Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
wget https://github.com/Bhanu−das/EE3900
    −2022/blob/main/filter/codes/
    Cancel_noise.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: python code

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/codes/Q3/X_Y.py
```

sketch

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/fig/Q3.png
```

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3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The following code yields Fig. fig:xnyn.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

```
wget https://github.com/Bhanu−das/EE3900
    −2022/blob/main/filter/figs/xnyn.pdf
```

3.3 Repeat the above exercise using a C code.
Solution: c code

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/codes/Q3/x_y.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.7)$$

But

$$x(n) = \{1, 2, 3, 4, 2, 1\} \quad (4.8)$$

so,

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.9)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.10)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.11)$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.12)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.13)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.15)$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \quad (4.16)$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.17)$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.18)$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.19)$$

Solution: Solution: let

$$f(n) = a^n u(n) \quad (4.20)$$

$$f(n) = \left\{ \begin{array}{ll} a^n, & \text{if } n > 0 \\ 0, & \text{otherwise} \end{array} \right\} \quad (4.21)$$

Now the Z- Transform of f(n) is

$$F(z) = \mathcal{Z}\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)z^{-n} \tag{4.22}$$

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.23}$$

This forms a infinite Geometric Progression.

$$F(z) = \frac{1}{1 - az^{-1}} \text{ for } z < a. \tag{4.24}$$

4.6 Let

$$H(e^{\jmath\omega}) = H\left(z = e^{j\omega}\right). \tag{4.25}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: The following code plots

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/dtft.
    py
```

```
wget https://github.com/Bhanu−das/EE3900
    −2022/blob/main/filter/figs/dtft.pdf
```

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$. **Solution:** We have,

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k} \tag{4.26}$$

However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.27}$$

and so,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{j\omega n}d\omega \tag{4.28}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} h(k)e^{\jmath\omega(n-k)}d\omega \tag{4.29}$$

$$= \frac{1}{2\pi}2\pi h(n) = h(n) \tag{4.30}$$

which is known as the Inverse Discrete Fourier Transform. Thus,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.31}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}e^{\jmath\omega n}d\omega \tag{4.32}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12). **Solution:** from (4.12)

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$\begin{array}{l} 1 + \frac{1}{2}z^{-1})\overline{1 + z^{-2}}(2z^{-1} \\ \quad 2z^{-1} + z^{-2} \\ 1 + \frac{1}{2}z^{-1})\overline{1 - 2z^{-1}}(- 4 \\ \quad -4 - 2z^{-1} \\ \quad \overline{5 + 0z^{-1}} \end{array}$$

Hence by long division will be

$$H(z) = 2z^{-1} - 4 + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.3}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.4}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.5}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.6}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hn.py
```

```
weget https://github.com/Bhanu−das/EE3900
    −2022/blob/main/filter/figs/dtft.pdf
```

From the graph we can say that maximim value of h(n) is 1.25. The minimum value of graph is around -0.6.

$$-0.6 > h(n) \geq .25$$

The function is bounded between 1.25 and -0.6.

5.4 Convergent? Justify using the ratio test.
Solution:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.7)$$

Ratio test

$$L = \lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| \quad (5.8)$$

$$h(n+1) = \left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1) \quad (5.9)$$

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.10)$$

$$as \quad n \to \infty \quad u(n) = u(n-1) = u(n+1) = u(n- \quad (5.11)$$

$$h(n+1) = \{-\frac{1}{2}\}h(n) \quad (5.12)$$

$$\left|\frac{h(n+1)}{h(n)}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2} \quad (5.13)$$

$$L = \lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} \quad (5.14)$$

L is less than 1 h(n) is convergent

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.15)$$

Is the system defined by (3.2) stable for the impulse response in (5.4)? **Solution:** from **??**

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.16)$$

then

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \quad (5.17)$$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3} \quad (5.18)$$

since

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.19)$$

$h(n)$ is stable.

5.6 Verify the above result using a python code.
Solution: the following code plotes

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/codes/Q5/h_nstable
```

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/fig/output.png
```

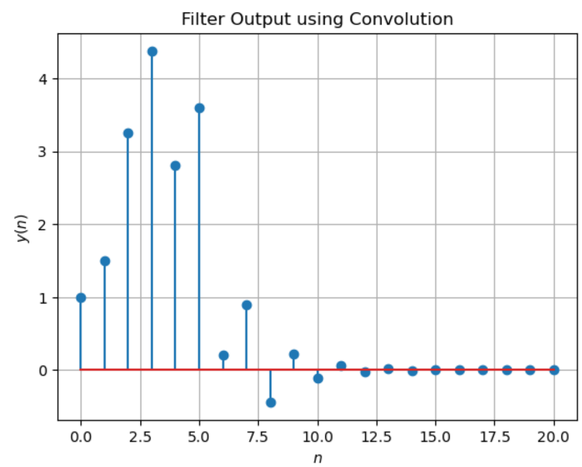

Fig. 5.6

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.20)$$

This is the definition of $h(n)$.
Solution: The following code plots .

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hndef
    .py
```

```
weget https://github.com/Bhanu−das/EE3900
    −2022/blob/main/filter/figs/hndef.pdf
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.21)$$

Comment. The operation in (5.21) is known as *convolution*.
Solution: The following code plots Note that this is the same as $y(n)$ in Fig.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.py
```

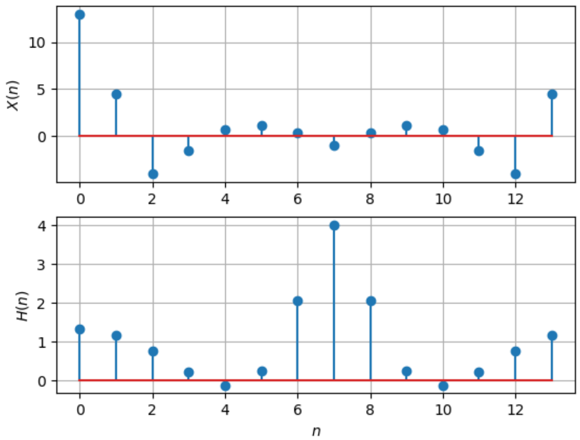

Fig. 5.8

5.9 Express the above convolution using a Teoplitz matrix.
Solution: The following code does convolution

```
weget https://github.com/Bhanu-das/EE3900/
    blob/main/Assign1/codes/Q5/teoplitz.py
```

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.22}$$

Solution: from 5.21 ww know that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.23}$$

now consider

$$t = n - k \tag{5.24}$$

will transform into

$$y(n) = \sum_{n-t=-\infty}^{\infty} x(n-t)h(t) \tag{5.25}$$

since n is finite and $-\infty < \infty$, 5.25 is equivalent to

$$y(n) = \sum_{t=-\infty}^{\infty} x(n-t)h(t) \tag{5.26}$$

hence proved.

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots

```
weget https://github.com/Bhanu-das/EE3900/
    blob/main/Assign1/codes/XkHk_dft.py
```

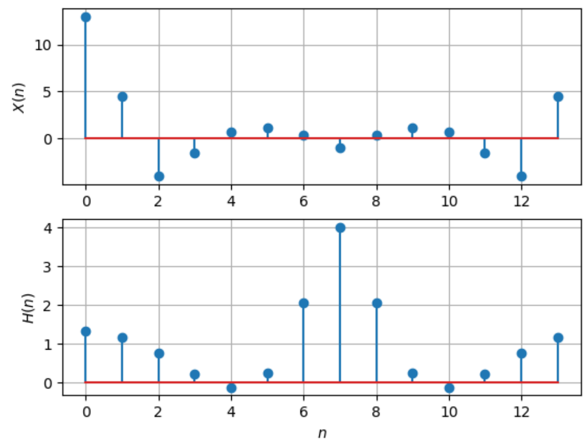

Fig. 6.1

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots

```
weget https://github.com/Bhanu-das/EE3900/
    blob/main/Assign1/codes/yk.py
```

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plot. Note that this is the same as $y(n)$ in Fig.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

```
weget https://github.com/Bhanu-das/EE3900
    -2022/blob/main/filter/figs/yndft.pdf
```

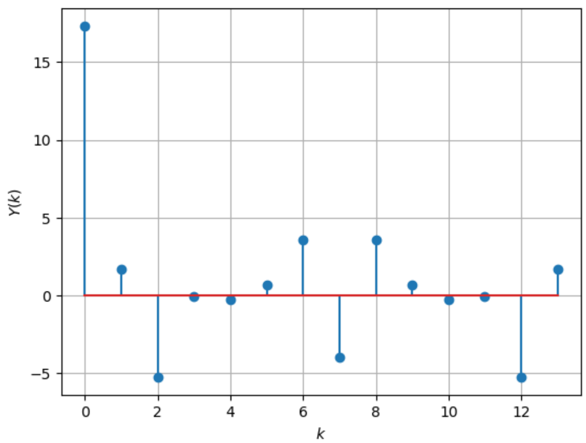

Fig. 6.2

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The following code plots $X(n)$, $H(n)$ and $y(n)$ by fft.

```
weget https://github.com/Bhanu−das/EE3900/
    blob/main/Assign1/codes/fft.py
```

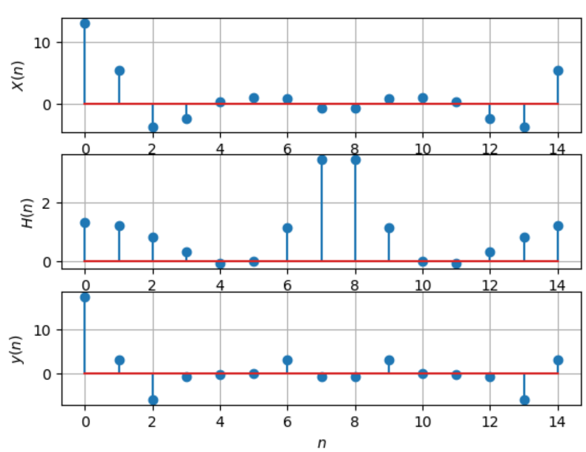

Fig. 6.4

6.5 Wherever possible, express all the above equations as matrix equations.

Solution:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 \\ e^{\frac{-2\pi j(1)k}{N}} \\ e^{\frac{-2\pi j(2)k}{N}} \\ . \\ . \\ . \\ e^{\frac{-2\pi j(N-1)k}{N}} \end{pmatrix}$$

$$X(k) = x^T J \tag{6.4}$$

$$h = \begin{pmatrix} h[0] \\ h[1] \\ . \\ . \\ . \\ h[N-1] \end{pmatrix}$$

$$H(k) = h^T J \tag{6.5}$$

$$y = \begin{pmatrix} h[0]x[0] \\ h[1]x[1] \\ . \\ . \\ . \\ h[N-1]x[N-1] \end{pmatrix}$$

$$Y(k) = y^T J \tag{6.6}$$

7 FFT

7.1 *Definitions*

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = [W_N^{mn}] \quad (7.3)$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let

$$\vec{I}_4 = \vec{e}_4^1 \vec{e}_4^2 \qquad \vec{e}_4^3 \vec{e}_4^4 \quad (7.4) ref$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \vec{e}_4^1 \vec{e}_4^3 \qquad \vec{e}_4^2 \vec{e}_4^4 \quad (7.5) ref$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag W_N^0 W_N^1 \qquad W_N^2 W_N^3 \quad ref(7.6)$$

7.2 Problems

1. Show that

$$W_N^2 = W_{N/2} \quad (7.7)$$

Solution: We know that.

$$W_N = e^{-j2\pi/N} \quad (7.8)$$

Then

$$W_{N/2} = e^{-2*j2\pi/N} \quad (7.9)$$
$$W_{N/2} = {W_N}^2 \quad (7.10)$$

Hence Proved.

2. Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.11)$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (**??**),

$$\vec{D}_2 \vec{F}_2 = W_4^0 \qquad 0 \quad (7.12)$$
$$0 W_4^1 W_2^0 \qquad W_2^0 \quad (7.13)$$
$$W_2^0 W_2^1 \quad ref(7.14)$$
$$= W_4^0 \qquad 0 \quad (7.15)$$
$$0 W_4^1 W_4^0 \qquad W_4^0 \quad (7.16)$$
$$W_4^0 W_4^2 \quad ref(7.17)$$
$$= W_4^0 \qquad W_4^0 \quad (7.18)$$
$$W_4^1 W_4^3 \quad ref(7.19)$$
$$\implies -\vec{D}_2 \vec{F}_2 = W_4^2 \qquad W_4^6 \quad (7.20)$$
$$W_4^3 W_4^9 \quad ref(7.21)$$

and

$$\vec{F}_2 = W_2^0 \qquad W_2^0 \quad (7.22)$$
$$W_2^0 W_2^1 \quad ref(7.23)$$
$$= W_4^0 \qquad W_4^0 \quad (7.24)$$
$$W_4^0 W_4^2 \quad ref(7.25)$$

Hence,

$$\vec{W}_4 = W_4^0 \qquad W_4^0 W_4^0 \qquad W_4^0 \quad (7.26)$$
$$W_4^0 W_4^2 \qquad W_4^1 W_4^3 \quad (7.27)$$
$$W_4^0 W_4^4 \qquad W_4^2 W_4^6 \quad (7.28)$$
$$W_4^0 W_4^6 \qquad W_4^3 W_4^9 \quad ref(7.29)$$
$$= \vec{I}_2 \vec{F}_2 \qquad \vec{D}_2 F_2 \quad (7.30)$$
$$\vec{I}_2 \vec{F}_2 - \vec{D}_2 F_2 \quad ref(7.31)$$
$$= \vec{I}_2 \qquad \vec{D}_2 \quad (7.32)$$
$$\vec{I}_2 \vec{D}_2 \vec{F}_2 \qquad 0 \quad (7.33)$$
$$0 \vec{F}_2 \quad ref(7.34)$$

Multiplying (7.34) by $\vec{P}_4$ on both sides, and noting that $\vec{W}_4 \vec{P}_4 = \vec{F}_4$ gives us.

3. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \quad (7.35)$$

Solution: Observe that for even N and letting $\vec{f}_N^i$ denote the i^{th} column of $\vec{F}_N$, from (7.19) and (7.21),

$$\vec{D}_{N/2} \vec{F}_{N/2} \quad (7.36)$$
$$-\vec{D}_{N/2} \vec{F}_{N/2} = \vec{f}_N^2 \vec{f}_N^4 \qquad \ldots \vec{f}_N^N \quad ref(7.37)$$

and

$$\vec{I}_{N/2} \vec{F}_{N/2} \quad (7.38)$$
$$\vec{I}_{N/2} \vec{F}_{N/2} = \vec{f}_N^1 \vec{f}_N^3 \qquad \ldots \vec{f}_N^{N-1} \quad ref(7.39)$$

Thus,

$$\vec{I}_2 \vec{F}_2 \qquad \vec{D}_2 \vec{F}_2 \quad (7.40)$$
$$\vec{I}_2 \vec{F}_2 - \vec{D}_2 \vec{F}_2 = \vec{I}_{N/2} \qquad \vec{D}_{N/2} \quad (7.41)$$
$$\vec{I}_{N/2} - \vec{D}_{N/2} \vec{F}_{N/2} \qquad 0 \quad (7.42)$$
$$0 \vec{F}_{N/2} \quad ref$$
$$= \vec{f}_N^1 \qquad \ldots \vec{f}_N^{N-1} \quad \vec{f}_N^2 \ldots \quad \vec{f}_N^N \quad ref \quad (7.43)$$

and so,

$$\vec{I}_{N/2} \qquad \vec{D}_{N/2} \tag{7.44}$$

$$\vec{I}_{N/2} - \vec{D}_{N/2}\vec{F}_{N/2} \qquad 0 \tag{7.45}$$

$$0\vec{F}_{N/2}\vec{P}_N \qquad ref$$

$$= \vec{f}_N^1 \qquad \vec{f}_N^2 \ldots \quad \vec{f}_N^N = \vec{F}_N \quad ref(7.46)$$

4. Find

$$\vec{P}_6\vec{x} \tag{7.47}$$

Solution: We have,

$$\vec{P}_4\vec{x} = \vec{e}_4^1\vec{e}_4^3 \qquad \vec{e}_4^2\vec{e}_4^4 x(0) \tag{7.48}$$

$$x(1) \tag{7.49}$$

$$x(2) \tag{7.50}$$

$$x(3) = x(0) \tag{7.51}$$

$$x(2) \tag{7.52}$$

$$x(1) \tag{7.53}$$

$$x(3) \qquad ref(7.54)$$

5. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.55}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.56}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.57}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.58}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.59}$$

and so, we can represent each of these terms as a matrix product

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.60}$$

where $\vec{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

6. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.62}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.63}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.65}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.66}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.67}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.68}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.69}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.70}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.71}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.72}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.73}$$

Solution: We write out the values of performing an 8-point FFT on $\vec{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.74}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.75}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.76}$$

where $\vec{X}_1$ is the 4-point FFT of the even-numbered terms and $\vec{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.77}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.78}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.79}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.80}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.81}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.82}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.83}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.84}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.85}$$

But observe that from (7.54),

$$\vec{P}_8 \vec{x} = \vec{x}_1 \tag{7.86}$$

$$\vec{x}_2 \tag{7.87}$$

$$\vec{P}_4 \vec{x}_1 = \vec{x}_3 \tag{7.88}$$

$$\vec{x}_4 \tag{7.89}$$

$$\vec{P}_4 \vec{x}_2 = \vec{x}_5 \tag{7.90}$$

$$\vec{x}_6 \tag{7.91}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

$$\vec{x} = 1 \tag{7.92}$$

$$2 \tag{7.93}$$

$$3 \tag{7.94}$$

$$4 \tag{7.95}$$

$$2 \tag{7.96}$$

$$1 \tag{7.97}$$

compte the DFT using (7.55)

7. Repeat the above exercise using (**??**)
8. Write a C program to compute the 8-point FFT.
   Solution: The following code calculates the 8-point fft of $x(n)$ in 3.1

```
wget https://github.com/Bhanu-das/EE3900/blob/main/Assign1/codes/8_point_fft.py
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a, input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m) y(n-m) = \sum_{k=0}^{N} b(k) x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace

signal.filtfilt with your own routine and verify.
Solution:

https://github.com/Bhanu−das/EE3900/blob/main/Assign1/codes/8.1.py

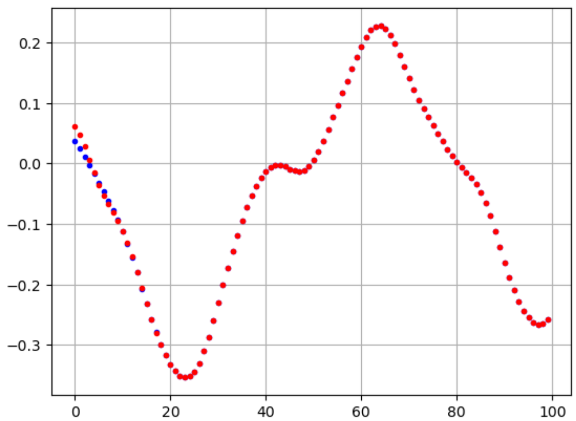

Fig. 8.1

8.2 Repeat all the exercises in the previous sections for the above a and b. **Solution:** For the given values, the difference equation is

$$\begin{aligned} &y(n) - (4.44)\, y(n-1) + (8.78)\, y(n-2) \\ &- (9.93)\, y(n-3) + (6.90)\, y(n-4) \\ &- (2.93)\, y(n-5) + (0.70)\, y(n-6) \\ &- (0.07)\, y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\ &+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\ &+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\ &+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\ &+ \left(5.02 \times 10^{-5}\right) x(n-7) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.19),

$$h(n) = \sum_i r(i) [p(i)]^n u(n) + \sum_j k(j) \delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) = &[(2.76)\,(0.55)^n \\ &+ (-1.05 - 1.84\jmath)\,(0.57 + 0.16\jmath)^n \\ &+ (-1.05 + 1.84\jmath)\,(0.57 - 0.16\jmath)^n \\ &+ (-0.53 + 0.08\jmath)\,(0.63 + 0.32\jmath)^n \\ &+ (-0.53 - 0.08\jmath)\,(0.63 - 0.32\jmath)^n \\ &+ (0.20 + 0.004\jmath)\,(0.75 + 0.47\jmath)^n \\ &+ (0.20 - 0.004\jmath)\,(0.75 - 0.47\jmath)^n]\, u(n) \\ &+ \left(-6.81 \times 10^{-4}\right) \delta(n) \end{aligned} \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

https://github.com/Bhanu−das/EE3900/blob/main/Assign1/codes/8_2_1.py

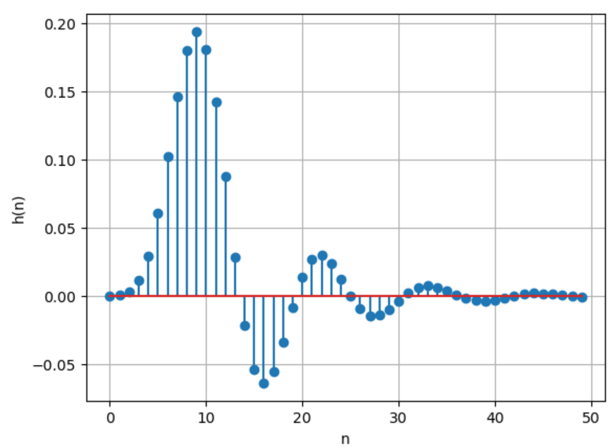

Fig. 8.2

The filter frequency response is plotted at

https://github.com/dhanushpittala11/EE3900−2022/blob/main/**filter**/codes/8_2_2.ipynb

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We note that observe that $|p(i)| < 1$ and so, as $h(n)$ is the sum of convergent series, we see that $h(n)$ converges. From Fig. (**??**), it is clear that $h(n)$ is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (**??**), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (**??**).

```
https://github.com/Bhanu-das/EE3900/blob/
    main/Assign1/codes/8_2_3.py
```

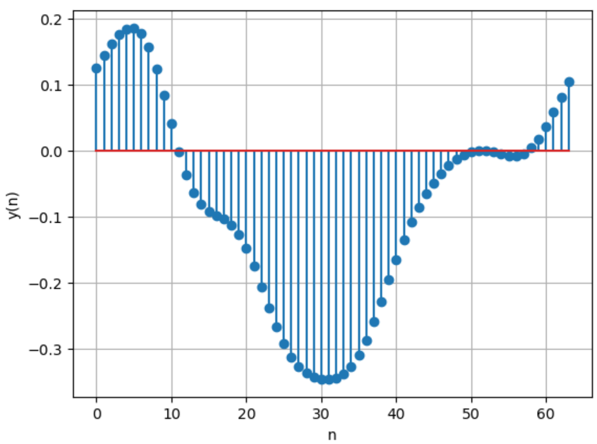

Fig. 8.2

8.3 What is the sampling frequency of the input signal?
Solution: run the following code to Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** a better filtering was found on changing the order of filter to 7.