

AI1103: Assignment 5

Damaragidda Bharadwaja Rao - CS20BTECH11012

Download all latex-tikz codes from

<https://github.com/Bharadwaja-rao-D/AI1103/blob/main/assignment5/assignment5.tex>

PROBLEM UGC-MATH 2019 Q 105:

Consider a simple symmetric random walk on integers, where from every state i you move to $i-1$ and $i+1$ with the probability half each. Then which of the following are true?

- 1) The random walk is aperiodic
- 2) The random walk is irreducible
- 3) The random walk is null recurrent
- 4) The random walk is positive recurrent

SOLUTION:

The simple symmetric random walk is a Markov chain with state space $S = \{i | i \in \mathbb{Z}\}$ and with transition matrix P where,

$$P(i, j) = \begin{cases} 0, & |i - j| > 1 \\ p = \frac{1}{2}, & j = i + 1 \\ q = 1 - p = \frac{1}{2}, & j = i - 1 \end{cases} \quad (0.0.1)$$

and $P^n(i, j)$ denotes the probability of being in state j , starting from state i after n steps or transitions.

0.1 Aperiodic

For a Markov chain to be aperiodic there should exist an integer k such that

$$P^n(j, j) > 0 \quad \forall n \geq k \quad (0.1.1)$$

to return to same state after n steps, number of forward and backward steps should be same, that is number of steps should be even.

$$P^n(j, j) = 0 \quad \forall n \in \text{Odd} \quad (0.1.2)$$

\therefore Equation (0.1.1) is not satisfied and Option (1) is **incorrect**.

0.2 Irreducible

For a Markov chain to be **irreducible** all pairs i, j should communicate with each other. Let us assume that the chain starts from state i and let us assume that it requires m forward and $(n-m)$ backward steps to reach j . Let $i < j$ wlog

$$j - i = m - (n - m) = 2m - n \quad (0.2.1)$$

$$m = \frac{(j - i) + n}{2} \quad (0.2.2)$$

$$P^n(i, j) = \binom{n}{m} p^m q^{n-m} \quad (0.2.3)$$

$$P^n(i, j) = \binom{n}{\frac{(j - i) + n}{2}} p^m q^{n-m} \quad (0.2.4)$$

$$P^n(i, j) > 0 \quad n = (j - i) + 2k \quad \forall k \in \mathbb{W} \quad (0.2.5)$$

Here i and j are general, hence all pairs i and j communicate with each other.

\therefore Option (2) is **correct**

0.3 Recurrent

In a Markov Chain for state i to be **recurrent** it should satisfy,

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t P^n(i, i) = \infty \quad (0.3.1)$$

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t P^n(i, i) = \lim_{t \rightarrow \infty} \left(\sum_{k=1}^t P^{2k}(i, i) + \sum_{k=1}^t P^{2k-1}(i, i) \right) \quad (0.3.2)$$

$$= \lim_{t \rightarrow \infty} \sum_{k=1}^t P^{2k}(i, i) \quad (0.3.3)$$

$$P^{2k}(i, i) = \binom{2k}{k} p^k q^k = \frac{2k!}{k!k!} p^k q^k \quad (0.3.4)$$

By using Stirling approximation to (0.3.4) we get

$$P^{2k}(i, i) = \frac{\left((2k)^{2k+\frac{1}{2}}\right) \times \exp(-2k) \times (2\pi)^{\frac{1}{2}}}{\left(k^{k+\frac{1}{2}} \times \exp(-k)\right)^2 \times 2\pi} p^k q^k \quad (0.3.5)$$

$$= \frac{(4pq)^{2k}}{(k\pi)^{\frac{1}{2}}} = \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.3.6)$$

$$\lim_{t \rightarrow \infty} \sum_{k=1}^t P^{2k}(i, i) = \lim_{t \rightarrow \infty} \sum_{k=1}^t \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.3.7)$$

Since $\frac{1}{k^{\frac{1}{2}}}$ is divergent, Equation (0.3.1) is satisfied

∴ The random walk is **recurrent**

The first-passage time probability is

$$f_{i,j}(n) = \Pr(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots, X_1 \neq j | X_0 = i) \quad (0.3.8)$$

The first-passage time $T_{i,j}$ from state i to j has the PMf $f_{i,j}(n)$ and the distribution function

$$F_{i,j}(n) = \sum_{k=0}^n f_{i,j}(k) \quad (0.3.9)$$

For the Markov Chain to be null recurrent

$$\overline{T_{j,j}} = \infty \quad (0.3.10)$$

and for positive recurrent

$$\overline{T_{j,j}} < \infty \quad (0.3.11)$$

where $\overline{T_{j,j}}$ represents the mean time to enter j starting from j. We can calculate the mean by using the distribution function

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^n (1 - F_{j,j}(k)) \quad (0.3.12)$$

From (0.3.9) we get (0.3.10) condition to be satisfied

∴ Option (3) is **correct**.

Answer : option2, option3