Vector Space Concept

Orthogonality :- The inner/ dot ~~vectors~~ product b/w 2 vectors must be zero.

$\langle v_1 \cdot v_2 \rangle = 0$

Orthonormality :- Orthogonal plus the vector components must have unit norm (magnitude = 1)

Requirement of orthogonality & orthonormality in digital communication :

→ Helps to generate the idea of proper signal mapping over space.

→ If the information bearing signals are orthogonal, it is easy to analyze the performance of those information modulation & demodulation ~~....~~ schemes

→ Orthogonality over a symbol duration helps to achieve efficient demodulation scheme.

Signal Space Concept

$\left.\begin{array}{l} x_1(t) \\ x_2(t) \end{array}\right\}$ Orthogonality check

$$\langle x_1(t), x_2(t) \rangle = \int_a^b x_1(t)\, x_2^*(t)\, dt = 0$$

$[a, b]$ limit of the signal, $*$ → complex conjugate.

Orthonormality → Norm check $= ||x(t)|| = \left| \int_a^b |x(t)|^2 dt \right|^{1/2}$

For real valued functions $\langle f_1(x), f_2(x) \rangle = \int_{-\infty}^{\infty} f_1(x) f_2(x) dx$

$$||f(x)|| = \left| \int_{-\infty}^{\infty} |f(x)|^2 dx \right|^{1/2} = 1$$

Basis Function - Minimum no. of function that are required to represent any given signal over space.

Basis functions are orthogonal to each other & .

independent to each other. None of the basis func can be derived by the linear combination of any other basis function.

Date:
Page No.

$\phi(t) = \{\phi_1(t)\ \phi_2(t) \cdots \phi_i(t)\ \phi_j(t) \cdots \phi_n(t)\}$

For basis :- $\int_{-\infty}^{\infty} \phi_i(t)\, \phi_j^*(t)\, dt = 0$

$\phi_i(t) \neq a_1\phi_1(t) + a_2\phi_2(t) \cdots a_j\phi_j(t) \cdots a_n\phi_n(t)$

Gram Schmidt Orthogonalization
(GSO)

$s_i(t) = \{s_1(t), s_2(t) \cdots s_M(t)\}$ Signal Set

$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) + \cdots s_{1n}\phi_n(t)$

$s_M(t) = s_{M1}\phi_1(t) + s_{M2}\phi_2(t) + \cdots s_{Mn}\phi_n(t)$

Orthogonality check $= \int_a^b f_m(x) f_n^*(x)\, dx$ for $m \neq n$

For orthonormality of real valued functions:-

$\Rightarrow \left[\int_a^b |x(t)|^2 dt\right]^{1/2} = 1$

We can also write $\Rightarrow S_i(t) = \sum_{j=1}^{N} S_{ij}\, \phi_j(t)$ where, $i = 1, 2 \cdots M$

S_{ij} = Scalar coefficient

$S_{ij} = \frac{1}{k_j}\int_a^b S_i(t)\,\phi_j(t)\, dt$ $\quad k_j$ = norm of $\phi_j(t)$

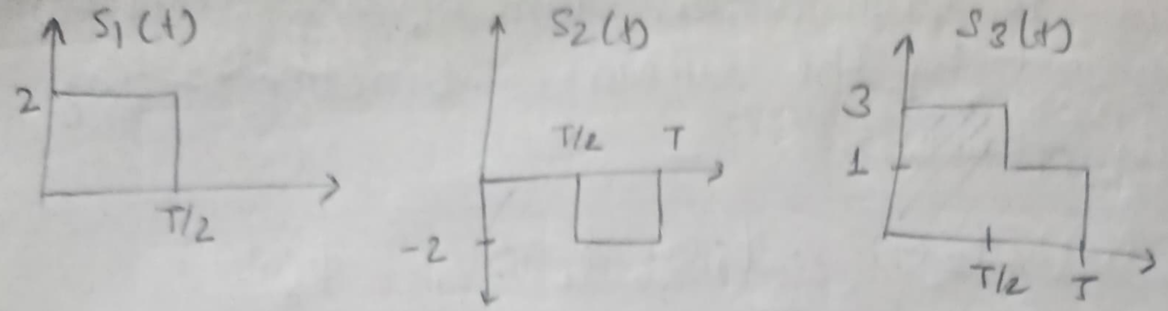
$f_1(t) = 2t+3$; $f_2(t) = 45t^2 + 9t - 17$ $\quad -1 \leq t \leq 1$

$\int_{-1}^{1} f_1(t) f_2(t)\, dt \Rightarrow \int_{-1}^{1} (2t+3)(45t^2+9t-17)\, dt$

$= \int_{-1}^{1} (90t^3 + 18t^2 - 34t + 135t^2 + 27t - 51)\, dt$

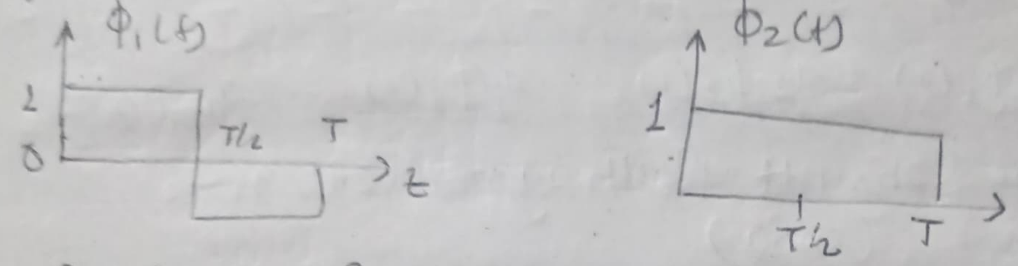
$\Rightarrow \int_{-1}^{1} (90t^3 + 153t^2 - 7t - 51)\, dt \Rightarrow \left| \frac{90t^4}{4} + \frac{153t^3}{3} - \frac{7t^2}{2} - 51t \right|_{-1}^{1}$

$= 51(2) - 51(2) = 0$

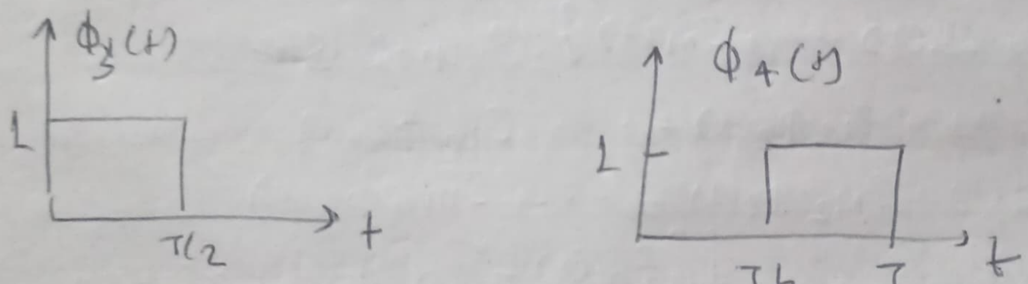


$S(t) = \{S_1, S_2, S_3\}$

$\Phi_A = \{\phi_1(t), \phi_2(t)\}$



$\Phi_B = \{\phi_3(t), \phi_4(t)\}$



Check orthogonality

$\phi_1(t)$ & $\phi_2(t)$ form a basis set

Express $S_i(t)$ in terms of $\phi_1(t)$, $\phi_2(t)$ ←

Verify $\phi_3(t)$, $\phi_4(t)$ form basis set

Express $S_i(t)$ in terms of $\phi_3(t)$ & $\phi_4(t)$

S_1 & $S_2 \rightarrow 0$; S_1 & S_3, S_2 & $S_3 \rightarrow$ NO

$S_i(t)$ is not orthogonal

Φ_A forms basis ~~for T = 1~~

Φ_B forms basis ~~for T = 2~~

$S_1(t) = \phi_1(t) + \phi_2(t)$

$S_2(t) = \phi_1(t) - \phi_2(t)$

$S_3(t) = \phi_1(t) + 2\phi_2(t)$

Analytical Method :-

$S_1(t) = \phi_3(t)/2$

$S_2(t) = -2\phi_4(t)$

$S_3(t) = 3\phi_3(t) + \phi_4(t)$

$s_{ij} = \frac{1}{k_j}\int s_i(t)\phi_j(t)\,dt$ Calculation Method:-

$s_{11} \Rightarrow \frac{1}{k_1}\int s_1(t)\phi_1(t)\,dt = T/k_1$

$k_1 = \int_0^T 1\,dt = T$ $\quad s_{11} = T \times 1/T = 1$

$s_{12} = \frac{1}{k_2}\int s_1(t)\phi_2(t)\,dt = T/k_2 = 1/T = 1$

$k_2 = \int_0^T 1^2\,dt = T$

Gram Schmidt Orthogonalisation

If a signal set $s_i(t)$, $i = 1, 2 \dots M$; can be represented by a basis set $\phi_j(t)$ where $j = 1, 2, \dots N$; where $N \le M$.

All the basis function are orthogonal / orthonormal to each other.

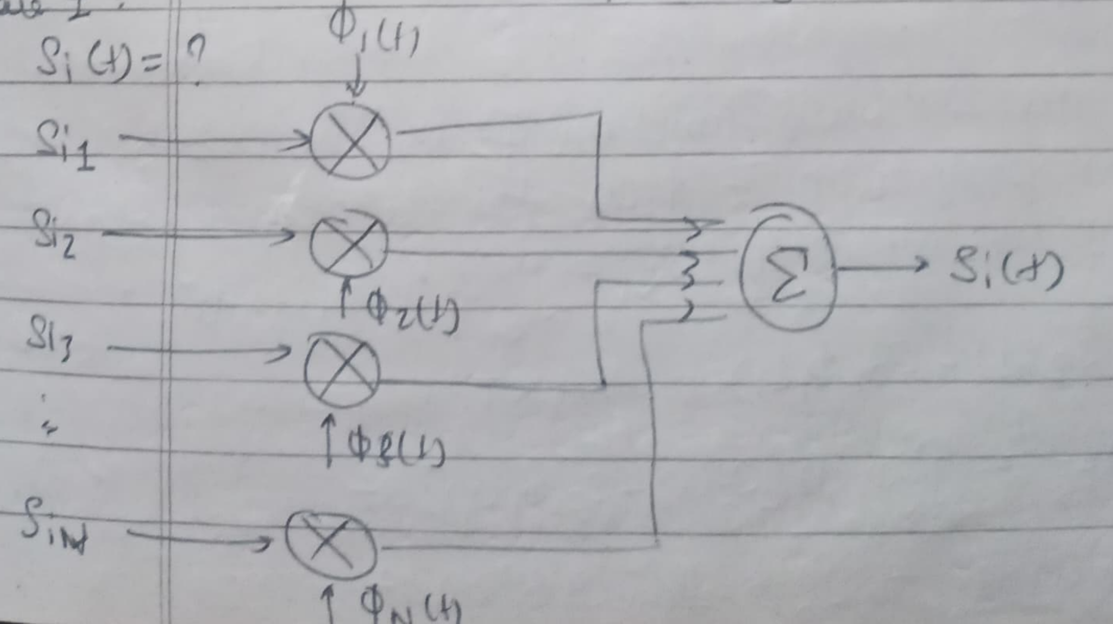
$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \quad ; \quad i = 1, 2 \dots M$$

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt \quad [k_j = 1 \text{ (Orthonormal)}]$$

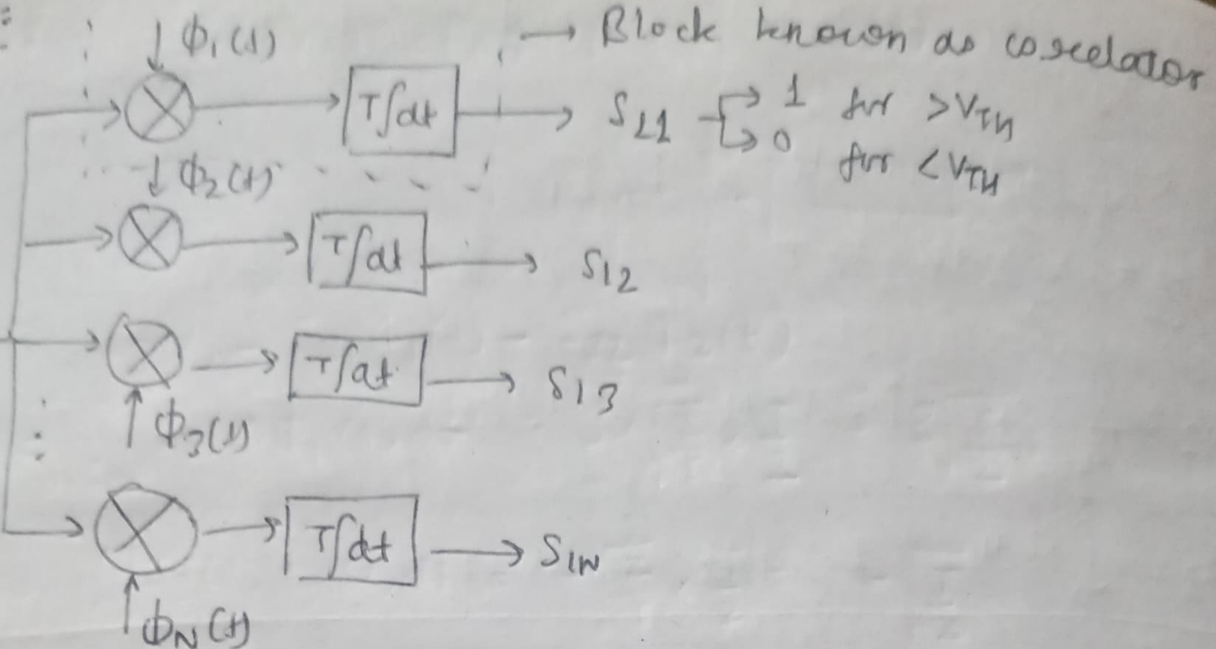
$$\int_0^T \phi_i(t)\phi_j(t)\,dt = \begin{cases} 0 & \text{for } i \ne j \\ 1 & \text{for } i = j \end{cases}$$

Case I:

$s_i(t) = ?$



Case 2:



$$\int_0^T S_i(t)\times\phi_1(t)\,dt = \int_0^T (S_{11}\phi_1(t) + S_{12}\phi_2(t) \dots S_{1N}(t)\phi_N(t))\,\phi_1(t)\,dt$$

$$= \int_0^T S_{11}\phi_1^2(t)\,dt + \int_0^T S_{12}\phi_1(t)\phi_2(t)\,dt + \dots \int_0^T S_{1N}\phi_1(t)\phi_N(t)\,dt$$

$$= S_{11}\left[\int_0^T \phi_1^2(t)\,dt = 1\right] + 0 + 0 \dots \quad \int_0^T \phi_4(t)\phi_M(t)\,dt = 0 \text{ for } N \neq M$$

Case 3: $\phi_j(t) = ?$

for $S_i(t) = \{S_1(t)\}$ $\Rightarrow$ $\phi_1(t) = S_i(t)/\sqrt{E}$

$E = \int S_1^2(t)\,dt \Rightarrow S_1(t) = \phi_1(t)\sqrt{E}$

$S_1(t) = S_{11}\phi_1(t)$

$= S_{11}\sqrt{E}$ $\quad(\sqrt{E} = \phi_1(t))$

To compute the 2nd basis function $\phi_2(t)$, we need to compute the projection of $\phi_1(t)$ onto the second signal waveform $S_2(t)$ which is $S_{21} = \int_0^T S_2(t)\,\phi_1(t)\,dt$

Let $\phi_2'(t) = S_2(t) - S_{21}\phi_1(t)$; $0 < t < T$

$$\int_0^T \phi_1(t)\,\phi_2'(t)\,dt = \int_0^T \phi_1(t)\left[S_2(t) - S_{21}\phi_1(t)\right]dt$$

$$= \int_0^T S_2(t)\phi_1(t)\,dt - \int_0^T S_{21}\phi_1^2(t)\,dt$$

$\Rightarrow S_{21} - S_{21} = 0$ Energy of $\phi_2'(t) = \int_0^T |\phi_2'(t)|^2 dt$

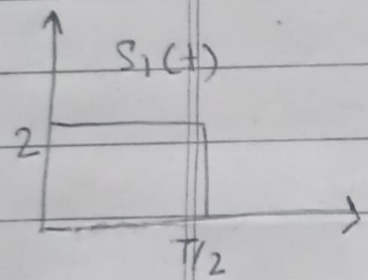
$\Rightarrow \int_0^T |S_2(t) - S_{21}\phi_1(t)|^2 dt = \int_0^T S_2^2(t) dt - 2S_{21}\int_0^T S_2(t)\phi_1(t) dt$
$+ S_{21}^2 \int_0^T \phi_1^2(t) dt$

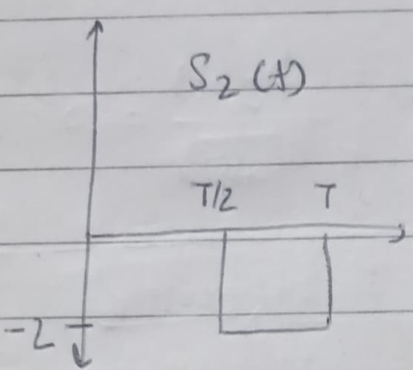
$= E_2 - 2S_{21}^2 + S_{21}^2 = E_2 - S_{21}^2$; $\phi_2(t) = \dfrac{\phi_2'(t)}{\sqrt{\text{Energy of } \phi_2'(t)}}$

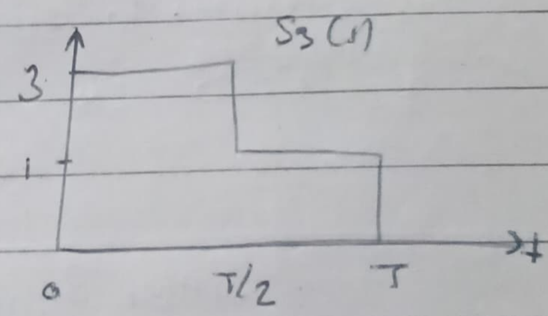
$\phi_2(t) = \dfrac{S_2(t) - S_{21}\phi_1(t)}{\sqrt{E_2 - S_{21}^2}}$

$\phi_i'(t) = S_i(t) - \sum_{j=1}^{i-1} S_{ij}\phi_j(t)$

$\phi_3'(t) = S_3(t) = S_{31}\phi_1(t) - S_{32}\phi_2(t)$



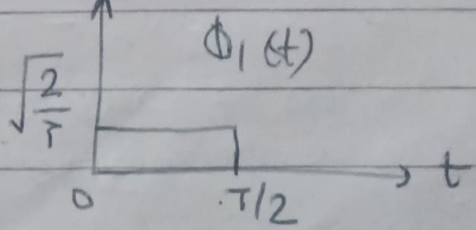




Find the basis functions for the above signal set

$\phi_1(t) = S_1(t)/\sqrt{E_1}$; $E_1 = \int_0^T S_1(t)^2 dt = 2T$

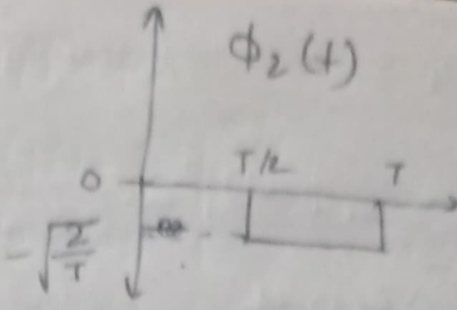
$\phi_1(t) = \sqrt{\dfrac{2}{T}}$ for $0 < t < T/2$



$\phi_2'(t) = S_2(t) - S_{21}\phi_1(t)$ $\therefore \phi_2'(t) = S_2(t)$

$S_{21} = \int_0^T S_2(t)\phi_1(t) dt = 0$ $\phi_2(t) = \dfrac{S_2(t)}{\sqrt{E_2}}$; $E_2 = 2T$

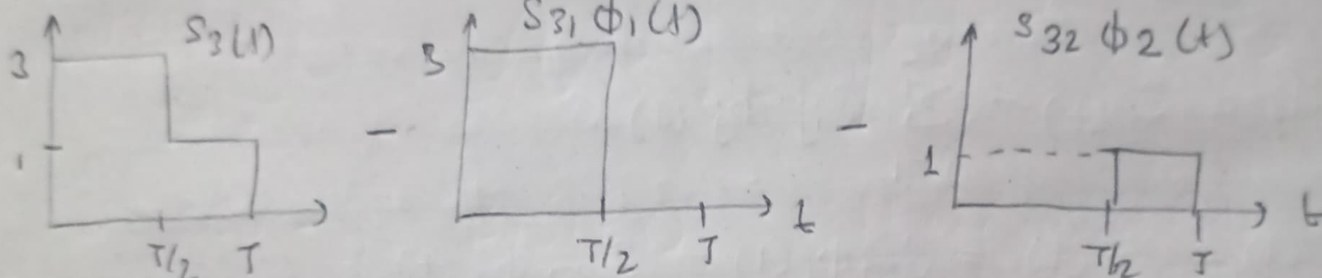
$\phi_2(t) = \frac{-\sqrt{2}}{\sqrt{T}}$ for $T/2 < t < T$



$$\phi_3'(t) = S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_2(t)$$

$$S_{31} = \int_0^T S_3(t)\phi_1(t)\,dt = 3 \times \sqrt{\frac{2}{T}} \times \frac{T}{2} = 3\sqrt{\frac{T}{2}}$$

$$S_{32} = \int_0^T S_3(t)\phi_2(t)\,dt = -\sqrt{\frac{2}{T}} \times \frac{T}{2} = -\sqrt{\frac{T}{2}}$$



This results in zero;

$\therefore \phi_3'(t) = 0$ so $\phi_3(t) = \frac{\phi_3'(t)}{\sqrt{E_3'(t)}} = 0$

Inter Symbol Interference

Channel $H(f) = |H(f)| \angle H(f)$

B_c = Coherent Bandwidth (Region of Channel Bandwidth where the channel response is static (magnitude & phase) or even if it changes, it changes linearly.

[B_t = Transmission Bandwidth

$|H(f)|$ → changes → leads to attenuation
can be overcome by amplifier

If $B_t > B_c$
↓

$\angle H(f)$ → changes → leads to shape change
to overcome is difficult

This leads to interference due to channel imperfection, due to which the magnitude & phase variations are non-linear

Date:

Page No.

$x(f) \rightarrow$ Transmitter $\rightarrow$ Channel $\rightarrow$ Receiver $\rightarrow$ $\rightarrow$ Detector

$H_T(f)$ $\quad$ $H_C(f)$ $\quad$ $H_R(f)$ $\quad$ $y(f)$

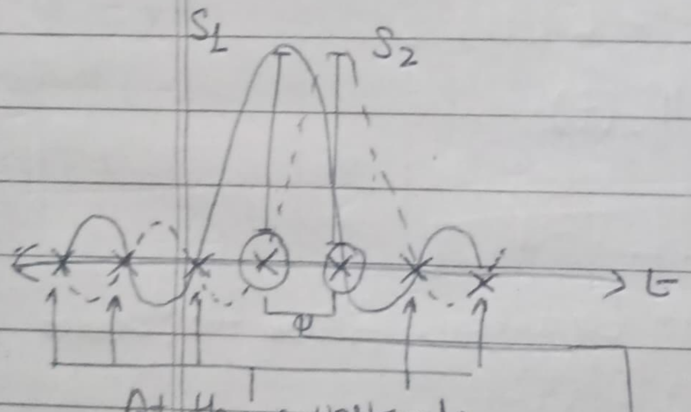
$$y(f) = X(f)\, H_T(f)\, H_C(f)\, H_R(f)$$

$p(t) \rightarrow \alpha\, p(t-\tau)$ $\quad$ $\alpha < 1 \rightarrow$ attenuation factor; $\tau \rightarrow$ Delay.

$$\sum_{k=-\infty}^{\infty} \alpha p(i-k)T_b = \alpha p(0) + \underbrace{\sum_{\substack{k=-\infty \\ k\neq i}}^{\infty} \alpha p(i-k)T_b}_{\text{Residual Pulse / ISI effect}}$$

Pulse Shaping Method:-

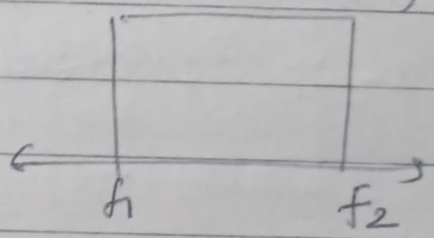
◦) Ideal Solution:



At these instants, sampling gives zero level

At these instants we get the max of 1 signal, and zero of the other

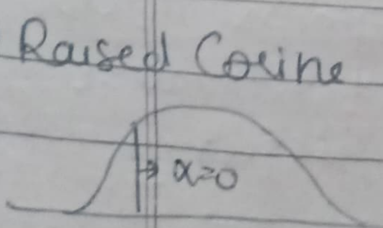
$\Longleftrightarrow$ (Nyquist filter)

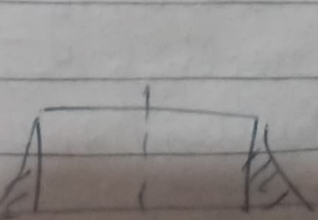


Not possible (ideal) as such sharp cut-off filters are not realisable.

(◦) Depends on exact sampling instants, slight change will cause large error.

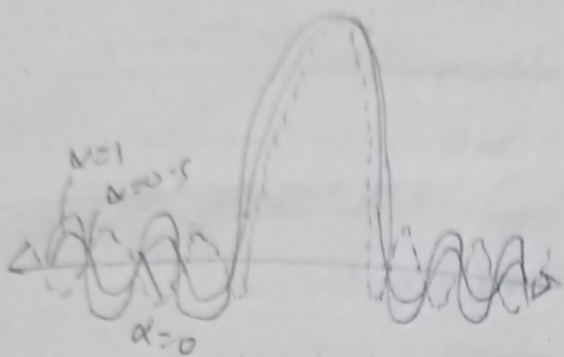
◦) Raised Cosine Filter

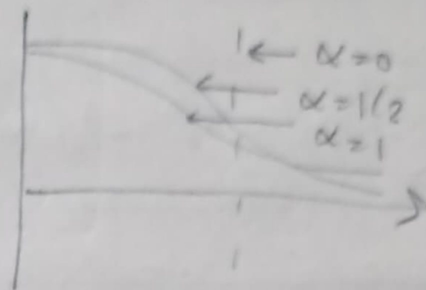




$$P(f) = \begin{cases} 1/2B_0 \; ; & 0 \le |f| < f_1 \\ \frac{1}{4B_0}\left\{1 + \cos\left[\frac{\pi(|f| - f_1)}{2B_0 - 2f_1}\right]\right\} \; ; & f_1 < |f| < 2B_0 - f_1 \\ 0 \; ; & 2B_0 - f_1 < |f| \end{cases}$$

α = (0 to 1) Roll off factor $= 1 - \frac{f_1}{B_0}$





$B = 2B_0 - f_1 = B_0(1+\alpha) = \frac{(1+\alpha)\text{ Symbol Rate}}{2}$

•) Equalizer:

Response of Equalizer $= 1/H_c(f)$

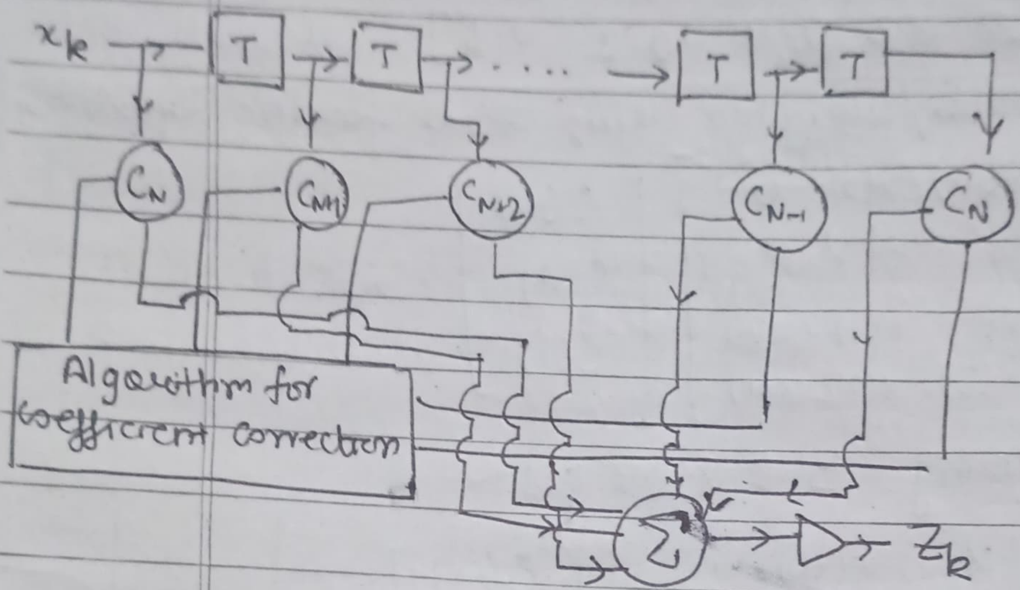
- Maximum Likelihood Sequence Estimator (MLSE)
- Equalizer filter → Preset
  - Transversal Equalizer
  - Decision feedback equalizer (DFE)
  - Adaptive

MLSE adjusts receiver to the transmission environment by measuring channel response $h_c(t)$, which enables receiver to make good estimation from the demodulated / distorted pulse sequences.

In this process the distorted samples are not reshape but the receiver adjusts itself in such a way so that it can better deal with the distorted pulses.

Ex:- Viterbi Equalizer

- Transversal Equalizer



$Z = XC$

$$\begin{bmatrix} z(-2N) \\ \vdots \\ z(0) \\ \vdots \\ z(2N) \end{bmatrix} = \begin{bmatrix} x(-N) & 0 & \cdots & 0 \\ x(-N+1) & x(-N) & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 \cdots & x(N) \end{bmatrix} \begin{bmatrix} C_{-N} \\ \vdots \\ 0 \\ \vdots \\ C_N \end{bmatrix}$$

$(4N+1)$ $\quad (4N+1) * (2N+1)$ $\quad (2N+1)$

Zero Forcing Solution:- A deterministic solution where the matrix X is transformed into $(2N+1)(2N+1)$ matrix.
By eliminating upper & lower 'N' no. of rows, we can transform X into $(2N+1) * (2N+1)$
Similarly the vector Z needs to be transformed into a vector of dimension $2N+1$
The weightage factor C_N is chosen such that it forces equalizer output to be zero at 'N' sample pts. of either side

of the desired pulse.

$$z(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

Consider a three tapped equalizer with input sample pts. $x(k) = [0 \;\; 0.2 \;\; \underset{\uparrow}{0.9} \;\; -0.3 \;\; 0.1]$. Calculate the filter coefficients/weights using zero forcing equalizer.

$$\begin{bmatrix} z(-2) \\ z(-1) \\ z(0) \\ z(1) \\ z(2) \end{bmatrix} = \begin{bmatrix} x(-1) & \cancel{x(0)} & 0 \\ x(0) & x(-1) & 0 \\ x(1) & x(0) & x(-1) \\ x(2) & x(1) & x(0) \\ x(3) & x(2) & x(1) \end{bmatrix} \begin{bmatrix} c(-1) \\ c(0) \\ c(1) \end{bmatrix}$$

By transforming z & x we get :-

$$\begin{bmatrix} z(-1) \\ z(0) \\ z(1) \end{bmatrix} = \begin{bmatrix} x(0) & x(-1) & 0 \\ x(1) & x(0) & x(-1) \\ x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} c(-1) \\ c(0) \\ c(1) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x(0) & x(-1) & 0 \\ x(1) & x(0) & x(-1) \\ x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} c(-1) \\ c(0) \\ c(+1) \end{bmatrix}$$

$$0 = x(0)c(-1) + x(-1)c(0)$$

$$1 = x(1)c(-1) + x(0)c(0) + x(-1)c(0)$$

$$0 = x(2)c(-1) + x(1)c(0) + x(0)c(1)$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 & 0 \\ -0.3 & 0.9 & 0.2 \\ 0.1 & -0.3 & 0.9 \end{bmatrix} = \begin{bmatrix} c(-1) \\ c(0) \\ c(1) \end{bmatrix}$$

$$0 = 0.9C_{-1} + 0.2C_0 \quad \Rightarrow \quad \boxed{C_{-1} = -\frac{0.2}{0.9}C_0}$$

$$1 = -0.3C_{-1} + 0.9C_0 + 0.2C_1$$

$$0 = 0.1C_{-1} + -0.3C_0 + 0.9C_1 \qquad -\frac{0.2 \times 0.1}{0.9}C_0 + 0.3C_0 = +0.9C_1$$

$$C_1 = \left(\frac{0.02 + 0.27}{(0.9)^2}\right)C_0 = \frac{0.29}{0.81}C_0$$

$1 = -0.3\left(\frac{0.29}{0.81}\right)C_0 - 0.3C_0 + 0.9\left(-\frac{0.2}{0.9}\right)C_0$

Date:

Page No.

$\Rightarrow C_0 = -1.646$

$C_0 = 0.9631$

$\Rightarrow C_{-1} = \left(\frac{0.2}{0.9}\right)\left(1.646\right) = 0.366$

$C_{-1} = 0.214$

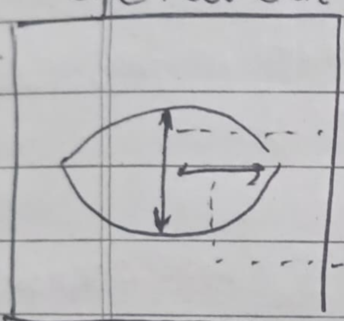
$C_1 = 0.3448$

$\Rightarrow C_1 = +0.589$

Drawbacks:

→ Finite filter length for which initial eye opening is required, (we remove some sample values from X & Z.)

Eye Pattern → feed digital data to vertical i/p of the CRO/DSO



Minimum Mean Squared Error Solution (MMSE)

$Z = XC$

$X^T Z = X^T X C$

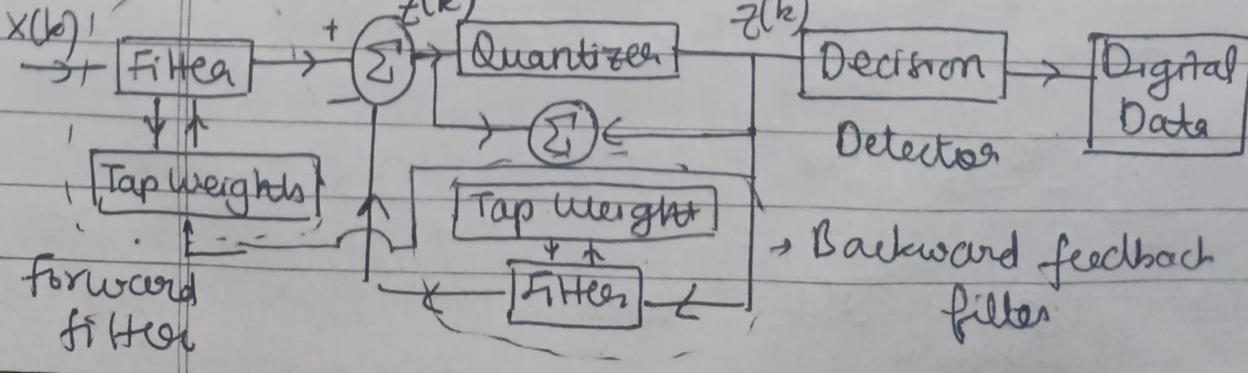
$R_{XZ} = R_{XX} C$

$\Rightarrow C = R_{XZ} R_{XX}^{-1}$

R_{XZ} = Cross correlation of X & Z

R_{XX} : Auto correlation b/w i/p.

Decision Feedback Equalizer:-



The basic idea of decision feedback equalizer is that if the values of previously detected symbols are known (past detections are correct) then the ISI contribution by these symbols can be canceled out at the o/p of forward filter by subtracting the past symbol values with appropriate weightage.

Optimum Receiving Filter :-

Noise is assumed to be AWGN
(Additive White Gaussian Noise)

Additive :- $n_1(t) + n_2(t) = n_{eq}(t)$

$(\eta/2) - \sigma^2$

White :- Uniform Power Spectral Density in freq. domain.

Gaussian :- Gaussian Distribution.

- Correlator Receiving Filter
- Matched Filter (preferred in excessive noise scenario)

$\{S_m(t)\}$ where $m = 1, 2 \cdots M$ $\quad \{\phi_n(t)\}$ where $n = 1, 2, \cdots N$

$S_1(t) = S_{11}\phi_1(t) + S_{12}\phi_2(t) \cdots S_{1N}\phi_N(t)$

$S_2(t) = S_{21}\phi_1(t) + S_{22}\phi_2(t) \cdots S_{2N}\phi_N(t)$

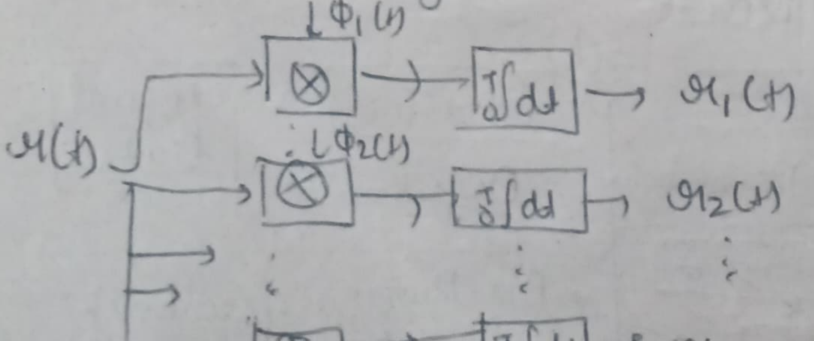
$\vdots$

$S_M(t) = S_{M1}\phi_1(t) + S_{M2}\phi_2(t) \cdots S_{MN}\phi_N(t)$

$r(t) = S_m(t) + n(t)$

$r(t) \rightarrow$ Received Signal; $S_m(t)$ = Transmitted Signal

$n(t) \rightarrow$ noise Signal



$$\int_0^T r(t)\,\phi_k(t)\,dt = \int_0^T (S_m(t)+n(t))\,\phi_k(t)\,dt$$

$$= \int_0^T S_m(t)\,\phi_k(t)\,dt + \int_0^T n(t)\,\phi_k(t)\,dt$$

$r_k(t)$: $S_{mk} + n_k(t) \rightarrow \eta/2$

$$E[r_k(t)] = E[S_{mk}] + E[n_k(t)]$$

$$0 + \eta/2$$

$$P\left[{}^{r_k}/_{S_M}\right] = \prod_{k=1}^{N} P\left[{}^{r_k}/_{S_{mk}}\right] = \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(r_k - S_{mk})^2}{2\sigma^2}\right)$$

$$= \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\eta}} \exp\left[-\frac{(r_k - S_{mk})^2}{2\sigma^2}\right]$$

$$= \frac{1}{(2\pi\eta)^{N/2}} \exp \sum_{k=1}^{N} \left[-\frac{(r_k - S_{mk})^2}{2\sigma^2}\right]$$

$$\ln\left[P\left({}^{r_k}/_{S_m}\right)\right] = -\frac{N}{2}\ln(2\pi\eta) + \sum_{k=1}^{N}\left[-\frac{(r_k - S_{mk})^2}{2\sigma^2}\right]$$

For max probability, we need $r_k - S_{mk}$ to be minimum this is nothing but the distance b/w r_k and S_{mk} represented by $D(r_k, S_{mk})$

Matched Filter:-

$s(t) \rightarrow \boxed{h(t)} \rightarrow r(t)$

$\uparrow$ $n(t)$ (AWGN).

$$r(t) = [s(t) + n(t)] * h(t)$$

$$= s(t) * h(t) + n(t) * h(t)$$

$$= s_o(t) + n_o(t)$$

$$R(f) = S_o(f) + N_o(f)$$

$$= S(f)H(f) + N(f)H(f)$$

$$S_o(f) = s(t)H(f)$$

$G_{xx}(f) \rightarrow \boxed{H(f)} \rightarrow G_{yy}(f)$

$$G_{yy}(f) = |H(f)|^2 G_{xx}(f)$$

$G_{xx}(f) = \eta/2 \Rightarrow G_{yy}(f) = \frac{\eta}{2}\int_{-\infty}^{\infty}|H(f)|^2 df$

$$SNR = \left| \frac{|\int_{-\infty}^{\infty} S(f)H(f)\, e^{j2\pi f t}\, df|^2}{\frac{\eta}{2}\int_{-\infty}^{\infty}|H(f)|^2\, df} \right|_{t=T}$$

~~Sum~~ Schwarz's Inequality :-

$$\int_{-\infty}^{\infty}|a(t)\, b(t)|^2 dt \leq \int_{-\infty}^{\infty}|a(t)|^2 dt \int_{-\infty}^{\infty}|b(t)|^2 dt$$

$\boxed{a(t) = k b^*(t)}$ → Condition for this inequality.

using the above inequality, put $H(f) = a(t)$, and $S(f)\, e^{j2\pi f t}$ as $b(t)$.

$$\text{so, } SNR \leq \left| \frac{\int_{-\infty}^{\infty}|H(f)|^2 df \times \int_{-\infty}^{\infty}|S(f)e^{j2\pi f T}|^2 df}{\frac{\eta}{2}\int_{-\infty}^{\infty}|H(f)|^2 df} \right|_{t=T}$$

$$= \left| \frac{2}{\eta}\left(\int_{-\infty}^{\infty}|S(f)\, e^{j2\pi f t}|^2 df \right) \right|_{t=T}$$

$$= \left| \frac{2}{\eta}\int_{-\infty}^{\infty}|S(f)|^2 df \right|_{t=T} \quad \text{as} \left(|e^{j2\pi f T}| = 1 \right)$$

$$= \left. \frac{2}{\eta}E \right|_{t=T} \quad \text{where } E = \int_{-\infty}^{\infty}|S(f)|^2 df$$

$$SNR_{max} = \frac{2E}{\eta} \text{ at } \boxed{t = T}$$

The purpose of matched filter is to maximize the SNR at the filter's output.

Date:

Page No.

$$a(t) = \int_{-\infty}^{\infty} |H(f)|^2 df \qquad b^*(t) = \int_{-\infty}^{\infty} |S(f) e^{-j2\pi fT}|^2 df$$

$a(t) = b^*(t)$ for $k = 1$

$$\Rightarrow |H(f)|^2 = |S(f) e^{-j2\pi fT}|^2$$

$$\Rightarrow H(f) = S(f) e^{-j2\pi fT}$$

$$\Rightarrow H(t) = \int_{-\infty}^{\infty} S(f) e^{-j2\pi fT} e^{+j2\pi ft} df$$

$$= s(t-T)$$

$$h(t) = \begin{cases} S(T-t) & \text{for } 0<t<T \\ 0 & \text{elsewhere} \end{cases}$$

The filter response is a shifted version of transmitted signal, so its a matched filter.

Signal Detector:-

$P(x)$, $P(y)$ are priori probabilities, as we can determine them before the experiment.
$P(x|y)$ are posterior probabilities, as we can't determine them until the completion of experiment.

MAP → Maximum A Posteriori
ML → Maximum Likelihood

{ Signal Detection Schemes }

Digital Signal Detection

$S_i(t) \rightarrow$ [Equalizer] $\rightarrow$ [Rx Filter] $\rightarrow$ [Detector], $z(t)$

$\left\{ \begin{matrix} S_1(t) = 1 \\ S_2(t) = 0 \end{matrix} \right\}$

Condition of Signal Detection:-

$$\boxed{P(S_1/z) \underset{H_2}{\overset{H_1}{\gtrless}} P(S_2/z)}$$

If $P(S_1/z) > P(S_2/z) \Rightarrow H_1$ hypothesis is correct i.e. S_1 is received or '1'.

If $P(S_1/z) < P(S_2/z) \Rightarrow H_2$ hypothesis is correct i.e. S_2 is received or '0'.

By application of Baye's Theorem:-

$$\Rightarrow \frac{P(z/S_1)P(S_1)}{P(z)} \underset{H_2}{\overset{H_2}{\gtrless}} \frac{P(z/S_2)P(S_2)}{P(z)}$$

$$\Rightarrow \frac{P(z/S_1)}{P(z/S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(S_2)}{P(S_1)}$$

By this equation, if $P(S_2)/P(S_1)$ ratio is more than 1, the $P(z/S_1) > P(z/S_2)$ therefore S_1 is transmitted, and vice-versa for S_2.

This mathematical model is known as MAP (Maximum A-Posteriori Technique).

Limitations:- Periori probabilities of signal must be known which is not practical. Also, the accuracy of this scheme is very less.

Maximum Likelihood Detector:- We take the probability of '1' and '0' to be equal, thus reducing the ratio of $P(S_2)$ and $P(S_1)$ to 1.

using Gaussian Distribution we get :-

$$\frac{P(z/s_1)}{P(z/s_2)} \Rightarrow \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{z-s_1}{\sigma}\right)^2\right]}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{z-s_2}{\sigma}\right)^2\right]} \underset{H_2}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow \frac{\exp\left[-\frac{z^2}{2\sigma^2} + \frac{zs_1}{\sigma^2} - \frac{s_1^2}{2\sigma^2}\right]}{\exp\left[\frac{-z^2}{2\sigma^2} + \frac{zs_2}{\sigma^2} - \frac{s_2^2}{2\sigma^2}\right]} \Rightarrow \exp\left[-\frac{z^2}{2\sigma^2} + \frac{zs_1}{\sigma^2} - \frac{s_1^2}{2\sigma^2} + \frac{z^2}{2\sigma^2} - \frac{zs_2}{\sigma^2} + \frac{s_2^2}{2\sigma^2}\right]$$

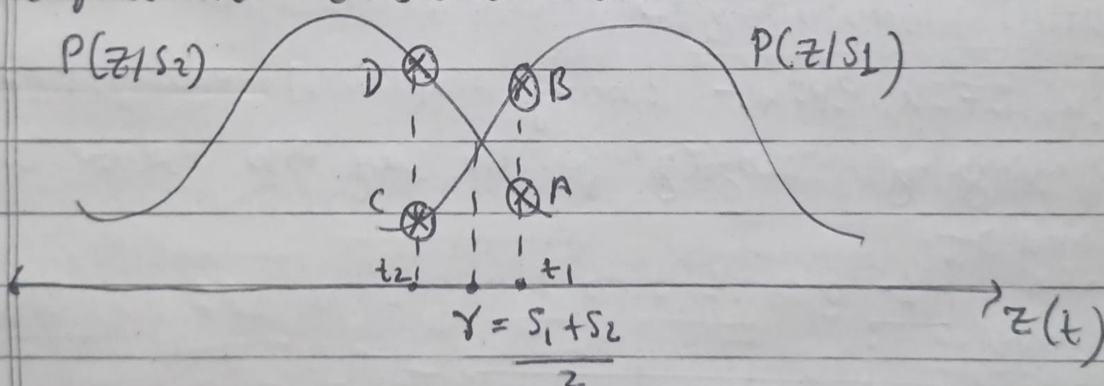
$$\Rightarrow \exp\left[\frac{z}{\sigma^2}(s_1 - s_2) - \frac{1}{2\sigma^2}(s_1^2 - s_2^2)\right] \underset{H_2}{\overset{H_1}{\gtrless}} 1$$

Taking natural logarithm on both sides :-

$$\frac{z}{\sigma^2}(s_1 - s_2) - \frac{1}{2\sigma^2}(s_1^2 - s_2^2) \underset{H_2}{\overset{H_1}{\gtrless}} 0$$

$$\Rightarrow z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{1}{2}\frac{(s_1^2 - s_2^2)}{(s_1 - s_2)} \Rightarrow z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{(s_1 + s_2)}{2} \Rightarrow z \underset{H_2}{\overset{H_1}{\gtrless}} \gamma$$

where γ is the threshold value $(s_1 + s_2)/2$



~~at $t_1 > \gamma$ A < B $\Rightarrow$ so $P(z/s_1) > P(z/s_2) \rightarrow s_1$ transmitted.~~

~~at $t_2 > \gamma$ D > C $\Rightarrow$ so $P(z/s_2) > P(z/s_1) \rightarrow s_2$ transmitted.~~

If $t = \gamma$, either the receiver discards the bit and requests for retransmission, or it takes the probabilities of '1' & '0' from previous samples and assigns the value of the max (1, 0) i.e. arbitrary selection.

Probability of Bit-Errors:-

If in $(-\infty, \gamma)$ S_1 is chosen or,
in (γ, ∞) S_2 is chosen

$$P_e = \int_{\frac{S_1+S_2}{2}}^{\infty} p(z/S_2)\,dz \Rightarrow \int_{\frac{S_1+S_2}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{z-S_2}{\sigma}\right)^2\right] dz$$

Put $\frac{z-S_2}{\sigma} = u$; $dz = \sigma\, du$

at $z = \frac{S_1+S_2}{2}$ $\quad u = \frac{S_1-S_2}{2\sigma}$

at $z = \infty$ $\quad u = \infty$

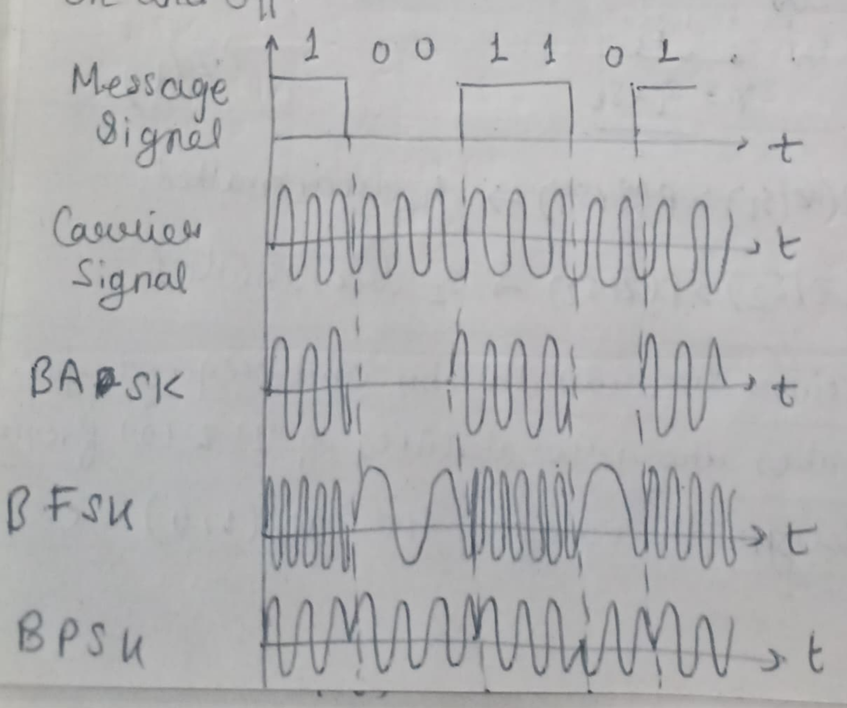
$$\Rightarrow \int_{\frac{S_1-S_2}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}u^2\right] \sigma\, du \Rightarrow P_e = \phi(x) \; ; \; x = \frac{S_1-S_2}{2\sigma}$$

where $\phi(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$

and $\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} \exp(-u^2)\,du$ [Complementary error function]

Digital Modulation :-

BASK, BFSK, BPSK - Binary (Amplitude/Frequency/Phase) Shift Keying. 'Keying' for the key like nature turning on and off.



Multiplication removes signal at '0' amplitude.

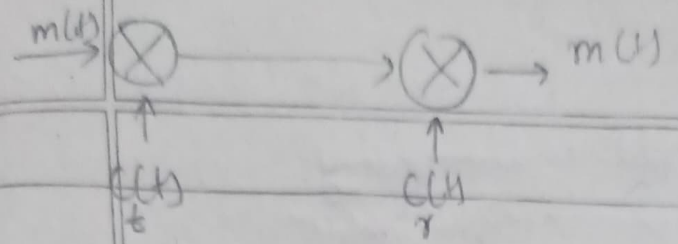
Frequency increases in '1' amplitude, decreases at '0' amplitude.

Phase change of $\frac{360}{2} = 180$ at each 1-0 or 0-1 transition. '2' is for binary.

Coherent and Non Coherent Modulations:-

Date:

Page No.



If the phase of $c_t(t)$ and $c_r(t)$ carrier signals at the modulation & de-modulation sides, then it is a coherent/synchronous modulation, else it is non-coherent / asynchronous modulation.

ASK - On/off keying.

$$S_i(t) = \begin{cases} s_1(t) = A_m \cos 2\pi f_c t & \rightarrow 1 \\ s_2(t) = 0 & \rightarrow 0 \end{cases}$$

Just one function in the signal set, so only one Basis function.

$\phi(t) = S_1(t) / \sqrt{\text{Energy of } S_1(t)}$

$$= \frac{A_m \cos 2\pi f_c t}{\sqrt{A_m^2/2 \; T_b}} \qquad \rightarrow A_m^2 \int_0^{T_b} \left(\frac{1+\cos 4\pi f_c t}{2}\right)$$

$$\rightarrow \frac{A_m^2}{2} T_b = \text{Energy}$$

$A_m = \sqrt{\dfrac{2E_b}{T_b}} \Rightarrow S_1(t) = \sqrt{\dfrac{2E_b}{T_b}} \cos 2\pi f_c t$

$\phi(t) = \sqrt{\dfrac{2}{T_b}} \cos 2\pi f_c t$

$S_1(t) = S_{11}(t)\, \phi(t)$

$S_1(t) = \sqrt{E_b}\, \phi_1(t)$

$S_{11} = \sqrt{E_b}$

Signal Space Diagram & Constellation Diagram.

$\phi_2(t)$ $\quad S_{11}$

$0 \qquad \sqrt{E_b} \qquad \rightarrow \phi_1(t)$

The receiver will plot the energy of a sampled signal at this diagram, if its closer to S_{1L} it takes $S_1(t)$ value otherwise it takes $S_2(t)$ value.

$m(t) \longrightarrow \otimes \longrightarrow s(t)$ BASK ~~Sampling~~ Waveform

UNRZ $\quad \uparrow$

$c(t) = \cos 2\pi f_c t$

Coherent Detector →

$m(t) \rightarrow \otimes \xrightarrow{\sqrt{\frac{2E_b}{T_b}} \cos^2 2\pi f_c t} \boxed{\int_0^T dt} \longrightarrow$ Unipolar NRZ

$= \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f t \quad \uparrow$

$\cos 2\pi f_c t$

$$\int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \left(\frac{1 + \cos 4\pi f_c t}{2}\right) dt \Rightarrow \sqrt{\frac{E_b T_b}{2}} > 0$$

as E_b & T_b are always +ve.

If $m(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t$

→ The receiver receives $\sqrt{\frac{E_b T_b}{2}}$ value which is +ve, so result is '1'.

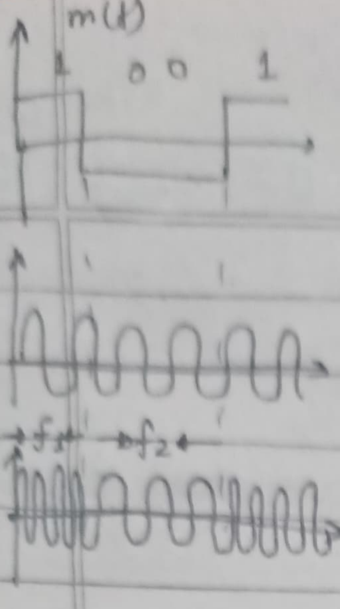
→ If $m(t) = S_2(t) = 0$, Receiver receives '0' so the result is '0'.

Envelope Detector :-

BASK Waveform → $\boxed{\text{BPF}}$ → $\boxed{\text{Rectifying Diode}}$ → $\boxed{\text{LPF}}$ → $\boxed{\text{Detector}}$ → 1 / 0

BPF ↓ Tuned at 'f_c'

Rectifying Diode + LPF ↓ Envelop Detector



$s_i(t) = A\cos(2\pi f_i t)$

Date:

$s_1(t)$ $s_2(t)$

'1' '0'

$\left.\begin{aligned} s_1(t) &= A\cos(2\pi f_1 t) \\ s_2(t) &= A\cos(2\pi f_2 t) \end{aligned}\right\}$ $\phi_1(t)$ $\phi_2(t)$

$\phi_1(t) = s_1(t) / \sqrt{\text{Energy of } s_1(t)}$

we know, $A = \sqrt{\frac{E_b \times 2}{T_b}}$

$f_1 > f_2$

$\Rightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_1 t$

$\Rightarrow \phi_1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_1 t$

$\Rightarrow \phi_2(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_2 t$

General Equations :-

$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$

$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$

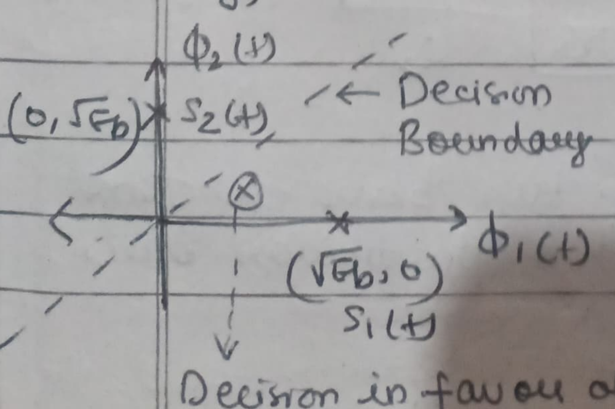
Actual Equations :-

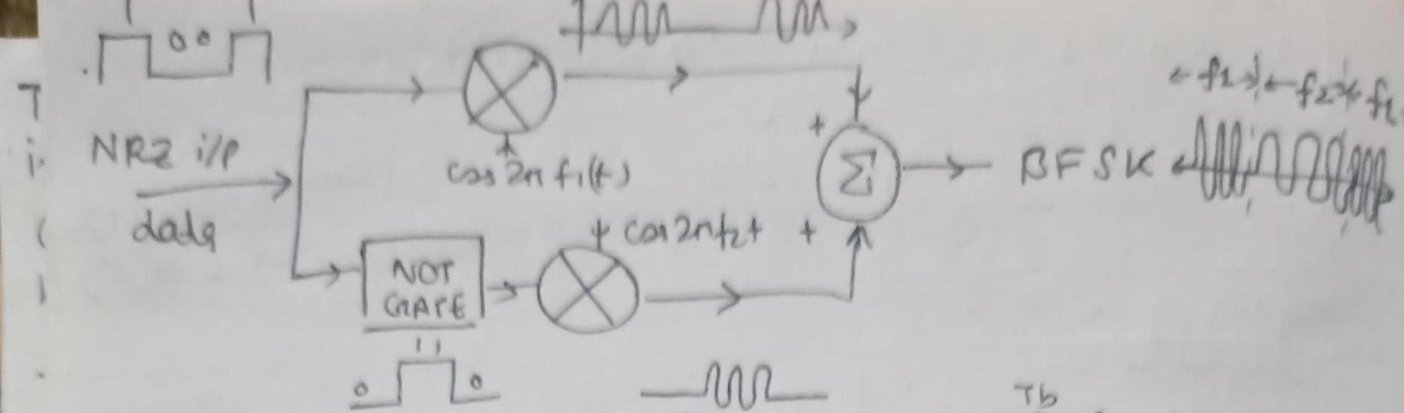
$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_1 t$; $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_2 t$

Comparing the general & actual equations we get :-

$s_{11} = \sqrt{E_b}$ and $s_{12} = 0$

Similarly, $s_{21} = 0$ and $s_{22} = \sqrt{E_b}$

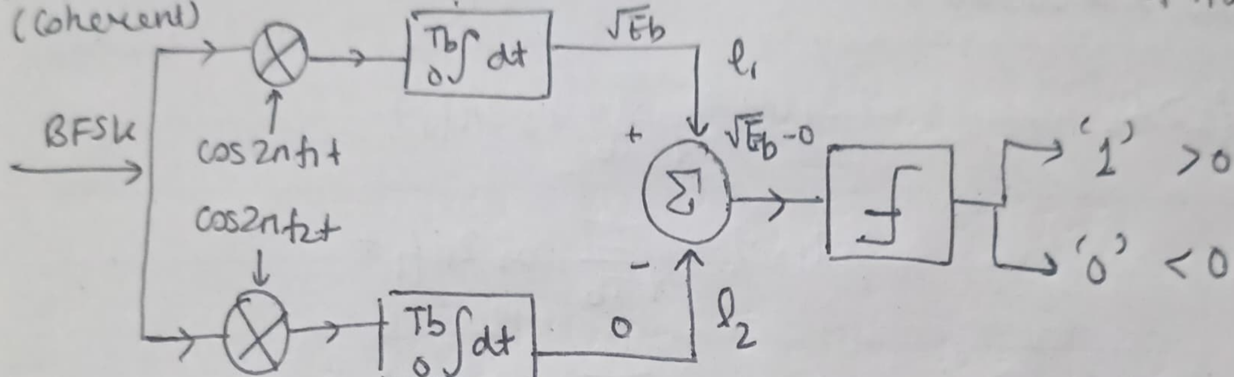




$$\int_0^{T_b} S_{11}\,\phi_1(t)\,\phi_1(t)\,dt \Rightarrow S_{11}\int_0^{T_b} \phi_1^2(t)\,dt \Rightarrow S_{11}$$

(norm of $\phi_1(t) = 1$)

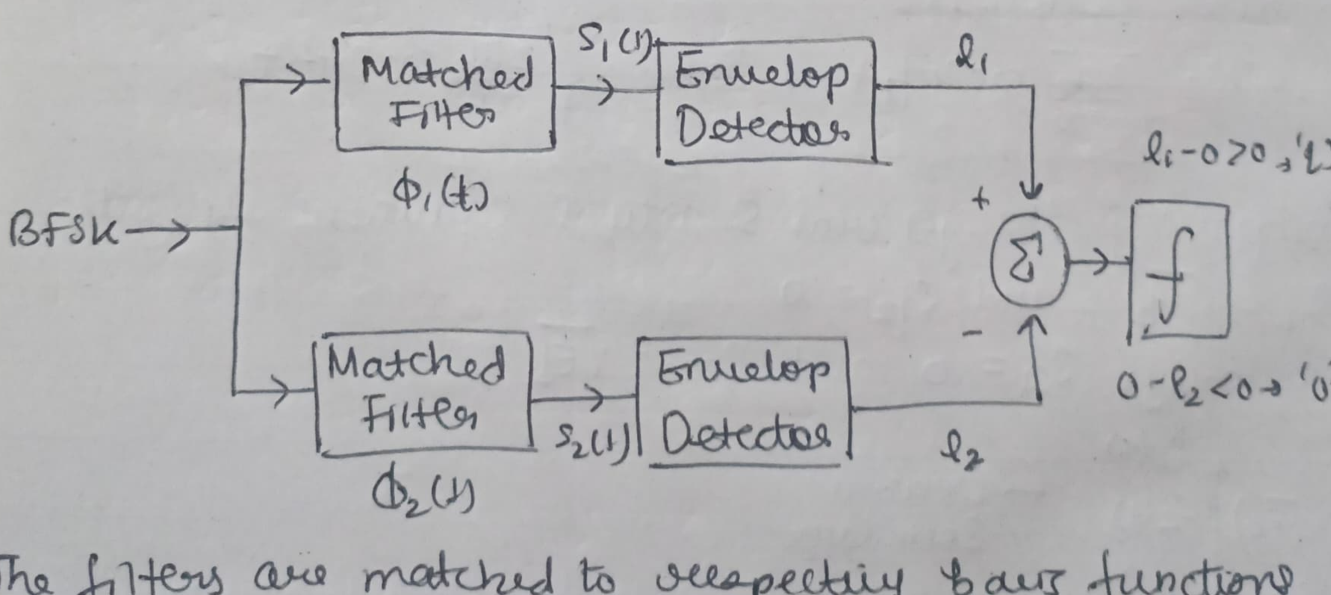
Demodulation :-
(Coherent)



$$\int_0^{T_b} S_{11}\,\phi_1(t)\,\phi_2(t)\,dt = 0$$

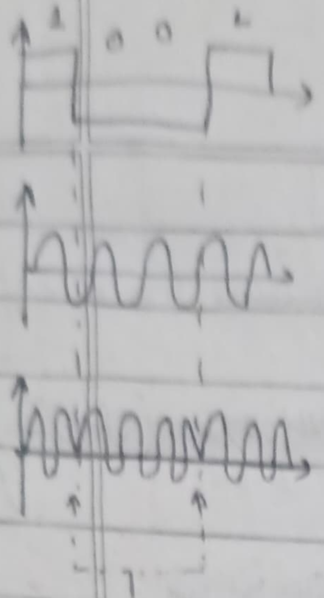
($\phi_1(t)$ & $\phi_2(t)$ are orthogonal)

Demodulator (Non Coherent) :-



The filters are matched to respective basis functions so that they may pass only that particular basis function & reject the rest.

BPSK (Binary Phase Shift Key)



$S_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \theta_i)$

$S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$ $\theta_1 = 0°$

$S_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + 180°)$ $\theta_2 = 180°$

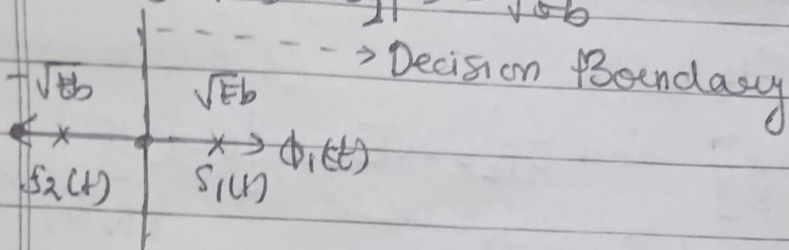
$= -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

180° phase shift

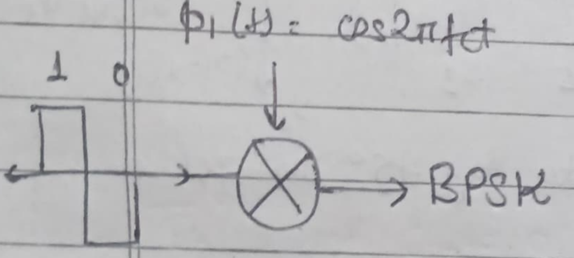
$\phi_1 = \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t$

$S_1(t) = \sqrt{E_b}\,\phi_1(t)$ $S_{11} = \sqrt{E_b}$

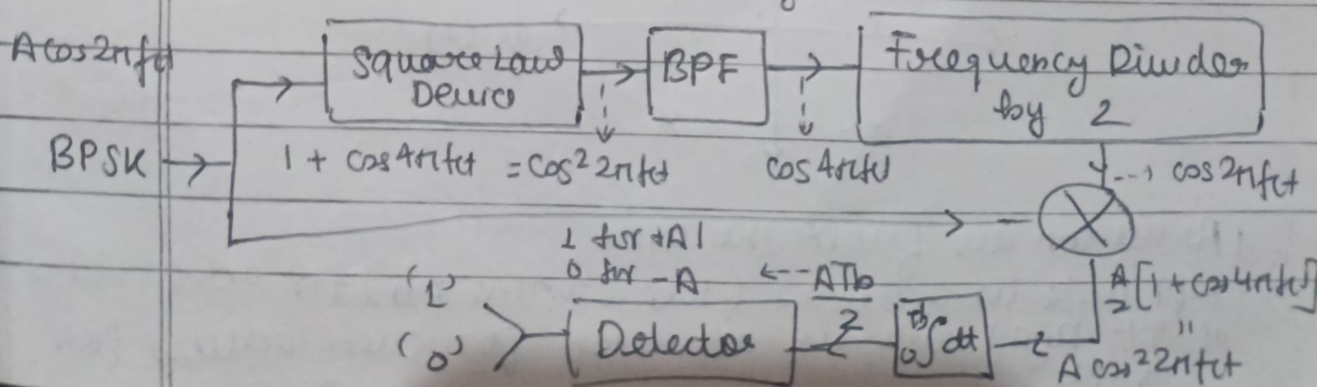
$S_2(t) = -\sqrt{E_b}\,\phi_1(t)$ $S_{21} = -\sqrt{E_b}$



In this case we take BNRZ as the input.



Carrier Phase Recovery Ckt.



Quadrature Phase Shift Keying
(QPSK)

$$S_{QPSK}(t) = \sqrt{\frac{2E_s}{T_s}} \cos[(2\pi f_c t) + (i-1)\pi/2]$$

$E_s = 2E_b$ and $T_s = 2T_b$ since each bit carries twice info

$$S_{QPSK}(t) = \sqrt{\frac{2E_s}{T_s}}\left(\cos(2\pi f_c t)\cos((i-1)\pi/2) - \sin(2\pi f_c t)\sin((i-1)\tfrac{\pi}{2})\right)$$

$$= \sqrt{\frac{2E_s}{T_s}} \cos\left[(i-1)\frac{\pi}{2}\right]\cos 2\pi f_c t - \sqrt{\frac{2E_s}{T_s}}\sin\left[(i-1)\frac{\pi}{2}\right]\sin 2\pi f_c t$$

$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) \qquad \phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t)$$

$$S_i(t) = \sqrt{E_s}\cos\left[(i-1)\frac{\pi}{2}\right]\phi_1(t) - \sqrt{E_s}\sin\left[(i-1)\frac{\pi}{2}\right]\phi_2(t)$$

$$S_i(t) = S_{i1}\phi_1(t) + S_{i2}\phi_2(t)$$

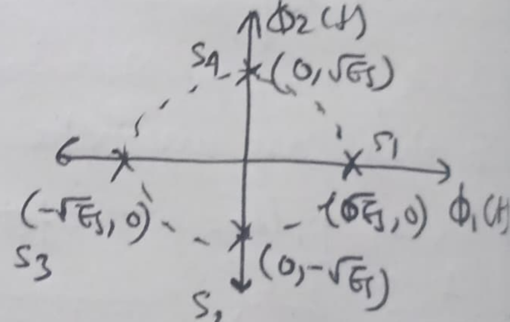
$S_{i1} = \sqrt{E_s}\cos\left[(i-1)\frac{\pi}{2}\right]$ and $S_{i2} = -\sqrt{E_s}\sin\left[(i-1)\frac{\pi}{2}\right]$

for $i=1$, $S_{11} = \sqrt{E_s}$ and $S_{12} = 0$

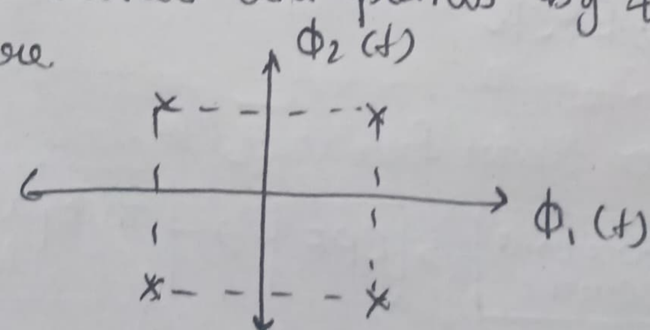
$i=2$, $S_{21} = 0$ and $S_{22} = -\sqrt{E_s}$

$i=3$, $S_{31} = -\sqrt{E_s}$ and $S_{32} = 0$

$i=4$, $S_{41} = 0$ and $S_{42} = \sqrt{E_s}$



We may also rotate the points by 45° to get a square figure



As we can see from the eqn,
Left side is activated (has a value $\neq 0$) for odd values of i. Right side is activated for even values for of (i).