Assignment 14

AVVARU BHARAT - EE20MTECH11008

Download the latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment14

1 Problem

(UGC,Dec 2015,74) :
Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{R}$. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation such that $rank(\mathbf{T}^2) = rank(\mathbf{T})$. Then,

1) $Kernel(\mathbf{T}^2) = Kernel(\mathbf{T})$
2) $Range(\mathbf{T}^2) = Range(\mathbf{T})$
3) $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$.
4) $Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$.

2 Explanation

$Range(\mathbf{T})$	It is column-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v} \tag{2.0.1}$$ where $\mathbf{x},\mathbf{v} \in \mathbf{V}$ and We can also say that $$Range(\mathbf{T}) = C(\mathbf{A}) \tag{2.0.2}$$ where $C(\mathbf{A})$ is column space of $\mathbf{A}$.
$Kernel(\mathbf{T})$	It is null-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = 0 \implies \mathbf{A}\mathbf{x} = 0 \tag{2.0.3}$$ where $\mathbf{x} \in \mathbf{V}$ and matrix $\mathbf{A}$ is same as before. We can also say that $$Kernel(\mathbf{T}) = N(\mathbf{A}) \tag{2.0.4}$$ where $N(\mathbf{A})$ is null space of $\mathbf{A}$.
$rank(\mathbf{T})$	$$rank(\mathbf{T}) = rank(\mathbf{A}) \tag{2.0.5}$$
$\mathbf{T}^2$	$$\mathbf{T}^2(\mathbf{x}) = \mathbf{A}^2\mathbf{x} \qquad \mathbf{x} \in \mathbf{V} \tag{2.0.6}$$ $$rank(\mathbf{T}^2) = rank(\mathbf{A}^2) \tag{2.0.7}$$
$\mathbf{A}$ and $\mathbf{A}^2$	The basis vectors of column-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same. The basis vectors of null-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same.

TABLE 1: Definitions and theorem used

3 Solution

Statement	Observations
Given	$\mathbf{V}$ is a finite dimensional space over $\mathbb{R}$ and $T : \mathbf{V} \to \mathbf{V}$ $$rank(\mathbf{T}) = rank(\mathbf{T}^2) \quad (3.0.1)$$ According to rank-nullity theorem. $$dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T}) \quad (3.0.2)$$ $$dim(\mathbf{V}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (3.0.3)$$ from (3.0.2) and (3.0.3). we get $$\implies rank(\mathbf{T}) + nullity(\mathbf{T}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (3.0.4)$$ $$\implies nullity(\mathbf{T}) = nullity(\mathbf{T}^2) \quad (3.0.5)$$

TABLE 2: Observations

Option	Solution	True/False
1	From (3.0.5), let $$nullity(\mathbf{T}) = nullity(\mathbf{T}^2) = n \quad (3.0.6)$$ Therefore, from table 1 and (3.0.6) we can say that both null space of linear operator $\mathbf{T}$ and null space of linear operator $\mathbf{T}^2$ will have same n number of basis. $$\implies Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \quad (3.0.7)$$	True
2	From (3.0.1), let $$rank(\mathbf{T}) = rank(\mathbf{T}^2) = r \quad (3.0.8)$$ Therefore, from table 1 and (3.0.8) we can say that both column space of linear operator $\mathbf{T}$ and column space of linear operator $\mathbf{T}^2$ will have same r number of basis. $$\implies Range(\mathbf{T}) = Range(\mathbf{T}^2) \quad (3.0.9)$$	True
3	From (3.0.6), (3.0.8) and also we can say that column space $C(\mathbf{A})$ and null space $N(\mathbf{A})$ are r-dimensional space and n-dimensional space respectively which will intersect only at origin(zero vector). And also from (2.0.2) and (2.0.4), we get $$\implies Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (3.0.10)$$	True
4	From table (3.0.7), (3.0.9) and (3.0.10), we get $$\implies Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (3.0.11)$$	True

TABLE 3: Solution

4 Example

Statement	**Calculations and observations**
Consider vector space $\mathbf{V} = \mathbb{R}^3$ Let matrix $\mathbf{A}$ be	$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \tag{4.0.1}$$
$\mathbf{A}^2$	$$\mathbf{A}^2 = \begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \tag{4.0.2}$$
Convert both $\mathbf{A}$ and $\mathbf{A}^2$ to Row Reduced echelon form	For matrix $\mathbf{A}$, $$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 - 2R_2]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \tag{4.0.3}$$ $$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.4}$$ For matrix $\mathbf{A}^2$, $$\begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \xleftrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ -5 & 13 & 18 \end{pmatrix} \tag{4.0.5}$$ $$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & -7 & -7 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R3 + R_1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.6}$$ $$\xleftrightarrow[R_1 \leftarrow -R_1]{R_2 \leftarrow \frac{R_2}{7}} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R1 \leftarrow R_1 + 4R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.0.7}$$
$Range(\mathbf{T}) = Range(\mathbf{T}^2)$	Therefore, from (4.0.4) and (4.0.7) we can say that the basis vectors of $Range(\mathbf{T})$ and $Range(\mathbf{T}^2)$ are same as shown below $$\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \mathbf{b_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.0.8}$$ and also we can say $$Range(\mathbf{T}) = Range(\mathbf{T}^2) \tag{4.0.9}$$
$Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2)$	Lets find the basis for null-space of linear operator $\mathbf{T}$ or $N(\mathbf{A})$. It is the solution of the equation $\mathbf{Ax} = 0$. From (4.0.4) we have,

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$$\mathbf{A}\mathbf{x} = 0 \tag{4.0.10}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{4.0.11}$$

Setting the value of the free variable $x_3 = 1$ we get the solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.0.12}$$

Hence, the basis vector of the $Kernel(\mathbf{T})$ is given by,

$$\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.0.13}$$

Now, lets find the basis for null-space of linear operator $\mathbf{T}^2$ or $N(\mathbf{A}^2)$.It is the solution of the equation $\mathbf{A^2x} = 0$. From (4.0.7) we have,

$$\mathbf{A}^2\mathbf{x} = 0 \tag{4.0.14}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{4.0.15}$$

Setting the value of the free variable $x_3 = 1$ we get the solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{4.0.16}$$

Hence, from (4.0.13) and (4.0.16) we got the basis vector of $Kernel(\mathbf{T}^2)$ same as the basis vector of $Kernel(\mathbf{T})$ which is $\mathbf{p}$. Therefore, we can say that

$$Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \tag{4.0.17}$$

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<td>$Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$</td>
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From (4.0.8) and (4.0.13), we got 2 basis vectors $\mathbf{b_1}$, $\mathbf{b_2}$ for $Range(\mathbf{T})$and 1 basis vector $\mathbf{p}$ for $Kernel(\mathbf{T})$. Here $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent which can be proven as below.
Let columns of matrix $\mathbf{M}$ are filled with vectors $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$.

$$\implies \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \tag{4.0.18}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \tag{4.0.19}$$

From (4.0.19), we got $rank(\mathbf{M}) = 3$.Therefore $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent

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	Range($\mathbf{T}$) is a 2-dimensional space which is a plane in $\mathbb{R}^3$ and *Kernel*($\mathbf{T}$) is a 1-dimensional space which is a line in $\mathbb{R}^3$. Since $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent then plane and line intersect at origin(zero vector). And we can say that $$Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (4.0.20)$$
$Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$	From (4.0.9), (4.0.17), (4.0.20) we get $$\implies Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (4.0.21)$$

TABLE 4: Example