UNIT-I – LINEAR PROGRAMMING

Principal components of decision problem – Modeling phases – LP Formulation and graphic solution – Resource allocation problems – Simplex method – Sensitivity analysis.

PART-A

1. What is linear programming?

Linear programming is a technique used for determining optimum utilization of limited resources to meet out the given objectives. The objective is to maximize the profit or minimize the resources(men, machine, materials and money).

2.What are the characteristics of Standard form of LPP?

- The objective function is to maximization type.
- All the constraint equation must be equal type by adding slack or surplus variables.
- RHS of the constraint equation must be positive type.
- All the decision variables are of positive type

3. A firm manufactures two types of product A and B and sells them at profit of Rs2 on type A and Rs3 on type B. Each products is processed on two machines M1 and M2.Type A requires 1 minute of processing time onM1 and 2 minutes on M2 Type B requires 1 minute of processing time on M1 and 1 minute on M2.Machine M1 is available for not more than 6 hours 40 minutes while machine M2 is available for 10 hours during working day. Formulate the problem as a LPP so as to maximize the profit.

Maximize $z= 2x_1+3x_2$.

Subject to the constraints:

$x_1+x_2 <=400$

$2x_1+x_2<=600$

$x_1,x_2 >=0$

4. Define feasible solution.

Any Solution to a LPP which satisfies the non negativity restrictions of LPP's called the feasible solution.

5. Define optimal solution of LPP.

Any feasible solution which optimizes the objective function of the LPP's called the optimal solution.

6. State the applications of linear programming.

- Work scheduling
- Production planning and production process
- Capital budgeting
- Financial planning
- Blending
- Farm planning
- Distribution

7. State the limitations of LP.

- LP treats all functional relations as linear.
- LP does not take into account the effect of time and uncertainty
- No guarantee for integer solution. Rounding off may not feasible or optimal solution
- Deals with single objective, while in real life the situation may be difficult.

8. What is slack variable?

If the constraint as general LPP be <= type then a non negative variable is introduced convert the inequalities into equalities are called slack variables. The values of these variable are interpreted as the amount of unused resources.

9.Define basic solution.

Given a system of m linear equations with n variables($m<n$).The solution obtained by setting($n-m$) variables equal to zero and solving for the remaining m variables is called abasic solution.

10.Define basic variable and non-basic variable in linear programming.

A basic solution to the set of constraints is a solution obtained by setting any n variables equal to zero and solving for remaining m variables not equal to zero. Such m variables are called basic variables and remaining n zero variables are called non-basic variables.

11.Define surplus variable.

If the constraint as general LPP be >=type then a non negative is introduced to convert the inequalities into equalities are called the surplus variables.

12. Write the standard form of LPP in the matrix solution.

Maximize Z=CX(objective function)

Subject to AX <=b(constraints) and X>=0(non negative restrictions)

Where $C=(C_1,C_2,\ldots C_n)$

$$A = \begin{matrix} a_{11} \ldots\ldots a_{12} \ldots\ldots a_{1n} \\ a_{21} \ldots\ldots a_{22} \ldots\ldots a_{2n} \\ . \\ . \\ a_{m1} \ldots\ldots a_{m2} \ldots\ldots a_{mn} \end{matrix}$$

$$X = \begin{matrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{matrix} \qquad b = \begin{matrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{matrix}$$

13. What is sensitivity analysis? What does it signify? What is the purpose of sensitivity analysis?

After formulating mathematic model to linear programming problems and then attaining the optimal solution of the problem, it may be required to study the effect of changes in the different parameters of the problem,on the optimum solution, that is it may be desirable to see the sensitiveness of the feasible optimal solution corresponding to the variations in the parameters. The investigations that deal with changes in the optimal solutions due to discrete variations in the parameters a_{ij},b_j and c_j are called sensitivity analysis.The purpose of sensitivity analysis is to find, how to preserve , to a minimum ,the additional computational efforts which arise in solving the problem as a new one.

14.**What do you understand by degeneracy?**

The concept of obtaining a degenerate basic feasible solution in LPP is known as degeneracy. This may occur in the initial stage when atleast one basic variable is zero in the initial basic feasible solution

PART B

UNIT-I

Linear Programming.

① Principal Components of Decision Problem.

② Modeling Phases.

③ LP Formulation & Graphic Solution.

④ Resource Allocation Problem – Simplex Method.

⑤ Sensitivity Analysis.

Definition:-

Linear Programming (also called linear optimization) is a method to achieve the best outcome (such as maximum profit (or) lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear Programming is a Special case of mathematical programming (mathematical optimization).

① Principal Components of Decision Problem:-

a) Decision variables:-

Decision variables represent quantities to be determined. In any linear programming model, the decision variables should completely describe the decisions to be made to achieve the optimal Solutions.

b) Objective Function:-

It represents how the decision variables affect the cost (or) value to be optimized (minimized (or) maximized). The decision maker to maximize (or) minimize some function of the decision variables. The function to be maximized (or) minimized is called the objective function.

c) Constraints:-

Constraints represent how the decision variables use resources, which are available in limited quantities. In a linear program, the objective function and the constraints are linear relationships, meaning that the effect of changing a decision variable is proportional to its magnitude.

d) Data:-

It quantifies the relationships represented in the objective function and the constraints.

e) Sign Restrictions:-

To complete the formulation of a linear programming problem, the following question must be answered for each decision variable

Can the decision variable only assume non-negative values, (or) its the decision variable allowed to assume both positive and negative values? 1-3

(f) Linear Inequalities:-

It is an inequality which involves a linear function. A linear inequality contains one of the symbols of inequality, < is less than, > is greater than, d" is less than or equal to, e" is greater than (or) equal to, " is not equal to.

(g) The proportionality and Additivity Assumptions

The fact that the objective function for an LP must be a linear function of the decision variables has 2 implications.

- The contribution of the objective function from each decision variable is proportional to the value of the decision variable.
- The contribution to the objective function for any variable is independent of the values of the other decision variables.

(h) The Divisibility Assumption:-

The Divisibility Assumption requires that each decision variable be allowed to assume fractional values. A linear programming problem in which some (or) all of the variables must be nonnegative integers is called an "Integer Programming Problem".

(i) The Certainty Assumption:-

The Certainty Assumption is that each parameter (objective function coefficient, right hand side, and technological coefficient) is known with certainty.

(j) Feasible Region and Optimal Solution:-

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution.

The feasible region is a set of ordered pairs that satisfy a system of inequalities. It is the region which satisfies the restrictions (or) constraints, imposed in LPP.

② Modeling Phases:-

Steps involved in modeling phases are

Step 1: Formulate the Problem

Step 2: Observe the System

Step 3: Formulate a mathematical model of the Problem.

Step 4: Verify the model and Use the model for prediction.

Step 5: Select a suitable Alternative Given a model and a set of alternatives.

Step 6: Present the Results & Conclusion of the Study to the Organization.

Step 7: Implement & Evaluate Recommendations.

Linear Programming:

Linear Programming is a technique used for determining optimum utilization of limited resources to meet out the given objectives. The objective is to maximize the profit (or) minimize the resources (men, machine, materials and money)..

General Form of LPP:

① Objective Function

Max (or) min $Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n$.

② Subject to the constraints

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq \geq)\ b_2$

$\cdots$

③ Non Negative Constraints

$x_1, x_2 \ldots x_m \geq 0$

Characteristics of standard form of LPP:-

* The Objective function is of maximization type.
* All the constraint equation must be of equal type by adding slack (or) surplus variables.
* RHS of the constraint equation must be positive type. All the decision variables are of positive type.

Characteristics of Canonical form of LPP:-

In Canonical form if the objective function is of maximization type, then all constraints are of $\leq$ type. Similarly if the objective function is of minimization type, then all Constraints are of $\geq$ type. But non-negative Constraints are $\geq$ type for both cases.

Procedure for forming LPP model:-

① Identify the decision variables.

② Identify the objective function to be optimized (maximized (or) minimized) and express it as a linear function of decision variables.

③ Identify the condition of the problems such as resource limitation market demand etc.,

Problem 1:-

1) A firm manufactures two types of product A and B and sells them at a profit of Rs 2. on Type A and Rs 3 on type B. Each product is processed on 2 machines m_1 and m_2. Type A requires 1 minutes of processing time on m_1 and 2 minutes on m_2. Type B requires 1 minute on m_1 and 1 minute on m_2. Machine m_1 is available for not more than 6 hours 40 minutes while machine m_2 is available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the problem.

Solution:-

Let the firm decide to produce x_1 units of product A and x_2 units of product B to maximize in profit.

	Type A	Type B	machine available Time
m_1	1	1	6 hours 40 min
m_2	2	1	10 hours
Profit →	2	3	

* The total profit is

$$2x_1 + 3x_2.$$

The Objective function is,

$$\boxed{\text{maximize } Z = 2x_1 + 3x_2}$$

* Subject to the Constraints :-

$$x_1 + x_2 \leq 400$$

$$2x_1 + 1x_2 \leq 600$$

* And the Non -ve constraints

$$x_1, x_2 \geq 0.$$

Problem 2 :-

2) Reddy Mikks company produces both interior and exterior paints from two raw materials, M_1 and M_2. The following table provides the basic data of the problem.

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
M_1	6	4	24
M_2	1	2	6
profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also the maximum daily demand for interior paint is 2 tons. 1.9

Reddy mikks wants to determine the optimum (best) product mix of interior and exterior paints the maximizes the total daily profit.

Solution:-

$x_1 \Rightarrow$ Tons produced daily of exterior paint

$x_2 \Rightarrow$ Tons produced daily of interior paint.

* The total profit is

$$5x_1 + 4x_2$$

The objective function is

$$\boxed{\text{Max } Z = 5x_1 + 4x_2}$$

* Subject to constraints

$\Rightarrow \quad 6x_1 + 4x_2 \leq 24 \longrightarrow$ ① (Material m_1)

$\quad x_1 + 2x_2 \leq 6 \longrightarrow$ ② (Material m_2)

⇒ The 1st demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$ should not exceed 1 ton, with translates

$$\boxed{x_2 - x_1 \leq 1} \longrightarrow (3)$$

⇒ The 2nd demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates

$$\boxed{x_2 \leq 2} \longrightarrow (4)$$

∴ The subject to constraints are

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_2 \leq 2$$

* The Non negative constraints

$$x_1, x_2 \geq 0.$$

3) A manufacturer has two products P_1 and P_2, both of which are produced in 2 steps by machines M_1 and M_2. The process time per hundred for the products on the machines are,

	M_1	M_2	contribution (Per 100 unit)
P_1	4	5	10
P_2	5	2	5

The manufacturer is in a market upswing and can sell as much as he can produce of both products. Formulate the problem as LP model and determine optimum product mix.

Solution:-

* The objective function is,

$\boxed{\text{Maximize (Total contribution)}}$

$$\text{Max } Z = 10x_1 + 5x_2.$$

* Subject to the constraints

$$4x_1 + 5x_2 \leq 100$$

$$5x_1 + 2x_2 \leq 100$$

* The Non -ve constraints

$$x_1, x_2 \geq 0.$$

Graphical Method:-

Linear Programming Problem:-

Linear Programming problems which involve only two variables can be solved graphically.

Step 1:-

Consider a set of rectangular cartesian axes ox and oy in the plane. Take the decision variable x on ox and y on oy. Any point (x,y) which satisfy the conditions $x \geq 0$ and $y \geq 0$ lies in the 1st Quadrant only for any point (x,y) in the 1st Quadrent $x \geq 0$ and $y \geq 0$.

Step 2:-

Consider each inequality constraint as equation (straight line).

Convert all the inequality signs in the given constraints into equality signs.

Step 3:-

Plot each straight line on the graph. the region above the line in the 1st Quadrant in shaded.

the region below the line in the 1st Quadrant in shaded.

easible Region:- [Common Region

The common region in which all the points lying in it will simultaneously satisfy all the constraints.

This common region is called the "feasible region
The feasible region refers to the area containing those solutions which satisfy all the constraints of the problem.

Step 4:-
Assign an arbitrary value, say zero for the objective function.

Step 5:
Draw the straight line to represent the objective function with the arbitrary value.

Step 6:-
As the objective function is increased from zero, move the objective function line to the right parallel to itself until the extreme points of the feasible region.

In the maximization problem:-
This line will stop farthest away from the origin and passing through atleast one corner of the feasible region. This is the point where the maximum is obtained.

In the minimization problem:-
This line will stop nearest the origin and passing through atleast one corner of the feasible region.

1.4

Problem 1:-

Solve the following LPP graphically.

Maximize $z = 4x + 7y$.

subject to

$$x + y \leq 60$$
$$x \leq 40$$
$$y \leq 40 \quad \text{and} \quad x, y \geq 0.$$

Solution:-

Step 1:-

Replace all the inequalities of the constraints by equality signs.

$$\boxed{\begin{aligned} x + y &= 60 \\ x &= 40 \\ y &= 40 \end{aligned}}$$

Step 2:-

eqn ①

$x + y = 60$

put $x = 0$ $\boxed{y = 60}$ $(0,60)$ | Put $y = 0$ $\boxed{x = 60}$ $(60,0)$

$\therefore x + y = 60$ is a line passing through $(0,60)$ & $(60,0)$

eqn ②

$x = 40$ so $\boxed{y = 0}$

$\therefore x = 40$ is a line passing through $(40,0)$

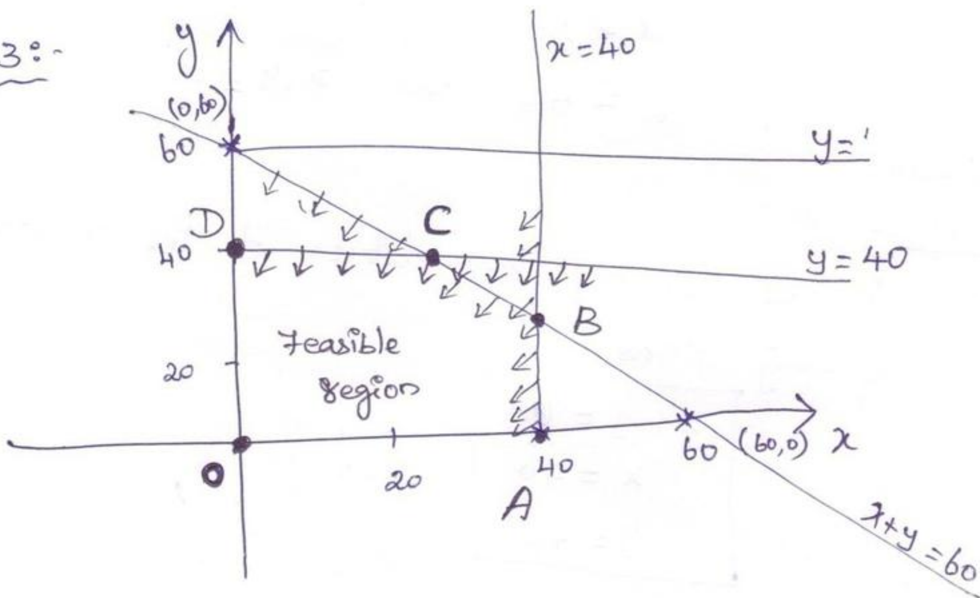
eqn (3)

$y = 40$

so $\boxed{x = 0}$

$y = 40$ is a line passing through $(0, 40)$

Step 3:-



Step 4:-

To find the optimum values.

O - $(0,0)$

A - $(40,0)$

B - ?

C - ?

D - $(0,40)$

To find B,

Solve the equations $x+y=60$ & $\boxed{x=40}$

$$x+y=60$$

$$40+y=60$$

$$y=60-40$$

$$\boxed{y=20}$$

B value $(40,20)$

To find C,

Solve the equations $x+y=60$ & $\boxed{y=40}$

$$x+y=60$$

$$x+40=60$$

$$x=60-40)$$

$$\boxed{x=20}$$

C value $(20,40)$

Step 5:-

The objective function becomes

Corner Points	$Z=4x+7y$
O $(0,0)$	0
A $(40,0)$	160
B $=(40,20)$	300
C $=(20,40)$	360 (Maximum value)
D $=(0,40)$	280

The objective function has a maximum value 360 at $(20,40)$

$\therefore$ The optimal solution is $(20,40)$

Problem 2:-

Solve the LPP using graphical method

Minimize $z = -x_1 + 2x_2$.

Subject to

$$-x_1 + 3x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2 \quad \text{and } x_1, x_2 \geq 0.$$

Solution:-

Step 1:-

Replace all the inequalities of the constraints by equality.

$$\boxed{\begin{aligned} -x_1 + 3x_2 &= 10 \\ x_1 + x_2 &= 6 \\ x_1 - x_2 &= 2 \end{aligned}}$$

Step 2:-

eqn ① $-x_1 + 3x_2 = 10$

put $\boxed{x_1 = 0}$	put $\boxed{x_2 = 0}$
$3x_2 = 10$	$-x_1 = 10$
$\boxed{x_2 = 10/3}$	$\boxed{x_1 = -10}$
$(0, 10/3)$	$(-10, 0)$

$\therefore -x_1 + 3x_2 = 10$ is a line passing through $(0, 10/3)$ & $(-10, 0)$

$(0, 3.33)$ & $(-10, 0)$

qn ②

$$x_1 + x_2 = 6$$

put $\boxed{x_1 = 0}$

$x_2 = 6$

$(0, 6)$

put $x_2 = 0$

$\boxed{x_1 = 6}$

$(6, 0)$.

$\therefore x_1 + x_2 = 6$ is a line passing through $(0,6)$ & $(6,0)$

qn ③

$$x_1 - x_2 = 2$$

put $x_1 = 0$

$x_2 = -2$

$(0, -2)$

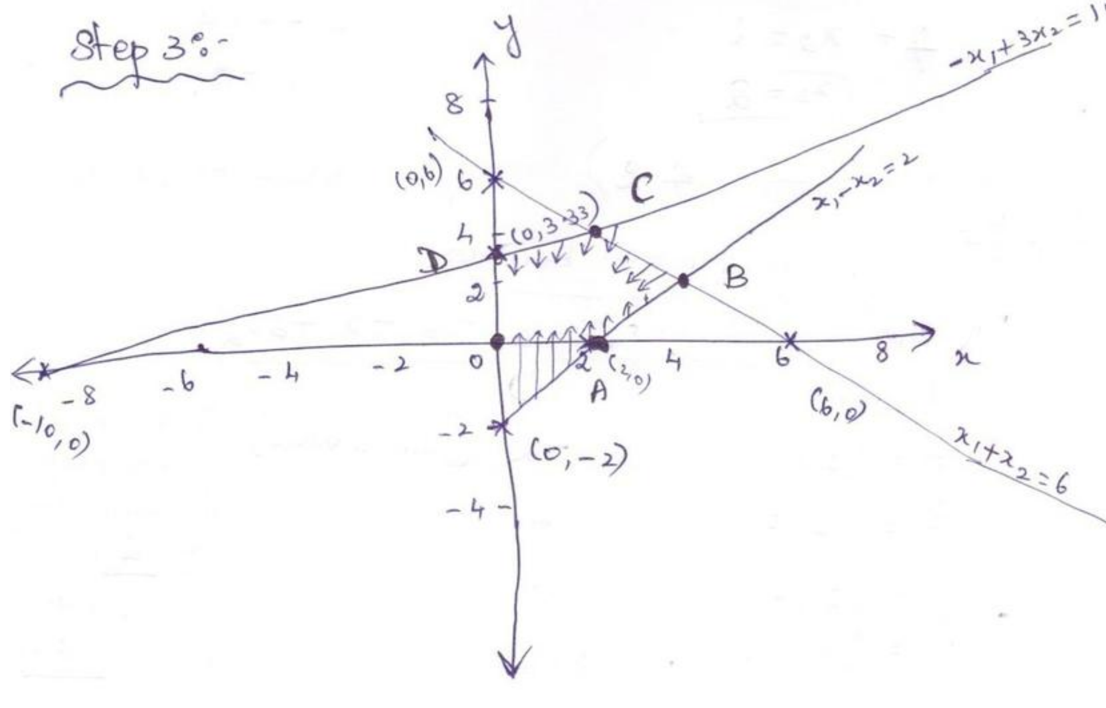
put $x_2 = 0$

$x_1 = 2$

$(2, 0)$

$\therefore x_1 - x_2$ is a line passing through $(0,-2)$ & $(2,0)$

Step 3:-



Step 4 :-

To find the optimum values.

O - (0,0)

A - (2,0)

B - ?

C - ?

D - (0,3.3)

To find B,

Solve the equations

$$\cancel{x_1} + \cancel{x_2} = 6$$
$$x_1 - \cancel{x_2} = 2$$
$$2x_1 = 8$$
$$\boxed{x_1 = 4}$$

put $\boxed{x_1 = 4}$ in $x_1 + x_2 = 6$.

$$4 + x_2 = 6$$
$$\boxed{x_2 = 2}$$

B value $\Rightarrow$ (4, 2)

To find C,

solve the equation

$$-\cancel{x_1} + 3x_2 = 10$$
$$\cancel{x_1} + x_2 = 6$$
$$4x_2 = 16$$
$$\boxed{x_2 = 4}$$

$$x_1 + 4 = 6$$
$$\boxed{x_1 = 2}$$

C value $\Rightarrow$ (2, 4)

Step 5 :-

The objective function becomes,

Vertex	Minimize $Z = -x_1 + 2x_2$
O = (0,0)	0
A = (2,0)	-2 [Small value]
B = (4,2)	4
C = (2,4)	6
D = (0,3.3	6.6

The objective function has a minimum value of 2.

The optimum solution is (2,0)

Problem 3:-

Solve graphically the following LPP

maximize $Z = 2x + 3y$.

Subject to
$$x + y \le 30$$
$$y \ge 3$$
$$0 \le y \le 12$$
$$x - y \ge 0$$
$$0 \le x \le 20$$

and $x, y \ge 0$

Solution:-

equation ①
$$x + y = 30$$

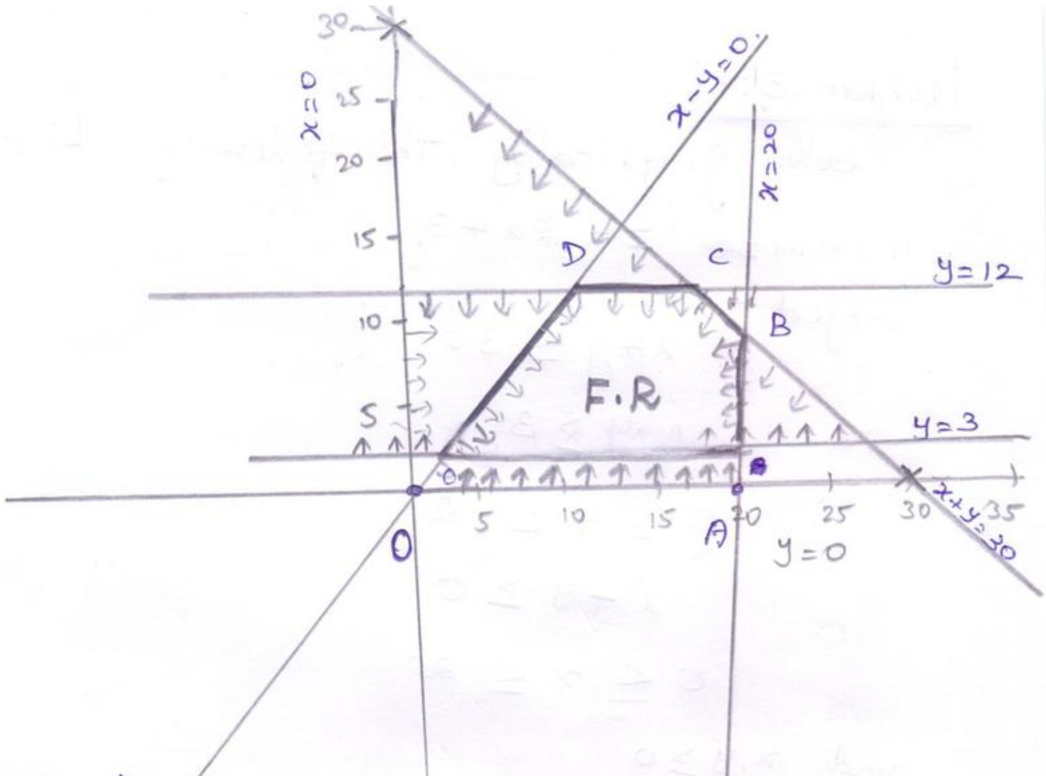
x	0	30
y	30	0

equation ②
$$y = 3$$

equation ③
$$y = 0,\ y = 12$$

equation ④
$$x - y = 0$$
$$\boxed{x = y}$$

equation ⑤
$$x = 0,\ x = 20$$



$\Rightarrow$ O (0,0)

$\Rightarrow$ A (20,0)

B ?

To find B Solve $\boxed{\begin{array}{r} x+y=30 \\ x=20 \end{array}}$

$x=20 \quad y=10$

$\Rightarrow$ B = (20,10)

To find C solve $\begin{array}{r} x+y=30 \\ y=12 \end{array}$

$\Rightarrow$ C (18,12)

To find D solve $\begin{array}{r} x-y=0 \\ y=12 \end{array}$

$\Rightarrow$ D (12,12)

Corner Points	Value of $z = 2x + 3y$.
O (0, 0)	0
A (20, 0)	40
B (20, 10)	70
C (18, 12)	72 (maximum value)
D (12, 12)	60.

$\therefore$ The optimal Solution is

$$\boxed{\begin{aligned} x &= 18 \\ y &= 12 \\ \text{Max } Z &= 72. \end{aligned}}$$

Feasible solution:-

* At any set of x_j, $j = 1, 2 \ldots n$ which satisfies the constraints is called a solution of LPP.

* Any solution which satisfies the non negative restrictions $[x_j > 0, j = 1, 2 \ldots]$ is called a feasible solution to the LPP.

Infeasible solution:-

In some LPP problems there are "no points" that satisfy all the constraints of the given problem (ie) there is no feasible region (ie) the feasible region is empty.

Optimal Solution:-

Any feasible solution which maximize (or) minimize the objective function z is called an "Optimal solution.

General Linear Programming Problem (LPP):-

The LPP involve more than 2 linear variables may be expressed as follows:-

Maximize (or) Minimize $Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$

subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \le \text{(or)} = \text{(or)} \ge b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \le \text{(or)} = \text{(or)} \ge b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = \le \text{(or)} = \text{(or)} \ge b_n$$

and the Non -ve restriction,

$$x_1, x_2, x_3 \ldots\ldots x_n \ge 0.$$

A Set of values $x_1, x_2, \ldots\ldots x_n$ which satisfy the constraints of LPP its called its solution.

Any solution to LPP which satisfies the Non negative distinction of the LPP is called its "feasible Solution".

Any feasible solution its optimizes (either max (or) min) the objective of the function is called as the "optimum solution".

Slack Variables:-

If the constraints of LPP is

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \; (i = 1, 2, 3 \ldots k)$$

then the non-ve variable S_i which a introduce to convert the inequivalities to equalities

$$\boxed{\sum_{j=1}^{n} a_{ij} x_j + S_i = b_i}$$ are called the slo va

Surplus Variable:-

If the constraints of LPP is

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \; (i = k+1, k+2 \ldots)$$

then the non-ve variables S_i which are introduce to convert the inequivalities to equ

$$\boxed{\sum_{j=1}^{n} a_{ij} x_i - S_i = b_i}$$ are called the Surpl

Standard form of LPP:-

1. The objective function is of maximize typ.
2. All constraints are expressed as "equatio
3. The RHS of each equations are +ve.
4. All variables are non -ve.

Note:-

1.26

For the minimization function, convert its into maximization function.

ie)

$$\boxed{\text{Min } f(x) = -\textbf{Max}\left\{-f(x)\right\}}$$

Ex

$$\boxed{\text{Min } Z = -\text{Max}(-Z)}$$

Example 1:-

Write the standard form of the following LPP

$$\text{Min}(Z) = 5x_1 + 7x_2$$

Subject to

$x_1 + x_2 \leq 8$ → slack

$3x_1 + 4x_2 \geq 3$ → surplus

$6x_1 + 7x_2 \geq 5$

and

$x_1, x_2 \geq 0$

Solution:-

Step 1 [Convert Min into Max]

$$\text{Min } Z = -\text{Max}(-Z) \rightarrow (*)$$
$$= -\max(z^*)$$

Step 2 [Introduce slack & surplus variable]

Introduce the slack variable S_1 to 1st constraint,

$$x_1 + x_2 \leq 8$$

$$\boxed{x_1 + x_2 + S_1 = 8} \rightarrow (*)$$

Introduce the surplus variable S_2 & S_3 in the 2^{nd} & 3^{rd} constraints

$$\boxed{\begin{aligned} 3x_1 + 4x_2 - S_2 &= 3 \\ 6x_1 + 7x_2 - S_3 &= 5 \end{aligned}} \longrightarrow (*)$$

$\therefore$ The standard form is

$$\text{Max } (z^*) = -5x_1 - 7x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

subject to constraints

$$x_1 + x_2 + S_1 + 0 \cdot S_2 + 0 \cdot S_3 = 8$$

$$3x_1 + 4x_2 + 0 \cdot S_1 - S_2 + 0 \cdot S_3 = 3.$$

$$6x_1 + 7x_2 + 0 \cdot S_1 + 0 \cdot S_2 - S_3 = 5.$$

where $x_1, x_2, S_1, S_2, S_3 \geq 0$.

Example 2:-

Express the following LPP in standard form & also write in matrix form.

$$\text{Maximize } Z = 4x_1 + 2x_2 + 6x_3.$$

Subject to

$$2x_1 + 3x_2 + 2x_3 \geq 6.$$

$$3x_1 + 4x_2 = 8$$

$$6x_1 - 4x_2 + x_3 \leq 10$$

and

$$x_1, x_2, x_3 \geq 0.$$

Solution:-

Step 1

$$Min(z) = -Max(z^*)$$

Here the Objective function is maximize. So n
change.

Step 2

By introducing "Surplus variable" in S_1 into 1^{st} Co

$$\boxed{2x_1 + 3x_2 + 2x_3 - S_1 = 6} \longrightarrow (*)$$

By Introducing "slack variable" in S_2 into 3^{rd} Co

$$\boxed{6x_1 - 4x_2 + x_3 + S_2 = 10} \longrightarrow (*)$$

→ 2^{nd} constraint equivallent sign. So no change.

$$\boxed{3x_1 + 4x_2 + 0 \cdot x_3 + 0 \cdot S_1 + 0 \cdot S_2 = 8} \rightarrow ($$

Step 3

The final Standard form is,

$$Max\ Z = 4x_1 + 2x_2 + 6x_3 + 0 \cdot S_1 + 0 \cdot S_2$$

Subject to

$$2x_1 + 3x_2 + 2x_3 - 1S_1 + 0 \cdot S_2 = 6.$$

$$3x_1 + 4x_2 + 0 \cdot x_3 + 0 \cdot S_1 + 0 \cdot S_2 = 8$$

$$6x_1 - 4x_2 + x_3 + 0 \cdot S_1 + 1 \cdot S_2 = 10$$

here

$$x_1, x_2, x_3, S_1, S_2 \geq 0.$$

The Matrix Form is,

$$\text{Max } Z = Cx.$$

Subject to

$$Ax = b$$

and

$$x \geq 0.$$

$$C = (4, 2, 6, 0, 0)$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \end{pmatrix}$$

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & S_1 & S_2 \end{array} \\ \begin{pmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Simplex Method:- [Resource Allocation Problem]

Step 1:

Check whether the objective function is to max (or) min. If it is to be minimized, then convert it into a problem of "Maximization method".

$$\boxed{\text{Min}(z) = -\text{max}(-z)}$$

Step 2:

Check whether all be b_i's are +ve. If any of the b_i's are -ve, multiply both sides of the constraints by -1. So as to make its RHS +ve.

Step 3:

By introducing slack (or) surplus variables, convert the inequivallity constraints into equations and express the given LPP into its standard form.

Step 4:

Find an initial Basic feasible solution and express the above information in the following simplex Table.

	C_j	c_1	c_2	c_3	$\cdots$	0	0	0
C_B Y_B	X_B	x_1	x_2	x_3	$\cdots$	S_1	S_2	S_3
C_{B_1} S_1	b_1	a_{11}	a_{12}	a_{13}	$\cdots$	1	0	0
C_{B_2} S_2	b_2	a_{21}	a_{22}	a_{23}	$\cdots$	0	1	0
C_{B_3} S_3	b_3	a_{31}	a_{32}	a_{33}	$\cdots$	0	0	1
$\vdots$	$\vdots$	$\vdots$						
$Z_j - C_j$	Z_0	Z_1-C_1	Z_2-C_2	Z_3-C_3	$\cdots$			

where

$C_j \Rightarrow$ Row denote the coefficient of the variable in Objective function.

$C_B \Rightarrow$ Column denote the coefficient of the Basic variables in Objective function.

$Y_B \Rightarrow$ Column denote the Basic variables.

$X_B \Rightarrow$ Column " the value of Basic variables.

The Row $Z_j - C_j$ the net evaluation.

Step 5:-

Compute the net evaluation $Z_j - C_j$, $(j = 1,2,3 \ldots n)$ by using the relation $\boxed{Z_j - C_j = C_B a_j - C_j}$

Examine the sign of $Z_j - C_j$

① If all $Z_j - C_j$ are ≥ 0 then the current basic feasible solution X_B is optimal.

② If atleast one $Z_j - C_j \leq 0$ then the current basic feasible solution is not optimal, Go to the next step.

Step 6:-

To find the entering variable.

* The entering variable is the non basic variable corresponding to the most -ve value of $Z_j - C_j$

Let it be X_r for some $j = r$. The entering variable column is known as "Key column". which is shown marked with an arrow at the bottom. If more than one variable has the same most variable $Z_j - C_j$ any of these variables may be selected arbitary has the entering variable.

Step 7:

To find the leaving variable

Compute the ratio $\theta = \text{Min}\left\{ \frac{X_{Bi}}{a_{ir}}, a_{ir} > 0 \right\}$

(ie) the ratio between the solution column and the entering variable column by considering only the +ve denominators.

a) If all $a_{ir} \leq 0$ then there is an "unbounded solution" to the given LPP.

b) If atleast one $a_{ir} > 0$ then the leaving variable is the basic variable corresponding to the minimum ratio θ.

If $\theta = \frac{X_{BK}}{a_{Kr}}$, then the basic variable X_K leaves the basics. In leaving variable row is called the "Key Row" & the element of the intersection of the Keycolumn & Key Row is called as "Keyelement". (or) loading element.

Step 8:-

Drop the leaving variable & introduce the entering variable along with its associate variable along with its associate value under C_B column.

"Convert the Keyelement unity by dividing the Key row by Key element". All other elements in its column to Zero by making use of

① $\text{New Key equation} = \dfrac{\text{Old Key equation}}{\text{Key element}}$.

② $\text{New equation} = \text{Old equation} - \begin{pmatrix}\text{Corresponding} \\ \text{column Coefficient}\end{pmatrix} \times \text{New Key equation}$

step 9

Go to step 5 & repeat the procedure until either an optimal solution is obtained.

1.34

Problem 1 :-

Use Simplex method to solve the LPP

Maximize $Z = 3x_1 + 2x_2 + 5x_3$.

Subject to Constraints

$$x_1 + 4x_2 \le 420$$

$$3x_1 + 2x_3 \le 460$$

$$x_1 + 2x_2 + x_3 \le 430.$$

and $x_1, x_2, x_3 \ge 0$.

Solution :-

Step 1 :

The Objective function is maximize Type.

So no change.

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3$$

Step 2:

All b_i's in RHS are +ve (420, 460, 430) ✓.

Step 3:

Introducing Non -ve slack variables S_1, S_2, S_3

$$x_1 + 4x_2 + S_1 = 420$$

$$3x_1 + 2x_3 + S_2 = 460$$

$$x_1 + 2x_2 + x_3 + S_3 = 430$$

$\therefore$ The standard Form for Given LPP is

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

Subject to

$$x_1 + 4x_2 + 1 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 = 420$$

$$3x_1 + 2x_3 + 0 \cdot S_1 + 1 \cdot S_2 + 0 \cdot S_3 = 460$$

$$x_1 + 2x_2 + x_3 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 = 430$$

and

$$x_1\, x_2\, x_3\, S_1\, S_2\, S_3 \geq 0.$$

Write these into matrix form,

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 420 \\ 460 \\ 430 \end{pmatrix}$$

Since there are 3 equations with 6 variables, initial Basic feasible solution is obtained by equating

$\Rightarrow$ no. of variables − no. of eqns.

$$\boxed{6 - 3 = 3 \text{ Variables to Zero}}$$

i) Put $\boxed{x_1 = x_2 = x_3 = 0}$ in the above standard form eqn.

$$\boxed{\begin{aligned} S_1 &= 420 \\ S_2 &= 460 \\ S_3 &= 430 \end{aligned}} \longrightarrow \circledast.$$

Basic Variable $\Rightarrow S_1\, S_2\, S_3$

Non " " $\Rightarrow x_1\, x_2\, x_3$

Initial Basic feasible soln $\Rightarrow$ $S_1 - 420$, $S_2 - 460$, $S_3 - 430$

Step 4 :-

Construct the table.

	C_j		3	2	5	0	0	0	
C_B	y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0	S_1	420	1	4	0	1	0	0	—
0	S_2	460	3	0	$\boxed{2}$	0	1	0	$\frac{460}{2} \Rightarrow 230$
0	S_3	430	1	2	1	0	0	1	$\frac{430}{1} \Rightarrow 430$
$Z_j - C_j$		0	-3	-2	-5	0	0	0	

(most -ve)

$$Z_j - C_j = \Sigma (C_B a_j) - C_j$$

$$= (0 \times 1 + 0 \times 3 + 0 \times 1) - 3$$

$$= -3$$

To find the entering variable :-

Since $\boxed{Z_3 - C_3 = -5}$ is the most -ve.

The corresponding Non Basic Variable x_3 enters into the basic coefficient variable. The column corresponding to x_3 is called "Key column".

To find the leaving variable :-

$$\theta = \frac{X_B}{\text{Row wise key column value}}$$

$$\theta_1 = \frac{420}{0} = 0$$

$$\theta_2 = \frac{460}{2} \Rightarrow 230$$

$$\theta_3 = \frac{430}{1} \Rightarrow 430$$

$$\theta = \text{Min}(0, 230, 430).$$

$$\boxed{\theta = 230}$$

The leaving is the basic variable S_2 which Correspondent to the minimum ratio of $\theta = 230$. The leaving variable Row is called as "Key Row".

The element of the intersection of the Key column & Key Row is 2. So 2 is Key element.

Step 5:- 1st Row → No change.

2nd Row

⇒ We have to Convert the key element into 1. $\boxed{R_2 \to \frac{R_2}{2}}$

So We have to divide all the Element in the Key Row (R_2) by 2. [R_2 becomes $\frac{3}{2}, 0, \frac{2}{2}, 0, \frac{1}{2}, 0$]

⇒ Key Column value C_j is entered into C_B on that Row (R_2).

S_2 is converted into x_3.

3rd Row

* Our main aim is Key element Convert to 1
* Except Key element all the remaining element in the Key column is converted into zero.

So convert R_3 key column element is 0.

So R_3 becomes,

		C_j	3	2	5	0	0	0	
C_B	y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0	S_1	420	1	4	0	1	0	0	$\frac{420}{4}=105$
5	x_3	$\frac{460}{2}=230$	$\frac{3}{2}$	0	$\frac{2}{2}=1$	0	$\frac{1}{2}$	0	–
0	S_3	200	$-\frac{1}{2}$	$\boxed{2}$	0	0	$-\frac{1}{2}$	1	$\frac{200}{2}=100$
$z_j - C_j$		1150	$9/2$	-2	0	0	$5/2$	0	

(most -ve)

3rd Row becomes,

3rd Row in Table 1	430	1	2	1	0	0	1
New 2nd Row in Table 2	230	$3/2$	0	1	0	$1/2$	0
	200	$-1/2$	2	0	0	$-1/2$	1

$$z_j - C_j = \Sigma (C_B a_j) - C_j$$

$$\theta = \frac{X_B}{\text{Row Key column}}$$

Step 5:-

* In Table 2 convert the key element 2 as 1.
  So divide Row 3 by 2.
* Except the key element, the remaining elements in column are converted as zero.
  So Row 1 values are change.
  $[R_1 \rightarrow R_1 - 4R_3]$

C_B	y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0	S_1	20	2	0	0	1	1	-2	
5	x_3	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	
2	x_2	$\frac{200}{2}=100$	$\frac{-1/2}{2} \Rightarrow -\frac{1}{4}$	1	0	0	$\frac{-1/2}{1/2} \Rightarrow -\frac{1}{4}$	$\frac{1}{2}$	
$Z_j - C_j$		1350	4	0	0	0	2	1	

1st Row in Table 2	420	1	4	0	1	0	0
$R_1 \to R_1 - 4R_3$	400	-1	4	0	0	-1	2
(−)	20	2	0	0	1	1	-2

Since all $Z_j - C_j \geq 0$.

Current Basic feasible solution is optimal

$\therefore$ Optimal Solution is

$$\boxed{\begin{gathered} x_2 = 100 \\ x_3 = 230 \\ x_1 = 0 \end{gathered}}$$

$$\boxed{\text{Max } Z = 1350}$$

Substitute

$Z = 3x_1 + 2x_2 + 5x_3$

$= 3(0) + 2(100) + 5(230)$

$\boxed{Z = 1350}$ ✓

1.40

Problem 2:-

Use Simplex method to minimize $Z = x_1 - 3x_2 + 2x_3$

Subject to constraints

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

and $x_1, x_2, x_3 \geq 0$.

Solution:-

Step 1:-

Since the given objective function is <u>minimize</u>. So we should convert it into maximize type.

$$\min(z) = -\max(-z)$$

$$= -\max(z)^*$$

$$\boxed{z^* = -z}$$

$$= -(x_1 - 3x_2 + 2x_3)$$

$\therefore$ The given problem becomes maximize to

$$\boxed{\text{Max } Z = -x_1 + 3x_2 - 2x_3} \longrightarrow (*)$$

Subject to constraints

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10 \quad \text{and} \quad x_1, x_2, x_3 \geq 0.$$

Step 2:-

All b_i's in RHS are +ve. So no change.

Step 3:-

Introduce slack variables S_1, S_2 & S_3.

The standard form of LPP is,

$$3x_1 - x_2 + 2x_3 + 1\cdot S_1 + 0\cdot S_2 + 1\cdot S_3 = 7$$

$$-2x_1 + 4x_2 + 0\cdot x_3 + 0\cdot S_1 + 1\cdot S_2 + 0\cdot S_3 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0\cdot S_1 + 0\cdot S_2 + 1\cdot S_3 = 10$$

The Standard Give LPP is

$$\max Z = -x_1 + 3x_2 - 2x_3 + 0\cdot S_1 + 0\cdot S_2 + 0\cdot S_3$$

Matrix form is

$$\begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$$

Since there are 3 equations with 6 variables.

$6-3 \Rightarrow 3$ variables to Zero.

put $\boxed{x_1 = x_2 = x_3 = 0}$ in the above eq^n.

$$S_1 = 7$$

$$S_2 = 12$$

$$S_3 = 10.$$

Step 4:-

Simplex Table I is,

		C_j	-1	3	-2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	$\theta = \frac{X_B}{\text{Key column}}$
0	S_1	7	3	$\cancel{-1}$	2	1	0	0	–
0	S_2	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4} = 3$
0	S_3	10	-4	$\cancel{3}$	8	0	0	1	$\frac{10}{3} = 3.3$
$Z_j - C_j$		0	1	-3	2	0	0	0	

(most -ve)

-3 is most -ve. It's Key Column.

$\theta = 3$ is minimum +ve. It's Key Row. So 4 is key element.

Step 5:-

* The non Basic variable x_2 enters into Basics. The Basic variable S_2 leaves the coefficient value.
* The key element 4 is converted into the value 1. So it should by divide by 4. And all the elements in the Key Row are divided by 4.
* Except key element remaining in the key column elements are converted by Zero. It becomes,

Table 2

	c_j		-1	3	-2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
0	S_1	10	$\boxed{5/2}$	0	2	1	$1/4$	0	$\frac{20}{5}=4$
3	x_2	$\frac{12}{4}=3$	$\frac{-2}{4} \rightarrow -1/2$	1	0	0	$1/4$	0	$-$
0	S_3	1	$-5/2$	0	8	0	$-3/4$	1	$-$
$Z_j - C_j$		9	$-1/2$	0	2	0	$3/4$	0	

↑ (most -ve)

$R_1 \rightarrow R_1 - (-1) R_2$

$\boxed{R_1 \rightarrow R_1 + R_2}$

1st Row in Table 1	7	3	-1	2	1	0	0
2nd Row in Table 2	3	$-1/2$	1	0	0	$1/4$	0 (+)
	10	$5/2$	0	2	1	$1/4$	0

$R_3 \rightarrow R_3 - 3(R_2)$

$\boxed{R_3 \rightarrow R_3 - 3R_2}$

10	-4	3	8	0	0	1
9	$-3/2$	3	0	0	$\frac{3}{4}$	0 (−)
1	$-5/2$	0	8	0	$-3/4$	1

$-1/2$ is most −ve. It's Key column.

$\theta = 4$ is maximum +ve. It's Key Row. So $5/2$ is Key element.

Step 6:-

* In R_1 Convert the key element as 1. So all the elements in R_1 are divided by $5/2$.
* In key column except the key element remaining all the elements are converted by Zero.

So $R_2 \rightarrow R_2 - \left(-\frac{1}{2}\right)R_1$

$R_3 \rightarrow R_3 + \frac{5}{2}(R_1)$

Table 3

	C_j		-1	3	-2	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	θ
-1	x_1	$\frac{10}{5/2} = 4$	$\frac{5/2}{5/2} = 1$	0	$\frac{2}{5/2} = 4/5$	$\frac{1}{5/2} = 2/5$	$\frac{1/4}{5/2} = \frac{1}{10}$	0	
3	x_2	5	0	1	$2/5$	$1/5$	$3/10$	0	
0	S_3	11	0	0	10	1	$-1/2$	1	
$Z_j - C_j$		11	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	

$R_2 \rightarrow R_2 + \left(\frac{1}{2}\right) R_1$

2nd Row in Table 2	3	$-\frac{1}{2}$	1	0	0	$1/4$	0
1st Row in Table 3 * $\left(\frac{1}{2}\right)$ R_1	$\frac{4}{2}$	$\frac{1}{2}$	0	$\frac{2}{5}$	$1/5$	$\frac{1}{20}$	0
$R_2 \Rightarrow$	5	0	1	$2/5$	$1/5$	$3/10$	0

3rd Row in Table2	1	$-5/2$	0	8	0	$-3/4$	1
Table3 $R_1 * \frac{5}{2}$	10	$5/2$	0	2	1	$1/4$	0
	11	0	0	10	1	$-1/2$	1

Since all $Z_j - C_j \geq 0$.

The current Basic feasible solution is optimal.

The optimal Solution is given by,

$$\boxed{\begin{aligned} x_1 &= 4 \\ x_2 &= 5 \\ x_3 &= 0 \end{aligned}} \rightarrow (1)$$

$$\min Z = -\max(Z^*)$$

$$\boxed{\min Z = -(11)} \rightarrow (2)$$

$$\text{Max } Z^* = x_1 - 3x_2 + 2x_3$$

$$= (4) - 3(5) + 2(0)$$

$$= 4 - 15$$

$$\boxed{\text{Max } Z^* = -11}$$

UNIT-II – DUALITY AND NETWORKS

Definition of dual problem – Primal – Dual relationships – Dual simplex methods – Post optimality analysis – Transportation and assignment model - Shortest route problem.

PART-A

1.What are the methods used in transportation problem to obtain the initial basic feasible solution.

- North-west corner rule
- Lowest cost entry method
- Vogel's approximation method.

2. What is balanced transportation problem and unbalanced transportation problem?

- When the sum of supply is equal to demands, then the problem is said to be balanced transportation problem.
- When the sum of supply is not equal to demands, then the problem is said to be unbalanced transportation problem.

3. Define unbounded assignment problem and describe the steps involved in solving it?

If the number of rows is not equal to the number of column in the given cost matrix the problem is said to be unbalanced. It is converted to a balanced one by adding the dummy row or dummy column with zero cost.

4.What is a travelling salesman problem?

A salesman normally must visit a number of cities starting from his head quarters. The distance between every pair of cities are assumed to be known. The problem of finding the shortest distance if the salesman starts from his head quarters and passes through each city exactly once and returns to the headquarters is called Travelling salesman problem.

5. Give two areas of operation of assignment problem.

- Assigning jobs to machines
- Allocating men to jobs/machines
- Route scheduling for a travelling salesman

6. Distinguish between transportation model and assignment model

Transportation problems	Assignment problems
Supply at any source may be any positive quantity	Supply at any source will be 1
Demand at any destination maybe a positive quantity	Demand at any destination will be 1
One or more source to any number of destination	One source one destination.

7. Explain the steps in the Hungarian method used for solving assignment problems.

Step1: Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix.

Step2: Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step1.

Step3:Assigning zeros

Step4:Apply optimal test

Step5:Cover all the zeros by drawing a minimum number of straight lines.

Step 6: Determine the smallest cost element not covered by the straight lines.Subtract this smallest cost element from all the uncovered elements and add this to all those element which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7: Repeat step (1) to (6),until optimum assignment is attained.

8. Give the mathematical formulation of assignment problem.

The assignment model is then given by the following LPP

Minimize Z= $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\ x_{ij}$

Subject to the constraints

$\sum_{i=1}^{n} x_{ij}$ =1,j=1,2…n

$\sum_{j=1}^{n} x_{ij}$ =1,i=1,2…n

And x_{ij} =0 or 1.

9. What is the purpose of MODI method?

Modified Distribution Method is the optimization technique used to find optimal transportation method. In MODI method,we modify our existing initial basic feasible solution through series of optimality tests to find the optimal solution.

10. What is a dual problem in linear programming?

For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optima solutions. The original problem then called primal solution where the other is called its dual problem . In general , the two problems are said to be duals of each other.

11. State the fundamentality theorem of duality

Dual to an LP in standard form

(P) maximize $c^T x$

subject to $Ax \leq b,\ 0 \leq x$

is the LP

(D) minimize $b^T y$

subject to $A^T y \geq c$, $0 \leq y$.

12. Explain the primal-dual relationship.

		x_1	x_2	x_3	…	x_n		
	y_1	a_{11}	a_{12}	a_{13}	…	a_{1n}	$<= b_1$	R.H.S
Dual	y_2	a_{21}	a_{22}	a_{23}	…	a_{2n}	$<= b_2$	Of
Variabl	…						…	primal
es	y_m	a_{m1}	a_{m2}	a_{m3}	…	a_{mn}	$<= b_m$	consta
								rints
		>=	>=	>=	….	>=		
		c_1	c_2	c_3	….	c_n		

RH.S of dual constraints

Minimize $W=b^TY$

Subject to the constraints $A^TY >= C^T$

And $Y >= 0$

13. What is the difference between regular simplex method and dual simplex method?

- The Simplex method will be the basic technique, exactly where linear programming techniques are usually derived. Within dual simplex the first schedule will be primal infeasible, due to the fact some all RHS tend to be non positive.
- In simplex method our aim is to find optimality condition using feasibility condition.But in dual method we are trying to achieve feasibility condition using optimality condition.

14. What do you mean by shadow prices?

Shadow prices are the estimated price of a good or service for which no market price exists.

15. Write down the symmetric form of dual problem.

Maximize $Z=c_1x_1+ c_2x_2+ c_3x_3+\ldots. c_nx_n$

Subject to the constraints

$a_{11}x_1+ a_{12}x_2+\ldots+ a_{1n}x_n <= b_1$

$a_{21}x_1+ a_{22}x_2+\ldots+ a_{2n}x_n <= b_2$

……………………………..

$a_{m1}x_1+ a_{m2}x_2+\ldots+ a_{mn}x_n <= b_m$

and $x_1+ x_2+\ldots\ldots..+ x_n >=0$

ie., Max $Z = CX$

subject to $AX <= b$

and $X >= 0$

PARTーB

UNIT - 2
Duality and Networks

① Dual Problem → Using Dual method & Solve LPP. → Dual Simplex Method.

② Transportation Model.

③ Assignment model.

④ Shortest Route Problem.

* ——— *

Topic 1:-

Dual Problem Introduction:-

* For every linear programming problem there is a unique linear programming problem associated with it, involving the same data and closely related optimal solutions.

* The original (given) problem is called the "primal problem". while the other is called its "dual problem".

* But in general the two problems are said to be "duals" of each other.

Formulation of dual problems:-

There are 2 important forms of primal-dual pairs namely

① Symmetric form.

② Unsymmetric form.

a) Symmetric form:-

To construct the dual problem,

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) The maximization problem has ($\leq$) constraints while the minimization problem has ($\geq$) constraints.

(iii) If the primal constains m constraints and n variables, then the dual will contain n constraints and m variables. ie) the transpose of the body matrix of the primal problem gives the body matrix of the dual and vice versa.

(iv) The constants $c_1, c_2, c_3 \ldots c_n$ in the objective function of the primal appear in the constraints of the dual.

(v) The constants $b_1, b_2 \ldots b_m$ in the constraints of the primal appear in the objectivie function of the dual.

(vi) The variables in both problems are non negative.

Example 5:

Write the dual of the following primal LPP.

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

Subject to

$$x_1 + x_2 + x_3 = 22$$

$$3x_1 + 5x_2 - 2x_3 \leq 65$$

$$x_1 + 7x_2 + 4x_3 \geq 120$$

and

$x_1 \geq 0$, $x_2 \geq 0$ and x_3 unrestricted.

Solution:

Step 1

Here the 3rd primal variable x_3 is unrestricted in sign, the corresponding 3rd dual constraint will be an equality sign.

Step 2:

The dual LPP is,

$$\max W = 22y_1 - 65y_2 + 120y_3$$

Subject to

$$y_1 - 3y_2 + y_3 \leq 4$$

$$y_1 - 5y_2 + 7y_3 \leq 5$$

$$y_1 + 2y_2 + 4y_3 = -3$$

and

$y_2, y_3 \geq 0$ and y_1 is unrestricted.

Example 1:-

Using dual simplex method solve the LPP,

$$\text{Minimize } Z = 2x_1 + x_2$$

Subject to
$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$

and $x_1, x_2 \geq 0$

Solution:-

Step 1:-

Convert the above problem into maximization type.

$$\text{Max } (Z^*) = -2x_1 - x_2$$

Subj to
$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$

and $x_1, x_2 \geq 0$.

Step 2:-

By introducing the non-negative slack variables, S_1, S_2 and S_3 the LPP becomes,

$$\text{Max } (z^*) = -2x_1 - x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

Subject to
$$-3x_1 - x_2 + 1 \cdot S_1 + 0 S_2 + 0 \cdot S_3 = -3$$
$$-4x_1 - 3x_2 + 0 \cdot S_1 + 1 \cdot S_2 + 0 \cdot S_3 = -6$$
$$-x_1 - 2x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 1 \cdot S_3 = -3$$

and $x_1, x_2 \geq 0$.

the initial basic solution is given by,

$$S_1 = -3$$
$$S_2 = -6$$
$$S_3 = -3 \qquad \left[\begin{array}{l}\text{where,}\\ x_1 = x_2 = 0\end{array}\right].$$

Step 3:-

Initial Iteration.

		C_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-3	-3	-1	1	0	0
⇒ 0	S_2	$\boxed{-6}$	-4	-3	0	1	0
0	S_3	-3	-1	-2	0	0	1
$Z_j - C_j$		0	2	1	0	0	0

* In X_B the most -ve value is -6. That the Row is "key Row".
* In that Row the -ve values are $-4, -3$ are using the following formula,

$$\theta = \text{Max}\left\{\frac{(Z_j - C_j)}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \max\left\{\frac{2}{-4}, \frac{1}{-3}\right\}$$

$$= -\frac{1}{3}$$ (ie) -3 is "key Column"

2nd Table:- Drop S_2 and introduce x_2.

	C_j		-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-1	-5/3	0	1	-1/3	0
-1	x_2	2	4/3	1	0	-1/3	0
0	S_3	1	5/3	0	0	-2/3	1
$Z_j - C_j$		-2	2/3	0	0	1/3	0

Key Row

R_1 old $\Rightarrow -3 \quad -1 \quad 1 \quad 0 \quad 0$

R_2 New $\Rightarrow 4/3 \quad 1 \quad 0 \quad -1/3 \quad 0$ (+)

$R_1 \Rightarrow -5/3 \quad 0 \quad 1 \quad -1/3 \quad 0$

R_3 old $\Rightarrow -1 \quad -2 \quad 0 \quad 0 \quad 1$

R_2 New $\times 2 \Rightarrow 8/3 \quad 2 \quad 0 \quad -2/3 \quad 0$

$5/3 \quad 0 \quad 0 \quad -2/3 \quad 1$

* In X_B the most -ve value is -1. That the Row is "Key Row".
* In that Row the -ve values are $-5/3$, $-1/3$ using the formula,

$$\theta = \max\left\{ \frac{(z_j - c_j)}{a_{ik}}, \ a_{ik} < 0 \right\}$$

$$= \max\left\{ \frac{2/3}{-5/3}, \frac{1/3}{-1/3} \right\}$$

$$= \frac{-2}{5}$$

$\boxed{-5/3 \text{ is "Key element}}$

3^{rd} Table:- Drop S_1 and introduce x_1.

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
-2	x_1	$3/5$	1	0	$-3/5$	$1/5$	0
-1	x_2	$6/5$	0	1	$4/5$	$-3/5$	0
0	S_3	0	0	0	1	-1	1
$Z_j - C_j$		$-12/5$	0	0	$2/5$	$1/5$	0

R_2 old $\Rightarrow$ $+4/3$ $\;1\;$ 0 $\;-1/3\;$ 0

R_1 New $* -4/3$ $\Rightarrow$ $-4/3$ $\;0\;$ $4/5$ $\;-4/15\;$ 0

$R_2 \Rightarrow$ 0 $\;1\;$ $4/5$ $\;-3/5\;$ 0

R_3 old $\Rightarrow$ $5/3$ $\;0\;$ 0 $\;-2/3\;$ 1

R_1 New $* -5/3$ $\Rightarrow$ $-5/3$ $\;0\;$ 1 $\;-1/3\;$ 0

$R_3 \Rightarrow$ 0 $\;0\;$ 1 $\;-1\;$ 1

In X_B all the values are +ve.

$\therefore$ The Optimum solution is

$$\text{Max } Z^* = \frac{-12}{5}, \quad \boxed{x_1 = \frac{3}{5}, \quad x_2 = 6/5}$$

But $\text{Min } Z = -\text{Max } Z^*$

$$\text{Min } Z = -\left(\frac{-12}{5}\right)$$

$$\boxed{Z = \frac{12}{5}}$$

Example 2:-

Using dual simplex method solve the LPP.

Minimize $Z = x_1 + x_2$.

subject to

$$2x_1 + x_2 \geq 2.$$

$$-x_1 - x_2 \geq 1$$

and $x_1, x_2 \geq 0$.

Solution:-

Step 1:-

Convert the given objective function Min to Max

$$\text{Max}(Z^*) = -x_1 - x_2.$$

subject to

$$-2x_1 - x_2 \leq 2.$$

$$x_1 + x_2 \leq 1.$$

and $x_1, x_2 \geq 0$.

Step 2:-

Introducing non-negative slack variables,

$$\text{Max}(Z^*) = -x_1 - x_2 + 0 \cdot S_1 + 0 \cdot S_2.$$

subject to

$$-2x_1 - x_2 + 1 \cdot S_1 + 0 \cdot S_2 = -2.$$

$$x_1 + x_2 + 0 \cdot S_1 + 1 \cdot S_2 = -1.$$

and $x_1, x_2, S_1, S_2 \geq 0$.

The Basic Initial solution is,

$S_1 = -2$, $S_2 = -1$ (where $x_1 = x_2 = 0$)

Step 3:-

1st Table:-

	C_j		-1	-1	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
0	S_1	-2	-2	-1	1	0
0	S_2	-1	1	1	0	1
$Z_j - C_j$		0	1	1	0	0

In X_B the most -ve value is -2.
It's Entering Row.

$$\theta = \max\left\{\frac{(Z_j - C_j)}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \max\left\{\frac{1}{-2}, \frac{1}{-1}\right\}$$

$$= \frac{-1}{2}$$ (The corresponding non-basic variable x_1 enters the basis)

2nd Table:-

C_B	Y_B	X_B	x_1	x_2	S_1	S_2
-1	x_1	1	1	1/2	-1/2	0
0	S_2	-2	0	1/2	1/2	1
$Z_j - C_j$		-1	0	1/2	1/2	0

In X_B the most -ve value is -2.
It's Entering Row.

Now $\theta = \max \left\{ \frac{(z_j - c_j)}{a_{ik}}, a_{ik} < 0 \right\}$

Since all the entries in the key row are positive, we cannot find the ratio θ with negative denominators. So there is no feasible solution to the given LPP.

Example 3:

Using dual simplex method to solve the LPP.

Maximize $Z = -3x_1 - 2x_2$.

Subject to

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3$$

and $x_1, x_2 \geq 0$.

Solution:-

step 1:

The given LPP is

$$\text{Max } Z = -3x_1 - 2x_2.$$

Subject to

$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$-x_1 - 2x_2 \leq -10$$
$$0x_1 + x_2 \leq 3.$$

and

$$x_1, x_2 \geq 0.$$

Step 2:

By introducing non -ve slack variables S_1, S_2, S_3 and S_4. the LPP becomes,

$$\text{Max } Z = -3x_1 - 2x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 + 0 \cdot S_4$$

$$-x_1 - x_2 + 1 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 + 0 \cdot S_4 = -1$$

$$x_1 + x_2 + 0 S_1 + 1 \cdot S_2 + 0 \cdot S_3 + 0 \cdot S_4 = 7$$

$$-x_1 - 2x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 1 \cdot S_3 + 0 \cdot S_4 = -10.$$

$$0 \cdot x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 + 1 \cdot S_4 = 3.$$

and

$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0.$$

The initial basic solution is,

$S_1 = -1$, $S_2 = 7$, $S_3 = -10$, $S_4 = 3$.

(where $x_1 = x_2 = 0$)

Step 3:

Table 1

		C_j	-3	-2 ⇓	0	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	-1	-1	-1	1	0	0	0
0	S_2	7	1	1	0	1	0	0
0	S_3 ⇒	-10	-1	-2	0	0	1	0
0	S_4	3	0	1	0	0	0	1
$Z_j - C_j$		0	3	2	0	0	0	0

In $X_B = -10$ is most -ve. (It's Key Row)

$$\theta = \max\left\{\frac{3}{-1}, \frac{2}{-2}\right\}$$

Table 2:- (Drop S_3 and Introduce x_2)

	C_j		-3	-2	0	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	4	$-1/2$	0	1	0	$-1/2$	0
0	S_2	2	$1/2$	0	0	1	$1/2$	0
-2	x_2	5	$1/2$	1	0	0	$-1/2$	0
0	S_4	-2	$-1/2$	0	0	0	$1/2$	1
$Z_j - C_j$		-10	2	0	0	0	1	0

In $\boxed{X_B = -2}$ is most -ve. (It's key Row)

$$\theta = \max\left\{\frac{2}{-1/2}\right\}$$

$$\boxed{\theta = -3}$$

Table 3:- (Drop S_4 and introduce x_1)

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
0	S_1	2	0	0	1	0	-1	-1
0	S_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
$Z_j - C_j$		-18	0	0	0	0	3	4

Since all $X_B \geq 0$, the current solution is an Optimum basic feasible solution.

$\max Z = -18$, $x_1 = 4$, $x_2 = 3$.

TRANSPORTATION MODEL

* Transportation deals with the transportation of a commodity from 'm' Sources to 'n' destination.

(orgin, Supply) $\leq$ (demand) $\geq$

* It assumes that,

(1) Level of supply at each source and the amount of demand at each destination.

(2) The Unit transportation cost of commodity from each source to each destination are known.

Mathematical formulation of a transportation problem:-

Let us assume that there are sources and destinations

Source	D_1	D_2	Destination	...	D_n	Supply
S_1						a_1
S_2						a_2
S_3						⋮
⋮						
S_m						a_m
Demand	b_1	b_2	b_3	...	b_n	

$$\boxed{\sum(a_i) = \sum(b_j)}$$

Let a_i be the supply at Souce i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from souce i to destination j and x_{ij} be the number of units shifted from souce i to destination j.

Then the transportation problem can be expressed mathematically,

$$\boxed{\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \; i = 1, 2, 3 \ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; \; j = 1, 2, 3 \ldots n$$

and

$$x_{ij} \geq 0$$

$m \rightarrow$ no. of. souces

$n \rightarrow$ no. of. destinations.

$a_i \rightarrow$ Supply to souce i

$b_j \rightarrow$ demand to destination j.

$c_{ij} \rightarrow$ Cost of Transportation unit from Souce to destination

$x_{ij} \rightarrow$ no. of. Units to be transported from Souce i to destination j.

Definition :-

A set of non-negative values,

x_{ij} $i = 1, 2, \ldots\ldots m$
$j = 1, 2, \ldots\ldots n$

that satisfies the constraints is called "feasible - solution" to the transportation problem.

A balanced transportation problem will always have a feasible solutions.

Balanced Transportation Problem:-

If the sum of the supplies of all sources are equal to the sum of the demands of all destinations. Then the problem is called as "Balanced Transportation problem".

$$\boxed{\underbrace{\sum_{i=1}^{m} a_i}_{\text{Supply}} = \underbrace{\sum_{j=1}^{n} b_j}_{\text{Demand}}}$$

UnBalanced Transportation problem:-

$$\boxed{\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j}$$

Methods for finding initial Basic feasible solution using Transportation model:-

1. Northwest corner cell Method.
2. Least Cost cell method
3. Vogels Approximation Method (VAM)

Method 1:- [North West Corner Rule]

Case(i)

If $\min\{a_1, b_1\} = a_1$ then put $x_{11} = a_1$, decreases b_1 by a_1 and move vertically to the 2nd Row.

ie) The cell (2,x) Crossout the first row.

Case(ii)

If $\min\{a_1, b_1\} = b_1$ then put $x_{11} = b_1$ and decrease a_1 by b_1 and move horizontally right. ie) to the cell (1,2) cross out the first column.

Case(iii)

If $\min\{a_1, b_1\} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Method 2:-

Least Cost Method (or) Matrix Minimum Method (or) Lower cost entry method:-

step 1:-

Identify the cell with Smallest cost and allocate $x_{ij} = \text{Min}\{a_i; b_j\}$

Case(i)

If $\min\{a_i, b_j\} = a_i$ then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i go to step 1.

Case (ii)

If $\min\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ Cross out the column and decrease a_i by b_j go to step2.

Case (iii)

If $\min\{a_i, b_j\} = a_i = b_j$ then put $x_{ij} = a_i = b_j$ cross out either i^{th} Row (or) j^{th} column but not both. goto step 2.

Step 2 :-

Repeat step 1 for the resulting reduced transportation tables until all the rim requirements are satisfied.

Method 3 :-

Vogels approximation method (or)

Unit Cost Penalty method.

Step 1 :-

Find the difference between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row [column] ie) Penalty.

Step 2 :-

Identify the row (or) column with largest penalty. choose the cell with smallest cost and cross out the satisfied row (or) column and goto step (3).

Step 3:

Again compute the column and row penalities for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Problem 1:-

Consider the following Transportation problem involving 3 sources and 4 destinations. The cell entries represent the cost of transportation per unit.

Destination

Source	1	2	3	4	Supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
Demand	250	350	400	200	

Find the initial basic feasible solution using

1. Northwest corner cell method
2. Least cost cell method
3. Vogels Approximation (or) Penalty Method.

Solution :-

Step 1 :-

$$\sum_{i=1}^{3} a_i = 300 + 400 + 500$$

$$\Rightarrow 1200 .$$

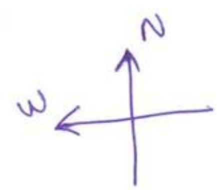
$$\sum_{j=1}^{4} b_j = 250 + 350 + 400 + 200$$

$$\Rightarrow 1200 .$$

$$\therefore \boxed{\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j}$$

So we can proceed .

(i) Northwest corner cell method :-



$\boxed{50}$ 1	7	4	50
6	5	9	400
3	3	2	500
350	400	200	

$$\begin{array}{r} 350 \\ 50 - \\ \hline 300 \end{array}$$

[300] 6	5	9	400 300 − 100
3	3	2	500
300	400	200	

[100] 5	9	100
3	2	500
400 100 − 300	200	

[300] 3	2	500 300 − 200
300	200	

[200] 2	200
200	

Final Table :-

[250] 3	[50] 1	7	4
2	[300] 6	[100] 5	9
8	3	[300] 3	[200] 2

Here the no. of. Positive independent allocations is equal to $m+n-1$.

$7 = 7$

∴ The solution is "Non degenerate Basic feasible".

The total cost is,

$\Rightarrow 3 \times 250 + 1 \times 50 + 6 \times 300 + 5 \times 100 + 3 \times 300 + 2 \times 200$.

$\Rightarrow 4400$.

Total Transportation cost $\Rightarrow 4400$.

② Least Cost cell Method :-

3	$\boxed{300}$ 1	7	4	300
2	6	5	9	400
8	3	3	2	500
250	350 300 50	400	200	

2 comes twice. So we can take any one of them

$\boxed{250}$ 2	6	5	9	400 250 − 150
8	3	3	2	500
250	50	400	200	

6	5	9	150
3	3	$\boxed{200}$ 2	500 200 − 300
50	400	200	

6	5	150
$\boxed{50}$ 3	3	300 − 50 = 250
50	400	

5	150
$\boxed{250}$ 3	250
400 − 250 = 150	

$\boxed{150}$ 5	150

Total Transportation cost is, $\Rightarrow$ no. of. positive independent allocations equal to m+n−1 So it is Non degenerate soln

3	$\boxed{300}$ 1	7	4	300
$\boxed{250}$ 2	6	$\boxed{150}$ 5	9	400
8	$\boxed{50}$ 3	$\boxed{250}$ 3	$\boxed{200}$ 2	500
250	350	400	200	

$\Rightarrow 1 \times 300 + 250 \times 2 + 50 \times 3 + 250 \times 3 + 200 \times 2 + 5 \times 150$

$\Rightarrow \underline{2850}$

③ Vogels approximation Method (VAM)

(or)

Penalty method.

				Supply	Row Penalty
3	1	7	4	300	(2)
2	6	5	9	400 250 - 150	← (3)
8	3	3	2	500	(1)
Demand 250	350	400	200		
Column Penalty (1)	(2)	(2)	(2)		

[* Take minimum of two elements in the Row & column
And subtract that elements ie) Penalty]

* [Take the maximum row (or) col Penalty. In that Row (or) C
Select the minimum cost.]

				Row Penalty
$\boxed{300}$ 1	7	4	300	← (3)
6	5	9	150	(1)
3	3	2	500	(1)
350 300 50	400	200		
Column Penalty (2)	(2)	(2)		

				Penalty
6	5	9	150	(1)
3	3	$\boxed{200}$ 2	500 200 - 300	(1)
50	400	200		
Penalty (3)	(2)	(7)		

6	5	150	(1)
[50] 3	3	300 50 − 250	(1)
50	400		
Penalty (3)	(2)		

		Penalty
5	150	(0)
[250] 3	250	(0)
400 250 − 150		
Penalty (2)		

[150] 5	150
150	

The total cost $\Rightarrow 2\times 250 + 1\times 300 + 2\times 200 + 3\times 50$

$+ 3\times 250 + 5\times 150$

$\Rightarrow \underline{\underline{2850}}$

$(m+n-1)$
$\Rightarrow 4+3-1$
$\Rightarrow 6$

Here also the number of possitive independent allocations is equal to $m+n-1$.
$\therefore$ The solution is "Non degenerate Basic feasible"

(2) Find the initial basic feasible solution of the following transportation problem using

(1) Northwest corner

(2) Least cost method

(3) Vogel's method

$$
\begin{array}{c|c}
 & \text{supply} \\
\begin{bmatrix} 2 & 3 & 11 & 7 \\ 1 & 0 & 6 & 1 \\ 5 & 8 & 15 & 9 \end{bmatrix} & \begin{matrix} 6 \\ 1 \\ 10 \end{matrix} \\
\text{Demand} \quad \begin{matrix} 7 & 5 & 3 & 2 \end{matrix} &
\end{array}
$$

<u>Solution:</u>

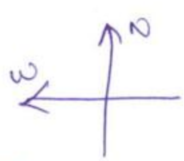
$$\sum a_i = 6 + 1 + 10 = 17.$$

$$\sum b_j = 7 + 5 + 3 + 2 = 17$$

$$\boxed{\sum a_i = \sum b_j}$$

$\therefore$ The given transportation problem is Balanced.

(i) Northwest Corner method:-



~~(6) 2~~	~~3~~	~~11~~	~~7~~	~~6~~
1	0	6	1	1.
5	8	15	9	10
7 / 6 / 1	5	3	2	

$\boxed{1}$ 1	0	6	1	~~1~~
5	8	15	9	10
~~1~~	5	3	2	

$\boxed{5}$ 8	15	9	$\frac{10}{5}$ 5
5	3	2	

$\boxed{3}$ 15	9	~~5~~ $3/2$
3	2	

$\boxed{2}$ 9	2
2	

Final Table

$\boxed{6}$ 2	3	11	7
$\boxed{1}$ 1	0	6	1
5	$\boxed{5}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9

$\boxed{\text{allocations} = m+n-1}$ ✓

$m+n-1 \Rightarrow 3+4-1$

$\Rightarrow 6$

∴ The Transportation cost

$\Rightarrow 2\times 6 + 1\times 1 + 8\times 5 + 15\times 3 + 9\times 2$

$\Rightarrow \underline{\underline{116}}$

ii) Least cost method:

2	3	11	7	6
1	1 $\boxed{0}$	6	1	1
5	8	15	9	10
7	$\frac{5}{\frac{1}{4}}$	3	2	

6 $\boxed{2}$	3	11	7	6
5	8	15	9	10
$\frac{7}{\frac{6}{1}}$	4	3	2	

1 $\boxed{5}$	8	15	9	$\frac{10}{\frac{1}{9}}$
1	4	3	2	

4 $\boxed{8}$	15	9	$\frac{9}{\frac{4}{5}}$
4	3	2	

15	$\boxed{9}$ 2	$5 \frac{2}{3}$
3	2	

15 $\boxed{3}$	3
3	

$\boxed{\text{allocations} = m+n-1}$ ✓

The transportation cost $\Rightarrow 1\times0 + 2\times6 + 1\times5 + 8\times4 + 9\times2 + 15\times3$

$\Rightarrow 0+12+5+32+18+45$

$\Rightarrow 112$

iii) Vogels Approximation Method:-

					Penalty
2	3	11	7	6	(1)
1	0	6	$\boxed{1}$ 1	1	(1)
5	8	15	9	10	(3)
7	5	3	~~2~~, 1		
Penalty (1)	(3)	(5)	(6)		

					Penalty
2	$\boxed{5}$ 3	11	7	~~6~~, 1	(1)
5	8	15	9	10	(3)
7	5	3	1		
Penalty (3)	(5)	(4)	(2)		

$\boxed{1}$ 2	11	7	1	(5)
5	15	9	10	(4)
~~7~~ 6	3	1		
(3)	(4)	(2)		

[6] 5	15	9	~~10~~ 6 4	Penalty (4)
6	3	1		
Penalty (0)	(0)	(0)		

15	[1] 9	4, 3
3	1	

[3] 15	3
3	

The final Table is

[1] 2	[5] 3	11	7
1	0	6	[1] 1
[6] 5	8	[3] 15	[1] 9

$\boxed{\text{allocation} = m+n-1}$ ✓

The transportation cost is

$\Rightarrow 2\times1 + 3\times5 + 1\times1 + 5\times6 + 15\times3 + 9\times1$

$\Rightarrow 2 + 15 + 1 + 30 + 45 + 9$

$\Rightarrow 102.$

3) Solve the transportation

Destination (columns); Source (rows)

11	20	7	8	50
21	16	20	12	40
8	12	18	9	70
30	25	35	40	

Solution:-

~~By using VAM method.~~

$$\sum a_i = 50 + 40 + 70 \Rightarrow 160$$

$$\sum b_j = 30 + 25 + 35 + 40 \Rightarrow 130.$$

$\boxed{\sum a_i \neq \sum b_j}$ It's unbalanced.

So change the problem become as,

						Penalty
11	20	35 7	8	0	50 35/15	1
21	16	20	12	0	40	4
8	12	18	9	0	70	1
30	25	35	40	30	160	
Penalty 3	8	11	1	0		

By using VAM method,

					Penalty
11	20	8	0	15	8
21	16	12	0	40	4
8	$\boxed{25}$ 12	9	0	70 25/45	1
30	25	40	30		
Penalty 3	8	1	0		

				Penalty
11	8	0	15	8
21	12	$\boxed{30}$ 0	40-10	← 12
8	9	0	45	8
30	40	30		
Penalty 3	1	0		

			Penalty
11	8	15	3
21	$\boxed{10}$ 12	10	← 9
8	9	45	1
30	~~40~~ 30		
Penalty 3	1		

11	$\boxed{15}$ 8	15	← 3
8	9	45	1
30	~~30~~ 15		
3	1		

[30] 8	9	~~45~~ 30 15	← 1
30	15		
0	0		

[15] 9	15
15	

∴ The initial transportation cost is

$\Rightarrow 7\times35 + 12\times25 + 0\times30 + 12\times10 + 8\times15 +$
$8\times30 + 9\times15$

$\Rightarrow$ Rs $\underline{1160}$

Maximization case in Transportation Problems:

If we have a transportation problem where the objective is to maximize the total profit first we have to convert the maximization problem into a minimization problem by subtracting all the entries from the highest entry in the given transportation table.

The modified minimization problem can be solved in the usual manner.

Problem 1:-

Solve the following transportation problem to maximize profit.

					Supply
	40	25	22	33	100
	44	35	30	30	30
	38	38	28	30	70
Demand	40	20	60	30	

Solution:-

Step 1: Since the given problem is maximization type. So convert this into minization problem by subtracting the cost elements from the highest cost element $(c_{ij} = 44)$

					Supply
	4	19	22	11	100
	0	9	14	14	30
	6	6	16	14	70
Demand	40	20	60	30	

The minimization problem is unbalanced.

$$\sum a_i = 100 + 30 + 70 \Rightarrow 200.$$

$$\sum b_j = 40 + 20 + 60 + 30 \Rightarrow 150.$$

$$\boxed{\sum a_i \neq \sum b_j}$$

So the Table becomes,

						Supply
	4	19	22	11	0	100
	0	9	14	14	0	30
	6	6	16	14	0	70
Demand	40	20	60	30	50	200

<u>step 2</u> :-

Using VAM method,

							Penalty
	4	19	22	11	0	100	4
	0	9	14	14	0	30	0
	6	6	16	14	0 $\boxed{50}$	70 50 20	← 6
	40	20	60	30	50		
Penalty	4	3	2	3	0		

					Penalty
4	19	22	11	100	7
$\boxed{30}$ 0	9	14	14	30	9
6	6	16	14	20	0
~~40~~ 10	20	60	30		
Penalty 4	3	2	3		

					Penalty
4	19	22	11	100	7
6	$\boxed{20}$ 6	16	14	~~20~~ 0	0
10	20	60	30		
Penalty 2	13	6	3		

				Penalty
4	22	11	100	7
$\boxed{0}$ 6	16	14	0	8
10	60	30		
Penalty 2	6	3		

$\boxed{10}$ 4	22	11	~~100~~ 90	7
10	60	30		

22	11 $\boxed{30}$	~~90~~ 60 ⇐ 11
60	30	

22 $\boxed{60}$	60
60	

The final Table becomes.

$\boxed{10}$ 4	19	22 $\boxed{60}$	11 $\boxed{30}$	0
0 $\boxed{30}$	9	14	14	0
$\boxed{0}$ 6	6 $\boxed{20}$	16	14	0 $\boxed{50}$

Substitute the allocation in 1st Table,

40 $\boxed{10}$	25	22 $\boxed{60}$	33 $\boxed{30}$	0
44 $\boxed{30}$	35	30	30	0
38 $\boxed{0}$	38 $\boxed{20}$	28	30	0 $\boxed{50}$

The Total cost is

$\Rightarrow (40\times10) + (22\times60) + (33\times30) + (44\times30) + (0\times38) + (38\times20) + (0\times50)$

$\Rightarrow 400 + 1320 + 990 + 1320 + 0 + 760 + 0$

$\Rightarrow \underline{4790}$

Transportation Algorithm using MODI method [modified distribution method] (or) UV method (or) Test for Optimal solution.

Step1:

Find the initial basic feasible solution of the given problem by North west corner (or) Least cost (or) VAM method.

Step2:

Check the number of occupied calls, the number of occupied cells is exactly equal to $m+n-1$.

Step 3:

Find the set of values u_i, v_j $(i=1,2 \cdots m)$ $(j=1,2 \cdots n)$

Step 4:

Find u_i+v_j for each unoccupies cell (i,j) and enter at the upper right corner of the corresponding cell (i,j)

Step 5:

Find the cell evaluations $\boxed{d_{ij} = c_{ij} - (v_i + v_j)}$

d_{ij} = upperleft − upper right.

Step6:

Examine the cell evaluations d_{ij} for all unoccupied cells (i,j)

* If all $d_{ij} > 0$ then the solution is optimal.
* If all $d_{ij} > 0$ with atleast one $d_{ij} = 0$ then the solution is optimal and an alternative optimal solution exists.
* If atleast one $d_{ij} < 0$ then the solution is not optimal.

Go to the next step.

Step 7:

Form a new Basic feasible solution by giving maximum allocation to the cell.

Step 8:

Repeat steps ② to ⑥ to test the optimally.

Step 9:

Continue the above procedure till an optimum solution is attained.

Problem 1:-

Solve the transportation problem using MODI method (or) finding the optimal solution.

				Supply
21	16	25	13	11
17	18	14	23	13
32	27	18	41	19
Demand 6	10	12	15	

Solution:-

$$\sum a_{ij} = 11 + 13 + 19 \Rightarrow 43.$$

$$\sum b_j = 6 + 10 + 12 + 15 \Rightarrow 43.$$

$$\boxed{\sum a_{ij} = \sum b_j}$$

The given problem is Balanced.

So proceed further.

By using VAM method,

						Penalty
	21	16	25	$\boxed{11}$ 13	~~11~~	(3)
	17	18	14	23	13	(3)
	32	27	18	41	19	(9)
	6	10	12	~~15~~ 4		
Penalty	(4)	(2)	(4)	(10)		

						Penalty
	17	18	14	$\boxed{4}$ 23	~~13~~ 9	(3)
	32	27	18	41	19	(9)
	6	10	12	4		
Penalty	(15)	(9)	(4)	(18)		

				Penalty
$\boxed{6}$ 17	18	14	~~9~~ 3	(1)
32	27	18	19	(9)
6	10	12		
(15)	(9)	(4)		

				Penalty
	18	14	3	(4)
	27	18 $\boxed{12}$	~~19~~ 7	(9)
	10	12		
Penalty	(9)	(4)		

18 $\boxed{3}$	3	(0)
27	7	(0)
~~10~~ 7		
(9)		

27 $\boxed{7}$	7
7	

By Vogel's method the Initial solution is

						Allocations
	21	16	25	13 $\boxed{11}$	11	$\boxed{1}$
	17 $\boxed{6}$	18 $\boxed{03}$	14	23 $\boxed{4}$	13	$\boxed{3}$ (maximum)
	32	27 $\boxed{7}$	18 $\boxed{12}$	41	19	$\boxed{2}$
	6	10	12	15		
Allocations	$\boxed{1}$	$\boxed{2}$	$\boxed{1}$	$\boxed{2}$		

Here $\Rightarrow m + n - 1 \Rightarrow 3 + 4 - 1 \Rightarrow 6$ ✓

The total cost is

$$\Rightarrow (13\times 11) + (17\times 6) + (18\times 3) + (23\times 4) + (27\times 7) + (18\times 12)$$

$$\Rightarrow 143 + 102 + 54 + 92 + 189 + 216$$

$$\Rightarrow \underline{796}$$

To find the optimal Solution :- [MODI method].

21	16	25	13 [11]	u_1
17 [6]	18 [3]	14	23 [4]	$u_2 = 0$
32	27 [7]	18 [12]	41	u_3
V_1	V_2	V_3	V_4	

Now we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij} = u_i + v_j$. As the 2^{nd} row contains maximum number of allocations, we choose $\boxed{u_2 = 0}$.

Therefore,

$$C_{21} = u_2 + V_1 \Rightarrow 17 = 0 + V_1 \Rightarrow \boxed{V_1 = 17}$$

$$C_{22} = u_2 + V_2 \Rightarrow 18 = 0 + V_2 \Rightarrow \boxed{V_2 = 18}$$

$$C_{24} = u_2 + V_4 \Rightarrow 23 = 0 + V_4 \Rightarrow \boxed{V_4 = 23}$$

$$C_{14} = u_1 + V_4 \Rightarrow 13 = u_1 + 23 \Rightarrow \boxed{u_1 = -10}$$

$$C_{32} = u_3 + V_2 \Rightarrow 27 = u_3 + 18 \Rightarrow \boxed{u_3 = 9}$$

$$C_{33} = u_3 + V_3 \Rightarrow 18 = 9 + V_3 \Rightarrow \boxed{V_3 = 9}$$

Thus we have the transportation table,

21	16	25	13 [11]	$u_1 = -10$
17 [6]	18 [3]	14	23 [4]	$u_2 = 0$
32	27 [7]	18 [12]	41	$u_3 = 9$
$V_1 = 17$	$V_2 = 18$	$V_3 = 9$	$V_4 = 23$	

Then we find the remaining unallocatted cells, using

$$\boxed{d_{ij} = C_{ij} - (u_i + V_j)}$$

1st Row:

$$d_{11} = C_{11} - (u_1 + V_1)$$
$$= 21 - (-10 + 17)$$
$$= 21 - 7$$
$$\boxed{d_{11} = 14}$$

$$d_{12} = C_{12} - (u_1 + V_2)$$
$$= 16 - (-10 + 18)$$
$$= 16 - 8$$
$$\boxed{d_{12} = 8}$$

$$d_{13} = C_{13} - (u_1 + V_3)$$
$$= 25 - (-10 + 9)$$
$$\boxed{d_{13} = 26}$$

IInd Row:

$$d_{23} = C_{23} - (u_2 + v_3)$$

$$= 14 - (0 + 9)$$

$$= 14 - 9$$

$$\boxed{d_{23} = 5}$$

IIIrd Row:

$$d_{31} = C_{31} - (u_3 + v_1)$$

$$= 32 - (9 + 17)$$

$$= 32 - 26$$

$$\boxed{d_{31} = 6}$$

$$d_{34} = C_{34} - (u_3 + v_4)$$

$$= 41 - (9 + 23)$$

$$= 41 - 32$$

$$\boxed{d_{34} = 9}$$

In above all the $d_{ij} > 0$.

$\therefore$ the optimal solution is 796.

21 $\boxed{14}$	16 $\boxed{8}$	25 $\boxed{26}$	13 $\boxed{11}$
17 $\boxed{6}$	18 $\boxed{3}$	14 $\boxed{15}$	23 $\boxed{4}$
32 $\boxed{6}$	27 $\boxed{7}$	18 $\boxed{12}$	41 $\boxed{9}$

ASSIGNMENT PROBLEM

(Hungarian method)

The Assignment problem is a special case of transportation with n-sources and n-destinations.

The assignment problem can be expressed as,

$$\boxed{\min \ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \ x_{ij}}$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \ , \quad j = 1, 2 \ldots n.$$

$$\sum_{j=1}^{n} x_{ij} = 1 \ , \quad i = 1, 2 \ldots n.$$

and

$$x_{ij} = 0 \ (\text{or}) \ 1.$$

Hungarian Algorithm :-

* It is also called as "Assignment algorithm.

First check whether number of rows = no. of. columns If it is so the assignment problem is said to be "Balanced". Then proceed to step 1.

* If it is not balanced, then we should convert it into balanced one before applying the algorithm.

Step 1:-

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

Step 2:
Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step 3: (Assigning the zero's)

a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zero's in the column of this encircled zero, as these will not be considered in any future assignment. Continue in this way until all the row have been examined.

Step 4: (Apply optimal Test)

a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

b) If atleast one row/column is without an assignment [(ie) if there is atleast one row/column is without one encircled zero] then the current assignment is not optimal. Go to step 5.

Step 5:-
Cover all the zeros by drawing a minimum number of straight lines as follows.

ⓐ Mark (✓) the rows that do not have assignment.

ⓑ mark (✓) the columns (not already marked) that have zeros in marked rows.

ⓒ Mark (✓) the rows (not already marked) that have assignments in marked columns.

ⓓ Repeat (b) and (c) until no more marking is required.

ⓔ Draw lines through all unmarked rows and columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution. Otherwise not.

Step 6:-

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7:

Repeat step ① to ⑥ until an Optimum assignment is attained.

* ——— *

Example 1:-

Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows.

$$\begin{array}{c|ccccc} & \multicolumn{5}{c}{\text{Job}} \\ & 1 & 2 & 3 & 4 & 5 \\ \hline A & 8 & 4 & 2 & 6 & 1 \\ B & 0 & 9 & 5 & 5 & 4 \\ C & 3 & 8 & 9 & 2 & 6 \\ D & 4 & 3 & 1 & 0 & 3 \\ E & 9 & 5 & 8 & 9 & 5 \end{array}$$

Determine the optimum assignment schedule.

Solution:-

The cost matrix of the given assignment problem is

$$\begin{bmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{bmatrix}$$

Since the number of rows is equal to the number of columns.

∴ The given assignment problem is Balanced.

Step 1:-

Select the smallest cost element in each row and subtract from all the elements of the corresponding row, we get the reduced matrix.

7	3	1	5	0
0	9	5	5	4
1	6	7	0	4
4	3	1	0	3
4	0	3	4	0

Step 2:-

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

7	3	0	5	0
0	9	4	5	4
1	6	6	0	4
4	3	0	0	3
4	0	2	4	0

Since each row and each column contains atleast one zero, we shall make assignments in the reduced matrix.

Step 3:-

$$\begin{bmatrix} 7 & 3 & \cancel{0} & 5 & \boxed{0} \\ \boxed{0} & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & \boxed{0} & 4 \\ 4 & 3 & \boxed{0} & \cancel{0} & 3 \\ 4 & \boxed{0} & 2 & 4 & \cancel{0} \end{bmatrix}$$

Step 4:

* Since each row and each column Contains exactly one assignment. (ie exactly one encircled zero) $\therefore$ The current assignment is optimal.

* The optimum assignment Schedule is given by $A \to 5$, $B \to 1$, $C \to 4$, $D \to 3$, $E \to 2$.

The Optimum (minimum) assignment cost $= (1 + 0 + 2 + 1 + 5)$ cost units

$\Rightarrow$ 9 units of cost

Example 2:-

The processing time in hours for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

Jobs \ machine	m_1	m_2	m_3	m_4	m_5
J_1	9	22	58	11	19.
J_2	43	78	72	50	63
J_3	41	28	91	37	45
J_4	74	42	27	49	39
J_5	36	11	57	22	25

Solution:-

* Since the number of rows is equal to the number of columns in the cost matrix.

* ∴ The assignment problem is balanced.

Step1:-

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Step 2:

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{bmatrix}$$

Step 3:

$$\begin{array}{c c} \begin{bmatrix} \cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 35 & 26 & 5 & 10 \\ 13 & \boxed{0} & 63 & 7 & 7 \\ 47 & 15 & \boxed{0} & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix} & \begin{array}{c} \\ \\ \checkmark \ (3) \\ \\ \checkmark \ (1) \end{array} \\ \begin{array}{ccccc} & \checkmark\ (2) & & & \end{array} & \end{array}$$

① Since 5^{th} row & 5^{th} column do not have any assignment. So put a ✓ mark on that row.

② Mark ✓ the column that have zero's in marked row. Thus column 2 is marked.

③ Mark ✓ the row that have assignments in marked columns. Thus row 3 is marked.

Draw the lines through all unmarked rows (row 1, 2 and 4) and marked columns(2).

Step 4:-

Here in step 3 The ④ is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of those straight lines and do not change the remaining elements which lie on these straight lines. We get,

0	17	49	0	0
0	39	29	5	10
9	0	59	3	3
47	19	0	20	2
21	0	42	5	0

Since each row and each column contains atleast one zero, we examine the rows and columns successively.

	M_1	M_2	M_3	M_4	M_5
J_1	$\cancel{0}$	17	49	$\boxed{0}$	$\cancel{0}$
J_2	$\boxed{0}$	39	29	5	10
J_3	9	$\boxed{0}$	59	3	3
J_4	47	19	$\boxed{0}$	20	2
J_5	21	$\cancel{0}$	42	5	$\boxed{0}$

The optimum assignment schedule is,

$J_1 \rightarrow M_4$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_3$, $J_5 \rightarrow M_5$

The optimum (minimum) processing time is

$\Rightarrow 11 + 43 + 28 + 27 + 25$ hours

$\Rightarrow$ 134 hours.

Example 3:-

Four different jobs can be done on four different machines. The set up and take down time costs are assumed to be prohibitively high for change overs. The matrix below gives the cost in rupees of processing job i on machine j.

Machines

	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized?

Solution:-

Since the $\boxed{\text{no of rows} = \text{no. of. columns}}$

$\therefore$ The assignment problem is balanced.

Step 1:

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row.

0	2	6	1
3	0	4	1
0	3	6	3
7	1	5	0

Step 2:-

Select the smallest cost element in each column and subtract this from all the elements of the corresponding <u>column</u>.

$$\begin{array}{cc}
\left[\begin{array}{cccc}
\boxed{0} & 2 & 2 & 1 \\
3 & \boxed{0} & \cancel{0} & 1 \\
\cancel{0} & 3 & 2 & 3 \\
7 & 1 & 1 & \boxed{0}
\end{array}\right] & \begin{array}{c} \checkmark \, (3) \\ \\ \checkmark \, (1) \\ \\ \end{array} \\
\begin{array}{cccc} \checkmark & & & \\ (2) & & & \end{array} &
\end{array}$$

(1) Since 3rd Row donot have any assingnment. So put a ✓ mark on that row.

(2) mark ✓ the column that have zero's in ✓ marked row. Thus column (1) is marked.

(3) mark ✓ the row that have assignmats in marked columns. Thus Row (1) is marked.

Draw the lines through all unmarked rows (row 4 & row 2) and marked column 1.

<u>Step 3:-</u>

Here 1 is the smallest cost element not covered by these straight lines. Add this 1 to those elements which lie in the intersection of those straight lines, subtract this 1 from all the uncovered elements and do not change the remaining elements which lie on the straight lines.

$$
\begin{array}{cccccl}
\cancel{0} & 1 & 1 & \cancel{0} & & \checkmark\ 1 \\
4 & \boxed{0} & \cancel{0} & 1 & & \\
\boxed{0} & 2 & 1 & 2 & & \checkmark\ 3 \\
8 & 1 & 1 & \boxed{0} & & \checkmark\ 3 \\
\checkmark\ 2 & & & \checkmark\ 2 & &
\end{array}
$$

Step 4:-

Here 1 is the smallest element. So subtract 1 from all uncovered elements and add 1 to the elements line in intersection of straight lines. do not change the remaining elements.

$$
\begin{array}{c|cccc|}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\
J_2 & 5 & \boxed{0} & \cancel{0} & 2 \\
J_3 & \cancel{0} & 1 & \boxed{0} & 2 \\
J_4 & 8 & \cancel{0} & \cancel{0} & \boxed{0} \\
\hline
\end{array}
$$

Since each row and each column contains exactly one assignment.

The optimum assignment Schedule is,

$J_1 \rightarrow M_1$, $J_2 \rightarrow M_2$, $J_3 \rightarrow M_3$, $M_4 \rightarrow M_4$.

the optimum (minimum) assignment cost is,

$\Rightarrow 5 + 5 + 10 + 3$

$\Rightarrow$ Rs 23/-

Unbalanced Assignment models:-

If the number of rows is not equal to number of columns in the cost matrix of the given assignment problem, then it is said to be "unbalanced".

Example:-

A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Jobs \ Machines	1	2	3	4
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

what are jobs assignments which will minimize the costs?

Solution:-

Step1: The $\boxed{\text{no. of row} \neq \text{no. of column}}$

The given assignment problem is unbalanced.

$\therefore$ The balanced cost matrix is given by

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Step 2:

Select the smallest element in each row & subtract this from all the elements of the row.

$$\begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3:

Select the smallest element in each column & Subtract this from all the elements of the column.

$$\begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4:

We shall assign in row & column having single zero.

$$\begin{array}{c c} \begin{bmatrix} \boxed{0} & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & (5) & 9 & 12 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \end{bmatrix} & \begin{array}{l} \\ \checkmark \; 1 \\ \checkmark \; 1 \\ \\ \end{array} \\ \begin{array}{l} 2 \end{array} & \end{array}$$

Step 5:

Here minimum element is 5. Subtract this from all the uncovered elements and add 5 to those element lie in the intersection part.

$$\begin{array}{cccc|l}
\boxed{0} & 11 & 5 & 9 & \\
0 & \boxed{0} & 4 & 6 & \\
\cancel{0} & \cancel{0} & 4 & 7 & \checkmark \ 1 \\
5 & \cancel{0} & \boxed{0} & \cancel{0} & \\
\hline
\checkmark & \checkmark & & & \\
2 & 2 & & &
\end{array}$$

Step 6:

Here minimum element is 4. Subtract this from all the uncovered elements and add 4 to those element lie in the intersection part.

$$\begin{bmatrix}
\boxed{0} & 1 & 1 & 5 \\
\cancel{0} & \boxed{0} & \cancel{0} & 2 \\
\cancel{0} & \cancel{0} & \boxed{0} & 3 \\
9 & 4 & \cancel{0} & \boxed{0}
\end{bmatrix}$$

Since each row & each column contains exactly one assignment.

∴ The optimum schedule is $A \to 1$, $B \to 2$, $C \to 3$, $D \to 4$.

The optimum (minimum) assignment cost is,

$\Rightarrow 18 + 13 + 19 + 0$

$\Rightarrow 50/-$ units of costs.

Maximization case in Assignment problems:-

The Conversion of maximization problem into an equivalent minimization problem can be done by any one of the following methods.

(i) Since $\max Z = -\min(-Z)$, multiply all the cost matrix by -1.

(ii) Subtract all the cost elements (c_{ij}) of the cost matrix from the highest cost element in that cost matrix.

Example:-

A Company has a team of four Salesman and there are four districts where the Company wants to start its business. After taking into account the capabilities of Salesman and the nature of districts, the Company estimates that the profit per day in rupees for the Salesman in each district is as below.

Districts

Salesman	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Find the assignment of Salesman to Various districts which will yield Maximum profit.

Solution:-

Step 1:

$\boxed{\text{no of Row} = \text{no. of column}}$

∴ The given problem is Balanced.

Since this maximization problem, can be converted into minimization by subtracting all the cost elements in the cost matrix from highest cost element 16.

∴ The minimization problem is,

0	6	2	5
2	5	1	1
1	1	3	4
3	4	2	1

Step 2:

Select smallest element of each row & subtract this from all the elements of that row.

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

Step 3:

Select the smallest element of each column & subtract this from all the elements of that column.

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

Step 4:

We shall make the assignment in rows & columns having single zero. We get,

$\boxed{0}$	6	2	5
1	4	$\boxed{0}$	$\cancel{0}$
$\cancel{0}$	$\boxed{0}$	2	3
2	3	1	$\boxed{0}$

∴ The optimum assignment schedule is,

A → 1 , B → 3 , C → 2 , D → 4 .

The optimum (maximum) profit.

⇒ Rs (16 + 15 + 15 + 15)

⇒ Rs 61/-

Shortest Route Problem
(Travelling Salesman Problem)

A travelling salesman problem is very similar to the assignment problem with the additional constraints.

a) The salesman should go through every city exactly once except the starting city.

b) The salesman starts from one city and comes back to that city.

c) Obviously going from any city to the same city directly is not allowed.

The necessary basic steps to solve a travelling salesman problem are,

(i) Assigning an infinitely large element (∞) in each of the squares along the diagonal line in the cost matrix.

(ii) Solving the problem as a routine assignment problem.

(iii) Scrutinizing the solution obtained under (ii) to see if the route conditions are satisfied.

(iv) If not making adjustments in assignments to satisfy the condition with minimum increase in total cost.

Example:-

Solve the following travelling salesman problem.

		To		
From	A	B	C	D
A	—	46	16	40
B	41	—	50	40
C	82	32	—	60
D	40	40	36	—

Solution:-

Step 1: The cost matrix is

$$\begin{bmatrix} \infty & 46 & 16 & 40 \\ 41 & \infty & 50 & 40 \\ 82 & 32 & \infty & 60 \\ 40 & 40 & 36 & \infty \end{bmatrix}$$

$\boxed{\text{no. of Row} = \text{no. of column}}$

Step 2: Subtract the smallest cost element in each row

$$\begin{bmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{bmatrix}$$

Step 3:

Subtract the smallest cost element in each column.

$$\begin{bmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ 3 & 4 & 0 & \infty \end{bmatrix}$$

Step 4:

Make the assignment in rows & columns having single zero.

$$\begin{bmatrix} \infty & 30 & \boxed{0} & 24 \\ \boxed{0} & \infty & 10 & \cancel{0} \\ 49 & \boxed{0} & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{bmatrix} \checkmark$$

$\checkmark$

Step 5:

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline A & \infty & 27 & \boxed{0} & 21 \\ B & 0 & \infty & 13 & \boxed{0} \\ C & 49 & \boxed{0} & \infty & 28 \\ D & \boxed{0} & 1 & 0 & \infty \\ \hline \end{array}$$

each Row & column contains exactly one zero.

The optimum assignment schedule is,

$A \to C$, $B \to D$, $C \to B$, $D \to A$.

$A \to C$, $C \to B$, $B \to D$, $D \to A$.

$A \to C \to B \longrightarrow D \longrightarrow A$

∴ The required minimum costs

$= (16 + 32 + 40 + 40)$ units of cost

$= 128$/- Units of cost.

2) Solve the following travelling salesman problem so into minimize the cost per cycle.

From \ To	A	B	C	D	E
A	–	3	6	2	3
B	3	–	5	2	3
C	6	5	–	6	4
D	2	2	6	–	6
E	3	3	4	6	–

Post Optimality Analysis:-

The following table lists the cases that can arise in post-optimal analysis and the actions needed to obtain the new solution.

Conditions after Parameter Change	Recommended actions
* Current Solution remains optimal & feasible.	* No further action is necessary.
* Current solution becomes Infeasible.	* Use dual simplex to recover feasibility.
* Current solution becomes non-optimal	* Use primal simplex to recover optimality.
* Current Solution becomes non-optimal and infeasible.	* Use the generalized Simplex method to obtain new solution.

① Changes affecting Feasibility:-

The feasibility of the current optimum solution may be affected only if,

a) The Right Hand side of the constraints are changed (or)

b) A new constraint is added to the model.

In both the cases, infeasibility occurs when at least one of the element of the right-hand side of the optimal table becomes negative - that is one (or) more of the current basic Variable becomes negative.

Changes in Right - Hand Side :-

This change requires recomputing the Right Hand side of the table as,

$$\begin{pmatrix} \text{New right Hand side} \\ \text{of table in iteration } i \end{pmatrix} = \begin{pmatrix} \text{Inverse of} \\ \text{Iteration } i \end{pmatrix} * \begin{pmatrix} \text{New} \\ \text{Right Hand} \\ \text{side of} \\ \text{constraints} \end{pmatrix}$$

Addition of New constraints :-

The addition of new constraints to the existing model can lead to one of 2 cases

- The new constraint is redundant, meaning that it is satisfied by the current optimum solution and hence can be dropped from the model together.
- The current solution violates the new constraints, in which case the dual simplex method is used to restore feasibility.

② Changes affecting optimality :-

The 2 situations that could affect the optimality of the current solution are,

- Changes in the original objective coefficients.
- Addition of the new economic activity to the model.

Changes in the Objective function Coefficients:

These changes affect only the optimality of the solution. Such changes thus require recomputing the z-row coefficients according to the following procedure.

- Compute the dual values using method 2 in optimal dual solution.
- Use the new dual values in the formula 2 of simplex dual computations to determine the new reduced costs.

Two cases will result

- New z-row satisfies the optimality condition. The solution remains unchanged (the optimum objective value may change).
- The optimality condition is not satisfied. Apply the (primal) simplex method to recover optimality.

UNIT-III – INTEGER PROGRAMMING

Cutting plan algorithm – Branch and bound methods, Multistage (Dynamic) programming.

PART A

1. What is integer programming? What are the types?

A linear programming problem in which some or all of the variables in the optimal solution are restricted to assume non-negative integer values is called an integer programming problem or integer linear programming.

Types : Mixed IPP,Pure IPP

2.Differentiate between pure and mixed programming problems.

- In a linear programming problem, if all the variables in the optimal solution are restricted to assume nonnegative integer values,then it is called the pure(all) integer programming problem.
- In a linear programming problem, if only some of the variables in the optimal solution are restricted to assume non-negative integer values, while the remaining variables are free to take any nonnegative values,then it is called mixed programming problems

3. Give any two applications of integer programming.

- All transportation, assignment and travelling salesman problems are integer programming problems, since the decision variables are either 0 or 1.
- All sequencing and routing decisions problems are integer programming problems, as it requires the integer values of the decision variables.
- Integer programming problems occur quite frequently in business and industry.

4. Write down the Gomory's fractional cut corresponding to the equation $x_1 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{-2}{3}$ That appears in the non-integer optimal simplex table of an integer programming problem.

$$\frac{-2}{3}x_3 - \frac{2}{3}x_4 + s_1 = \frac{-2}{3}$$

5. What is search method?(Branch and Bound Technique)

It is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the Branch and Bound Technique. It also starts with the continuous optimum, but systematically partitions the solution space into sub problems that eliminate parts that contain no feasible integer solution. It was originally developed by A.H.Land and A.G.Doig.

6. What is dynamic programming?

Dynamic programming is a mathematical technique of optimization using multi-stage decision process.The dynamic programming technique decomposes the original problem in n-variables in to n-sub problems(stages) each in one variable.The solution is obtained in an orderly manner by starting from one stage to the next and is completed after the final stage is reached.

7.State Bellman's principle of optimality

An optimal policy(set of decisions) has the property that whatever be the initial state and initial decisions, the remaining decisions must constitute an optimal policy for the state resulting from the first decisions.

8. State the application of the principle of optimality in dynamic programming.

- In the population area, this technique has been used for production, scheduling and employment smoothening problems.

- It is used to determine the optimal combination of advertising media (TV, Radio, Newspapers) and frequency of advertising.
- It can be used in replacement theory to determine at which age the equipment is to be replaced for optimal return from the facilities.
- Spare part level determination to guarantee high efficiency utilization of expensive equipment.

9. State any two merits of dynamic programming techniques.

Dynamic programming enables you to develop sub solutions of a large program. The sub solutions are easier to maintain,use and debug. And they possess overlapping also that means we can reuse them, these sub solutions are optimal solutions for the problem.

10. What is Zero-one problem?

If all the variables in the optimum solution are allowed to take values either 0 or 1 as in 'do' or 'not to do' type decisions, then the problem is called Zero-one problem or standard discrete programming problem.

11. Why not round off the optimum values instead of resorting to IP? (MAY '08)

There is no guarantee that the integer valued solution (obtained by simplex method) will satisfy the constraints. i.e. ., it may not satisfy one or more constraints and as such the new solution may not feasible. So there is a need for developing a systematic and efficient algorithm for obtaining the exact optimum integer solution to an IPP.

12.What are methods for IPP? (MAY '08)

Integer programming can be categorized as

(i) Cutting methods

(ii) Search Methods.

13. What is cutting method?

A systematic procedure for solving pure IPP was first developed by R.E.Gomory in 1958. Later on, he extended the procedure to solve mixed IPP, named as cutting plane algorithm, the method consists in first solving the IPP as ordinary LPP.By ignoring the integrity restriction and then introducing additional constraints one after the other to cut certain part of the solution space until an integral solution is obtained.

14. Give the general format of IPP?

The general IPP is given by Maximize Z = CX

Subject to the constraints $AX \leq b$,

$X \geq 0$ and some or all variables are integer.

15. Write an algorithm for Gomory's Fractional Cut algorithm?

1.Convert the minimization IPP into an equivalent maximization IPP and all the coefficients and constraints should be integers.

2.Find the optimum solution of the resulting maximization LPP by using simplex method.

3.Test the integrity of the optimum solution.

4.Rewrite each X_{Bi}

5. Express each of the negative fractions if any, in the k^{th} row of the optimum simplex

table as the sum of a negative integer and a non-negative fraction.

6.Find the fractional cut constraint

7. Add the fractional cut constraint at the bottom of optimum simplex table obtained in

step 2.

8. Go to step 3 and repeat the procedure until an optimum integer solution is obtained.

16. **A manufacturer of baby dolls makes two types of dolls, doll X and doll Y. Processing of these dolls is done on two machines A and B. Doll X requires 2 hours on machine A and 6 hours on Machine B. Doll Y requires 5 hours on machine A and 5 hours on Machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. The profit is gained on both the dolls is same. Format this as IPP?**

Let the manufacturer decide to manufacture x_1 the number of doll X and x_2 number of doll Y so as to maximize the profit. The complete formulation of the IPP is given by

Maximize $Z = x_1+x_2$

Subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

and ≥ 0 and are integers.

PART-B

Unit - 3

Integer Programming (IPP)

① Cutting plane Method (or) Gomory's fractional cut algorithm.

② Branch & Bound Technique (or) Search method.

③ Multistage (Dynamic Programming - DPP).

Introduction:-

* A linear programming problem (LPP) in which some (or) all of the variables in the optimal solution are restiricted to assume non-negative integer values is called an "integer programming" problem (IPP).

* The general integer programming problem is given by,

Maximize $Z = CX$.

Subject to constraints,

$AX \leq b$.

and $X \geq 0$ and some (or) all variables are integer.

Applications of IPP:-

① All transportation problems.

② Assignment and Travelling salesman problems.

Methods of Integer Programming:-

It can be categorized as,

(1) Cutting methods (or) Gomory's Fractional cut method

(2) Search methods (or) Branch & Bound method

Topic 1:

Cutting Plane Method for pure (all) integer program (IPP) (or)

Gomory's Fractional cut Algorithm:-

Step 1:-
Convert the minimization IPP into an equivalent maximization IPP.

Step 2:
Find the optimum solution using simplex method.

Step 3:-
Test the integerality of the optimum solution.

(i) If all $X_B \geq 0$ and are integers, an optimum integer solution is obtained.

(ii) If all $X_{Bi} \geq 0$ and atleast one X_{Bi} is not an integer, then go to the next step.

Step 4:
Rewrite each X_{Bi} as

$$\boxed{X_{Bi} = [X_{Bi}] + f_i}$$

Step 5:-

Express each of the negative fractions.

Step 6:-

Find the fractional cut constraints,

from the Souce row $\sum_{j=1}^{n} a_{kj} x_j = X_{BK}$

ie) $\sum_{j=1}^{n} ([a_{kj}] + f_{kj}) x_j = [X_{BK}] + f_K$.

in the form $\sum_{j=1}^{n} f_{kj} \geq f_K$.

(or)

$-\sum_{j=1}^{n} f_{kj} x_j \leq -f_k$

(or)

$-\sum_{i=1}^{n} f_{kj} + S_i = -f_K$.

S_1 is the Gomorian slack.

Step 7:-

Add the fractional cut constraints Obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step 8:-

Goto step 3 and repeat the procedure until an optimum integer solution is obtained.

Procedure to Solve Cutting Plane Method :-

Step 1 :-

Solve the given problem in Simplex method.

Step 2 :-

If Answer is integer means stop the problem.

<u>Ex</u> : $x = 2$, $x = 4$.

If the Answer is <u>floating</u> point means go to next step. (<u>Ex</u>: $x = 2$, $x = 4.4$)

Step 3 :-

Take the fraction point as "Gomory's Constraint

And solve it in dual simplex method.

Till all the <u>values</u> becomes <u>integer.</u>

Example 1 :-

Find the optimum integer solution to the following LPP.

$$\text{Max } Z = x_1 + x_2$$

Subject to constraints

$$3x_1 + 2x_2 \leq 5$$

$$x_2 \leq 2.$$

and $x_1 > 0$, $x_2 \geq 0$ and are integers.

Solution:-

Step 1 [simplex Method] :-

The given problem is maximize. So no problem.

Now standard form becomes,

$$\text{Max } Z = x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4 .$$

subject to

$$3x_1 + 2x_2 + x_3 + 0x_4 = 5.$$

$$0x_1 + x_2 + 0x_3 + x_4 = 2.$$

and $x_1, x_2, x_3, x_4 \geq 0$.

The initial basic feasible solution is given by $x_3 = 5$, $x_4 = 2$, $(x_1 = x_2 = 0)$

Step 2:-

1st Table:

	C_j		1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	$\boxed{3}$	2	1	0	$5/3$ ← most +ve.
0	x_4	2	0	1	0	1	–
$Z_j - C_j$		0	-1	-1	0	0	
			↑ (most –ve)				

Table 2: (Introduce x_1 and Drop x_3)

	C_j		1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$5/3$	1	$2/3$	$1/3$	0	$5/2$
0	x_4	2	0	$\boxed{1}$	0	1	2 ← most +ve.
$Z_j - C_j$		$5/3$	0	$-1/3$	$1/3$	0	
				↑ (most –ve)			

Table 3: (Introduce x_2 and Drop x_4)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	
1	x_2	2	0	1	0	1	
$(Z_j - C_j)$		$7/3$	0	0	$1/3$	$1/3$	

$$\text{Max } Z = 7/3 \qquad x_1 = 1/3 \qquad x_2 = 2$$

Step 3:- [fractional cut - Gomory method]

To Obtain the optimum integer solution, we have to add a fractional cut constraint in the optimum simplex table.

Since $x_1 = \frac{1}{3}$, from the souce row (first row) we have,

$$\frac{1}{3} = x_1 + \frac{1}{3} x_3 - \frac{2}{3} x_4$$

Take the -ve fractional part & split it into,

$$\left(-1 + \frac{1}{3} x_4\right)$$

$$\frac{1}{3} = \cancel{x_1} + \frac{1}{3} x_3 + \left(\cancel{-1} + \frac{1}{3} x_4\right)$$

The fractional cut (Gomorian) Constraint is given by,

$$\frac{1}{3} x_3 + \frac{1}{3} x_4 \geq \frac{1}{3}$$

$$-\frac{1}{3} x_3 - \frac{1}{3} x_4 \leq -\frac{1}{3}$$

$$\boxed{-\frac{1}{3} x_3 - \frac{1}{3} x_4 + S_1 = -\frac{1}{3}} \rightarrow (*)$$

where $S_1 \rightarrow$ Gomorian slack.

$\Downarrow$

Add this fractional cut constraint at the bottom of the above optimum simplex table.

Step 4 :-

New Table 1 [using Dual Simplex Method] add New constraints &

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
c_j			1	1	0	0	0
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	0
1	x_2	2	0	1	0	1	0
0	S_1	$-1/3$	0	0	$-1/3$	$-1/3$	1
$(Z_j - C_j)$		$7/3$	0	0	$1/3$	$1/3$	0

To obtain the feasible optimal solution, we have to use "Dual simplex method".

Since $S_1 = -\frac{1}{3}$, S_1 leaves the basis.

To find the entering variable :

$$\text{Let Max} \left\{ \frac{Z_j - C_j}{a_{ik}}, \; a_{ik} < 0 \right\}$$

$$= \text{Max} \left\{ \frac{1/3}{-1/3}, \frac{1/3}{-1/3} \right\}$$

$$= \text{Max} \{-1, -1\}$$

$= -1$ [which corresponds to both x_3 & x_4. we choose x_3 as the entering variable arbitrarily.

3.9

New Table 2 : [Drop S_1 and Introduce x_3]

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
1	x_1	0	1	0	0	−1	1
1	x_2	2	0	1	0	1	1
0	x_3	1	0	0	1	1	−3
$(z_j - c_j)$		2	0	0	0	0	1

Since all $(z_j - c_j) \geq 0$ and all $X_{Bi} \geq 0$, the current solution is feasible and optimal and integer.

$\therefore$ The optimum integer solution is

$$\text{Max } Z = 2, \; x_1 = 0, \; x_2 = 2.$$

Example 2:

Using Gomory's cutting plane method

Maximize $Z = 2x_1 + 2x_2$.

Subject to

$$5x_1 + 3x_2 \leq 8$$

$$2x_1 + 4x_2 \leq 8.$$

and $x_1, x_2 \geq 0$ and are all integers.

Solution:-

Step 1:

The given problem is maximize. So no change.

The standard form becomes,

$$\text{Max } Z = 2x_1 + 2x_2 + 0S_1 + 0S_2.$$

Subject to

$$5x_1 + 3x_2 + 1S_1 + 0S_2 = 8$$

$$2x_1 + 4x_2 + 0S_1 + 1S_2 = 8.$$

and

$$x_1, x_2, S_1, S_2 \geq 0.$$

The initial basic feasible solution is given by,

Step 2 [simplex]: $S_1 = 8$, $S_2 = 8$ $(x_1 = x_2 = 0)$

Table 1:-

	C_j		2	2	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	S_1	8	5	3	1	0	8/5
0	S_2	8	2	4	0	1	8/2
$Z_j - C_j$		0	−2	−2	0	0	

Table 2: [Introduce x_1 and drop x_3]

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
2	x_1	$8/5$	1	$3/5$	$1/5$	0	$8/3$
0	x_4	$24/5$	0	$\boxed{14/5}$	$-2/5$	1	$12/7$ ← most +ve
$Z_j - C_j$		$16/5$	0	$-4/5$	$2/5$	0	
				↑ (most -ve)			

Table 3: [Introduce x_2 and drop x_4]

		C_j	2	2	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$4/7$	1	0	$2/7$	$-3/4$
2	x_2	$12/7$	0	1	$-1/7$	$5/14$
$Z_j - C_j$		$32/7$	0	0	$2/7$	$2/7$

To obtain the optimum integer solution, we have to construct a fractional cut constraint.

Now $x_1 = 4/7 = 0 + 4/7 = [X_{B1}] + f_1$

$x_2 = 12/7 = 1 + 5/7 = [X_{B2}] + f_2$

$$\text{Max}\{f_1, f_2\} = \text{Max}\left\{\frac{4}{7}, \frac{5}{7}\right\}$$

$= \frac{5}{7}$ which corresponds to the Second row (called souce row).

Step 3: [fractional - cut Gomory method]:

From this source row, we have,

$$\frac{12}{7} = x_2 - \frac{1}{7} x_3 + \frac{5}{14} x_4 .$$

To split the -ve integer

$$1 + \frac{5}{7} = x_2 + \left(-1 + \frac{6}{7}\right) x_3 + \frac{5}{14} x_4 .$$

∴ The fractional cut (Gomorian) constraint is given by,

$$\frac{6}{7} x_3 + \frac{5}{14} x_4 \geq \frac{5}{7}$$

$$-\frac{6}{7} x_3 - \frac{5}{14} x_4 \leq -\frac{5}{7}$$

$$-\frac{6}{7} x_3 - \frac{5}{14} x_4 + S_1 = -\frac{5}{7}$$

Add this fractional cut constraint at the bottom of the above optimum simplex table,

Step 4: [Dual simplex method]

New Table 1

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
2	x_1	$4/7$	1	0	$2/7$	$-3/4$	0
2	x_2	$12/7$	0	1	$-1/7$	$5/14$	0
0	S_1	$-5/7$	0	0	$\boxed{-6/7}$	$-5/14$	1
$Z_j - C_j$		$32/7$	0	0	$2/7$	$2/7$	0

3.15

In dual simplex $X_B = -5/7$ is most $-ve$.

$\therefore S_1$ leaves the basis.

$$\theta = \text{Max}\left\{\frac{z_j - C_j}{a_{ik}},\ a_{ik} < 0\right\}$$

$$= \max\left\{\frac{2/7}{-6/7},\ \frac{2/7}{-5/14}\right\}$$

$$= \max\left\{-\frac{1}{3},\ -\frac{4}{5}\right\}$$

$= \frac{-1}{3}$ which corresponds to x_3. So x_3 enters the basis.

New Table 2: (Drop S_1 and introduce x_3).

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
2	x_1	$1/3$	1	0	0	$-1/3$	$1/3$
2	x_2	$11/6$	0	1	0	$5/12$	$-1/6$
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$
$Z_j - C_j$		$13/3$	0	0	0	$1/6$	$1/3$

Now $x_1 = \frac{1}{3} = 0 + \frac{1}{3} = [X_{B1}] + f_1$

$x_2 = \frac{11}{6} = 1 + \frac{5}{6} = [X_{B2}] + f_2$

$X_3 = \frac{5}{6} = 0 + \frac{5}{6} = [X_{B3}] + f_3$

$\therefore \text{Max}\{f_1, f_2, f_3\} = \max\left\{\frac{1}{3}, \frac{5}{6}, \frac{5}{6}\right\}$

$= \frac{5}{6}$ which corresponds to both second and third rows, we select the <u>second row</u> arbitary as the <u>source row</u>.

From this source row, we have,

$$\frac{11}{6} = x_2 + \frac{5}{12}x_4 + \frac{-1}{6}s_1$$

$$1 + \frac{5}{6} = x_2 + \frac{5}{12}x_4 + \left(-1 + \frac{5}{6}\right)s_1$$

$\therefore$ The fractional cut (Gomorian) construct is given by,

$$\frac{5}{12}x_4 + \frac{5}{6}s_1 \geq \frac{5}{6}$$

$$-\frac{5}{12}x_4 - \frac{5}{6}s_1 \leq \frac{-5}{6}$$

$$\frac{-5}{12}x_4 - \frac{5}{6}s_1 + s_2 = \frac{-5}{6}.$$

where s_2 is the Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table.

New Table 3:

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1	S_2
2	x_1	$1/3$	1	0	0	$-1/3$	$1/3$	0
2	x_2	$11/6$	0	1	0	$5/12$	$-1/6$	0
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$	0
0	S_2	$-5/6$	0	0	0	$-5/12$	$-5/6$	1
$Z_j - C_j$		$13/3$	0	0	0	$1/6$	$1/3$	0

Here the solution is optimal but infeasible. So we have to use dual simplex method to obtain the feasible solution.

Since $S_2 = -5/6$, S_2 leaves the basis.

Also $\max\left\{\dfrac{z_j - c_j}{a_{ik}},\ a_{ik} < 0\right\}$

$= \max\left\{\dfrac{1/6}{-5/12},\ \dfrac{1/3}{-5/6}\right\}$

$= \max\left\{-2/5,\ -2/5\right\}$

$= -2/5$ which corresponds to both

x_4 and S_1.

we select x_4 arbitary as the

New Table 5 [Introduce x_4 & drop s_2).

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
2	x_1	1	1	0	0	0	1	$-\frac{4}{5}$
2	x_2	1	0	1	0	0	-1	1
0	x_3	0	0	0	1	0	-2	1
0	x_4	2	0	0	0	1	2	$\frac{-12}{5}$
$Z_j - C_j$		4	0	0	0	0	0	$\frac{2}{5}$

The optimal solution to the new problem is,

$$\boxed{\begin{aligned} \text{Max } Z &= 4 \\ x_1 &= 0 \\ x_2 &= 2 \end{aligned}}$$

Gomory's Mixed Integer Method:-

In this mixed Integer problem, the fractional cut (is -ve fraction) must be split by

$$\boxed{f_k = \frac{f_k}{f_{k-1}}}.$$

Example 1:-

Solve the following mixed integer programming problem,

$$\text{Max } Z = x_1 + x_2$$

Subject to constraints,

$$2x_1 + 5x_2 \leq 16.$$

$$6x_1 + 5x_2 \leq 30.$$

and $x_2 \geq 0$, x_1 is non-negative integer.

Solution:-

Step 1: The given problem is Maximum. So no change.

The standard LPP form is,

$$\text{Max } Z = x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2$$

subject to

$$2x_1 + 5x_2 + 1 \cdot S_1 + 0 \cdot S_2 = 16.$$

$$6x_1 + 5x_2 + 0 \cdot S_1 + 1 \cdot S_2 = 30.$$

and $x_1, x_2, S_1, S_2 \geq 0$. $\boxed{\begin{matrix} S_1 = 16 \\ S_2 = 30 \end{matrix}}$

Step 2: [Simplex Method]

Table 1:-

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ	
		C_j	1	1	0	0		
0	S_1	16	2	5	1	0	8	
0	S_2	30	$\boxed{6}$	5	0	1	5	← min +ve
$Z_j - C_j$		0	-1 ↑	-1	0	0		

Table 2:- (Introduce x_1 and drop S_2)

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ	
		C_j	1	1	0	0		
0	S_1	6	0	$\boxed{\left(\frac{10}{3}\right)}$	1	$\left(\frac{-1}{3}\right)$	$\left(\frac{9}{5}\right)$	← min +ve
1	x_1	5	1	$\left(\frac{5}{6}\right)$	0	$\left(\frac{1}{6}\right)$	6	
$Z_j - C_j$		5	0	$\frac{-1}{6}$ ↑ most -ve	0	$\frac{1}{6}$		

Table 3:- (Introduce x_2 & drop S_1)

C_B	Y_B	X_B	x_1	x_2	S_1	S_2
1	x_2	$\frac{18}{10}$	0	1	$3/10$	$-1/10$
1	x_1	$\frac{7}{2}$	1	0	$-1/4$	$1/4$
$Z_j - C_j$		$\frac{53}{10}$	0	0	$\frac{1}{20}$	$\frac{3}{20}$

Since the integer constrained variable x_1 is not an integer.

So we have from the 2^{nd} (Source) row:-

$$\frac{7}{2} = x_1 + 0 \cdot x_2 - \frac{1}{4} S_1 + \frac{1}{4} S_2$$

$$3 + \frac{1}{2} = x_1 + 0 \cdot x_2 - \frac{1}{4} S_1 + \frac{1}{4} S_2$$

The Gomorian constraint is given by,

$$\boxed{\frac{f_k}{f_k - 1}}$$

$$\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2} - 1\right)} \left(\frac{-1}{4}\right) S_1 + \frac{1}{4} S_2 \geq \frac{1}{2}$$

$$+ \frac{1}{4} S_1 + \frac{1}{4} S_2 \geq \frac{1}{2}$$

$$- \frac{1}{4} S_1 - \frac{1}{4} S_2 \leq \frac{-1}{2}$$

$$\boxed{\frac{-1}{4} S_1 - \frac{1}{4} S_2 + G_1 = \frac{-1}{2}}$$

where G_1 is the Gomorian slack.

Add this Gomorian constraint at the bottom of the above optimum simplex table,

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	G_1
		C_j	1	1	0	0	0
1	x_2	$9/5$	0	1	$3/10$	$-1/10$	0
1	x_1	$7/2$	1	0	$-1/4$	$1/4$	0
0	G_1	$-1/2$	0	0	$-1/4$	$-1/4$	1
$Z_j - C_j$		$53/10$	0	0	$1/20$	$3/20$	0

Here the solution is optimal. But infeasible.

$\therefore$ Use the dual simplex Method.

Since $G_1 = -1/2$, G_1 leaves the basis.

$$\theta = \max\left\{\frac{-4}{20}, \frac{-12}{20}\right\}$$

$$= \max\left\{\frac{-1}{5}, \frac{-3}{5}\right\}$$

$= \frac{-1}{5}$ which corresponds to the Variable S_1. So S_1 enters the basis.

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	G_1
1	x_2	$6/5$	0	1	0	$-2/5$	$6/5$
1	x_1	4	1	0	0	$1/2$	-1
0	S_1	2	0	0	1	1	-4
$Z_j - C_j$		$26/5$	0	0	0	$1/10$	$1/5$

Since x_1 is integer. So Max $Z = 26/5$, $x_1 = 4$, $x_2 = 6/5$

3.21

Example 2:-

Solve the following mixed integer programming problem by Gomory's cutting plane algorithm:

$$\text{Max } Z = x_1 + x_2.$$

subject to

$$3x_1 + 2x_2 \leq 5.$$

$$x_2 \leq 2.$$

and $x_1, x_2 \geq 0$ and x_1 an integer.

Solution:-

step 1 The given problem is maximize. So no change.

The standard LPP becomes,

$$\text{Max } Z = x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2.$$

subject to

$$3x_1 + 2x_2 + 1 \cdot S_1 + 0 \cdot S_2 = 5.$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot S_1 + 1 \cdot S_2 = 2.$$

and

$$x_1, x_2, S_1, S_2 \geq 0.$$

The initial basic feasible solution is

$$\boxed{\begin{array}{l} x_3 = 5 \\ x_4 = 2 \end{array}}$$

where $x_1 = x_2 = 0$.

Step 2: [Using Simplex Method]

Initial Iteration:

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
		C_j	1	1	0	0	
0	x_3	5	3	2	1	0	$5/3$
0	x_4	2	0	1	0	1	–
$Z_j - C_j$		0	-1	-1	0	0	

First Iteration [Introduce x_1 & drop S_1]

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
1	x_1	$5/3$	1	$2/3$	$1/3$	0	$5/2$
0	S_2	2	0	1	0	1	2
$Z_j - C_j$		$5/3$	0	$-1/3$	$1/3$	0	

Second Iteration [Introduce x_2 and drop S_2]

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	
1	x_2	2	0	1	0	1	
$Z_j - C_j$		$7/3$	0	0	$1/3$	$1/3$	

Here $(Z_j - C_j)$ all are +ve.

But x_1 is not an integer. So proceed further.

From the Source row (1^{st} row),

$$\frac{1}{3} = x_1 + 0 \cdot x_2 + \frac{1}{3} S_1 - \frac{2}{3} S_2$$

∴ The gomorian construct is,

$$\frac{1}{3} S_1 + \left(\frac{\frac{1}{3}}{1/3 - 1}\right)\left(-\frac{2}{3}\right) S_2 \geq \frac{1}{3}$$

$$\frac{1}{3} S_1 + \frac{1}{3} S_2 \geq \frac{1}{3}$$

$$-\frac{1}{3} S_1 - \frac{1}{3} S_2 \leq -\frac{1}{3}$$

$$\boxed{-\frac{1}{3} S_1 - \frac{1}{3} S_2 + G_1 = -\frac{1}{3}}$$

Add this Gomorian constraint at the bottom of the above optimum simplex table.

		C_j	1	1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	G_1
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	0
1	x_2	2	0	1	0	1	0
0	G_1	$-1/3$	0	0	$-1/3$	$-1/3$	1
$Z_j - C_j$		$7/3$	0	0	$1/3$	$1/3$	0

The solution is optimal but infeasible. So use dual simplex method.

Sin $G_1 = \frac{-1}{3}$, G_1 leaves the basis.

$$\text{Max}\left\{ \frac{z_j - c_j}{a_{ik}}, a_{ik} < 0 \right\}$$

$$= \text{Max}\left\{ \frac{\frac{1}{3}}{-1/3}, \frac{1/3}{-1/3} \right\}$$

$$= \text{Max}\{-1, -1\}$$

$= -1$ which corresponds to both s_1 & s_2.

we choose s_1 arbitary as the entering variable.

3rd Iteration :- Drop G_1 and introduce s_1.

		c_j	1	1	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	G_1
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	s_1	1	0	0	1	1	-3
$z_j - c_j$		2	0	0	0	0	1

all $z_j - c_j$ are +ve.

x_1 is an integer.

$\therefore$ $\boxed{\text{Max } Z = 2, \quad x_1 = 0, \quad x_2 = 2}$.

BRANCH AND BOUND METHOD

This method is applicable to both pure as well as mixed integer programming problems.

Let the given IPP be,

Maximize $Z = CX$.

Subject to

$AX \leq b$.

$X \geq 0$ and Integers.

If the optimal solution some of the variables say x_r is not an integer then,

$$x_r^* < x_r < x_r^* + 1.$$

where x_r^* and $x_r^* + 1$ are consecutive non-negative Integers.

We form 2 different Sub Problems:-

Sub Problem I	Sub Problem II
Max $Z = CX$	Max $Z = CX$
subject to	subject to
$AX \leq b$	$AX \leq b$
$x \leq x_r^*$	$x_r \geq x_r^* + 1$
and $x \geq 0$	and $x \geq 0$.

Example 1:-

Use Branch and Bound technique to solve the following:

$$\text{Maximize } Z = 2x_1 + 2x_2$$

Subject to constraints

$$5x_1 + 3x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0 \text{ and integers.}$$

Solution:-

Step 1 The given problem is maximize. So no change

The standard LPP form is,

$$\text{Maximize } Z = 2x_1 + 2x_2 + 0\cdot S_1 + 0\cdot S_2$$

Subject to

$$5x_1 + 3x_2 + 1\cdot S_1 + 0\cdot S_2 = 8$$

$$x_1 + 2x_2 + 0\cdot S_1 + 1\cdot S_2 = 4$$

The initial basic feasible solution is,

$$S_1 = 8,\ S_2 = 4 \quad [\text{where } x_1 = x_2 = 0]$$

Step 2 [Simplex]

Initial Iteration:-

C_B	Y_B	X_B	x_1	x_2	S_1	S_2	θ
0	S_1	8	5	3	1	0	8/5
0	S_2	4	1	2	0	1	4
$Z_j - C_j$		0	-2	-2	0	0	

First Iteration : [Introduce x_1 & drop x_3]

C_B	Y_B	X_B	x_1 (2)	x_2 (2)	x_3 (0)	x_4 (0)	θ
2	x_1	$8/5$	1	$3/5$	$1/5$	0	$8/3$
0	x_4	$12/5$	0	$7/5$	$-1/5$	1	$12/7$ ← +ve
$Z_j - C_j$		$16/5$	0	$-4/5$ ↑ most -ve.	$2/5$	0	

Second Iteration :- [Introduce x_2 & drop x_4]

C_B	Y_B	X_B	x_1 (2)	x_2 (2)	x_3 (0)	x_4 (0)
2	x_1	$4/7$	1	0	$2/7$	$-3/7$
2	x_2	$12/7$	0	1	$-1/7$	$5/7$
$Z_j - C_j$		$32/7$	0	0	$2/7$	$4/7$

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal, but non-integer.

$$\text{Max } Z = 32/7$$

$$\left.\begin{aligned} x_1 &= 4/7 \\ x_2 &= 12/7 \end{aligned}\right\} \Rightarrow \text{Take } x_1 \,\&\, x_2 \text{ the maximum value } (x_2)$$

In order to obtain the integer optimal solution, we have to branch this problem into two sub-problem.

Now from $x_2 = \frac{12}{7} \Rightarrow 1 < x_2 < 2$.

$$\Rightarrow \underbrace{x_2 \le 1}_{\text{Subproblem ①}} \text{ (or) } \underbrace{x_2 \ge 2}_{\text{Subproblem ②}}.$$

Applying these 2 conditions seperately in the Continuous LPP, we have 2 sub problems.

Subproblem 1:-

$$\text{Max } Z = 2x_1 + 2x_2.$$

subject to

$$5x_1 + 3x_2 \le 8 \longrightarrow ①$$

$$x_1 + 2x_2 \le 4 \longrightarrow ②$$

$$x_2 \le 1 \longrightarrow ③$$

and $x_1, x_2 \ge 0$.

By using graphical method subproblem becomes,

eqn ① $5x_1 + 3x_2 = 8$

x_1	0	1.6
x_2	2.66	0

eqn ②

$x_1 + 2x_2 \le 4$

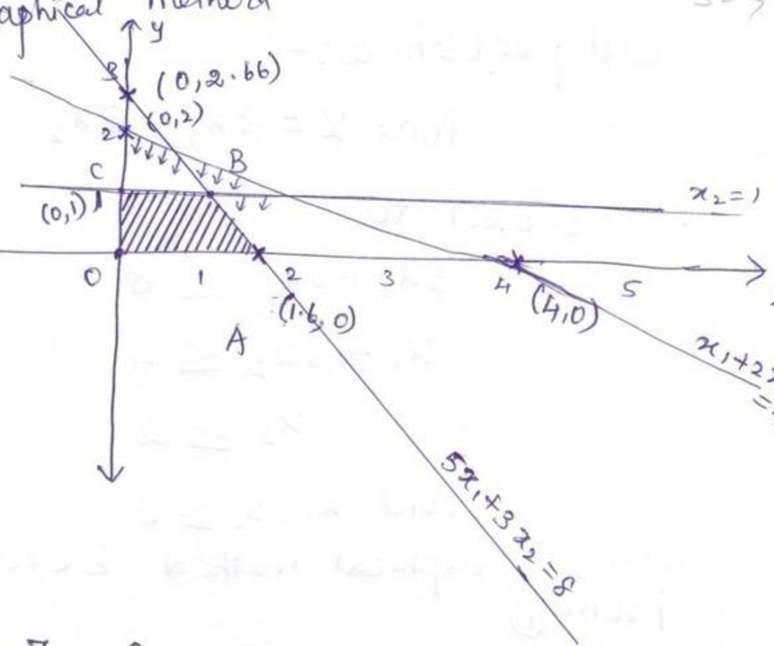
x_1	0	4
x_2	2	0

eqn ③

$x_2 = 1$

$x_2 = 1$

By using graphical method



To find B :-

Solve

$$5x_1 + 3x_2 = 8$$
$$x_2 = 1$$

$\boxed{x_1 = 1}$ $\boxed{x_2 = 1}$

	Max $Z = 2x_1 + 2x_2$
O (0,0)	$Z = 0$
A (1.6, 0)	$Z = 3.2$
B (1, 1)	$\boxed{Z = 4}$ max
C (0, 1)	$Z = 2$

Final answer is

Max $\boxed{\begin{array}{l} Z = 4 \\ x_1 = 1 \\ x_2 = 1 \end{array}}$

So this subproblem ① is fathomed.

Subproblem 2:-

$$\text{Max } Z = 2x_1 + 2x_2.$$

subject to

$$5x_1 + 3x_2 \le 8$$

$$x_1 + 2x_2 \le 4$$

$$x_2 \ge 2.$$

and $x_1, x_2 \ge 0$.

Using graphical method becomes,
From ①.

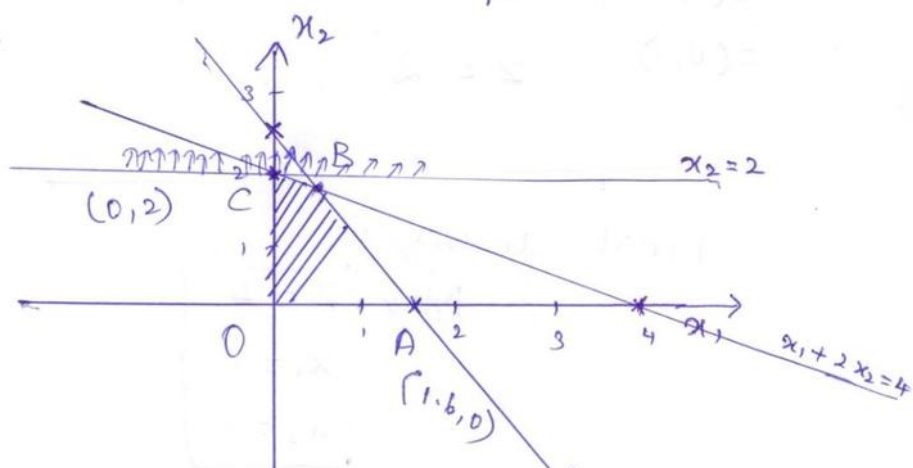
$5x_1 + 3x_2 = 8$

x_1	0	1.6
x_2	2.66	0

$x_1 + 2x_2 = 4$

x_1	0	4
x_2	2	0

$x_2 = 2$.



According to this 3 constraints the point C (0,2) only is common region.

So $x_1 = 0$, $x_2 = 2$.

$$Z = 4$$

Hence from both the sub problem ① & ② the integer optimum solution is given by,

$$\text{Max } Z = 4.$$

with $x_1 = 1$, $x_2 = 1$ (or) $x_1 = 0$, $x_2 = 2$.

Original Problem

$$\text{Max } Z = 2x_1 + 2x_2$$

subject to

$$5x_1 + 3x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$

and $x_1, x_2 \geq 0$

$$\text{Max } Z = \frac{32}{7} \quad x_1 = \frac{4}{7}, \quad x_2 = \frac{12}{7}$$

$x_2 \leq 1$ | $x_2 \geq 1$

Sub Problem (1)	Sub Problem (2)
Max $Z = 4$	Max $Z = 4$
$x_1 = 1$, $x_2 = 1$	$x_1 = 0$, $x_2 = 2$
Fathomed	Fathomed.

Hence the integer Optimum solution is,

$$\text{Max } Z = 4.$$

with $x_1 = 1$, $x_2 = 1$ (or) $x_1 = 0$, $x_2 = 2$.

UNIT-IV – CLASSICAL OPTIMISATION THEORY

Unconstrained external problems, Newton – Ralphson method – Equality constraints – Jacobean methods – Lagrangian method – Kuhn – Tucker conditions – Simple problems.

PART-A

1. Write down the sufficient condition for the Extrema?

A sufficient condition for a stationary point X_0 to be extremum is that the Hessian matrix H evaluated at is

i) Positive definite when X_0 is a minimum point.

ii) Negative definite when X_0 is a maximum point

2. Write the Hessian matrix for f (x_1, $x_{2,}$, x_3)

The Hessian Matrix is $\begin{matrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial 3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{matrix}$

3. State the order of convergence of Newton's Raphson Method.

The order of convergence of Newton's Raphson Method is of order2.

4. Find an iterative formula of Newton's Raphson method.

$$X_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

5. State the criterion for the Newton's Raphson method.

The criterion for convergence of Newton's Raphson is $|f(x)f''(x)| < |f'(x)|^2$

6. What is the order of Convergence for fixed point iteration?

The Convergence is linear and the convergence is of order 1.

7. Write the two methods of Equality Constraints.

1. Jacobian Method, 2. Lagrangean Method.

8. Write the method of Inequality Constraints.

Kuhn-Tucker Method.

9. Write down the sufficient condition for Kuhn-Tucker Method.

If the objective function and the solution space satisfy certain conditions regarding convexity and concavity.

10. Write the constraint gradient vector of f with respect to Z.

$$\nabla_c f = \frac{\partial_c f(Y,Z)}{\partial_c Z} = \nabla_z f - \nabla_Y f J^{-1} C$$

11. Write the necessary conditions for Kuhn tucker method.

a. $\emptyset_i \geq 0, i = 1,2,3 \ldots m$

b. $\frac{\partial L}{\partial X_i} = 0 , i = 1,2,3 \ldots n$

c. $\emptyset_i \; g_i \; (X_1, X_2, \ldots X_n) = 0, i = 1,2, \ldots m$

d. $g_i \; (X_1, X_2, \ldots X_n) \leq 0, i = 1,2, \ldots m$

PART B

UNIT - IV

Classical Optimization Theory.

I * Newton Ralphson Method

1) Find the root of $x^4 - x = 10$ correct to three decimal places using Newton - Ralphson Method [N-R method]

Solution :-

Given $x^4 - x = 10$

let $f(x) = x^4 - x - 10$.

$f(0) = 0 - 0 - 10 = -10 = -ve$.

$f(1) = 1^4 - 1 - 10 = -10 = -ve$.

$f(2) = 2^4 - 2 - 1^0 = 4 = +ve$.

The root lies between 1 and 2.

Here $|f(1)| > |f(2)|$

$\therefore$ The root is nearer to 2.

Let $x_0 = 2$.

N-R formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

We have $f(x) = x^4 - x - 10$.

$f'(x) = 4x^3 - 1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left(\frac{2^4 - 2 - 10}{4 \cdot 2^3 - 1}\right)$$

$$= 2 - \frac{4}{31}$$

$$= 1.8709 \Rightarrow 1.871$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_2)}$$

$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \left(\frac{0.3835}{25.199}\right) \Rightarrow 1.856.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{f(1.856)}{f'(1.856)}$$

$$= 1.856 - \left(\frac{0.010}{24.574}\right) \Rightarrow 1.856.$$

Here $x_2 = x_3 = 1.856$.

$\therefore$ The req. root is 1.856.

2) Using Newton's Raphson method, find the root of $x^3 = 6x - 4$. Correct two decimal places.

Solution:

Given $x^3 = 6x - 4$.

Let $f(x) = x^3 - 6x + 4$.

$f(0) = 4 = +ve$.

$f(1) = 1 - 6 + 4 = -1 = -ve$.

Here $|f(0)| > |f(1)|$

$\therefore$ The root is nearer to 1.

N-R formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

we have $f(x) = x^3 - 6x + 4$.

$f'(x) = 3x^2 - 6$.

Let $x_0 = 1$

$$x_1 = x_0 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \left[\frac{1^3 - 6(1) + 4}{3(1)^2 - 6}\right]$$

$$= 1 - \left[\frac{(-1)}{-3}\right] = 1 - \tfrac{1}{3} \Rightarrow 0.666$$

$= 0.67$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.67 - \frac{f(0.67)}{f'(0.67)}$$

$$= 0.67 - \left(\frac{0.28}{-4.65}\right)$$

$= 0.73$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73 - \frac{f(0.73)}{f'(0.73)}$$

$$= 0.73 - \left[\frac{0.009}{-4.4013}\right]$$

$= 0.7320$

$= 0.73$

Here $x_2 = x_3$

$\therefore$ The req. root is 0.73

II * Jacobian Method.

Formula:

If $Y = (x_1, x_2)$

$$\nabla_y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right)$$

$$\nabla_x (g) = J = \frac{\partial (g_1, g_2\, g_3)}{\partial (x_1, x_2\, x_3)}$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{bmatrix}$$

$$\boxed{\nabla_c f = \nabla_z f - \nabla_y f\, J^{-1} C}$$

where $C = \nabla_z g = \begin{pmatrix} \nabla_z g_1 \\ \vdots \\ \nabla_z g_m \end{pmatrix}$

$\nabla_c f$ - Constrained gradient vector of f with respect to Z.

1) Determine the constrained extreme point of the problem

minimize $f(x) = x_1^2 + x_2^2 + x_3^2$

Subject to

$g_1(x) = x_1 + x_2 + 3x_3 - 2 = 0 \rightarrow ①$

$g_2(x) = 5x_1 + 2x_2 + x_3 - 5 = 0 \rightarrow ②$

Solution:-

Let $Y = (x_1, x_2)$ $\quad Z = x_3$.

$$\nabla_y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) = (2x_1, 2x_2)$$

$$\nabla_z f = \frac{\partial f}{\partial x_3} = 2x_3$$

$$\nabla_y (g) = J = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}$$

$$J^{-1} = \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix}$$

$$\nabla_z g = C = \begin{pmatrix} \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_3} \end{pmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Hence

$$\nabla_c f = \frac{\partial_c f}{\partial_c x_3} = \nabla_z f - \nabla_y f \, J^{-1} C$$

$$= 2x_3 - (2x_1, 2x_2) \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= 2x_3 - (2x_1, 2x_2) \begin{bmatrix} -6/3 & +1/3 \\ 15/3 & -1/3 \end{bmatrix}$$

$$= 2x_3 - (2x_1, 2x_2) \begin{bmatrix} -5/3 \\ 14/3 \end{bmatrix}$$

$$= 2x_3 - \left[\frac{-10x_1}{3} + \frac{28x_2}{3} \right]$$

$$\nabla_c f = \frac{\partial_c f}{\partial_c x_3} = \frac{10x_1}{3} - \frac{28x_2}{3} + 2x_3 \longrightarrow (3)$$

At a stationary point $\nabla_c f = 0$

$$\therefore \frac{10x_1}{3} - \frac{28x_2}{3} + 2x_3 = 0.$$

$$10x_1 - 28x_2 + 6x_3 = 0 \longrightarrow (4)$$

By Solving ①, ② and ④ we get,

$x_1 = 0.81$, $x_2 = 0.35$, $x_3 = 0.28$

$$\boxed{X^\circ = (0.81, 0.35, 0.28)}$$

Partially differenciate ③ wrt x_3

$$\frac{\partial_c^2 f}{\partial c\, x_3^2} = \frac{10}{3}\left(\frac{dx_1}{dx_3}\right) - \frac{28}{3}\left(\frac{dx_2}{dx_3}\right) + 2$$

$$= \left(\frac{10}{3}, \frac{-28}{3}\right)\begin{pmatrix} \frac{dx_1}{dx_3} \\ \frac{dx_2}{dx_3} \end{pmatrix} + 2 \rightarrow ⑤$$

From the development of the Jacobian, method,

$$\begin{pmatrix} dx_1/dx_2 \\ dx_2/dx_3 \end{pmatrix} = -JC^{-1}$$

$$= \begin{pmatrix} 5/3 \\ -14/3 \end{pmatrix}$$

Substitute in ⑤ we get,

$$\frac{\partial^2 f}{\partial_c^2 x_3^2} = \frac{460}{9} > 0$$

$\therefore X^\circ$ is a minimum point.

Lagrangean Method:

1) Solve the following Non linear programming using Lagrangean method.

$$\text{Minimize } Z = 2x_1^2 - 3x_2^2 + 18x_2$$

Subject to

$$2x_1 + x_2 = 8$$

$$x_1 \text{ and } x_2 \geq 0.$$

Solution:-

The general form of the given non-linear programming

$$\text{Minimize } Z = 2x_1^2 - 3x_2^2 + 18x_2$$

Subject to

$$2x_1 + x_2 - 8 = 0$$

$$x_1 \text{ and } x_2 \geq 0.$$

Number of variables $n = 2$.

Number of Constraints $m = 1$ then

$$L = 2x_1^2 - 3x_2^2 + 18x_2 - \phi(2x_1 + x_2 - 8) \rightarrow (1)$$

Partial derivative w.r.t x_1, x_2 and ϕ

$$\frac{\partial L}{\partial x_1} = 4x_1 - 2\phi = 0 \longrightarrow (2)$$

$$\frac{\partial L}{\partial x_2} = -6x_2 + 18 - \phi = 0$$

$$\Rightarrow 6x_2 + \phi = 18 \longrightarrow (3)$$

$$\frac{\partial L}{\partial \phi} = -2(2x_1 + x_2 - 8) = 0$$

$$\Rightarrow 2x_1 + x_2 = 8 \longrightarrow (4)$$

$$(2) \Rightarrow 4x_1 = 2\phi \qquad (3) \Rightarrow 6x_2 = 18 - \phi$$

$$2x_1 = \phi \qquad x_2 = \frac{18-\phi}{6}$$

$$x_1 = \phi/2$$

$$(4) \Rightarrow 2\left(\frac{\phi}{2}\right) + \frac{18-\phi}{6} = 8$$

$$\frac{6\phi + 18 - \phi}{6} = 8$$

$$5\phi + 18 = 48$$

$$5\phi = 30$$

$$\boxed{\phi = 6}$$

$$x_1 = 6/2 = 3 \qquad x_2 = \frac{18-6}{6} = 2.$$

The solution of the system is

$$(x_1^*, x_2^*, \phi^*) = (3, 2, 6) \; \& \; z^* = 42.$$

2) Solve the following non-linear programing problem using Lagrangean method.

$$\text{Maximize } Z = x_1^2 + 2x_2^2 + x_3^2$$

Subject to

$$2x_1 + x_2 + 2x_3 = 30.$$

$$x_1, x_2 \text{ and } x_3 \geq 0.$$

Solution:

Given that

$$\text{Maximize } Z = x_1^2 + 2x_2^2 + x_3^2$$

Subject to

$$2x_1 + x_2 + 2x_3 - 30 = 0.$$

$$x_1 \text{ and } x_2 \geq 0.$$

No. of. Variables $n = 3$.

No. of Constraints $m = 1$.

$$L = x_1^2 + 2x_2^2 + x_3^2 - \phi(2x_1 + x_2 + 2x_3 - 30) \rightarrow (1)$$

Partial derivatives w.r.t x_1, x_2, x_3 and ϕ

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2\phi = 0 \longrightarrow (2)$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \phi = 0 \longrightarrow (3)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 2\phi = 0 \longrightarrow (4)$$

Now

$$\frac{\partial^2 L}{\partial x_1^2} = 4 \quad \frac{\partial^2 L}{\partial x^2} = -6.$$

$$\frac{\partial^2 L}{\partial x_1, \partial x_2} = 0 \quad \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$$

Borded Hession matrix of the problem

0	2	1
2	4	0
1	0	-6

The value of $n-m=1$ and the Corresponding last one principal minor determinant of Hessian matrix is

$$2(-12-0)+1(0-4) = -24-4$$
$$= -28$$
$$= -ve.$$

The sign of $(-1)^m = (-1)^{+1} = -1 = -ve$.

$\therefore$ The solution (x, x, ϕ) corresponds to the minimum object function.

The result is $(x_1^*, x_2^*) = (3, 2)$

and z (Minimum) $= 42$.

3) Solve the following Non linear programming problem using Lagrangean method

$$\text{Maximize } Z = X_1^2 + 2X_2^2 + X_3^2$$

Subject to

$$2X_1 + X_2 + 2X_3 = 30.$$

$$X_1, X_2 \geq 0.$$

Solution:-

Given that

$$\text{Maximize } Z = X_1^2 + 2X_2^2 + X_3^2$$

Subject to

$$2X_1 + X_2 + 2X_3 - 30 = 0.$$

$$X_1 \text{ and } X_2 \geq 0.$$

No. of variables $n = 3$.

No. of Constrains $m = 1$

Then

$$L = X_1^2 + 2X_2^2 + X_3^2 - \phi(2X_1 + X_2 + 2X_3 - 30) \rightarrow (1)$$

Partial derivative (1) w.r.t X_1, X_2, X_3 & ϕ,

$$\frac{\partial L}{\partial X_1} = 2X_1 - 2\phi = 0 \longrightarrow (2)$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \phi = 0 \longrightarrow (3)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 2\phi = 0 \longrightarrow (4)$$

$$\frac{\partial L}{\partial \phi} = -(2x_1 + x_2 + 2x_3 - 30) = 0.$$

$$2x_1 + x_2 + 2x_3 = 30 \longrightarrow (5).$$

$(2) \Rightarrow 2x_1 = 2\phi$ $\qquad$ $(3) \Rightarrow 4x_2 = \phi$

$x_1 = \phi$ $\qquad$ $x_2 = \phi/4$

$(4) \Rightarrow 2x_3 = 2\phi$

$x_3 = \phi$

$(5) \Rightarrow 2\phi + \frac{\phi}{4} + 2\phi = 30$

$17\phi = 120$

$\phi = 120/17.$

The req. solution is

$$(x_1^* \; x_2^* \; x_3^* \; \phi^*) = \left(\frac{120}{17}, \frac{30}{17}, \frac{120}{17}, \frac{120}{17}\right)$$

$$z^* = 105.88$$

Now

$$\frac{\partial^2 L}{\partial x_1^2} = 2 \quad \frac{\partial^2 L}{\partial x_2^2} = 4 \quad \frac{\partial^2 L}{\partial x_3^2} = 2$$

$$\frac{\partial L}{\partial \phi} = -(2x_1 + x_2 + 2x_3 - 30) = 0$$

$$\Rightarrow 2x_1 + x_2 + 2x_3 = 30 \longrightarrow (5)$$

(2) $\Rightarrow$ $2x_1 = 2\phi$ (3) $\Rightarrow$ $4x_2 = \phi$

$\boxed{x_1 = \phi}$ $\boxed{x_2 = \phi/4}$

(4) $\Rightarrow$ $2x_3 = 2\phi$ (5) $\Rightarrow$ $2\phi + \frac{\phi}{4} + 2\phi = 30$

$\boxed{x_3 = \phi}$

$$\frac{8\phi + \phi + 8\phi}{4} = 30$$

$$17\phi = 120$$

$$\boxed{\phi = 120/17}$$

$$\therefore x_1 = \frac{120}{17}, \quad x_2 = \frac{30}{17}, \quad x_3 = \frac{120}{17}$$

The Solution of the above System

$$(x_1^*, x_2^*, x_3^*, \phi^*) = \left(\frac{120}{17}, \frac{30}{17}, \frac{120}{17}, \frac{120}{17}\right)$$

$$z^* = 105.88$$

Now

$$\frac{\partial^2 L}{\partial x_1^2} = 2 \quad \frac{\partial^2 L}{\partial x_2^2} = 4 \quad \frac{\partial^2 L}{\partial x_3^2} = 2$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = 0 \qquad \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_2 \partial x_1} = 0 \qquad \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0 \qquad \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0 .$$

Bordered Hessian Matrix is

$$\left[\begin{array}{c|ccc} 0 & 2 & 1 & 2 \\ \hline 2 & 2 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{array}\right]$$

The value of $n-m=2$ and the Corresponding last 2 principal minor determinants of the Hessian matrix

Third Principal Minor Determinant

$$\begin{matrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 4 \end{matrix}$$

$$\text{Value} = 2(8-0) + 1(-2)$$
$$= 16 - 2 \Rightarrow 14.$$

Fourth (last) principal minor Determinant

$$\begin{vmatrix} 0 & 2 & 1 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix}$$

$$\boxed{\text{Value} = 4}$$

The sign of both principal minor determinant are positive.

The sign of $(-1)^{m+1} = (-1)^{m+1} = (-1)^{1+1}$

$= 1 = +ve$.

The solution $(x_1^*, x_2^*, x_3^*, \phi^*)$ corresponds to the maximum objective function,

The results are

$$(x_1^*, x_2^*, x_3^*) = \left(\frac{120}{17}, \frac{30}{17}, \frac{120}{17}\right)$$

$$Z\,(\text{Maximum}) = 105.88$$

IV Kuhn - Tucker method:

Solve the following non-linear programming using Kuhn - Tucker conditions.

Maximize $Z = X_1^2 + X_1 X_2 - 2X_2^2$

subject to

$$4X_1 + 2X_2 \leq 24$$

$$X_1, X_2 \geq 0.$$

Solution:-

The given problem is modified.

maximize $Z = X_1^2 + X_1 X_2 - 2X_2^2$

Subject to

$$4X_1 + 2X_2 - 24 \leq 0$$

$$X_1 X_2 \geq 0.$$

$$L = X_1^2 + X_1 X_2 - 2X_2^2 - \phi\,(4X_1 + 2X_2 - 24) \longrightarrow (1)$$

The four set of Kuhn - Tucker condition,

a) $\phi \geq 0 \longrightarrow (2)$

b) $\frac{\partial L}{\partial X_1} = 2X_1 + X_2 - 4\phi = 0 \longrightarrow (3)$

$\frac{\partial L}{\partial X_2} = X_1 - 4X_2 - 2\phi = 0 \longrightarrow (4)$

c) $\phi\,(4x_1 + 2x_2 - 24) = 0 \longrightarrow$ (5)

d) $4x_1 + 2x_2 - 24 \leq 0 \longrightarrow$ (6).

In (5), if ϕ is equated to 0 then x_1 & x_2 are zero, which is not true.

$$\therefore 4x_1 + 2x_2 - 24 = 0.$$

$$4x_1 + 2x_2 = 24 \longrightarrow (7)$$

Solving (3), (4) and (7) we get,

$x_1^* = 6$.

$x_2^* = 0$.

$x_3^* = 3$

and

Z^* (Maximum) $= 36$.

2) Solve the following Non-linear program problem using Kuhn-Tucker condition.

Maximize $Z = 3x_1^2 + 14x_1x_2 - 8x_2^2$

Subject to

$3x_1 + 6x_2 \le 72$.

x_1 and $x_2 \ge 0$.

Solution:-

The given problem is modified,

Maximize $Z = 3x_1^2 + 14x_1x_2 - 8x_2^2$

Subject to

$3x_1 + 6x_2 - 72 \le 0$.

x_1 and $x_2 \ge 0$.

$$L = 3x_1^2 + 14x_1x_2 - 8x_2^2 - \phi(3x_1 + 6x_2 - 72) \rightarrow (1)$$

The four set of Kuhn-Tucker condition

a) $\phi \ge 0 \longrightarrow (2)$

b) $\frac{\partial L}{\partial x_1} = 6x_1 + 14x_2 - 3\phi = 0 \rightarrow (3)$

$\frac{\partial L}{\partial x_2} = 14x_1 - 16x_2 - 6\phi = 0 \rightarrow (4)$

c) $\phi(3x_1 + 6x_2 - 72) = 0 \longrightarrow (5)$

d) $3x_1 + 6x_2 - 72 \le 0 \longrightarrow (6)$

In ⑥, If ϕ is equated to 0 then X_1 and X_2 zero. which is not true.

$$\therefore 3X_1 + 6X_2 - 72 = 0.$$

$$3X_1 + 6X_2 = 72 \longrightarrow ⑦$$

$$③ \times 2 \Rightarrow 12X_1 + 28X_2 - 6\phi = 0.$$

$$④ \Rightarrow \underline{\begin{array}{l} 14X_1 - 16X_2 - 6\phi = 0 \\ - \quad\quad + \quad\quad\quad + \end{array}}$$

$$-2X_1 + 44X_2 = 0 \longrightarrow ⑧$$

$$⑦ \times 2 \Rightarrow 6X_1 + 12X_2 = 144$$

$$⑧ \times 3 \Rightarrow \underline{-6X_1 + 132X_2 = 0}$$

$$144X_2 = 144$$

$$X_2 = \frac{144}{144}$$

$$\boxed{X_2 = 1}$$

Subs $X_2 = 1$ in ⑧,

$$⑧ \Rightarrow -2X_1 + 44 = 0.$$

$$-2X_1 = -44$$

$$\boxed{X_1 = 22}$$

The results are $X_1 = 22$ $X_2 = 1$

$$Z\,(\text{maximum}) = 1752.$$

③ Solve the following Non linear programming problem using Kuhn - Tucker method.

$$\text{Maximize } Z = x_1^2 + x_1 x_2 - 2x_2^2.$$

Subject to

$$4x_1 + 2x_2 \leq 24.$$

$$5x_1 + 10x_2 \leq 30.$$

$$x_1, x_2 \geq 0.$$

Solution:

The Lagrangean function of this model is,

$$L = x_1^2 + x_1x_2 - 2x_2^2 - \phi_1(4x_1 + 2x_2 - 24) - \phi_2(5x_1 + 10x_2 - 30) \rightarrow ①$$

The four set of Kuhn-Tucker conditions are,

a) $\phi_1 \geq 0 \longrightarrow ②$

~~b~~ $\phi_2 \geq 0 \longrightarrow ③$

(b) $\dfrac{\partial L}{\partial x_1} = 2x_1 + x_2 - 4\phi_1 - 5\phi_2 = 0 \rightarrow ④$

$\dfrac{\partial L}{\partial x_2} = x_1 - 4x_2 - 2\phi_1 - 10\phi_2 = 0 \rightarrow ⑤$

c) $\phi_1((4x_1) + 2x_2 - 24) = 0 \longrightarrow$ (6)

$\phi_2(5x_1 + 10x_2 - 30) = 0 \longrightarrow$ (7)

d) $4x_1 + 2x_2 - 24 \leq 0 \longrightarrow$ (8)

$5x_1 + 10x_2 - 30 \leq 0 \longrightarrow$ (9)

In (6), If $\phi_1 = 0$ then x_1 & x_2 must be equal to zero which is not true.

$\therefore 4x_1 + 2x_2 - 24 = 0 \longrightarrow$ (10)

In (7) If $\phi_2 = 0$ then x_1 & x_2 must be equal to zero which is not true.

$\therefore 5x_1 + 10x_2 - 30 = 0 \longrightarrow$ (11).

(4) $\times 2 \Rightarrow 4x_1 + 2x_2 - 8\phi_1 - 10\phi_2 = 0$.

(5) $\Rightarrow x_1 - 4x_2 - 2\phi_1 - 10\phi_2 = 0$

$\quad - \quad + \quad + \quad + \quad -$

$3x_1 + 6x_2 + 10\phi_1 = 0 \longrightarrow$ (12)

Solving (10), (11) & (12) we get,

$x_1^* = 6$

$x_2^* = 0$

$\phi_1^* = 3 \quad \phi_2^* = 0$

z^* (Maximum) $= 36$.

UNIT-V – OBJECT SCHEDULING

Network diagram representation – Critical path method – Time charts and resource levelling – PERT.

PART-A

1. Define project. What are the three main phases of project

A project is defined as a combination of interrelated activities,all of which must be executed in a certain order to achieve a set of goal.

Phases of project: Planning, Scheduling and Control

3.What are the two basic planning and controlling techniques in a network analysis?

- Critical Path Method (CPM)
- Programme Evaluation and Review Technique (PERT)

4.What are the advantages of CPM and PERT techniques?

- It encourages a logical discipline in planning, scheduling and control of projects
- It helps to effect considerable reduction of project times and the cost
- It helps better utilization of resources like men, machines, materials and money with reference to time
- It measures the effect of delays on the project and procedural changes on the overall schedule.

5.What is resource leveling?

Resource leveling attempt to reduce peak resource requirements and smooth out period to period assignments without changing the constraints on project duration.

6.What is heuristic programming?

Lacking time or inclination to pursue more thorough problem solving procedures, one employs a rule of thumb arising out of experience, expertise and common sense. In some cases rule of thumb is insufficient. It must be combined with other rules to take into additional factors or exceptional circumstances. A collection of such rules for solving a particular problem is called a heuristic program,

7.What are the two main costs for a project? Illustrate with examples.

- Direct costs are the costs directly associated with each activity such as machine costs, labour costs etc for each activity.
- Indirect costs are the costs due to management services ,rentals, insurance including allocation of fixed expenses, cost of security etc.

8. What are the three time estimates of PERT?

- **Optimistic (least) time estimate:(t_0 or a)** is the duration of any activity when everything goes on very well during the project ie., labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed etc.
- **Pessimistic (greatest) time estimate:(t_p or b)** is the duration of any activity when almost everything goes against our will and a lot of difficulties is faced while doing a project.
- **Most likely time estimate :(t_m or m)** is the duration of any activity when sometimes things go on very well, sometimes things go on very bad while doing the project

9. Difference between PERT and CPM.

PERT	CPM
1.PERT was developed in a brand new R and D project it had to consider and deal with uncertain ties associated with such projects	1.CPM was developed for conventional projects like construction project which consists of well known routine tasks whose resource requirements and

	duration were known with certainty.
2.PERT is usually used for projects in which time estimates are uncertain. Example: R &D activities which are usually non-repetitive.	2.CPM is used for projects involving well known activities of repetitive in nature.
3.Emphasis is given to important stages of completion of task rather than the activities required to be performed to reach a particular event or task in the analysis of network ie.,PERT network is essentially an event-oriented network.	3.CPM is suited to establish a trade off for optimum balancing between schedule time and cost of the project.
4.PERT is Probabilistic	4.CPM is Deterministic

10. What is the formula to compute the cost slope for each activity?

Cost slope = (Crash cost-Normal cost)/(Normal duration-Crash duration)

11. What is the crash time?

Crash time is the duration upto which the normal time of an activity can be shortened by adding extra resources.

12. Define free float, independent float and total float.

Total float of an activity(T.F) is defined as the difference between the latest finish and the earliest finish of the activity or the difference between the latest start and the earliest start of the activity.

Free float of an activity is that portion of the total float which can be used for rescheduling that activity without affecting the succeeding activity

Free float of an activity i-j=(total float of i-j)- Slack of the head event j

Independent float of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activities of that activity.

I.F=Free float i - j -slack of the tail event i.

13.What is critical paths?

Path connecting the first initial node to the very last terminal node, of longest duration in any project network is called critical path. All the activities in the critical path is called critical activities.

14.What is standard deviation and variance in PERT network? (NOV ’07)

The expected time of an activity in actual execution is not completely reliable and is likely to vary. If the variability is known we can measure the reliability of the expected time as determined from three estimates. The measure of the variability of possible activity time is given by standard deviation, their probability distribution Variance of the activity is the square of the standard deviation

15. Define float or slack? (MAY ’08)

Slack is with respect to an event and float is with respect to an activity. In other words, slack is used with PERT and float with CPM. Float or slack means extra time over and above its duration which a non-critical activity can consume without delaying the project.

PART-B

UNIT – V

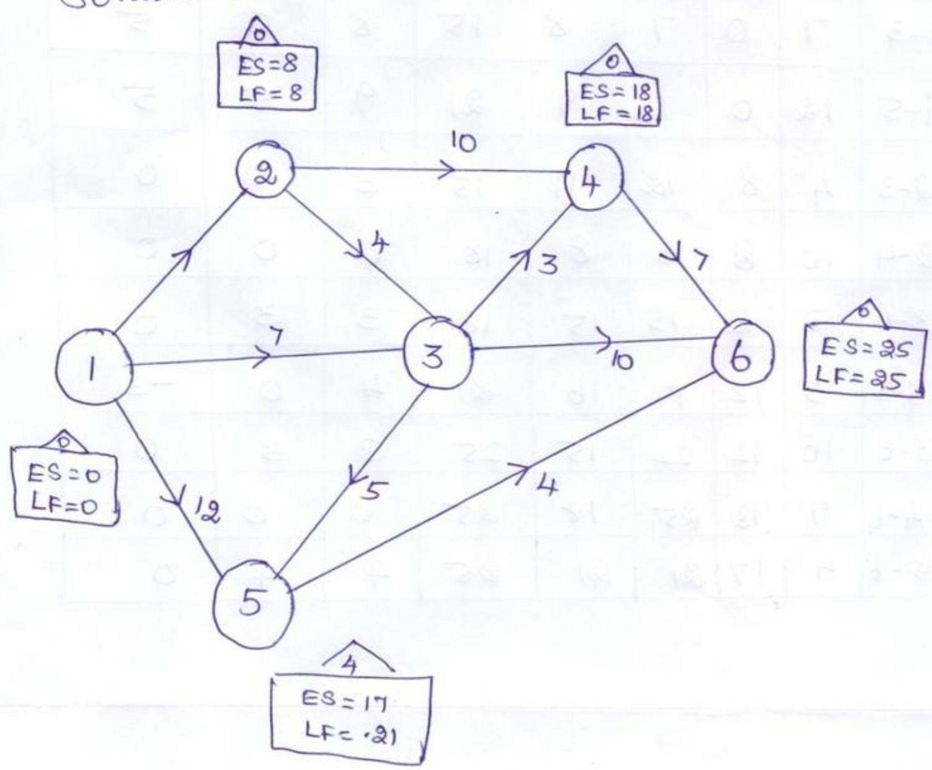
Object Scheduling

CPM:

1) Calculate the total float, free float and independent float for the project whose activities are given below

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration (in weeks)	8	7	12	4	10	3	5	10	7	4

Solution:-



$$\text{Total float} = (LF)_{ij} - (EF)_{ij} \text{ (or) } (LS)_{ij} - (ES)_{ij}$$

$$\text{Free Float} = \text{Total Float} - \text{Slack of the head event } j$$

$$\left.\begin{array}{l}\text{Independent}\\ \text{Float}\end{array}\right\} = \text{free float} - \text{slack of the tail event } i.$$

Activity	Duration (weeks) t_{ij}	Earliest		Latest		Floats		
		Start (ES)	Finish (EF)	Start (LS)	Finish (LF)	TF	FF	IF
1-2	8	0	8	0	8	0	0	0
1-3	7	0	7	8	15	8	5	5
1-5	12	0	12	9	21	9	5	5
2-3	4	8	12	11	15	3	0	0
2-4	10	8	18	8	18	0	0	0
3-4	3	12	15	15	18	3	3	0
3-5	5	12	17	16	21	4	0	-3
3-6	10	12	22	15	25	3	3	0
4-6	7	18	25	18	25	0	0	0
5-6	4	17	21	21	25	4	4	0

activity
(1-2)
TF = LF - EF = 8 - 8 = 0
(1-3) ⇒ = 15 - 7 = 8
(1-5) ⇒ 21 - 12 = 9
(2-3) ⇒ 15 - 12 = 3
(2-4) ⇒ 18 - 18 = 0
(3-4) ⇒ 18 - 15 = 3
(3-5) ⇒ 21 - 17 = 4
(3-6) ⇒ 25 - 22 = 3
(4-6) ⇒ 25 - 25 = 0
(5-6) ⇒ 25 - 21 = 4

FF = Total float - Slack of head j

Activity
(1-2) ⇒ 0 - 0 = 0
(1-3) ⇒ 8 - 3 = 5
(1-5) ⇒ 9 - 4 = 5
(2-3) → 3 - 3 = 0
(2-4) → 0 - 0 = 0
(3-4) → 3 - 0 = 0
(3-5) → 4 - 4 = 0
(3-6) → 3 - 0 = 3
(4-6) → 0 - 0 = 0
(5-6) → 4 - 0 = 4

IF = Free float - slack of the tail i

Activity
(1-2) → 0 - 0 = 0
(1-3) → 5 - 0 = 5
(1-5) → 5 - 0 = 5
(2-3) → 0 - 0 = 0
(2-4) → 0 - 0 = 0
(3-4) → 0 - 0 = 0
(3-5) → 0 - 3 = -3
(3-6) → 3 - 3 = 0
(4-6) → 0 - 0 = 0
(5-6) → 4 - 4 = 0.

$$EF = ES + t_{ij}$$

$$LS = LF - t_{ij}$$

Use the above equation and calculate EF and LS.

PERT - EXAMPLE 1

A Project consists of the following activities and time estimates:

Activity	Least time (days)	Greatest time (days)	Most likely time (days)
1-2	3	15	6
1-3	2	14	5
1-4	6	30	12
2-5	2	8	5
2-6	5	17	11
3-6	3	15	6
4-7	3	27	9
5-7	1	7	4
6-7	2	8	5

a) Draw the network

b) What is the probability that the project will be completed in 27 days?

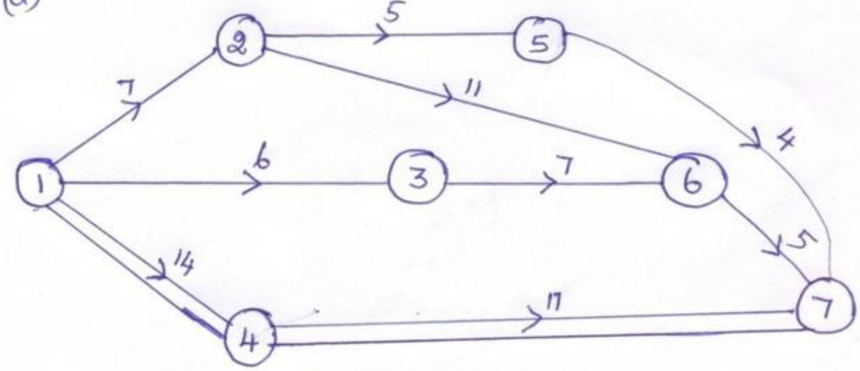
Solution:-

Greatest time = Pessimistic time = t_p

Least time = Optimistic time = t_o

Most likely time = t_m

(a)



Critical path : 1 - 4 - 7

Expected Project duration = 25 days.

Activity	t_0	t_p	t_m	$t_e = \frac{t_0 + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$
1-2	3	15	6	7	4
1-3	2	14	5	6	4
$\boxed{1-4}$	6	30	12	14	16
2-5	2	8	5	5	1
2-6	5	17	11	11	4
3-6	3	15	6	7	4
$\boxed{4-7}$	3	27	9	11	16
5-7	1	7	4	4	1
6-7	2	8	5	5	1

Expected variance of the project length = Sum of the expected variances of all the critical activities

$= 16 + 16 \Rightarrow 32$

σ_c = Standard deviation of the project length

$= \sqrt{32} = 4\sqrt{2} = 5.656$

(b) $\frac{T_s - T_E}{\sigma_c} = \frac{27 - 25}{5.656} = \frac{2}{5.656} = 0.35$

Probability that the project will be completed in 27 days $= P(T_s \leq 27)$

$= P(Z \le 0.35)$ ($\because$ See the table Area Under a Normal curve)

$= 0.1368 + 0.5$

$= 0.6368$

$= 68\%$

PERT - EXAMPLE - 2

Three time estimates (in months) of all activities of a project are as given below:

Activity	a	m	b
1-2	0.8	1.0	1.2
2-3	3.7	5.6	9.9
2-4	6.2	6.6	15.4
3-4	2.1	2.7	6.1
4-5	0.8	3.4	3.6
5-6	0.9	1.0	1.1

a) Find the expected duration and standard deviation of each activity

b) Construct the project network

c) Determine the critical path, expected project length and expected variance of the project length.

d) what is the probability that the project will be completed (i) two months later than expected

ii) not more than 3 months earlier than expected

iii) What due date has about 90% chance of being met?

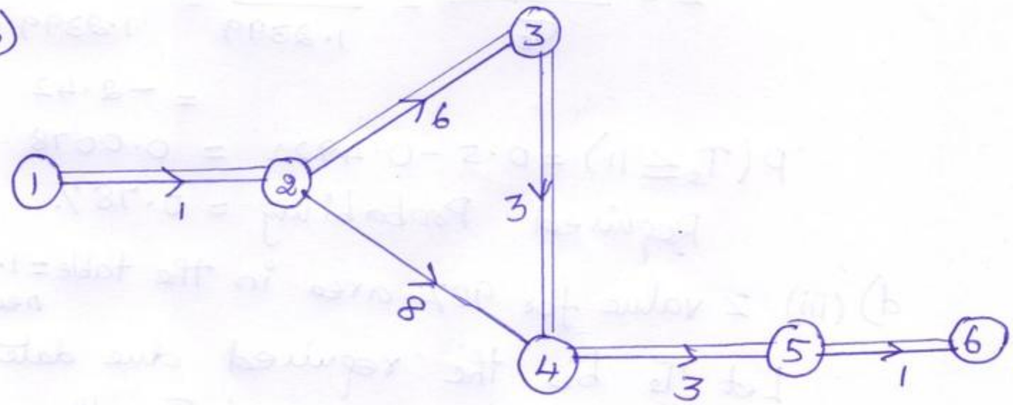
Solution:-

$$a = t_o, \quad m = t_m, \quad b = t_p.$$

a)

Activity	a	m	b	$t_e = \frac{a+4m+b}{6}$	$\sigma = \frac{t_p - t_o}{6}$
$\boxed{1-2}$	0.8	1.0	1.2	1	0.067
$\boxed{2-3}$	3.7	5.6	9.9	6	1.03
2-4	6.2	6.6	15.4	8	1.53
$\boxed{3-4}$	2.1	2.7	6.1	3	0.5
$\boxed{4-5}$	0.8	3.4	3.6	3	0.47
$\boxed{5-6}$	0.9	1.0	1.1	1	0.033

b)



c) Critical path : 1-2-3-4-5-6.

Expected Project length = 14 months

Expected variance $= (0.067)^2 + (1.03)^2 + (0.5)^2 + (0.47)^2 + (0.033)^2$

$= 1.5374$

$\sigma_c = \sqrt{1.5374} = 1.2399$.

d)(i) $T_s = 16$, $T_E = 14$ $\sigma = 1.2399$

$= \dfrac{16-14}{1.2399} = \dfrac{2}{1.2399} = 1.61$

$P(T_s \leq 13) = 0.4463 + 0.5$
$= 0.9463$
$= 94.63\%$ Required Probability

d)(ii) $T_s = 11$, $T_E = 14$ $\sigma_c = 1.2399$

$Z = \dfrac{T_s - T_E}{\sigma_c} = \dfrac{11-14}{1.2399} = \dfrac{-3}{1.2399}$
$= -2.42$

$P(T_s \leq 11) = 0.5 - 0.4922 = 0.0078$.

Required Probability $= 0.78\%$.

d)(iii) z value for 90% area in the table = 1.28 nearly

Let T_s be the required due date

$Z = \dfrac{T_s - T_E}{\sigma_c}$ ie) $1.28 = \dfrac{T_s - 14}{1.2399}$

$T_s = 14 + 1.28 \times 1.2399 = 15.59$ months nearly.

CRASHING :

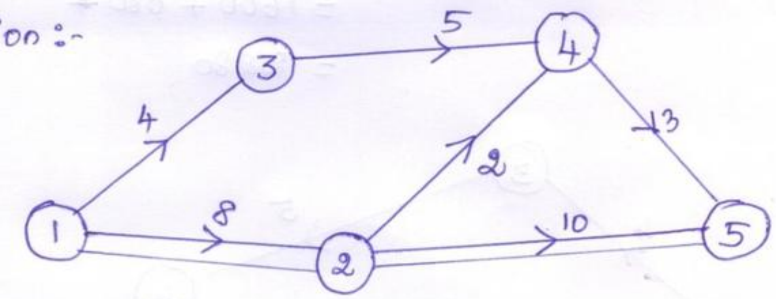
The following data is pertaining to a project with normal time and crash time.

	Normal		Crash	
	Time	Cost	Time	Cost
1-2	8	100	6	200
1-3	4	150	2	350
2-4	2	50	1	90
2-5	10	100	5	400
3-4	5	100	1	200
4-5	3	80	1	100

a) If the indirect cost is Rs 100 per day. find the least cost schedule (optimum duration)

b) What is the minimum duration?

Solution :-



Critical path : 1-2-5

Normal duration 18 days.

Total cost = Indirect cost + direct cost

$= 18 \times 100 + 580$

$=$ Rs 2380 /-

Cost - Slope Table

$$\text{Cost slope} = \frac{\begin{pmatrix}\text{Crash}\\ \text{cost}\end{pmatrix} - \begin{pmatrix}\text{Normal}\\ \text{cost}\end{pmatrix}}{\begin{pmatrix}\text{Normal}\\ \text{duration}\end{pmatrix} - \begin{pmatrix}\text{Crash}\\ \text{duration}\end{pmatrix}}$$

Activity	slope	
1-2	50	C_r
1-3	100	
2-4	40	
2-5	60	C_r
3-4	25	
4-5	10	

$\boxed{C_r = \text{Critical}}$

Stage 1:-

1-2 is the critical path of least cost slope.

Crash 1-2 by 2 days.

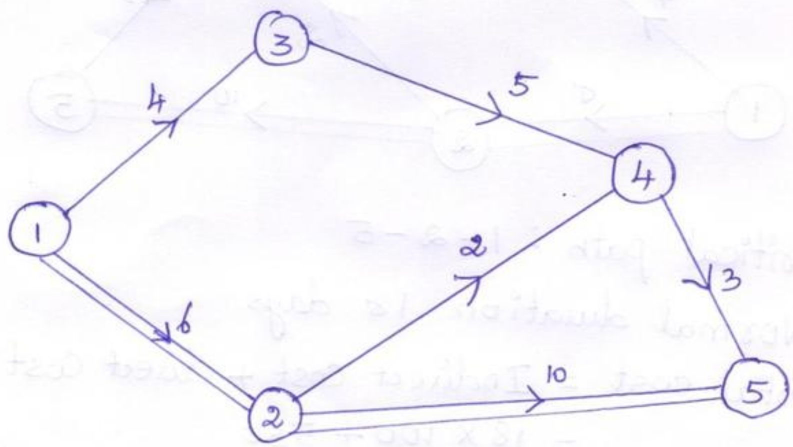
Current critical path : 1-2-5

Current duration = 18 - 2 = 16 days.

Current Total cost = 16 × 100 + 580 + 2 × 50

= 1600 + 680

= 2280



Stage 2:

Critical activities 1-2 and 2-5

Crash 2-5 by 4 days, since the duration of the path 1-3-4-5 is 12 days.

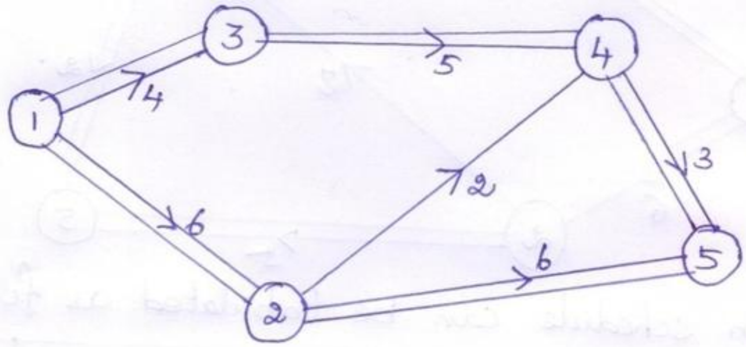
Current Critical paths (i) 1-2-5 (ii) 1-3-4-5

Current duration = 16 - 4 = 12 days

Current total cost = Rs 12 × 100 + Rs 680 + Rs 4×60

= 1200 + 680 + 240

= Rs 2120



Stage 3:

Critical activities 1-2, 1-3, 2-5, 3-4 & 4-5

Crash 2-5 and 4-5 by 1 day each.

Current Critical paths (i) 1-2-5 and (ii) 1-3-4-5

Current duration : 12 - 1 = 11 days

Current total cost = Rs 11 × 100 + 920 + 1 × 60 + 1 × 10

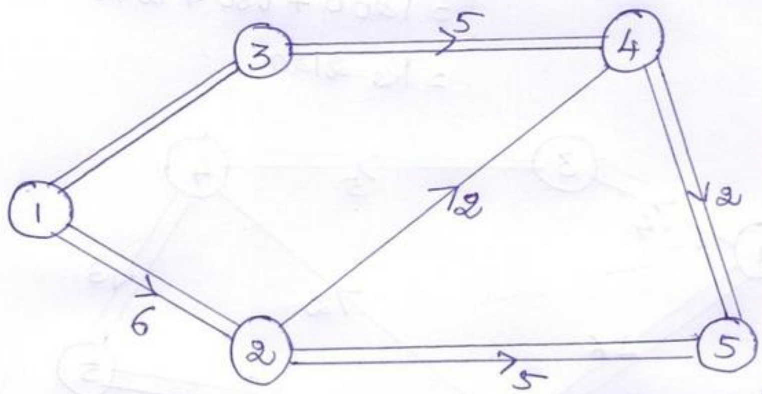
= Rs 2090.

No further crashing is possible since all the activities on the critical path 1-2-5 have been crashed to the maximum event.

Hence the optimum duration = 11 days.

Least cost = 2090

Least (or) minimum duration is also 11 days.



Crash schedule can be tabulated as follows:

Stage	Crash	Current duration	Direct Cost	Indirect Cost	Total cost
(0)	0	18	580	1800	2380
(1)	1-2 by 2 days	16	680	1600	2280.
(2)	2-5 by 4 days	12	920	1200	2120
(3)	2-5 and 4-5 by 1 day	11	990	1100	2090

Resource Leveling:

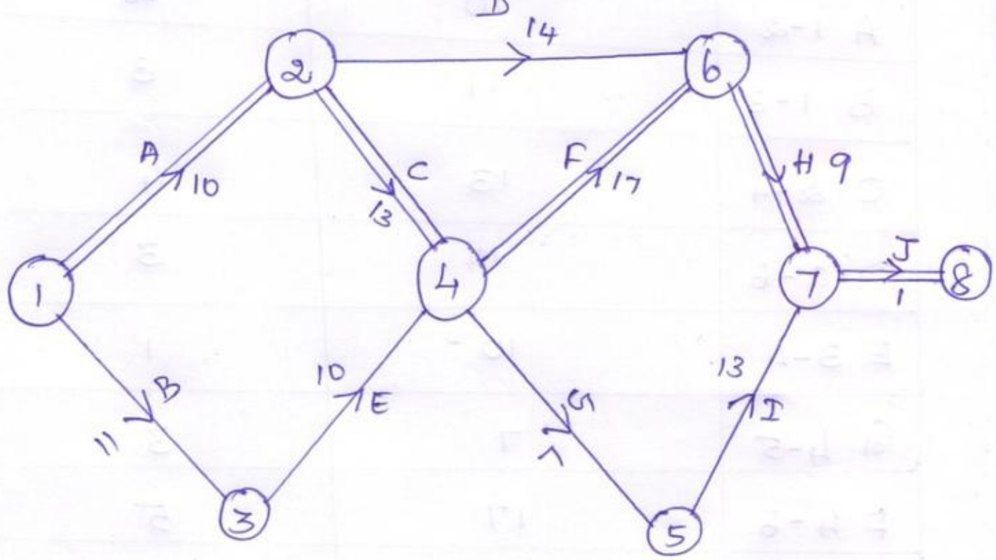
5(i) The manpower required for each activity of a project is given in the following table:

Activity	Normal Time (days)	Manpower required per day
A 1-2	10	2
B 1-3	11	3
C 2-4	13	4
D 2-6	14	3
E 3-4	10	1
G 4-5	7	3
F 4-6	17	5
I 5-7	13	3
H 6-7	9	8
J 7-8	1	11

The contractor stipulates that the first 26 days, only 4 to 5 men and during the remaining days 8 to 11 men only are available. Find whether it is possible to rearrange the activity suitably for levelling the manpower resources satisfying the above condition.

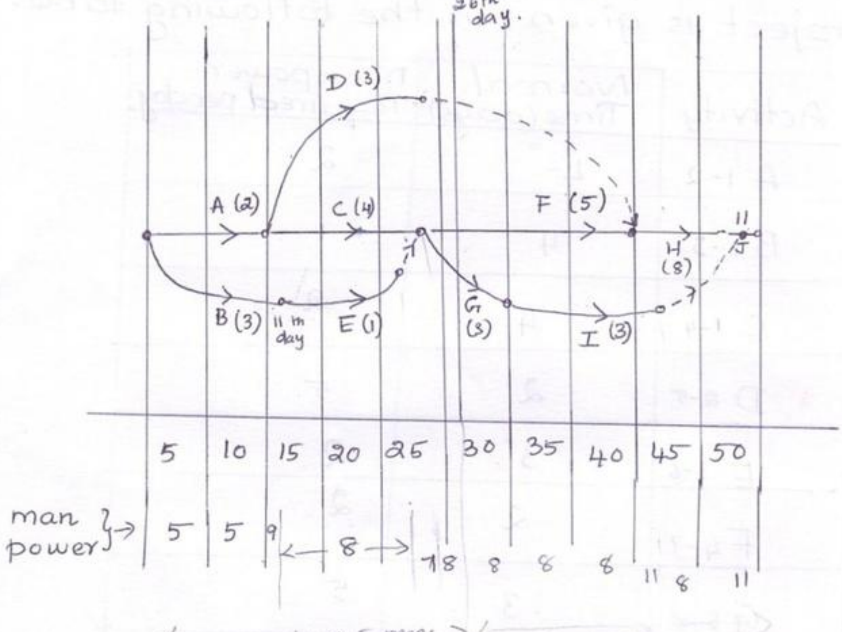
Solution:-

Draw the network and schedule graph.

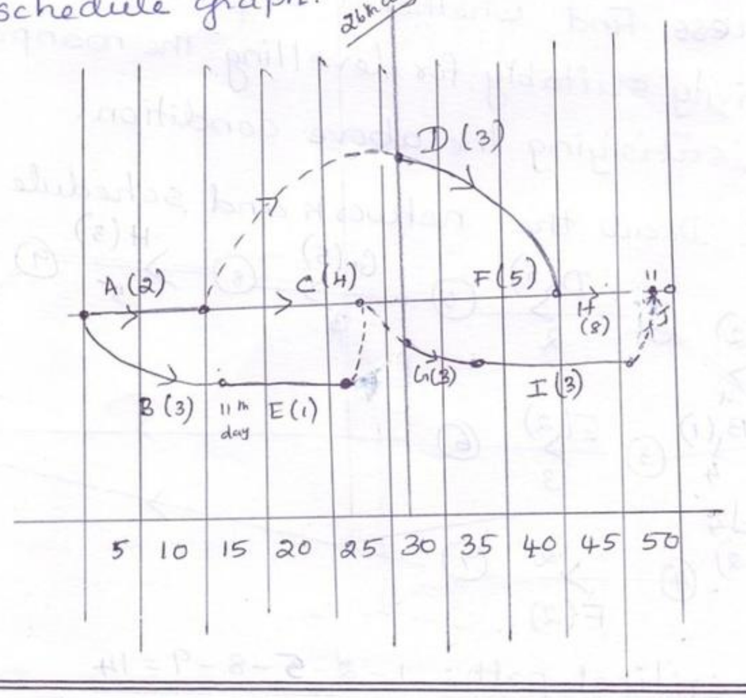


Critical path 1-2-4-6-7-8.

project duration is 50 days



only 4 or 5 mens are available till 11 days after 11th days 11 mens available. we need to reallocate the schedule graph.

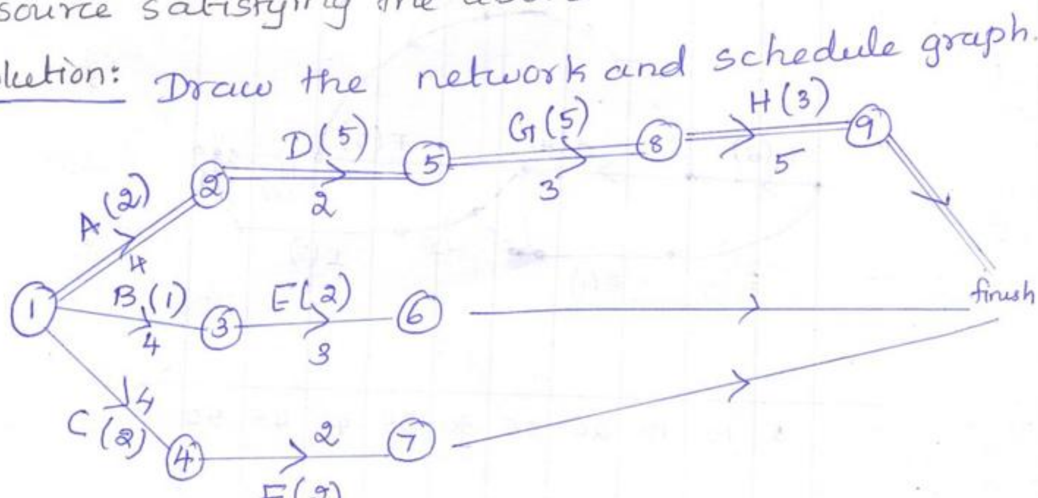


5 (ii) The manpower required for each activity of a project is given in the following table:

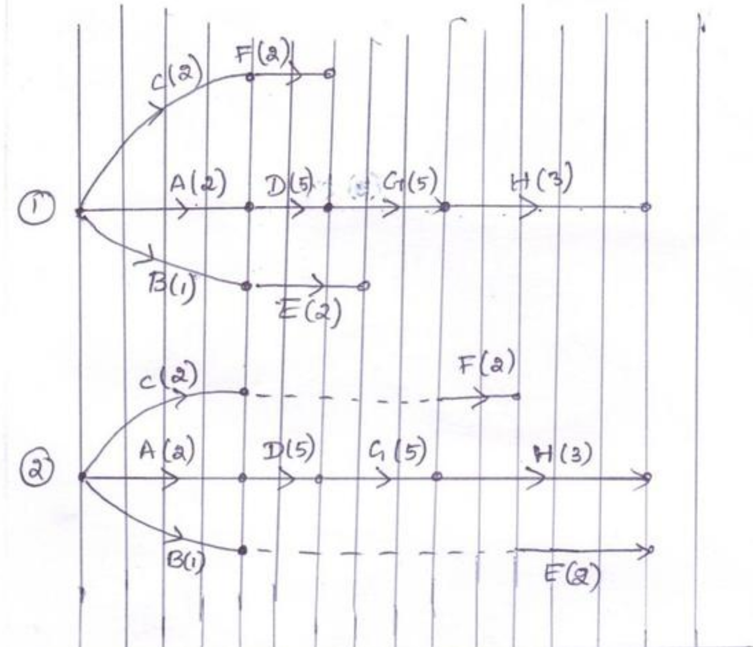
Activity	Normal Time (days)	Manpower required per day
A 1-2	4	2
B 1-3	4	1
C 1-4	4	2
D 2-5	2	5
E 3-6	3	2
F 4-7	2	2
G 5-8	3	5
H 8-9	5	3

Only 5 mens are available althrough the process. Find whether it is possible to rearrange the activity suitably for levelling the manpower resource satisfying the above condition.

<u>Solution:</u> Draw the network and schedule graph.



Critical path: 1-2-5-8-9 = 14



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
man power for ①	5	5	5	5	9	9	7	5	5	3	3	3	3	3	
man power for ②	5	5	5	5	5	5	5	5	5	5	5	5	5	5	

Thus the resource (man power) was satisfied by using resource levelling.