_/_/_

THE LOGIARITHMIC TEST:

If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and the limit

$$\lim_{n\to\infty}\left[n\log\left(\frac{a_n}{a_{n+1}}\right)\right] = l$$

exists, then the series converges when $l > 1$ and diverges when $l < 1$. The test fails when $l = 1$.

PROBLEM:

1) Test the convergence of the series

$$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots, \quad x > 0$$

Soln:

Ignoring the first term, let $\sum_{n=1}^{\infty} a_n$ be the given series.

Then, $a_n = \frac{n^n x^n}{n!}$

and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n}$$

$$= \frac{(n+1)^n x}{n^n} = \left(1+\frac{1}{n}\right)^n x$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = xe$$

By Ratio test, the given series is convergent when $xe<1$ (ie) $x<\frac{1}{e}$ and divergent when $x>\frac{1}{e}$

when $x=\frac{1}{e}$, the test fails. So, we will apply logarithmic test

$$n\left[\ln\left[\frac{a_n}{a_{n+1}}\right]\right] = n\left[\ln\left[\frac{n^n e}{(n+1)^n}\right]\right]$$

$$= n\left[n\ln\left(\frac{n}{n+1}\right)+1\right]$$

$$= n + n^2\ln\left(\frac{n}{n+1}\right)$$

$$= n - n^2\ln\left(\frac{n+1}{n}\right)$$

$$= n - n^2\ln\left(1+\frac{1}{n}\right)$$

$$= n - n^2\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right]$$

$$= n - n + \frac{1}{2} + \frac{1}{3n} + \frac{1}{4n^2} - \cdots$$

$$= \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots$$

$$\therefore \lim_{n\to\infty} n\ln\left[\frac{a_n}{a_{n+1}}\right] = \frac{1}{2} < 1$$

$\therefore$ By the logarithmic test, this series is divergent when $x=\frac{1}{e}$

__/__/__

$\therefore$ The given series is convergent for $0 < x < \frac{1}{e}$

and divergent for $x \geq \frac{1}{e}$