Radioactive Decay
CBE 5790 (Autumn 2018)

Radioactive decay is believed to be a truly stochastic (random) process. Consider radon as an example. The half-life of the radon isotope ^{222}Rn is 3.8235 days, meaning that given some quantity of ^{222}Rn we expect approximately half of the atoms to decay in slightly less than 4 days; however, if we look at a single atom of ^{222}Rn there is no way to know whether or not that particular atom will decay during any given time period.

In general, given the half-life $t_{1/2}$ of a radioactive element, the mean lifetime τ is

$$\tau = \frac{t_{1/2}}{\ln(2)} \tag{1}$$

There's nothing special about the half-life of a radioactive isotope – we could just as easily talk about quarter-life or tenth-life or etc. In general, given the $1/m^{th}$ life of a radioactive isotope, $t_{1/m}$, the mean lifetime τ is

$$\tau = \frac{t_{1/m}}{\ln(m)} \quad \text{where } m > 1 \tag{2}$$

A continuous deterministic model of radioactive decay

Suppose at time zero we have n_0 atoms of a radioactive element with half life $t_{1/2}$. As you may have learned in introductory physics, radioactive decay is described by a simple exponential model:

$$n = n_0 e^{-t/\tau} = n_0 2^{-t/t_{1/2}} \tag{3}$$

This model predicts that n decreases smoothly and continuously with time, in exponential fashion, approaching but never quite reaching zero. Note that this model is semi-empirical (because $t_{1/2}$ is determined experimentally), continuous (t and n are treated as continuous variables), and deterministic (for given n_0 and $t_{1/2}$, there is one and only one possible plot of n vs. t). No need to run multiple simulations – a single simulation of the model will suffice. This model works very well for most applications, which upon reflection is somewhat surprising – because it's a complete fantasy! Think about it – n is actually a discrete variable, not continuous, and in fact all of the atoms eventually will decay in a finite time and the variation is neither smooth nor predictable – it is stochastic. The differences between continuous-deterministic and discrete-stochastic approaches become apparent when n_0 is small.

A discrete stochastic model of radioactive decay

Once we've calculated τ (generally from half-life data, since this is what is commonly available) we can rearrange eq 2 to compute the fraction $1/m$ of radioactive atoms expected to "survive" (i.e., not decay) during any general time interval $t_{1/m}$:

$$\frac{1}{m} = \exp\left(-\frac{t_{1/m}}{\tau}\right) \tag{4}$$

For the stochastic model, we start by choosing a time step size $t_{1/m}$ and then apply eq 4 to calculate the expected proportion ($1/m$) that does not decay during this interval. For a single radioactive atom, $1/m$ is thus the *probability* that it will not decay during the specified time step. An interesting feature of radioactive decay is that it is a constant-probability process – the probability of decay during a given time interval is the same for all atoms and does not vary with time or with the concentration (proportion) of radioactive nuclei. Given *n* remaining radioactive atoms at any time *t*, the `binomial` function in the `random` module of the `numpy` package for Python can be used to generate probabilities for *n* trials, each with probability 1/*m* of "success" (no decay) and probability (1-1/*m*) of decaying during the next $t_{1/m}$ time step.

References:

https://en.wikipedia.org/wiki/Radon
https://en.wikipedia.org/wiki/Half-life
https://en.wikipedia.org/wiki/Exponential_decay

The Python program below simulates the decay of radon 222 (222Rn) in two ways: using a deterministic continuous-variable model and using a stochastic discrete-variable model. You can download this program (radioactive_decay.py) from the Python Examples page in Carmen.

```python
# -*- coding: utf-8 -*-
"""
Comparison of different models for radioactive decay
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Created: 2015-07-27
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"""

import numpy as np
import matplotlib.pyplot as plt

nAtomsZero = 20 # number of radioactive atoms at time zero
halfLife = 3.8235 * 24 # halflife of radon isotope 222Rn (hr)
meanLifetime = halfLife / np.log(2)
maxTime = 5 * meanLifetime # simulation time limit (hr)

#Determinisitc continuous-variable model
time = np.linspace(0, maxTime, num = 50)
nAtoms = nAtomsZero * np.power(2, -time / halfLife)
plt.plot(time, nAtoms, 'r--', label = 'deterministic')

#Stochastic discrete-variable model
timeStep = 5.0 # time step size for stochastic model (hr)
time = np.arange(0., maxTime, timeStep)
probNotDecay = np.exp(-timeStep / meanLifetime) #probability of NOT decaying
nAtoms = np.zeros(time.size, dtype = np.int)
nAtoms[0] = nAtomsZero
plt.hold(True)

for i in range(1, time.size):
    #calculate number that remain (i.e., do not decay)
    nAtoms[i] = np.random.binomial(nAtoms[i-1], probNotDecay)

#stair step plot to properly show that nAtoms is discrete, not continuous
plt.step(time, nAtoms, 'b-', label = 'stochastic')


#Add title, axis labels and legend
plt.title('Radioactive decay of $\mathregular{^{222}Rn}$')
plt.xlabel('time (hr)')
plt.ylabel('number')
plt.legend(loc = 'best')
plt.hold(False)
```