Lecture Notes

August 27, 2023

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1 Lecture 3

1.1 Properties of abstract reduction systems

In this lecture, we explore some properties of abstract reduction systems and look at the relationships between them.

Definition 1.1. An abstract reduction system $(A, \underset{A}{\rightarrow})$ is said to be **Church-Rosser** if

$$\forall x, y \in A, \quad x \overset{*}{\underset{A}{\leftrightarrow}} y \implies x \downarrow_A y$$

Consider abstract reduction systems A and B as shown in the figure below. System A is not Church-Rosser, since $a \overset{*}{\underset{A}{\leftrightarrow}} b$ but a and b are not joinable. System B is a simple example of a Church-Rosser system.

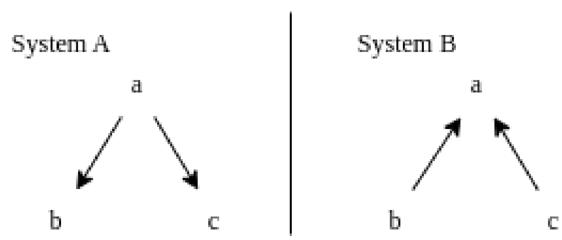

Figure 1: Examples of abstract reduction systems

Definition 1.2. An abstract reduction system $(A, \underset{A}{\rightarrow})$ is said to be **Confluent** if

$$\forall a, b, c \in A, \quad a \overset{*}{\underset{A}{\rightarrow}} b \text{ and } a \overset{*}{\underset{A}{\rightarrow}} c \implies b \downarrow_A c$$

Definition 1.3. An abstract reduction system $(A, \underset{A}{\rightarrow})$ is said to be **Semi-confluent** if

$$\forall a, b, c \in A, \quad a \underset{A}{\rightarrow} b \text{ and } a \overset{*}{\underset{A}{\rightarrow}} c \implies b \downarrow_A c$$

Theorem 1.4. For an abstract reduction system, **Church-Rosser, Confluence and Semi-confluence are equivalent**.

Proof: We will prove this theorem in the following three stages. Clearly, the three of them combined result in the theorem stated above.

1. Church-Rosser $\implies$ Confluence
2. Confluence $\implies$ Semi-confluence
3. Semi-confluence $\implies$ Church-Rosser

First, we will prove that **if a system is Church-Rosser, it is confluent**. Let $(A, \underset{A}{\rightarrow})$ be an abstract reduction system that is Church-Rosser.

Let $a, b, c \in A : a \xrightarrow[A]{*} b$ and $a \xrightarrow[A]{*} c$

By definition, $b \xleftrightarrow[A]{*} c$

Since A is Church-Rosser, $b \xleftrightarrow[A]{*} c \implies b \downarrow_A c$

i.e. $a \xrightarrow[A]{*} b$ and $a \xrightarrow[A]{*} c \implies b \downarrow_A c$, proving that A is confluent

Next, we will prove that **if a system is Confluent, it is also semi-confluent**. Let $(A, \xrightarrow[A]{})$ be a confluent abstract reduction system.

Let $a, b, c \in A : a \xrightarrow[A]{} b$ and $a \xrightarrow[A]{*} c$

Since $A \subseteq A^*$, $a \xrightarrow[A]{} b \implies a \xrightarrow[A]{*} b$

Since A is confluent, $a \xrightarrow[A]{*} b$ and $a \xrightarrow[A]{*} c \implies b \downarrow_A c$

i.e. $a \xrightarrow[A]{} b$ and $a \xrightarrow[A]{*} c \implies b \downarrow_A c$, proving that A is semi-confluent

Finally, we will prove that **if a system is semi-confluent, it is also Church-Rosser**. Let $(A, \xrightarrow[A]{})$ be a semi-confluent abstract reduction system.

Let $a, b \in A : a \xleftrightarrow[A]{*} b$

Let p be the shortest path connecting a and b in $A^{\leftrightarrow^*}$. We will use induction on $|p|$ to prove that a and b are joinable.
Base case: For $p = 0$, we have $a = b$ which makes them trivially joinable.
Induction step: Let it be true that if the shortest path connecting a and b in $A^{\leftrightarrow^*}$ is $|p|$, then a and b are joinable in A. We will prove that this is also true for $|p| + 1$.

Let $a, b' \in A$ such that the shortest path connecting them in $A^{\leftrightarrow^*}$ is of length $|p| + 1$. Then, $\exists b \in A$: the shortest path connecting a and b in $A^{\leftrightarrow^*}$ is $|p|$ and $b \xleftrightarrow[A]{} b'$. Since our induction hypothesis holds true for $|p|$, a and b are joinable (they both reduce to some $c \in A$).

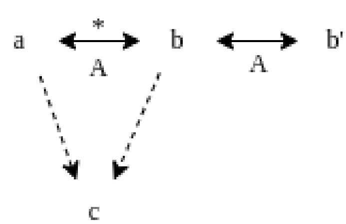

Figure 2: Abstract reduction system representing the induction step

We now have 2 cases.
Case 1: $b \xleftarrow[A]{} b'$

$b' \xrightarrow[A]{} b \xrightarrow[A]{*} c$. Therefore, $a \downarrow_A b'$.
Case 2: $b \xrightarrow[A]{} b'$

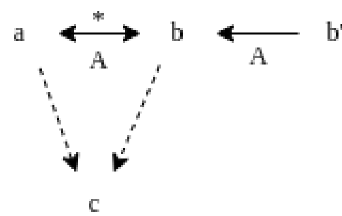

Figure 3: Abstract reduction system in case 1

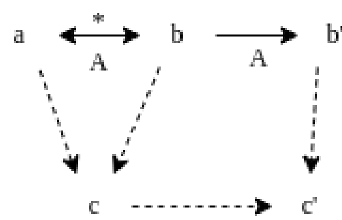

Figure 4: Abstract reduction system in case 2

Here we have $b \underset{A}{\longrightarrow} b'$ and $b \overset{*}{\underset{A}{\longleftrightarrow}} c$. Since A is semi-conflient, b' and c must be joinable. That is, $\exists c' \in A : b' \overset{*}{\underset{A}{\longrightarrow}} c'$ and $c \overset{*}{\underset{A}{\longrightarrow}} c'$. Since $a \overset{*}{\underset{A}{\longrightarrow}} c \overset{*}{\underset{A}{\longrightarrow}} c'$, we have $a \overset{*}{\underset{A}{\longrightarrow}} c'$. Hence, $a \downarrow_A b'$

In either case, we have shown that $a \downarrow_A b'$, which completes our induction. We have proven that $a \overset{*}{\underset{A}{\longleftrightarrow}} b \implies a \downarrow_A b$, which means A is also Church-Rosser.

This completes our proof that Church-Rosser, Confluence and Semi-confluence are equivalent properties of an abstract reduction system. While it may seem redundant to have multiple terms to refer to the same thing, they each give us a different perspective of looking at the same property which can prove to be helpful.

1.2 Address space

Data structures provide a way to organize and address data. For a data structure, a valid set of addresses form its address space. This is **not** a formal definition of address spaces and is only meant to give a broad idea. We will look at an example below to illustrate one way of addressing a binary tree. Let us consider the following addressing of a binary tree: each edge is labelled 1 or 2 depending on whether it leads to the left or right descendant of a node; the address of each node is obrained by appending the label of the edge leading into it to the address of its parent, with the root being ϵ. Look at the figure below to bettter understand this.

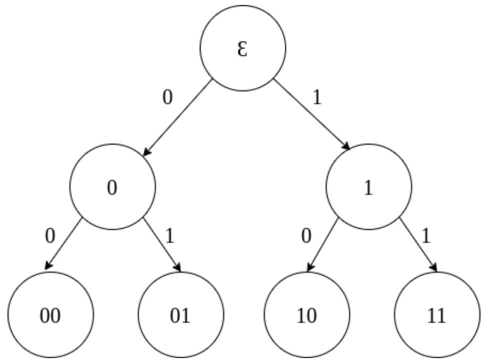

Figure 5: Addressing a binary tree

The address space for this binary tree would be the set $S = \{\epsilon, 0, 00, 01, 1, 10, 11\}$. The set $S_1 = \{\epsilon, 0, 00, 01, 1\}$ would also be a valid address space for some binary tree but the set $S_2 = \{0, 00, 1\}$ would not, since it does not contain ϵ, the address of the root.