Challenging Problem1

Shivangi Parashar

Lines and Planes

***Abstract*—This documnet contains the solution to find the points on the lines that are closest to each other.Given Lines are skew**

Download latex-tikz codes from

https://github.com/shivangi−975/Challenge_1/blob/master/Challenge_1.tex

1 Problem

Find the points on the skew lines that are closest to eachother in 3-Dimensions? skew line 1 passing through the point $A(1,1,0)$ with directional vector $v_1(2,-1,1)$ and skew line 2 passing through the point $B(2,1,-1)$ with directional vector $v_2(3,-5,2)$

$$L_1 : x = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.0.1}$$

and

$$L_2 : x = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.0.2}$$

2 Solution

Let the closest points be $P(p_1, p_2, p_3)$ on skew line1 and $Q(q_1, q_2, q_3)$ on skew line2,
Let $\boldsymbol{p}$,$\boldsymbol{q}$ be two points on the lines L_1,L_2

$$\boldsymbol{p} = x_1 + \lambda_1 \boldsymbol{v_1} \tag{2.0.1}$$

$$\boldsymbol{q} = x_2 + \lambda_2 \boldsymbol{v_2} \tag{2.0.2}$$

$$\boldsymbol{pq} = x_2 + \lambda_2 \boldsymbol{v_2} - x_1 - \lambda_1 \boldsymbol{v_1}$$

$$\boldsymbol{pq} = \begin{pmatrix}x_2 & \boldsymbol{v_2}\end{pmatrix}\begin{pmatrix}1\\\lambda_2\end{pmatrix} - \begin{pmatrix}x_1 & \boldsymbol{v_1}\end{pmatrix}\begin{pmatrix}1\\\lambda_1\end{pmatrix} \tag{2.0.3}$$

points P and Q are closest points,p-q will be perpendicular to both the skew lines,
Therefore,

$$\mathbf{v}_1^T(\mathbf{q} - \mathbf{p}) = 0 \tag{2.0.4}$$

$$\mathbf{v}_2^T(\mathbf{q} - \mathbf{p}) = 0 \tag{2.0.5}$$

From 2.0.4 and 2.0.5 we have: The dot product of $\boldsymbol{v_1}$ with the line $\boldsymbol{pq}$ is

$$\boldsymbol{v_1^T}\begin{pmatrix}x_2 & \boldsymbol{v_2}\end{pmatrix}\begin{pmatrix}1\\\lambda_2\end{pmatrix} - \boldsymbol{v_1^T}\begin{pmatrix}x_1 & \boldsymbol{v_1}\end{pmatrix}\begin{pmatrix}1\\\lambda_1\end{pmatrix} = 0 \tag{2.0.6}$$

The dot product of $\boldsymbol{v_2}$ with the line $\boldsymbol{pq}$ is

$$\boldsymbol{v_2^T}\begin{pmatrix}x_2 & \boldsymbol{v_2}\end{pmatrix}\begin{pmatrix}1\\\lambda_2\end{pmatrix} - \boldsymbol{v_2^T}\begin{pmatrix}x_1 & \boldsymbol{v_2}\end{pmatrix}\begin{pmatrix}1\\\lambda_1\end{pmatrix} = 0 \tag{2.0.7}$$

solving 2.0.6 and 2.0.7we get

$$\begin{pmatrix}\boldsymbol{v_1^T}x_2 & \boldsymbol{v_1^T v_2} & -\boldsymbol{v_1^T}x_1 & -\boldsymbol{v_1^T v_1}\\ \boldsymbol{v_2^T}x_2 & \boldsymbol{v_2^T v_2} & -\boldsymbol{v_2^T}x_1 & -\boldsymbol{v_2^T v_1}\end{pmatrix}\begin{pmatrix}1\\\lambda_1\\1\\\lambda_2\end{pmatrix} = 0 \tag{2.0.8}$$

simplifying it further

$$\begin{pmatrix}\boldsymbol{v_1^T v_1} & -\boldsymbol{v_1^T v_2}\\ \boldsymbol{v_2^T v_1} & -\boldsymbol{v_2^T v_2}\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{v_1^T(x_2 - x_1)}\\ \boldsymbol{v_2^T(x_2 - x_1)}\end{pmatrix} \tag{2.0.9}$$

Substituting values we have λ_1=25/59 and λ_2=7/59 and coordinates of points would be.

$$P = \begin{pmatrix}109/59\\34/59\\23/59\end{pmatrix} \tag{2.0.10}$$

$$Q = \begin{pmatrix}139/59\\24/59\\-45/59\end{pmatrix} \tag{2.0.11}$$