Assignment 15

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Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment15

1 Problem

(UGC,JUNE 2015,68) :
Let $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{Ax}, \mathbf{y} \rangle$, where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$ and $\mathbf{A}$ is a $n \times n$ real matrix. Here D denotes the total derivative. Which of the following statements are correct?

1) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = \langle \mathbf{Au}, \mathbf{y} \rangle + \langle \mathbf{Ax}, \mathbf{v} \rangle$.
2) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(0, 0) = 0$.
3) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ may not exist for some $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.
4) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ does not exist at $(\mathbf{x}, \mathbf{y}) = (0, 0)$.

2 Explanation

Inner product	Inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} \quad (2.0.1)$$ Where $\mathbf{x}$,$\mathbf{y} \in \mathbb{R}^n$
Total Derivative D	Total derivative is a linear transformation. For function $\mathbf{F}(\mathbf{x}, \mathbf{y})$, the total derivative is given as $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ which says that total derivative of function $\mathbf{F}$ at $(\mathbf{x}, \mathbf{y})$.

TABLE 1: Definitions and theorem used

3 Solution

Statement	**Observations**
Given	Function $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, it is given as $$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{Ax}, \mathbf{y} \rangle \quad (3.0.1)$$ where $\mathbf{x}$,$\mathbf{y} \in \mathbb{R}^n$, let $$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad (3.0.2)$$ Given matrix $\mathbf{A}$ is a $n \times n$ real matrix, Let it be as

	$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{3.0.3}$$ From (2.0.1),(3.0.1),(3.0.2) we get $$\mathbf{F}(\mathbf{x},\mathbf{y}) = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \tag{3.0.4}$$ $$\implies \mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{3.0.5}$$ It is in quadratic form, On solving we get $$\implies \mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} a_{11}x_1 & a_{21}x_1 & \dots & a_{n1}x_1 \\ +a_{12}x_2 & +a_{22}x_2 & & +a_{n2}x_2 \\ +\dots & +\dots & & +\dots \\ +a_{1n}x_n & +a_{2n}x_n & & +a_{nn}x_n \end{pmatrix}_{1\times n} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{3.0.6}$$ $$\implies \mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i & \sum_{i=1}^{n} a_{2i}x_i & \dots & \sum_{i=1}^{n} a_{ni}x_i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{3.0.7}$$ $$\implies \mathbf{F}(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} a_{1i}x_i\right)y_1 + \left(\sum_{i=1}^{n} a_{2i}x_i\right)y_2 + \dots + \left(\sum_{i=1}^{n} a_{ni}x_i\right)y_n \tag{3.0.8}$$
Total Derivative D	Now we will calculate $D\mathbf{F}(\mathbf{x},\mathbf{y})$ $$D\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial x_1} & \frac{\partial \mathbf{F}}{\partial x_2} & \dots & \frac{\partial \mathbf{F}}{\partial x_n} & \frac{\partial \mathbf{F}}{\partial y_1} & \frac{\partial \mathbf{F}}{\partial y_2} & \dots & \frac{\partial \mathbf{F}}{\partial y_n} \end{pmatrix}_{1\times n^2} \tag{3.0.9}$$ Lets represent $D\mathbf{F}(\mathbf{x},\mathbf{y})$ using block matrix as below $$D\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \mathbf{S} & \mathbf{T} \end{pmatrix} \tag{3.0.10}$$ $$\mathbf{S} = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial x_1} & \frac{\partial \mathbf{F}}{\partial x_2} & \dots & \frac{\partial \mathbf{F}}{\partial x_n} \end{pmatrix}_{1\times n} \tag{3.0.11}$$ $$\mathbf{T} = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial y_1} & \frac{\partial \mathbf{F}}{\partial y_2} & \dots & \frac{\partial \mathbf{F}}{\partial y_n} \end{pmatrix}_{1\times n} \tag{3.0.12}$$ Using (3.0.8), (3.0.11), (3.0.12) we get $$\mathbf{S} = \begin{pmatrix} a_{11}y_1 & a_{12}y_2 & \dots & a_{1n}y_n \\ +a_{21}y_2 & +a_{22}y_2 & & +a_{2n}y_2 \\ +\dots & +\dots & & +\dots \\ +a_{n1}y_n & +a_{n2}y_n & & +a_{nn}y_n \end{pmatrix}_{1\times n} \tag{3.0.13}$$

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$$\implies \mathbf{S} = \begin{pmatrix} \sum_{i=1}^{n} a_{i1} y_i & \sum_{i=1}^{n} a_{i2} y_i & \dots & \sum_{i=1}^{n} a_{in} y_i \end{pmatrix}_{1 \times n} \tag{3.0.14}$$

$$\mathbf{T} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i} x_i & \sum_{i=1}^{n} a_{2i} x_i & \dots & \sum_{i=1}^{n} a_{ni} x_i \end{pmatrix}_{1 \times n} \tag{3.0.15}$$

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TABLE 2: Observations

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<tr><th>Option</th><th>Solution</th><th>True/
False</th></tr>
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First we calculate $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v})$
Here $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, Let

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{3.0.16}$$

Using (3.0.10), (3.0.16) and block matrix multiplication we get

$$(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = \begin{pmatrix} \mathbf{S} & \mathbf{T} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \tag{3.0.17}$$

$$\implies (D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = \mathbf{S}\mathbf{u} + \mathbf{T}\mathbf{v} \tag{3.0.18}$$

Now substituting (3.0.14), (3.0.15),(3.0.16) we get

$$\begin{aligned} (D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = &\begin{pmatrix} \sum_{i=1}^{n} a_{i1} y_i & \sum_{i=1}^{n} a_{i2} y_i & \dots & \sum_{i=1}^{n} a_{in} y_i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\ &+ \begin{pmatrix} \sum_{i=1}^{n} a_{1i} x_i & \sum_{i=1}^{n} a_{2i} x_i & \dots & \sum_{i=1}^{n} a_{ni} x_i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \end{aligned} \tag{3.0.19}$$

$$\begin{aligned} (D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = &\left(\sum_{i=1}^{n} a_{i1} y_i\right) u_1 + \left(\sum_{i=1}^{n} a_{i2} y_i\right) u_2 + \dots + \left(\sum_{i=1}^{n} a_{in} y_i\right) u_n \\ &+ \left(\sum_{i=1}^{n} a_{1i} x_i\right) v_1 + \left(\sum_{i=1}^{n} a_{2i} x_i\right) v_2 + \dots + \left(\sum_{i=1}^{n} a_{ni} x_i\right) v_n \end{aligned} \tag{3.0.20}$$

Now we will calculate

$$\langle \mathbf{A}\mathbf{u}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{A}^T \mathbf{y} + \mathbf{x}^T \mathbf{A}^T \mathbf{v} \tag{3.0.21}$$

lets, first Consider $\mathbf{u}^T \mathbf{A}^T \mathbf{y}$ and calculate by substituting (3.0.3),(3.0.2), (3.0.16) we get

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$$\mathbf{u}^T\mathbf{A}^T\mathbf{y} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{3.0.22}$$

$$\implies \mathbf{u}^T\mathbf{A}^T\mathbf{y} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} a_{11}y_1 + a_{21}y_2 + \dots + a_{n1}y_n \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{n2}y_n \\ \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{nn}y_n \end{pmatrix}_{n\times 1} \tag{3.0.23}$$

$$\implies \mathbf{u}^T\mathbf{A}^T\mathbf{y} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} a_{i1}y_i \\ \sum_{i=1}^{n} a_{i2}y_i \\ \vdots \\ \sum_{i=1}^{n} a_{in}y_i \end{pmatrix} \tag{3.0.24}$$

$$\implies \mathbf{u}^T\mathbf{A}^T\mathbf{y} = \left(\sum_{i=1}^{n} a_{i1}y_i\right)u_1 + \left(\sum_{i=1}^{n} a_{i2}y_i\right)u_2 + \dots + \left(\sum_{i=1}^{n} a_{in}y_i\right)u_n \tag{3.0.25}$$

Now we will calculate $\mathbf{x}^T\mathbf{A}^T\mathbf{v}$ by substituting (3.0.3), (3.0.2), (3.0.16)

$$\mathbf{x}^T\mathbf{A}^T\mathbf{v} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{3.0.26}$$

$$\implies \mathbf{x}^T\mathbf{A}^T\mathbf{v} = \begin{pmatrix} a_{11}x_1 & a_{21}x_1 & \dots & a_{n1}x_1 \\ +a_{12}x_2 & +a_{22}x_2 & & +a_{n2}x_2 \\ +\dots & +\dots & & +\dots \\ +a_{1n}x_n & +a_{2n}x_n & & +a_{nn}x_n \end{pmatrix}_{1\times n} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{3.0.27}$$

$$\implies \mathbf{x}^T\mathbf{A}^T\mathbf{v} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i & \sum_{i=1}^{n} a_{2i}x_i & \dots & \sum_{i=1}^{n} a_{ni}x_i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{3.0.28}$$

$$\implies \mathbf{x}^T\mathbf{A}^T\mathbf{v} = \left(\sum_{i=1}^{n} a_{1i}x_i\right)v_1 + \left(\sum_{i=1}^{n} a_{2i}x_i\right)v_2 + \dots + \left(\sum_{i=1}^{n} a_{ni}x_i\right)v_n \tag{3.0.29}$$

Now substitute (3.0.25) and (3.0.29) in (3.0.21) we get

$$\begin{aligned} \langle \mathbf{Au}, \mathbf{y} \rangle + \langle \mathbf{Ax}, \mathbf{v} \rangle = &\left(\sum_{i=1}^{n} a_{i1}y_i\right)u_1 + \left(\sum_{i=1}^{n} a_{i2}y_i\right)u_2 + \dots + \left(\sum_{i=1}^{n} a_{in}y_i\right)u_n \\ &+ \left(\sum_{i=1}^{n} a_{1i}x_i\right)v_1 + \left(\sum_{i=1}^{n} a_{2i}x_i\right)v_2 + \dots + \left(\sum_{i=1}^{n} a_{ni}x_i\right)v_n \end{aligned} \tag{3.0.30}$$

From (3.0.20) and (3.0.30) we can say that

	$$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \langle \mathbf{Au},\mathbf{y}\rangle + \langle \mathbf{Ax},\mathbf{v}\rangle \quad (3.0.31)$$	True
2.	Using (3.0.18) and (3.0.20), if $\mathbf{u} = 0$ and $\mathbf{v} = 0$ then we can get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(0,0) = 0 \quad (3.0.32)$$	True
3.	Since from (3.0.10), (3.0.14), (3.0.15) we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist for any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.	False
4.	From (3.0.10), (3.0.14), (3.0.15), if $(\mathbf{x},\mathbf{y}) = (0,0)$ we get $$D\mathbf{F}(\mathbf{x},\mathbf{y})\|_{(0,0)} = 0 \quad (3.0.33)$$ Therefore we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist at $(\mathbf{x},\mathbf{y}) = (0,0)$.	False

TABLE 3: Solution