Assignment 15

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Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment15

1 Problem

(UGC,JUNE 2015,68) :
Let $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle$, where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$ and $\mathbf{A}$ is a $n \times n$ real matrix. Here D denotes the total derivative. Which of the following statements are correct?

1) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = \langle \mathbf{A}\mathbf{u}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{v} \rangle$.
2) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(0, 0) = 0$.
3) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ may not exist for some $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.
4) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ does not exist at $(\mathbf{x}, \mathbf{y}) = (0, 0)$.

2 Explanation

Inner product	Inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} \quad (2.0.1)$$ Where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
Inner Product Property used	$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle \quad (2.0.2)$$
Total Derivative D	Total derivative is a linear transformation. For function $\mathbf{F}(\mathbf{x}, \mathbf{y})$, the total derivative is given as $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ which says that total derivative of function $\mathbf{F}$ at $(\mathbf{x}, \mathbf{y})$.

TABLE 1: Definitions and theorem used

3 Solution

Statement	**Observations**
Given	Function $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, it is given as $$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{A}^T \mathbf{y} \quad (3.0.1)$$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ Using property (2.0.2), we can also get $$\implies \mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle \quad (3.0.2)$$ $$\implies \mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{A}\mathbf{x} \quad (3.0.3)$$

Total Derivative D	Now we will calculate $D\mathbf{F}(\mathbf{x},\mathbf{y})$ $$D\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix}\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\end{pmatrix} \tag{3.0.4}$$ From (3.0.1),(3.0.3) we get $$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \mathbf{y}^T\mathbf{A} \tag{3.0.5}$$ $$\frac{\partial \mathbf{F}}{\partial \mathbf{y}} = \mathbf{x}^T\mathbf{A}^T \tag{3.0.6}$$ Substitute (3.0.5) and (3.0.6) in (3.0.4) $$D\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix}\mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T\end{pmatrix}_{1\times n^2} \tag{3.0.7}$$

TABLE 2: Observations

Option	**Solution**	**True/ False**
1	First we calculate $(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v})$ where $\mathbf{u}$,$\mathbf{v} \in \mathbb{R}^n$ Using (3.0.7)and block matrix multiplication we get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \begin{pmatrix}\mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \tag{3.0.8}$$ $$\implies (D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \mathbf{y}^T\mathbf{A}\mathbf{u} + \mathbf{x}^T\mathbf{A}^T\mathbf{v} \tag{3.0.9}$$ $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \langle \mathbf{y},\mathbf{A}\mathbf{u}\rangle + \langle \mathbf{A}\mathbf{x},\mathbf{v}\rangle \tag{3.0.10}$$ Using property (2.0.2) we get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \langle \mathbf{A}\mathbf{u},\mathbf{y}\rangle + \langle \mathbf{A}\mathbf{x},\mathbf{v}\rangle \tag{3.0.11}$$	
2.	Using (3.0.9), if $\mathbf{u} = 0$ and $\mathbf{v} = 0$ then we get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(0,0) = 0 \tag{3.0.12}$$	True
3.	Since from (3.0.7) we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist for any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.	False
4.	From (3.0.7), if $(\mathbf{x},\mathbf{y}) = (0,0)$ we get $$D\mathbf{F}(\mathbf{x},\mathbf{y})\|_{(0,0)} = 0 \tag{3.0.13}$$ Therefore we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist at $(\mathbf{x},\mathbf{y}) = (0,0)$.	False

TABLE 3: Solution