

Assignment 14

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Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment14

1 PROBLEM

(UGC,Dec 2015,74) :

Let \mathbf{V} be a finite dimensional vector space over \mathbb{R} . Let $T : \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation such that $\text{rank}(\mathbf{T}^2) = \text{rank}(\mathbf{T})$. Then,

- 1) $\text{Kernel}(\mathbf{T}^2) = \text{Kernel}(\mathbf{T})$
- 2) $\text{Range}(\mathbf{T}^2) = \text{Range}(\mathbf{T})$
- 3) $\text{Kernel}(\mathbf{T}) \cap \text{Range}(\mathbf{T}) = \{0\}$.
- 4) $\text{Kernel}(\mathbf{T}^2) \cap \text{Range}(\mathbf{T}^2) = \{0\}$.

2 EXPLANATION

$\text{Range}(\mathbf{T})$	<p>It is column-space of linear operator \mathbf{T}.</p> $\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v} \quad (2.0.1)$ <p>where $\mathbf{x}, \mathbf{v} \in \mathbf{V}$ and We can also say that</p> $\text{Range}(\mathbf{T}) = C(\mathbf{A}) \quad (2.0.2)$ <p>where $C(\mathbf{A})$ is column space of \mathbf{A}.</p>
$\text{Kernel}(\mathbf{T})$	<p>It is null-space of linear operator \mathbf{T}.</p> $\mathbf{T}(\mathbf{x}) = 0 \implies \mathbf{A}\mathbf{x} = 0 \quad (2.0.3)$ <p>where $\mathbf{x} \in \mathbf{V}$ and matrix \mathbf{A} is same as before. We can also say that</p> $\text{Kernel}(\mathbf{T}) = N(\mathbf{A}) \quad (2.0.4)$ <p>where $N(\mathbf{A})$ is null space of \mathbf{A}.</p>
$\text{rank}(\mathbf{T})$	$\text{rank}(\mathbf{T}) = \text{rank}(\mathbf{A}) \quad (2.0.5)$
\mathbf{T}^2	$\mathbf{T}^2(\mathbf{x}) = \mathbf{A}^2\mathbf{x} \quad \mathbf{x} \in \mathbf{V} \quad (2.0.6)$ $\text{rank}(\mathbf{T}^2) = \text{rank}(\mathbf{A}^2) \quad (2.0.7)$
\mathbf{A} and \mathbf{A}^2	<p>The basis vectors of column-space of \mathbf{A} and \mathbf{A}^2 are same. The basis vectors of null-space of \mathbf{A} and \mathbf{A}^2 are same.</p>

TABLE 1: Definitions and theorem used

3 SOLUTION

Statement	Observations
Given	<p>\mathbf{V} is a finite dimensional space over \mathbb{R} and $T : \mathbf{V} \rightarrow \mathbf{V}$</p> $\text{rank}(\mathbf{T}) = \text{rank}(\mathbf{T}^2) \quad (3.0.1)$ <p>According to rank-nullity theorem.</p> $\dim(\mathbf{V}) = \text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) \quad (3.0.2)$ $\dim(\mathbf{V}) = \text{rank}(\mathbf{T}^2) + \text{nullity}(\mathbf{T}^2) \quad (3.0.3)$ <p>from (3.0.2) and (3.0.3). we get</p> $\implies \text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \text{rank}(\mathbf{T}^2) + \text{nullity}(\mathbf{T}^2) \quad (3.0.4)$ $\implies \text{nullity}(\mathbf{T}) = \text{nullity}(\mathbf{T}^2) \quad (3.0.5)$

TABLE 2: Observations

Option	Solution	True/False
1	<p>From (3.0.5), let</p> $\text{nullity}(\mathbf{T}) = \text{nullity}(\mathbf{T}^2) = n \quad (3.0.6)$ <p>Therefore, from table 1 and (3.0.6) we can say that both null space of linear operator \mathbf{T} and null space of linear operator \mathbf{T}^2 will have same n number of basis.</p> $\implies \text{Kernel}(\mathbf{T}) = \text{Kernel}(\mathbf{T}^2) \quad (3.0.7)$	True
2	<p>From (3.0.1), let</p> $\text{rank}(\mathbf{T}) = \text{rank}(\mathbf{T}^2) = r \quad (3.0.8)$ <p>Therefore, from table 1 and (3.0.8) we can say that both column space of linear operator \mathbf{T} and column space of linear operator \mathbf{T}^2 will have same r number of basis.</p> $\implies \text{Range}(\mathbf{T}) = \text{Range}(\mathbf{T}^2) \quad (3.0.9)$	True
3	<p>From (3.0.6), (3.0.8) and also we can say that column space $C(\mathbf{A})$ and null space $N(\mathbf{A})$ are r-dimensional space and n-dimensional space respectively which will intersect only at origin(zero vector). And also from (2.0.2) and (2.0.4), we get</p> $\implies \text{Kernel}(\mathbf{T}) \cap \text{Range}(\mathbf{T}) = \{0\} \quad (3.0.10)$	True
4	<p>From table (3.0.7), (3.0.9) and (3.0.10), we get</p> $\implies \text{Kernel}(\mathbf{T}^2) \cap \text{Range}(\mathbf{T}^2) = \{0\} \quad (3.0.11)$	True

TABLE 3: Solution

4 EXAMPLE

Statement	Calculations and observations
Consider vector space $V = \mathbb{R}^3$ Let matrix \mathbf{A} be	$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \quad (4.0.1)$
\mathbf{A}^2	$\mathbf{A}^2 = \begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \quad (4.0.2)$
Convert both \mathbf{A} and \mathbf{A}^2 to Row Reduced echelon form	<p>For matrix \mathbf{A},</p> $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 - 2R_2]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - 5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.3)$ <p>For matrix \mathbf{A}^2,</p> $\begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ -5 & 13 & 18 \end{pmatrix} \quad (4.0.4)$ $\xleftrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & -7 & -7 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.5)$ $\xleftrightarrow{R_2 \leftarrow \frac{R_2}{7}} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1 + 4R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.0.6)$
$Range(\mathbf{T}) = Range(\mathbf{T}^2)$	<p>Therefore, from (4.0.3) and (4.0.6) we can say that the basis vectors of $Range(\mathbf{T})$ and $Range(\mathbf{T}^2)$ are same as shown below</p> $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (4.0.7)$ <p>and also we can say</p> $Range(\mathbf{T}) = Range(\mathbf{T}^2) \quad (4.0.8)$
$Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2)$	<p>Lets find the basis for null-space of linear operator \mathbf{T} or $N(\mathbf{A})$. It is the solution of the equation $\mathbf{Ax} = 0$. From (4.0.3) we have,</p> $\mathbf{Ax} = 0 \quad (4.0.9)$ $\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (4.0.10)$

	<p>Setting the value of the free variable $x_3 = 1$ we get the solution,</p> $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (4.0.11)$ <p>Hence, the basis vector of the $Kernel(\mathbf{T})$ is given by,</p> $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (4.0.12)$ <p>Now, lets find the basis for null-space of linear operator \mathbf{T}^2 or $N(\mathbf{A}^2)$. It is the solution of the equation $\mathbf{A}^2\mathbf{x} = 0$. From (4.0.6) we have,</p> $\mathbf{A}^2\mathbf{x} = 0 \quad (4.0.13)$ $\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (4.0.14)$ <p>Setting the value of the free variable $x_3 = 1$ we get the solution,</p> $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (4.0.15)$ <p>Hence, from (4.0.12) and (4.0.15) we got the basis vector of $Kernel(\mathbf{T}^2)$ same as the basis vector of $Kernel(\mathbf{T})$ which is \mathbf{p}. Therefore, we can say that</p> $Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \quad (4.0.16)$
$Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$	<p>From (4.0.7) and (4.0.12), we got 2 basis vectors $\mathbf{b}_1, \mathbf{b}_2$ for $Range(\mathbf{T})$ and 1 basis vector \mathbf{p} for $Kernel(\mathbf{T})$. Here $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$ are linearly independent. $Range(\mathbf{T})$ is a 2-dimensional space which is a plane in \mathbb{R}^3 and $Kernel(\mathbf{T})$ is a 1-dimensional space which is a line in \mathbb{R}^3. Since $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$ are linearly independent then plane and line intersect at origin(zero vector). And we can say that</p> $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (4.0.17)$
$Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$	<p>From (4.0.8), (4.0.16), (4.0.17) we get</p> $Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (4.0.18)$

TABLE 4: Example