# Introduction to Frequency domain

- Here we are processing signals (images) in frequency domain.
- Since this Fourier series and frequency domain is purely mathematics, so we will try to minimize that math's part and focus more on its use in DIP.

# Frequency domain analysis

- All the domains in which we have analyzed a signal , we analyze it with respect to time.
- But in frequency domain we don't analyze signal with respect to time, but with respect of frequency.

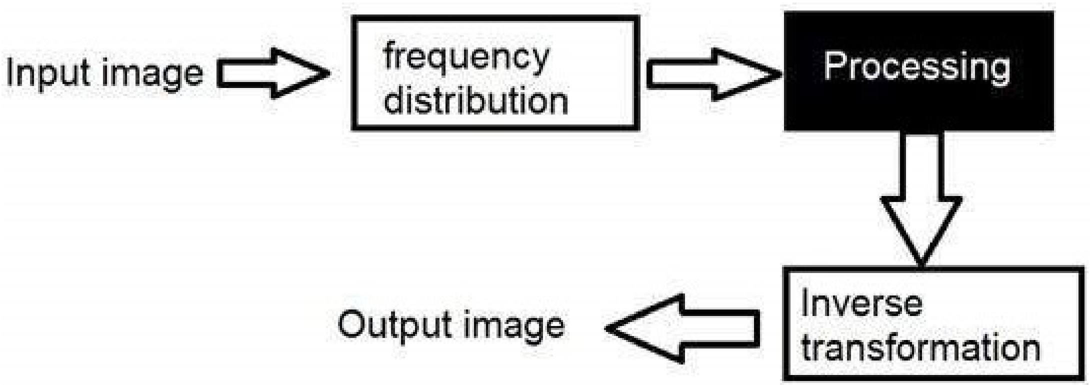
**Difference between spatial domain and frequency domain**

**Spatial Domain**

- In spatial domain, we deal with images as it is. The value of the pixels of the image change with respect to scene.
- Whereas in frequency domain, we deal with the rate at which the pixel values are changing in spatial domain.


**Frequency Domain**

- We first transform the image to its frequency distribution.
- Then our black box system perform what ever processing it has to performed, and the output of the black box in this case is not an image, but a transformation.




- After performing inverse transformation, it is converted into an image which is then viewed in spatial domain.
- Here we have used the word transformation. What does it actually mean?

**Transformation**

- A signal can be converted from time domain into frequency domain using mathematical operators called transforms.
- There are many kind of transformation that does this. Some of them are given below.
  - Fourier Series
  - Fourier transformation
  - Laplace transform
  - Z transform
- Now, we will discuss Fourier series and Fourier transformation.

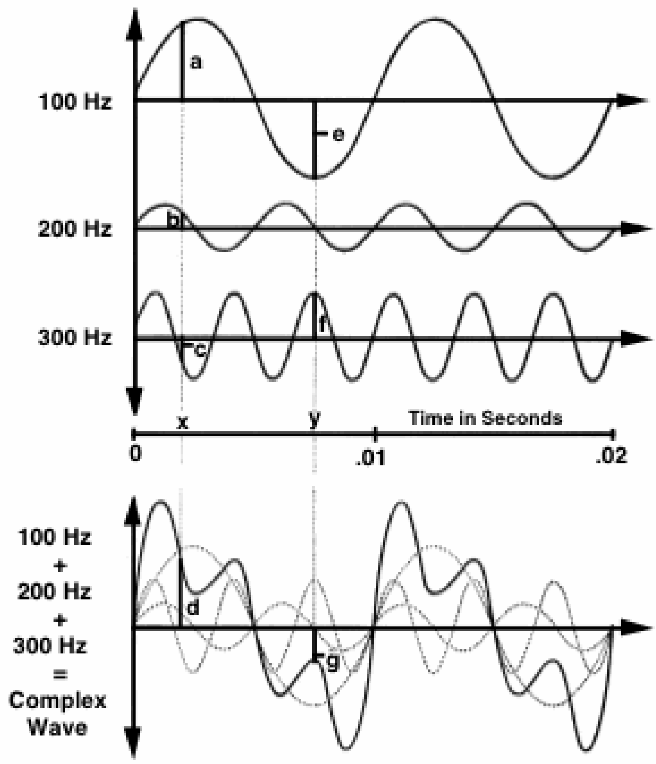
**Fourier Series and Fourier Transforms**

- Fourier series and Fourier transform are used to convert a signal to frequency domain.
- Jean-Baptiste Joseph **Fourier** was a mathematician in 1822.
- He give Fourier series and Fourier transform to convert a signal into frequency domain.

**Fourier Series**

- Fourier series simply states that, periodic signals can be represented into sum of sines and cosines when multiplied with a certain weight.
- It further states that periodic signals can be broken down into further signals with the following properties:
  - The signals are sines and cosines
  - The signals are harmonics of each other

It can be pictorially viewed as




- In the above signal, the last signal is actually the sum of all the above signals.
- This was the idea of the Fourier.
- The Fourier series can be denoted by the formula given as:

$$s(x) = \sin(2\pi f x + \phi_i) = \sin(\omega x + \phi_i)$$

- If we convolve the sinusoidal signal s(x) with a filter whose impulse response is h(x), we get another sinusoid of the same frequency but different magnitude and phase,

$$o(x) = h(x) * s(x) = A\sin(\omega x + \phi_o)$$

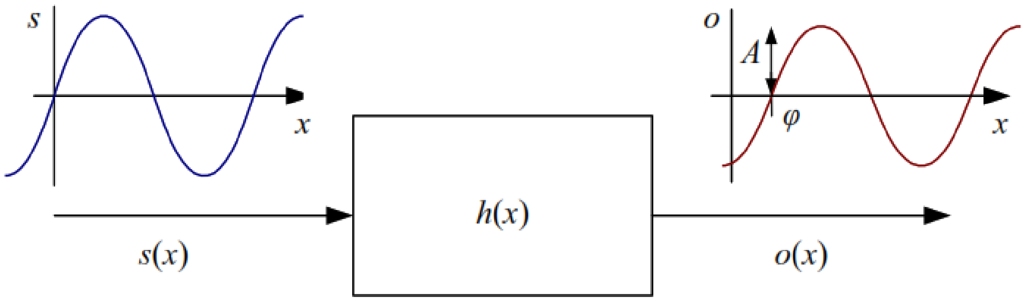



**Figure 3.24** *The Fourier Transform as the response of a filter $h(x)$ to an input sinusoid $s(x) = e^{j\omega x}$ yielding an output sinusoid $o(x) = h(x) * s(x) = Ae^{j(\omega x+\phi)}$.*

- The new magnitude A is called the gain or magnitude of the filter, while the phase difference is called the shift or phase
- The Fourier transform is simply a tabulation of the magnitude and phase response at each frequency
- closed form equations for the one dimensional Fourier transform exist both in the continuous domain,

$$H(\omega) = \int_{-\infty}^{\infty} h(x)e^{-j\omega x}dx,$$

- and in the discrete domain,

$$H(k) = \frac{1}{N}\sum_{x=0}^{N-1} h(x)e^{-j\frac{2\pi kx}{N}},$$

where N is the length of the signal or region of analysis. These formulas apply both to filters, such as h(x), and to signals or images, such as s(x) or g(x).

- The discrete form of the Fourier transform is known as the Discrete Fourier Transform (DFT).
- DFT takes O($N^2$) operations (multiply-adds) to evaluate.

- There exists a faster algorithm called the Fast Fourier Transform (FFT), which requires only $O(N \log_2 N)$ operations

**Two-dimensional Fourier transforms**

The formulas and insights we have developed for one-dimensional signals and their transforms translate directly to two-dimensional images. Here, instead of just specifying a horizontal or vertical frequency $\omega_x$ or $\omega_y$, we can create an oriented sinusoid of frequency $(\omega_x; \omega_y)$,

$$s(x, y) = \sin(\omega_x x + \omega_y y)$$

The corresponding two-dimensional Fourier transforms are then

$$H(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) e^{-j(\omega_x x + \omega_y y)} dx\, dy$$

and in the discrete domain,

$$H(k_x, k_y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x, y) e^{-j2\pi(k_x x/M + k_y y/N)}$$

where M and N are the width and height of the image.

All of the Fourier transform properties from 1D carry over to two dimensions if we replace the scalar variables x, $\omega$, $x_0$ and a with their 2D vector counterparts:
x = (x, y), $\omega = (\omega_x, \omega_y)$, $x_0 = (x_0, y_0)$, and $a = (a_x, a_y)$, and use vector inner products instead of multiplications

**Wiener filtering**

While the Fourier transform is a useful tool for analyzing the frequency characteristics of a filter kernel or image, it can also be used to analyze the frequency spectrum of a whole class of images.

A simple model for images is to assume that they are random noise fields whose expected magnitude at each frequency is given by this power spectrum $P_s(\omega_x, \omega_y)$, i.e.,

$$\left\langle [S(\omega_x, \omega_y)]^2 \right\rangle = P_s(\omega_x, \omega_y)$$

where the angle brackets denote the expected (mean) value of a random variable

The observation that signal spectra capture a first-order description of spatial statistics is widely used in signal and image processing.

In particular, assuming that an image is a sample from a correlated Gaussian random noise field combined with a statistical model of the measurement process yields an optimum restoration filter known as the **Wiener filter**.