Colby College

Honors Thesis

A Coder's Guide to Elliptic Curve Cryptography

Author:
Stephen Morse

Supervisor:
Fernando Gouvêa

A thesis submitted in fulfilment of the requirements for graduating with Honors in Mathematics at

Colby College

May 2014

COLBY COLLEGE

Abstract

Fernando Gouvea
Colby College - Department of Mathematics and Statistics

Bachelors of Arts

A Coder's Guide to
Elliptic Curve Cryptography

by Stephen Morse

Many software applications and websites today require a basic understanding of cryptographic protocols. This work, which is aimed at software engineers and mathematicians interested in Elliptic Curve cryptography, may serve as a guide to understanding the mathematics and abstract protocols in Elliptic Curve cryptography. This document is broken up into four sections. The first describes fundamental concepts in cryptography and gives necessary background information. The second covers the mathematics of Elliptic Curves and their complex group structure. The third describes ECC and implementation details for the ElGamal system. The last section introduces Diffie-Hellman key exchange and shows how to implement it using Java.

Acknowledgements

To professor Fernando Gouvêa, thank you for overseeing my project. Your articulate feedback and patient guidance was and is extremely appreciated. I have learned so much from you and I am thankful for the time you committed to helping me with this project.

To professor Justin Sukiennik, I am grateful for your feedback on my thesis and for your help with LaTeX.

To Stephen Jenkins, Byoungwook Jang, and Matt Burton, it has been a great 4 years. I thank you for keeping my spirits up and my mind sharp.

To Nicole Hewes, your baked goods and heartfelt encouragements were priceless. You kept me going even when I wanted to quit.

To my family, thank you all for supporting me by coming to my presentation. You convincingly feigned interest for quite some time and I appreciate it!

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Preface

> *Science is what we understand well enough to explain to a computer. Art is everything else we do.*
>
> - Donald Knuth

As Knuth suggests, real understanding only comes when we can break a complex process down into its simplest components. Programming languages, the tools for explaining procedures to a computer, are some of the best tools we have for breaking processes down and systematically describing each component clearly and concisely.

With this in mind, this work will try to break Elliptic Curve cryptography down into its simplest components for the reader. By the end, the reader will hopefully understand ECC well enough that he or she could write an implementation of ECC in a programming language. However, this work does not cover the syntax for any real programming languages. Instead, we are striving in this guide for a level of understanding of Elliptic Curve cryptography that is sufficient to be able to explain the entire process to a computer.

This is guide is mainly aimed at computer scientists with some mathematical background who are interested in learning more about Elliptic Curve cryptography. It is an introduction to the world of Elliptic Cryptography and should be supplemented by a more thorough treatment of the subject. See section 1.2 for a summary of what background material this guide assumes the reader has already covered.

Happy reading!

Chapter 1

What is Public Key Cryptography?

1.1 A Historical Introduction

World War II was unique for several reasons, among them being the monumental role that Cryptography took in the communication between military groups, largely due to new cryptographic tools which the technology of the time enabled. Naturally, the opposing military forces desired fast and accurate communication between their respective allies, while simultaneously wishing to ensure complete secrecy from their enemies. It is a common theme in the world of Cryptography, however, that these two features (convenience and security) are in opposition to each other. When opposing military forces became privy to confidential information the repercussions were severe, so there was a huge need to protect one's algorithms for sharing secret information. Great effort was put into doing just that, and thus World War II became not just a war between physical forces but also between mental and mathematical ones.

One example of an emerging technology that gave groups the power to communicate securely, for a time at least, was the Enigma machine. It was a small machine that looked somewhat like a typewriter, and it worked by scrambling the letters of the alphabet according to its current settings. The trick, however, was that with each keystroke the settings would change, creating a seemingly random sequence of letters. To decode a message, all one had to do was start the enigma machine with the exact same settings as the initial encoding settings and then type the ciphertext into it. Many thought this system to be unbreakable due to the sheer number of possible settings that the machine could have (well into the trillions) [1]. However, due to the efforts of Polish mathematicians and many British code-breakers, methods to reliably decode

messages created by the Enigma machine were discovered in 1940. These efforts were not made public knowledge until the 1970s [2, p. 36].

Do you notice something strange about the Enigma machine's procedure? It required both the sender and the receiver to already have a bit of shared information: what settings to start the Enigma machine at in order encode and decode messages. That is, *in order to share secret information one had to first share secret information!* This raises the question: how does one share the first information securely? This type of cipher, where both the sender and receiver have a small bit of shared information in order to share larger amounts of secret information, has become known as a Symmetric Cipher. While typically more secure and efficient, Symmetric Ciphers are very inconvenient, and in particular is not directly suitable for many web-based applications where the two parties which need to exchange sensitive information have never met and so have not had a chance to exchange their own private key.

How can one exchange information securely with someone whom they have never met? The solution is to create a system by which any two users can share their secret information over an insecure line of communication. This seems impossible, but, surprisingly, it is not! That is, it is possible if you assume that anyone listening in on your communication has a realistic amount of computing power and has not developed any radically new cryptanalysis (code-breaking) procedures. In most cases, these are reasonable assumptions.

These procedures to share information between users which have never met and over communication channels that may be monitored fall under the category of Public Key Cryptography. Such systems are based on mathematical problems that are presumed to be hard for their security. That is, to crack these systems would require solving a mathematical problem which is believed to be computationally difficult. In these systems, users will typically each have two keys: a private key and a public key. The procedure for using these keys is also made public knowledge so that anyone may send a message to any other individual. This has the unfortunate side effect that anyone who may be eavesdropping may be assumed to know the exact method for encoding a secret message using a public key, but fortunately there are algorithms that are extremely hard to undo, even if one knows how they were performed.

Ciphers which use Public Key Cryptography are called Asymmetric ciphers because the sender and the receiver do not share the same information. Users are typically aware of all of the other users' public keys but each user's private key is kept secret. While less efficient than Symmetric ciphers, Asymmetric ciphers have a distinct advantage in that they allow users to share secret

information even if someone is listening to the entire 'dialogue'. Typically, these types of ciphers may be combined: the Asymmetric cipher allows the sharing of a private key and the private key may then be used with a Symmetric cipher for further communication.

1.2 A Short Note to the Reader

This guide will make an attempt to explain things as carefully as possible, but studying theoretical Public Key Cryptography, and Elliptic Curve Cryptography in particular, really does require some mathematical background. It will be assumed that the reader has at least a basic understanding of Number Theory and Abstract Algebra. In particular, the reader should be familiar with modular arithmetic, Fermat's Little theorem, the Chinese Remainder Theorem, Groups, and Finite Field theory. For a basic introduction to these topics I recommend looking at [2]. This great book on Mathematical Cryptography contains short chapters specifically devoted to bringing the reader up to speed on some of the background information necessary for studying Cryptography.

1.3 A Mathematical Introduction

Before we jump right into the mathematics behind Public Key Cryptography, some introductions are in order. In studying Cryptography, one regularly talks about a few fictional characters named Alice, Bob, and Eve. As depicted below, the usual story is that Alice and Bob would like to be able to share confidential messages with each other, but Eve (the eavesdropper) is listening to every word they say. Having these characters in one's pocket can help to simplify the description of crypto-systems.

Now suppose, as we always do, that Alice would like to send Bob a message. Let's say she wants to send the message "Hi!". The first step is for Alice to turn her message into a number that can then be mathematically manipulated. There are many ways to do this, one being to use the standard ASCII or Unicode mapping, both of which are built into computers. Another, would be to simply replace "A" by 10, "B" by 11... and so on, so that the message "ABC" would become the number 101112. This translation from strings to integers and back is not particularly complicated, so for the remainder of this chapter we will assume that the messages

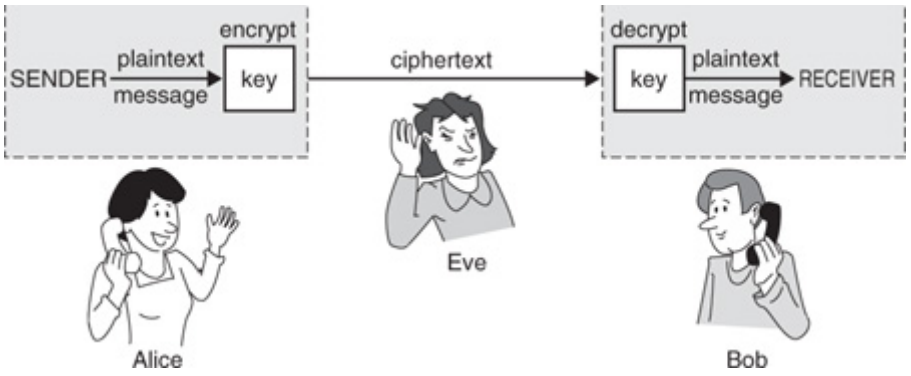

FIGURE 1.1: Alice uses a symmetric cipher to send secret messages to Bob, which Eve is unable to decipher. Source: [3]

that Alice and Bob are trying to send to one another are numerical integers in some defined range. However, when we get to Elliptic Curve cryptography, we will need to find more clever ways of embedding our messages (see section 3.3 for more details).

Now we will describe, in an abstract way, the components necessary for Public Key Encryption. To do this, we needs some definitions.

Definition 1.1. A **message** is simply information which one person would like to send to another. Typically these are integers within a certain range. We will let $\mathcal{M}$ be the set of all possible messages for a given cipher.

Definition 1.2. A **key** is a value, which may either be secret or public knowledge, that is used in the encryption or decryption of a message. We will let $\mathcal{K}$ be the set of all possible keys for a given cipher.

Definition 1.3. A **ciphertext** is a value that is created using an encryption algorithm. We will let $\mathcal{C}$ be the set of all possible ciphertexts for a given cipher.

The first step in creating a Public Key cipher is for every one to agree on two functions, an Encryption function $E(k_{\text{pub}}, m) : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$ and a Decryption function $D(k_{\text{priv}}, c) : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$. After this, each user picks a private key from $\mathcal{K}$ and they make sure that this key is not shared with anyone else. Using their private key, each user can then calculate which public key corresponds to their private key and post their that key to a pubic database. After Alice and Bob have completed this procedure, if Alice would like to send a message m to Bob she looks up his public Key k^B_{pub}, encrypts the message by calculating $E(k^B_{\text{pub}}, m)$. When she sends this encrypted message, only Bob, with his secret information k^B_{priv}, will be able to decode her message.

Clearly, if our encryption and decryption functions are to be made properly they must satisfy a certain set of conditions. These conditions are summarized below.

(1) First, we require that the decryption undoes the encryption, as this is necessary for the entire process to even work at all. That is, it must be the case that for a user's set of keys $(k_{\text{priv}}, k_{\text{pub}})$, we have that $D(k_{\text{priv}}, E(k_{\text{pub}}, m)) = m$. Note that this does not mean that the users may pick any $(k_{\text{priv}}, k_{\text{pub}})$ pair they like, but rather that the E and D functions need to be functions such that many such pairs exist. Typically, one chooses a private key first, computes what the public key should be and then publishes the public key. Note that for the system to be secure at all, knowing k_{pub} should not reveal k_{priv}.

(2) Second, we need our functions to be easy to compute, as this process will be no good at all if we cannot share many messages quickly. The determination of how quick one needs it to be depends one's own constraints, such as whether the message sender or receiver is using a computer.

(3) Lastly, we do not want any E functions which reveal information about k_{priv} or m. In other words, we require that it be extremely difficult to compute both k_{priv} and m, even if one knows the values of k_{pub} and $E(k_{\text{pub}}, m)$.

Now, suppose that we have given to us two such functions as described above. First we make them public for all the world to see. Then anyone who would like to receive messages may choose a $k_{\text{priv}} \in \mathcal{K}$, calculate the associated public key k_{pub} and publish it. Suppose that Alice and Bob would like to share a message, and their keys are as given in Figure 1.1. If Alice wants to send her message m to Bob, then all she must do is compute $c = E(k^B_{\text{pub}}, m)$ and send it to Bob. Bob, who is the only one who knows k^B_{priv}, will then be able to decode and receive Alice's message by computing $m = D(k^B_{\text{priv}}, c)$.

The remarkable thing about this process is that after Alice has encrypted her message, not even she can decode it! Granted, yes, she probably remembers what the message originally was, but the point is that even she cannot come up with an algorithm to decode her own message. This is because, by property (3), she cannot is not able to gain any information about k_{priv}, which is necessary for the decryption.

	Alice	Bob
Private Key	k^A_{priv}	k^B_{priv}
Public Key	k^A_{pub}	k^B_{pub}

TABLE 1.1: All members have a their own public key and their own private key.

1.4 Example - RSA

After much talk about these mysterious E and D functions, it is time for an example. The reader should note that such a system is not easy to create. It was actually postulated that such a system could exist for quite some time before anyone was actually able to produce an example. RSA was one of the first examples provided of such a system. The original patent for this system can be found at [4].

Standard RSA (as created by Rivest, Shamir, and Adleman) is one the simpler examples of a Public Key system. RSA has also withstood the test of time. That is, no one has been able to devise an efficient method to break all cases of RSA [5]. Some parameters must be chosen wisely to thwart known methods of attack, but being careful as to what parameters we choose is a small inconvenience.

Suppose that someone gives you two prime numbers, 37 and 71, and asks you to compute their product. You would likely pull the pencil out from behind your ear, scratch away at a piece of paper for a moment, and then come up with the answer $2,627$. Now suppose that instead of the numbers 37 and 71, someone gives you the number $2,627$ and asks you to factor it. What would be your first step? You would likely sit down and try, one by one, each prime less than $\sqrt{2627} \approx 51.3$ until you got to 37, but this would likely take you much longer than multiplying the two numbers did! This illustrates how much harder factorization is than multiplying, and this sort of a one way process (a function that easy to compute but difficult to invert) can be useful for cryptographic purposes.

RSA encryption takes advantage of the fact that even if someone knows the product $N = p \cdot q$, where p and q are two large primes, it would be very difficult for him or her to gain any knowledge of p or q. Thus, if Bob wants to allow others to send him messages, he picks two large primes p and q, calculates $N = p \cdot q$, and now N can safely be included in Bob's public key. We will need one more bit of information for encryption: a public exponent e chosen in the range $1 < e < \alpha$ with the property that $\gcd(e, \alpha) = 1$, where $\alpha = (p-1)(q-1)$. Since Bob is

the only one who knows p and q, only he will be able to calculate $\alpha = (p-1)(q-1)$. He is also the only one who can correctly choose the parameter e. Notice that we are also safe publishing e because even if someone were able to factor this public exponent, it does little more than tell a few factors α cannot have (because $\gcd(e, \alpha) = 1$).

Now, to encrypt a message m in the range $1 < m < N$, all Alice must do is calculate $c = E(k^B_{\text{pub}}, m) = m^e \pmod N$. There are fast powering algorithms that make this encryption very efficient, on the order of $O(\log(e))$ time complexity. Section 3.2 describes one such algorithm. She sends this ciphertext to Bob.

To decode Alice's message, Bob must calculate the inverse, $d = e^{-1}$, of e in $(\mathbb{Z}/\alpha\mathbb{Z})^\times$, which he can do because he knows α. In other words, we find a number d such that $ed \equiv 1 \pmod \alpha$. We know this d exists because we chose the value e such that $\gcd(e, \alpha) = 1$, and calculating d can be done using the Euclidean algorithm. This d is Bob's private key, along with the secret information p and q. All we must do to decode c, then, is to calculate $D(k^B_{\text{priv}}, c) = (c^d \mod N)$ and this will be Alice's value m. To see this, notice that:

$$\begin{aligned} c^d &\equiv m^{ed} && \pmod N \\ &\equiv m^{1+r\alpha} && \pmod N \quad \text{for some } r \in \mathbb{Z} \\ &\equiv m \cdot m^{r\alpha} && \pmod N \\ &\equiv m \cdot (m^\alpha)^r && \pmod N. \end{aligned}$$

Now, if we know that $(m^\alpha) \equiv 1 \pmod N$, then we are done. To show this, we will need to use Fermat's Little theorem and the Chinese Remainder theorem.

Theorem 1.4. *If $N = p \cdot q$ for two primes p and q and $\alpha = (p-1)(q-1)$, then we have that*

$$m^\alpha \equiv 1 \pmod N.$$

Proof. We will first show that $m^\alpha \equiv 1 \pmod p$ and $m^\alpha \equiv 1 \pmod q$. This is not hard because Fermat's Little theorem gives us that

$$m^{p-1} \equiv 1 \pmod p \quad \text{and} \quad m^{q-1} \equiv 1 \pmod q.$$

After raising both sides of the first equation by $(q-1)$, we get that $m^\alpha \equiv 1 \pmod p$. Similarly, after raising both sides of the second equation by $(p-1)$, we have that $m^\alpha \equiv 1 \pmod q$.

Now, we know from the Chinese Remainder Theorem that all solutions to the relations $m^\alpha \equiv 1 \pmod{p}$ and $m^\alpha \equiv 1 \pmod{q}$ give values for m^α that equivalent mod N. Since $m^\alpha = 1$ is a solution to these two relations, it must me that $m^\alpha \equiv 1 \pmod{N}$. This completes the proof that $m^\alpha \equiv 1 \pmod{N}$. □

Therefore, the decryption undoes the encryption process, just as we would like it to. This completes the description of the RSA Public Encryption scheme. To summarize, the encryption and decryption functions are as below.

$$E(k^B_{\text{pub}}, m) = m^e \pmod{N}$$
$$D(k^B_{\text{priv}}, c) = c^d \pmod{N}.$$

As a final note, the security of this system does not only depend on being able to factor N. If Eve could factor N, she would definitely break the system because then she would easily be able to compute α and d, which would allow her to decode ciphertexts. But what if someone could regain the value of m just from knowing $c = m^e \pmod{N}$? Many people have looked for ways to do this efficiently, but no one has found an efficient algorithm to always solve this problem. Hence, we say that RSA relies on the difficulty of mathematical problems which are presumed to be computationally hard because no one has been able to solve them yet. In fact, no attack on RSA has proved to be easier than factoring N and it can be shown that any algorithm to break RSA will also factor N. Thus, as long as we have no efficient method for factoring N, RSA will remain secure.

1.5 Summary

The Public Key encryption process is a counter-intuitive scheme. It a strange system in which one cannot even decode one's own ciphertext, but rather the ciphertexts are furnished in such a way that only the person they were created for can decode them. It also relies on the *presumption* that certain problems are computationally difficult to solve, unless one has some extra information, for its security. For most encryption schemes we do not actually have a proof that a breaking the system is hard.

Public Key encryption is useful because it allows parties that have never met to share small amounts of confidential information. Many times that small amount of information is a private key which can be used in a more efficient symmetric cipher.

RSA is an example of an Asymmetric cipher. Although the parameters must be chosen wisely, RSA seems to be very secure if implemented correctly. Part of the beauty of Public Key encryption schemes like RSA is that even though the value of the encrypted message is completely determined by the values of the ciphertext and public key, (assuming the parameters are chosen well) no one will likely have the computing power to be able to find it!

Chapter 2

What are Elliptic Curves?

2.1 Introduction to Elliptic Curves

Elliptic curves, as they are unfortunately named, do not have much to do with Ellipses. Although Elliptic Curves were originally discovered while studying the properties of Ellipses, they are now important enough mathematical objects in their own right that a whole field of mathematics is devoted to them and can be studied without knowledge of how they relate to Ellipses at all.

An Elliptic Curve is the curve given by the set of points which satisfy an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where the coefficients must satisfy a complicated polynomial condition to guarantee that the curve is nonsingular (having no cusps or self-intersections). The above form is known as the generalized Weierstrass form for Elliptic Curves. The subscripts on the coefficients are chosen so that the above equation is homogeneous (all terms have the same weight) when x is given weight 2 and y is given weight 3.

If $a_1 = 0$ then when working over fields with characteristic 2 (which we will mainly be concerned with), E is what is known as a supersingular curve. This is a property which we will avoid for reasons to be explained in section 3.5. There are Elliptic curves with $a_1 = 0$, but we are just not going to work with them as they allow certain attacks that do not work on other types of curves.

The reader should be careful not to confuse singular and supersingular. These two conditions are less related than their names might suggest. Singularity is a condition on the shape of the curve which guarantees that it has no cusps or self-intersections. All Elliptic Curves are by definition non-singular to ensure certain properties of the curves. Supersingular curves are a special class of Elliptic Curves which we will cover in chapter 3.

For $a_1 \neq 0$ (the non-supersingular case over fields of characteristic 2), there is a change of variables that will take us from the generalized Weierstrass form to a simplified form:

$$y^2 + xy = x^3 + ax^2 + b.$$

In this simplified form, the nonsingularity condition is easier to state: we must be using a curve with non-zero b. This ensures that certain operations with points of the curve work well. The change of variables, which assumes that $a_1 \neq 0$, is given by:

$$x \rightarrow a_1^2 x + \frac{a_3}{a_1}$$

$$y \rightarrow a_1^3 y + a_1^{-3}(a_1^2 a_4 + a_3^2).$$

Although in practice we will use curves written in the simplified form $y^2 + xy = x^3 + ax^2 + b$, in examples I will use equations of various other forms that are easier to work with and to visualize over $\mathbb{R}$. Elliptic curves come in two main varieties, curves that have two parts and curves that only have one part. See the figures below for examples of the two varieties.

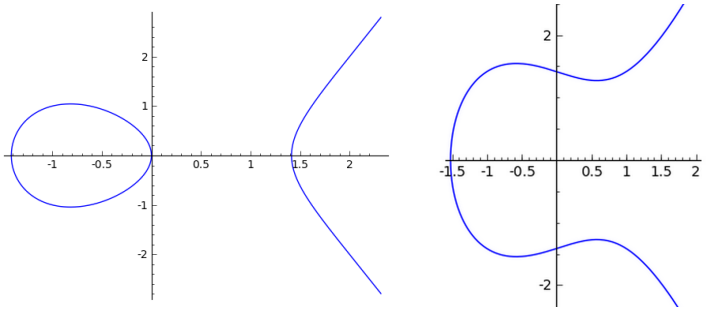

FIGURE 2.1: The two types of Elliptic Curves. The left curve is given by $y^2 = x^3 - 2x$ and the right curve is given by $y^2 = x^3 - x + 2$.

2.2 The Group Law on points of an Elliptic Curve

These curves are special because they come with an operation under which the points on the curve form a group. The operation is most easily described geometrically. Take two points on the curve with distinct x-coordinates, call them P and Q. If we extend a line through P and Q, this line will intersect the curve at a third point, call it R'. We will define the sum $P+Q$ to be the second point on the curve that has the same x-coordinate as R' has. This point R will either be directly above or directly below R'. For an illustration of this process, see the figure below (figure 2.2).

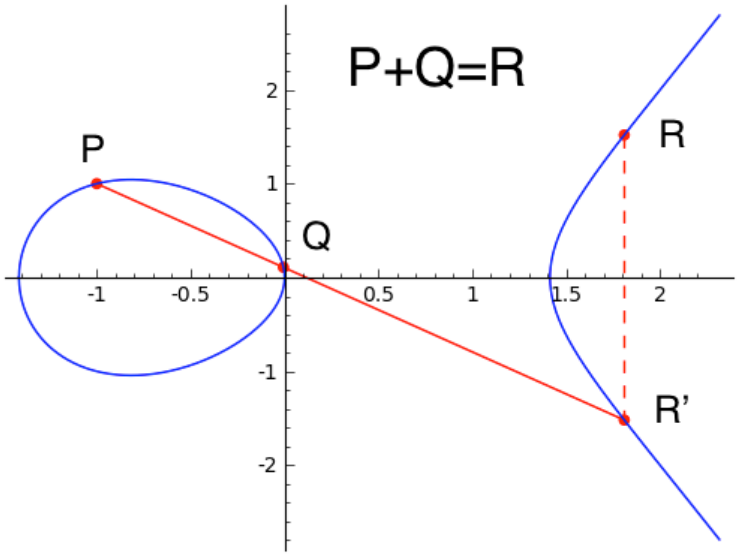

FIGURE 2.2: Elliptic Curve given by $y^2 = x^3 - 2x$ with addition of points.

There are still quite a few details to be sorted out, such as how to add a two points with the same x-coordinate and how we can be sure that there will always be a third point of intersection, but think about how surprising this should be: under this strange operation, the points of the curve form a commutative group! That means there is a zero-element, there are inverses, and that the addition is even associative. This seems to be a bizarre coincidence that there is such a nice operation on these points, but there is some deep mathematics hiding in the background. For the interested reader, there is actually an association between the points of an elliptic curve and the points in the fundamental domain of a lattice in $\mathbb{C}$. In addition, adding modulo the lattice actually corresponds to adding points on the Elliptic curve. This addition in the lattice is somewhat more natural, the surprising part is that there is a geometric description of adding corresponding points on an Elliptic Curve at all. For more on this, see [6].

Now we will sort out some of the subtleties in Elliptic Curve point addition. We first check that the points on the curve with respect to this operation really do form a commutative group. It is clear that this operation is commutative, since the operation is not defined in terms of which point is used first in the addition, but seeing that the points of an Elliptic Curve actually a group is not quite as easy. Along the way we will further clarify how the addition works, and at the end of this section we will provide the general algorithm.

(Identity Element) First, there must be an identity element $\mathcal{O}$ such that for any point P on the curve, $P + \mathcal{O} = P$. One way to think about where this element must be is to think of it as living on the curve, but all the way at at the end of the curve, infinitely far away. For example, in figure 2.3, the arrows point along the curve, approaching the identity element, sometimes called the point at infinity.

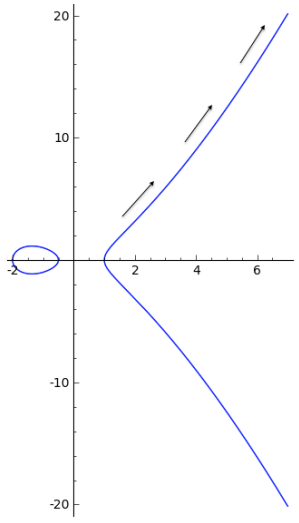

FIGURE 2.3: Elliptic Curve together with its point at infinity.

Then, one might think of adding a point P to the point at infinity $\mathcal{O}$ as:

$$\lim_{Q \to \mathcal{O}} (P + Q).$$

Eventually, Q will be so far above P that the horizontal distance that it has travelled will be insignificant compared to how large the vertical gap is between the points. Thus, it looks just as though Q is right above P, and so the third point of intersection will be extremely close to the point directly opposite from P. Then, according to the geometric definition of the addition law, we just get back the point P because it is directly opposite the third point of intersection.

One may equivalently think of the point at infinity as being on every vertical line, which makes it easier to visualize the addition but makes it less clear that the point is actually on the curve.

(Inverse Elements) Verifying the existence of inverses for any point P on the curve is easy now that we know what our zero element is: $-P$ is just the point with the same x - coordinate as P, directly above or below it. The line which intersects P and $-P$ will always be vertical, extending all the way to the point at infinity. Reflecting the point at infinity simply gives back the same point at infinity ($-\mathcal{O} = \mathcal{O}$), showing that $P + (-P) = \mathcal{O}$.

(Associativity) The last part of the group law to verify is the associativity of the operation. This can be done one of two ways. The first is to use a brute force approach where one breaks the addition up into many cases, computes the general formula for $P+(Q+R)$ and $(P+Q)+R$ for each of the cases, and verifies that they are equal in each case. Verifying this proof would be an extremely tedious and unenlightening task. The second is to used advanced methods to relate lattices and Elliptic Curves to prove the associativity, but this is beyond the scope of this project. We will take this result as known. The reader may look to [6] to see some of the theory relating Elliptic Curves to lattices. See figure 2.4, to see an example of the associativity of Elliptic Curve point addition.

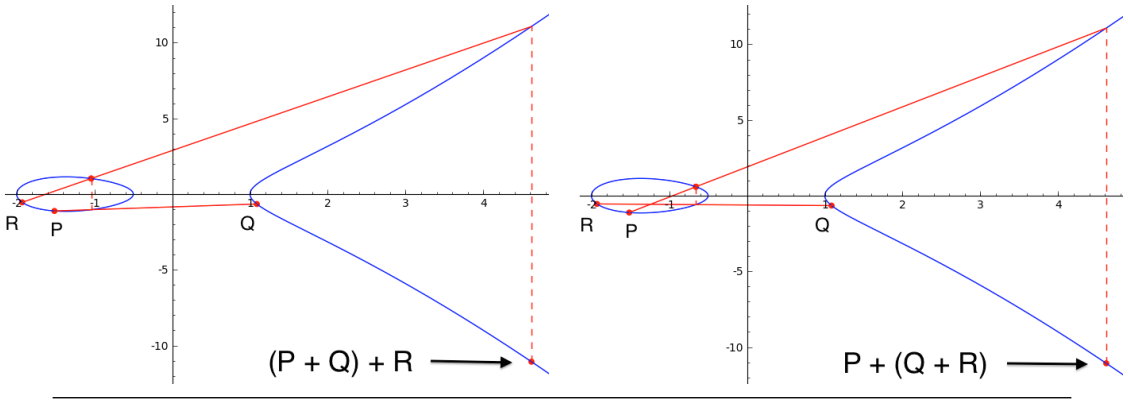

FIGURE 2.4: Associativity of Elliptic Curve Point Addition

The two remaining loose ends are how to add a point to another point with the same x-coordinate and how we can be sure that there will always be a third point of intersection.

(1) There are two possible cases when adding a point P which has the same x-coordinate as a point Q. If they have different y-coordinates, then the points are in fact additive inverses of each other and the third point of intersection on the curve is the point at infinity. In the other case, P and Q have he same y-coordinate and we are simply looking to find $2P$. Similar to how

the addition with the point at infinity was defined, we can define $2P$ as:

$$\lim_{Q \to P} (P + Q).$$

Eventually P and Q are very close to one another, and in the limit, the line between P and Q approaches the tangent line of the curve at P. Accounting for multiplicity of intersections, the tangent line will intersect the curve twice at P and will intersect the curve in one other place, call it $-R$ (this will be $\mathcal{O}$ if P has a vertical tangent line). The addition of the point P to itself, denoted $2P$, is R. See figure 2.5 for a visual.

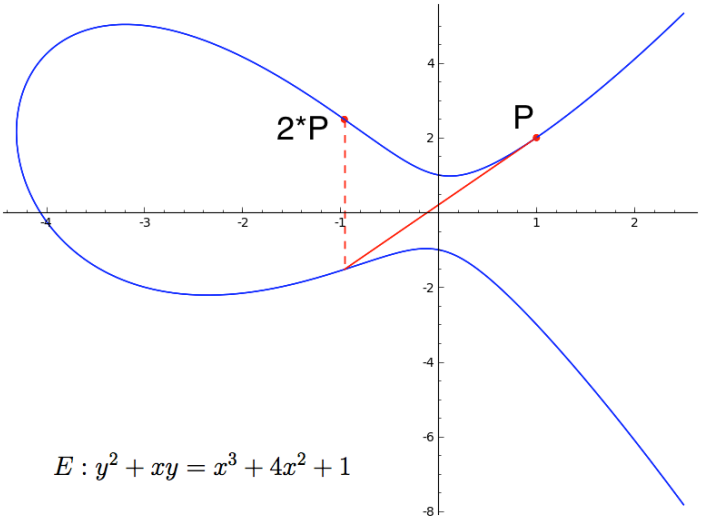

FIGURE 2.5: Doubling a point on an Elliptic Curve

(2) It is not extremely difficult to see that the any line that intersects the curve in two places must intersect it in a third when we include the point at infinity. Suppose we are given the points P and Q. Let L given by $ax+by+c=0$ be the line that intersects the curve at P and Q. If L is vertical (in which case $b=0$) then the third point of intersection is the point at infinity, $\mathcal{O}$.

Otherwise, we know that $b \neq 0$ and we may write our line in the form $y = \lambda x + v$. Note that λ may be obtained as the slope between two distinct points or as the slope of a tangent line of the curve. In this case, to find the third point of intersection we must find all solutions to the equation:

$$(\lambda x + v)^2 + x(\lambda x + v) = x^3 + Ax^2 + B.$$

Or, equivalently, after some rearranging:

$$0 = x^3 + (A - \lambda^2 - \lambda)x^2 + (-\lambda v - v)x + (B - v^2).$$

This is a cubic which we already know two of the roots of, namely the x-coordinates of P and Q, because we already know two of the points of intersection between the curve and the line. This means that there must be a third point of intersection because a cubic can only have either 1 or 3 real roots, and since we know it has at least 2, it must be that this cubic actually has 3 real roots.

2.3 Derivations and Algorithms for Addition and Negation

So far we have developed some intuition for Elliptic Curve point operations, but it will be helpful for automating elliptic curve addition to have explicit formulas for the addition and negation of points. The derivation and pseudocode for each of these operations is summarized below.

(Negation) Given a point P, we aim to find the point $-P$ such that $P + (-P) = \mathcal{O}$.

If $P = \mathcal{O}$ then $-P$ is also $\mathcal{O}$. Otherwise, P has x and y coordinates P_x and P_y that satisfy the given Elliptic Curve equation. The negative of P is the second point on the curve with the same x-coordinate as P. We can find the two points on the Elliptic Curve that have x-coordinate P_x by plugging in P_x for x in our Elliptic Curve Equation:

$$y^2 + P_x y - (P_x^3 + AP_x^2 + B) = 0.$$

We know that the sum of the roots of this equation is $-P_x$ and we know that one of the roots is P_y. Thus, $P_y + (-P)_y = -P_x$, which tells us that:

$$(-P)_y = -P_x - P_y.$$

And if we are working in a field K with characteristic 2, meaning that for any $k \in K$ $2k = 0$, then the above equation is equivalent to:

$$(-P)_y = P_x + P_y.$$

We will use this definition because we will be working with fields of characteristic 2, where there is no difference between adding or subtracting an element of the field. The pseudocode summary of this calculation is below. Note that in this pseudocode, we use 0 both as the identity element of the field we are working with and as the identity element of the group of points on an Elliptic Curve. The reader should be able to tell from the context which additive identity it is that 0 is playing the role of in each line of pseudocode.

```
def negative(Point P):
    if (P = 0):
        return 0
    else:
        return (P_x, P_x + P_y)
```

FIGURE 2.6: Elliptic Curve Point Negation Pseudocode for Points with Coordinates in a Field of Characteristic 2.

(Addition) Given P and Q, which are any two points on an Elliptic Curve together with the point at infinity, we aim to find a third point R such that $P + Q = R$.

By our definition of the addition, if either one of our points is the identity element $\mathcal{O}$, then the addition is just the other point. In addition, if the points have the same x-coordinates and different y-coordinates then we know the points are negatives of each other. If they have both the the same x and y-coordinates and the tangent line at the point is vertical, then the addition also results in $\mathcal{O}$. These are the base cases which we must get out of the way, the real work comes in when we are either doubling a point with non-vertical tangent line or we are adding points with different x-coordinates. We derive the addition formulae for these cases below.

We return to the equation that we saw at the end of section 2.2. This equation has three roots, each of which corresponds to an intersection between a line and the curve. The line, of course, is the line with slope λ that connects the two points we are adding (or the non-vertical tangent line of a point that we are doubling), and we are looking to find the x-coordinate of the third point of intersection. We have:

$$0 = x^3 + (A - \lambda^2 - \lambda)x^2 + (-\lambda v - v)x + (B - v^2).$$

We will use a trick similar to the one that we used in calculating the negative of a point. If we factor this polynomial completely, which we know we can do because it has 3 real roots, then

we get an equation that contains the three x-coordinates that we are interested in:

$$0 = (x - P_x)(x - Q_x)(x - R_x).$$

If we expand this and then equate the x^2 term of both of these polynomials, we will see that it must be the case that:

$$-P_x - Q_x - R_x = A - \lambda^2 - \lambda.$$

Therefore, we can find the x-coordinate of the third point of intersection R_x by calculating:

$$R_x = \lambda^2 + \lambda - A - P_x - Q_x.$$

The y-coordinate of $-R$ is clearly given by $(-R)_y = \lambda R_x + v$, as we know it lies on the $y = \lambda x + v$ line. All that is left is to negate the point we have found, $-R$, using the previously defined formula for point negation and we have found our point R such that $P + Q = R$.

Just as before, if we are working with a field of characteristic 2, then these formulas may actually be adjusted slightly. We have:

$$\begin{aligned} (-R)_x &= \lambda^2 + \lambda + A + P_x + Q_x \\ (-R)_y &= \lambda R_x + v \\ R &= \text{negative}(-R). \end{aligned}$$

The negative function used here is taken to mean the Elliptic Curve point negation algorithm, which was derived at the beginning of this section. Also as before, 0 is used both in comparisons with points and in comparisons with point coordinates. The reader should be able to tell from the context if a 0 is referring to the additive identity of the points on the elliptic curve (pairs of coordinates) or the additive identity of the field in which the coordinates lie.

Note that there are many computational improvements that could be made for the addition algorithm described in this section.

2.4 Elliptic Curves Over Finite Fields

The algorithms described in section 2.3 were designed for a field of characteristic 2 in anticipation of needing such a field for computation gains, but our intuition for addition has been gleaned

```
def sum(Point P, Point Q):
    if (P = 0):
        return Q
    else if (Q = 0):
        return P

    Declare lambda
    if (Q_x = P_x):
        if (Q_y = P_y):
            if (P_x = 0):
                return 0
            else:
                lambda = (P_x + P_y)/P_x
        else:
            return 0
    else:
        lambda = (P_y + Q_y)/(P_x + Q_x)

    Rx = lambda^2 + lambda + A
    if (Q_x not = P_x):
        Rx = Rx + P_x + Q_x

    Ry = lambda*Rx + lambda*P_x + P_y
    R = (Rx, Ry)
    return negative(R)
```

FIGURE 2.7: Elliptic Curve Point Addition Pseudocode for Points with Coordinates in a Field of Characteristic 2.

from pictures of the geometric construction over the reals $\mathbb{R}$. One of the remarkable aspects of Elliptic Curves is that the points of the curve still form a commutative group even when we are working with a finite field [2, p. 286]. That is, all of the operations that we have described over $\mathbb{R}$ actually still make sense in the language of algebraic geometry, and this language is valid over *any* field. The only loss in working over a finite field is that we may no longer nicely visualize the points on a smooth Elliptic Curve in $\mathbb{R}^2$. The gain, however, is that with finite fields we do not need to have the same amount of precision necessary for ECC over $\mathbb{R}$. In addition we may even choose our finite fields so as to make the addition on those fields much more efficient in computer automated implementations.

Clearly, to make use of the massive computing power available to us today we seek to find a field which we may represent with 0s and 1s. The fields that may be expressed this way most

efficiently are the fields $\mathbb{F}_{2^m}$ for some $m \in \mathbb{Z}$. Elements of a field $\mathbb{F}_{2^m}$ may be represented as a polynomial of degree $m-1$ with m coefficients that are each either 0 or 1. We will use a as the variable when describing such polynomials. For example, $a^2+1 \in \mathbb{F}_{2^3}$. Equivalently, we may use the placement of the bits to denote the the power of the term, so we may write $a^2+1 = 101 \in \mathbb{F}_{2^3}$. Notice that using this notation it only takes m bits to store an element of $\mathbb{F}_{2^m}$.

Addition in this field is polynomial addition modulo 2, and multiplication is done by regular polynomial multiplication followed by reduction modulo an irreducible polynomial of degree m. The operations in used in calculating Elliptic Curve point addition are done using these two fundamental operations. It does not matter which irreducible polynomial over $\mathbb{F}_2$ of degree m we choose, different choices will result in the same finite field. However, the representation of certain elements will look very different for different choices of irreducible polynomials. Standards have been created so that anyone following these standards will be using the same representation of the field. These standards are known as the Conway polynomials [7]. We will always use the standard Conway polynomials for creating the field representation. In addition, we will use the notation $E(\mathbb{F})$ to denote the group of points with coordinates in $\mathbb{F}$ over the Elliptic Curve E.

Example 2.1. *Suppose we are working with* $\mathbb{F}_{2^3} \cong \mathbb{F}_2[a]/(a^3+a+1)$. *We now show an example of addition with two points of* $E(\mathbb{F}_{2^3})$, $P = (a+1, a+1)$ *and* $Q = (a, a^2+a)$, *where* E *is given by* $y^2 + xy = x^3 + (a)x^2 + (a^2+1)$.

Note that the two points above are points on the curve because:

$$\begin{aligned}(a+1)^2 + (a+1)(a+1) &= (a+1)^3 + (a)(a+1)^2 + (a^2+1)\\ 0 &= a^2 + 1 + (a^2+1)\\ 0 &= 0\end{aligned}$$

and,

$$\begin{aligned}(a^2+a)^2 + (a)(a^2+a) &= (a)^3 + (a)(a)^2 + (a^2+1)\\ a + (a^2+a+1) &= (a^2+1)\\ a^2+1 &= a^2+1.\end{aligned}$$

These points have different x-coordinates so we set

$$\lambda = \frac{P_y + Q_y}{P_x + Q_x} = \frac{(a+1)+(a^2+a)}{(a+1)+(a)} = a^2+1.$$

Then, to calculate the third point of intersection, we first calculate the x-coordinate:

$$\begin{aligned}(-R)_x &= \lambda^2 + \lambda + A + P_x + Q_x \\ &= (a^2+1)^2 + (a^2+1) + (a) + (a+1) + (a) \\ &= 1.\end{aligned}$$

And then we the calculate the y-coordinate of $(-R)$ as:

$$\begin{aligned}(-R)_y &= \lambda \cdot (-R)_x + \lambda \cdot P_x + P_y \\ &= (a^2+1) \cdot (1) + (a^2+1) \cdot (a+1) + (a+1) \\ &= a.\end{aligned}$$

And finally, we negate $-R = (1, a)$ to conclude that $P + Q = (1, a+1)$. This could also be written as $(011, 011) + (010, 110) = (001, 011)$. The last thing we should do is to check is that this point in fact is a point on the curve:

$$\begin{aligned}(a+1)^2 + (1)(a+1) &= (1)^3 + (a)(1)^2 + (a^2+1), \\ (a^2+1) + (a+1) &= 1 + a + (a^2+1), \\ a^2 + a &= a^2 + a.\end{aligned}$$

2.5 Summary

The points that satisfy the equation of an Elliptic curve form a group, and the addition law for that group may be described geometrically. Using this geometric definition of addition, we may define laws and algorithms for negation and addition of points that extend to finite fields, where there is not a simple geometric definition of addition. In addition, if we choose to work with coordinates that are in a field of characteristic 2, then we may efficiently store and manipulate points with computers.

Chapter 3

What is Elliptic Curve Public Key Cryptography?

3.1 The Elliptic Curve Discrete Logarithm Problem

Public Key cryptography is built upon problems which are assumed, although not proven, to be very computationally difficult. In RSA, this problem was factoring a large product of two primes. Many other Pubic Key encryption schemes are built on the witnessed difficulty of a problem known as the discrete logarithm problem. The Elliptic Curve Discrete Logarithm Problem (ECDLP) is a special case of the discrete logarithm problem that is acknowledged to be even more computationally difficult than the standard discrete logarithm problem [2, p. 296].

Definition 3.1. The discrete logarithm over a group G: Given an element $b \in G$ and a power of b given by $g \in G$, find an integer k such that $b^k = g$.

This called the discrete logarithm problem because of the analogous continuous problem where one might seek to find a number $r \in \mathbb{R}$ such that $g^r = h$, where $g, h \in \mathbb{R}$ and $g, h > 0$. A solution to this problem would be denoted as the logarithm $\log_g h$. The continuous problem is much easier than the discrete one, however, because the Taylor series for the natural logarithm gives a simple converging formula for calculating the r. This formula does not extend to finite fields, and in general the number e such that $g^e \equiv h \pmod{p}$ looks random. Therefore, the most natural way to try to solve this problem would be to try every possible value for e, which is typically computationally infeasible.

Example 3.1. *Let us use the group* $\mathbb{Z}/11\mathbb{Z}$ *under multiplication. It is true that* $g = 2$ *is a generator for this field. We seek to find an* e *such that* $2^e \equiv 5 \pmod{11}$.

Without any advanced tools in our tool box (such as the Index Calculus or Sieve methods), we are stuck trying each value for e until we find one that works.

$$\begin{aligned} 2^1 &\equiv 2 \pmod{11} \times \\ 2^2 &\equiv 4 \pmod{11} \times \\ 2^3 &\equiv 8 \pmod{11} \times \\ 2^4 &\equiv 5 \pmod{11} \checkmark \end{aligned}$$

Definition 3.2. The Elliptic Curve Discrete Logarithm Problem over $E(\mathbb{F}_{2^m})$: Given P, a point of $E(\mathbb{F}_{2^m})$, and another point Q which is a multiple of P, find an integer n such that $nP = Q$.

Note that here the group operation is addition, but this is just a notational difference from the original DLP. Also notice that we only say an integer n (instead of *the* integer n) because there will in fact be infinitely many such integers. This was not necessarily the case in the standard discrete logarithm problem introduced above because we did not necessarily assume that the group was finite. In the ECDLP, however, the group is a finite group and so we can show that there will in fact be infinitely many integers that relate P and Q.

To see this, note that the set $\{nP : n \in \mathbb{Z}\}$ must be finite because $E(\mathbb{F}_{2^m})$ is finite (having at most $(2^m)^2$ elements). Thus, there will be distinct a, b such that $aP = bP$. Without loss of generality, let b be the larger value. Then $(b-a)P = \mathcal{O}$, and so for any one n such that $nP = Q$ (which we are assuming exist), we also have that $(n+c(b-a))P = nP+c(b-a)P = Q+c\mathcal{O} = Q$, for any $c \in \mathbb{Z}$. This shows that there are in fact infinitely many such integers that will solve any particular Elliptic Curve DLP, and in fact any one of them will allow an eavesdropper to decrypt a system which was built using a specific multiple of P.

Example 3.2. *Suppose we are working over the field* $\mathbb{F}_{2^3} \cong \mathbb{F}_2[a]/(a^3 + a + 1)$ *and let* E *be given by the equation* $y^2 + xy = x^3 + (a)x^2 + (a^2 + 1)$. *Given* $P = (1, a + 1)$ *and* $Q = (a, a^2)$, *We seek to find an integer* n *such point such that* $nP = Q$.

As an example of what a brute force attack on the ECDLP would entail, we try each integer until we find an n that relates the two points.

$$\begin{aligned} & & 1P &= (1, a+1) & \times \\ P+P &= 2P &= (a^2, a^2+a) & \times \\ P+2P &= 3P &= (a^2+a+1, a) & \times \\ P+3P &= 4P &= (a, a^2) & \checkmark \end{aligned}$$

3.2 Fast Addition Algorithms

Before we can introduce how to make a crypto-system based on the difficulty of the ECDLP we need to have an efficient method of calculating nP, where n is some large integer. Obviously, if encryption involved calculating nP by n distinct additions of P, then it would take just as long as it would to solve the ECDLP by brute force. So, in order to have any chance of making a public key system based on the DLP, we must have methods to calculate nP very efficiently.

One such method is known as the Double-and-Add algorithm, which is similar to the method of successive squaring in modular arithmetic. It works by using the base 2 representation of n, and is more clearly described with an example than with formulae, although I will provide a pseudo-code algorithm after the following example.

We will continue working with E be given by the equation $y^2 + xy = x^3 + (a)x^2 + (a^2+1)$, but we change the field we are working with to be $\mathbb{F}_{2^5} \cong \mathbb{F}_2[a]/(a^5+a^2+1)$. It is not hard to check that (a^2+1, a^4+a+1) is a point on this curve. We will calculate $26P$ using the base two representation for 26, which is 11010. Now, notice that:

$$\begin{aligned} 26P &= (2^4 + 2^3 + 2^1)P \\ &= 2^4P + 2^3P + 2^1P. \end{aligned}$$

Start with what will eventually be the result $R = 26P$ by setting $R = \mathcal{O}$. Since the base 2 representation of 26 has a 0 in its first digit (the least-significant digit), we do not add P to R. Now double P to calculate $2P$. Since the base 2 representation of 26 does have a 1 in its second digit, we do add $2P$ to R. Then we double $2P$ to find $4P$, which we do not add to R because there is a zero in the next digit. Double again to find $8P$ and add to R and then double again to find $16P$ and add to R. After this procedure is completed, we find that $26P = (a^2+a, a^3+a^2+a)$. The results at each step of the process can be seen in table 3.1.

R	Power of 2	Digit of 26	Doubles of P
$\mathcal{O}$	0	0	$P = (a^2 + 1, a^4 + a + 1)$
$\mathcal{O}$	1	1	$2P = (a^2 + a, a^3)$
$(a^2 + a, a^3)$	2	0	$4P = (a^4 + a^3 + a^2 + 1, a^2)$
$(a^2 + a, a^3)$	3	1	$8P = (a^4 + a^3 + a + 1, a^3 + a^2 + a + 1)$
$(a^4, 1)$	4	1	$16P = (a^4 + a, a^2 + 1)$
$(a^2 + a, a^3 + a^2 + a + 1)$	NA	NA	NA

TABLE 3.1: An example of the Double-and-Add algorithm, step by step.

Notice that instead of the 26 additions that it would have taken us to calculate $26P$ naively, it only took 4 different doublings and 5 additions, for a total of 9 EC point operations, which is much better than 26. On average, we will need $\log_2 n$ doublings and $\frac{1}{2}\log_2 n$ additions. The difference between the the two methods becomes much more impressive for larger values of n. For example, to calculate $1869P$ for an arbitrary point P would take 17 point operations instead of approximately $1,900$. This method is very efficient but there are still many improvements that can be made. For more, see [2].

```
def multiple(n, P):
    R = 0
    while (n > 0):
        if (n mod 2 = 1):
            R = R + P
        P = 2*P
        n = n/2 (integer division without the remainder)
    return R
```

FIGURE 3.1: Double-and-Add pseudo code.

Remember, this algorithm relies on the fact that Elliptic Curve point addition is associative. That is, calculating 26 distinct additions of P as $P + (P + (P + \cdots + P)\cdots)$ is the same as $16P + (8P + 2P)$.

3.3 Embedding Messages into $E(\mathbb{F}_{2^k})$

The first step in any cryptographic system is usually to turn one's message into a number that can then be mathematically manipulated and encrypted. In ECC, however, we use a set of

points of an Elliptic curve for manipulation and encryption. In this section, we will show a method to relate simple number messages and the points of an Elliptic curve.

What we would like to have is a one-to-one function that turns a number message into a point on an Elliptic Curve, and can then be easily inverted after the encrypted point has been decrypted. If we had such a function, the sender would take his or her message, put it through the function to get the corresponding point for that message, encrypt it and then send the encrypted point. Then the receiver would decrypt the message using his or her private key to get a point on the Elliptic Curve, which they then can invert back into the original number message.

In practice, it is hard to create a fail-safe method for creating a correspondence between number messages and points of an Elliptic curve. The alternative is to create a probabilistic system which is extremely likely to work. We will present one way of doing this here, but this is just one of many possible methods. This method is the characteristic 2 version of one of the solutions proposed in Koblitz' paper on Elliptic Curve Cryptosystems [8].

Before a more concrete description of the system, I will do an example. The idea is to sacrifice a few bits in message length in order to be able to replace those bits with other values so that the resulting message is an x-coordinate of a point that is on the curve. Remember that elements of $\mathbb{F}_{2^k}$ can be represented as a sequence of k bits (where each bit is a 0 or a 1). Suppose we are working over $\mathbb{F}_{2^{277}}$, where the coordinates of a point on an Elliptic Curve each take 277 bits to store. But instead of sending 277 bit messages, we are going to only send 256 bit messages (a nice even 32 bytes). Now, when finding a point on an Elliptic Curve that has an x-coordinate that matches a message m, we only need m and the x-coordinate to match in 256 of the 277 bits. The remaining 21 bits will be used freely to make an x-coordinate for which there is a corresponding y-coordinate of a point on our Elliptic Curve. If Alice uses this method to embed, encrypt, and then send her message, then after Bob decrypts the message using his private key he will just throw away the last 21 bits of the message, knowing that Alice made them whatever she needed to be to find a point on the curve.

This sounds like a reasonable solution, but there are a few practical matters to work out. For example, how can we be sure that this method will work? That is, if we send messages which each take i bits to store and there are j bits left over in the k bits of our field (so $i + j = k$), how can we be sure there is a sequence of j bits such that m concatenated with the j bits forms the x-coordinate of a point on the curve? If this method will not always work, then how often

with it fail? How many bits do we need to sacrifice in order to make this message embedding scheme be reliable? We will work out those details now.

To determine the likelihood that this process will succeed, we need to know when it is that a random x-coordinate in $\mathbb{F}_{2^k}$ has at least 1 corresponding y coordinate on the curve. To find a point on the Elliptic curve, we start with a proposed x-coordinate $w \in \mathbb{F}_{2^k}$, which matches m for its first i bits. We first start with the remaining j bits of w bits all being zero. We seek to find a $y \in \mathbb{F}_{2^k}$ such that:

$$y^2 + wy = w^3 + aw^2 + b.$$

Or, equivalently, that:

$$y^2 + (w)y + (w^3 + aw^2 + b) = 0.$$

Multiplying each size by w^{-2} (assuming we are not trying to send a string of zeroes), we have that:

$$\left(\frac{y}{w}\right)^2 + \left(\frac{y}{w}\right) + \left(w + a + \frac{b}{w^2}\right) = 0.$$

We now make a change of variables which we can undo later in order to recover the y-coordinate. The change of variables is $y \to wY$ gives:

$$Y^2 + Y + \left(w + a + \frac{b}{w^2}\right) = 0.$$

Given that $w \neq 0$, this quadratic must have either 0 or 2 roots. In [9], Pommerening shows a method for determining if this quadratic has 0 or 2 roots in $\mathbb{F}_{2^k}$. It uses the trace of w in $\mathbb{F}_2$. The trace is defined by:

$$\mathrm{Tr}(w) = \sum_{i=1}^{k} w^{2^{i-1}} = w + w^2 + w^4 + \cdots + w^{2^{k-1}}.$$

This is an element of $\mathbb{F}_2$. Pommerening shows that a quadratic of the above form has roots if and only if $\mathrm{Tr}\left(w + a + \frac{b}{w^2}\right) = 0$. I will denote this value $\left(w + a + \frac{b}{w^2}\right)$ as $\Gamma_E(w)$. This gives us a relatively easy algorithm to check whether or not we can use the sequence of j bits that was used to form w to form a point on E: we just check if $\mathrm{Tr}(\Gamma_E(w)) = 0$. If it is not, then we increment the value in the j bits and test the algorithm again.

A randomly chosen $w \in \mathbb{F}_{2^k}$ will have $\mathrm{Tr}(\Gamma_E(w)) = 0$ about half the time. This is not proven, but rather is more of an experimentally observed truth. When every possible w that matches m for the first i bits has $\mathrm{Tr}(\Gamma_E(w)) = 1$ we fail to turn our number message into a point on an

Elliptic curve. The goal is to make this improbable enough that it will not happen for as long as the curve is in use.

We can estimate the chance that this probabilistic method will fail by using the estimate that any given w is likely to have $\mathrm{Tr}(\Gamma_E(w)) = 0$ about $\frac{1}{2}$ of the time. In order for a message to fail the encryption process, it must fail 2^j times, meaning that we have a probability of $\left(\frac{1}{2}\right)^{2^j}$ of failing to encrypt our message. In the example that we used before where $k = 277$ and $i = 256$, the probability of failing to decrypt a message would be astronomically low:

$$\left(\frac{1}{2}\right)^{2^{21}} \approx 2.201 \cdot 10^{-631,306}$$

In practice, it would be wasteful to use $j = 21$ because we could be using some of those bits to exchange message data. It is usually sufficient to use j such that $5 \leq j \leq 8$, but individual specifications on the need for a fail-safe embedding must be used to determine j.

Example 3.3. *Suppose we are working with the field $\mathbb{F}_{2^7} \cong \mathbb{F}_2[a]/(a^7 + a + 1)$ on the curve E given by $y^2 + xy = x^3 + (a)x^2 + (a^2 + 1)$. Also suppose that our messages have length $i = 5$ bits and so we have $j = 2$ free bits. We will find the point on the curve that corresponds to the number message $m = 12$.*

Write m in base two using $i = 5$ bits: 01100. We try our first sequence of j bits: 00. Then our proposed x-coordinate is 0110000, or $a^5 + a^4$. We first calculate $\Gamma_E(w)$ below:

$$\begin{aligned} \Gamma_E(w) &= w + a + \tfrac{b}{w^2} \\ &= (a^5 + a^4 + 1) + (a) + (a^2 + 1)(a^5 + a^4 + 1)^{-2} \\ &= a^3 + a^2 + 1. \end{aligned}$$

It is not hard to check that this has trace 1. Since $\Gamma_E(w)$ has trace 1, there is no point on the curve E that has this x-coordinate, so we need not waste our time looking. Next we try 01 as our sequence of j bits. Our new proposed x-coordinate is 0110001, or $w = a^5 + a^4 + 1$. We calculate

$$\begin{aligned} \Gamma_E(w) &= w + a + \tfrac{b}{w^2} \\ &= (a^5 + a^4) + (a) + (a^2 + 1)(a^5 + a^4)^{-2} \\ &= a^6 + a^5, \end{aligned}$$

which has trace 0. Now we may look for y-coordinates that can be matched with our w using algorithms to solve the quadratic in a finite field. For algorithms to do this, see [10]. We find two points that have $w = a^5 + a^4 + 1$ as their x-coordinate:

$$(a^5 + a^4 + 1, a^6 + 1) \text{ and } (a^5 + a^4 + 1, a^6 + a^5 + a^4).$$

Either of these points may serve as the message point for use in our ECC system. After dropping the last $j = 2$ bits, they both give the message 01100, or 12.

3.4 Elliptic Curve Cryptography

So far, the key point in this chapter has been that calculating nP is relatively easy (for a computer) compared to finding a value of n relating P and $Q = nP$. The goal of this section is to describe how we can turn this hard problem (the ECDLP) into a Public Key Cryptography system. The system presented here is known as the Elliptic ElGamal System, and was independently extended from the standard ElGamal system by Neal Koblitz and by Victor Miller [11].

In the Elliptic ElGamal system, the group which wishes to communicate securely should publicly choose an Elliptic Curve, a Finite Field representation and a point which has large order (the next section goes over how to choose some of these parameters to avoid known attacks on ECC). We will denote the finite field we are using as $\mathbb{F}_{2^k}$ for some integer k and the starting point on the curve as P. Each person that would like to receive messages picks a large integer k_{priv} and computes $Q_{\text{pub}} = k_{\text{pub}} = k_{\text{priv}}P$, which is their public key. Remember that making this information public is only considered safe because the ECDLP is considered difficult.

The encryption process in Elliptic ElGamal consists of creating a pair of cipher texts from which only the person with the secret integer k_{priv} will be able to regain the original message, which is a point on the chosen Elliptic Curve. The first cipher text, c_1, is created by taking a random large value r and the message point M that one would like to send, and then computing $c_1 = M + rQ_{\text{pub}}$. In order to be able to recover the message M, the receiver must be able to subtract rQ_{pub} from c_1. The problem, however, is that r cannot be sent by itself or and eavesdropper would be able to decode the message easily. But rQ_{pub} is the value the receiver really needs in order to decode c_1, and $rQ_{\text{pub}} = r(k_{\text{priv}}P) = k_{\text{priv}}(rP)$. So if we sent the value

$c_2 = rP$ (which we are assuming does not reveal r), then the receiver could use their secret value k_{priv} to calculate $k_{\text{priv}} \cdot c_2$, subtract this from c_1, and they will recover the value M.

In summary, encryption is done by randomly generating r and then calculating:

$$\begin{aligned} c_1 &= M + rQ_{\text{pub}} \\ c_2 &= rP. \end{aligned}$$

After the receiver has the values c_1 and c_2, he or she may undo the decryption by calculating:

$$\begin{aligned} c_1 - k_{\text{priv}} \cdot c_2 &= M + r \cdot Q_{\text{pub}} - k_{\text{priv}} \cdot r \cdot P \\ &= M + (r \cdot k_{\text{priv}})P - (r \cdot k_{\text{priv}})P \\ &= M. \end{aligned}$$

Example 3.4. *We will do a sample encryption and decryption using the following parameters. Suppose we are working over the field* $\mathbb{F}_{2^7} \cong \mathbb{F}_2[a]/(a^7+a+1)$ *and let* E *be given by the equation* $y^2 + xy = x^3 + (a)x^2 + (a^2+1)$*. Then* $E(\mathbb{F}_{2^7})$ *consists of 116 distinct points, including the point at infinity. The point*

$$P = (a^6 + a^5 + a^3 + 1, a^6 + a^5 + a^3 + a + 1)$$

on this curve will be used in our encryption system. Suppose Alice has the message $m = 12$*, which was shown in the previous section to correspond to the point* $M = (a^5 + a^4 + 1, a^6 + 1)$*, that she would like to send to Bob. Bob has the private key* $n_{priv} = 97$*, so his public key is* $Q_{pub} = 97P = (a^6 + a^4 + a^3 + a + 1, a^4 + a^3 + a^2 + a)$*.*

Alice calculates her two cipher texts, which are custom made for Bob using his public key Q_{pub}. She picks a random $r = 49$ and calculates c_1 as:

$$\begin{aligned} c_1 &= M + rQ_{\text{pub}} \\ &= (a^5 + a^4 + 1, a^6 + 1) + 49 \cdot (a^6 + a^4 + a^3 + a + 1, a^4 + a^3 + a^2 + a) \\ &= (a^5 + a^4 + 1, a^6 + 1) + (a^6 + a^4 + 1, a^5 + a^4 + a^2) \\ &= (a, a^6 + a^5 + a^4 + a^2 + a). \end{aligned}$$

Using $r = 49$, she also calculates c_2 as:

$$\begin{aligned} c_2 &= rP \\ &= 49 \cdot (a^6 + a^5 + a^3 + 1, a^6 + a^5 + a^3 + a + 1) \\ &= (a^4 + a^3 + a^2 + a + 1, a^4 + a^2 + a). \end{aligned}$$

She sends (c_1, c_2) to Bob over the insecure channel. Bob can then do the following calculations to regain Alice's message, M:

$$\begin{aligned} M &= c_1 - k_{\text{priv}} \cdot c_2 \\ &= (a, a^6 + a^5 + a^4 + a^2 + a) - 97 \cdot (a^4 + a^3 + a^2 + a + 1, a^4 + a^2 + a) \\ &= (a, a^6 + a^5 + a^4 + a^2 + a) - (a^6 + a^4 + 1, a^5 + a^4 + a^2) \\ &= (a^5 + a^4 + 1, a^6 + 1). \end{aligned}$$

Knowing that M actually only holds 5 bits of useful message information, Bob drops the last two bits of the x-coordinate to regain the message $m = 01100$, which is Alice's original message.

3.5 Choosing your Curve

The previous section described how to use an Elliptic Curve and other parameters in an Elliptic ElGamal System in order to secretly share points. But choosing the parameters for an ECC system is a very delicate process and requires care to avoid certain known attacks on systems with parameters that have certain properties. The parameters also need to be of a certain size to have a sufficient security level, and different security levels are required depending on the sensitivity of the information and on the processing power of modern CPUs (i.e. how much time and money would an attacker need to invest in order to break the system). This section briefly summarizes a few known attacks, explains how the existence of each attack should regulate the choice of parameters, and describes the relationship between the size of the parameters chosen and their security levels.

We need a measure of security of a certain set of parameters. Usually these measurements of security are:

(1) Security level - The security level is the number of bits in the number of operations that would be necessary to break the system using the best (publicly) known algorithm for

breaking the system. This is a measure of how hard it would be to break the system using the best known methods; the more bits necessary the better the security level. Notice that if more efficient algorithms are developed, the security for a particular system may decrease even though the system has not changed. In this sense, the security level is a reflection of how hard an adversary would have to work to break the system if they were using the best publicly known algorithm.

(2) Key size - For a given system with specified parameters, the Key size is the number of bits necessary to store a key for the system, usually the private key in Public Key Cryptography. This also represents the highest possible security level, because any system can be broken by testing every possible private key (see 3.5.1). Since there are usually faster algorithms than the brute force approach, however, the Security level will likely be less than the Key size. The higher the key size, the harder encryption and decryption will be. Essentially, the Key size is a measure of how hard the users of the system have to work and how much data they have to store in order to implement the cryptographic system.

Cryptographic systems always have to balance the key size with the security level. If the parameters are chosen to be too small then the Security level will be too small and too easy for an attacker to break. If the parameters are chosen to be too large, however, then the Key size will be too large to be able to use the system efficiently or on a large scale.

An Elliptic Curve E over the finite field $\mathbb{F}_{2^m}$ has key size m because that is how many bits it will roughly take to store a private key in ECC. The best known algorithm to solve the ECDLP in a well-chosen Elliptic Curve finishes in roughly $\sqrt{2^m} = 2^{\frac{m}{2}}$ steps, so the Security Level for such a system is $\frac{m}{2}$ [12]. Thus, if we require a Security level of at least 120 (a good baseline Security level), then we need to use $m \geq 240$.

3.5.1 Brute Force Attacks

In this attack, the attacker simply tries every possible value of a private key until the message has been decoded. One natural problem in this attack is how to tell when a message has been decoded. The attacker can solve this problem by using one's own encrypted message, so that they know what the correct answer will be when they attempt decryption. After the attacker finds a value that decrypts their own message, they can use that key to decrypt any other messages they intercept. Note that this method of encrypting ones own seed data only applies

in Public Key systems because here one may encrypt their own message without knowledge of the private key.

Defense: Make the size of the finite field large enough that a brute force attack is not realistic. When working over a field $\mathbb{F}_{2^m}$, any value of $m \geq 80$ will usually make this attack unfeasible. This will usually not be the limiting factor, however, as there are much more efficient attacks than the naive Brute force approach.

3.5.2 Baby-step Giant-Step Algorithm Attacks

The BSGS algorithm works by creating two lists of points (one list created using a "Baby" step and the other using a "Giant" step) and then searching for a common point between the two lists. As soon as a common point is found, it is easy to determine a constant that relates P and $Q = nP$. This algorithm takes roughly $O(\sqrt{N})$ steps, where N is the order of the publicly chosen parameter P.

Defense: As described before, the preventive measure that must be taken against attacks that run in $O(\sqrt{N})$ time is to double the number of bits in order to achieve the same Security level. This is a manageable inconvenience, however, and is much better than the equivalent overhead necessary in other Public Key systems. This method is not actually the best known method of attack since it actually requires a significant amount of storage to keep and maintain the two lists of values. It has the same running time as the best known generic method of attack, but the overhead in storing the list of values makes this attack unmanageable for even a small key size.

3.5.3 Pollard's ρ Algorithm Attacks

Pollard's ρ Algorithm is a generic attack that can be formulated to work on any system that relies on some version of the Discrete Logarithm Problem. It is very similar to the Baby-step Giant-step algorithm, except that instead of keeping two lists it only needs to keep track of two values at any given time. Like the BSGS algorithm, Pollard's ρ algorithm also works in $O(\sqrt{N})$ steps, where N is the order of the publicly chosen parameter P, but it has the advantage of not needing the same amount of memory that the BSGS algorithm requires.

Defense: The same precaution described in the Baby-step Giant-Step Algorithm defense will prevent a successful attack using Pollard's ρ Algorithm.

3.5.4 Pohlig-Hellman Algorithm Attacks

Let G be the cyclic group generated by P, our publicly chosen parameter. The Pohlig-Hellman method works by finding the order of G, factoring it, and then using its factorization gain information about the private key mod each of its factors. The Chinese remainder Theorem is then used to piece the information back together to determine the private key. This method runs in $O(\sqrt{l})$, where l is the largest factor of the order of G.

Defense: The natural defense against this attack is to pick an element whose order has a large prime divisor, so as to make it infeasible to run any algorithm that will take $O(\sqrt{l})$ time. For ways of finding such curves, see [13].

3.5.5 MOV Algorithm Attacks

The MOV (Menezes-Okamoto-Vanstone) algorithm uses the Weil pairing to transport the ECDLP over the group $E(\mathbb{F}_q)$ to a DLP over a larger group $\mathbb{F}_{q^k}^{\times}$ for some integer k. The smallest integer k for which this can be done is called the embedding degree of E with respect to the order of P (the point used in the ECDLP). Solutions to the DLP in the larger group $\mathbb{F}_{q^k}^{\times}$ correspond to solutions to the ECDP in $E(\mathbb{F}_q)$. Since sub-exponential algorithms are known when working over $\mathbb{F}_{q^k}^{\times}$, this can decrease the time to break the system if k is low.

Defense: This attack is extremely effective if the embedding degree is small but becomes unwieldy for any (E, P) pairs for which there is not an extremely low embedding degree. Supersingular curves are a type of curve with typically low embedding degrees, and so this class of curves must be avoided in cryptography. An Elliptic Curve E is called supersingular if the number of points in the group $E(\mathbb{F}_{p^k})$ is equivalent to 1 $(\text{mod } p)$. Typically, parameter choices with embedding degree less than or equal to 6 should be avoided. Most curves have much higher embedding degrees [2, p. 328], so the precaution that must be taken here is to avoid using curves with extremely low embedding degrees.

3.5.6 Anomalous Curve Attacks

In [14], N.P. Smart introduced a linear time algorithm for solving the ECDLP on anomalous curves, curves where the number of points in the group $E(\mathbb{F}_p)$ is just p. Notice that this attack only applies when working over a prime field $\mathbb{F}_p$.

Defense: The natural defense against this attack is to not use anomalous curves. This brings us now to 2 general classes of curves that should not be used for cryptographic purposes, supersingular curves and anomalous curves. These classes are both fairly rare and are unlikely to be used unintentionally.

3.5.7 Weil Descent Attacks

When the finite field we are working with $\mathbb{F}_{2^m}$ uses a power m that is not prime, the Weil Descent Attacks may be successful. Attacks that use the Weil Descent method transport the ECDLP over $E(\mathbb{F}_{2^m})$ to a Hyperelliptic Curve DLP over the smaller field $\mathbb{F}_{2^k}$, where k divides m. [2, p. 328]

Defense: The obvious precaution which is easy to implement is to use finite fields $\mathbb{F}_{2^m}$ where m is prime.

3.6 Summary

Just as the ElGamal system relies on the difficulty of the Discrete Logarithm Problem to ensure that no one will be able to crack the system, the Elliptic ElGamal system relies on the Elliptic Curve Discrete Logarithm Problem. The ECDLP seems to be much harder than the standard DLP, though, because the group structure is far more complex. The fastest known method to solve any ECDLP takes $O(\sqrt{N})$ steps, where N is is the order of the publicly chosen parameter P.

Although it is hard to determine n from P and $Q = nP$, there are very efficient methods to determine nP using P and n. This is convenient since it makes encryption and decryption very easy.

Elliptic Curve Cryptography (for the moment) seems to be very secure. There are a few classes of curves that must be avoided and a few parameters that need to be chosen carefully to thwart

known attacks, but these are not unmanageable problems. This problem is by no means unique to ECC, however, as every cryptographic system has certain classes of parameters that should be avoided for greatest security.

Since there are so many different attacks for which we must prepare, one often can use a set of highly analyzed and trusted parameter choices. NIST, the National Institute of Standards and Technology, keeps an updated list of secure Elliptic Curve parameter choices for whatever Security level is desired [15].

The reader should hopefully now be convinced that ECC is a viable Public Key Cryptography system. One reason it is gaining traction in the world of information security is that for a specified level of security, ECC requires a much smaller key size than other methods of encryption. For example, in order to have 80 bits of security using ECC we need only to have a key size of 160 bits. In RSA, however, in order to obtain 80 bits of security we would need to have use a key size of 1024 bits [12]. This is because there are more efficient algorithms to break RSA than there are to break ECC. Essentially, the best known mathematical algorithms for factoring are much better than those for solving the ECDLP. In addition, there are many choices of Elliptic Curves and many Finite fields that we may use in our system. This freedom makes it easy to change parameters more regularly and enlarges the space of allowable Cryptography systems.

As a final note, we should say that many features of Elliptic Curve Cryptography are not in the public domain. Security companies, such as Certicom, have many patents on Elliptic Curve Cryptography algorithm features. Anyone wishing to incorporate ECC into their business for added security should ensure that they follow any guidelines laid out by Certicom, NIST, and the NSA in order to be in compliance with all patents and laws.

Chapter 4

Elliptic Curve Cryptography in JavaSE 7

4.1 National Security

Java has a cryptographic library which is quite extensive and useful. However, Oracle, the owner of Java, is currently prohibited by law from including any ECC libraries which will encrypt and decrypt data using ECC because Java distributes its JDK (Java Development Kit) internationally [16]. Thus, although it would be great to be able to do an example here of ECC in action, we will have to work around this inconvenience. Instead, I will show how to use Java's ECC libraries to implement EC Diffie-Hellman key exchange. This can still be useful for incorporating security into one's own projects, it just must be used in conjunction with a symmetric cipher like AES or DES.

4.2 ECDH Key Exchange

Diffie-Hellman key exchange leverages the difficulty of the discrete logarithm problem in order to allow two users to share a secret shared key even if attackers are listening to every message they send. The only catch is that the users don't get to pick what their shared value will be ahead of time! In addition, although it works over any group, it is more secure when working with some groups than others. For example, the ECDH key exchange is acknowledged to be more secure than DH key exchange over $\mathbb{F}_p^\times$.

To use Diffie-Hellman key exchange over a generic group G, Alice and Bob must first agree upon an element $g \in G$ such that g has fairly large (and prime, if possible) order in G. Next, Alice picks a secret integer a and Bob picks a secret integer b. Alice calculates $A = g^a$ and Bob calculates $B = g^b$. There are very fast algorithms to do this, such as the fast powering (or addition, whichever the operation used in the group is) algorithms described in section 3.2. They each send their newly calculated values to the other. Alice then calculates $B^a = g^{ab}$ and Bob calculates $A^b = g^{ab}$. They are the only ones who can do this because they are the only ones that have their private constants a and b, and the DLP prohibits anyone else from being able to solve for a or b efficiently just having A and B. Thus, after the process is completed, Alice and Bob both have the shared value g^{ab}.

Example 4.1. *We will do a short example of ECDH. Suppose we are working over the field* $\mathbb{F}_{2^7} \cong \mathbb{F}_2[a]/(a^7 + a + 1)$ *and let* E *be given by the equation* $y^2 + xy = x^3 + (a)x^2 + (a^2 + 1)$. *We will use the following point to be our generator:*

$$P = (a^6 + a^5 + a^3 + 1, a^6 + a^5 + a^3 + a + 1).$$

Alice has the private key $a = 22$ *and Bob has the private key* $b = 39$.

In order for Alice and Bob to share a secret key, Alice calculates:

$$A = aP = 22(a^6 + a^5 + a^3 + 1, a^6 + a^5 + a^3 + a + 1) = (a^6 + a^5 + a^3 + a, a).$$

And, similarly, Bob calculates:

$$B = bP = 39(a^6 + a^5 + a^3 + 1, a^6 + a^5 + a^3 + a + 1) = (a^6 + a^5 + a + 1, a^2 + 1).$$

They each send their new value to the other over an insecure channel. After Alice receives Bob's message, she calculates:

$$aB = 22(a^6 + a^5 + a + 1, a^2 + 1) = (a^6, a^6 + a^2).$$

Similarly, Bob calculates:

$$bA = 39(a^6 + a^5 + a^3 + a, a) = (a^6, a^6 + a^2).$$

They now have a shared value which they can manipulate and use for a private shared key in a symmetric cipher.

4.3 ECDH using Java

One of the companies that is in charge of researching Elliptic Curve parameters and publishing standards for ECC is the Standards for Efficient Cryptography Group (SECG). Their most recent published standards were in 2010 and can be found here [17]. They recommend many sets of Elliptic Curve domain parameters, which offer a wide range of security levels. For showing how to implement ECDH with Java, I will used the set of curve and associated parameters labeled as "sect233r1".

There are two files in this ECDH example. The main file is ECDH.java. This is just the controller for a short simulation where two users share a value. In order to make this code as readable as possible, I have also made an accompanying ECHDUser.java file which just contains some useful methods for anyone who might want to use ECDH. The code is fairly well commented, explaining each step it is doing as it goes.

```
 1 import java.security.KeyPairGenerator;
 2 import java.security.spec.ECGenParameterSpec;
 3
 4 /**
 5  * File: ECHD.java
 6  *
 7  * This file simulates the Elliptic Curve
 8  * Diffie-Hellman key exchange between two
 9  * ECDH users, Alice and Bob.
10  *
11  * @author Stephen Morse
12  */
13 public class ECDH {
14
15     public static void main(String[] args) throws Exception {
16
```

```
17          // The first step in using ECDH in Java is to create a
18          // KeyPairGenerator. This lets us make private keys and
19          // public keys in ECDH. To do this, we may use the static
20          // getInstance() method.
21          KeyPairGenerator kpg = KeyPairGenerator.getInstance("EC","SunEC");
22
23          // Initialize the EC using "sect233r1" curve specifications.
24          // This just defines a set of parameters where the key size
25          // is 233 bits and the parameters are chosen (somewhat)
26          // randomly (hence the r1).
27          ECGenParameterSpec ecsp = new ECGenParameterSpec("sect233r1");
28          kpg.initialize(ecsp);
29
30          // We create two ECDH users, Alice and Bob. Their public keys
31          // will print uponing running this code.
32          ECDHUser alice = new ECDHUser("Alice", kpg);
33          System.out.println(alice.getName() + "'s public key is: " +
34                  alice.getPubKey().toString());
35          ECDHUser bob = new ECDHUser("Bob", kpg);
36          System.out.println(bob.getName() + "'s public key is: " +
37                  bob.getPubKey().toString());
38
39          // The users exchange a private key using ECDH protocol.
40          // The values that they each receive will be printed.
41          alice.sendValueTo(bob);
42          bob.sendValueTo(alice);
43      }
44
45  }
```

And here is the second file, ECDHUser.java, which just has a few useful methods and fields for implementing ECDH.

```
1 import java.math.BigInteger;
```

```
 2 import java.security.KeyPair;
 3 import java.security.KeyPairGenerator;
 4 import java.security.PrivateKey;
 5 import java.security.PublicKey;
 6
 7 import javax.crypto.KeyAgreement;
 8
 9 /**
10  * File: ECHDUser.java
11  *
12  * This class represents a user of the Elliptic
13  * Curve Diffie-Hellman system. It has a sendValueTo
14  * method that can be used for sharing a value with
15  * another ECDH user.
16  */
17 class ECDHUser {
18     /** This ECDH implementer's private key. */
19     private PrivateKey privKey;
20
21     /** This ECDH implementer's public key. */
22     private PublicKey pubKey;
23
24     /** The name of this ECDH User. */
25     private String name;
26
27     /**
28      * A Constructor.
29      *
30      * @param name - The name of the user.
31      *
32      * @param kpg
33      */
34     public ECDHUser(String name, KeyPairGenerator kpg) {
35         this.name = name;
```

```
36
37          KeyPair kpU = kpg.genKeyPair();
38          this.privKey = kpU.getPrivate();
39          this.pubKey = kpU.getPublic();
40      }
41
42      public void sendValueTo(ECDHUser receiver) throws Exception {
43          // Get A key agreement controller for ECDH
44          KeyAgreement ecdh = KeyAgreement.getInstance("ECDH");
45
46          // Initialize it so it knows the private key (a for Alice, b for Bob)
47          ecdh.init(this.privKey);
48
49          // DoPhase actually computes the shared value A^b = B^a
50          // The second parameter lets the ECDH instance know that we are
51          // all done using this instance
52          ecdh.doPhase(receiver.getPubKey(), true);
53
54          // To get the shared value, we have to user our private ecdh
55          // object to call ecdh.generateSecret(). Here we print so
56          // that we can see we have the same value.
57          System.out.println("Secret computed by " + receiver.getName() +
58                  ": 0x" + (new BigInteger(1, ecdh.generateSecret()
59                          ).toString(16)).toUpperCase());
60      }
61
62      /**
63       * @return the pubKey
64       */
65      public PublicKey getPubKey() {
66          return pubKey;
67      }
68
69      /**
```

```
70      * @return the name
71      */
72     public String getName() {
73         return name;
74     }
75
76 }
```

The below is a sample output for the program. Note that this output will be different every time it is run, since randomness is used whenever we create a private key.

```
Alice's public key is: Sun EC public key, 233 bits
  public x coord:
  782391194665206226941068075891590725854185216224833879038969678501882O
  public y coord:
  881597063692424336914800405388966204813844847874770219179555952715633 6
  parameters: sect233r1 [NIST B-233] (1.3.132.0.27)
Bob's public key is: Sun EC public key, 233 bits
  public x coord:
  755587373896629746077466012295923815437315350906605547434011470641348 6
  public y coord:
  759544664141142912597886413790475963498375726564201894658947200972177 8
  parameters: sect233r1 [NIST B-233] (1.3.132.0.27)
Secret computed by Bob:
0x556B85D6AE7D782971F6E6BF450A776B9B96977C44214A00120AAF7710
Secret computed by Alice:
0x556B85D6AE7D782971F6E6BF450A776B9B96977C44214A00120AAF7710
```

At the end of the output, we can see that Alice and Bob have the same shared value. And that's it! You can now implement ECDH in any small coding projects you might have. For example, a JavaFX web application for your small business.

4.4 Summary

Cryptography is a body of mathematics with a very rich history. Securing information has been a real concern for centuries, and this need for security has recently become even more essential with the introduction of the internet.

The two main branches of Cryptography are Private Key Cryptography and Public Key Cryptography. Private Key Cryptography deals with symmetric ciphers, in which both parties that would like to share confidential information have the same shared private key. Symmetric ciphers are extremely secure and can be made even more secure through the use of a one-time pad. Conversely, Public Key Cryptography deals with asymmetric ciphers, in which the two parties each have their own private key and their own public key. When using an asymmetric cipher, messages have to be encrypted for specific users of the system using their public key. Asymmetric ciphers are usually the more mathematically interesting since they rely on mathematical problems which are acknowledged to be computationally hard for their security.

RSA is one type of asymmetric cryptosystem. It is older than ECC and there are more efficient algorithms to break RSA than there are ECC. Thus, ECC is slowly becoming accepted as the next generation default for information security, partially because mobile devices require smaller key sizes and ECC offers more security per bit of key length. There are a few known attacks on ECC, but most of the attacks require a specific type of Elliptic Curve and do not work for a general Elliptic Curve. Care are must be taken to ensure that the parameters chosen for an Elliptic Curve to be used in a cryptosystem do not form an insecure curve. Typically, however, we may just use a set of standard parameters that have been inspected thoroughly by companies whose mission is to develop security standards.

In this document, we have only had a glimpse into the massive massive body of mathematical and computational research that has been done and is being done on Elliptic Curves. Although I believe these topics represent a good sample of the work that is being done, there are many other important topics in ECC to be studied. For the interested reader, the next step up from this introduction would likely be Hoffstein, Pipher, and Silverman's book entitled An Introduction to Mathematical Cryptography [2].

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