Assignment 5

AVVARU BHARAT

Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment5

1 Question

(loney 13.8) Q. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (1.0.1)$$

2 Explanation

Comparing the given equation with the general equation of second degree given as below:

$$ax^2 + 2bxy + cy^2 + +2dx + 2ey + f = 0 \quad (2.0.1)$$

we will get $a = 12$, $b = \frac{k}{2}$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

The general equation can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (2.0.4)$$

The given equation represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.5)$$

$$\implies \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (2.0.6)$$

The matrix in (2.0.6) must be singular matrix and in echelon form of the matrix should consist a row with all zeros.

$$\implies \begin{pmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{pmatrix} \quad (2.0.7)$$

$$\implies \begin{pmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{pmatrix} \quad (2.0.8)$$

$$\xleftrightarrow{R_2 \leftarrow 24R_2 - kR1} \begin{pmatrix} 24 & k & 11 \\ 0 & 96 - k^2 & -120 - 11k \\ 11 & -5 & 4 \end{pmatrix} \quad (2.0.9)$$

$$\xleftrightarrow{R_3 \leftarrow 24R_3 - 11R1} \begin{pmatrix} 24 & k & 11 \\ 0 & 96 - k^2 & -120 - 11k \\ 0 & -120 - 11k & -25 \end{pmatrix} \quad (2.0.10)$$

$$\xleftrightarrow{R_3 \leftarrow (96-k^2)R_3 - (-120-11k)R2}$$

$$\begin{pmatrix} 24 & k & 11 \\ 0 & 96 - k^2 & -120 - 11k \\ 0 & 0 & -96k^2 - 2640k - 16800 \end{pmatrix} \quad (2.0.11)$$

In (2.0.11), the elements in last row must consist all zeros. For this to happen we should find k value.

$$\implies -96k^2 - 2640k - 16800 = 0 \quad (2.0.12)$$

$$\implies 2k^2 + 55k + 350 = 0 \quad (2.0.13)$$

$$\implies (10 + k)(2k + 35) = 0 \quad (2.0.14)$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \quad (2.0.15)$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.

Now Lets find equation of lines for $k = -10$.

Substitute $k = -10$ in (1.0.1).we get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (2.0.16)$$

Comparing above equation with (2.0.1), we will get $a = 12$, $b = -5$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.18}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \tag{2.0.19}$$

$$\implies \left|\mathbf{V}\right| = -1 \tag{2.0.20}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.21}$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{2.0.22}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{2.0.23}$$

Equating their product with (2.0.2)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.24}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \{a, 2b, c\} = \{12, -10, 2\} \tag{2.0.25}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.26}$$

$$c_1c_2 = f = 2 \tag{2.0.27}$$

The slopes of the lines are given by roots of polynomial

$$cm^2 + 2bm + a = 0 \tag{2.0.28}$$

$$\implies 2m^2 - 10m + 12 = 0 \tag{2.0.29}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.30}$$

$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \tag{2.0.31}$$

$$\implies m_1 = 3 \tag{2.0.32}$$

$$m_2 = 2 \tag{2.0.33}$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.0.34}$$

$$\implies \mathbf{n_1} = k_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.35}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.0.36}$$

Substituting (2.0.35),(2.0.60) in (2.0.25). we get

$$k_1k_2 = 2 \tag{2.0.37}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.38}$$

$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{2.0.39}$$

Substitute (2.0.38), (2.0.39) in (2.0.26) we get

$$c_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{2.0.40}$$

$$-4c_1 - 3c_2 = -11 \tag{2.0.41}$$

$$2c_1 + c_2 = -5 \tag{2.0.42}$$

Solving equations (2.0.41) ,(2.0.42) we get

$$c_1 = -13 \tag{2.0.43}$$

$$c_2 = 21 \tag{2.0.44}$$

Substituting (2.0.38),(2.0.39),(2.0.41),(2.0.42) in (2.0.22) and (2.0.23). we get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -13 \tag{2.0.45}$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix}\mathbf{x} = 21 \tag{2.0.46}$$

Now Lets find equation of lines for $k = -\frac{35}{2}$.

Substitute $k = -\frac{35}{2}$ in (1.0.1).we get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \tag{2.0.47}$$

Comparing above equation with (2.0.1), we will get $a = 12$, $b = -\frac{35}{4}$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \tag{2.0.48}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.49}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \tag{2.0.50}$$

$$\implies \left|\mathbf{V}\right| = -\frac{841}{16} \tag{2.0.51}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.52}$$

Therefore the lines will intersect.

Now from (2.0.25),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \{a, 2b, c\} = \{12, -\frac{35}{2}, 2\} \tag{2.0.53}$$

The slopes of the lines are given by roots of polynomial in (2.0.28)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \tag{2.0.54}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.55}$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \tag{2.0.56}$$

$$\implies m_1 = 8 \tag{2.0.57}$$

$$m_2 = \frac{3}{4} \tag{2.0.58}$$

The normal vector for two lines is given by (2.0.34)

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{2.0.59}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{2.0.60}$$

Substituting (2.0.59),(2.0.60) in (2.0.53). we get

$$k_1 k_2 = 2 \tag{2.0.61}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{2.0.62}$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \tag{2.0.63}$$

Substitute (2.0.62), (2.0.63) in (2.0.26) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{2.0.64}$$

$$-3c_1 - 16c_2 = -22 \tag{2.0.65}$$

$$2c_1 + c_2 = -5 \tag{2.0.66}$$

Solving equations (2.0.65) ,(2.0.66) we get

$$c_1 = -\frac{102}{29} \tag{2.0.67}$$

$$c_2 = \frac{59}{29} \tag{2.0.68}$$

Substituting (2.0.62),(2.0.63),(2.0.65),(2.0.66) in (2.0.22) and (2.0.23). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \tag{2.0.69}$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \tag{2.0.70}$$