Illustration of Nonlinear Robust Optimization Models in Engineering Design

A Thesis

Presented to

the Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

in Mechanical Engineering

by

Bhanu Kiran Susarla

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Abstract

The main focus of robust optimization has been put on linear models with uncertain parameters for many years. However, many real-life optimization problems are nonlinear. A robust-optimization formulation for nonlinear programming has been recently introduced by Yin Zhang (2007), which deals with first order robustness. But the effect of designation of different variables as state and control on the optimization results is unanswered. In this study, the above robust-optimization formulation is first applied to a structural nonlinear optimization problem and a spring-mass-damper system to understand its effectiveness. Later, a third case study in which the designer has to differentiate the state and control variables is discussed to understand its implications on the optimization results. Finally, the famous Golinski's speed reducer problem is used to show that even a very small uncertainty in parameters could significantly alter the optimal solution and explain the advantage of robust optimization over design using factor-of-safety.

Keywords: Robust Nonlinear Optimization, Factor of Safety, Data perturbations.

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Chapter 1 Introduction

1.1 Motivation

Optimization in general is the process of picking the best solution out of a set of available solutions. An optimization problem usually takes the form

$$\begin{aligned} &\textbf{minimize} \;\; f(x,p) \\ &\textbf{s.t.} \;\; F(x,p) = 0, \\ &\qquad\;\; G(x,p) \leq 0, \end{aligned} \tag{1.1}$$

where x represents the variables involved in the problem and p is the set of known parameters. $f(x,p)$ is the objective function which has to be minimized. $F(x,p)$ and $G(x,p)$ are the equality and inequality constraints respectively. If the objective function and all the constraints are linear, the problem is said to be a linear optimization problem and if at least one of the constraints or the objective function is nonlinear, the problem is said to be a nonlinear optimization problem.

Some optimization problems have multiple objective functions which make them more complicated. An example would be optimizing a structural design problem where one needs the structure to be both light and rigid. As these are two conflicting objectives, the optimum solution has to be a trade-off between them. However, multiple objective function problems can *sometimes* be treated as single objective optimization problems by adding an artificial constraint which would serve the purpose of one of the objective functions.

The optimum solution can be obtained using calculus equations for fairly simple problems and by iterative algorithms for complicated optimization problems. Whatever optimum solution obtained, holds good until the parameters assumed to be constant are indeed constant. But, in most of the real life problems parameters are subject to uncertainties and we can only assume that they belong to some uncertainty set. For example, the actual availability of raw materials and other resources, actual demand for the products are not precisely known in a supply chain optimization problem when critical decisions need to be made. An optimization problem with uncertainty in parameters can be represented as

$$\begin{aligned} \textbf{minimize} \quad & f(x,p) \\ \textbf{s.t.} \quad & F(x,p) = 0, \\ & G(x,p) \leq 0, \\ & p \in P. \end{aligned} \tag{1.2}$$

The uncertain variable p can take any value in the uncertainty set P.

An optimum solution obtained without considering the uncertainty in parameters could be highly infeasible and difficult to implement. Some examples were worked out in the latter chapters of this study to explain it. Several optimization techniques have been developed to account for this uncertainty in the problem data. Some of the prominent ones are:

i. Sensitivity analysis

ii. Stochastic programming

iii. Robust programming

Sensitivity analysis is the study of the effect of various inputs on the output of an optimization problem[1]. It merely analyzes the goodness of a solution and doesn't actually generate solutions that are robust to data uncertainties. Moreover, if a model has many parameter uncertainties, a joint sensitivity analysis may not give a reasonable output.

Stochastic programming takes advantage of the probability distributions of the uncertain data[2]. But, we may not be able to estimate the probability distributions of the uncertain parameters in all the cases. Even if the probability distributions are available, it is still computationally challenging to evaluate the constraints. It means that stochastic programming cannot be applied to optimization problems where the parameters are only known to vary within certain bounds.

Robust programming or *Robust optimization* on the other hand provides a robust optimal solution with the knowledge of uncertainty bounds of the parameters. It is in general considered as an improvement over the previous optimization techniques that were formulated to deal with data uncertainty. In robust optimization, instead of immunizing the solution in some probabilistic sense as in stochastic optimization, we construct a solution that is feasible for any realization of uncertainty in the given set.

The other advantages of robust optimization over the previous optimization techniques are: *Computational tractability* and *Flexibility*. Robust counterparts developed using robust optimization techniques are in general solvable in polynomial-time. Robust optimization techniques can be used in a wide variety of applications which make them more attractive. Some of the applications of robust optimization in engineering[3] are discussed in the following section.

1.2 Applications of Robust Optimization in engineering

StructuralDesign: Ben-Tal and Nemirovski[4] proposed a robust model for truss topology design in which the resulting truss remains stable over a set of loading scenarios.

Circuit design: Boyd et al. [5] and Patil et al. [6] considered the problem of minimizing delay in digital circuits when the underlying gate delays are not known exactly. They showed how to approach such problems using geometric programming.

Simulation Based Optimizations: Many engineering design problems do not have characteristics captured by an easily-evaluated and manipulated functional form. Instead, for many problems, the physical properties of a system can often only be described by numerical simulation. In [7], Bertsimas, Nohadani and Teo presented a framework for robust optimization in exactly this setting, and described an application of their robust optimization method for electromagnetic scattering problems.

Some other applications include *Power control in wireless channels* by Hsiung et al. [8], *Antenna design* by Lorenz and Boyd [9] and *Robust Control* by Bertsimas and Brown [10].

In the present study, recent developments in robust nonlinear optimization are discussed. The main focus is put on the robust optimization methodology developed by Zhang[11] for uncertain nonlinear optimization problems. Four case studies are taken up to explain how this methodology could be applied to engineering design problems. The first case study considered is the weight minimization of a two bar structure acted upon by an out-of-plane load, with uncertainty in load and modulus of elasticity of the bar material. The second one deals with minimizing the maximum acceleration of a spring-mass-damper system supplied with an initial velocity, with uncertainty in mass of the system and initial velocity supplied.

A reactor-separator system is taken up as the third case study in which the reaction rates are uncertain. This is a special case of nonlinear optimization problems where state variables and control variables are not distinctly separated. Some suggestions are made on how to split the variables in such problems, and the effect of the split on the optimum solution is discussed.

The final case study is the weight minimization of a speed reducer with uncertainty in power transmitted and pinion speed. This problem is being studied for a very long time due to the difficulty of finding a feasible space for it, with all the constraints satisfied. It helps us understand how drastically different, robust optimization is from design using factor-of-safety.

Chapter 2 Literature Review

Even though the concept of optimization has been studied by many mathematicians through out the world for many years, much of the theory for solving linear optimization problems was developed by Kantorovich[12] in 1939. The main focus of his study was optimizing the industrial production during the third five-year plan in Russia. He considered relatively simple production examples to explain writing the mathematical models for such problems. To solve these mathematical models, he used a method called the *method of resolving multipliers* which drastically reduced the number of variables to be solved. The multipliers can be found by successive approximations finally leading to an optimal solution. Although, a direct optimal solution cannot be found using this method, it reduced the mathematical complexity of dealing with large number of variables.

Dantzig[13] in 1951 proposed the *Simplex method* to optimize a linear function subject to linear equalities. This method is also iterative and it uses slack variables to convert linear inequalities in to equalities. In this method a basic feasible solution(BFS) is first found, and then an adjacent feasible solution is searched by changing the slack variables in the optimal direction. This process continues until the optimal feasible solution is found. In some cases, the optimal solution is unbounded or it simply doesn't exist. The simplex method was a great improvement over earlier methods and is highly efficient in practice. However, it is time consuming, because of which the *revised simplex method* was later introduced in 1953 by Dantzig[14]. In the revised simplex method, the computation time is significantly less when the number of constraints is small compared to the number of variables of the problem. In addition to it, the inaccuracies due to rounding errors in the original simplex method were avoided in the revised simplex method.

Soyster[15](1972) formulated a convex mathematical programming problem in which the usual convex inequalities, $f_i(x) \leq a_i$, i = 1,2,....n were replaced by convex activity sets(K_i, i = 1,2,..m). The ordinary linear programming problem is a special case of this formulation where the convex sets contain a single vector. This was also extended to inexact linear programming where the convex activity vectors are not precisely known. All that is known is that they belong to a single convex set K. This inexact linear programming formulation developed by Sosyter, even though technically sound, seemed too conservative as too much optimality is lost to counter the uncertainty in the convex activity vectors.

Mulvey and Vanderbei[16](1994) coined the term *Robust optimization*, as a method of solving mathematical programming models with noisy, erroneous or incomplete data. They used a set of scenarios(Ω) and a set of controlled constraints(uncertain parameters) associated with each scenario. Robustness of the optimal solution was described in two ways:

- *Solution Robust*: "The optimal solution of the mathematical program will be robust with respect to optimality if it remains close to optimal for any realization of the scenario."

- *Model Robust*: "The solution is robust with respect to feasibility if it remains almost feasible for any realization of the scenario."

As it is unlikely that any solution will remain both feasible(Model Robust) and optimal(Solution Robust), Mulvey formulated a robust optimization model that allows us to measure the trade off between solution and model robustness. The optimization model considered was shown as

$$\begin{aligned} \min_{x \in R^{n1}, y \in R^{n2}} \quad & c^T x + d^T y \\ s.t. \quad & Ax = b, \\ & Bx + Cy = e, \\ & x, y \geq 0. \end{aligned} \tag{2.1}$$

A set of control variables $(y_1, y_2, \ldots, y_s)$ for each scenario s $\in \Omega$, and a set of error vectors $(z_1, z_2, \ldots, z_s)$ which will measure the infeasibility allowed in control constraints were introduced in the robust model. The probability of each scenario was taken as p_s. The robust model formulated was shown to be

$$\begin{aligned} \min \quad & \sigma(x, y_1, \ldots, y_s) + \omega \rho(z_1, \ldots, z_s) \\ s.t. \quad & Ax = b, \\ & B_s x + C_s y + z_s = e_s, \quad x \geq 0, \; y_s \geq 0, \quad \forall s \in \Omega. \end{aligned} \tag{2.2}$$

The second term in the objective function $\rho(z_1, \ldots, z_s)$ is a feasibility penalty function which is used to penalize violations of the control constraints under some of the scenarios. The first term of the objective function takes care of the optimality robustness, while the second term is a measure of model robustness. The penalty function distinguishes this robust optimization method from the previous methods that deal with uncertain data. Also, the advantages of RO over sensitivity analysis and stochastic linear programming were discussed by Mulvey.

The main drawback of this RO formulation is that it allows the solution to violate the scenario realizations of the constraints. Even the scenarios considered in Ω are just one possible set of realizations of the uncertain data. Also, these formulations are complex and require heavy computational time.

A major breakthrough in the field of robust optimization was made by the independent work of Ben-Tal and Nemirovski[17]. Their robust counterpart for an uncertain linear program is based on the geometry of the uncertain set as the geometry determines the computational tractability of the problem. Ellipsoidal uncertainty sets were proposed to avoid the conservatism of Soyster's formulations. Also, ellipsoidal sets are easy to handle numerically, and most of the uncertainty sets can be approximated to ellipsoids or the intersection of a finite number of ellipsoids.

The conditions under which the robust counterpart is solvable, and the proximity of the robust optimal solution and the optimal solution in the nominal case were discussed in detail. It was demonstrated that the robust counterpart of an uncertain LP(Linear program) is an explicit conic quadratic program, of an uncertain conic quadratic program is an explicit semidefinite program and that of an uncertain semidefinite program is an NP-hard (non-deterministic polynomial-time hard). A tractable approximate robust counterpart of a general uncertain semidefinite program was also proposed.

Ben-Tal and Nemirovski[18] in 1999 suggested a modeling methodology for uncertain linear programming(LP) problems associated with hard constraints. Hard constraints are those which must be satisfied for all the realizations of the data. It was shown that the robust counterpart of an LP problem is computationally tractable which can be solved in polynomial time. To demonstrate that optimal solutions of linear programming problems become highly infeasible due to slight perturbations in the nominal data, several LPs were studied from NETLIB and robust optimization methodology[19] was applied to them to get robust solutions. The resulting robust solutions were observed to be very close to the nominal optimal solutions.

All the above mentioned robust optimization methods deal with linear models with uncertain parameters. But, many practical optimization problems are nonlinear. Diehl et al.[20](2006) proposed an approximate technique for nonlinear optimization which linearizes the uncertainty set. The worst case of the objective function for each uncertain parameter was used to linearize both the state(model) equations and the objective function. However, this method failed for problems where the worst case occurs due to a combination of the uncertain parameters.

Zhang.Y[11](2007) proposed a general robust optimization formulation for nonlinear programming with parameter uncertainty involving both equality and inequality constraints. This formulation also covers non-convex problems. The uncertain parameters are bounded with upper and lower bounds and the robust solution satisfies all the realizations of the uncertain data between the bounds. He also proposed a mechanism to automatically and adaptively choose safety margins(bounds) based on constraint sensitivity towards the uncertain parameters. The optimization model he considered was

$$\begin{aligned} \min_{y,u \in U} \quad & \phi(y,u,s) \\ s.t. \quad & F(y,u,s) = 0, \\ & G(y,u,s) \leq 0, \end{aligned} \tag{2.3}$$

where y is the set of state variables, u - control variables and s - uncertain parameters.

The robust counterpart of the above optimization model was given as

$$
\begin{aligned}
\min_{y,u \in U} \quad & \phi(y,u,\hat{s}) \\
s.t. \quad & F(y,u,\hat{s}) = 0, \\
& \tau(F_y y_s + F_s) = 0, \\
& \operatorname{diag}(G)E \pm \tau(G_y y_s + G_s) \leq 0,
\end{aligned}
\tag{2.4}
$$

where $\hat{s}$ is the best estimate of the uncertain parameter s and τ is the best estimate on the maximum deviation of s from $\hat{s}$. It can be observed that when $\tau = 0$, the robust optimization problem reduces to the original optimization model(eq 2.3). The proposed formulation is valid only in a neighborhood of the nominal value($\hat{s}$) and is robust only to the first-order. It was also assumed that reasonable parameter estimates(nominal values) are available for the uncertain parameters.

Hale and Zhang[21](2007) studied three nonlinear robust optimization case studies to assess the effectiveness of the robust optimization formulation proposed by Zhang[11]. They observed that the first-order method produced reasonable solutions when the parameter uncertainty was small to moderate. The third case study was a special case of nonlinear optimization where the variables were not differentiated as state and control. The effect of the split of variables on the robust solution was not studied.

Chapter 3 Robust Nonlinear Optimization

Diehl et al.[20](2006) proposed an approximate technique for nonlinear optimization which linearizes the uncertainty set. In this method, the worst case can be approximately computed by linearizing the model equations at a point corresponding to the nominal value of the parameter. It illustrated that approximated robust optimization leads to reduced dependence of the constraints on uncertain parameters. However, the robust solution achieved is approximate and may not be feasible in problems where worst case occurs due to a combination of the uncertain parameters.

Yin Zhang[11](2007) extended the concept of robust optimization to general nonlinear programming setting with uncertain parameters in both equality and inequality constraints. This extension was proposed to give out a solution to non-convex and non-linear problems, as the previous robust optimization approach was limited to linear models of parameter uncertainty with only inequality constraints. Another restriction of the previous optimization approach was in the fact that they could only deal with problems where the data elements themselves were the uncertain parameters and were independent of each other. The proposed formulations were stated to be valid for applications with moderate variations in their parameters, whose reasonable parameter estimates are available and are robust to the first-order in a neighborhood of a given nominal parameter value. Mathematical tools such as linearization (linear approximation of a function at a given point, used to approximate the value at that point using the derivative of the function and its value at a point near the desired point) and implicit function theorem (tool to approximate a relation to a function or to find the value of an implicit function in a small interval) were used to derive these formulations.

3.1 Problem Description

The following general form of a nonlinear optimization problem was considered in the proposed Robust Optimization method

$$\begin{aligned} \min_{y,u \in U} \quad & \phi(y,u,s) \\ s.t. \quad & F(y,u,s) = 0, \\ & G(y,u,s) \leq 0, \end{aligned} \tag{3.1}$$

where $s \in \mathbb{R}^{N_s}$ is the uncertain system parameter vector, $y \in \mathbb{R}^{N_y}$ the state variable (representing the states of the system at different instances) and $u \in \mathbb{R}^{N_u}$ is the design variable in the feasibility set $U \subset \mathbb{R}^{N_u}$. F(y, u, s) = 0 is the state equation which always needs to be satisfied for any value of the uncertain parameter vector 's'. This equation implicitly (the dependent variable has not been explicitly stated in terms of the independent variable) defines the state variable function y = y(u, s). The inequality constraint G(y, u, s) $\leq$ 0 represents the safety constraints for $G \in \mathbb{R}^m$. It was assumed that the functions F and G were continuously differentiable. This proposed approach dealt only with local optimization unlike the case of convex programming.

3.2 Methodology

The following robust counterpart was proposed for the above general nonlinear optimization problem

$$
\begin{aligned}
\min_{y,u\in U} \quad & \phi(y,u,\hat{s}) \\
\text{s.t.} \quad & F(y,u,\hat{s}) = 0, \\
& \tau(F_y y_s + F_s) = 0, \\
& \text{diag}(G)E \pm \tau(G_y y_s + G_s) \leq 0.
\end{aligned}
\tag{3.2}
$$

Since the inequality constraint is less than or equal to zero, taking the maximum value of the left hand side only, the constraint becomes $diag(G)E + \tau|(G_y y_s + G_s)| \leq 0$. $y_s \in \mathbb{R}^{N_y \times N_s}$ denotes the Jacobian matrix of y w.r.t. s. $E \in \mathbb{R}^{m \times N_s}$ is the matrix with all 1's used with the intention to convert the safety constraint 'G' into an $m \times N_s$ matrix with the help of diag(G). All the Jacobian's were evaluated at $\hat{s}$, which is the best estimate of the uncertain parameter 's' and τ denotes the maximum deviation of 's' from '$\hat{s}$' measured using the L-1 norm (sum of the absolute values of the vector entities). The non-negative term in the safety constraint plays the role of safety margin. It was observed by the author that when $\tau = 0$, the robust counterpart reduced to the original problem. In addition to the N_y equations given by the state equation, there are extra $N_y N_s$ equations in the robust counterpart. Also, there are $2mN_s$ element wise inequality constraints.

3.3 Understanding the Derivation

In the derivation of the robust counterpart, the inequality only case is first considered, where the equality constraint (state equation) is used to eliminate the state variable y(u,

s) giving the following form of the original optimization problem

$$\begin{aligned} \min_{u \in U} \quad & \phi(u, s) \\ s.t. \quad & G(u, s) \leq 0. \end{aligned} \tag{3.3}$$

It was assumed that the inequality constraint (safety constraint) is strictly satisfiable i.e., $\exists$ a pair (u, s) $\in \mathbb{R}^{N_u} \times \mathbb{R}^{N_s}$ such that G(u, s) $<$ 0. The constraint has to be satisfied for every possible realization of the uncertain parameter from the uncertainty set $s \in S$. This is equivalent to the satisfying of $\max_{s \in S} g_i(u, s) \leq 0,\ i = 1, \ldots, m$ where G $= (g_1, \ldots, g_m)^T \in \mathbb{R}^m$.

It was stated that in order to obtain solutions (at least approximate) to the above maximization problem, linearization of the functions g_i w.r.t. s, at the nominal parameter value $\hat{s}$ (to obtain approximate values of g_i in neighborhood of $\hat{s}$) was required along with the uncertainty set S to have a simple form. In this regard, the uncertainty set was formulated ($\tau > 0$, p $\geq$ 1) as

$$S_\tau := \{\hat{s} + \tau D \delta : \|\delta\|_p \leq 1\}, \tag{3.4}$$

where $\delta \in \mathbb{R}^{N_d}$, $N_d \leq N_s$ is the parameter variation within the unit ball in a p-norm ($\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$) centered at $\hat{s}$. The δ gives the direction of variations (in unit ball) only in the components of s which are subject to variations (some entities may be positive and some negative at a time) and this makes its dimension less than s. In general terms, δ gives information regarding which components of s are subject to variations and their signs. $\tau > 0$ represents the magnitude of variations in the parameter s calculated using the L-1 norm. This value could be obtained by prior experience or sampling methods. The D matrix is flexible in its components and dimensions, with it taking positive diagonal form when there are parameter variations of different scales, takes $N_s \times N_d$ basis matrix form if there are variations only in certain components of s ($N_d < N_s$). For simplicity, D can be taken as an identity matrix of size $N_s \times N_s$.

In order to linearize the constraints g_i and to obtain its value at a neighborhood point of $\hat{s}$, the given first order Taylor's approximation was used,

$$g_i(u, \hat{s} + \tau D\delta) \approx g_i(u, \hat{s}) + \tau\langle \nabla_s g_i(u, \hat{s}, D\delta\rangle. \tag{3.5}$$

The inner product of the gradient of g_i and the parameter variation vector along with the value of the constraint at $\hat{s}$ gives the constraint value at a point in the neighborhood of $\hat{s}$. According to the Holder's inequality, for two vectors c and x,

$$|\langle c, x\rangle| \leq \|x\|_p \|c\|_q \;\; for \;\; \frac{1}{p} + \frac{1}{q} = 1, \; 1 \leq p, q \leq +\infty. \tag{3.6}$$

That is, the absolute value of the inner product of two vectors is $\leq$ the product of the p and q norms of the two vectors such that $\frac{1}{p} + \frac{1}{q} = 1, \; 1 \leq p, q \leq +\infty$. Using the Holder's inequality, the above given approximation of the constraint g_i can be again formulated as

$$\begin{aligned} \max_{s \in S_\tau} g_i(u, s) &\approx g_i(u, \hat{s}) + \tau \max_{\|\delta\|_p = 1} \langle D^T \nabla_s g_i(u, \hat{s}), \delta\rangle \leq 0 \\ &= g_i(u, \hat{s}) + \tau \|D^T \nabla_s g_i(u, \hat{s})\|_q \leq 0. \end{aligned} \tag{3.7}$$

The maximization problem on the right reduces to the q-norm of the gradient of g_i at $\hat{s}$ when the parameter uncertainty δ is at the boundary its unit ball (since this value of δ gives the max. value of the safety margin). Thus the Robust counterpart of the uncertain nonlinear optimization problem with inequality only constraint was proposed as

$$
\begin{aligned}
&\min_{u \in U} \quad \phi(u, \hat{s}) \\
&s.t. \quad g_i(u, \hat{s}) + \tau \| D^T \nabla_s g_i(u, \hat{s}) \|_q \leq 0, \; i = 1, \ldots, m.
\end{aligned}
\tag{3.8}
$$

In this proposed formulation, the added non-negative term is the safety margin of the safety constraint, which is directly proportional to the magnitude of variations τ and the sensitivity of the constraint w.r.t. s at the point $\hat{s}$. This shows that in the counterpart, only the nominal value of the parameter $\hat{s}$ is required to calculate the robust optimal solution. Also, the optimization gives local optimal values due to the absence of convexity. The size of the robust feasibility set depends on the value of τ, q and the sensitivity of the constraint w.r.t. the uncertain parameter s at the nominal value $\hat{s}$. Using an example it was demonstrated that the size of feasibility set was inversely proportional to the value of the magnitude (τ) of the uncertain parameter and directly proportional to the value of q. It was maximum for q = ∞ (infinity norm) and minimum for q = 1(taxicab or L-1 norm).

Addressing the general case where equality constraints are also involved, $F_y(y, u, s)$ was taken as the partial Jacobian of the state equation function w.r.t. the state variable y, given as

$$
\Big[F_y(y, u, s)\Big]_{ij} = \frac{\partial F_i(y,u,s)}{\partial y_j}, \; i, j = 1, \ldots, N_y. \tag{3.9}
$$

According to the Implicit Function theorem, an implicit function (y in this case) implicitly defined by a state equation (F(y, u, s) = 0 in this case) is well defined in a neighborhood of (u_0, s_0) if $\exists$ a y_0 (value of the implicit function at the sample/nominal point) which satisfies the state equation F(y_0, u_0, s_0) = 0 and also satisfies certain mild conditions on its partial derivatives ($\nabla_y F(y_0, u_0, s_0)$ should be non-singular in this case). The non-singularity (determinant $\neq 0$) condition needs to be satisfied since if $\nabla_y F(y_0, u_0, s_0)$ denotes the coefficients of a linear system of equations, then this condition ensures that system to have a unique solution.

The differentiations of the state equation (F) and safety constraints (G) w.r.t. the state variables would have the form

$$\begin{aligned} F_y(y,u,s)y_s + F_s(y,u,s) &= 0, \\ \nabla_s G(y,u,s) &= G_y(y,u,s)y_s + G_s(y,u,s). \end{aligned} \tag{3.10}$$

In this relation, the robust counterpart for the general nonlinear optimization problem was formulated as

$$\begin{aligned} \min_{y,u\in U} \quad & \phi(y,u,\hat{s}) \\ s.t. \quad & F(y,u,\hat{s}) = 0, \\ & \tau(F_y y_s + F_s) = 0, \\ & g_i(y,u,\hat{s}) + \tau \| e_i^T (G_y y_s + G_s) D \|_q \leq 0, \ i = 1,\ldots,m. \end{aligned} \tag{3.11}$$

Different values for τ, D and q for different constraints can be used to increase the flexibility of the formulation.

3.4 Robustness Characteristics

The (p, q)-Lipschitz continuity of a function caps the rate of change of the function to a Lipschitz constant (L, or modulus of uniform continuity). For a function $f : C \subset \mathbb{R}^n \to \mathbb{R}^m$ with $\mathbb{R}^n$ and $\mathbb{R}^m$ endowed with the conjugate pair of p and q norms respectively, it is said to be (p, q)-Lipschitz continuous in C if

$$\|f(x) - f(y)\|_q \leq L\|x - y\|_p, \forall x, y \in C. \tag{3.12}$$

It was stated that if the nominal optimization problem was strictly feasible at $\hat{s}$ (i.e., if there was atleast one solution in the relative interior of the feasibility set of the nominal optimization problem at the nominal parameter value $\hat{s}$), then the robust formulation stated earlier was also strictly feasible at $\hat{s}$ provided that the value of maximum parameter variations τ was sufficiently small. Addressing the characteristic of the robust solution obtained, it was stated that the solution was first-order robust and the following theorem was proved by Zhang [11].

• If $(\hat{y}, \hat{u})$ was a strictly feasible solution to the robust optimization problem at the nominal value $\hat{s}$ and $\tau > 0$, then on assuming in the set S_T that the function y($\hat{u}$, s) is implicitly defined by the state equation F(y, $\hat{u}$, s) = 0 as a differentiable function in s and that all the safety margin functions $[G_y y_S + G_S]$(y(s), $\hat{u}$, s) were (p, q)-Lipschitz continuous modulo to L, then the cap on the maximum constraint violation in the uncertainty set S_T was given as being proportional to the square of τ by the expression

$$G(y(\hat{u}, s), \hat{u}, s) \leq \frac{L}{2}\tau^2, \forall s \in S_T. \tag{3.13}$$

The result of the above stated theorem was addressed as first-order robustness. This characterizes the robustness of the solution obtained after solving the robust optimization problem. The first-order robust characteristic of the formulation is that there is no guarantee offered by the methodology of satisfying all the safety constraints $G(y(\hat{u}, s), \hat{u}, s) \leq 0 \forall s \in S_T$, it ensures that the maximum constraint violation when the uncertain parameter takes any value from the uncertainty set is capped by the product of a second order term in τ and the bound on the rate of change of the safety margin in the desired neighborhood of nominal parameter value. This indicates that the lower the value of τ, the lesser the constraint violation can be expected. Also it was observed that the state variable $\hat{y}$ is only good at the nominal parameter value $\hat{s}$. But since the goal of design or optimal control robust optimization problems is to obtain a robust design variable, this limitation can be tolerated.

Finally, it was stated that the proposed robust nonlinear optimization methodology was a generalization of the previous robust counterpart approach which dealt with inequality only, convex constraints. If the functions are linear in terms of the design variables with no uncertain equality constraints , then the problem gets reduced to a robust linear program. This methodology involves rigorous mathematics but provides a simple mechanism for choosing a safety margin (depending on the sensitivity of the constraints to the uncertain parameters). The limitation of this formulation is that it is robust only in a neighborhood of the nominal parameter value and the solutions are robust only to the first-order. This means that a feasible solution can be obtained only when the uncertainty in parameter(τ) is small.

The first-order formulation used in Zhang's[11] work would be more suitable for larger problems than most other proposed approaches as it requires the solution of just a single NLP with $n_y(1+n_s)+n_u$ variables, $n_y(1+n_s)$ equality constraints and m inequality constraints. On the other hand, the previous iterative approaches proposed in the literature require the solution of a much larger non linear programming(NLP) problem.

Chapter 4 Case Studies

4.1 Out of Plane Loading Problem

4.1.1 Introduction

Consider the design of a two-member frame subjected to out-of-plane loads[22] as shown in Fig 4.1. The volume of the frame has to be minimized with stress and size limitations as constraints which gives us a normal non-linear optimization problem. We then assume uncertainty in the load applied(P) on the members and the modulus of elasticity(E) of the material. Thus, a Robust counterpart is developed and the merits of the robust solution are discussed.

4.1.2 Design of the optimization problem

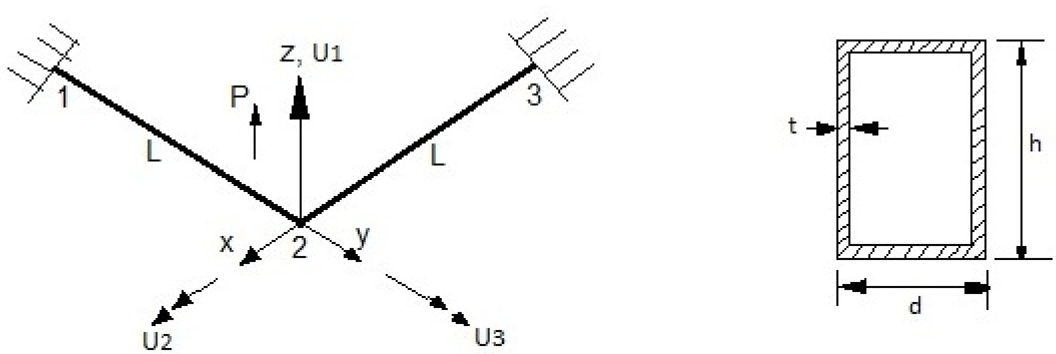

Figure 4.1: Two member frame

Since the optimum structure will be symmetric, the two members of the frame are identical. The members are considered to be hollow rectangular sections with design variables defined as d=width of the member(in), h=height of the member(in), and t=wall thickness(in). The volume(V) of the structure, which is the objective function of the problem is an explicit function of the design variables given as

$$V = 2L(2dt + 2ht - 4t^2), \tag{4.1}$$

where L is the length of the members(in). The failure criterion for the member is based on a combined stress theory, known as von Mises yield condition. According to it, the effective stress is given as $\sqrt{\sigma^2 + 3\tau^2}$ or

$$\frac{\sigma^2 + 3\tau^2}{\sigma_a^2} - 1.0 \leq 0, \tag{4.2}$$

where σ and τ are the maximum bending and shear stresses in the member, respectively. And, σ_a is the allowable design stress. Thus the stress constraints at points 1 and 2 are given as

$$\begin{aligned} g_1 &= \frac{\sigma_1^2 + 3\tau^2}{\sigma_a^2} - 1.0 \leq 0, \\ g_2 &= \frac{\sigma_2^2 + 3\tau^2}{\sigma_a^2} - 1.0 \leq 0. \end{aligned} \tag{4.3}$$

σ_1, σ_2 and τ are further explained in the Appendix. The constraints on design variables are

$$\begin{aligned} &2.5 \leq d \leq 10.0, \\ &2.5 \leq h \leq 10.0, \\ &0.1 \leq t \leq 1.0. \end{aligned} \tag{4.4}$$

4.1.3 Optimal solution without uncertainty

The above optimization problem when solved using $Matlab$ gives an optimum volume of 703.9467(in^3), and the design variables thus obtained are

$$d = 7.7987(in), h = 10(in), t = 0.1(in). \tag{4.5}$$

Later, the load applied(P) and modulus of elasticity(E) are subjected to an uncertainty of 0.1% retaining the above dimensions. As a result, the maximum constraint value is increased to 3.472e+06(which otherwise has to be zero), making the above dimension set highly infeasible.

4.1.4 Robust counterpart

The robust counterpart of an uncertain optimization problem[11] applied here gives us

$$\begin{aligned} &min \qquad V = 2L(2dt + 2ht - 4t^2) \\ &s.t. \qquad \text{diag}(G)E_1 \pm \tau_S(G_S) \leq 0. \end{aligned} \tag{4.6}$$

Here G =

$$\begin{pmatrix} \frac{\sigma_1^2+3\tau^2}{\sigma_a^2} - 1.0 \\ \frac{\sigma_2^2+3\tau^2}{\sigma_a^2} - 1.0 \\ d - 10 \\ h - 10 \\ t - 1 \\ 2.5 - d \\ 2.5 - h \\ 0.1 - t \end{pmatrix},$$

$E_1(m \times n)$ is the matrix of all ones, where m = no. of rows of matrix G and n=no. of uncertain parameters.τ_S is the best estimate of maximum deviation of uncertain parameter s, and G_S is the Jacobian of G with respect to s.

Now, consider 0.1% uncertainty in load applied (P= -10000(lb)) and modulus of elasticity (E=3 $\times$ $10^7 psi$), i.e., $\tau_S = 0.001$. The robust counterpart when solved gives an optimum volume of $704.3367(in^3)$ with the design variables as

$$d = 7.8084(in), h = 10(in), t = 0.1(in). \tag{4.7}$$

4.1.5 Inference

- The above dimension set satisfies all the constraints over a load varying from -9990(lb) to -10010(lb) and modulus of elasticity varying from $(2.997 to 3.003) \times 10^7 psi$.
- The optimum volume obtained from the robust counterpart is greater than or equal to the optimum volumes obtained from normal optimization over the given uncertainty range.
- A sudden jump in the maximum constraint value can be observed when E varies from 3.0029×10^7 to 3.0031×10^7, which proves the validity of the solution obtained.

4.2 Spring Mass Damper System

4.2.1 Introduction

For the spring-mass-damper system shown in Fig 4.2, the spring constant(k) and damping coefficient(c) need to be determined such that the maximum acceleration of the system over a period of 10 sec when subjected to an initial velocity of 5m/sec is minimized. The mass is considered to be 5kg. The displacement of the mass, spring constant and damping coefficient are constrained.

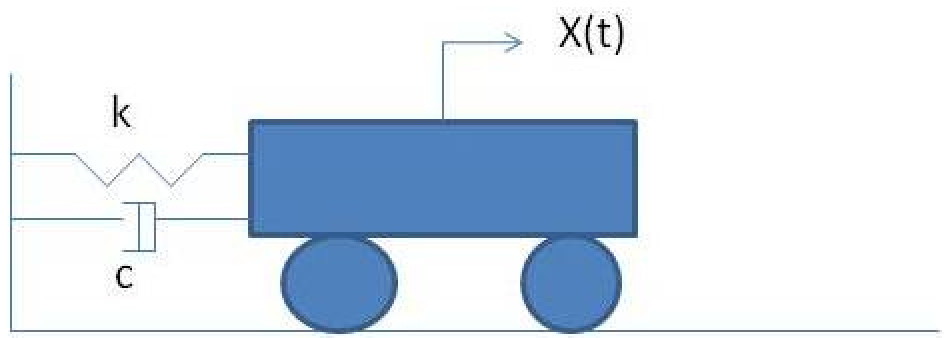

Figure 4.2: Spring mass damper system

An example of how the acceleration varies with time for this system is shown in Fig 4.3. Each pair of k and c gives some maximum acceleration(Point A) and our objective is to find (k*,c*) for which this point is the lowest. When no damping is considered(i.e., c=0), the system continuously oscillates and the maximum acceleration becomes maximum for such a case. The acceleration vs time graph of a no-damping case is shown in Fig 4.4.

This min-max problem can be converted to a nonlinear programming problem by introducing an artificial design variable A such that the acceleration $|a(t)|$ is always less

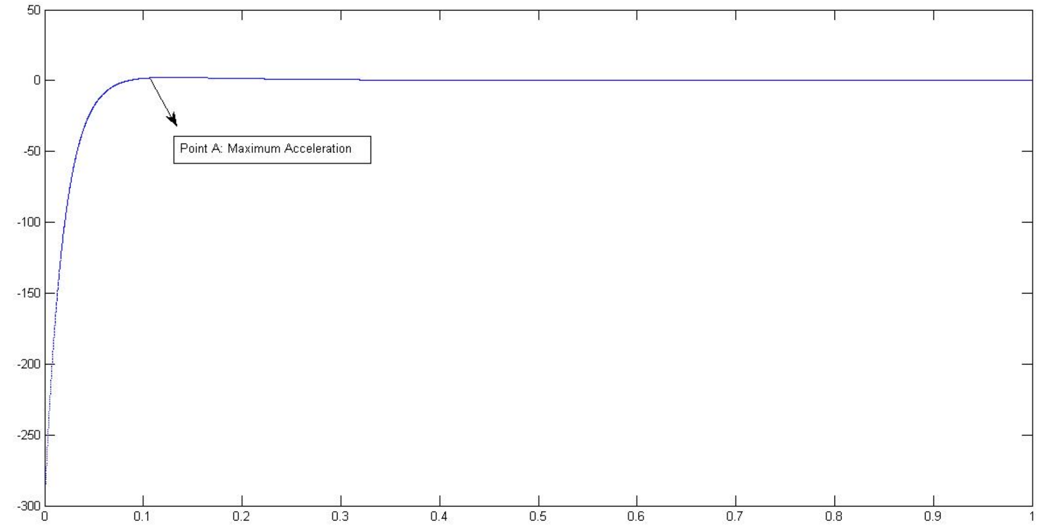

Figure 4.3: Acceleration-vs-time

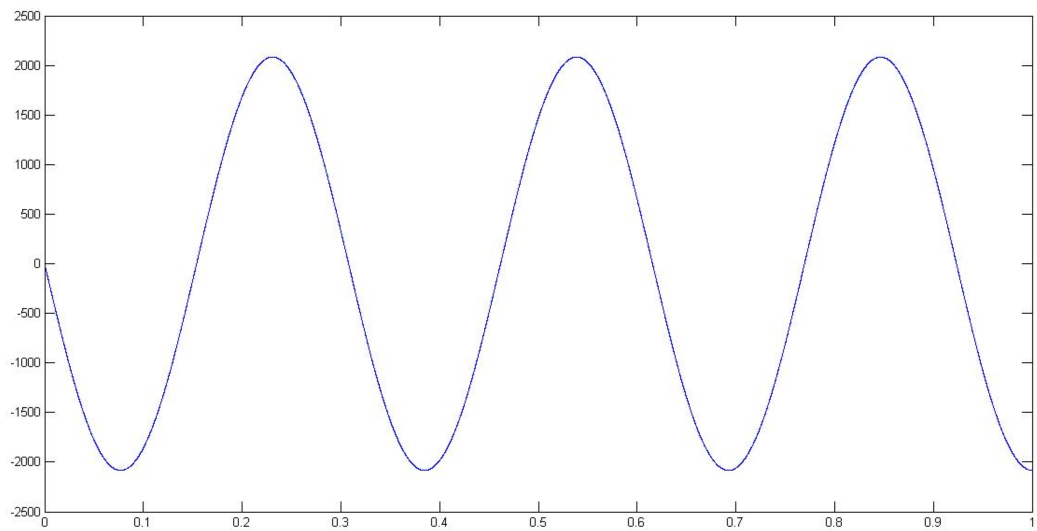

Figure 4.4: Acceleration-vs-time when c=0

than or equal to A. Later we introduce uncertainty in mass(m) and the initial velocity(v_O) and the optimum solution is calculated from the robust counterpart.

4.2.2 Optimal solution without uncertainty

The damping coefficient(c), spring constant(k) and time(t) are considered as the control variables and the displacement(x) is taken as the design variable. $y = [x],\ u = [c\ k\ t]$. The displacement(x) is defined in terms of c,k,t,m and v_o and it has different expressions for different viscous damping factors (ζ). Damping factor is expressed in terms of mass and natural frequency of the system(w_n) as

$$\begin{aligned} \zeta &= c/(2mw_n), \\ w_n &= \sqrt{(}k/m). \end{aligned} \tag{4.8}$$

For under-damped systems($0 \leq \zeta \leq 1$), over-damped systems($\zeta \geq 1$), and critically-damped systems($\zeta = 1$), displacement(x) [23] can be expressed as

$$\begin{aligned} x &= \frac{v_o \times e^{-\zeta \times w_n \times t} \times \sin(w_d \times t)}{w_d} \quad \text{for} \ \ 0 \leq \zeta \leq 1, \\ x &= \frac{e^{-\zeta \times w_n \times t} \times v_o \times \sinh(\sqrt{(}\zeta^2 - 1) \times w_n \times t)}{\sqrt{(}\zeta^2 - 1) \times w_n} \quad \text{for } \zeta \geq 1, \\ x &= v_o \times t \times e^{-w_n \times t} \quad \text{for } \zeta = 1, \end{aligned} \tag{4.9}$$

where $w_d = \sqrt{(}1 - \zeta^2) \times w_n$. The optimization problem is given as

$$
\begin{aligned}
& min \quad A \\
& s.t. \quad x \leq 0.05, \\
& \qquad -x \leq 0.05, \\
& \qquad 1000 \leq k \leq 300, \\
& \qquad 0 \leq c \leq 300, \\
& \qquad 0 \leq t \leq 10.0, \\
& \qquad |a| \leq A.
\end{aligned} \tag{4.10}
$$

Acceleration a(t) is taken as $a(t) = -(c*v+k*x)/m$, where v and x are the velocity and displacement of mass(m) at time t. Without uncertainty the optimum solution obtained is

$$
(c, k) = (300, 1264.7), \qquad (a) = 0.9589 \ m/sec^2, \tag{4.11}
$$

c in N.s/m and k in N/m.

4.2.3 Robust counterpart

The robust counterpart for this problem considering uncertainty in mass and initial velocity given to the mass can be written from [11] as

$$
\begin{aligned}
& min \quad A \\
& s.t. \quad \text{diag}(G)E \pm \tau(G_y * y_S + G_S) \leq 0.
\end{aligned} \tag{4.12}
$$

G is the constraint matrix, $E(m \times n)$ is the matrix of all ones, where m = no. of rows of matrix G and n=no. of uncertain parameters.τ is the best estimate of maximum deviation of uncertain parameter s, and G_S is the Jacobian of G with respect to s. G_y is the Jacobian of G with respect to the design variable y.

With 0.1% uncertainty in mass(m) and initial velocity(v0), the optimum solution obtained is

$$(c, k) = (300, 1270), \qquad (a) = 0.9657\ m/sec^2. \tag{4.13}$$

The variation of acceleration with time for this particular optimum set of c and k is shown in Fig 4.5.

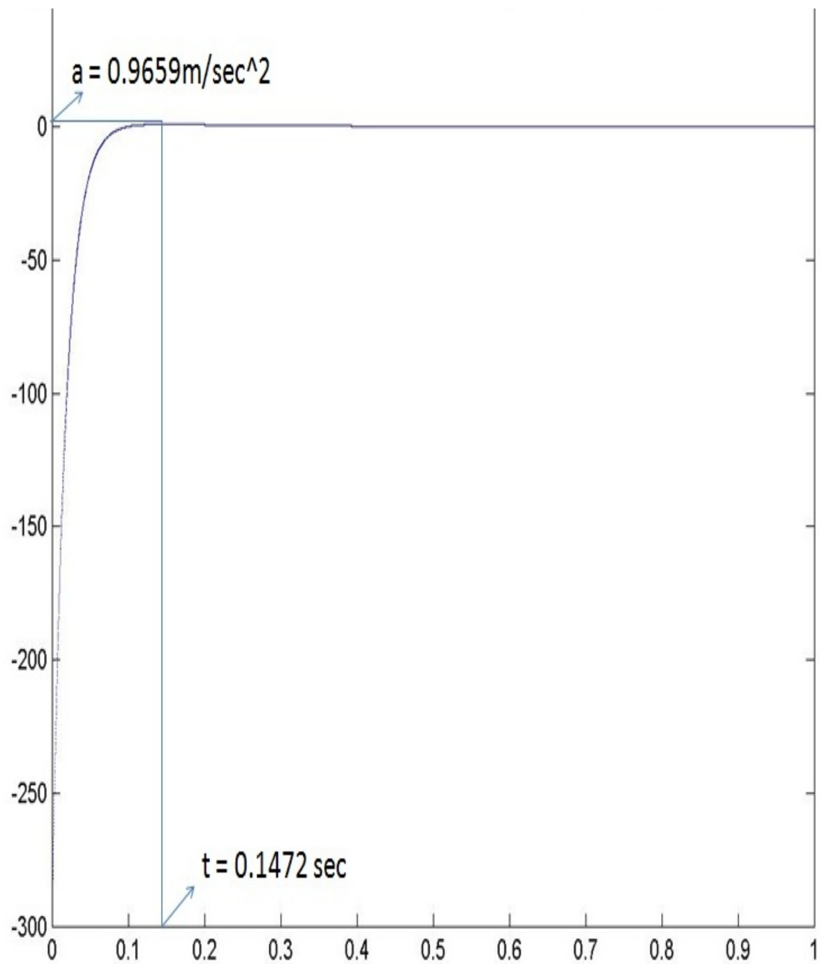

Figure 4.5: Acceleration-vs-time @ optimum(c,k)

4.2.4 Inference

- The spring constant and damping coefficient from the normal optimization problem violated the constraints when uncertainty is induced.

- The optimum acceleration has obviously increased with uncertainty in mass and initial velocity.

- Damping coefficient(c) remained the same and it is at its upper limit(300). It proved that the damping has to be maximum to minimize the acceleration of the mass.

4.3 Design of a reactor-separator system

In this case study, we consider the reactor-separator system used in the works of William C.Rooney and Lorenz T.Biegler[24]. The purpose of the system is to chemically convert reactant A to product C via the following four reactions

$$A \xrightarrow{k1} B$$

$$B \xrightarrow{k2} C$$

$$A \xrightarrow{k3} D$$

$$B \xrightarrow{k4} E.$$

The variables considered are V(volume of the reactor in m^3), F(outlet flow rate in mol/time), F_{max}, δ(fraction of species A and B that is recycled back in to the reactor), β(fraction of species D and E that is recycled back in to the reactor) and mole fraction of each species at the reactor outlet - x_a, x_b, x_c, x_d, x_e. The uncertain parameters are the reaction rates k1, k2, k3 and k4. The flow sheet of the reactor-separator system is shown in Fig 4.6.

It is desired to produce at least 40 mol/time of C, while the rest of the components are either sent down stream or recycled. The variables are linked with each other in the form of six equations. Number of inequality constraints is six. The known parameters are F_{a0} = 100 mol/time (inlet molar flow rate), and c_{a0} = 10 mol/m^3 (concentration of species A at the inlet.)

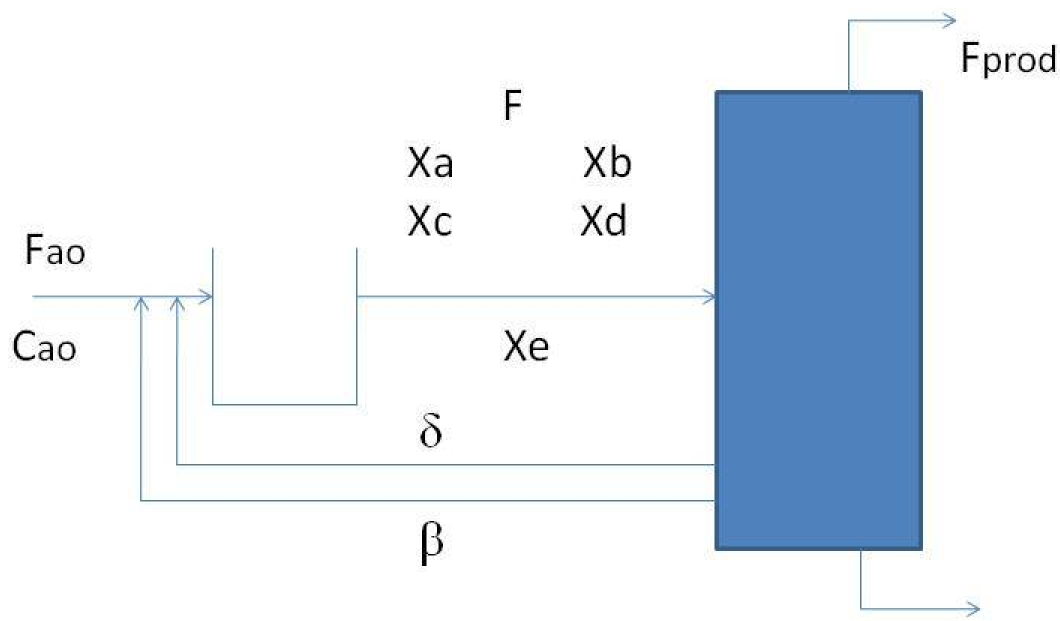

Figure 4.6: Reactor-Separator flow sheet

The nonlinear optimization problem is given as

$$
\begin{aligned}
min \quad & 10V^2 + 5F_{max} \\
s.t. \quad & F_{a0} - x_a F(1-\delta) - c_{a0} V (k_1 + k_3) x_a = 0 \\
& -x_b F(1-\delta) + c_{a0} V (k_1 x_a - (k_2 + k_4) x_b) = 0 \\
& -x_c F + c_{a0} V k_2 x_b = 0 \\
& -x_d F(1-\beta) + c_{a0} V k_3 x_a = 0 \\
& -x_e F(1-\beta) + c_{a0} V k_4 x_b = 0 \\
& x_a + x_b + x_c + x_d + x_e = 1 \\
& x_c F \geq 0 \\
& F \leq Fmax \\
& 0 \leq \delta \leq 1 \\
& 0 \leq \beta \leq 1 \\
& \forall (k_1, k_2, k_3, k_4) \in K.
\end{aligned}
\tag{4.14}
$$

The rough estimates of k_1, k_2, k_3, k_4 considered were

$$
\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0.9945 \\ 0.5047 \\ 0.3866 \\ 0.3120 \end{pmatrix}. \tag{4.15}
$$

Without considering uncertainty in the reaction rates, the nominal optimum solution obtained is given in Table 4.1.

Table 4.1: Nominal design for the reactor separator system with control variables V, δ, β

V(m^3)	F_{max} (mol/t)	Cost($\$\times 10^3$)	δ	β	x_a	x_b	x_c	x_d	x_e
19.08	413.9	570.8	0.974	0.000	0.364	0.414	0.097	0.065	0.060

This optimization problem with uncertainties was solved by Y.Zhang and E.T.Hale [21] using the upper and lower bounds on k1-k4 obtained from Table 5 of [24], which are

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0.9541 \\ 0.4792 \\ 0.3638 \\ 0.2908 \end{pmatrix}, \quad \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 1.0373 \\ 0.5329 \\ 0.4109 \\ 0.3348 \end{pmatrix}. \tag{4.16}$$

They used the first order robust nonlinear programming(1-rNLP) methodology proposed by Zhang [11]. The resulting optimum solution is given in Table 4.2.

Table 4.2: Robust design for the reactor separator system

V(m^3)	F_{max} (mol/t)	Cost($\$\times 10^3$)	δ	β	x_a	x_b	x_c	x_d	x_e
19.57	430.8	598.3	0.978	0.000	0.358	0.412	0.101	0.067	0.062

The distinction between state variables and control variables is natural in optimal control and design problems. For example, in the first case study of out of plane loading problem, the dimensions (d, h, t) were the control variables. In the second case study of spring-mass-damper system, it was obvious that the displacement(x) was the state variable and (c,k,t) were the control variables.

However, such a clear distinction may not be present in other applications such as this case study. In such cases, the variables are split in to two groups. In this reactor-separator system, (x_a,x_b,x_c,x_d,x_e,F) were considered as the state variables - y, and (V, δ, β) as the control variables - u. But, what if consider a different set of variables as state variables? Does it affect the robust solution?

To answer these questions, first we have to look in to the robust counterpart proposed in [11]. We have two terms $F_y y_s$ and $G_y y_s$ in the constraints which are dependent on the state variable vector y. y_s is solved using the equation $F_y y_s + F_s = 0$. So, F_y has to be a square matrix and nonsingular since we have to deal with the inverse of F_y. F_y is the partial derivative equality constraints(F) w.r.t. state variable(y).

So, the number of state variables has to be chosen in such a way that it equals the number of equality constraints in the optimization problem. In this reactor-separator system, we have nine variables and six equality constraints. So , they have to be split up as six state variables and three design/control variables. The robust solution obtained by taking (x_a,x_b,x_c,x_d,x_e,F) as state variables is shown in Table 4.2 . Now, let us consider y = (x_a,x_b,x_c,x_d,x_e,V) and u = (F,δ,β). The robust solution obtained in this case is given in Table 4.3.

Table 4.3: Robust design for the reactor separator system with control variables F, δ, β

V(m^3)	F_{max} (mol/t)	Cost(\$$\times$ 10^3)	δ	β	x_a	x_b	x_c	x_d	x_e
19.46	426.63	591.98	0.981	0.000	0.361	0.419	0.096	0.064	0.060

We can observe that the cost has decreased compared to the previous case where V, δ, β are the control variables. So, this is a less conservative design than the first design as we have a minimization problem. We could also try other combinations of the variables to make the different splits, but it would be meaningless to have some mole fractions as state variables and some other as the control variables. In the same way, the fractions of reactants A,B,D,E recycled back need to be in one group. Hence, the only possible interchange could be between F(outlet flow rate) and V(volume of the reactor).

From the two possible splits between with the variables, the one with (x_a,x_b,x_c,x_d,x_e,F) as state variables was more conservative and hence it is recommended to follow that design as our problem is a nonlinear *minimization* problem.

4.3.1 Inference

- The number of state variables should be equal to the number of equality constraints in an optimization problem when no clear distinction is available to split the variables as state and control.

- State variables(y) should be chosen in such a way that the Jacobian of equality constraints w.r.t. state variables (F_y) is nonsingular.

- The designation of different variables as state and control does affect the robust solution.

- The design with the most conservative solution turns out to be the safest design if it's a minimization problem and vice-versa.

4.4 Golinski's Speed Reducer Problem

4.4.1 Introduction

A speed reducer(commonly known as a gear reducer) is designed for speed reduction, and minimization of its weight is highly important when used in aircraft engines. It consists of a gear, pinion, their shafts and the casing. A schematic of a speed reducer is shown in Fig 4.7. The objective is to find the values of the parameters defining the system such as teeth module, diameters and lengths of the shafts, etc. which minimize the weight(volume) of the speed reducer and also obey a number of dimensional and stress constraints. We later introduce uncertainty in the power supplied to the speed reducer and the pinion speed.

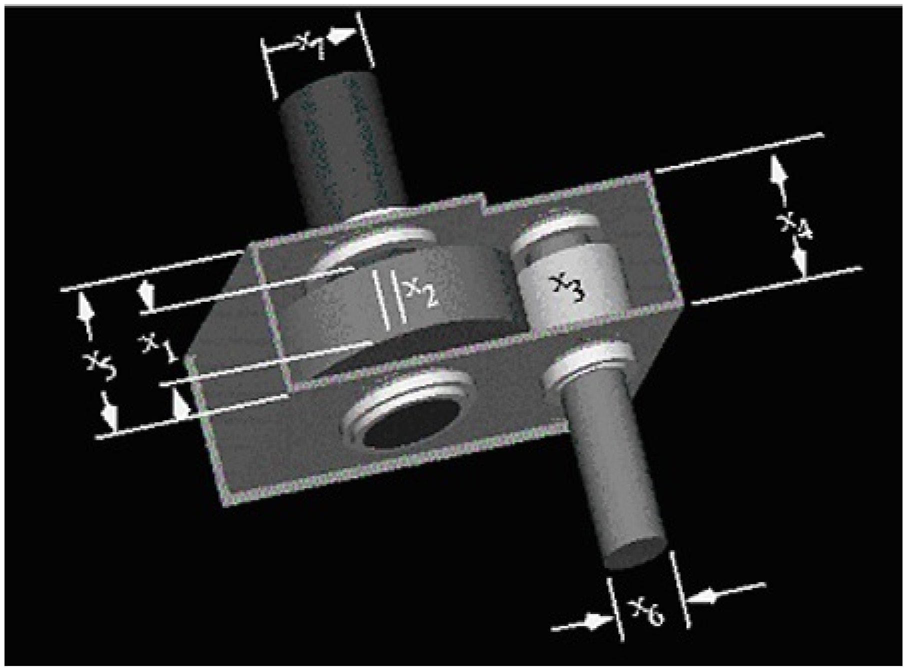

Figure 4.7: Schematic of a speed reducer[25]

Golinski[26], in 1969 used the following speed reducer problem with 7 design variables and 13 constraints which was later extended by him in 1973[27] to 25 constraints. More constraints were placed on the parameters from experience and partly due to space limitations. Most of the optimization techniques couldn't find an optimal solution to this problem satisfying all the constraint equations. Hence, this gear reducer problem is generally used as one of the testing platforms for new optimization methods. The seven design variables are

x_1	$- b -$ **face width**, (cm)
x_2	$- m -$ **teeth module**, (cm)
x_3	$- z -$ **number of pinion teeth**
x_4	$- l_1 -$ **shaft-length 1(between bearings)**, (cm)
x_5	$- l_2 -$ **shaft-length 2(between bearings**, (cm)
x_6	$- d_1 -$ **shaft diameter 1**, (cm)
x_7	$- d_2 -$ **shaft diameter 2**, (cm).

The values of the other parameters are assumed as

Transmitted power	$N = 100$ km
Pinion speed	$n = 1500$ 1/min
Transmission ratio	$i = 3$
Permissible bending stress of gear teeth	$k_g = 900$ kg cm^{-}2
Permissible surface compressive stress	$p_d = 5800$ kg cm^{-}2
Permissible bending stress(for shaft 1)	$k_{g1} = 1100$ kg cm^{-}2
Permissible bending stress(for shaft 2)	$k_{g2} = 850$ kg cm^{-}2
Tooth form factor	$q = 2.54$
Modulus of elasticity	$E = 30$Mpsi
Elastic co-efficient	$B = 1400785$ kg$^{0.5}$ cm^{-}1.

km is unit of horsepower in Poland equivalent to 735.5Watt. A transmission ratio of 3 implies that the number of teeth on the gear is three times the number of teeth on the pinion. This relation is used to simplify one of the constraint equations.

4.4.2 Optimal solution without uncertainty

The objective function is the combined volume of gear and pinion, shafts and the casing which is simplified as

$$f(x) = 0.7854x_1x_2^2(3.3333x_3^2 + 14.9334x_3 - 43.0934) - 1.508x_1(x_6^2 + x_7^2) \\ +7.477(x_6^3 + x_7^3) + 0.7854(x_4x_6^2 + x_5x_7^2). \tag{4.17}$$

It has to be noticed that the objective function is independent of the parameters assumed and depends only on the seven design variables.

Constraints:

1. Bending condition

$$\sigma_g = \frac{2Mq}{bm^2z} \leq k_g, \quad M = N/(2\pi n) \tag{4.18}$$

2. Compressive stress limitation

$$P_s^2 = \frac{2BM}{m^2z^2b} \leq P_d^2 \tag{4.19}$$

3. Minimum number of pinion teeth $z \geq 17$

4,5. Relative face width conditions $5 \leq \frac{b}{m} \leq 12$

6. Overall dimensions condition $m(z_1 + z_2) \leq 160$

This constraint is later simplified as $mz \leq 40$, since it is given that $z_2 = 3z_1$.

7. Transverse deflection of shaft 1 due to load P

$$f_1 = \frac{Pl_1^3}{48EI_1} \leq 0.001 \tag{4.20}$$

P = 2M/(pitch diameter) i.e., P = 2M/(mz)

8. Transverse deflection of shaft 1 due to load P

$$f_1 = \frac{Pl_2^3}{48EI_2} \leq 0.001 \tag{4.21}$$

9. Substitute stress condition for shaft 1

$$\sigma_{g1} = \frac{M_{z1}}{W_{x1}} \leq k_{g1} \tag{4.22}$$

10. Substitute stress condition for shaft 2

$$\sigma_{g2} = \frac{M_{z2}}{W_{x2}} \leq k_{g2} \tag{4.23}$$

11. Dimension condition $d_2 \geq 5cm$

12,13. Design conditions $1.5d_1 + 1.9 \leq l_1$, $1.5d_2 + 1.9 \leq l_2$

The constraints on the design variables later developed are

$$2.6 \leq b \leq 4.4$$

$$0.7 \leq m \leq 0.8$$

$$z \leq 28$$

$$7.3 \leq l_1 \leq 8.3$$

$$7.3 \leq l_2 \leq 8.3$$

$$2.8 \leq d_1 \leq 3.9$$

$$d_2 \leq 5.5$$

In the above constraint equations, σ_g denotes actual bending stress of gear teeth, P_s denotes actual surface compressive stress, B denotes elastic coefficient dependent on modulus of elasticity, P denotes transmitted force, M_g, M_s denote bending and torsional moments for shafts 1 & 2 and, W_x denotes the strength section modulus. The maximum bending moment(M_z) and section modulus are defined as

$$
\begin{aligned}
M_z &= \sqrt{M_g^2 + 0.75 M_s^2} \\
W_{x1} &= \pi d_1^3/32 \\
W_{x1} &= \pi d_2^3/32.
\end{aligned}
\tag{4.24}
$$

Without any uncertainty in the parameters, and with an initial point (2.6, 0.7, 17, 7, 7, 2.9, 5) the optimal volume obtained is 2815.3 (cm^3). The final set of design variables obtained is given in Table 4.4. This initial point is taken from the work of Azarm and Li [28] (1989) involving Multi-level design optimization, and our optimal set of design variables closely match their result.

Table 4.4: Optimal set of design variables without uncertainty

b	m	z	l_1	l_2	d_1	d_2	$V(cm^3)$
3.5	0.7	17	7.3	7.3	3.3563	5.0	2815.3

4.4.3 Robust Counterpart

Now, consider uncertainty in the Power transmitted(N) and Pinion speed(n). As there are no equality constraints and state variables, the robust counterpart reduces to the form

$$
\begin{aligned}
&\min \quad f(x) \\
&s.t. \quad \mathrm{diag}(G)E \pm \tau(G_s) \leq 0.
\end{aligned}
\tag{4.25}
$$

0.0001% of uncertainty in both Power transmitted and Pinion speed gives an optimum volume of 2924.4 (cm^3). The final set of design variables obtained is given in Table 4.5.

Table 4.5: Optimal set of design variables with 0.0001% uncertainty

b	m	z	l_1	l_2	d_1	d_2	$V(cm^3)$
3.5	0.7	17	7.3	7.3	3.738	5.0	2924.4

0.00011% of uncertainty in the parameters gives an optimum volume of 2944.3 (cm^3) and the corresponding design variables are given in Table 4.6.

Table 4.6: Optimal set of design variables with 0.00011% uncertainty

b	m	z	l_1	l_2	d_1	d_2	$V(cm^3)$
3.5	0.7	17	7.3	7.3	3.7999	5.0	2944.3

But, there are hardly any instruments in the world which can identify a 0.0001% change in the parameters. So, we consider a rather practical value of uncertainty(say 0.01%), and see how the optimal solution is affected. Table 4.7 shows the optimum set of design variables when there's an uncertainty of 0.01% in the parameters.

Table 4.7: Optimal set of design variables with 0.01% uncertainty

b	m	z	l_1	l_2	d_1	d_2	$V(cm^3)$
3.834	0.7963	17	6.9584	7.2515	4.5905	5.4284	4204.2

From the above results, it can be said that

- The optimal solution is extremely sensitive to even very small uncertainty in parameters.
- As the uncertainty in the parameters increases, the value of the optimum solution increases and vice-versa.
- A constraint involving d_1, is always being active, as it is the only design variable changing with uncertainty in parameters(here).

4.4.4 Factor of Safety

The age old method of designing mechanical structures/systems is to design them for a load much higher than what they are normally intended to carry. For example, a beam which has to normally withstand a load of 10KN, is designed with a factor of safety of 1.2 so that it could carry up to 12KN. This method may not be applicable/profitable in the following cases:

- When there are two parameters that oppositely affect the failure criteria.
- When there's a restriction on the material to be used.

We shall use the present case study to understand why design using Factor of Safety is not always recommended. Let's consider the speed reducer problem, and see what would happen if the Power transmitted and Pinion speed are increased/decreased by 0.01%, and the set of design variables obtained from normal optimization is used as the starting point. Table 4.8 shows the maximum deviation of constraints.

Initial point - (3.5, 0.7, 17, 7.3, 7.3, 3.5193, 5.0689)

Table 4.8: Maximum deviation of constraints under increased power and pinion speed

Power(km) Pinion speed($1/min$)	Max.deviation of constraints
100.01, 1500.15	3.214 e-03
99.99, 1499.85	3.692 e-03
100, 1500	0.431

The maximum deviation in the constraints in both the extreme cases is almost negligible. Does it mean that this solution is feasible for all the points in between them? In engineering examples like this, the failure criterion lies somewhere in between the extremes and it is impossible to find it out by guess work or trial & error. This example explains why solutions obtained using factor of safety could go completely wrong and become infeasible. In order to check the feasibility at each and every point, and keep looking for a better solution, a strong optimization technique with emphasis on mathematical principles like continuity is required, which is what robust optimization is.

Chapter 5 Summary and Conclusions

The evolution of different optimization methodologies and their advantages/ disadvantages are studied. As most of the real-life engineering problems are nonlinear, and contain uncertain data, Zhang's(2007) robust nonlinear optimization technique is given much attention. The ability of this technique to solve both linear and nonlinear uncertain optimization problems, with equality and inequality constraints gives it an edge over the traditional optimization methods. Four engineering case studies are taken up, to clearly explain how the Robust Optimization method could be implemented. Finally, the advantage of a mathematically proven optimization technique over traditional design methods such as factor of safety is discussed in the last case study.

The four case studies considered are the design of a two-member frame subjected to out of plane loads, spring mass damper system, reactor-separator system and a speed reducer. In all these case studies, it should be noted that the robust formulations developed are valid only in a neighborhood of the nominal value of the uncertain parameter. It means that the success of the robust solution obtained, is dependent on the estimate of the uncertain parameter and the magnitude of its variation.

In the first case study, the two-member frame has to be designed with stress and size limitations as constraints. Volume of the frame has to be minimized against uncertainty in parameters. Uncertainty is considered in the load applied on the members, and the modulus of elasticity of the material of the members. This is a basic nonlinear optimization problem with no equality constraints and state variables. The robust solution obtained is greater than equal to the solutions obtained from all the combinations of load and modulus of elasticity in the given neighborhood.

The design of the spring-mass-damper system is different from the first case study as it has a state variable in the form of displacement of the mass. Here, the maximum acceleration of the system over a period of time needs to be minimized, when subjected to an initial velocity. The robust solution is a set of the spring constant and damping coefficient which minimize the objective function. The fact that damping needs to be maximum to get the least acceleration possible, is proven once again with robust solution of this case study.

The presence of both equality constraints and state variables, makes the design of reactor-separator system different from the first two case studies. It is shown that the split we choose between the variables affects the robust solution, when no clear distinction as state and control variables is present. The split chosen should be in such a way that the state variables equal the number of equality constraints, and the Jacobian of equality constraints with respect to state variables is non-singular. When all the criteria are met, the split that gives the most conservative solution is recommended.

Golinski's speed reducer(1970) problem, which forms the basis of the final case study, is used to differentiate the design using robust optimization techniques and design using factor of safety. Uncertainty is considered in the power supplied and the pinion speed, against which the volume of the speed reducer has to be minimized. This is a peculiar case of nonlinear optimization problem, as the optimal solution is extremely sensitive to the uncertainty in parameters. Robust Optimization method helped solve this problem satisfying all the constraints, which has been a difficult task for many other optimization methods.

It is proved that Factor of Safety(FOS), which is basically the design using extreme load condition doesn't always give feasible and practical solutions. The mechanical design and manufacturing industry still largely depends on the FOS models when it comes to uncertainty in the parameters, where considerable increase in the productivity and quality could be achieved using robust optimization techniques. They could also be hugely helpful in designing mechanical systems that are both robust and flexible within a range of varying parameters.

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Appendix

Data required for the Out of Plane loading problem (Case study 1).

E = modulus of elasticity, (3.0E+07) psi

L = member length, 100 in

G = shear modulus, (1.154E+07) psi

P = load at node 2, -10000 lbs

I = moment of inertia = $\frac{dh^3-(d-2t)(h-2t)^3}{12},\ in^4$

J = polar moment of inertia = $\frac{2t(d-t)^2(h-t)^2}{d+h-2t},\ in^4$

A = area for calculation of torsional shear stress = $(d-t)(h-t),\ in^2$

$$\tau = \frac{T}{2At}, \qquad \text{psi}$$

$$\sigma_1 = \frac{M_1 h}{2I}, \qquad \text{psi (bending stress at end 1)}$$

$$\sigma_2 = \frac{M_2 h}{2I}, \qquad \text{psi (bending stress at end 2)}$$

$$T = \frac{-GJU_3}{L}, \ \text{lb-in}$$

$$M_1 = \frac{2EI(-3U_1+U_2L)}{L^2}, \ \text{lb-in(moment at end 1)}$$

$$M_2 = \frac{2EI(-3U_1+2U_2L)}{L^2}, \ \text{lb-in(moment at end 2)}$$

$$\frac{EI}{L^3}\begin{pmatrix} 24 & -6L & 6L \\ -6L & (4L^2 + \frac{GJL^2}{EI}) & 0 \\ 6L & 0 & (4L^2 + \frac{GJL^2}{EI}) \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} P \\ 0 \\ 0 \end{pmatrix}$$