AI1103: Assignment 5

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Download all latex-tikz codes from

https://github.com/Bharadwaja−rao−D/AI1103/blob/main/assignment5/assignment5.tex

Problem UGC-MATH 2019 Q 105:

Consider a simple symmetric random walk on integers, where from every state i you move to i-1 and i+1 with the probability half each. Then which of the following are true?

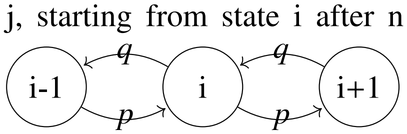
1) The random walk is aperiodic
2) The random walk is irreducible
3) The random walk is null recurrent
4) The random walk is positive recurrent

Solution:

The simple symmetric random walk is a Markov chain with state space S = $\{i|i \in \mathbb{Z}\}$ and with transition matrix P where,

$$P(i,j) = \begin{cases} 0, & |i-j| > 1 \\ p = \frac{1}{2}, & j = i+1 \\ q = 1-p = \frac{1}{2}, & j = i-1 \end{cases} \tag{0.0.1}$$

and $P^n(i,j)$ denotes the probability of being in state j, starting from state i after n steps or transitions.



0.1 Aperiodic

For a Markov chain to be aperiodic there should exist an integer k such that

$$P^n(j,j) > 0 \quad \forall n \geq k \tag{0.1.1}$$

to return to same state after n steps, number of forward and backward steps should be same, that is number of steps should be even.

$$P^n(j,j) = 0 \quad \forall n \in Odd \tag{0.1.2}$$

∴ Equation (0.1.1) is not satisfied and Option **(1)** is **incorrect**.

0.2 Irreducible

For a Markov chain to be **irreducible** all pairs i,j should communicate with each other.Let us assume that the chain starts from state i and let us assume that it requires m forward and (n-m) backward steps to reach j .Let $i < j$ wlog

$$j - i = m - (n-m) = 2m - n \tag{0.2.1}$$

$$m = \frac{(j-i)+n}{2} \tag{0.2.2}$$

$$P^n(i,j) = \binom{n}{m} p^m q^{n-m} \tag{0.2.3}$$

$$P^n(i,j) = \binom{n}{\frac{(j-i)+n}{2}} p^m q^{n-m} \tag{0.2.4}$$

$$P^n(i,j) > 0 \quad n = (j-i) + 2k \ \ \forall k \in \mathbb{W} \tag{0.2.5}$$

Here i and j are general , hence all pairs i and j communicate with each other.
∴Option **(2)** is **correct**

0.3 Recurrent

In a Markov Chain for state i to be **recurrent** it should satisfy ,

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n(i,i) = \infty \tag{0.3.1}$$

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n(i,i) = \lim_{t\to\infty} \left(\sum_{k=1}^{t} P^{2k}(i,i) + \sum_{k=1}^{t} P^{2k-1}(i,i) \right) \tag{0.3.2}$$

$$= \lim_{t\to\infty} \sum_{k=1}^{t} P^{2k}(i,i) \tag{0.3.3}$$

$$P^{2k}(i,i) = \binom{2k}{k} p^k q^k = \frac{2k!}{k!k!} p^k q^k \tag{0.3.4}$$

By using Stirling approximation to (0.3.4) we get

$$P^{2k}(i,i) = \frac{\left((2k)^{2k+\frac{1}{2}}\right) \times \exp(-2k) \times (2\pi)^{\frac{1}{2}}}{\left(k^{k+\frac{1}{2}} \times \exp(-k)\right)^2 \times 2\pi} p^k q^k \tag{0.3.5}$$

$$= \frac{(4pq)^{2k}}{(k\pi)^{\frac{1}{2}}} = \frac{1}{(k\pi)^{\frac{1}{2}}} \tag{0.3.6}$$

$$\lim_{t\to\infty} \sum_{k=1}^{t} P^{2k}(i,i) = \lim_{t\to\infty} \sum_{k=1}^{t} \frac{1}{(k\pi)^{\frac{1}{2}}} \tag{0.3.7}$$

Since $\frac{1}{k^{\frac{1}{2}}}$ is divergent, Equation (0.3.1) is satisfied
$\therefore$ The random walk is **recurrent**
The first-passage time probability is

$$f_{i,j}(n) = \Pr(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, ...X_1 \neq j | X_0 = i) \tag{0.3.8}$$

The first-passage time $T_{i,j}$ from state i to j has the PMf $f_{i,j}(n)$ and the distribution function

$$F_{i,j}(n) = \sum_{k=0}^{n} f_{i,j}(k) \tag{0.3.9}$$

For the Markov Chain to be null recurrent

$$\overline{T_{j,j}} = \infty \tag{0.3.10}$$

and for positive recurrent

$$\overline{T_{j,j}} < \infty \tag{0.3.11}$$

where $\overline{T_{j,j}}$ represents the mean time to enter j starting from j. We can calculate the mean by using the distribution function

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^{n} \left(1 - F_{j,j}(k)\right) \tag{0.3.12}$$

From (0.3.9) we get (0.3.10) condition to be satisfied
$\therefore$ Option **(3)** is **correct**.
Answer : option2, option3