CBE 5790 Modeling and Simulation

Worksheet #3 – Big trouble on little Elba

The small Tuscan island of Elba, population 32,000, is threatened by a particular deadly strain of flu. Drawing on our knowledge of chemical kinetics and reactor design, we propose the network of elementary "reactions" below to model how a contagious disease will spread among the hapless residents of Elba.

Event ("reaction")	Rate (events/day)	Description
$H + S \xrightarrow{k_1} 2S$	$r_1 = k_1 n_H n_S$	Healthy person becomes sick through contact with a sick person
$S \xrightarrow{k_2} I$	$r_2 = k_2 n_S$	Sick person recovers and is now immune
$S \xrightarrow{k_3} D$	$r_3 = k_3 n_S$	Sick person dies
$H + V \xrightarrow{k_4} I$	$r_4 = k_4 n_H n_V$	Healthy person is immunized by vaccination

a) List the set of ordinary differential equations that must be solved to determine how the number of people in each group (healthy, sick, immune, dead) and the number of available doses of vaccine vary with time. Hint: each ODE is a balance equation: accumulation = generation – consumption. Assume no one enters or leaves Elba during the time of interest, so there's no input or output terms in the balance equations. There are n_{V0} doses of vaccine available at time zero and no production of vaccine thereafter.

b) At time zero, there are 31,999 healthy Elba residents and one person sick with the H3N2 flu. The rate constants are: $k_1 = 1.76 \times 10^{-5}$ day^{-1}, $k_2 = 0.100$ day^{-1}, $k_3 = 0.010$ day^{-1}, and $k_4 = 3.52 \times 10^{-6}$ day^{-1}. Perform a simulation to determine what happens during the first 120 days if there are no doses of vaccine available.

c) Explore how the outcome of an H3N2 flu outbreak depends on the number of available doses of vaccine and the rate at which people are vaccinated.

d) Consider a similar scenario as part (b) except that, instead of the flu, the one person sick at time zero has Ebola hemorrhagic fever, an extremely severe disease that is usually fatal. The reaction rate constants are: $k_1 = 1.76 \times 10^{-5}$ day^{-1}, $k_2 = 0$ day^{-1}, $k_3 = 0.600$ day^{-1}, and there is no vaccine. Perform a simulation to determine what happens.

e) The U.S. Center for Disease Control (CDC) asks you to develop a simulation to explore the following question: Assuming 31,999 healthy Elba residents and one person sick with the H3N2 flu at time zero, how many doses of vaccine should be available at time zero so that the final death toll is no higher than 400?

   Note that this is a boundary value problem. To answer the question, use a trial-and-error approach: guess $n_V(0)$ until you find a value for which the final death toll is 400.

Solution:

Event ("reaction")	Rate (events/day)	Description
$H + S \xrightarrow{k_1} 2S$	$r_1 = k_1 n_H n_S$	Healthy person becomes sick through contact with a sick person
$S \xrightarrow{k_2} I$	$r_2 = k_2 n_S$	Sick person recovers and is now immune
$S \xrightarrow{k_3} D$	$r_3 = k_3 n_S$	Sick person dies
$H + V \xrightarrow{k_4} I$	$r_4 = k_4 n_H n_V$	Healthy person is immunized by vaccination

a)

$$\frac{dn_H}{dt} = -r_1 - r_4 = -k_1 n_H n_S - k_4 n_H n_V \qquad n_H(0) = 31999$$

$$\frac{dn_S}{dt} = -r_1 + 2r_1 - r_2 - r_3 = k_1 n_H n_S - k_2 n_S - k_3 n_S \qquad n_S(0) = 1$$

$$\frac{dn_I}{dt} = r_2 + r_4 = k_2 n_S + k_4 n_H n_V \qquad n_I(0) = 0$$

$$\frac{dn_D}{dt} = r_3 = k_3 n_S \qquad n_D(0) = 0$$

$$\frac{dn_V}{dt} = -r_4 = -k_4 n_H n_V \qquad n_V(0) = 0$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial n_H} & \frac{\partial f_1}{\partial n_S} & \frac{\partial f_1}{\partial n_I} & \frac{\partial f_1}{\partial n_D} & \frac{\partial f_1}{\partial n_V} \\ \frac{\partial f_2}{\partial n_H} & \frac{\partial f_2}{\partial n_S} & \frac{\partial f_2}{\partial n_I} & \frac{\partial f_2}{\partial n_D} & \frac{\partial f_2}{\partial n_V} \\ \frac{\partial f_3}{\partial n_H} & \frac{\partial f_3}{\partial n_S} & \frac{\partial f_3}{\partial n_I} & \frac{\partial f_3}{\partial n_D} & \frac{\partial f_3}{\partial n_V} \\ \frac{\partial f_4}{\partial n_H} & \frac{\partial f_4}{\partial n_S} & \frac{\partial f_4}{\partial n_I} & \frac{\partial f_4}{\partial n_D} & \frac{\partial f_4}{\partial n_V} \\ \frac{\partial f_5}{\partial n_H} & \frac{\partial f_5}{\partial n_S} & \frac{\partial f_5}{\partial n_I} & \frac{\partial f_5}{\partial n_D} & \frac{\partial f_5}{\partial n_V} \end{bmatrix} = \begin{bmatrix} -k_1 n_S - k_4 n_V & -k_1 n_H & 0 & 0 & -k_4 n_H \\ k_1 n_S & k_1 n_H - k_2 - k_3 & 0 & 0 & 0 \\ k_4 n_V & k_2 & 0 & 0 & k_4 n_H \\ 0 & k_3 & 0 & 0 & 0 \\ -k_4 n_V & 0 & 0 & 0 & -k_4 n_H \end{bmatrix}$$

c) Vary rate constant k_4 to explore the rate of vaccination