Assignment 16

AVVARU BHARAT - EE20MTECH11008

Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment16

1 Problem

(UGC,JUNE 2014,75) :
Let $\mathbf{A}$ be 5×5 matrix and let $\mathbf{B}$ be obtained by changing one element of $\mathbf{A}$. Let r and s be the ranks of $\mathbf{A}$ and $\mathbf{B}$ respectively. Which of the following statements is/are correct?

1) $s \leq r + 1$
2) $r - 1 \leq s$
3) $s = r - 1$
4) $s \neq r$

2 Explanation

Theorem	If $\mathbf{M}$ and $\mathbf{N}$ are two matrices whose ranks are $rank(\mathbf{M})$ and $rank(\mathbf{N})$ respectively. Then $$rank(\mathbf{M} + \mathbf{N}) \leq rank(\mathbf{M}) + rank(\mathbf{N}) \tag{2.0.1}$$

TABLE 1: Definitions and theorem used

3 Solution

Option	Solution	True/ False
1.	Given matrix $\mathbf{A}$ has rank r and $\mathbf{B}$ has rank s. Also given matrix $\mathbf{B}$ is obtained by changing only one element of $\mathbf{A}$. Lets assume another matrix $\mathbf{P}$ whose addition to matrix $\mathbf{A}$ results to matrix $\mathbf{B}$ as below. $$\mathbf{A} + \mathbf{P} = \mathbf{B} \tag{3.0.1}$$ Since matrix $\mathbf{P}$ consists only single element we can say that $rank(\mathbf{P}) = 1$ From (2.0.1), (3.0.1), we get $$rank(\mathbf{A} + \mathbf{P}) \leq rank(\mathbf{A}) + rank(\mathbf{P}) \tag{3.0.2}$$ $$\implies rank(\mathbf{B}) \leq rank(\mathbf{A}) + rank(\mathbf{P}) \tag{3.0.3}$$ $$\implies s \leq r + 1 \tag{3.0.4}$$ **Example:** Let matrices $\mathbf{A}$ and $\mathbf{B}$ be as below	True

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{3.0.5}$$

$$\mathbf{B} = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & 4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{3.0.6}$$

lets calculate rank of matrix **A**

$$\begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{3.0.7}$$

$$\xleftrightarrow[R_5 \leftarrow R_5 - 3R_1]{R_4 \leftarrow R_4 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \\ 0 & 0 & -6 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_5 \leftarrow R_5 - 2R_3]{R_4 \leftarrow R_4 + 3R_3} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.9}$$

$$\implies rank(\mathbf{A}) = 3 = r \tag{3.0.10}$$

Now lets calculate rank of matrix **B**

$$\begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & 4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{3.0.11}$$

$$\xleftrightarrow[R_5 \leftarrow R_5 - 3R_1]{R_4 \leftarrow R_4 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \\ 0 & 0 & -6 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_5 \leftarrow R_5 - 2R_3]{R_4 \leftarrow R_4 + 3R_3} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.12}$$

$$\implies rank(\mathbf{B}) = 4 = s \tag{3.0.13}$$

Now matrix **P** will be

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$$\mathbf{P} = \mathbf{B} - \mathbf{A} \tag{3.0.14}$$

$$\implies \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.15}$$

$$\implies rank(\mathbf{P}) = 1 \tag{3.0.16}$$

Now we will see equation (3.0.4) is satisfied or not

$$s \leq r + 1 \implies 4 \leq 3 + 1 \implies 4 \leq 4 \tag{3.0.17}$$

Hence satisfied

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From (3.0.1), If $\mathbf{P} = -\mathbf{Q}$ then we can get as below

$$\mathbf{A} - \mathbf{Q} = \mathbf{B} \tag{3.0.18}$$

$$\implies \mathbf{B} + \mathbf{Q} = \mathbf{A} \tag{3.0.19}$$

Since matrix $\mathbf{Q}$ also consists only single element we can say that $rank(\mathbf{Q}) = 1$
From (2.0.1), (3.0.19), we get

$$rank(\mathbf{B} + \mathbf{Q}) \leq rank(\mathbf{B}) + rank(\mathbf{Q}) \tag{3.0.20}$$

$$\implies rank(\mathbf{A}) \leq rank(\mathbf{B}) + rank(\mathbf{Q}) \tag{3.0.21}$$

$$\implies r \leq s + 1 \tag{3.0.22}$$

$$\implies r - 1 \leq s \tag{3.0.23}$$

Example:
Let matrix $\mathbf{A}$ and $\mathbf{B}$ are considered same as in (3.0.5), (3.0.6)
From (3.0.10) and (3.0.13) we got

$$rank(\mathbf{A}) = r = 3 \tag{3.0.24}$$

$$rank(\mathbf{B}) = s = 4 \tag{3.0.25}$$

$$\tag{3.0.26}$$

Here matrix $\mathbf{Q}$ will be

$$\mathbf{Q} = \mathbf{A} - \mathbf{B} \tag{3.0.27}$$

$$\implies \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies \mathbf{Q} = -\mathbf{P} \tag{3.0.28}$$

$$\implies rank(\mathbf{Q}) = 1 \tag{3.0.29}$$

Now we will see equation (3.0.23) is satisfied or not

$$r - 1 \leq s \implies 3 - 1 \leq 4 \implies 2 \leq 4 \tag{3.0.30}$$

Hence satisfied

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<td>True</td>
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3.	Let matrix $\mathbf{A}$ be identity matrix then $rank(\mathbf{A})$ is 5 and matrix $\mathbf{B}$ can be $$\mathbf{A} = \mathbf{I}_{5\times 5} \tag{3.0.31}$$ $$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.0.32}$$ Then $rank(\mathbf{B})$ is also 5.Therefore $s = r - 1$ is always not true.	False
4.	Similarly from (3.0.31),(3.0.32) we can say that $s \neq r$ is not true always.	False

TABLE 2: Solution