4.2. Graph Coloring

The m-Coloring problem concerns finding all ways to color an undirected graph using at most m different colors, so that no two adjacent vertices are the same color. We usually call the m-Coloring problem a unique problem for each value of m.

Example 1

Consider the graph in figure 1. There is no solution to the 2-Coloring problem for this graph because, if we can use at most two different colors, there is no way to color the vertices so that no adjacent vertices are the same color.

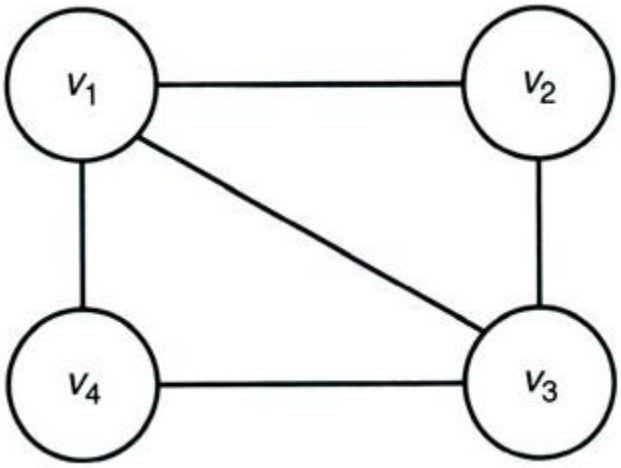


Figure 1: graph with 4 nodes

There are six solutions to the 3-Coloring problem as in the table below:

node	solutions					
	color	color	color	color	color	color
V1	1	2	2	3	3	1
V2	2	1	3	2	1	3
V3	3	3	1	1	2	2
V4	2	1	3	2	1	3

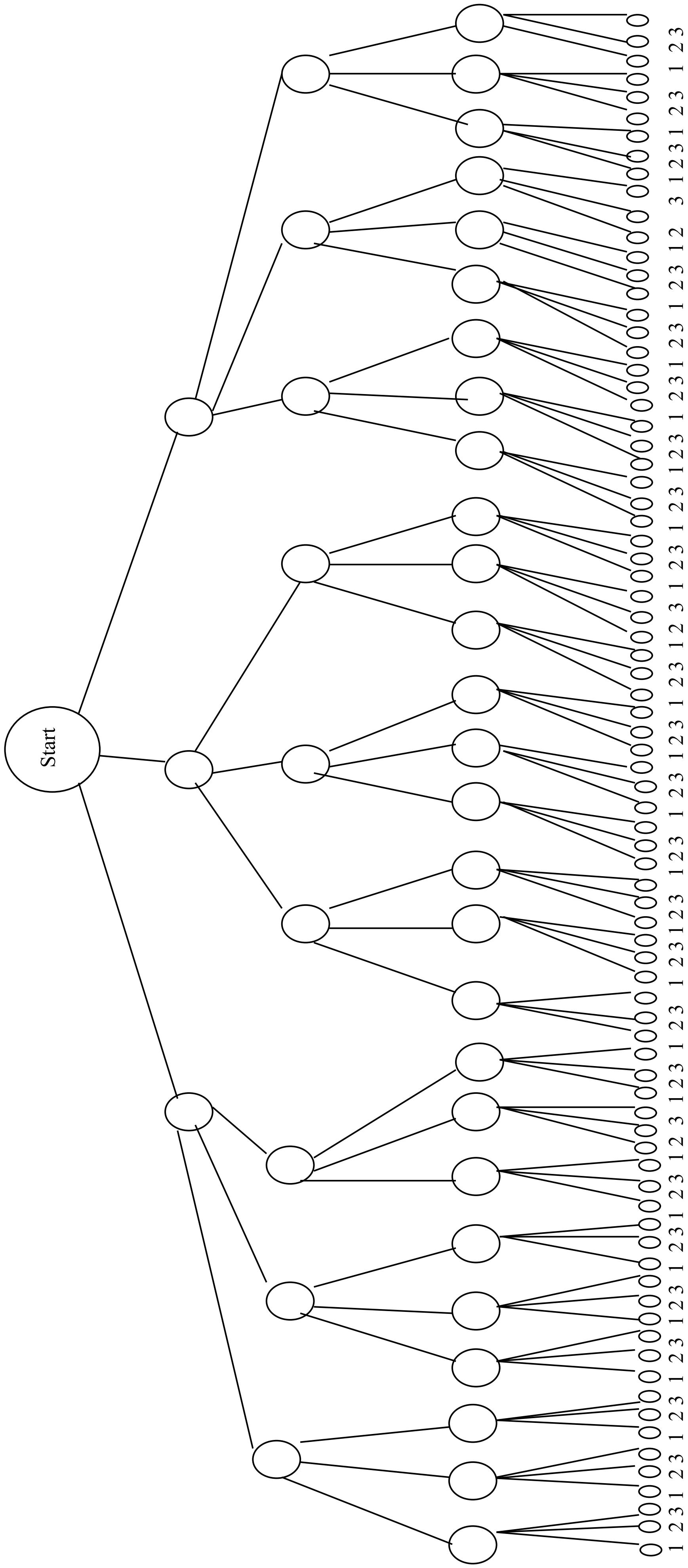


Figure 2: state space tree for 3-coloring problem in example 1.

A straightforward state space tree for the *m*-Coloring problem is one in which each possible color is tried for vertex v_1 at level 1, each possible color is tried for vertex v_2 at level 2, and so on until each possible color has been tried for vertex v_n at level *n*. Each path from the root to a leaf is a candidate solution. We check whether a candidate solution is a solution by determining whether any two adjacent vertices are the same color. To avoid confusion, remember in the following discussion that "node" refers to a node in the state space tree and "vertex" refers to a vertex in the graph being colored.

Figure 3 shows a portion of the pruned state space tree that results when this backtracking strategy is applied to a 3-coloring of the graph in Figure 1. The number in a node is the number of the color used on the vertex being colored at the node. The first solution is found at the shaded node. Nonpromising nodes are labeled with crosses. After v_1 is colored color 1, choosing color 1 for v_2 is nonpromising because v_1 is adjacent to v_2. Similarly, after v_1, v_2, and v_3 have been colored colors 1, 2, and 3, respectively, choosing color 1 for v_4 is nonpromising because v_1 is adjacent to v_4.

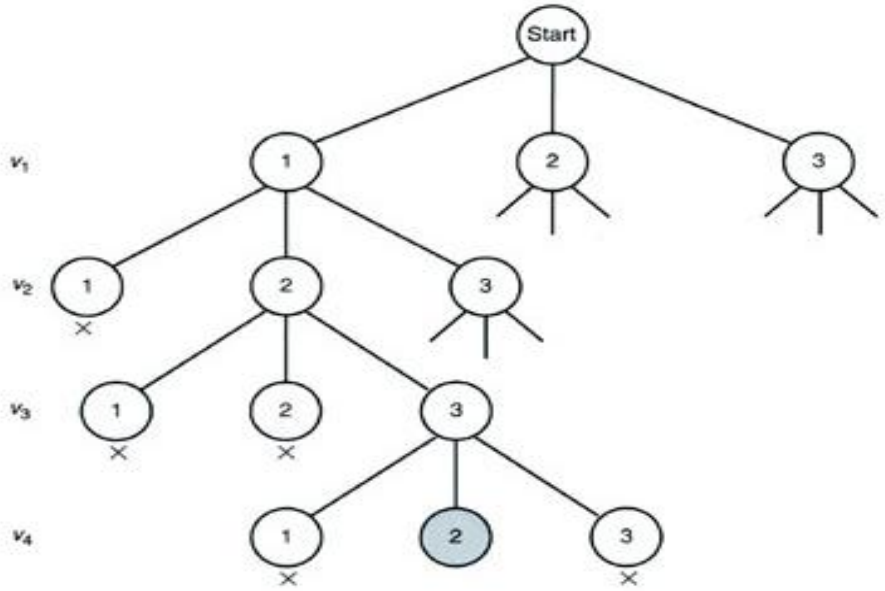


Figure 4: A portion of the pruned state space tree produced using backtracking to do a 3-coloring of the graph in Figure 1. The first solution is found at the shaded node. Each nonpromising node is marked with a cross.

An important application of graph coloring is the coloring of maps. A graph is called ***planar*** if it can be drawn in a plane in such a way that no two edges cross each other. The graph at the bottom of Figure 5 is planar. However, if we were to add the edges (v_1, v_5) and (v_2, v_4) it would no longer be planar. To every map there corresponds a planar graph. Each region in the map is represented by a vertex. If one region is adjacent to another region, we join their corresponding vertices by an edge. Figure 5 shows a map at the top and its planar graph representation at the bottom. **The *m*-Coloring problem for planar graphs is to determine how many ways the map can be colored, using at most *m* colors, so that no two adjacent regions are the same color**.

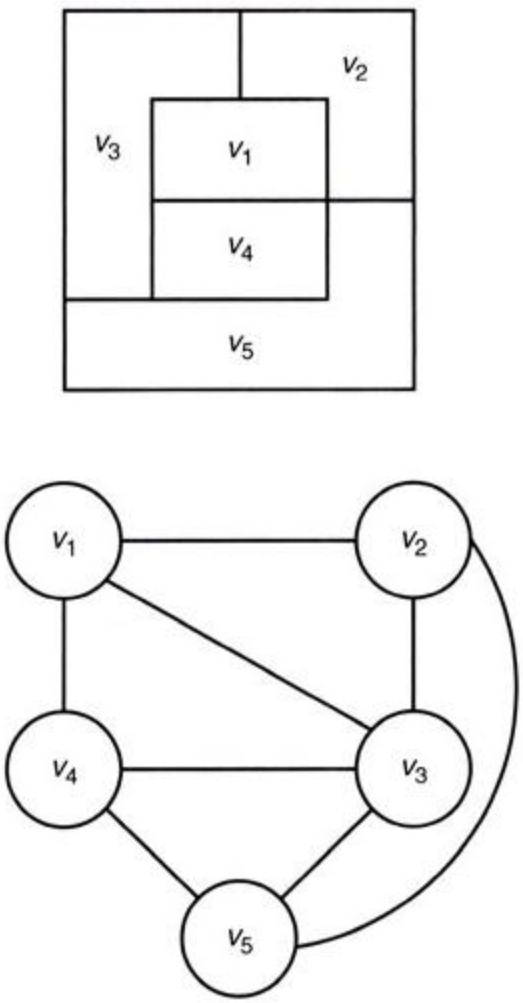


Figure 5: Map (top) and its planar graph representation (bottom)

Next we present an algorithm that solves the *m*-Coloring problem for all values of *m*. In this algorithm the graph is represented by an adjacency matrix,. However, each entry in the matrix is simply true or false depending on whether or not there is an edge between the two vertices.

Algorithm: The Backtracking Algorithm for the m-Coloring Problem

Problem: Determine all ways in which the vertices in an undirected graph can be colored, using only *m* colors, so that adjacent vertices are not the same color.

Inputs: positive integers *n* and *m*, and an undirected graph containing *n* vertices. The graph is represented by a two-dimensional array *W*, which has both its rows and columns indexed from 1 to *n*, where *W [i, j]* is true if there is an edge between *i*th vertex and the *j*th vertex and false otherwise.

Outputs: all possible colorings of the graph, using at most *m* colors, so that no two adjacent vertices are the same color. The output for each coloring is an array *vcolor* indexed from 1 to *n*, where *vcolor [i]* is the color (an integer between 1 and *m*) assigned to the *i*th vertex.

```
Procedure m_coloring (i : integer)
var color: integer;
begin
  if (promising (i)) then
     if (i = n) then
        Write  vcolor [1] through vcolor [n];
     else
       for color = 1 to  m                    { Try every }
        begin
           vcolor [i + 1] = color;             { color for }
           m_coloring (i + 1);                 { next vertex.}
         end;
 end;
```

```
Function promising (i : integer):Boolean;
var j: integer; switch:Boolean;
begin
   switch = true;
    j = 1;
   while (j  and  switch)  do                    { Check if an
    begin                                          adjacent vertex
       if (W[i,j] and vcolor[i] = vcolor[j])  then   is already
       switch = false;                              this color.}
       j := j+1;
     end;
     promising :=switch;
end;
```

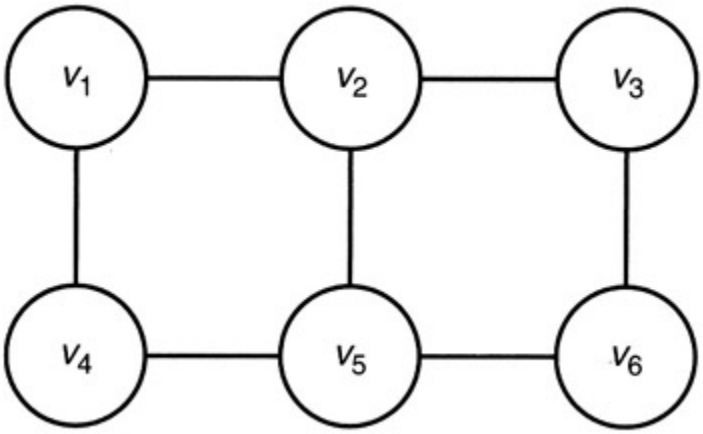
The parameters n, *m*, *W*, and *vcolor* are not inputs to either routine. In an implementation of the algorithm, the routines would be defined locally in a simple procedure that had *n*, *m*, and *W* as inputs, and *vcolor* defined locally. The top level call to *m*_coloring would be
m_coloring (0)
The number of nodes in the state space tree for this algorithm is equal to

$$1 + m + m^2 + \cdots + m^n = \frac{m^{n+1} - 1}{m - 1}.$$

Homework:

1. Modify the Backtracking algorithm for the *n*-Queens problem so that, instead of generating all possible solutions, it finds only a single solution.
2. Find at least two instances of the *n*-Queens problem that have no solutions.
3. Apply the Backtracking algorithm for the n-Queens problem to the problem instance in which n = 5, and show the actions step by step. Draw the state space tree and the pruned state space tree produced by this algorithm up to the point where the first solution is found.

4. Use the Backtracking algorithm for the m-Coloring problem to find all possible colorings of the graph below using the three colors red, green, and white. Show the actions step by step.



5. Suppose that to color a graph properly we choose a starting vertex and a color to color as many vertices as possible. Then we select a new color and a new uncolored vertex to color as many more vertices as possible. We repeat this process until all the vertices of the graph are colored or all the colors are exhausted. Write an algorithm for this greedy approach to color a graph of n vertices. Analyze this algorithm and show the results using order notation.
6. Suppose we are interested in minimizing the number of colors used in coloring a graph. Does the greedy approach of homework 5 guarantee an optimal solution? Justify your answer.
7. Compare the performance of the Backtracking algorithm for the m-Coloring problem and the greedy algorithm of homework 5. Considering the result(s) of the comparison and your answer to homework 6, why might one be interested in using an algorithm based on the greedy approach?
8. List some of the practical applications that are representable in terms of the m-Coloring problem.