Principle of Mathematical Induction

Important Points

If statement P(n) of natural variable $n \in N$ is given, the Principle of Mathematical Induction is useful to verify the validity of the given statement, $\forall\, n \in N$

The Principle of Mathematical Induction:

Let P(n) be a statement involving natural number n.

The statement P(n) is true $\forall\, n \in N$, if

(1) P(1) is true,

(2) P(k), $k \in N$ is true => P(k+1}, $k \in N$ is true, then P(n), $\forall\, n \in N$ is true.

Note:- I .Principle of Mathematical Induction verifies the validity of statements involving natural number variable only.

2. Formula involving natural number variable cannot be derived, but only its validity can be verified.

<u>Use of Principle of Mathematical Induction in some special types of variable:-</u>

(1) Variable type 1:

The statement P(n), $n \in N$ is given. If for positive integer k_0, P(ko) is true and for $k \geq k_0$, $k \in N$, P(k) is true => P(k+ 1) is true, then P(n) is true for all $n \geq k_0$, $k \in N$

(2) Variable type 2:

The statement P(n), n EN is given.

If (1) P(1) and P(2) are true and

(2) for positive integer k, P(k) and P(k+1) are true => P(k+2) is true, then $\forall\, n \in N$, P(n) is true.

Question Bank

(1) For all $n \in N$ - {1}, $7^{2n} - 48n - 1$ is divisible by ....

(a) 25 (b) 26 (c) 1234 **(d) 2304**

(2) $\forall\, n \in N$, P(n):$2.7^n + 3.5^n - 5$ is divisible by ....

(a) 64 (b) 676 (c) 17 **(d) 24**

(3) $\forall\, n \geq 2, n^2(n^4 - 1)$ is divisible by ....

(a) 60 (b) 50 (c) 40 (d) 70

(4) For $n \in N,\ 10^{n-2} > 81n$, if....

(a) $n > 5$ **(b)** $\boldsymbol{n \geq 5}$ (c) $n < 5$ (d) $n > 6$

(5) For each $n \in N$, the correct statement is .....

(a) $2^n < n$ (b) $n^2 > 2^n$ (c) $\boldsymbol{n^4 < 10^n}$ (d) $2^{3n} > 7n + 1$

(6) If $a_n = 2^{2^n} + 1$, then for n>1, $n \in N$, last digit of a_n is.....

(a) 3 (b) 5 (c) 8 (d) 7

(7) If $P(n)$: $4^n \,/\, (n + 1) < (2n)! \,/\, (n!)^2$, then P(n) is true for ......

(a) $n \geq 1$ (b) $n > 0$ (c) $n < 0$ (d) $\boldsymbol{n \geq 2},\ n \in N$

(8) By principle of mathematical induction,
$\forall n \in N\ cos\, 9\ cos\, 2\theta\ cos\, 4\theta \cdots cos[(2^{n-1})\theta] =$ ... ...

(a) $\boldsymbol{sin\, 2^n\theta \,/\, 2^n\, sin\, \theta}$ (b) $cos\, 2^n\theta \,/\, 2^n\, sin\, \theta$
(c) $sin\, 2^n\theta \,/\, 2^{n-1}\, sin\, \theta$ (d) None of these

(9) By principle of mathematical induction, $\forall\, n \in N$, $1/(1.2.3) + 1/(2.3.4) + \cdots + 1 \,/\, \{n(n + 1)(n + 2)\} =$.........

(a) n(n+l) /4(n+2)(n+3) **(b) n(n+3) /4(n+l)(n+2)**
(c) n{n+2) /4(n+1){n+3) (d) None of these

(10) By principle of mathematical induction, $\forall\, n \in N$, $5^{2n+1} + 3^{n+2}.2^{n-1}$ is divisible by.....

(a) 19 (b) 18 (c) 17 (d) 14

(11) The product of three consecutive natural numbers is divisible by ……

(a) 6 (b) 5 (c) 7 (d) 4

(12) $\forall n \in N, a^n - b^n$ is always divisible by….. (a and b are distinct rational nos) **(a) 2a-b** (b) a+b (c) a-b (d) a-2b

(13) If $x^{2n-1} + y^{2n-1}$ is divisible by x+y, then n is…

(a) Positive integer (b) only for an even positive integer

(c) an odd positive integer (d) $\forall n \in N, n \geq 2$

(14) The inequality $n! > 2^{n-1}$ is true for…..

(a) $n > 2,\ n \in N$ (b) $n < 2$ (c) $n \in N$ (d) None of these

(15) The smallest positive integer n for which $n! < \left\{\frac{n+1}{2}\right\}^n$ holds, is ….

(a) 1 **(b) 2** (c) 3 (d) 4

(16) The greatest positive integer, which divides (n+2)(n+3) (n+4)(n+5)(n+6) $\forall n \in N$ is….

(a) 120 (b) 4 (c) 240 (d) 24

(17) $x(x^{n-1} - n\alpha^{n-1}) + \alpha^n(n-1)$ is divisible by $(x-\alpha)^2$ for…..

(a) $n > 1$ (b) $n > 2$ **(c) $\forall n \in N$** (d) None of these

(18) For each $n \in N,\ 3^{2n} - 1$ is divisible by ……

(a) 8 (b) 16 (c) 32 (d) None of these

(19) For each $n \in N,\ 2^{3n} - 7n - 1$ is divisible by ……

(a) 64 (b) 36 **(c) 49** (d) 25

(20) For each $n \in N,\ 10^{2n-1} + 1$ is divisible by ……

(a) 11 (b) 13 (c) 9 (d) None of these

(21) For each $n \in N,\ 2.4^{2n+1} + 3^{n+1}$ is divisible by ……

(a) 2 (b) 9 (c) 3 (d) 11

(22) Let $P(n): n^2 + n + 1$ is and odd integer. If it is assumed that P(k) is true => P(k+1) is true. Therefore, P(n) is true…

(a) for n>1
(b) $\forall\, n \in N$
(c) for n>2
(d) None of these

(23) Let $P(n): 3^n < n!\,, n \in N$, then P(n) is true…

(a) for $n \geq 6$
(b) $\boldsymbol{for\ n \geq 7, n \in N}$
(c) for $n \geq 3$
(d) $\forall\, n$

(24) Let $P(n): 1 + 3 + 5 + \cdots + (2n-1) = n^2$, is…

(a) true for n>1
(b) $\boldsymbol{true\ \forall\, n \in N}$
(c) true for no n
(d) None of these

(25) If $\forall\, \boldsymbol{n} \in \boldsymbol{N}$, P(n) is a statement such that, if P(k) is true => P(k+1) is true for $\boldsymbol{k} \in \boldsymbol{N}$, then P(n) is true…

(a) $\forall\, n > 1$
(b) $\forall\, n \in N$
(c) $\forall\, n > 2$
(d) Nothing can be said

(26) Let $P(n): 1 + 3 + 5 + \cdots + (2^n - 1) = 3 + n^2$, then which of the following is true?

(a) P(1) is true
(b) P(k) is true=> P(k+1) is true
(c) P(k) is true, P(k+1) is not true
(d) both (a) and (b) are true

(27) If matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $and\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds $\forall\, n \in N$, (use PMI)

(a) $A^n = n.A - (n-1)I$
(b) $A^n = 2^{n-1}.A + (n-1)I$
(c) $A^n = n.A + (n-1)I$
(d) $A^n = 2^{n-1}.A - (n-1)I$

(28) $S_n = 2.7^n + 3.5^n - 5$, $n \in N$ is divisible by the multiple of……

(a) 5 (b) 7 **(c) 24** (d) None of these

(29) $10^n + 3(4^{n+2}) + 5$, $n \in N$ is divisible by……

(a) 7 (b) 5 **(c) 9** (d) 17

(30) $\forall\, \boldsymbol{n} \in \boldsymbol{N}, \left(3 + 5^{\frac{1}{2}}\right)^n + \left(3 - 5^{\frac{1}{2}}\right)^n$ is…

(a) Even natural number
(b) Odd natural number
(c) Any natural number
(d) Rational number

(31) The remainder, when 5^{99} is divided by 13, is ……

(a) 6 **(b) 8** (c) 9 (d) 10

(32) For all positive integral values of n, $n^{3n} - 2n + 1$ is divisible by ……

(a) 2 (b) 4 (c) 8 (d) 12

(33) If $n \in N, then\ 11^{n+2} + 12^{2^n+1}$ is divisible by ……

(a) 113 (b) 123 **(c) 133** (d) None of these

(34) If $n \in N, P(n)\colon 2^n(n-1)! < n^n$ is true, if ……

(a) n<2 **(b) n>2** (c) $n \geq 2$ (d) Never

Hints

(1)

P(1): 0 = 0×2304 P(2): 2304=1×2304 $\therefore$ P(1) and P(2) are true --(1)

Let P(k):$7^{2k} - 48k - 1 = m \times 2304$, $m \in N$ and

P(k+1) : $7^{2k+2} - 48(k+1) - 1 = m'$ (2304) , $m' \in N$ be true . ------ (2)

Now, P(k+2): $7^{2k+4} - 48(k+2) - 1 = 49 \times 7^{2k+2} - 48(k+1) - 49$

$= 49 \times 7^{2k+2} - 48(k+1) - 49 = 49(7^{2k+2} - 1) - 48(k+1)$

$= 49(2304m' + 48k + 48) - 48k - 48$ (.... (2))

$= 49 \times 2304m' + 49 \times 48k + 49 \times 48 - 48k - 48$

$= 49 \times 2304m' + 48 \times 48k + 48 \times 48 = 2304(49m' + k + 1)$

$= 2304 \times m''$, where $m'' = 49m' + k + 1$ is positive integer.

$\therefore$ Ans. (d) 2304

(2)

$\forall n \in N$ માટે P(n): $2.7^n + 3.5^n - 5$

P(1): 24 , P(2):98+75−5=168=7×24

Ans. (a) 24

(3)

For every positive integers $n \geq 2$, P(n): $n^2(n^4 - 1)$

P(2): 4 × 15 = 60, P(3): 9 × 80=60 × 12

$\therefore$ From option Ans. (a) 60

(4) For $n \in N$, P(n):$10^{n-2} > 81n$

P(1) : 0.1 > 81 isn't true , P(2): 1> 162 isn't true , P(3):10 > 243 isn't true, as the same way P(4) isn't true , but P(5): 1000 > 405 is true and P(6): 10000 > 486 is true $\therefore$ Ans. (b) $n \geq 5$

(5)

Here for $n = 1$, $2^n < n$ isn't true,

$n^2 > 2^n$ isn't true,

$n^4 < 10^n$ is true,

$n^{3n} > 7n + 1$ isn't true,

Ans. (c) $n^4 < 10^n$

(6)

For $n = 2$, $a_2 = 2^{2^2} + 1 = 17 = 10 + 7$

Let $a_k = 2^{2^k} + 1 = 10m + 7$ be true where $k > 1, m \in N \dots (1)$

Now, $a_{k+1} = 2^{2^{k+1}} + 1 = \left(2^{2^k}\right)^2 + 1 = (10m + 6)^2 + 1$ (by (1))

$= 10(10m^2 + 12m + 3) + 7$

$\therefore$ Digit of one's place of a_n is 7.

$\therefore$ Ans. (c) 7

(7)

P(n): $4^n / (n+1) < (2n)! / (n!)^2$, $n \in N$

P(1) isn't true and $n < 0$ isn't possible.

$\therefore$ (a), (b), (c) options are not possible .

$\therefore$ Ans. **(d)** $\mathbf{n \geq 2}$, $n \in N$

(8) For n=1, by P(n): $\cos\theta \cos 2\theta \cos 4\theta \dots \cos[(2^{n-1})\theta]$ $\therefore$ P(1): $\cos\theta$

in option (a) $n = 1$ we get $\cos\theta$. $\therefore$ Ans. **(a)** $\sin 2^n\theta / 2^n \sin\theta$

(9)

For $n = 1$,

$1/(1.2.3) = 1/6$

Now, for $n = 1$, value of only option **(b) $n(n+3)/4(n+1)(n+2)$ is** $1/6$

$\therefore$ Ans. **(b) $n(n+3)/4(n+1)(n+2)$**

(10)

For $n = 1$, $P(1): 5^{2+1} + 3^{1+2} . 2^{1-1}$

$= 125 + 27 = 152 = 19\times 8$

Let $P(k) = 5^{2k+1} + 3^{k+2} . 2^{k-1} = 19m,\ m \in N$ ---- (1)

$P(k+1) = 5^{2k+3} + 3^{k+3} . 2^{k} = 5^{2} . 5^{2k+1} + 3.3^{k+2} . 2.2^{k-1}$

$= 25(19m - 3^{k+2} . 2^{k-1}) + 6. 3^{k+2} . 2^{k-1}$ (by (1))

$= 25.19m - 19. 3^{k+2} . 2^{k-1}$

$= 19(25m - 3^{k+2} . 2^{k-1})$

$= 19m'$

$\therefore$ Ans. (a) 19

(11)

Product of three consecutive natural numbers $P(n) : n(n+1)(n+2)$

$P(1) = 6$ which is divisible by 6.

$P(2) = 24$ which is divisible by 6.

$\therefore$ Ans. (a) 6

(12)

For every $n \in N$, $P(n) : a^n - b^n$

$P(1) = a - b$ and $P(2) = a^2 - b^2 = (a - b)(a+b)$

$\therefore$ Ans. (c) $a - b$

(13)

$P(n)$: $x^{2n-1} + y^{2n-1} = \lambda(x + y)$ where λ is a polynomial.

$P(1)$: $x + y$ is divisible by $x+y$

$P(2)$: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$\therefore$ Ans. **(a) Positive integer**

(14)

$P(n)$: $n! > 2^{n-1}$

Now $P(1)$ and $P(2)$ are not true, but $P(3)$ is true.

Let $P(k)$:$k! > 2^{k-1}$, $k>2$ be true.

$P(k+1)$:$(k+1)! > 2^k$

L.H.S of $P(k+1) = (k+1)! = k!(k+1)$

$$> 2^{k-1}(k+1) = 2^k.(k+1)/2$$

$$> 2^k$$

$\therefore$ Ans. (a) $n > 2$

(15) For smallest positive integer n. $P(n)$:$n! < \{(n+1)/2\}^n$,

$P(1)$:$1 < 1$ isn't true, $P(2)$:$2 < 9/4$ is true. $P(3)$:$6<8$ is true $P(4)$ is true. $\therefore$ **Ans. (b) 2**

(16)

For $\forall n \in N$, for which greatest positive integer, does $(n+2)(n+3)(n+4)(n+5)(n+6)$ divide ?

$P(n): (n+2)(n+3)(n+4)(n+5)(n+6), \quad n \in N$

$P(1) = 3.4.5.6.7 = 120.21$ $\qquad$ $P(2) = 4.5.6.7.8 = 120.56$

$P(3) = 5.6.7.8.9 = 120.126$ $\qquad$ $P(4) = 6.7.8.9.10 = 120.252$

$P(5) = 7.8.9.10 = 120.42$ $\qquad$ $P(6) = 8.9.10.11.12 = 120.99.13$

$\therefore$ Ans. **(a) 120**

(17)

$$P(n) : x(x^{n-1} - n\alpha^{n-1}) + \alpha^n (n-1) = g(x).(x-\alpha)^2$$

$$P(1) = 0$$

$$P(k) : x(x^{k-1} - k\alpha^{k-1}) + \alpha^k (k-1) = g(x).(x-\alpha)^2$$

$$P(k+1) : x(x^k - (k+1)\alpha^k) + \alpha^{k+1}(k) = g'(x).(x-\alpha)^2$$

$$\text{L.H.S.} = x[kx\alpha^{k-1} - (k-1)\alpha^k + g(x).(x-\alpha)^2 - (k+1)\alpha^k] + \alpha^{k+1}k$$

$$= kx^2\alpha^{k-1} - 2kx\alpha^k + g(x).x.(x-\alpha)^2 + k\alpha^{k+1}$$

$$= g(x).x.(x-\alpha)^2 + (x^2 - 2x\alpha + \alpha^2)k\alpha^{k-1}$$

$$= (x-\alpha)^2 [g(x).x + k\alpha^{k-1}]$$

$$= g'(x).(x-\alpha)^2$$

$$= \text{R.H.S.}$$

$\therefore$ Ans. **(c) all $n \in N$**

(18) For each $n \in N$. $P(n) : 3^{2n} - 1$

$P(1) = 8, \quad P(2) = 80 = 10.8 \quad \therefore$ Ans.**(a) 8**

(19)

For each $n \in N$, $P(n) : 2^{3n} - 7n - 1$

$P(1) = 0 \quad P(2) = 49 \quad P(3) = 512 - 21 - 1 = 490 = 49.10$

$\therefore$ Ans. **(c) 49**

(20)

For each $n \in N$, $P(n) : 10^{2n-1} + 1$

$P(1) = 11$,

$P(2) = 1001 = 11.91$

$\therefore$ Ans. **(a) 11**

(21)

$\forall n \in N$, $P(n) : 2.4^{2n+1} + 3^{3n+1}$

$P(1) = 209 = 11.19$

$P(2) = 11.385$

$\therefore$ Ans. **(d) 11**

(22) $P(n) : n^2 + n + 1 = n(n+1) + 1$

$P(1) : 3$ which is true.

$P(n) : n^2 + n + 1 = n(n+1) + 1$ which is always odd number

$\therefore$ Ans. **(b)** $\boldsymbol{\forall n \in N}$

(23) $P(n) : 3^n < n!$, $n \in N$

$P(1) : 3^1 < 1$ is not true. $P(3) : 3^3 < 3!$ is not true.

$P(6) : 3^6 < 6!$ is not true.

$P(7) : 3^7 < 7!$ is true. $\therefore$ Ans. **(b)** $\boldsymbol{n \geq 7}$

(24)

$P(1): 1 = 1$

$P(k): 1+3+5+.......+(2k-1) = k^2.$

$P(k+1): 1+3+5+.......+(2k-1) + (2k+1) = (k+1)^2.$

$L.H.S. = 1+3+5+.......+(2k-1) + (2k+1)$

$= k^2 + 2k + 1 = (k+1)^2 = R.H.S.$

$\therefore$ Ans. **(b) true for all $n \in N$**

(25)

P(1) validity cannot be checked because statement P(n) is given

$\therefore$ Ans. **(d) nothing can be said**

(26)

$P(1) : 1 = 4$ is not true.

Let $P(k) : 1+3+5+ \dots + (2k-1) = 3 + k^2$ be true.

$P(k+1) = 1+3+5+ \dots + (2k-1)+(2k+1)$

$= 3 + k^2 + 2k+1 = (k+1)^2 + 3 = R.H.S.$

$\therefore$ Ans.**(b) P(k) is true $\Rightarrow$ P(k+1) is true.**

(27)

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P(1): A = A - (1-1) I = A$ $\therefore$ P(1) is true.

P(k) is true $\Rightarrow$ **P(k+1) is true, k $\in$**N

$\therefore$ Ans. (a) $A^n = n. A-(n-1) I$

(32)

$\forall n \in N,\ P(n): 3^{3n} - 2n + 1$

$P(1): 26 = 2 \times 13$

$P(2): 726 = 2 \times 343$

$P(3): 19683 - 6 + 1 = 19678 = 2 \times 9839$

$\therefore$ Ans. **(a) 2**

(33)

$\forall n \in N, P(n) = 11^{n+2} + 12^{2n+1}$

$P(1): 11^{1+2} + 12^{2+1} = 133 \times 23,$

$P(2): 11^{2+2} + 12^{4+1} = 14641 + 248832 = 263473 = 133 \times 1981$

$\therefore$ Ans. **(c) 133**

(34)

For $n \in N$, $P(n) = 2^{n} (n-1)! < n^{n}$

$P(1): 2 < 1$ is not true.

$P(2): 4 < 4$ is not true.

$P(3): 16 < 27$ is true.

Same as $P(4)$ is true.

$\therefore$ Ans. **(b) $n > 2$**