Assignment 8

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***Abstract*—This document explains the concept of finding the distance between a given point and a plane using Singular Value Decomposition.**

Download all python codes from

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https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment8/Codes
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and latex-tikz codes from

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https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment8
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1 Problem

Find the distance of the given point $\begin{pmatrix}-6\\0\\0\end{pmatrix}$ from the plane $\begin{pmatrix}2 & -3 & 6\end{pmatrix}\mathbf{x} = 2$.

2 Explanation

Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\-3\\6\end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 2a - 3b + 6c = 0 \tag{2.0.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\-\frac{1}{3}\end{pmatrix} \tag{2.0.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix} \tag{2.0.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Substituting (2.0.4) and (2.0.5) in (2.0.6),

$$\begin{pmatrix}1 & 0\\0 & 1\\-\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}-6\\0\\0\end{pmatrix} \tag{2.0.7}$$

To solve (2.0.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{MM}^T$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}\frac{10}{9} & -\frac{1}{6}\\-\frac{1}{6} & \frac{5}{4}\end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & -\frac{1}{3}\\0 & 1 & \frac{1}{2}\\-\frac{1}{3} & \frac{1}{2} & \frac{13}{36}\end{pmatrix} \tag{2.0.10}$$

Substituting (2.0.8) in (2.0.6),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & -\frac{1}{3}\\0 & 1-\lambda & \frac{1}{2}\\-\frac{1}{3} & \frac{1}{2} & \frac{13}{36}-\lambda\end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \tag{2.0.15}$$

From equation (2.0.15) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{2.0.16}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}-\frac{12}{13}\\\frac{18}{13}\\1\end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix}\frac{3}{2}\\1\\0\end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix}\frac{1}{3}\\-\frac{1}{2}\\1\end{pmatrix} \tag{2.0.17}$$

Normalizing the eigen vectors in equation (2.0.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{12}{7\sqrt{13}} \\ \frac{18}{7\sqrt{13}} \\ \frac{13}{7\sqrt{13}} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{7} \\ -\frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & -\frac{3}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \tag{2.0.19}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.20}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.21}$$

$$\implies \begin{pmatrix} \frac{10}{9} - \lambda & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{4} - \lambda \end{pmatrix} = 0 \tag{2.0.22}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{2.0.23}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{2.0.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{2.0.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.27}$$

From (2.0.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & -\frac{3}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{2.0.28}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.29}$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{72}{7\sqrt{13}} \\ -\frac{18}{\sqrt{13}} \\ -\frac{12}{7} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{432}{49\sqrt{13}} \\ -\frac{18}{\sqrt{13}} \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{270}{49} \\ -\frac{36}{49} \end{pmatrix} \tag{2.0.32}$$

Verifying the solution of (2.0.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.33}$$

Evaluating the R.H.S in (2.0.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \tag{2.0.34}$$

$$\implies \begin{pmatrix} \frac{10}{9} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \tag{2.0.35}$$

Solving the augmented matrix of (2.0.35) we get,

$$\begin{pmatrix} \frac{10}{9} & -\frac{1}{6} & -6 \\ -\frac{1}{6} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{9}{10}R_1} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ -\frac{1}{6} & \frac{5}{4} & 0 \end{pmatrix} \tag{2.0.36}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{1}{6}R_1} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ 0 & \frac{49}{40} & -\frac{9}{10} \end{pmatrix} \tag{2.0.37}$$

$$\xleftrightarrow{R_2 = \frac{40}{49}R_2} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ 0 & 1 & -\frac{36}{49} \end{pmatrix} \tag{2.0.38}$$

$$\xleftrightarrow{R_1 = R_1 + \frac{3}{20}R_2} \begin{pmatrix} 1 & 0 & -\frac{270}{49} \\ 0 & 1 & -\frac{36}{49} \end{pmatrix} \tag{2.0.39}$$

From equation (2.0.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{270}{49} \\ -\frac{36}{49} \end{pmatrix} \tag{2.0.40}$$

Comparing results of $\mathbf{x}$ from (2.0.32) and (2.0.40), we can say that the solution is verified.