Who Wants a Haircut?

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1 Introduction

This semester I started giving haircuts. It started when I jokingly asked my friend if I could give her a haircut, and to my surprise, she gave a very hesitant "yes." I really enjoyed giving my friend a haircut, and I found that I was actually pretty good at it (all I had to do was cut in a straight line, but apparently that's not a skill everyone has), so I started cutting more hair. The more I cut hair, the more I found it to be a chore—I wasn't going to charge any money for a haircut because I just wanted to do something nice for my friends, but an hour of cutting hair is an hour of not doing homework. While cutting hair is enjoyable to me, at a certain point I don't derive as much utility from it. Furthermore, I don't like doing multiple haircuts in a row, meaning that the time between haircuts will impact how much utility I derive from the current haircut.

In this paper, we will develop a model that can tell me how many haircuts I should give in a week. The model will take into account how much utility I get from a single haircut, and how the previous haircuts I've done in the week will influence the utility I get from doing more haircuts. The model will include several components, and we will go through each one individually. The first will be to look at my utility function as a function of the time it takes to give a haircut and of the other person's utility. From there, we will look at potential sequences of giving haircuts, that's when we observe how utility derived previously will influence my current utility level. Lastly, we will amalgamate the results to determine the utility-maximizing plan for giving haircuts.

The problem is multidimensional and also contains two separate temporal components. While this may seem like a complicated setup, we will incorporate various figures that will illuminate how the model works.

2 Dissecting the Model

2.1 My Utility Function

In this problem I have a total utility function (which we will not give a label to just yet), which depends on what my utility looks like in different time periods. If I do six haircuts in a week, then my total utility will be the sum of the utility I got from giving each individual haircut. But what do those individual utility functions look like?

Assume that I break up my haircuts up into weeks, and that the haircuts I did in the week before do not in any way impact the haircuts I'm giving in the current week. We do this so that we don't complicate the model. This is just the best way to develop the model—we can easily adapt it so that it takes the previous weeks into account, but let's not get ahead of ourselves.

We denote a sequence of haircuts by $t_0, t_1, \ldots, t_n$, which simply means that I spent t_0 minutes on the initial haircut, t_1 on the next one, and so on. Furthermore, associated with each t_i is a utility function, f_i; we denote the level of utility derived in haircut i by p_{t_i}. Except in the case of the initial haircut, each f_i is a function of the number minutes spent on a haircut, my friend's utility as a function of time spent on the haircut $u(t)$, the utility I derived from the previous haircut, $p_{t_{i-1}}$, and lastly the time between the current haircut and the previous one, s_i. To put it more compactly,

$$p_{t_0} = f_0(t, u)$$
$$p_{t_i} = f_i(t, u, p_{t_{i-1}}, s_i)$$

The only thing that hasn't been explained yet is the person's utility u. Firstly, we assume that every customer[1] has the same utility (this is done for the sake of simplicity). Their utility is nonnegative, because I'm not that bad at cutting hair (and it makes the model simpler).

With this in mind, we look at my utility function for the initial haircut, just to get an idea of what's going on in the model. We refer to the figure below. The dotted curves represent the upper and lower bounds of my utility function, and these are determined by the customer's utility level, so my utility function (for the initial period) can be anywhere between those dotted curves.

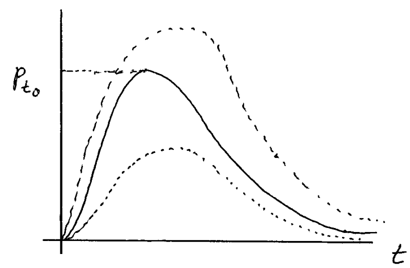

Figure 1: Initial utility function, f_0

This concept should be simple enough to understand, at least in the case of my initial utility function, where I don't have to include any information about prior utility levels. The models becomes complicated when we introduce that component, but the utility functions will always have this shape and structure. To end on a descriptive note, if we let $h(t)$ be a function that gives my utility function its shape, then we have that for $i = 1, \ldots, n$

$$p_{t_0} = f_0(t, u) = [h(t) \cdot k(u)]$$
$$p_{t_i} = f_i(t, u, p_{t_{i-1}}, s_i) = [h(t) \cdot k(u)] \cdot g(p_{t_{i-1}}, s_i),$$

Where $k(u)$ is simply a scaling factor for my utility function (it is what determines where the function lies between the dotted curves), and $g(p_{t_{i-1}}, s_i)$ is a function that changes the shape of my function in time period i in a way that accounts for the utility I had in the previous time period.

2.2 Considering Prior Utility

The most important variable to consider here is s_i, the time between haircuts. This introduces a second temporal component to the model, but unlike the variable t that appears in each utility function, is more global, in that it relates two different utility functions, f_i and f_{i+1}.

Like we said earlier, the utility functions generally look the same. Consider the graph in Figure 1; we will now see what the next utility function looks like. Figure 2 shows the two graphs at t_0 and at t_1, and the time between haircuts is denoted by s_1 (note that we included another value s_1', but we will explain that later).

[1]Although I'm not charging anybody, the easiest term to use is customer, so I am officially adopting this terminology.

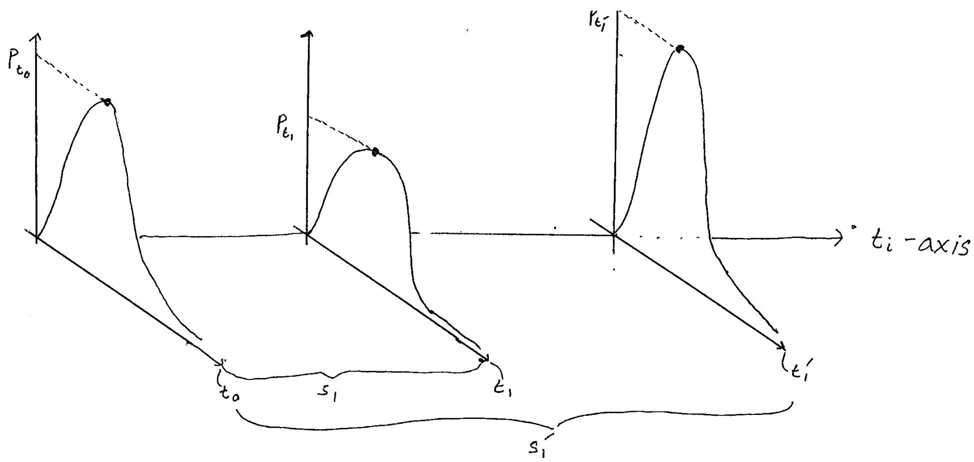

Figure 2: Tracking utility over time

We compare the two graphs at t_0 and t_1. Note that $s_1 = t_1 - t_0$, and the two functions are different. Assuming that my utility function at each time ends up being the ones drawn (the darker curves), then I want to optimize my utility at each one individually, so we put dots at the peaks of these curves. Then we're done—this is how we understand how prior utility levels impact current ones, by comparing the curves in this way.

To clarify exactly how the s_i variable works, I have included another graph at t_1' corresponding to s_1'. Notice that the peak of this curve is much higher than the one at t_1; this is the nature of the s_i variable. As s_i gets larger, p_{t_i} will be larger because I prefer to have haircuts spaced out. There are infinitely many graphs along the horizontal axis (the t_i-axis), and this is why this increases the complexity of the problem so much. There are many moving parts, like the utility functions at each t_i, but we also have to consider the possible interdependencies between each instance of a haircut. This concept is tricky, but makes the model robust. We will now see how all of this comes together.

2.3 Synthesizing Utilities

Remember the dots I made on the peaks of those curves in Figure 2? If we add those up, then we have the total utility function we mentioned earlier. We denote it by

$$P_{t_0 t_1 \cdots t_n} = \sum_{i=0}^{n} p_{t_i}.$$

All this means is that at each t_i, we make a dot at the peak of that graph, and that dot corresponds to utility level p_{t_i}. Now we have kept track of when I give a haircut (the t_i's) and the utility I derived from giving it (the p_{t_i}'s). All we need to do is add up the utilities to get my total utility.

This value $P_{t_0 \cdots t_n}$ is like the "profit expression," meaning that it simply makes a calculation, but it doesn't find the maximum total utility level. The goal is to find the sequence $t_0^*, \ldots, t_n^*$ that maximizes my total utility.

3 Putting it to Use

I make use of this model if I have an idea of what the functions in question look like, and if I can accurately capture how my utility changes as I do more haircuts. If I can get functions for all of these, then I can also add constraints, such as limiting myself to only three haircuts a week, then the model will tell me when I should give each haircut. I can also say that I want to spend no more than five hours cutting hair, and the model will also be able to deal with this constraint. Finding these solutions is a computational exercise, not an analytical one.

This problem reminded me of social welfare functionals in that I am allocating haircuts to myself at different times in order to maximize my utility, kind of like how a social dictator allocates resources to people in order to maximize their utility function. Like the social dictator, I am dealing with a variety of utility functions, although they are all mine, but they change quite a bit. With the proper setup and computational tools, the model could tell me when I should give haircuts; maybe I should spread them out evenly, or maybe I should do them all on the weekend.

The next step in refining my model would be to explore the relationship between my utility functions and the time between haircuts, s_i. It currently acts as a discounting value, because my utility function in one time period is very different than in another one.