

AI1103: Assignment 5

Damaragidda Bharadwaja Rao - CS20BTECH11012

Download all latex-tikz codes from

<https://github.com/Bharadwaja-rao-D/AI1103/blob/main/assignment5/assignment5.tex>

PROBLEM UGC-MATH 2019 Q 105:

Consider a simple symmetric random walk on integers, where from every state i you move to $i-1$ and $i+1$ with the probability half each. Then which of the following are true?

- 1) The random walk is aperiodic
- 2) The random walk is irreducible
- 3) The random walk is null recurrent
- 4) The random walk is positive recurrent

SOLUTION:

The simple symmetric random walk is a Markov chain with state space $S = \{i | i \in \mathbb{Z}\}$ and with transition matrix P where,

$$p_{ij} = \begin{cases} 0, & |i - j| > 1 \\ p = \frac{1}{2}, & j = i + 1 \\ q = 1 - p = \frac{1}{2}, & j = i - 1 \end{cases} \quad (0.0.1)$$

and p_{ij} denotes the probability of being in state j , starting from state i after n steps or transitions.

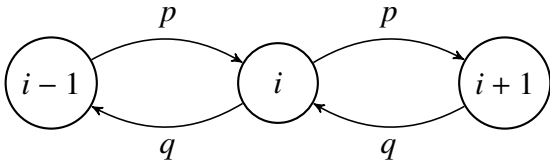


Fig. 1: Simple Symmetric random walk on integers

$$P = \begin{matrix} & \dots & -2 & -1 & 0 & 1 & 2 & \dots \\ \begin{matrix} \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & p & 0 & 0 & 0 & \dots \\ \dots & q & 0 & p & 0 & 0 & \dots \\ \dots & 0 & q & 0 & p & 0 & \dots \\ \dots & 0 & 0 & q & 0 & p & \dots \\ \dots & 0 & 0 & 0 & q & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{matrix} \quad (0.0.2)$$

Fig. 2: Transition matrix of the Markov chain

- 1) For a Markov chain to be aperiodic there should exist an integer k such that

$$p_{jj}^n > 0 \quad \forall n \geq k \quad (0.0.3)$$

to return to same state after n steps, number of forward and backward steps should be same, that is number of steps should be even.

$$p_{jj}^n = 0 \quad \forall n \in \text{Odd} \quad (0.0.4)$$

\therefore Equation (0.0.3) is not satisfied and Option **(1)** is **incorrect**.

- 2) For a Markov chain to be **irreducible** all pairs i, j should communicate with each other. Let us assume that the chain starts from state i and let us assume that it requires m forward and $(n-m)$ backward steps to reach j . Let $i < j$ wlog

$$j - i = m - (n - m) = 2m - n \quad (0.0.5)$$

$$m = \frac{(j - i) + n}{2} \quad (0.0.6)$$

$$p_{ij} = {}^nC_m p^m q^{n-m} \quad (0.0.7)$$

$$p_{ij} = {}^nC_{\left(\frac{(j-i)+n}{2}\right)} p^m q^{n-m} > 0$$

$$n = (j - i) + 2k \quad \forall k \in \mathbb{W} \quad (0.0.8)$$

Here i and j are general, hence all pairs i and j communicate with each other.

∴ Option (2) is **correct**

- 3) In a Markov Chain for state i to be **recurrent** it should satisfy ,

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t p_{ii}^n = \infty \quad (0.0.9)$$

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t p_{ii}^n = \lim_{t \rightarrow \infty} \left(\sum_{k=1}^t p_{ii}^{2k} + \sum_{k=1}^t p_{ii}^{(2k-1)} \right) \quad (0.0.10)$$

$$= \lim_{t \rightarrow \infty} \sum_{k=1}^t p_{ii}^{2k} \quad (0.0.11)$$

$$p_{ii}^{2k} = {}^{2k}C_k p^k q^k = \frac{2k!}{k!k!} p^k q^k \quad (0.0.12)$$

By using Stirling approximation to the (0.0.12) we get

$$p_{ii}^{2k} = \frac{\left((2k)^{2k+\frac{1}{2}} \times \exp(-2k) \times (2\pi)^{\frac{1}{2}} \right)}{\left(k^{k+\frac{1}{2}} \times \exp(-k) \right)^2 \times 2\pi} p^k q^k \quad (0.0.13)$$

$$= \frac{(4pq)^{2k}}{(k\pi)^{\frac{1}{2}}} = \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.0.14)$$

$$\lim_{t \rightarrow \infty} \sum_{k=1}^t p_{ii}^{2k} = \lim_{t \rightarrow \infty} \sum_{k=1}^t \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.0.15)$$

Since $\frac{1}{k^{\frac{1}{2}}}$ is divergent, Equation (0.0.9) is satisfied

∴ The random walk is **recurrent**

The first-passage time probability is

$$f_{i,j}(n) = \Pr(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots, X_1 \neq j | X_0 = i) \quad (0.0.16)$$

The first-passage time $T_{i,j}$ from state i to j has the PMf $f_{i,j}(n)$ and the distribution function

$$F_{i,j}(n) = \sum_{k=0}^n f_{i,j}(k) \quad (0.0.17)$$

For the Markov Chain to be null recurrent

$$\overline{T}_{j,j} = \infty \quad (0.0.18)$$

and for positive recurrent where $\overline{T}_{j,j}$ represents the mean time to enter j starting from j. We can calculate the mean by using the distribution

function

$$\overline{T}_{j,j} = 1 + \sum_{k=0}^n (1 - F_{j,j}(k)) \quad (0.0.19)$$

From (0.0.17) and (0.0.19) we get (0.0.18) condition to be satisfied

∴ Option (3) is **correct**.

- 4) For a Markov chain to be positive recurrent it should satisfy the following condition

$$\overline{T}_{j,j} < \infty \quad (0.0.20)$$

Since the Markov chain satisfies the (0.0.18) but not (0.0.20) it is not positive recurrent.

∴ Option (4) is **incorrect**

Answer : option2, option3