Assignment 14

AVVARU BHARAT - EE20MTECH11008

Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment14

1 Problem

(UGC,Dec 2015,74) :
Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{R}$. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation such that $rank(\mathbf{T}^2) = rank(\mathbf{T})$. Then,

1) $Kernel(\mathbf{T}^2) = Kernel(\mathbf{T})$
2) $Range(\mathbf{T}^2) = Range(\mathbf{T})$
3) $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$.
4) $Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$.

2 Explanation

$Range(\mathbf{T})$	It is column-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v} \tag{2.0.1}$$ where $\mathbf{x},\mathbf{v} \in \mathbf{V}$ and We can also say that $$Range(\mathbf{T}) = C(\mathbf{A}) \tag{2.0.2}$$ where $C(\mathbf{A})$ is column space of $\mathbf{A}$.
$Kernel(\mathbf{T})$	It is null-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = 0 \implies \mathbf{A}\mathbf{x} = 0 \tag{2.0.3}$$ where $\mathbf{x} \in \mathbf{V}$ and matrix $\mathbf{A}$ is same as before. We can also say that $$Kernel(\mathbf{T}) = N(\mathbf{A}) \tag{2.0.4}$$ where $N(\mathbf{A})$ is null space of $\mathbf{A}$.
$rank(\mathbf{T})$	$$rank(\mathbf{T}) = rank(\mathbf{A}) \tag{2.0.5}$$
$\mathbf{T}^2$	$$\mathbf{T}^2(\mathbf{x}) = \mathbf{A}^2\mathbf{x} \qquad \mathbf{x} \in \mathbf{V} \tag{2.0.6}$$ $$rank(\mathbf{T}^2) = rank(\mathbf{A}^2) \tag{2.0.7}$$
$\mathbf{A}$ and $\mathbf{A}^2$	The basis vectors of column-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same. The basis vectors of null-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same.

TABLE 1: Definitions and theorem used

3 Solution

Statement	Observations
Given	$\mathbf{V}$ is a finite dimensional space over $\mathbb{R}$ and $T : \mathbf{V} \to \mathbf{V}$ $$rank(\mathbf{T}) = rank(\mathbf{T}^2) \quad (3.0.1)$$ According to rank-nullity theorem. $$dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T}) \quad (3.0.2)$$ $$dim(\mathbf{V}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (3.0.3)$$ from (3.0.2) and (3.0.3). we get $$\implies rank(\mathbf{T}) + nullity(\mathbf{T}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (3.0.4)$$ $$\implies nullity(\mathbf{T}) = nullity(\mathbf{T}^2) \quad (3.0.5)$$

TABLE 2: Observations

Option	Solution	True/False
1	From (3.0.5), let $$nullity(\mathbf{T}) = nullity(\mathbf{T}^2) = n \quad (3.0.6)$$ Therefore, from table 1 and (3.0.6) we can say that both null space of linear operator $\mathbf{T}$ and null space of linear operator $\mathbf{T}^2$ will have same n number of basis. $$\implies Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \quad (3.0.7)$$	True
2	From (3.0.1), let $$rank(\mathbf{T}) = rank(\mathbf{T}^2) = r \quad (3.0.8)$$ Therefore, from table 1 and (3.0.8) we can say that both column space of linear operator $\mathbf{T}$ and column space of linear operator $\mathbf{T}^2$ will have same r number of basis. $$\implies Range(\mathbf{T}) = Range(\mathbf{T}^2) \quad (3.0.9)$$	True
3	From (3.0.6), (3.0.8) and also we can say that column space $C(\mathbf{A})$ and null space $N(\mathbf{A})$ are r-dimensional space and n-dimensional space respectively which will intersect only at origin(zero vector). And also from (2.0.2) and (2.0.4), we get $$\implies Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (3.0.10)$$	
4	From table (3.0.7), (3.0.9) and (3.0.10), we get $$\implies Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (3.0.11)$$	True

TABLE 3: Solution