Twenty-fifth International Olympiad,1984

1. Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, x, y and z are non-negaive real numbers for which $x + y + z = 1$.

2. Find one pair of positive integers a and b such that :
   (i) $ab\,(a + b)$ is not divisible by 7;
   $(ii)(a + b)^7 - a^7 - b^7$ is divisible by 7^7
   Justify your answer.

3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a\,(X)$ the measure of the angle between OA and OX in radians countrclockwise from $OA\,(O \leq a\,(X) < 2\pi)$. Let $C\,(X)$ be the circle with center O and radius of length $\frac{OX+a(X)}{OX}$. each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a\,(y) > 0$ such that color appears on the circumference of the circle $C\,(Y)$.

4. Let $ABCD$ be a convex quadrilateral such tha the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$,and let p be its perimeter.Prove that.

   $$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

   Where $\left[x\right]$ denotes the gratest integer not exceeding x

6. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$