Assignment 10

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Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment10

1 Problem

Let $\mathbf{V}$ be the vector space over the complex numbers of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1,\ f_2(x) = e^{ix},\ f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.

(b) Let $g_1(x) = 1,\ g_2(x) = \cos x,\ g_3(x) = \sin x$. Find an invertible 3×3 matrix $\mathbf{P}$ such that

$$g_j = \sum_{i=1}^{3} \mathbf{P}_{ij} f_i \tag{1.0.1}$$

2 Solution

Given,

$$f_1(x) = 1 \tag{2.0.1}$$
$$f_2(x) = e^{ix} \tag{2.0.2}$$
$$f_3(x) = e^{-ix} \tag{2.0.3}$$

For f_1, f_2, and f_3 to be linearly independent, the following condition must satisfy.

$$k_1 f_1 + k_2 f_2 + k_3 f_3 = 0 \tag{2.0.4}$$

$\forall k_i = 0$ and i =1,2,3

Substitute (2.0.1),(2.0.2), (2.0.3) in (2.0.4), we get

$$k_1 + k_2 e^{ix} + k_3 e^{-ix} = 0 \tag{2.0.5}$$

Differentiate (2.0.5), we get

$$k_2 i e^{ix} - k_3 i e^{-ix} = 0 \tag{2.0.6}$$

Differentiate (2.0.6), we get

$$-k_2 e^{ix} - k_3 e^{-ix} = 0 \tag{2.0.7}$$

(2.0.5),(2.0.6).(2.0.7) form system of linear equations as below

$$\mathbf{F}\mathbf{k} = 0 \tag{2.0.8}$$

$$\implies \begin{pmatrix} 1 & e^{ix} & e^{-ix} \\ 0 & ie^{ix} & -ie^{-ix} \\ 0 & -e^{ix} & -e^{-ix} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0 \tag{2.0.9}$$

Now if $\left|\mathbf{F}\right| \neq 0$, then we can say that columns of matrix $\mathbf{F}$ are linearly independent.

$$\implies \begin{vmatrix} 1 & e^{ix} & e^{-ix} \\ 0 & ie^{ix} & -ie^{-ix} \\ 0 & -e^{ix} & -e^{-ix} \end{vmatrix} = -2i \tag{2.0.10}$$

$$\implies \left|\mathbf{F}\right| \neq 0 \tag{2.0.11}$$

Therefore, coloumns of matrix $\mathbf{F}$ are linearly independent. Then we can also conclude that f_1, f_2, and f_3 are also linearly independent.

Given,

$$g_1(x) = 1 = f_1 \tag{2.0.12}$$

$$g_2(x) = \cos x = \frac{e^{ix} + e{-ix}}{2} = \frac{f_2}{2} + \frac{f_3}{2} \tag{2.0.13}$$

$$g_3(x) = \sin x = \frac{e^{ix} - e{-ix}}{2i} = \frac{f_2}{2i} - \frac{f_3}{2i} \tag{2.0.14}$$

Now (2.0.12), (2.0.13), (2.0.14) can be converted to matrix form as below.

$$\begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \tag{2.0.15}$$

Therefore, on comparing with (1.0.1) we get

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \tag{2.0.16}$$

Now we will verify $\mathbf{P}$ is invertible or not by row reduction.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & 0 & -\frac{1}{i} \end{pmatrix} \tag{2.0.17}$$

we got rank of matrix $\mathbf{P}$ is 3 and it is full rank matrix. Therefore, $\mathbf{P}$ is invertible matrix.

Hence verified it.