,
$A = \{n \mid n \in N,\ 4 < n < 12\}$
$B = \{n \mid n \in N,\ 8 < n < 15\}$,
$C = \{n \mid n \in N,\ 5 < n < 10\}$,

find $\bar{A} - \bar{B}$ and $\bar{C} - \bar{A}$.

Solution:

$$\bar{A} = \{1, 2, 3, 4, 12, 13, 14, 15\}$$
$$\bar{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 15\}$$
$$\bar{C} = \{1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15\}$$

$$\therefore\ \bar{A} - \bar{B} = \{12, 13, 14\}$$
$$\bar{C} - \bar{A} = \{5, 10, 11\}$$

Difference Example

Examples

E1 If $A = \{a, b, \{a, c\}, \phi\}$, determine the following sets.

i) $A - \{a, c\}$

ii) $\{\{a, c\}\} - A$

iii) $A - \{\{a, b\}\}$

iv) $\{a, c\} - A$

Solution:

i) $A - \{a, c\} = \{b, \{a, c\}, \phi\}$

ii) $\{\{a, c\}\} - A = \phi$

iii) $A - \{\{a, b\}\} = A$

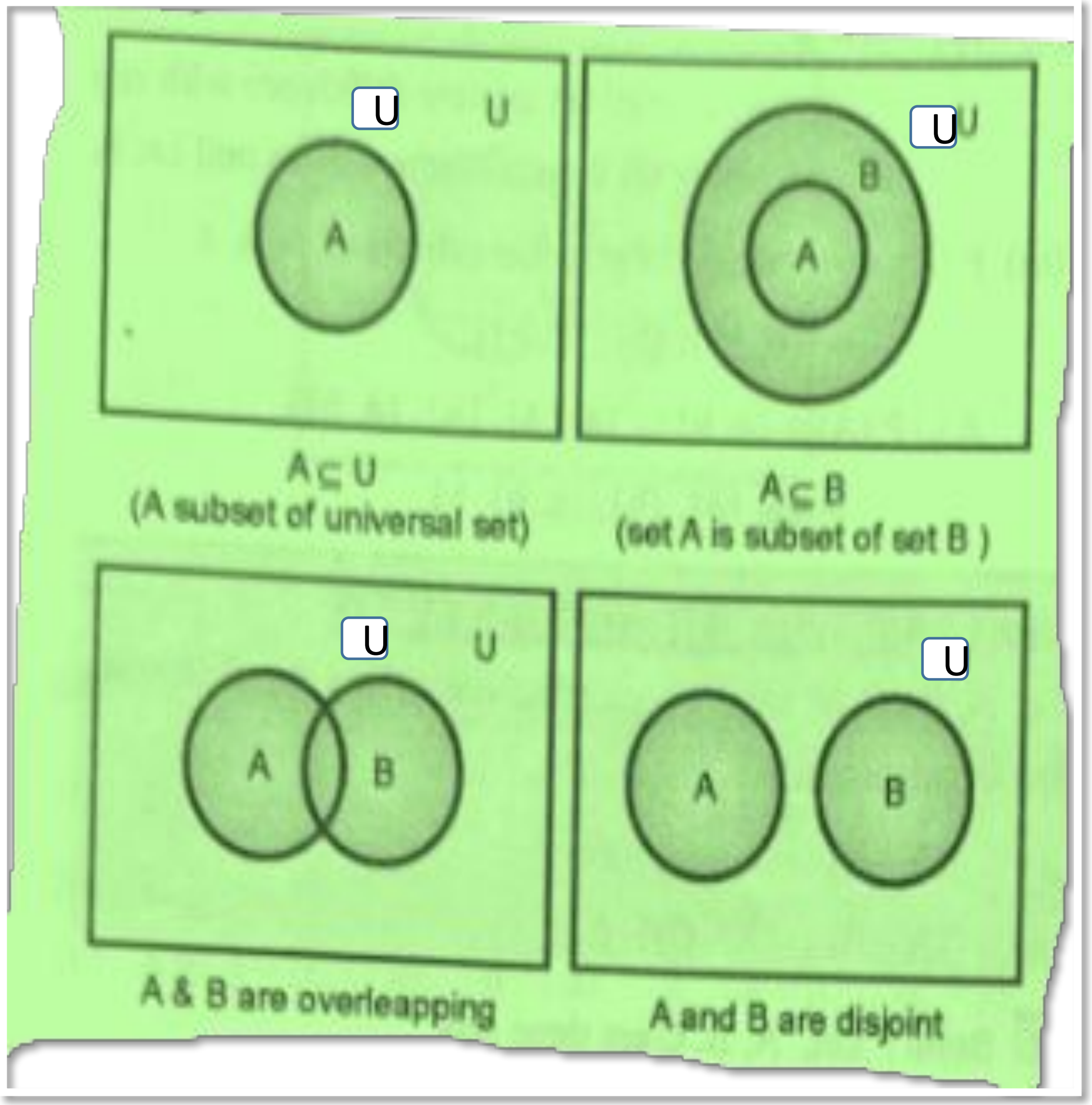
iv) $\{a, c\} - A = \{c\}$

What is Venn Diagram?

- Pictorial representation of **relation** between two sets and their **operations**
- Named after the British Logician **John Venn**

Conventions used in diagram

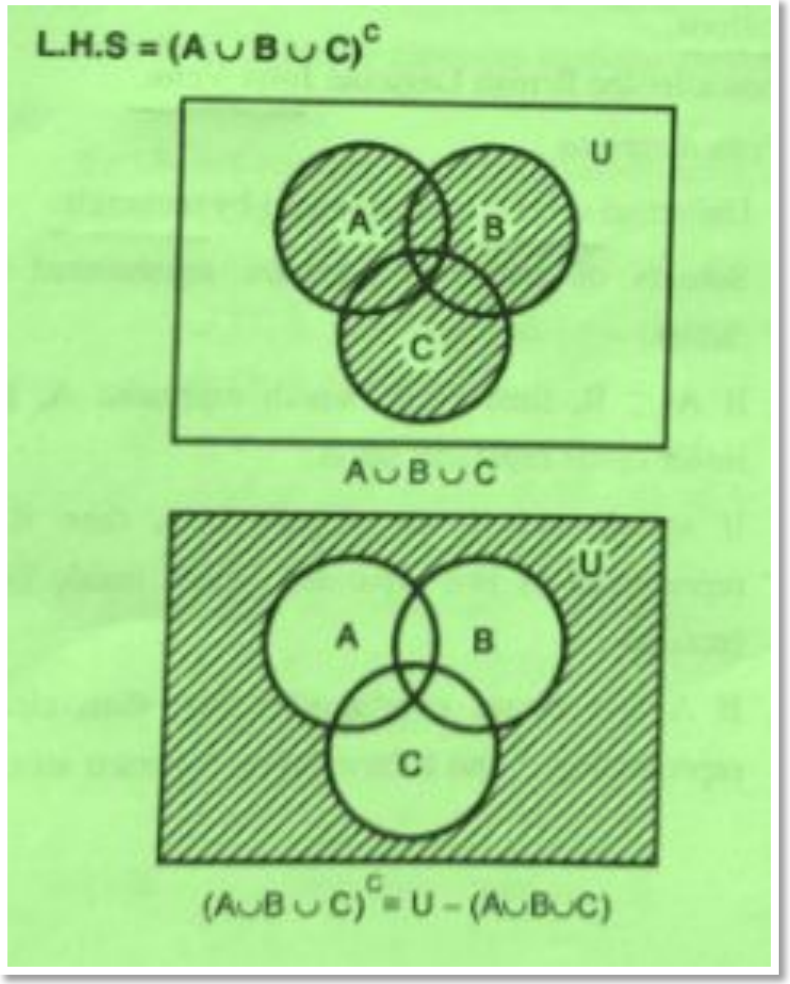
- **Rectangle** = Universal set (U)
- **Subsets** = Circles
- **Circle A inside B = A $\subseteq$ B**
- **Disjoint sets** = Two separate circles
- **Overlapping sets =** A and B will have common area



Draw the venn diagram and prove the expression:

$$(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$$

L . H . S .



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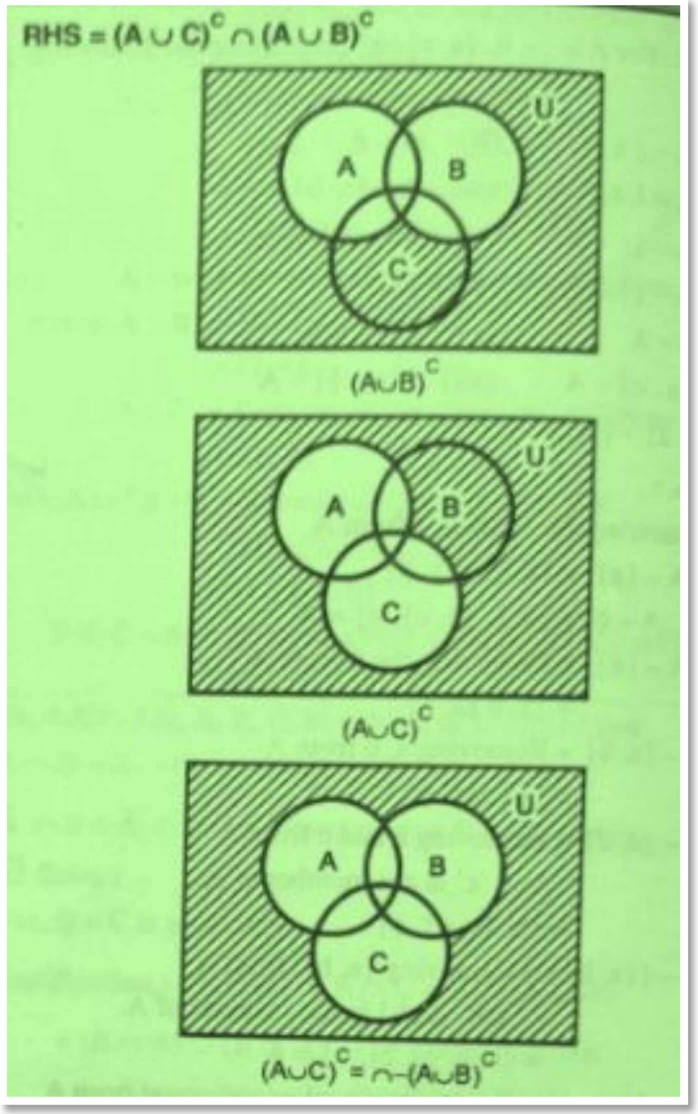
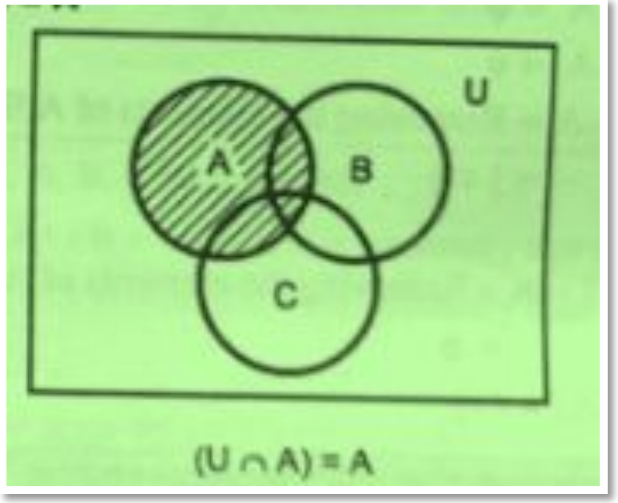
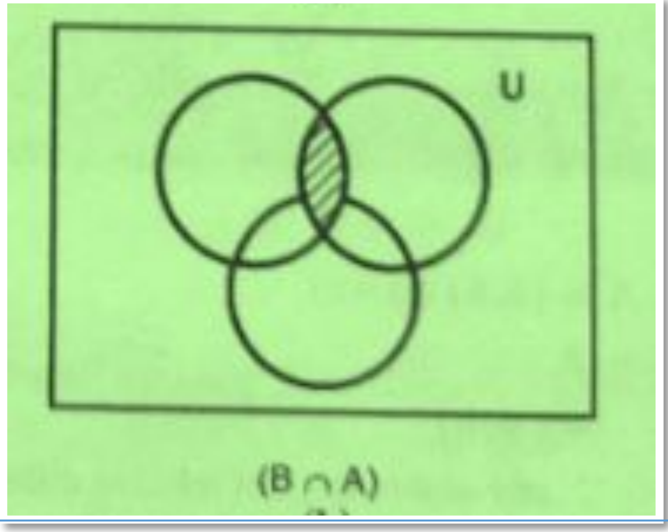


Draw the venn diagram and prove the expression:

$$(U \cap A) \cup (B \cap A) = A$$





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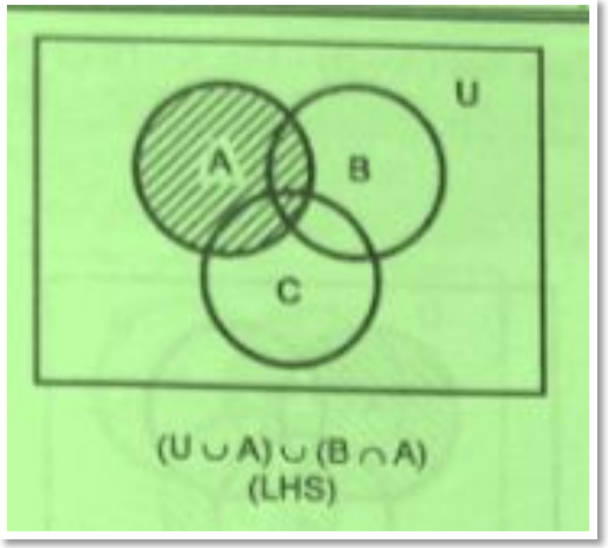

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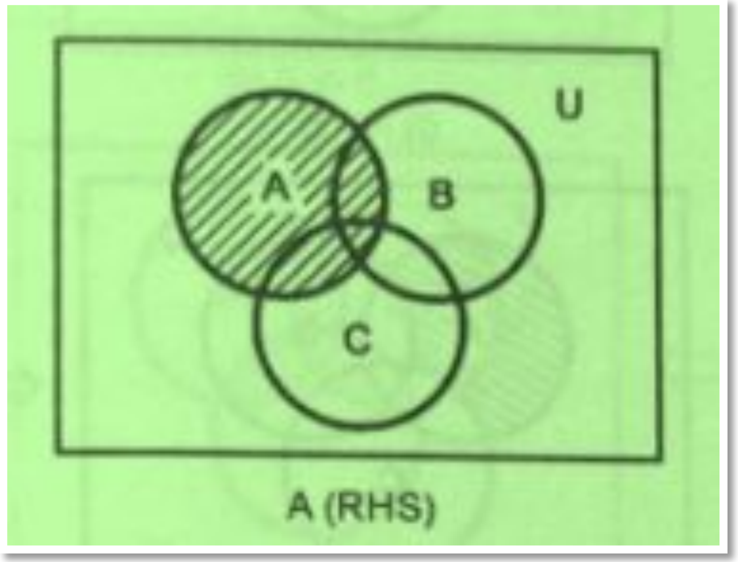


Let A, B, C be any set, then operations are as follows:

a) Idempotent Law

i) $A \cup A = A$

ii) $A \cap A = A$

Ex: $A = \{3, 4, 6\}$

$A \cup A = \{3, 4, 6\}$

$A \cap A = \{3, 4, 6\}$

b) Commutative Law

i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

Ex: $A = \{1, 2, 3\}$

$B = \{3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

$B \cup A = \{1, 2, 3, 4\}$

$A \cap B = \{3\}$

$B \cap A = \{3\}$

c) Associative Law

i) $A \cup (B \cup C) = (A \cup B) \cup C$

$= A \cup B \cup C$

ii) $A \cap (B \cap C) = (A \cap B) \cap C$

$= A \cap B \cap C$

Ex: $A = \{a, b, c, d\}$

$B = \{c, d, e, f\}$

$C = \{a, g, h, e\}$

$(B \cup C) = \{c, d, e, f, a, g, h\}$

$(A \cup B) = \{a, b, c, d, e, f\}$

$(A \cap B) = \{c, d\}$

$(B \cap C) = \{e\}$

$A \cup (B \cup C) = \{a, b, c, d, e, f, g, h\}$

$(A \cup B) \cup C = \{a, b, c, d, e, f, g, h\}$

$A \cap (B \cap C) = \{\phi\}$

$(A \cap B) \cap C = \{\phi\}$

d) Distributive law :

i) <u>union operation</u> →

Distributive over intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) <u>Intersection Operation</u> →

Distributive over union

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Algebra of SET OPERATIONS

e) Identity Law:

i) $A \cup \phi = A$

$A \cup U = U$

ii) $A \cap \phi = \phi$

$A \cap U = A$

f) Involution Law or Double Complement Law

i) $(\bar{\bar{A}}) = A$ or

$(A^c)^c = A$

g) Complement Law

i) $A \cup A^c = U$ $\quad U^c = \phi$

ii) $A \cap A^c = \phi$ $\quad \phi^c = U$

h) Absorption Law

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

Ex:

$A = \{3, 4, 8, 5\}$

$B = \{3, 5, 7, 6\}$

$(A \cap B) = \{3, 5\}$

$(A \cup B) = \{3, 4, 8, 5, 7, 6\}$

$A \cup (A \cap B) = \{3, 4, 8, 5\}$

$A \cap (A \cup B) = \{3, 4, 8, 5\}$

i) De Morgan's Law:

i) $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$

ii) $(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

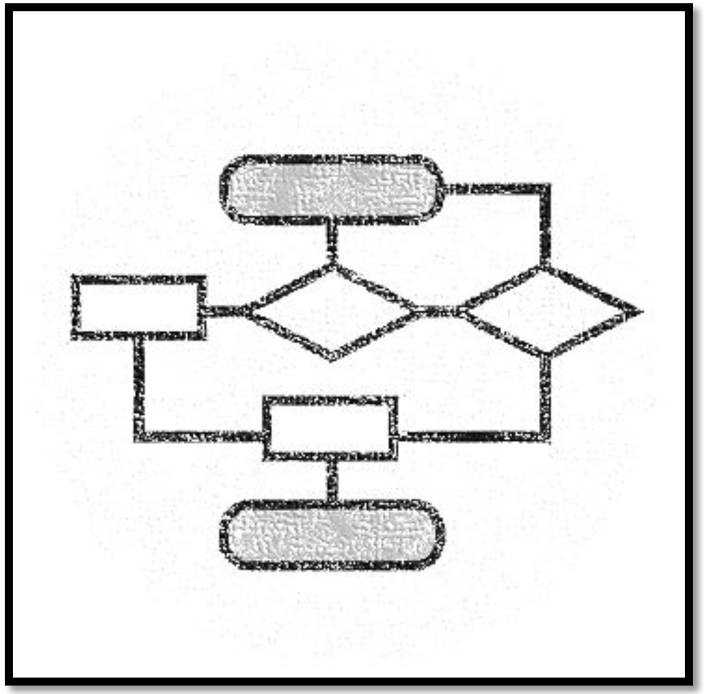
Cardinality:

- Cardinality here refers to number of elements in a set
- It is denoted as $|A|$ or $n(A)$

A = { 5 }	|A| = 1
B = { 7,2 }	|B| = 2
C = { 1,3,4 }	|C| = 3
D = { 9,1,5,8 }	|D| = 4

Use [Analysis of Algorithm]:

- Efficiency of an algorithm - It's important to count number of operations executed by algorithm,



Properties of Cardinality of Sets:

(a) If $A = \phi$ then $n(A) = 0$

(b) If $A \subseteq B$ then $n(A) \leq n(B)$

(c) Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and $|A \cup B| = |A| + |B|$

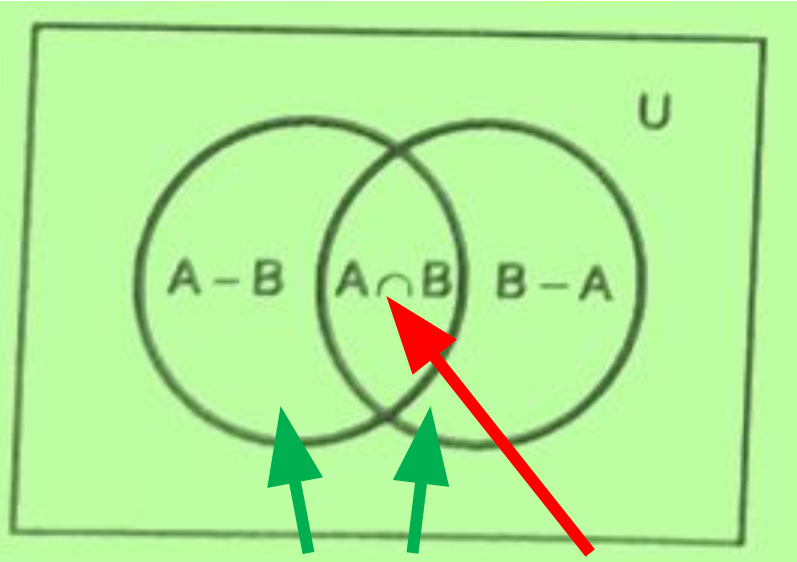
Session 3

- Inclusion and Exclusion Principle for Sets - Two sets, Three sets, Four sets
- Examples – based on exclusion principle
- Multiset- MSet
    - Multiplicity, Equality, Union, Intersection, Subset, Difference, Sum

Inclusion and Exclusion Principle - It's fundamental theorem of counting

Why its important? To overcome the "**over counting**"

Two Sets



$$|A \cup B| = \underbrace{|A| + |B|}_{\text{Inclusion of elements}} - \underbrace{|A \cap B|}_{\text{Exclusion of elements}}$$

Below equation will hold true only when A and B are two finite disjoint sets

$$|A \cup B| \cancel{=} |A| + |B|$$

Three Sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Four Sets

Let, A, B, C and D are finite sets then

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

Ex. of Inclusion and Exclusion Principle

Ex. In a survey 2000 People were asked whether they India today or Business Times. It was found that 1200 read India today, 900 read Business Times and 400 read both. Find how many read at least one magazine and how many read none.

? Universal set

? Assume as set A

?

? Assume as set B

Ex. of Inclusion and Exclusion Principle

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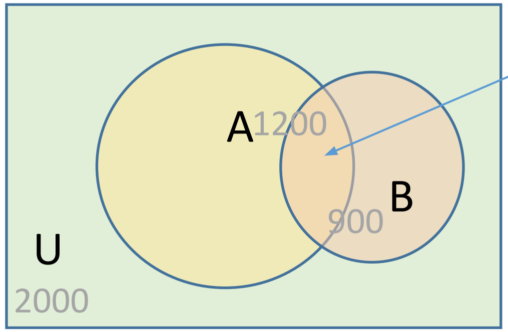
? $A \cap B$

? Assume as set B



$A \cap B$ = 400

Ex. of Inclusion and Exclusion Principle

Ex. In a survey 2000 People were asked whether they India today or Business Times. It was found that 1200 read India today, 900 read Business Times and 400 read both. Find how many read at least one magazine and how many read none.

Sol^n: Let $\underline{A}$ denote the set of people who read India Today

B denote the set of people who read Business Times

Now,

$$|A| = 1200 \quad , \quad |B| = 900$$

and $|A \cap B| = 400$

By Mutual Inclusion-Exclusion Principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 1200 + 900 - 400 = 1700$$

and

$$|U - (A \cup B)| = |U| - |A \cup B|$$

$$= 2000 - 1700 = 300$$

Hence 1700 read at least one magazine
and 300 read neither

we need to take union

To find out "how many read none", we need to subtract above from total strength

Ex: (3-set)

Inclusion-Exclusion Principle

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Ex. Consider a set of integers 1 to 500
Find:

i) How many of these numbers are divisible by 3 or 5 or by 11.

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ?

Soln:

$n(A)$ = no. of integers divisible by 3 $= \left\lfloor \frac{500}{3} \right\rfloor = 166$

$n(B)$ = no. of integers divisible by 5 $= \left\lfloor \frac{500}{5} \right\rfloor = 100$

$n(C)$ = No. of integers divisible by 11 $= \left\lfloor \frac{500}{11} \right\rfloor = 45$

No. of integers divisible by 3 & 5 = $n(A \cap B)$

$$= \left| \frac{500}{3 \times 5} \right| = 33$$

No. of integers divisible by 3 & 11 = $n(A \cap C)$

$$= \left| \frac{500}{3 \times 11} \right| = 15$$

No. of integers divisible by 5 & 11 = $n(B \cap C)$

$$= \left| \frac{500}{5 \times 11} \right| = 9$$

No. of integers divisible by 3, 5 and 11 = $n(A \cap B \cap C)$

$$= \left| \frac{500}{3 \times 5 \times 11} \right| = 3$$

i) According to Inclusion - Exclusion Principle

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 166 + 100 + 45 - 35 - 15 - 9 + 3$$

$$= 257$$

$\therefore$ The no. of integers divisible by 3 or 5 or by 11 under 500 are = 275

ii) The no. of integers divisible by 3 ~~or 5~~ or by 11 but not by all 3, 5, and 11, calculated as

$$n(A \cup C) - n(A \cap B \cap C)$$

$$= [n(A) + n(C) - n(A \cap C)] - n(A \cap B \cap C)$$

$$= [166 + 45 - 15] - 3$$

$$= 196 - 3 = 193$$

iii) The no. of integers divisible by 3 or 11 but not by 5 $= n(A \cup B \cup C) - n(B)$

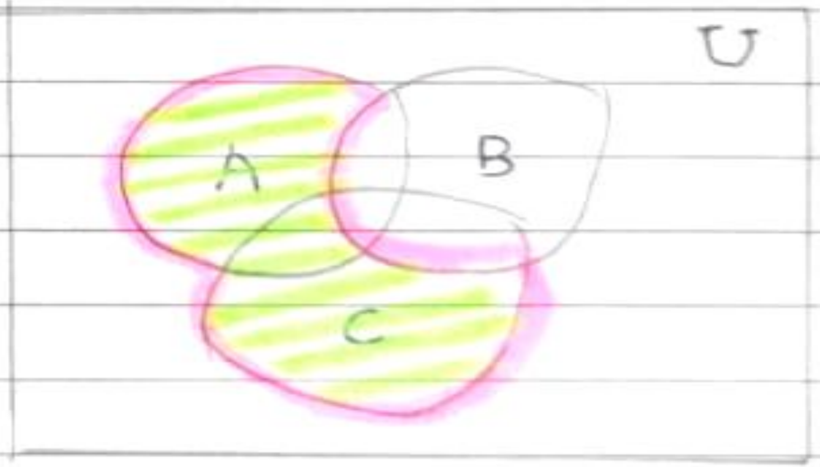
$= 257 - 100$

$= 157$

Venn Diagram

$n(A \cup C) - n(A \cap B \cap C)$

(ii)



$n(A \cup B \cup C) - n(B)$

(iii)

Sets are collection of distinct objects (unique elements) but in many cases collection of just distinct or unique object is not possible, e.g. birth month of peoples, copies of same book in library,

MULTISET

- Multiset is generalization of a set
- In **Multi**set there are multiple copies of same object
- Multisets is also called as **Msets**.
- Multisets are denoted by enclosing elements in a square bracket

  [**a, b, b**, c**,** **a**]

i) Multiplicity of an Element

Let, A be a multiset and $x \in A$. The multiplicity of x is the number of time the element x appears in the multiset.

In other words, multiplicity of an element in multiset is the number of time the element appears in mset.

$A = [a, b, c, a, a, b]$

$\mu(a) = 3, \quad \mu(b) = 2, \quad \mu(c) = 1$

ii) Equality of MSets

$A = [a, b, a, b], \quad B = [a, a, b, b]$

Where, multisets A and B are equal.

$C = [a, b, a], \quad$ Here, $A \neq C, B \neq C$

iii) Union of MSets

$A = [a, a, a, b, b, c, c, c]$

$B = [a, a, b, c, c, c, c]$

$A \cup B$ = Maximum occurrence of each element of A and B

$A \cup B = [a, a, a, b, b, c, c, c, c]$

iii) Intersection of MSets

The intersection of two multisets A and B denoted as $A \cap B$, is the multiset such that for each element $x \in A \cap B$ where.

$$\mu(x) = \min \{ \mu_A(x), \mu_B(x) \}$$

$A = [a, a, a, b, b, c, c, c]$

$B = [a, a, b, c, c, c, c]$

$A \cap B = [a, a, b, c, c, c]$

i) Subset of Msets

If A is a subset of B, if multiplicity of each element of A is less than or equal to its multiplicity in B.

$$[a, b, a, a, b] \subseteq [a, a, b, b, a, b]$$

iii) Sum of MSets

The sum of two multisets A and B, is denoted as $A + B$, such that, $x \in A + B$, where.

$$\mu(x) = \mu_A(x) + \mu_B(x)$$

$A = [c, c, a, b]$, $B = [b, b, c, c, a, a]$

$A + B = [c, c, c, c, b, b, b, a, a, a]$

ii) Difference of MSets

The difference of multisets A and B, denoted as, $A - B$, is a muliset such that $x \in A - B$

Where, $(\mu_A(x) - \mu_B(x)) \geq 1$

$A = [a, a, a, b, b, c]$, $B = [a, a, b, c]$

$A - B = [a, b]$

$A = [1, 2, 3, 4, 3, 3, 2, 2]$,

$B = [2, 2, 2, 1, 1, 1, 3, 3, 3, 3, 4]$

$A - B = [\ ]$ or ϕ

Session 4

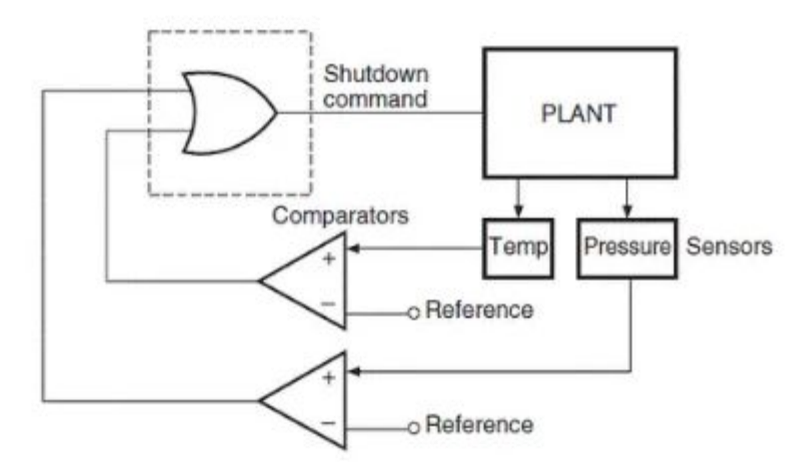
- Proposition – Intro & examples
- Notations & Statement Types
- Logical Connectives & Truth Tables
- Special Propositions
- Propositions Solve Examples 1,2,3
- Statement formulas (Tautology, Contradiction, Contingency)

Logics: Fundamental concept of mathematical reasoning

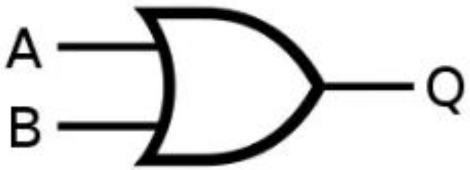
Proofs: Used to prove if mathematical statement is true or not.

- To verify if a computer program produces correct output for all different types of inputs
- To check if algorithm produces correct result or not

<< Based on your current enginnering knowledge > >
How you like to interpret below diagram & the table ?

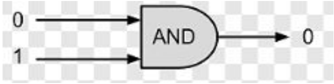


Inputs		Output
A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

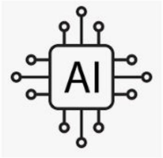


Mathematical Logic

Computer Architecture Design



Artificial Intelligence
(for Logical Inference)



Proposition

- It is also called as statement
- It is an declarative sentence
- It is either true or false, but not both at the same time

Lets explore more about declarative sentences

(i)	Mumbai is a capital of Maharashtra.	**True** / False
(ii)	Ice floats in water.	**True** / False
(iii)	$2 + 3 = 6$	True / **False**
(iv)	It will rain tomorrow.	? Can you guess → *Tomorrow we will know if its going to be True or False*
(v)	$3 + 2 = 5$	**True** / False

Proposition

- It is also called as statement
- It is an declarative sentence
- It is either true or false, but not both at the same time

Lets take another example

(i) What is your name ? → It is not proposition statement, its an question statement

(ii) Where are you going ? → It is not proposition statement, its an question statement

(iii) $x + 3 = 8.$ → ? Can you guess *Is this a declarative sentence?*

Notations used for Proposition

- Usually denoted by capital letters or small letters
  - A, B, C, …
  - a, b, c, …
- For True Statement
  - Truth value is denoted by ‘T’ or ‘1’
- For False Statement
  - False value is denoted by ‘F’ or ‘0’

Notations Examples

- P : 4 + 3 = 7, which holds true value
- q : 3 + 5 = 7, which holds false value
- x : Mumbai is capital of Maharashtra, which holds true value

Primitive statements

- If a statement can’t be broken down or split into simpler proposition or statement
- It is also called as primary or atomic statement

Examples

- 2 + 7 = 9
- Ice floats in water
- India is in Asia

All these statements are primitive as it can not split into simple propositions

Compound statements

- It is formed by primitive statements by using logical connectives

Example

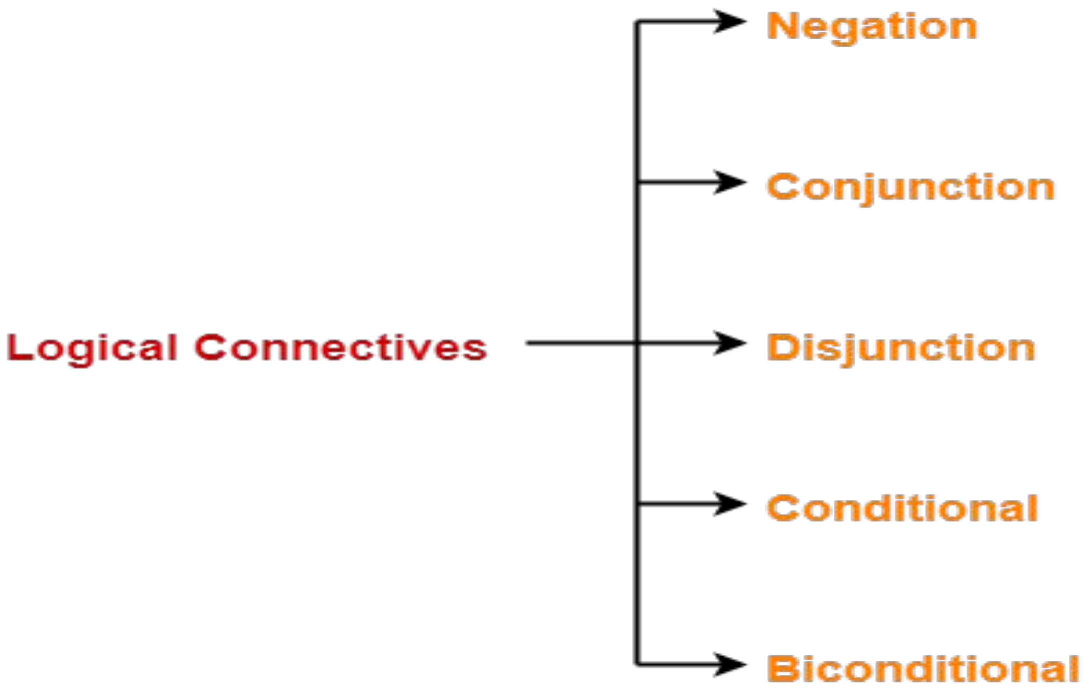
- Ajay is hockey player and he is engineer
- Atul is smart or he works very hard

Here we can see that compound statements are formed from primitive propositions using logical connectives like and , or

Logical Connectives

- To connect two or more statements we use the words like “not”, “and”, “or”, “but”, “if-then”, “while”
- Sometimes usage of these words may make the forming statement as ambiguous or inexact
- For DM subject, some special connectives are used which has precise meaning & are also having corresponding symbol

Sr. No.	Name of connectives	Notation
(i)	Negation	$\sim$ or $\neg$
(ii)	AND	$\wedge$
(iii)	OR	$\vee$
(iv)	If ...then	$\rightarrow$
(v)	If and only if (iff)	$\leftrightarrow$



Negation

1. Negation

"not p" $\sim p$

p	$\sim p$
T	F
F	T

Conjunction

2. Conjunction

"p and q" $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

3. Disjunction

"p or q" $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional Statement

4. Conditional Statement
(If… then) $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Statement

5. Biconditional Statement
(If and only If) $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Nor ($p \downarrow q$)

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

XOR ($p \veebar q$)

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

Logic & Proofs

GURUKUL Page No. Date

Proposition

Ex 1 Use p : I will Study Maths
q : I will go to a movie
r : I am in a good mood.

Write the English statement that corresponds to each of the following :

i) $\sim r \to q$ ii) $\sim q \wedge p$ iii) $q \to \sim p$ iv) $\sim p \to \sim r$

Sol^n: Let, p : I will study Maths
q : I will go to a movie
r : I am in a good mood

Sr. No.	Name of connectives	Notation
(i)	Negation	$\sim$ or $\neg$
(ii)	AND	$\wedge$
(iii)	OR	$\vee$
(iv)	If ...then	$\to$
(v)	If and only if (iff)	$\leftrightarrow$

i) $\sim r \rightarrow q$

If I am not in good mood then I will goto a movie

ii) $\sim q \wedge p$

I will not goto movie but I will study Maths.

iii) $q \rightarrow \sim p$

If I wil go to a movie then I will not study Maths

iv) $\sim p \rightarrow \sim r$

If I will not study Maths then I am not in a good mood.

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Ex 2 Using the following statements

p: Ram is hard worker
q: Ram is intelligent

Write the following statement in symbolic form.

i) Ram is hardworker and intelligent.
ii) Ram is hardworker but not intelligent.
iii) It is false that Ram is not hardworker nor intelligent
iv) Ram is hardworker or Ram is not hardworker and intelligent.

Soln: i) $p \wedge q$
ii) $p \wedge \sim q$
iii) $\sim(\sim p \vee q)$
iv) $p \vee (\sim p \wedge q)$

Sr. No.	Name of connectives	Notation
(i)	Negation	$\sim$ or $\neg$
(ii)	AND	$\wedge$
(iii)	OR	$\vee$
(iv)	If ...then	$\rightarrow$
(v)	If and only if (iff)	$\leftrightarrow$

Statement	If p, then q	$p \rightarrow q$
Converse	If q, then p	$q \rightarrow p$
Inverse	If not p, then not q	$\sim p \rightarrow \sim q$
Contrapositive	If not q, then not p	$\sim q \rightarrow \sim p$

<u>Conditional Statement</u>

p : It is raining

q : Grass is wet.

$p \rightarrow q$: "If it is raining then the grass is wet"

<u>Contra-Positive</u>

$\sim q \rightarrow \sim p$: If the grass is not wet then it is not raining.

<u>Converse</u>

$q \rightarrow p$: "If the grass is wet then it is raining."

<u>Inverse</u>

$\sim p \rightarrow \sim q$: "If it is not raining then the grass is not wet."

Conditional Statement
$p \rightarrow q$

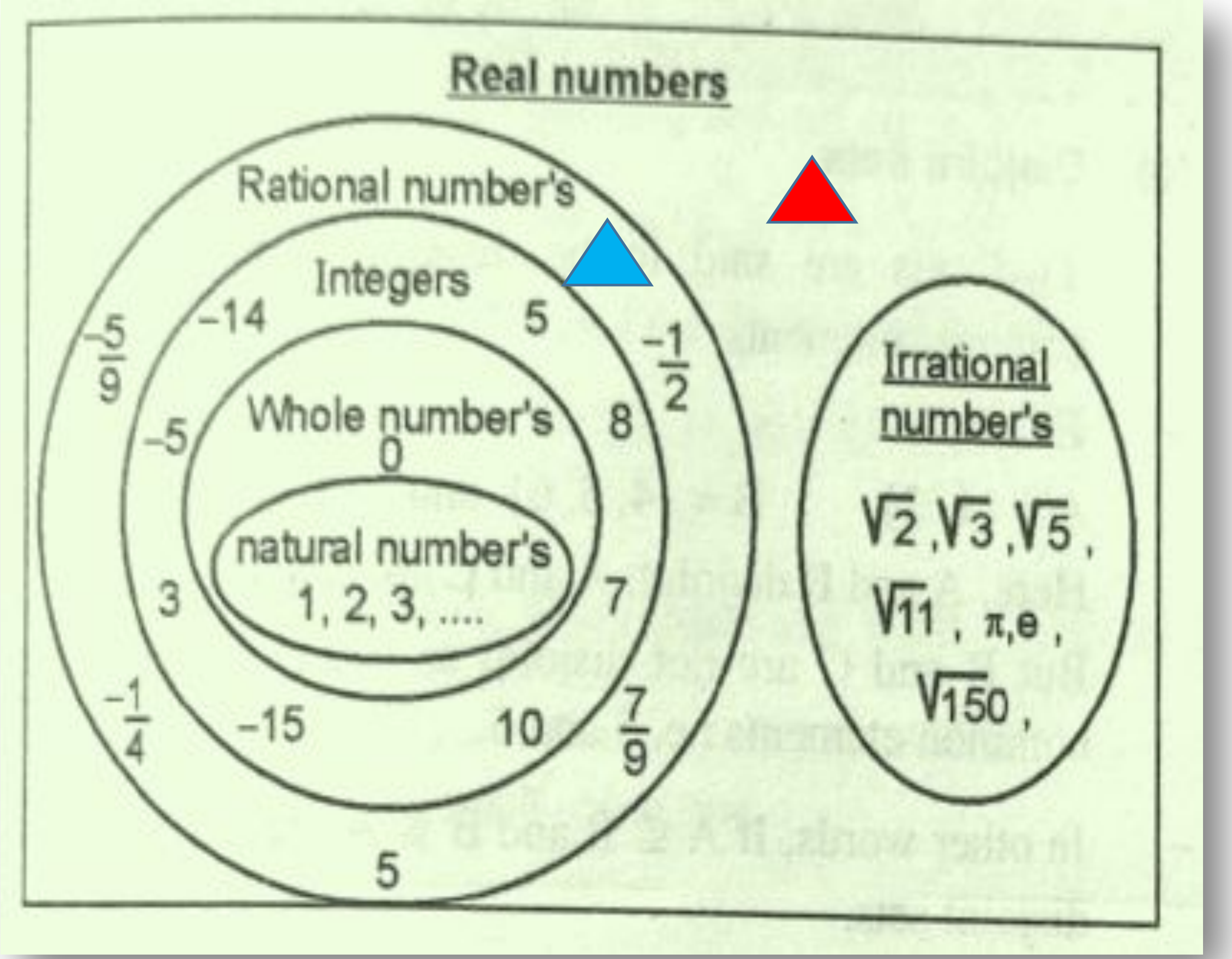
Converse Statement
$q \rightarrow p$

Inverse Statement
$\sim p \rightarrow \sim q$

Contrapositive Statement
$\sim q \rightarrow \sim p$

Write the contrapositive, inverse, converse and negation of the following statement

If x is rational then x is real



☑ **Soln. :**

Let, P : x is rational, q : x is real.

In symbolic form : $p \to q$

(i) **Contrapositive**

$(\sim q \to \sim p)$

That is, "If x is not real, then x is not rational".

(ii) **Inverse**

$\sim p \to \sim q$

That is, "If x is not rational, then x is not real".

(iii) **Converse**

$q \to p$

That is, "If x is real then x is rational".

(iv) **Negation**

$\sim (p \to q) \Rightarrow \sim p\ (\sim p \vee q) \Rightarrow p \wedge \sim q$

That is "x is rational and not real".

1. If today is Sunday, then it is a holiday.

We have-

- The given sentence is- “If today is Sunday, then it is a holiday.”
- This sentence is of the form- “If p then q”.

So, the symbolic form is **p → q** where-

p : Today is Sunday

q : It is a holiday

Converse Statement- If it is a holiday, then today is Sunday.

Inverse Statement- If today is not Sunday, then it is not a holiday.

Contrapositive Statement- If it is not a holiday, then today is not Sunday.

2. If 5x – 1 = 9, then x = 2.

We have-

- The given sentence is- “If 5x – 1 = 9, then x = 2.”
- This sentence is of the form- “If p then q”.

So, the symbolic form is **p → q** where-

p : 5x – 1 = 9

q : x = 2

Converse Statement- If x = 2, then 5x – 1 = 9.

Inverse Statement- If 5x – 1 ≠ 9, then x ≠ 2.

Contrapositive Statement- If x ≠ 2, then 5x – 1 ≠ 9.

?

1. You will qualify GATE only if you work hard

2. If you are intelligent, then you will pass the exam.

Topic: Different Statement Formulas

Tautology

It’s a Statement formula

Always TRUE

For all possible values

e.g. $p \vee \sim p$ is tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Contradiction

It’s a statement formula

Always FALSE

For all possible values

Example : $p \wedge \sim p$ is contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Contingency

It’s a statement formula

Neither Tautology nor Contradiction

For all possible values

Example : $p \wedge q$ is contingency.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition: A compound statement, that is always true regardless of the truth value of the individual statements, is defined to be a **tautology**.

Is $(p \wedge q) \rightarrow p$ a tautology?

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Solution : Yes

Is $(p \vee q) \rightarrow (p \wedge q)$ a tautology?

p	q	$(p \vee q)$	$(p \wedge q)$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Solution: No; the truth values of $(p \vee q) \rightarrow (p \wedge q)$ are {T, F, F, T}.

Solution : No

?

Is $\sim b \rightarrow b$ a tautology?

Is $[(p \rightarrow q) \wedge p] \rightarrow p$ a tautology?

Session 5

- Logical Equivalence – Definition & Example
- Fundamental Logical Identities & Examples
- The Principle of Duality
- Normal Forms – DNF, CNF & Examples

Logical Equivalence: In daily life we observe number of similar things with respect to some aspect.

Ex. Two persons are similar with respect to their height.

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Logical Equivalence

Defn: Two statements or propositions are said to be logically equivalent iff they have same truth value.

Ex.1 p and $p \wedge p$ are logically equivalent

p	$p \wedge p$
T	T
F	F

Ex 2 $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Both truth values are Same

Sr. No.	Name of Identity	Identity
1.	Idempotence of $\vee$	$p \equiv p \vee p$
2.	Idempotence of $\wedge$	$p \equiv p \wedge p$
3.	Commutativity of $\vee$	$p \vee q \equiv q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q = q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee r) \wedge (p \vee r)$
9.	Double negation	$p \equiv \sim(\sim p)$
10.	De-Morgan's law	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
11.	De-Morgan's law	$\sim(p \wedge q) \equiv \sim p \vee \sim q$
12.	Absorption law	$p \vee (p \wedge q) \equiv p$
13.	Absorption law	$p \wedge (p \vee q) \equiv p$
14.	Negation law of $\vee$	$p \vee \sim p = T$
15.	Negation law of $\wedge$	$p \wedge \sim p = F$
16	Identity law of $\vee$	$p \vee F = p$ $p \vee T = T$
17.	Identity law of $\wedge$	$p \wedge F = F$ $p \wedge T = p$

Date

Examples

Ex. Truth Table : Construct the truth tables for the following statements.

i) $p \to p$

Sol^n

p	p	$p \to p$
T	T	T
F	F	T

ii) $(p \to p) \vee (p \to \sim p)$

Sol^n

p	p	$\sim p$	$p \to p$	$p \to \sim p$	$(p \to p) \vee (p \to \sim p)$
T	T	F	T	F	T
F	F	T	T	T	T

Sol^n

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \rightarrow \sim q$
T	T	F	T	F
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

? iv) $(p \vee \sim q) \vee \sim p$

Let S_1 and S_2 be the two statements are said to be dual of each other if either one can be obtained from the other by interchanging $\wedge$ (AND), $\vee$ (OR), T (TRUE) and
F (FALSE) by $\vee$ (OR), $\wedge$ (AND), F (FALSE) and T (TRUE) respectively.

The dual of $(p \wedge q)$ will be $(p \vee q)$.

Similarly, dual of $p \wedge F$ will be $p \vee T$.

Challenges with Truth Table:

- If statement has n distinct variables then truth table will have 2^n possible combinations of truth values. Hence if number of variables are more then constructing Truth Table is not convenient

Normal Form

- In such cases (mentioned above) normal form is used to find the Truth Values.
- It is also used to declare given statement as tautology or contradiction or not

Fundamental Product = Conjunction (product) of Variables and their negation

$p, \sim p, \sim p \wedge q, p \wedge \sim q, \sim p \wedge \sim q$

Fundamental Sum = Disjunction (sum) of Variables and their negation

$p, \sim p, \sim p \vee q, p \vee q, p \vee \sim q$

Disjunctive Normal Form (DNF)

A statement form which is disjunction of fundamental conjunctions

$$\underbrace{(p \wedge r)}_{\text{(Conjunction)}} \underbrace{\vee}_{\text{disjunction}} \underbrace{(p \wedge q)}_{\text{Conjunction}}$$

$$(\sim p \wedge r) \vee (\sim q \wedge r) \vee (\sim r)$$

$$(p \wedge q) \vee (q) \vee (\sim q \wedge p)$$

Disjunctive Normal Form (DNF) - Rules to find DNF of a given statement

Conditional	($\rightarrow$)
Bi-conditional	($\leftrightarrow$)

Replace By

Logical Correctives

$\sim$, $\wedge$, $\vee$ etc.

$$p \rightarrow q \equiv \sim p \vee q$$

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$$

Use De-Morgans law to eliminate '$\sim$' before sum or product

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

Apply distributive laws repeatedly and eliminate products of variables to obtain required normal form

Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Ex: Find **DNF** of : $((p \to q) \wedge (q \to p)) \vee p$

Sol^n [**Note: Replace** conditional $(\to)$ or bi-conditional $(\leftrightarrow)$ by using logical connectives $\sim, \wedge, \vee$

ex: $p \to q \equiv \sim p \vee q$

$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

Let $((p \to q) \wedge (q \to p)) \vee p$

$\equiv ((\sim p \vee q) \wedge (\sim q \vee p)) \vee p$

Distributive Law: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\equiv ((\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)) \vee p$

$\equiv (\sim p \wedge \sim q) \vee (F) \vee (F) \vee (p \wedge q) \vee p$

Absorption Law: $p \vee (p \wedge q) = p$

$\equiv (\sim p \wedge \sim q) \vee (p)$

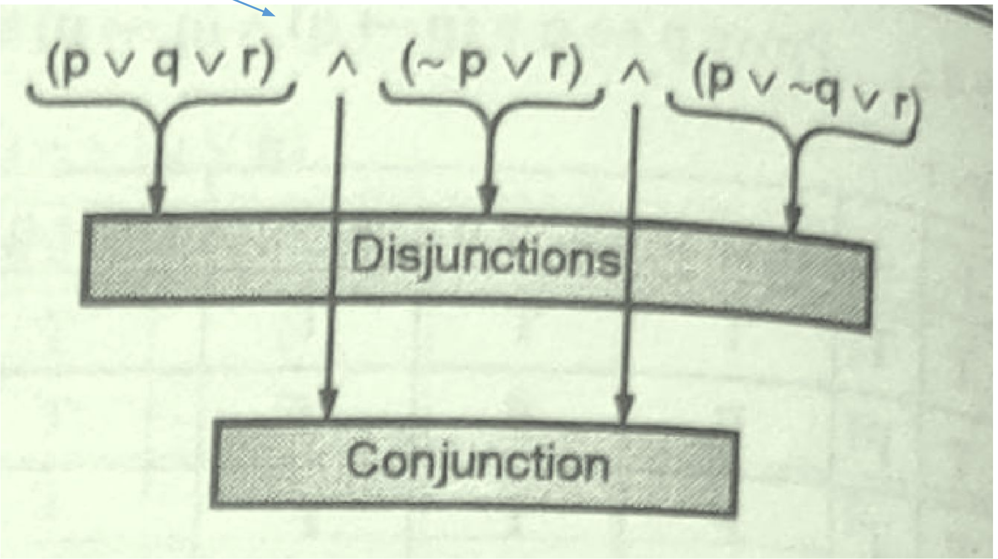
$\equiv$ Required DNF

Conjunctive Normal Form (CNF)

A statement form which consist of conjunctions of fundamental disjunction

$p \wedge (p \vee q)$

Here p and $(p \vee q)$ are combined using conjunction. Hence it is conjunction of disjunction.



Ex. Find CNF of: $p \leftrightarrow (\sim p \vee \sim q)$

$\equiv (\sim p \vee (\sim p \vee \sim q)) \wedge (p \vee \sim(\sim p \vee \sim q))$

$\equiv (\sim p \vee (\sim p \vee \sim q)) \wedge (p \vee (p \wedge q))$

$\equiv (\sim p \vee (\sim p \vee \sim q)) \wedge p$

$\equiv (\sim p \vee \sim p) \vee \sim q) \wedge p$

$= (\sim p \vee \sim q) \wedge (p)$

$=$ Required CNF

? Find the conjunctive and disjunctive normal forms for the following without using truth tables

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

Session 6

- Methods of Proof – Valid Argument & Examples
- Rules of Inference,
- Predicates, Quantifiers
- Rules of Inference for Predicate Calculus & Examples

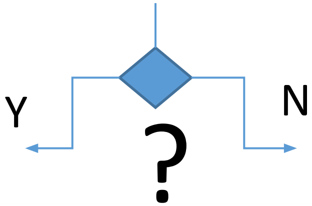
In mathematical world & real word as well we come across many arguments, theorems & statements

Arguments

Theorems

Statements

Are these valid?



It need to be proved, But How

Therefore, we need to learn methods of proof

?

Set of **Premises**
Which are **TRUE**

Use standard rules
of Inference

Derive a **conclusion**

Premises = Initial Collection of Statements

$\{ S_1, S_2, S_3, \ldots S_n \}$

Conclusion = It ultimately helps to prove the theorem

C is conclusion

Set of **Premises** Which are **TRUE**

Use standard rules of Inference

Derive a **conclusion**

Premises = Initial Collection of Statements

$$\{ S_1, S_2, S_3, \ldots S_n \}$$

Statement 1 : It is snowing today
Statement 2 : There is congestion on MG road
Statement 3 : I go to school via MG road
Statement 4 : School has given choice to join online class
Statement 5: School bus service is stopped
Statement 6 : I can't go to school today

Conclusion = It ultimately helps to prove the theorem

C is conclusion

Conclusion: When it snows I join online class

$$\{ S_1 \wedge S_2 \wedge S_3 \wedge \ldots S_n \} \rightarrow \mathbf{C}$$

Conclusion statement should be such that it is a **tautology**

Premises = Initial Collection of Statements

$\{ S_1, S_2, S_3, \ldots S_n \}$

Conclusion = S is conclusion

Statement 1 : S_1 : All my friends are musicians

Statement 2 : S_2 : John is my friend

Statement 3 : S_3 : None of my neighbors are musician

S: John is not my neighbor

Can you determine validity of the argument S ?

Statement 1 : S_1 : All my friends are musicians

Statement 2 : S_2 : John is my friend

Statement 3 : S_3 : None of my neighbors are musician

S: John is not my neighbor

Can you determine validity of the argument S ?

Step1 : Convert above statements into Propositions

p : All my friends are musicians

q : John is my friend

r : My neighbors are musician

s : John is not my neighbor

Step2 : Convert propositions in Symbolic form

p

q

~ r

S

$$\begin{array}{l} p \\ q \\ \sim r \\ \hline S \end{array}$$

Statement 1 : S_1 : All my friends are musicians

Statement 2 : S_2 : John is my friend

Statement 3 : S_3 : None of my neighbors are musician

S: John is not my neighbor

Can you determine validity of the argument S ?

Step3 : Validate

p : All my friends are musicians $\rightarrow$ p

q : John is my friend $\rightarrow$ q

r : None of my neighbors are musician $\rightarrow$ ~ r

s : John is not my neighbor $\rightarrow$ S

$\therefore\ p \wedge q \rightarrow$ John is musician

$\therefore\ p \wedge q \sim r \rightarrow$ John is musician **and** my neighbors are not musicians

$\therefore\ p \wedge q \sim r \rightarrow$ John is not my neighbour

Given Argument:

If I try hard and I have talent then I will become a musician.

If I become a musician then I will become happy.

Therefore if I will not be happy then I did not try hard or I do not have talent

Question: Convert the above statements into Symbolic forms

Given Argument:

If I try hard and I have talent then I will become a musician.

If I become a musician then I will become happy.

Therefore if I will not be happy then I did not try hard or I do not have talent

Question: Convert the above statements into Symbolic forms

Step1 : Look for simple English sentences & lets assume

p : I try hard → p

q : I have talent → q

r : I will become musician → r

s : I will become happy → S

Given Argument:

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Question: Convert the above statements into Symbolic forms

Step1 : Look for simple English sentences & lets assume

p : I try hard $\longrightarrow$ p

q : I have talent $\longrightarrow$ q

r : I will become musician $\longrightarrow$ r

s : I will become happy $\longrightarrow$ S

Step2 : Convert input statements (propositions) into symbolic form

If I try hard and I have talent then I will become a musician.

S1 : $p \wedge q \longrightarrow r$

If I become a musician then I will become happy.

S2 : $r \longrightarrow S$

if I will not be happy then I did not try hard or I do not have talent

S3 : $\sim S \longrightarrow \sim p \vee \sim q$

If I try hard and I have talent then I will become a musician.

If I become a musician then I will become happy.

if I will not be happy then I did not try hard or I do not have talent

S1 : $p \wedge q \longrightarrow r$

S2 : $r \longrightarrow S$

S3 : $\sim S \longrightarrow \sim p \vee \sim q$

P	Q	$P \longrightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

To verify the validity of above problem –

Lets assume S3 (conclusion) is invalid

S3 : $\sim S \longrightarrow \sim p \vee \sim q$

Lets assume S3 is invalid

Left Part Right Part

∴ $T \longrightarrow F$

∴ $\boxed{\sim p \vee \sim q}$

$\boxed{\sim p \vee \sim q}$

~p	~q	$\sim p \vee \sim q$
T	T	T
F	T	T
T	F	T
F	F	F

∴ p = T q = T

If I try hard and I have talent then I will become a musician.

If I become a musician then I will become happy.

if I will not be happy then I did not try hard or I do not have talent

S1 : $p \wedge q \longrightarrow r$

S2 : $r \longrightarrow S$

S3 : $\sim S \longrightarrow \sim p \vee \sim q$

p	q	$p \longrightarrow q$
T	T	T
F	T	T
T	F	F
F	F	T

Lets assume S2 is true

S2 : $r \longrightarrow S$

Left Part Right Part

$\therefore$ F $\longrightarrow$ F

$\therefore$ r is False

Lets assume S1 is true

S1 : $p \wedge q \longrightarrow r$

Right Part is False

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	F

If r is False, it implies that either p or q is False, This is a contradiction.

Let's recap: Earlier we have said S3 is invalid, $\therefore$ p = T q = T But from S1 we see that, p = F or q = F Contradiction

$\therefore$ Argument S3 is valid S3 : $\sim S \longrightarrow \sim p \vee \sim q$ ✔

Set of **Premises** Which are **TRUE** → Use standard rules of Inference → Derive a **conclusion**

Rules of Inference

- **Criteria** for finding validity of an argument
- Represented in the form of **Statements**
- There are four types of **Rules of Inference**.
  - Law of Detachment (Modus ponens)
  - Law of Contrapositive (Modus Tollen)
  - Disjunctive syllogism
  - Hypothetical Syllogism

Law of Detachment (Modus ponens) – This method is used **to prove** arguments

Law of Contrapositive (Modus tollens) – This method is used **to disprove** arguments

(p $\rightarrow$ q)	If Akshay has a password, then he can log on to facebook
p	Akshay has a password
q	He can log on to facebook

Law of Detachment
(Modus ponens)

$p \rightarrow q$

p

$\therefore q$

Premises Or Hypotheses

Conclusion

$(p \rightarrow q) \wedge p$

is Tautology

$(p \rightarrow q)$

If Akshay has a password, then he can log on to facebook

p : Akshay has a password

q : He can log on to facebook

p	q	$(p \rightarrow q)$	$(p \rightarrow q) \wedge p$	$(p \rightarrow q \wedge p) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Law of Contrapositive (Modus Tollen)

Premises: $\{ S_1, S_2 \}$

$p \rightarrow q \qquad \sim q$

Conclusion **C**

$\sim p$

$S_1 : p \rightarrow q$

$S_2 : \sim q$

$\therefore \boxed{\sim p}$

S_1 : If you have a password, then you can log on to facebook

S_2 : You cannot log on to facebook

C: You do not have a password

Disjunctive Syllogism Rule:

A logical argument of the form that if there are only two possibilities, and one of them is ruled out, then the other must take place

$S_1 : p \vee q$

$S_2 : \sim p$

$\therefore \; q$

S_1 : David is going to study Law or Engineering

S_2 : David is not going to study Law

C: Therefore he will study Engineering

Disjunctive Syllogism Rule:

This form of inference is also known as transitive rule

Transitive rule is “If a is equal to b and b is equal to c, then a is equal to c.”

$S_1 : p \rightarrow q$

$S_2 : q \rightarrow r$

$\therefore \; p \rightarrow r$

S_1 : James studies hard

S_2 : He obtains distinction in BE

C: He will get a good job

What Is Predicate?

- A predicate is a statement or **mathematical assertion** that contains one or more variables.
  - “**is a smart**”, “**intelligent student**” are the examples of assertion
  - P(x,y): "x + 2 = y" is a predicates
- It may be true or false depending on those variables’ value or values.

- **Example 1:**
- Let P(x): “x > 10”
- What are the truth values of P(11) and P(5)?

- **Example 1:**
- Let $P(x)$: "$x > 10$"
- What are the truth values of $P(11)$ and $P(5)$?
- **Solution:**
- $P(11)$ is equivalent to the statement $11 > 10$, which is True.
- $P(5)$ is equivalent to the statement $5 > 10$, which is False.

- **Example 2:**
- Let $R(x, y)$: “$x = y + 1$“.
- What is the truth value of the propositions $R(1,3)$ and $R(2,1)$?

- **Example 2:**
- Let R(x, y): “x = y + 1“.
- What is the truth value of the propositions R(1,3) and R(2,1)?
- **Solution:**
- R(1,3) is the statement 1 = 3 + 1, which is False.
- R(2,1) is the statement 2 = 1 + 1, which is True.

We can assign values to each variable — thus, creating a true or false proposition, as seen in the example below.

Question:

$P(x): x + y \geq 6$

Possible Solutions:

Let $P(7,1)$ $\quad P(7,1):\ (7)+(1) \geq 6 \Rightarrow 8 \geq 6$ $\quad$ True propositional statement

Let $P(3,2)$ $\quad P(3,2):\ (3)+(2) \geq 6 \Rightarrow 5 \ngeq 6$ $\quad$ False propositional statement

$8 \geq 6$, $5 \ngeq 6$

But <u>this</u> isn't always effective or helpful, as we ultimately want the predicate to be realistic over a range of elements, not just the ones that we've hand-selected.

Quantifier

- Quantifiers are used in predicate logic.
- They express the extent to which a predicate is true over a range of elements

Types

- ❑ Universal Quantifiers
  - “for all”, “for every”
- ❑ Existential Quantifiers
  - “there exist some”, “there are at least 10 members”

Universal Quantification $\forall P(x)$

- Mathematical statements sometimes assert that **a property is true for all the values** of a variable in a particular domain (domain of discourse). Such a statement is expressed using **universal quantification**.

For all values of x, the assertion P(x) is true

$P(x)$ Here P(x) is a predicate, where x is an argument

$\forall$ Here "for all" is a universal quantifier

$$\forall P(x)$$

Example:

All students from my college carry valid ID Card

Existential Quantification $\exists P(x)$

- “Some students are intelligent but not hardworking” ?

*In above example “**There exists at least one**” intelligent student OR “**there exist some**” intelligent students*

- Existential quantification can be used to form a proposition that is true if and only if P(x) is true for at least one value of x in the domain.

$P(x)$ Here P(x) is a predicate, where x is an argument

$\exists$ There is at least 1 or there exist some

$$\exists P(x)$$

Comparison between Universal & Existential Quantifiers

Statement	When True?	When False?
$\forall P(x)$	P(x) is true for all x	There is an x for which P(x) is false
$\exists P(x)$	There is an x for which P(x) is true	P(x) is false for all x

What is the correct translation of the following statement into mathematical logic?

“Some real numbers are rational”

(A) $\exists x\,(\text{real}(x) \vee \text{rational}(x))$
(B) $\forall x\,(\text{real}(x) \rightarrow \text{rational}(x))$
(C) $\exists x\,(\text{real}(x) \wedge \text{rational}(x))$
(D) $\exists x\,(\text{rational}(x) \rightarrow \text{real}(x))$

What is the correct translation of the following statement into mathematical logic?

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(B) $\forall x\ (\text{real}(x) \rightarrow \text{rational}(x))$
(C) $\exists x\ (\text{real}(x) \wedge \text{rational}(x))$
(D) $\exists x\ (\text{rational}(x) \rightarrow \text{real}(x))$

(A) "There exist some numbers which are either real OR rational"

(B) "All real numbers are rational"

(C) "There exist some numbers which are both real AND rational"

(D) "There exist some numbers for which rational implies real"

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all phones and Samsung A31 is a phone.

"All phones have camera."

"Therefore, Samsung A31 has a camera."

Universal Generalization (UG)

$$\frac{P(c)}{\therefore \forall x P(x)}$$ For an arbitrary c

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c)}$$ For some element c

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Over the universe of book defined propositions.

$B(x)$: x has blue cover

$M(x)$: x is maths book

$I(x)$: x published in India

Translate the following

(i) $\forall x\, (M(x) \wedge I(x) \rightarrow B(x))$

(ii) There are maths books published outside India.

Ans i) The maths book published in India have a blue color

Ans ii) $(\exists x)\, (M(x) \wedge \sim I(x))$

Negate each of the statement :

(i) $\forall x, |x| = x.$ (ii) $\exists x, x^2 = x,$

(i) $\forall x, |x| = x$

$\Rightarrow (\exists x)(|x| \neq x)$

(ii) $\exists x, x^2 = x$

$\Rightarrow \forall x\,(x^2 \neq x)$

End of Session 6

Session 7 – Mathematical Induction

- Mathematical Induction - introduction
- Domino effect
- Principles of mathematical induction & steps involved
- Sum of power of integers
- Solved examples

Mathematical induction is a mathematical proof technique

Have you heard of the "Domino Effect"?

Step 1. The first domino falls
Step 2. When any domino falls, the next domino falls
So ... all dominos will fall!

That is how Mathematical Induction works.

What’s is Induction About?

- Many statements states that a property is an universal true – i.e., all the elements of the universe exhibit that property;
- Examples:
  1. For every positive integer n: $n! \leq n^n$
  2. For every set with n elements, the cardinality of its power set is 2^n.
- Induction is one of the most important techniques for proving statements about universal properties.

We know that:

1. We can reach the first step of this ladder;
2. If we can reach a particular step of the ladder,

then we can reach the next step of the ladder.

Can we reach every step of this infinite ladder?

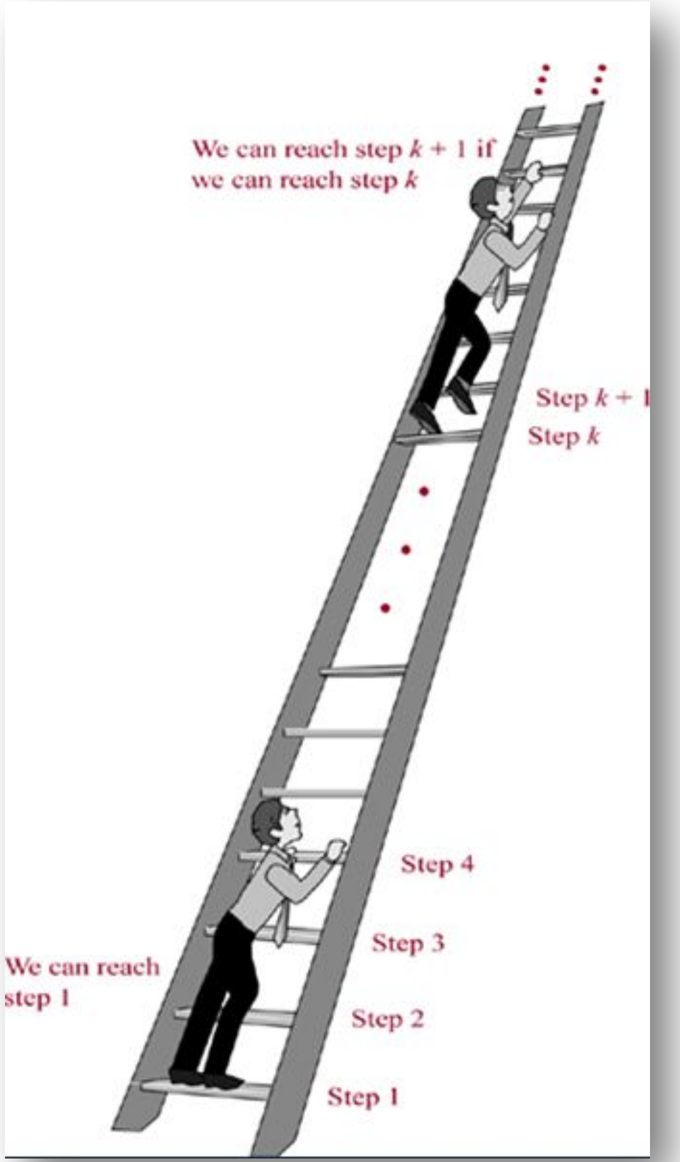
Yes, using Mathematical Induction which is

a rule of inference that tells us:

$$P(1)$$

$$\forall k\,(P(k) \rightarrow P(k+1))$$

$$\therefore \forall n\,(P(n)$$



- Hypothesis: $P(n)$ is true for all integers $n \geq b$

- To prove that $P(n)$ is true for all integers $n \geq b$ (*), where $P(n)$ is a propositional function, follow the steps:

- Basic Step or Base Case: Verify that $P(b)$ is true;

- Inductive Hypothesis: assume $P(k)$ is true for some $k \geq b$;

- Inductive Step: Show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers $k \geq b$. *This can be done by showing that under the inductive hypothesis that $P(k)$ is true, $P(k+1)$ must also be true.*

1. State the hypothesis very clearly:
    - P(n) is true for all integers $n \geq b$ – state the property P in English
2. Identify the base case
    - P(b) holds because …
3. Inductive Hypothesis
    - Assume P(k)
4. Inductive Step - Assuming the inductive hypothesis P(k), prove that P(k+1) holds; i.e.,

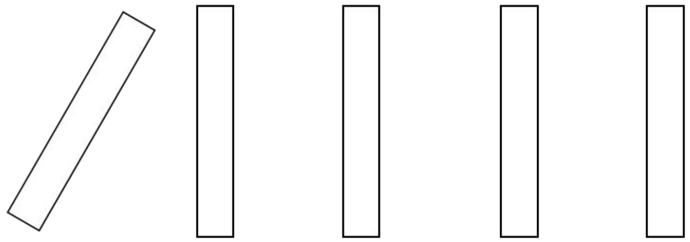
    $P(k) \rightarrow P(k+1)$

Conclusion

By induction we have shown that P(k) holds for all $k \geq b$ (b is what was used for the base case).

Domino Effect

- Mathematical induction works like domino effect:



- Let P(n) be "The n*th* domino falls backward".
- If (a) "P(1) is true";

  (b) "P(k) is true" implies "P(k+1) is true"

  Then P(n) is true for every n

Mathematical Induction

Method of proving — Theorems, Statements, Formulae

Steps Involved :

1) **Basis of Induction :**

Check validity of given statement say S(n) is True for the smallest integral value of n = 1 or 2 or 3 ...

2) **Induction Step :**

assume given st S(n) is true for n = k where k denotes any value of n, then it is also true that S(n) is true for n = k+1

3) **Conclusion :**

The statement is true for all integral values.

Ex. Adding up Odd numbers.

$$1+3+5+\cdots+(2n-1)=n^2$$

1. Show it is true for $n=1$

$1=1^2$ is **TRUE**

2. Assume it is true for $n=k$

$1+3+5+\cdots+(2k-1)=k^2$ is TRUE

Now, prove it is true for "$k+1$"

$$\underbrace{1+3+5+\cdots+(2k-1)}_{k^2}+(2(k+1)-1)=(k+1)^2$$

$k^2+2k+2-1=k^2+2k+1$

Simplify $k^2+2k+1=k^2+2k+1$

SAME $\therefore$ It is TRUE

So, $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ is TRUE

Mathematical Induction

Ex: Prove that $5^n - 1$ is divisible by 4 for $n \geq 1$

Sol^n

Basis of Induction:

for $n = 1$, $5^1 - 1 = 4$, divisible by 4

Induction Step:

Assume that $5^k - 1$ is divisible by 4

We have
$$5^{k+1} - 1 = (5^k \cdot 5 - 5) + 4$$
$$= 5(5^k - 1) + 4$$

By Induction, $5^k - 1$ is divisible by 4

$\therefore$ Each term on the RHS is divisible by 4

$\therefore$ $5^{k+1} - 1$ is divisible by 4

Hence, $5^n - 1$ is divisible by 4 for $n \geq 1$

Principle of Mathematical Induction

Let $P(n)$ be a predicate defined for integers n.

Suppose the following statements are true:

1. Basis step:

   $P(a)$ is true for some fixed $a \in \mathbf{Z}$.

2. Inductive step: For all integers $k \geq a$,

   if $P(k)$ is true then $P(k+1)$ is true.

Then for all integers $n \geq a$, $P(n)$ is true.

Example:

Use mathematical induction to prove

$$S_n = 2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$$

for every positive integer *n*.

1. Show that the formula is true when $n = 1$.

$$S_1 = n(n + 1) = 1(1 + 1) = 2$$ True

2. Assume the formula is valid for some integer k. Use this assumption to prove the formula is valid for the next integer, $k + 1$ and show that the formula $S_{k+1} = (k + 1)(k + 2)$ is true.

$$S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k + 1)$$ Assumption

Example continues.

Example continued:

$$S_{k+1} = 2 + 4 + 6 + 8 + \cdots + 2k + [2(k + 1)]$$

$$= 2 + 4 + 6 + 8 + \cdots + 2k + (2k + 2)$$

$$= S_k + (2k + 2)$$ Group terms to form S_k.

$$= k(k + 1) + (2k + 2)$$ Replace S_k by $k(k + 1)$.

$$= k^2 + k + 2k + 2$$ Simplify.

$$= k^2 + 3k + 2$$

$$= (k + 1)(k + 2)$$

$$= (k + 1)((k + 1)+1)$$

The formula $S_n = n(n + 1)$ is valid for all positive integer values of n.

Sums of Powers of Integers :

1. $$\sum_{i=1}^{n} i = 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$$

2. $$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

4. $$\sum_{i=1}^{n} i^4 = 1^4+2^4+3^4+4^4+\cdots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

5. $$\sum_{i=1}^{n} i^5 = 1^5+2^5+3^5+4^5+\cdots+n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Example:

Use mathematical induction to prove for all positive integers *n*,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$S_1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(2+1)}{6} = \frac{6}{6} = 1$$ True

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$ Assumption

$$S_{k+1} = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2$$

$$= S_k + (k+1)^2$$

$$= S_k + k^2 + 2k + 1$$ Group terms to form S_k.

$$= \frac{k(k+1)(2k+1)}{6} + k^2 + 2k + 1$$ Replace S_k by $k(k + 1)$.

Example continues.

Example continued:

$$= \frac{2k^3+3k^2+k}{6} + \frac{6k^2+12k+6}{6}$$

Simplify.

$$= \frac{2k^3+9k^2+13k+6}{6}$$

$$= \frac{(k^2+3k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

The formula $S_n = \dfrac{n(n+1)(2n+1)}{6}$ is valid for all positive integer values of *n*.

- **Proposition:** For any integer $n \geq 1$,

  $7^n - 2^n$ is divisible by 5. (P(n))

- **Proof** (*by induction*):

  1) Basis step:

  The statement is true for n=1: (P(1))

  $7^1 - 2^1 = 7 - 2 = 5$ is divisible by 5.

  2) Inductive step:

  Assume the statement is true for some $k \geq 1$ (P(k))

  (inductive hypothesis) ;

  show that it is true for k+1 . (P(k+1))

□ **Proof (cont.):** We are given that

P(k): $7^k - 2^k$ is divisible by 5. **(1)**

Then $7^k - 2^k = 5a$ for some $a \in \mathbf{Z}$. *(by definition)* **(2)**

We need to show:

P(k+1): $7^{k+1} - 2^{k+1}$ is divisible by 5. **(3)**

$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k$

$= 5 \cdot 7^k + 2 \cdot (7^k - 2^k) = 5 \cdot 7^k + 2 \cdot 5a$ *(by (2))*

$= 5 \cdot (7^k + 2a)$ which is divisible by 5. *(by def.)*

Thus, P(n) is true by induction. ■

End of Session 7

~ End of Unit I ~

Thank you