Assignment 5

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Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment5

1 Question

(loney 13.8) Q. Find the value of k so that the following equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \tag{1.0.1}$$

2 Explanation

Comparing the given equation with the general equation of second degree given as below:

$$ax^2 + 2bxy + cy^2 + +2dx + 2ey + f = 0 \tag{2.0.1}$$

we will get $a = 12$, $b = \frac{k}{2}$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

The general equation can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.4}$$

The equation (2.0.2) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

$$\implies \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \tag{2.0.6}$$

$$\implies \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies 24\begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k\begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11\begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies 2k^2 + 55k + 350 = 0 \tag{2.0.9}$$

$$\implies (10 + k)(2k + 35) = 0 \tag{2.0.10}$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \tag{2.0.11}$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.

Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (1.0.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{2.0.12}$$

Comparing above equation with (2.0.1), we will get $a = 12$, $b = -5$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$.

From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.14}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \tag{2.0.15}$$

$$\implies \left|\mathbf{V}\right| = -1 \tag{2.0.16}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.17}$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{2.0.18}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{2.0.19}$$

Equating their product with (2.0.2)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.20}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{2.0.21}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.22}$$

$$c_1 c_2 = f = 2 \tag{2.0.23}$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \tag{2.0.24}$$

$$\implies 2m^2 - 10m + 12 = 0 \tag{2.0.25}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.26}$$

$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \tag{2.0.27}$$

$$\implies m_1 = 3 \tag{2.0.28}$$

$$m_2 = 2 \tag{2.0.29}$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.0.30}$$

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.0.32}$$

Substituting (2.0.31),(2.0.32) in (2.0.21). we get

$$k_1 k_2 = 2 \tag{2.0.33}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{2.0.35}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{2.0.36}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.37}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (2.0.34), (2.0.35) in (2.0.22) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{2.0.38}$$

$$-4c_1 - 3c_2 = -11 \tag{2.0.39}$$

$$2c_1 + c_2 = -5 \tag{2.0.40}$$

Solving equations (2.0.39) ,(2.0.40) using row reduction technique.

$$\implies \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{2.0.41}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \tag{2.0.42}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \tag{2.0.43}$$

$$\implies \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \tag{2.0.44}$$

$$\implies c_1 = -13 \tag{2.0.45}$$

$$c_2 = 21 \tag{2.0.46}$$

Substituting (2.0.34),(2.0.35),(2.0.45),(2.0.46) in (2.0.18) and (2.0.19). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -13 \tag{2.0.47}$$

$$\begin{pmatrix} -4 & 2 \end{pmatrix}\mathbf{x} = 21 \tag{2.0.48}$$

The plot of these two lines is shown in Fig. 1.

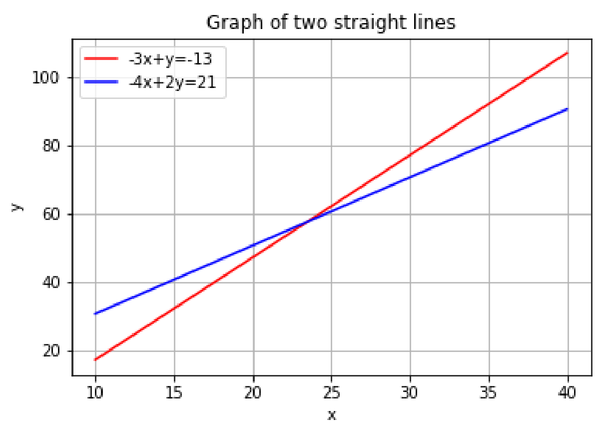

Fig. 1: Pair of straight lines for $k = -10$

Now Lets find equation of lines for $k = -\frac{35}{2}$.
Substitute $k = -\frac{35}{2}$ in (1.0.1). We get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \tag{2.0.49}$$

Comparing above equation with (2.0.1), we will get $a = 12$, $b = -\frac{35}{4}$, $c = 2$, $d = \frac{11}{2}$, $e = -\frac{5}{2}$, $f = 2$. From (2.0.2), (2.0.3), (2.0.4) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \tag{2.0.50}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{2.0.51}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \tag{2.0.52}$$

$$\implies \left|\mathbf{V}\right| = -\frac{841}{16} \tag{2.0.53}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{2.0.54}$$

Therefore the lines will intersect.
Now from (2.0.21),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{2.0.55}$$

The slopes of the lines are given by roots of equation (2.0.24)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \tag{2.0.56}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.57}$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \tag{2.0.58}$$

$$\implies m_1 = 8 \tag{2.0.59}$$

$$m_2 = \frac{3}{4} \tag{2.0.60}$$

The normal vector for two lines is given by (2.0.30)

$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{2.0.61}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{2.0.62}$$

Substituting (2.0.61),(2.0.62) in (2.0.55). we get

$$k_1 k_2 = 2 \tag{2.0.63}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{2.0.64}$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \tag{2.0.65}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{2.0.66}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.67}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (2.0.64), (2.0.65) in (2.0.22) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{2.0.68}$$

$$-3c_1 - 16c_2 = -22 \tag{2.0.69}$$

$$2c_1 + c_2 = -5 \tag{2.0.70}$$

Solving equations (2.0.69) ,(2.0.70) using row reduction technique.

$$\implies \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \tag{2.0.71}$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \tag{2.0.72}$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \tag{2.0.73}$$

$$\implies \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \tag{2.0.74}$$

$$\implies c_1 = -\frac{102}{29} \tag{2.0.75}$$

$$c_2 = \frac{59}{29} \tag{2.0.76}$$

Substituting (2.0.64),(2.0.65),(2.0.75),(2.0.76) in (2.0.18) and (2.0.19). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \tag{2.0.77}$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \tag{2.0.78}$$

The plot of these two lines is shown in Fig. 2.

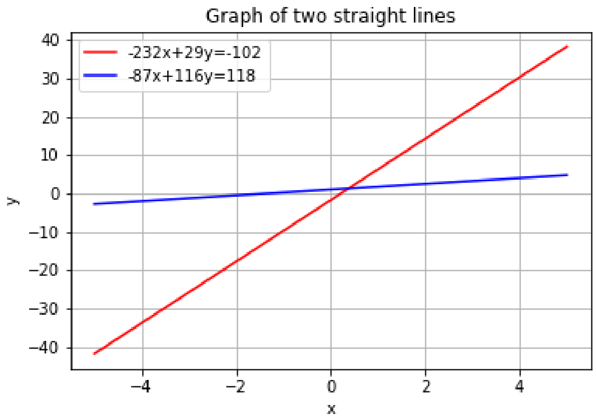

Fig. 2: Pair of straight lines for $k = -\frac{35}{2}$