

## Twenty-fifth International Olympiad, 1984

1. Prove that  $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$ ,  $x, y$  and  $z$  are non-negative real numbers for which  $x + y + z = 1$ .
2. Find one pair of positive integers  $a$  and  $b$  such that :
  - (i)  $ab(a + b)$  is not divisible by 7;
  - (ii)  $(a + b)^7 - a^7 - b^7$  is divisible by  $7^7$
 Justify your answer.
3. In the plane two different points  $O$  and  $A$  are given. For each point  $X$  of the plane, other than  $O$ , denote by  $a(X)$  the measure of the angle between  $OA$  and  $OX$  in radians counterclockwise from  $OA$  ( $0 \leq a(X) < 2\pi$ ). Let  $C(X)$  be the circle with center  $O$  and radius of length  $\frac{OX + a(X)}{OX}$ . Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point  $Y$  for which  $a(Y) > 0$  such that color appears on the circumference of the circle  $C(Y)$ .
4. Let  $ABCD$  be a convex quadrilateral such that the line  $CD$  is a tangent to the circle on  $AB$  as diameter. Prove that the line  $AB$  is a tangent to the circle on  $CD$  as diameter if and only if the lines  $BC$  and  $AD$  are parallel.
5. Let  $d$  be the sum of the lengths of all the diagonals of a plane convex polygon with  $n$  vertices ( $n > 3$ ), and let  $p$  be its perimeter. Prove that.

$$n - 3 < \frac{2d}{p} < \left[ \frac{n}{2} \right] \left[ \frac{n+1}{2} \right] - 2,$$

Where  $[x]$  denotes the greatest integer not exceeding  $x$

6. Let  $a, b, c$  and  $d$  be odd integers such that  $0 < a < b < c < d$  and  $ad = bc$ . Prove that if  $a + d = 2^k$  and  $b + c = 2^m$  for some integers  $k$  and  $m$ , then  $a = 1$