Assignment 6

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Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment6

1 Question

(Ramsey 3.4.4)Q. What does the equation

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\mathbf{x} = 2a^2 \tag{1.0.1}$$

become when the axes are turned through 30°

2 Explanation

The general second degree equation is expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) and (2.0.1), we get

$$\mathbf{V} = \begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.3}$$

$$f = -2a^2 \tag{2.0.4}$$

Now we find

$$\left|\mathbf{V}\right| = \begin{vmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{vmatrix} \tag{2.0.5}$$

$$\implies \left|\mathbf{V}\right| = -4 \tag{2.0.6}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.7}$$

Therefore the given equation (1.0.1) represents hyperbola.

Now from affine transformations,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.8}$$

We have to rotate the axes by $\theta = 30°$, Then using rotation matrix.

$$\mathbf{P} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{2.0.9}$$

$$\implies \mathbf{P} = \begin{pmatrix}\cos 30° & -\sin 30°\\ \sin 30° & \cos 30°\end{pmatrix} \tag{2.0.10}$$

$$\implies \mathbf{P} = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \tag{2.0.11}$$

We are not shifting the centre from origin, then

$$\implies \mathbf{x} = \mathbf{P}\mathbf{y} \tag{2.0.12}$$

From eigenvalue decomposition,

$$\mathbf{P}^{\mathbf{T}}\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.13}$$

$$\implies \mathbf{D} = \begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \tag{2.0.14}$$

$$\implies \mathbf{D} = \begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{2.0.15}$$

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} \tag{2.0.16}$$

$$\mathbf{P}^{T} = \mathbf{P}^{-1} \tag{2.0.17}$$

The equation (2.0.1) can be expressed as

$$\mathbf{y}^{T}\mathbf{D}\mathbf{y} = \mathbf{u}^{T}\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.18}$$

with

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.19}$$

Substitute (2.0.2),(2.0.3),(2.0.4),(2.0.15) in (2.0.18) and (2.0.19), we get

$$\mathbf{y}^{T}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\mathbf{y} = 2a^2 \tag{2.0.20}$$

with centre,

$$\mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.21}$$

Therefore the given equation (1.0.1) becomes (2.0.20) when the axes are turned through 30°.