Assignment 12

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Download latex-tikz codes from

https://github.com/Bharat437/Matrix_Theory/tree/master/Assignment12

1 Problem

(Hoffman, page 208, 4) :
Let $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ be $n \times n$ complex matrices which commute. Let $\mathbf{E}$ be the $2n \times 2n$ matrix

$$\mathbf{E} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{1.0.1}$$

prove that $\det(\mathbf{E}) = \det(\mathbf{AD} - \mathbf{BC})$

2 Solution

Given matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ commute.

Let $\mathbf{P}$ be an invertible matrix that can simultaneously diagonalize matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ as below

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda_a}\mathbf{P}^{-1} \tag{2.0.1}$$
$$\mathbf{B} = \mathbf{P}\mathbf{\Lambda_b}\mathbf{P}^{-1} \tag{2.0.2}$$
$$\mathbf{C} = \mathbf{P}\mathbf{\Lambda_c}\mathbf{P}^{-1} \tag{2.0.3}$$
$$\mathbf{D} = \mathbf{P}\mathbf{\Lambda_d}\mathbf{P}^{-1} \tag{2.0.4}$$

where $\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are diagonal matrices whose diagonal values are eigenvalues of matrices $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ respectively and matrix $\mathbf{P}$ is formed by n-linearly independent eigen vectors.

Now (1.0.1) can be written as

$$\mathbf{E} = \begin{pmatrix} \mathbf{P}\mathbf{\Lambda_a}\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda_b}\mathbf{P}^{-1} \\ \mathbf{P}\mathbf{\Lambda_c}\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda_d}\mathbf{P}^{-1} \end{pmatrix} \tag{2.0.5}$$

Using block matrix multiplication, we get

$$\implies \mathbf{E} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} \end{pmatrix} \tag{2.0.6}$$
$$\implies \mathbf{E} = \mathbf{MDM}^{-1} \tag{2.0.7}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \tag{2.0.8}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix} \tag{2.0.9}$$

Now we will calculate $\det(\mathbf{E})$,

$$\left|\mathbf{E}\right| = \left|\mathbf{MDM}^{-1}\right| \tag{2.0.10}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right|\left|\mathbf{D}\right|\left|\mathbf{M}^{-1}\right| \tag{2.0.11}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right|\left|\mathbf{D}\right|\left|\mathbf{M}\right|^{-1} \tag{2.0.12}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{D}\right| \tag{2.0.13}$$
$$\implies \left|\mathbf{E}\right| = \begin{vmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{vmatrix} \tag{2.0.14}$$

$$= \begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ \lambda_{1c} & 0 & \dots & 0 & \lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \tag{2.0.15}$$

Using row reduction,

$$\xleftrightarrow{R_{n+1}=R_{n+1}-\frac{\lambda_{1c}}{\lambda_{1a}}R_1}$$

$$\begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ 0 & 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \tag{2.0.16}$$

similarly doing elementary row operations for rows R_{n+2} to R_{2n}, we get

$$\left|\mathbf{E}\right| = \begin{vmatrix} \lambda_{1a} & \dots & 0 & \lambda_{1b} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na} & 0 & \dots & \lambda_{nb} \\ 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}} \end{vmatrix} \tag{2.0.17}$$

Since it is upper triangular matrix, then $\left|\mathbf{E}\right|$ will be multiplication of diagonal elements.

$$\implies \left|\mathbf{E}\right| = \lambda_{1a}\lambda_{2a}\dots\lambda_{na} \times \left(\lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}}\right)\dots\left(\lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}}\right) \tag{2.0.18}$$

$$\implies \left|\mathbf{E}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1c}\lambda_{1b}) \times (\lambda_{2a}\lambda_{2d} - \lambda_{2c}\lambda_{2b})\dots(\lambda_{na}\lambda_{nd} - \lambda_{nc}\lambda_{nb}) \tag{2.0.19}$$

Now we will calculate $\det(\mathbf{AD} - \mathbf{BC})$, substitute (2.0.1) to (2.0.4)

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{P\Lambda_a P^{-1} P \Lambda_d P^{-1}} - \mathbf{P\Lambda_b P^{-1} P\Lambda_c P^{-1}}\right| \tag{2.0.20}$$

$$= \left|\mathbf{P\Lambda_a\Lambda_d P^{-1}} - \mathbf{P\Lambda_b\Lambda_c P^{-1}}\right| \tag{2.0.21}$$

$$= \left|\mathbf{P}(\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c})\mathbf{P}^{-1}\right| \tag{2.0.22}$$

$$= \left|\mathbf{P}\right|\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right|\left|\mathbf{P}^{-1}\right| \tag{2.0.23}$$

$$= \left|\mathbf{P}\right|\left|\mathbf{P}\right|^{-1}\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \tag{2.0.24}$$

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \tag{2.0.25}$$

Since$\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are diagonal matrices, we get

$$\mathbf{\Lambda_a\Lambda_d} = \begin{pmatrix} \lambda_{1a}\lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2a}\lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na}\lambda_{nd} \end{pmatrix} \tag{2.0.26}$$

$$\mathbf{\Lambda_b\Lambda_c} = \begin{pmatrix} \lambda_{1b}\lambda_{1c} & 0 & \dots & 0 \\ 0 & \lambda_{2b}\lambda_{2c} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nb}\lambda_{nc} \end{pmatrix} \tag{2.0.27}$$

$$\tag{2.0.28}$$

Substitute (2.0.26) and (2.0.27) in (2.0.25), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \begin{vmatrix} \lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c} & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc} \end{vmatrix} \tag{2.0.29}$$

$$\implies \left|\mathbf{AD} - \mathbf{BC}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c}) \times (\lambda_{2a}\lambda_{2d} - \lambda_{2b}\lambda_{2c})\dots(\lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc}) \tag{2.0.30}$$

Comparing (2.0.19) and (2.0.30) we can say that

$$\left|\mathbf{E}\right| = \left|\mathbf{AD} - \mathbf{BC}\right| \tag{2.0.31}$$

Hence proved.