Assignment 10

AVVARU BHARAT

Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment10

1 Problem

Let $\mathbf{V}$ be the vector space over the complex numbers of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(1) Prove that f_1, f_2, and f_3 are linearly independent.

(2) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix $\mathbf{P}$ such that

$$g_j = \sum_{i=1}^{3} \mathbf{P}_{ij} f_i \tag{1.0.1}$$

2 Solution

1) Given,

$$f_1(x) = 1 \tag{2.0.1}$$
$$f_2(x) = e^{ix} \tag{2.0.2}$$
$$f_3(x) = e^{-ix} \tag{2.0.3}$$

For f_1, f_2, and f_3 to be linearly independent, the following condition must satisfy.

$$k_1 f_1 + k_2 f_2 + k_3 f_3 = 0 \tag{2.0.4}$$

$\forall k_i = 0$ and i =1,2,3

Substitute (2.0.1),(2.0.2), (2.0.3) in (2.0.4), we get

$$k_1 + k_2 e^{ix} + k_3 e^{-ix} = 0 \tag{2.0.5}$$

We know from Euler formula,

$$e^{ix} = \cos x + i \sin x \tag{2.0.6}$$

Substitute (2.0.6) in (2.0.5), we get

$$k_1 + k_2 \cos x + ik_2 \sin x + k_3 \cos x - ik_3 \sin x = 0 \tag{2.0.7}$$

Now equate real and imaginary parts of (2.0.7), we get

$$k_1 + k_2 \cos x + k_3 \cos x = 0 \tag{2.0.8}$$
$$k_2 \sin x - k_3 \sin x = 0 \tag{2.0.9}$$
$$\implies k_2 = k_3 \tag{2.0.10}$$

Substitute (2.0.10) in (2.0.8), we get

$$\implies k_1 + 2k_3 \cos x = 0 \tag{2.0.11}$$

Differentiating (2.0.11) with respect to x, we get

$$\implies -2k_3 \sin x = 0 \tag{2.0.12}$$
$$\implies k_3 = 0 \tag{2.0.13}$$

Substitute (2.0.13) in (2.0.10) and (2.0.11), we get

$$k_3 = k_2 = 0 \tag{2.0.14}$$
$$k_1 = 0 \tag{2.0.15}$$

Therefore, from (2.0.14) and (2.0.15), we can say that

$$k_1 f_1 + k_2 f_2 + k_3 f_3 = 0 \tag{2.0.16}$$

$\forall k_i = 0$ and i =1,2,3

Hence, $f_1, f_2, and f_3$ are linearly independent

2) Given,

$$g_1(x) = 1 = f_1 \tag{2.0.17}$$
$$g_2(x) = \cos x = \frac{e^{ix} + e{-ix}}{2} = \frac{f_2}{2} + \frac{f_3}{2} \tag{2.0.18}$$

$$g_3(x) = \sin x = \frac{e^{ix} - e{-ix}}{2i} = \frac{f_2}{2i} - \frac{f_3}{2i}$$
$$= -\frac{i}{2} f_2 + \frac{i}{2} f_3 \tag{2.0.19}$$

Now (2.0.17), (2.0.18), (2.0.19) can be converted to matrix form as below.

$$\begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \tag{2.0.20}$$

Therefore, on comparing with (1.0.1) we get

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \tag{2.0.21}$$

Now we will verify $\mathbf{P}$ is invertible or not by row reduction.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix} \xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & i \end{pmatrix} \tag{2.0.22}$$

we got rank of matrix $\mathbf{P}$ is 3 and it is full rank matrix. Therefore, $\mathbf{P}$ is invertible matrix.
Hence verified it.