

Assignment 5

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Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment5

1 QUESTION

(loney 13.8) Q. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.0.1)$$

2 EXPLANATION

Comparing the given equation with the general equation of second degree given as below:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

we will get $a = 12, b = \frac{k}{2}, c = 2, d = \frac{11}{2}, e = -\frac{5}{2}, f = 2$.

The general equation can be expressed as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (2.0.4)$$

The equation (2.0.2) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.5)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (2.0.6)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (2.0.7)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} 11 & -5 \\ 4 & 4 \end{vmatrix} + 11 \begin{vmatrix} 11 & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (2.0.9)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (2.0.10)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (2.0.11)$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.

Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.0.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (2.0.12)$$

Comparing above equation with (2.0.1), we will get $a = 12, b = -5, c = 2, d = \frac{11}{2}, e = -\frac{5}{2}, f = 2$.

From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (2.0.13)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (2.0.14)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (2.0.15)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (2.0.16)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (2.0.17)$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (2.0.18)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (2.0.19)$$

Equating their product with (2.0.2)

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (2.0.20)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (2.0.21)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (2.0.22)$$

$$c_1 c_2 = f = 2 \quad (2.0.23)$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \quad (2.0.24)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (2.0.25)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (2.0.26)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (2.0.27)$$

$$\Rightarrow m_1 = 3 \quad (2.0.28)$$

$$m_2 = 2 \quad (2.0.29)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.0.30)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (2.0.31)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2.0.32)$$

Substituting (2.0.31),(2.0.32) in (2.0.21). we get

$$k_1 k_2 = 2 \quad (2.0.33)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (2.0.34)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (2.0.35)$$

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (2.0.36)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.0.37)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (2.0.34), (2.0.35) in (2.0.22) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (2.0.38)$$

Solving using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (2.0.39)$$

$$\xleftrightarrow{R_2 \leftarrow -2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (2.0.40)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (2.0.41)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (2.0.42)$$

$$\Rightarrow c_1 = -13 \quad (2.0.43)$$

$$c_2 = 21 \quad (2.0.44)$$

Substituting (2.0.34),(2.0.35),(2.0.43),(2.0.44) in (2.0.18) and (2.0.19). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \quad (2.0.45)$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \quad (2.0.46)$$

The plot of these two lines is shown in Fig. 1.

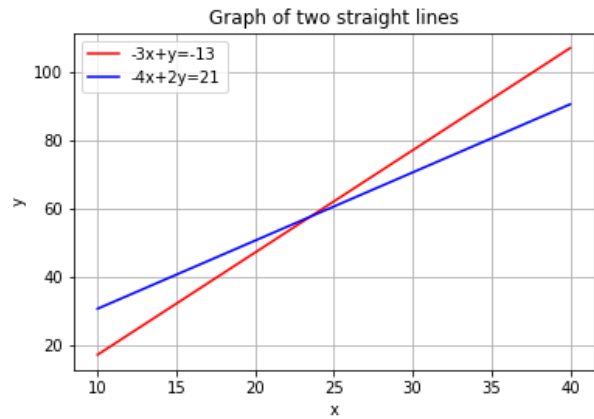


Fig. 1: Pair of straight lines for $k = -10$

Now Lets find equation of lines for $k = -\frac{35}{2}$.

Substitute $k = -\frac{35}{2}$ in (1.0.1). We get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (2.0.47)$$

Comparing above equation with (2.0.1), we will get $a = 12$, $b = -\frac{35}{4}$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (2.0.48) \quad \Rightarrow \quad \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (2.0.62)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (2.0.49) \quad \mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (2.0.63)$$

If $|\mathbf{V}| < 0$ then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (2.0.50)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (2.0.51)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (2.0.52)$$

Therefore the lines will intersect.

Now from (2.0.21),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (2.0.53)$$

The slopes of the lines are given by roots of equation (2.0.24)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (2.0.54)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (2.0.55)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (2.0.56)$$

$$\Rightarrow m_1 = 8 \quad (2.0.57)$$

$$m_2 = \frac{3}{4} \quad (2.0.58)$$

The normal vector for two lines is given by (2.0.30)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (2.0.59)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (2.0.60)$$

Substituting (2.0.59), (2.0.60) in (2.0.53). we get

$$k_1 k_2 = 2 \quad (2.0.61)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1, k_2 = 2$ we get

We verify obtained $\mathbf{n}_1, \mathbf{n}_2$ using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (2.0.64)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.0.65)$$

Therefore the obtained $\mathbf{n}_1, \mathbf{n}_2$ are correct.

Substitute (2.0.62), (2.0.63) in (2.0.22) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (2.0.66)$$

Solving using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (2.0.67)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (2.0.68)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (2.0.69)$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (2.0.70)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (2.0.71)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (2.0.72)$$

$$c_2 = \frac{59}{29} \quad (2.0.73)$$

Substituting (2.0.62), (2.0.63), (2.0.72), (2.0.73) in (2.0.18) and (2.0.19). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \quad (2.0.74)$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \quad (2.0.75)$$

The plot of these two lines is shown in Fig. 2.

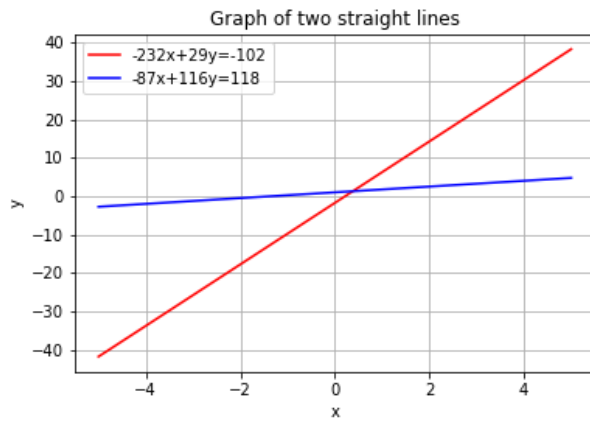


Fig. 2: Pair of straight lines for $k = -\frac{35}{2}$