Homework 3

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Question 1: Prove that Pygmalion is NP-Complete

We would follow the standard procedure that was discussed in class for showing that a problem is NP-complete.

Let’s define Pygmalion first, given an undirected graph G, where nodes represent towns and edges represent roads, is there a way to build k bunkers at k different towns so that every town either has its own bunker or connected (by a direct road) to a town that does have a bunker.

***Step 1***: Show that Pygmalion is in NP

To prove that NP Pygmalion is in NP, we need to show that given a certificate, we should be able to verify if it is a solution or not in polynomial time.

Let’s say we have a certificate in the form of a list of k bunkers (k bunker nodes) and we also have the knowledge of graph G (u,v) (each town is a node and each road between two towns is an edge). Now we need to verify if each town has a bunker of its own or is connected directly to a bunker town. Here is the algorithm for checking it:

* For each town, check if has a bunker.
* If yes, go to next town.
* If no, check if every neighbor of this town (i.e. other town of every road that this town is part of) has a bunker. If one of the neighboring towns has a bunker, we check the next town. If not, we return false i.e. given certificate is not a solution.

Let’s check out the time complexity of the above algorithm. We are doing checks for each town and for each town, we need to check all it’s neighbors. Each town at maximum can have n-1 neighboring towns, where n is the total number of towns. So our time complexity becomes O(n\*n-1) i.e. O(n2). Hence, a given certificate can be verified in polynomial time.

***Step 2***: Choose a NP-complete problem. We choose vertex cover as the given problem is same as vertex cover. As know vertex cover is known to be NP-complete. We will use this problem to show that Pygmalion is NP-complete as well.

Let’s define vertex cover, given a graph G, we have a set of vertices <=k such that this set set covers each edge i.e. for each edge(u,v), either u or v belongs to the set of k vertices.

**Step 3**: Now we need to prove that Vertex-Cover problem is poly-time reducible to Pygmalion. It can be done by following steps:

* Describe a procedure that converts the inputs of Vertex Cover (problem X) to inputs of Pygmalion (problem Y) in polynomial time.
* Show that if X(i) = YES then Y(i) = YES and vice-versa

X - Vertex Cover

Y - Pygmalion

So, first step is to convert inputs of X into inputs of Y. For vertex cover, we have a graph of nodes, if we consider each node as the town, then the set of k vertices becomes our k bunkers. We see that this conversion didn’t take any extra computation time which depends on the size of n and hence can be done in constant time. This takes care of the first step i.e. we have converted inputs of Vertex Cover to inputs of Pygmalion problem. **Please note that direct connection between towns means an edge for the graph problem.**

Now, we will show that ***if X(i) = YES then Y(i) = YES*** i.e. if an input of X (Vertex Cover) is a solution then we have a solution for Y (Pygmalion) as well.

We have a set of k vertices such that for each edge, either u or v lies in the set. Let’s proof that Y(i) = YES as well using proof of contradiction. Let’s say that given set of k vertices which equates to k towns which have bunkers is not the solution for Pygmalion problem. This means that there exists a town which does not have a bunker and is not connected to any town with a bunker. Using our input conversion (town ~ nodes), this equates to the case where we have a node and that node is not in the k set of vertices and it is neither part of any edge for which the other node is in k set of vertices. This means that given k set of vertices is not the solution for Vertex Cover problem and this contradicts the fact that we started with i.e. we have a solution for Vertex Cover. Hence, we have a valid solution for Pygmalion problem as well.

Now, we will show that ***if Y(i) = YES then X(i) = YES*** i.e. if an input of Y (Pygmalion) is a solution then we have a solution for X (Vertex Cover) as well.

We have a set of k towns with bumpers such that every town either has a bunker or is connected to a town with bunker. Let’s again do this with the help of contradiction. Now as we know that k towns equate to k vertices for vertex cover problem. We say that set of k vertices is not the solution for Vertex Cover problem. This means that we have a node such that it is not in the set of k vertices and this node is not a part of any edge for which the other node is part of the set. Using our input conversion (town ~ nodes), this also means that we have a town with no bunker and it is not connected to any town with a bunker. This contradicts the fact that we started with a solution for Pygmalion problem. Hence, we have a valid solution for Vertex Cover.

Hence, it is proved that Pygmalion problem is NP-Complete.