Assignment 12

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https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment12

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1 Problem

(Hoffman, page 208, 4) :
Let $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ be $n \times n$ complex matrices which commute. Let $\mathbf{E}$ be the $2n \times 2n$ matrix

$$\mathbf{E} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{1.0.1}$$

prove that $\det(\mathbf{E}) = \det(\mathbf{AD} - \mathbf{BC})$

2 Solution

Given matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ commute.

Let $\mathbf{P}$ be an invertible matrix that can simultaneously diagonalize matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ as below

$$\mathbf{A} = \mathbf{P\Lambda_a P}^{-1} \tag{2.0.1}$$
$$\mathbf{B} = \mathbf{P\Lambda_b P}^{-1} \tag{2.0.2}$$
$$\mathbf{C} = \mathbf{P\Lambda_c P}^{-1} \tag{2.0.3}$$
$$\mathbf{D} = \mathbf{P\Lambda_d P}^{-1} \tag{2.0.4}$$

where $\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are diagonal matrices whose diagonal values are eigenvalues of matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ respectively and matrix $\mathbf{P}$ is formed by n-linearly independent eigen vectors.

Now (1.0.1) can be written as

$$\mathbf{E} = \begin{pmatrix} \mathbf{P\Lambda_a P}^{-1} & \mathbf{P\Lambda_b P}^{-1} \\ \mathbf{P\Lambda_c P}^{-1} & \mathbf{P\Lambda_d P}^{-1} \end{pmatrix} \tag{2.0.5}$$

Using block matrix multiplication, we get

$$\implies \mathbf{E} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} \end{pmatrix} \tag{2.0.6}$$
$$\implies \mathbf{E} = \mathbf{MDM}^{-1} \tag{2.0.7}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \tag{2.0.8}$$
$$\mathbf{D} = \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix} \tag{2.0.9}$$

Now we will calculate $\det(\mathbf{E})$,

$$\left|\mathbf{E}\right| = \left|\mathbf{MDM}^{-1}\right| \tag{2.0.10}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right| \left|\mathbf{D}\right| \left|\mathbf{M}^{-1}\right| \tag{2.0.11}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right| \left|\mathbf{D}\right| \left|\mathbf{M}\right|^{-1} \tag{2.0.12}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{D}\right| \tag{2.0.13}$$
$$\implies \left|\mathbf{E}\right| = \begin{vmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{vmatrix} \tag{2.0.14}$$

$$= \begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ \lambda_{1c} & 0 & \dots & 0 & \lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \tag{2.0.15}$$

Using row reduction,

$$\xleftrightarrow{R_{n+1} = R_{n+1} - \frac{\lambda_{1c}}{\lambda_{1a}} R_1}$$

$$\begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ 0 & 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \tag{2.0.16}$$

similarly doing elementary row operations for rows R_{n+2} to R_{2n}, we get

$$\left|\mathbf{E}\right| = \begin{vmatrix} \lambda_{1a} & \dots & 0 & \lambda_{1b} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na} & 0 & \dots & \lambda_{nb} \\ 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}} \end{vmatrix} \tag{2.0.17}$$

Since it is upper triangular matrix, then $\left|\mathbf{E}\right|$ will be multiplication of diagonal elements.

$$\implies \left|\mathbf{E}\right| = \lambda_{1a}\lambda_{2a}\dots\lambda_{na} \times \left(\lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}}\right)\dots\left(\lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}}\right) \tag{2.0.18}$$

$$\implies \left|\mathbf{E}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1c}\lambda_{1b}) \times (\lambda_{2a}\lambda_{2d} - \lambda_{2c}\lambda_{2b})\dots(\lambda_{na}\lambda_{nd} - \lambda_{nc}\lambda_{nb}) \tag{2.0.19}$$

Now we will calculate det($\mathbf{AD} - \mathbf{BC}$) by substituting (2.0.1),(2.0.2),(2.0.3),(2.0.4)

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{P\Lambda_a P^{-1} P \Lambda_d P^{-1}} - \mathbf{P\Lambda_b P^{-1} P\Lambda_c P^{-1}}\right| \tag{2.0.20}$$

$$= \left|\mathbf{P\Lambda_a\Lambda_d P^{-1}} - \mathbf{P\Lambda_b\Lambda_c P^{-1}}\right| \tag{2.0.21}$$

$$= \left|\mathbf{P}(\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c})\mathbf{P}^{-1}\right| \tag{2.0.22}$$

$$= \left|\mathbf{P}\right|\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right|\left|\mathbf{P}^{-1}\right| \tag{2.0.23}$$

$$= \left|\mathbf{P}\right|\left|\mathbf{P}\right|^{-1}\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \tag{2.0.24}$$

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \tag{2.0.25}$$

Since $\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are diagonal matrices, we get

$$\mathbf{\Lambda_a\Lambda_d} = \begin{pmatrix} \lambda_{1a}\lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2a}\lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na}\lambda_{nd} \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{\Lambda_b\Lambda_c} = \begin{pmatrix} \lambda_{1b}\lambda_{1c} & 0 & \dots & 0 \\ 0 & \lambda_{2b}\lambda_{2c} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nb}\lambda_{nc} \end{pmatrix} \tag{2.0.27}$$

$$\tag{2.0.28}$$

Substitute (2.0.26) and (2.0.27) in (2.0.25), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \begin{vmatrix} \lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c} & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc} \end{vmatrix} \tag{2.0.29}$$

$$\implies \left|\mathbf{AD} - \mathbf{BC}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c}) \times (\lambda_{2a}\lambda_{2d} - \lambda_{2b}\lambda_{2c})\dots(\lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc}) \tag{2.0.30}$$

Comparing (2.0.19) and (2.0.30) we can say that

$$\left|\mathbf{E}\right| = \left|\mathbf{AD} - \mathbf{BC}\right| \tag{2.0.31}$$

Hence proved.

3 Example

Let us consider below matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ be $n \times n$ complex matrices which commute.

$$\mathbf{A} = \begin{pmatrix} 1 - i & -2 \\ 3 & -1 - i \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 - 3i & -6 \\ 9 & -3 - 3i \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{C} = \begin{pmatrix} 7 - 7i & -14 \\ 21 & -7 - 7i \end{pmatrix} \tag{3.0.3}$$

$$\mathbf{D} = \begin{pmatrix} -2 + 2i & 4 \\ -6 & 2 + 2i \end{pmatrix} \tag{3.0.4}$$

Lets find eigenvalues of matrix $\mathbf{A}$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{3.0.5}$$

$$\implies \begin{vmatrix} 1 - i - \lambda & -2 \\ 3 & -1 - i - \lambda \end{vmatrix} = 0 \tag{3.0.6}$$

$$\implies \lambda^2 + 2\lambda i + 4 = 0 \tag{3.0.7}$$

$$\implies \lambda_{1a} = -(1 + \sqrt{5})i \quad \lambda_{2a} = (-1 + \sqrt{5})i \tag{3.0.8}$$

(3.0.8) are eigenvalues of matrix $\mathbf{A}$.

The eigenvectors of matrix $\mathbf{A}$ are

$$\mathbf{v_1} = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-1-\sqrt{5}i}{3} \\ 1 \end{pmatrix} \tag{3.0.9}$$

Since matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ commute, from (3.0.1), (3.0.2), (3.0.3), (3.0.4) we can say that

$$\mathbf{B} = 3\mathbf{A} \tag{3.0.10}$$

$$\mathbf{C} = 7\mathbf{A} \tag{3.0.11}$$

$$\mathbf{D} = -2\mathbf{A} \tag{3.0.12}$$

Then the eigenvalues of matrices **B**,**C**,**D** are

$$\lambda_{1b} = -3(1+\sqrt{5})i \quad \lambda_{2b} = 3(-1+\sqrt{5})i \tag{3.0.13}$$

$$\lambda_{1c} = -7(1+\sqrt{5})i \quad \lambda_{2c} = 7(-1+\sqrt{5})i \tag{3.0.14}$$

$$\lambda_{1d} = 2(1+\sqrt{5}+1)i \quad \lambda_{2d} = -2(-1+\sqrt{5})i \tag{3.0.15}$$

But the eigenvectors of matrices **B**,**C**,**D** are same as of matrix **A**.
The eigenvalue decomposition of matrices **A**,**B**,**C**,**D** is done as in (2.0.1), (2.0.2), (2.0.3), (2.0.4). Here eigenvector matrix **P** and $\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are

$$\mathbf{P} = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 1 & 1 \end{pmatrix} \tag{3.0.16}$$

$$\mathbf{\Lambda_a} = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix} \tag{3.0.17}$$

$$\mathbf{\Lambda_b} = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix} \tag{3.0.18}$$

$$\mathbf{\Lambda_c} = \begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \tag{3.0.19}$$

$$\mathbf{\Lambda_d} = \begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \tag{3.0.20}$$

where

$$m = -(1+\sqrt{5}) \quad n = (-1+\sqrt{5}) \tag{3.0.21}$$

From (2.0.7), we got

$$\mathbf{E} = \mathbf{MDM}^{-1} \tag{3.0.22}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \tag{3.0.23}$$

$$= \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.0.24}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix}$$

$$= \begin{pmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{pmatrix} \tag{3.0.25}$$

Now we will calculate det(E), from (2.0.13) we got

$$\left|\mathbf{E}\right| = \left|\mathbf{D}\right| \tag{3.0.26}$$

$$= \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{3.0.27}$$

Using row reduction technique,

$$\begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{3.0.28}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 7R_1} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{3.0.29}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - 7R_2} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \tag{3.0.30}$$

Now

$$\left|\mathbf{E}\right| = \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \tag{3.0.31}$$

$$\implies \left|\mathbf{E}\right| = 529m^2n^2 \tag{3.0.32}$$

Substitute (3.0.21) in (3.0.32), we get

$$\left|\mathbf{E}\right| = 529(1+\sqrt{5})^2(-1+\sqrt{5})^2 = 8464 \tag{3.0.33}$$

Now we will calculate det(**AD** − **BC**), from (2.0.25) we got

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{\Lambda_a}\mathbf{\Lambda_d} - \mathbf{\Lambda_b}\mathbf{\Lambda_c}\right| \tag{3.0.34}$$

Using (3.0.17), (3.0.18), (3.0.19), (3.0.20) we get

$$\mathbf{\Lambda_a}\mathbf{\Lambda_d} = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix}\begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \tag{3.0.35}$$

$$\implies \mathbf{\Lambda_a}\mathbf{\Lambda_d} = \begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} \tag{3.0.36}$$

$$\mathbf{\Lambda_b}\mathbf{\Lambda_c} = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix}\begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \tag{3.0.37}$$

$$\implies \mathbf{\Lambda_b}\mathbf{\Lambda_c} = \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix} \tag{3.0.38}$$

Substitute (3.0.36) and (3.0.38) in (3.0.41), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} - \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix}\right| \tag{3.0.39}$$

$$= \begin{vmatrix} 23m^2 & 0 \\ 0 & 23n^2 \end{vmatrix} \tag{3.0.40}$$

$$= 529m^2n^2 \tag{3.0.41}$$

Substitute (3.0.21) in (3.0.41), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = 529(1 + \sqrt{5})^2(-1 + \sqrt{5})^2$$
$$= 8464 \tag{3.0.42}$$

Comparing (3.0.33) and (3.0.42), we get

$$\left|\mathbf{E}\right| = \left|\mathbf{AD} - \mathbf{BC}\right| \tag{3.0.43}$$