UNIT-III

GRAPHS

Graphs and Graph Models:

Defn: Graph

vertices (node)

A graph $G = (V(G), E(G))$ consists of V, a non empty set of vertices (nodes or points) and E, a set of edges (also called lines).

(vertices, edges)

i.e. A graph G is an ordered triple $(V(G), E(G), \phi)$ consists of a non empty set V called the set of vertices (nodes or points) of the graph G, E is said to be set of edges of the graph G, and ϕ is a mapping from set of edges E to a set of order or unordered pairs of elements of V.

e_1, e_2, e_3, e_4 edges

distance b/w two vertices

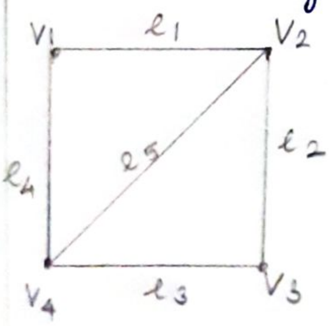
Loop

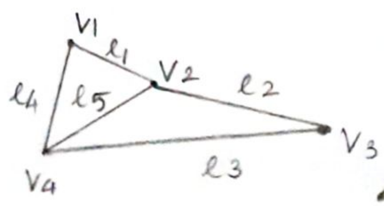
start and end in same vertice

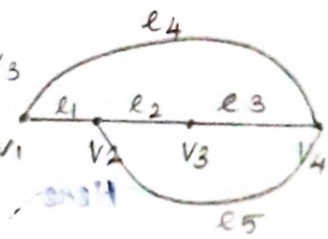
Example:

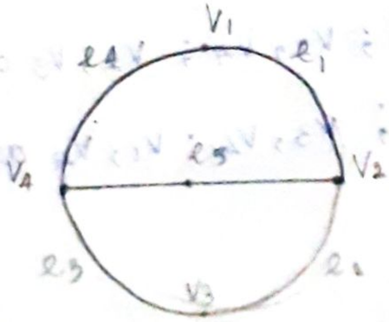
Let $G = (V(G), E(G), \phi)$ where $V(G) = \{V_1, V_2, V_3, V_4\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$ and ϕ is defined by $\phi(e_1) = \{V_1, V_2\}$, $\phi(e_2) = \{V_2, V_3\}$, $\phi(e_3) = \{V_3, V_4\}$, $\phi(e_4) = \{V_4, V_1\}$, $\phi(e_5) = \{V_1, V_3\}$.

Now the diagrammatic form of G is as follows.









It should be noted that, in drawing a graph, it is immaterial whether the edges are drawn straight or curved, long or short, the important point is how the vertices are joined up.

The above graphs are same

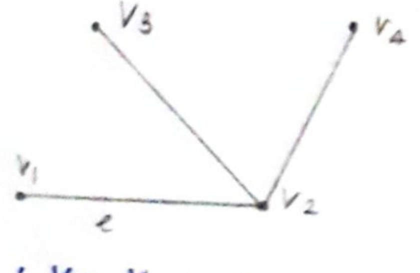
Note: ① We denote the graph G as $G(V,E)$ or simply as G.

② If $e \in E$ is an edge and $\phi(e) = \{v_1, v_2\}$ then we say that e is an edge joining v_1 and v_2, the vertices v_1 and v_2 are called the ends (end vertices) of e.

③ If graphs, an edge should not pass through any points (vertices) other than the two end vertices of the edge.

Defn: Adjacent Vertices:

Any pair of vertices which are connected by an edge in a graph is called adjacent vertices.

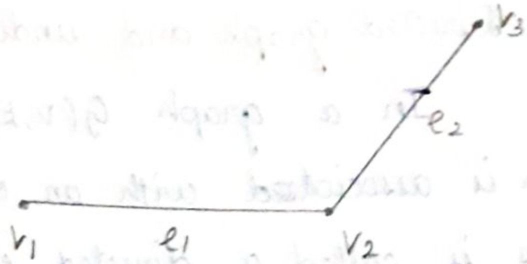


Here V_1, V_2 ; V_2, V_4 ; V_2, V_3 are adjacent vertices

V_1, V_3 ; V_3, V_4 ; V_1, V_4 are not adjacent

Defn: Adjacent edges:

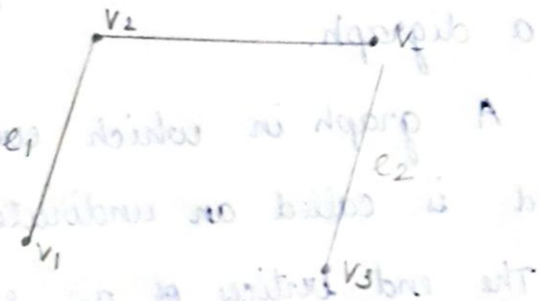
If two distinct edges are incident with a common vertex, then they are called adjacent edges.



Here e_1 and e_2 are incident with a common vertex v_2.

Defn: Isolated vertex:

In any graph, a vertex which is not adjacent to any other vertex is called an isolated vertex. Otherwise the vertex has no incident edge.



Here v_3 has no incident edge. Therefore the vertex v_3 is called isolated vertex.

Note:

① A graph with p vertices and q edges is called a (p,q) graph.

② The graph (p,0) is trivial or null graph

③ If any two edges are intersected then their intersection is not considered as a vertex

④ The set of edges in a null graph is empty

Defn: Label graph:

A graph in which each vertex is assigned a unique name or label, is called a label graph.

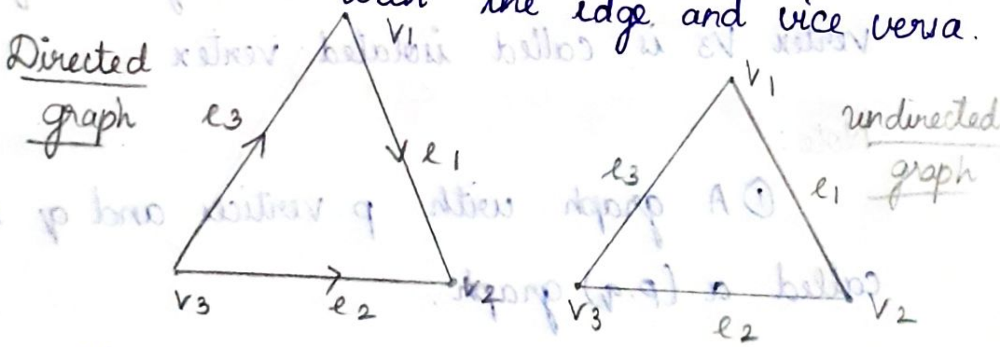
Defn: Directed graph and undirected graph:

In a graph $G(V,E)$, an edge which is associated with an ordered pair of vertices is called a directed edge of graph G, while an edge which is associated with an unordered pair of vertices is called an undirected edge.

A graph in which every edge is directed is called a directed graph or simply a digraph.

A graph in which every edge is undirected is called an undirected graph.

The end vertices of an edge are said to be incident with the edge and vice versa.

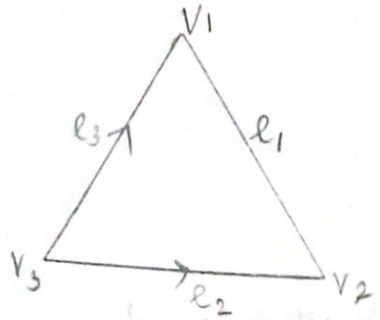


The edge e_1 is incident with the vertices v_1 and v_2. also the vertex v_1 is incident with e_1 and e_3.

The vertices v_1 and v_2 are also called the initial and terminal vertices of an edge e_1.

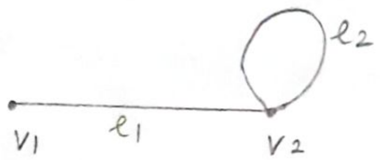
Defn: Mixed graph:

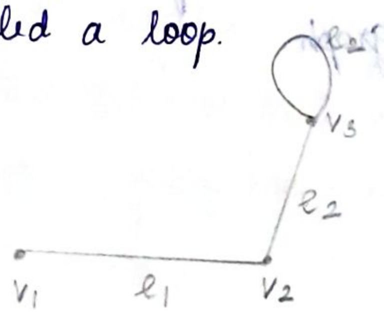
If some edges are directed and some are undirected in a graph, then the graph is a mixed graph.



Defn: Loop

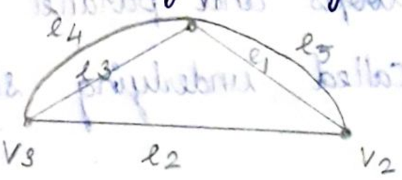
A loop is an edge whose vertices are equal. i.e.) An edge of a graph which joins a vertex to itself is called a loop.





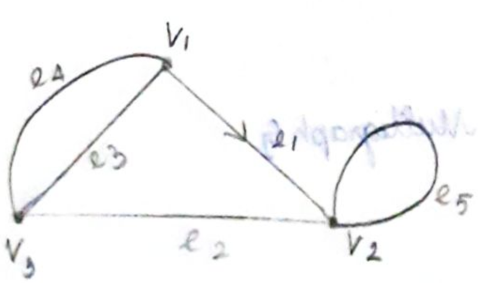
Defn: Parallel edges (Multiple edges)

Multiple edges are edges having the same pair of vertices.



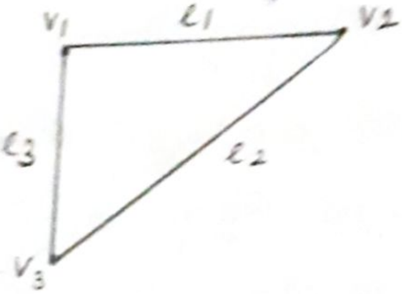
Defn: Multi graph:

Any graph which contains some parallel edges and loops is called as multi graph.



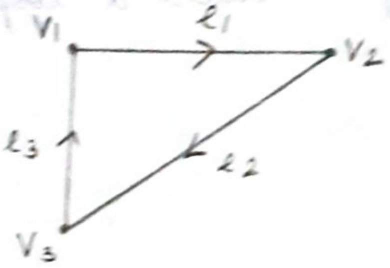
Defn: Simple graph.

A simple graph is a graph having no loops or multiple edges.



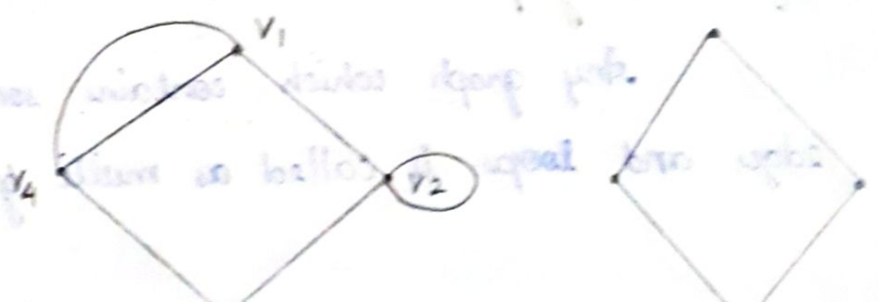
Defn: Simple directed graph:

When a directed graph has no loops and has no loops and has no multiple directed edges, it is called a simple directed graph.



Defn: Underlying simple graph:

A graph obtained by deleting all loops and parallel edges from a graph is called underlying simple graph.



Multigraph G

Underlying simple graph of G.

Defn: Finite graph:

A graph G is finite if and only if both the vertex set $V(G)$ and the edge set $E(G)$ are finite, otherwise the graph is infinite.

Ex: Let $V(G) = \mathbb{Z}$ and $E(G) = \{e_{ij} / |i-j| = 1\}$

clearly, the graph G is infinite

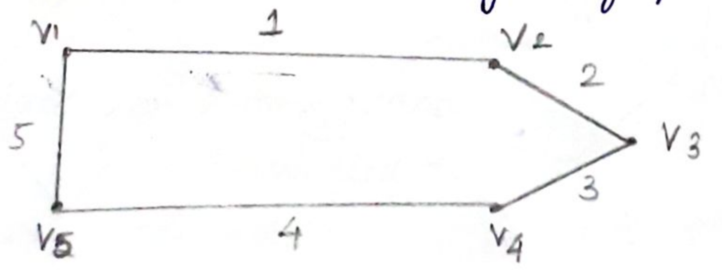
Note: Hereafter, a graph means that is a finite graph unless otherwise stated.

Defn: Multiplicity m.

When there are m directed edges, each associated to an ordered pair of vertices (u,v), we say that (u,v) is an edge of multiplicity m.

Defn: Weighted graph.

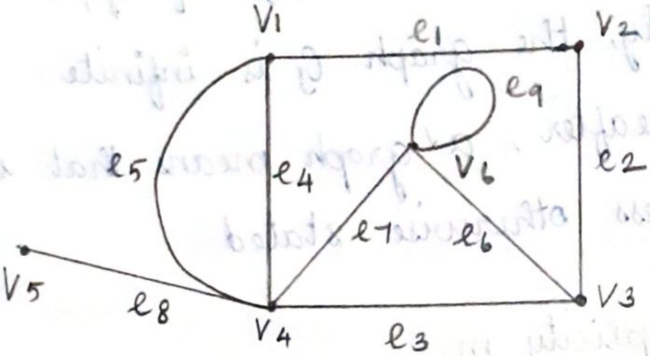
A graph in which weights are assigned to every edge is called a weighted graph.



Here 1, 2, 3, 4, 5 are weights assigned to each edge respectively.

Note:

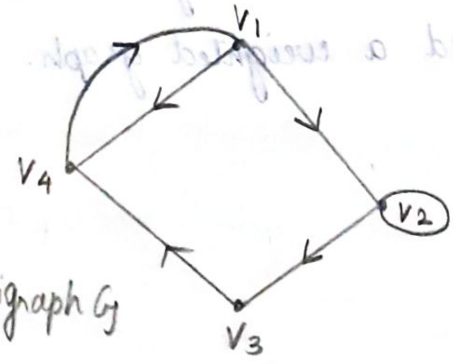
If the graph G is finite, $|V|$ denote the number of vertices of G known as order of G and $|E|$ denotes the number of edges of G, known as size of G.



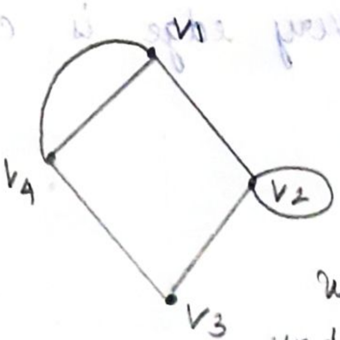
For this graph $|V| = 6$; $|E| = 9$.

Defn: underlying undirected graph

A graph obtained by ignoring the direction of edges in a directed graph is called underlying undirected graph.



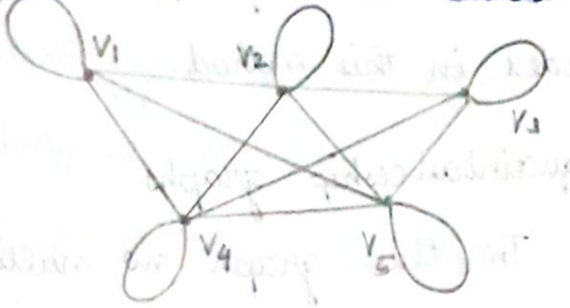
Digraph G



underlying undirected graph of G

Defn: Pseudographs:

Graphs that may include loops and possibility multiple edges connecting the same pair of vertices, are sometimes called Pseudographs.



S.No	Type	Edges	Multiple Edges	Loops.
1.	Simple graph	Undirected	No	No.
2.	Multigraph	Undirected	Yes	No.
3.	Pseudograph	Undirected	Yes	Yes.
4.	Simple directed graph	Directed	No	No.
5.	Directed multigraph	Directed	Yes	Yes.
6.	Mixed graph	Directed and undirected	Yes	Yes

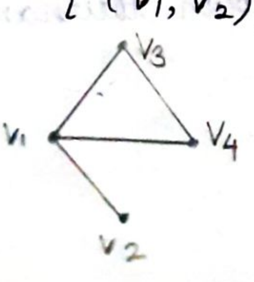
① Draw a diagram for the following graph.

$$G = G(V, E)$$

$$V = \{V_1, V_2, V_3, V_4\}$$

$$E = \{(V_1, V_2), (V_4, V_1), (V_3, V_1), (V_3, V_4)\}$$

Soln:



Graph Models:

1. Niche Overlap Graphs in Ecology

A Niche overlap graph is a simple graph because no loops or multiple edges are needed in this method.

2. Acquaintanceship graphs

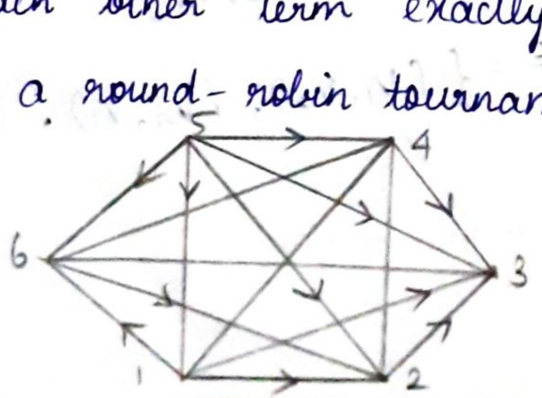
In this graph no multiple edge and no loops are used. The acquaintanceship graph of all people in the world has more than six billion vertices and probably more than one trillion edges.

3. The Hollywood graph:

The hollywood graph represents actors by vertices and connects two vertices when the actors represented by these vertices have acted together one a movie. This graph is a simple graph since its edges are undirected it contain no multiple edge and no multiple loops.

4. Round-Robin Tournaments

In a tournament where each term play each other term exactly once is called a round-robin tournament.



5. Call Graphs:

A directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge. The edge representing a call starts at the telephone number from which the call was made and ends at the telephone number of which the call was made. We need directed edges since the direction in which the call is made matters. We need multiple directed edges since we want to represent each call made from a particular telephone number to a second number.

6. Collabration Graphs:

This graph is a simple graph because it contain undirected edges and has no loops or multiple edges.

In collabration graph, vertices represent people and edge link two people if they have jointly written a paper.

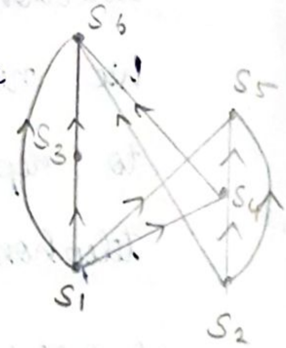
7. Precedence Graphs and Concurrent Processing:

The dependence of statements on previous statements can be represented by a directed graph. Each statement is represented by a vertex, and there is an edge from one vertex to a second vertex if the statement represented by the second vertex cannot be executed before the statement represented by the first vertex has been executed. This graph is called a precedence graph.

S_1 $a := 0$ S_4 $d := b + a$

S_2 $b := 1$ S_5 $e := d + 1$

S_3 $c := a + 1$ S_6 $f := c + d$



8. Roadmaps:

Roadmaps depicting only one-way roads and no loop roads, and where no two roads start at the same intersection and end at the same intersection, can be modeled using simple directed graphs. Mixed graphs are needed to depict roadmaps that include both one-way and two-way roads.

9. The Web Graph:

The World Wide Web can be modeled as a directed graph where each Web page is represented by a vertex and where an edge starts at the Web page a and ends at the Web page b if there is a link on a pointing to b. Since new Web pages are created and others removed somewhere on the Web almost every second, the Web graph changes on an almost.

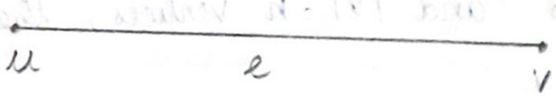
Graph Terminology and Special Types of Graphs:

Definition:

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u, v are endpoints of an edge of G.

If e is associated with {u, v} the edge e is called incident with the vertices u and v. The edge e is also said to connect u and v.

The vertices u and v are called endpoints of an edge associated with {u, v}.



Defn: The degree of a vertex:

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The degree of the vertex is denoted by deg( )

Example :

Let v be a vertex in a graph G then the degree d G(v) of the incident with v (each loop is counted twice). The d G(v) can also be denoted by deg G(v), (or explicitly, we use d(v) or deg(v) to denote the degree of v).

$\deg(v_1) = 6$

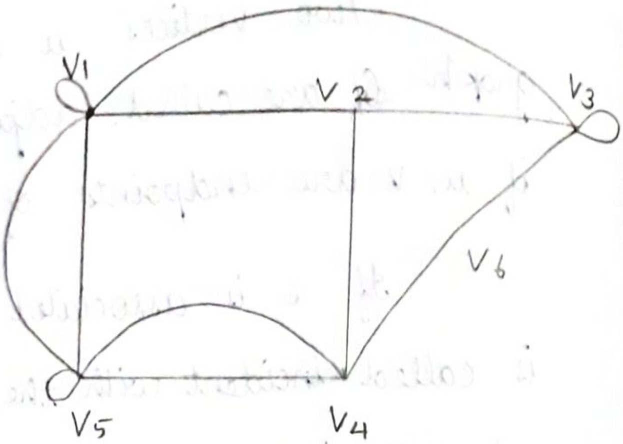
$\deg(v_2) = 3$

$\deg(v_3) = 5$

$\deg(v_4) = 4$

$\deg(v_5) = 6$

$\deg(v_6) = 0$



Note :

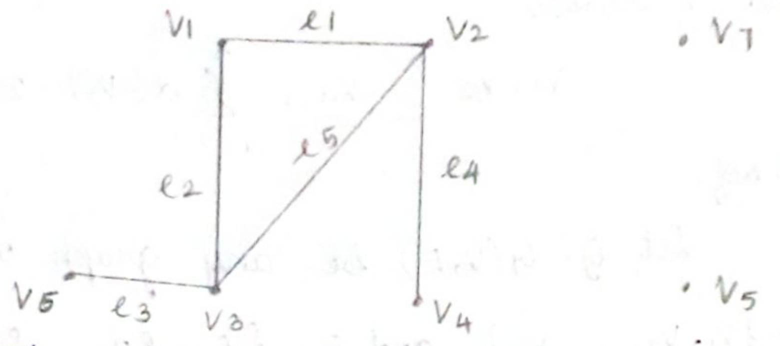
① Let G be an undirected graph with $|E|$ edges and $|V| = n$ vertices, then $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

② In any graph, the number of vertices of odd degree is even.

③ A vertex of degree one is called a pendant or end vertex in G.

④ A vertex of degree zero is called Isolated vertex in G.

⑤ Two adjacent edges are said to be in series if this common vertex is of degree two.



The vertices V_4, V_6 are pendant vertices.

The vertices V_5, V_7 are isolated vertices.

① How many edges are there in a graph with 10 vertices each of degree six?

Sum of degrees of the 10 vertices is

$(6)(10) = 60$

(i.e) $2e = 60$

$e = 30$.

② Show that the sum of degree of all the vertices in a graph G is even.

<u>Proof</u>: Each edge contribute two degree in a graph.

Also each edge contributes one degree to each of the vertices on which it is incident.

Hence, if there are N edges in G, then

$2N = d(V_1) + d(V_2) + \dots + d(V_N)$

thus $2N$ is always even.

[The Handshaking Theorem]

Theorem: For any graph G with E edges and v vertices.

$$v_1, v_2, \dots v_n, \quad \sum_{i=1}^{n} d(v_i) = 2E$$

Proof:

Let $G = G(V, E)$ be any graph where $V = \{v_1, v_2, \dots v_n\}$ and $E = \{e_1, e_2, \dots e_n\}$

Since, each edge contributes twice as a degree, the sum of the degree of all vertices in G is twice as the number of edges in G.

(i.e) $\sum_{i=1}^{n} d(v_i) = 2|E| = 2e$.

Note:

This theorem applies even if multiple edges and loops are presence.

Theorem:

The number of odd degree vertices is always even.

Let $G = \{V, E\}$ be any graph with 'n' number of vertices and 'e' number of edges.

Let $v_1, v_2, \dots v_k$ be the vertices of odd degree and $v_1', v_2', \dots v_m'$ be the vertices of even degree.

To prove, k is even

We know that $\sum_{i=1}^{n} d(v_i) = 2|E| = 2e$

$$\Rightarrow \sum_{i=1}^{k} d(v_i) + \sum_{j=1}^{m} d(v_j') = 2e$$

Each of $d(v_j')$ is even $\Rightarrow \sum_{j=1}^{m} d(v_j')$ and $2e$ are even numbers (being the sum of even no's).

$$\therefore \sum_{i=1}^{k} d(v_i) + \text{an even no} = \text{an even number}$$

$$\Rightarrow \sum_{i=1}^{k} d(v_i) = \text{an even number.}$$

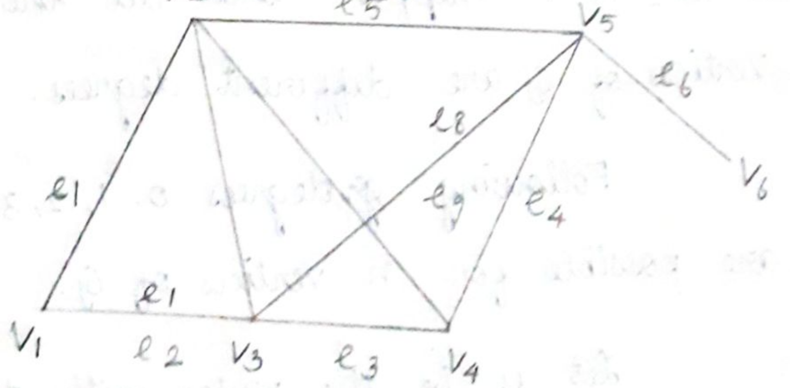
Since, each term $d(v_i)$ is odd

Therefore, the numbers of terms in the LHS sum must be even.

$\Rightarrow k$ is even

Hence the theorem.

① Verify the handshaking theorem for the graph



To prove $\sum deg(v_i) = (2)$ (No. of. edges)

(i.e) $\sum deg(v_i) = (2)(9) = 18$

$$\sum deg(v_i) = deg(v_1) + deg(v_2) + deg(v_3) + deg(v_4) + deg(v_5) + deg(v_6)$$

$$= 2 + 4 + 4 + 3 + 4 + 1$$

$$= 18$$

Hence the theorem is true.

Theorem:

A simple graph with atleast two vertices has at least two vertices of same degree.

Proof:

Let G be a simple graph with $n \geq 2$ vertices.

The graph G has no loop and parallel edges.

Hence the degree of each vertices is ~~n~~ $\leq n-1$. Suppose that all the vertices of G are different degrees.

Following ~~g~~ degrees $0, 1, 2, 3, \ldots n-1$ are possible for n vertices of G.

Let u be the vertex with degree 0. Then u has an isolated vertex.

Let v be the vertex with degree $n-1$ then v has $n-1$ adjacent vertices

Because v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G other than u is an adjacent vertex u

Hence u cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree

Note:

The converse of the above theorem is not true.

① Is there a simple graph corresponding to the following degree sequences.

(i) $(1, 1, 2, 3)$.

There are odd number (3) of odd degree vertices 1, 1 and 3. Hence there exist no graph corresponding to this degree sequence.

(ii) $(2, 2, 4, 6)$

Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6 which is not possible to as the maximum degree cannot exist one less than the number of vertices.

(2) Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Use handshaking theorem,

(i.e) $\sum_{i=1}^{n} d(v_i) = 2e$,

where e is the number of edges with n vertices in the graph G.

(i.e) $d(v_1) + d(v_2) + \dots d(v_n) = 2e \rightarrow (1)$

Since we know that the maximum degree of each vertex in the graph G can be $(n-1)$

$(1) \Rightarrow (n-1) + (n-1) + \dots$ to n terms $= 2e$

$\Rightarrow n(n-1) = 2e$

$$e = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Definition.

When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u.

The vertex u is called the initial vertex (u, v) and v is called the terminal or end vertex of (u, v).

The Initial vertex and terminal vertex of a loop are the same.

Definition:

In a graph with directed edges the in-degree of a vertex v, denoted by $deg^-(v)$ is the number of edges with v as their terminal vertex.

The out-degree of v, denoted by $deg^+(v)$ is the number of edges with v as their initial vertex.

Note

A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex

In degree:

In a directed graph G, the in degree of v denoted by in deg $G(v)$ or ~~deg~~ $deg^{-}_G(v)$, is the number of edges ending at v.

Out degree:

In a directed graph G, the out degree of vertex v of G denoted by out $deg_G(v)$ or $deg^{+}_G(v)$, is the number of edges beginning at v.

Note:

1) The Sum of the in degree and out degree of a vertex is called the total degree of the vertex.

2) A vertex with zero in degree is called a source and a vertex with zero out degree is called a sink.

Theorem:

If $G = (V, E)$ be a directed graph with e edges, then

$$\sum_{v \in V} deg^{+}_G(v) = \sum_{v \in V} deg^{-}_G(v) = e.$$

i.e. the sum of the out edges of the vertices of a diagraph G equal the sum of in degrees of the vertices which equal the number of edges in G.

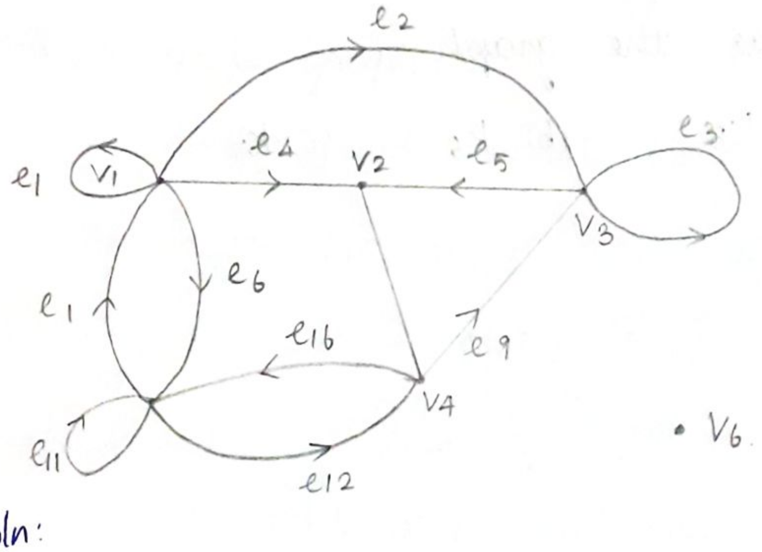
Proof:

Each edge has an initial vertex and a terminal vertex.

$\Rightarrow$ Each edge contributes one out degree to its initial vertex and one in degree to its terminal vertex.

Thus the sum of the in degrees and the sum of the out degrees of all vertices in a directed graph are same.

① Verify $\sum_{i=1}^{n} deg^{+}(v_i) = \sum_{i=1}^{n} deg^{-1}(v_i) = |E| = e$ in the following graph.



Soln:

deg	Out degree, deg^{+1} deg^{-1}	In degree deg^{-}
V_1	4	2
V_2	1	2
V_3	2	3

deg	Out degree deg^{+}	in degree deg^{-}
V_4	2	2
V_5	3	9
V_6	0	0

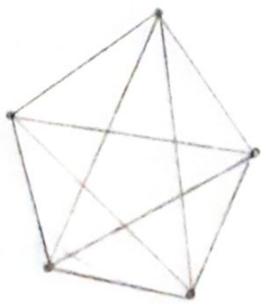
$$\sum_{i=1}^{n} deg^{+}(v_i) = \sum_{i=1}^{n} deg^{-}(v_i) = e = 12$$

Definition:

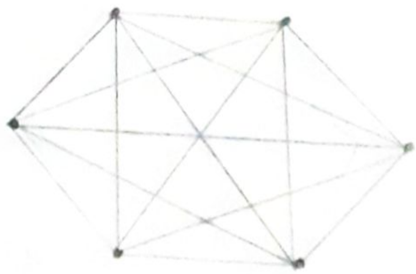
underlying undirected graph. The directed graph that results from ignoring directions of edge is called the underlying undirected graph.

Draw the graph

(a) K_5 (b) K_6 (c) K_7

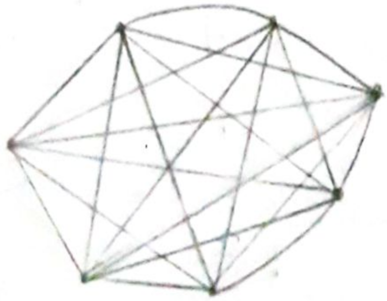


(a)



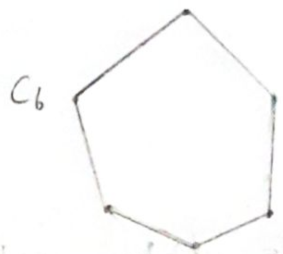
(b)

(c) K_7



Definition:

A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n.



Note:

In a graph a cycle that is not a loop must have length atleast 3, but there may be cycle of length 2 in a multigraph.

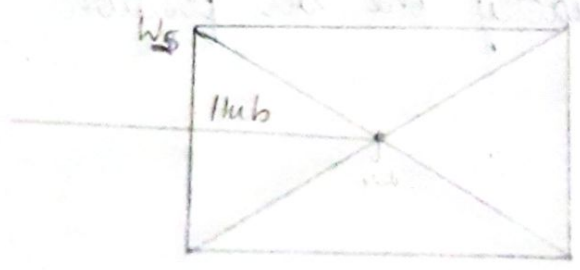
A simple diagraph having no cycles is called a cyclic graph.

A cyclic graph cannot have any loop.

The cycle C_n, $n \geq 3$, consists of n vertices $1, 2, \ldots n$ and edges $\{1,2\}, \{2,3\} \ldots \{n-1, n\}, \{n,1\}$

Wheel Graph: Definition

A wheel graph of order n is obtained by joining a new vertex called 'Hub' to each vertex of a cycle graph of order $n-1$, denoted by W_n.



Note:

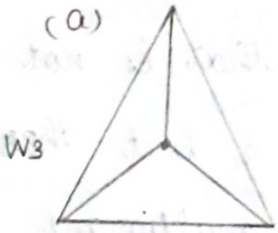
We obtained the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect ~~lines~~ this new vertex to each of the n vertices in C_n, by new edges.

Draw the graphs:

(a) W_3 (b) W_4 (c) W_5 (d) W_6 (e) W_7

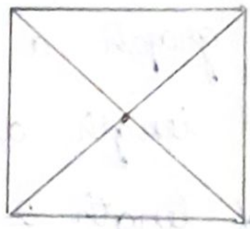
$\boxed{W_n}$ → $(n+1)$ vertices, $2n$ edges

(a)



W_3

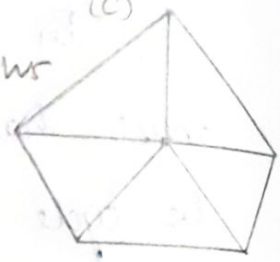
(b)



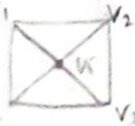
W_4

(c)

W_5



The degree sequence of W_4 is



$d(v_1) = 3$
$d(v_2) = 3$
$d(v_3) = 3$
$d(v_4) = 3$
$d(v_5) = 4$

4, 3, 3, 3, 3

It can be of any order

(d)

W_6

(e)

W_7

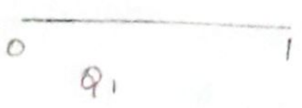
Definition: n-Cubes

The n-dimensional hypercube or n-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position

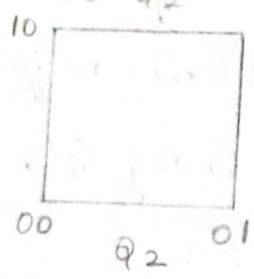
Draw the graphs :

(a) Q_1 b) Q_2 (c) Q_3.

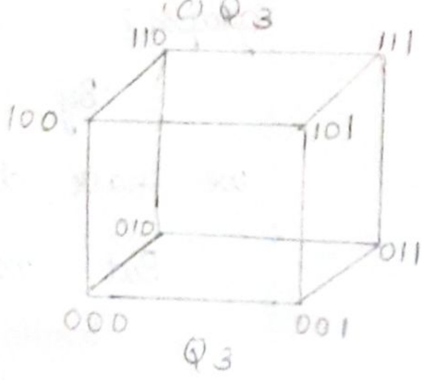
(a) Q_1



(b) Q_2



(c) Q_3



How many vertices and how many edges do those graph here.

(a) Q_n (b) Q_3.

(a) 2^n vertices , $n(2^{n-1})$ edges.

(b) $2^3 = 8$ vertices $(3)(2^2) = 12$ edges.

Regular graph:

A graph in which all vertices are of equal degree is called a regular graph.

If the degree of each vertex is r, then the graph is called a regular graph of degree r.

Note:

(1) Every null graph is regular of degree zero

(2) The complete graph K_n of degree $n-1$.

(3) If G has n vertices and is regular of degree r, then G has $(1/2)(rn)$ edges.

$\frac{rn}{2}$

What is the size of a n-regular (p, q) graph?
edges $\quad$ V, E

By the definition of regularity of G we have $\deg_G(v_i) = n$ for all $v_i \in V(G)$

But $2q = \sum \deg_G(v_i)$

$= \sum n$

$2q = pn$

$q = \frac{pn}{2}$

How many vertices does a regular graph of degree four with 10 edges have?
n $\quad$ e

Given: $n = 4$

$q = 10$.

edges $= \frac{hn}{2}$ → degree

To find : p.

$e = \frac{nn}{2}$

We know that $2q = pn$

$10 = \frac{n(4)}{2}$

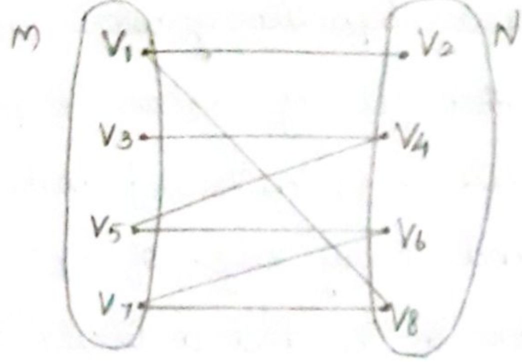
$p = \frac{2q}{n}$ $\quad$ $\boxed{n = 5}$

$p = \frac{2(10)}{4}$

$p = 5$.

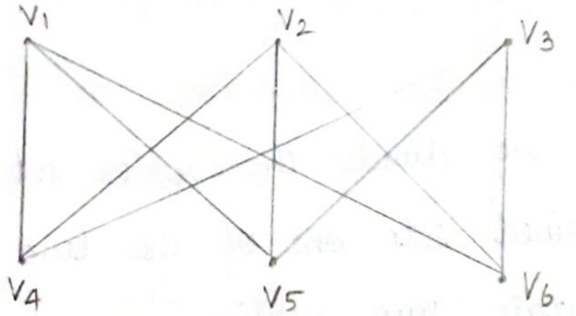
Definition: Biparite graph:

A bipartite graph is an undirected graph whose set vertices can be partitioned into two set, M and N is such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N.



Definition: Complete Diparetite graph

A complete bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N. The complete bipartite graphs that may be partitioned into sets M and N as above Such that S.t $|M| = m$ and $|N| = n$ are denoted by $K_{m,n}$. ($K_{m,n}$)



Definition : Star graph

Any graph that is $K_{1,n}$ is called a star graph.

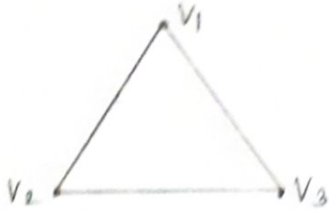
① Show that C_6 is a bipartite graph.

The vertex set of C_6 can be partioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edges of C_6 connects a vertex in V_1 and a vertex in V_2.

Here C_6 is a bipartite graph.

② Is K_3 is bipartite?

No, the complete graph K_3 is not bipartite.



If we divide the vertex set of K_3 into two disjoined sets one of the two sets must contain two vertices.

If the graphs is bipartite, these two vertices should not be connected by an edges, but in K_3 each vertex is connected to every other vertex by an edge.

K_3 is not bipartite.

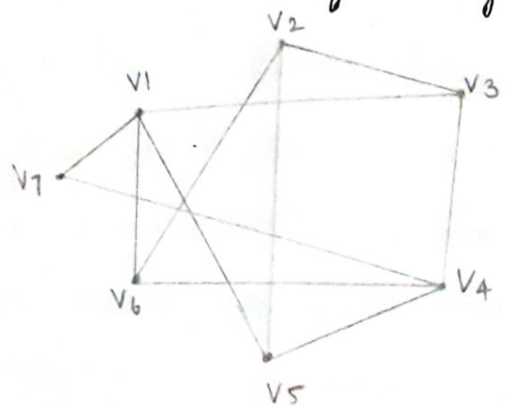
③ How many vertices and how many edges of $K_{m,n}$ graph have?

$m+n$ vertices, mn edges.

④ Find the degree sequence of the following graph $K_{2,3}$.

3, 3, 2, 2, 2.

⑤ Show that the following graph G is bipartite.



Graph G is bipartite since its vertex set is the union of two disjoint set $\{V_1, V_2, V_3\}$ and $\{V_3, V_5, V_6, V_7\}$ and each edges connects a vertex in one of these subsets to a vertex in the other subject.

⑥ For which values of m and n is $K_{m,n}$ regular?

A complete bipartite graph $K_{m,n}$ is not a regular if $m \neq n$

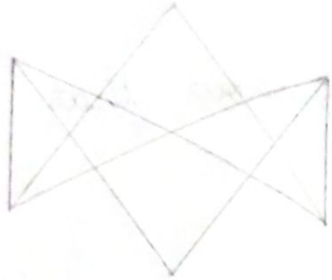
$\Rightarrow$ If $m = n$ then $K_{m,n}$ is regular.

⑦ Prove that a graph which contains a triangle can not be bipartite.

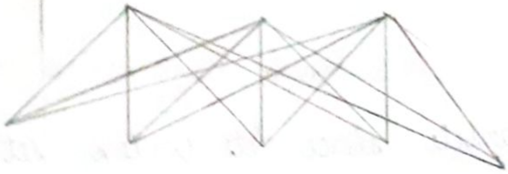
Atleast two of the three vertices must be in one of the bipartite ~~ga~~ sets because these two are joined by edges the graph cannot be bipartite.

⑧ Draw the complete bipartite graph $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$.

$K_{2,3}$



$K_{3,3}$



⑨ Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq \frac{v^2}{4}$

Let G be a complete bipartite graph with v vertices.

Let v_1 and v_2 be the number of vertices in the partitions V_1 and V_2 of vertex set of G.

Since G is complete bipartite, each vertex in V_1, in joined to each vertex in V_2 by exactly one edges

Thus G has $v_1 v_2$ edges when $v_1 + v_2 = v$. But we know that maximum value of $v_1 v_2$ subject to

$$V_1 + V_2 = V \text{ is } \frac{V^2}{4}.$$

Thus the maximum number of edges in G is $V^2/4$. i.e) $e \leq V^2/4$.

Graph coloring:

The assign of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors is called the proper coloring of G or simply vertex coloring.

If G has n coloring, then G is said to be n-colorable.

Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof: Let $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

Suppose that it is possible to assign colors to the vertices of the graph using just two colors.

$\Rightarrow$ No two adjacent vertices are assigned the same color.

Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$.

i.e every edge connects a vertex in V_1 and a vertex in V_2 since no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite.

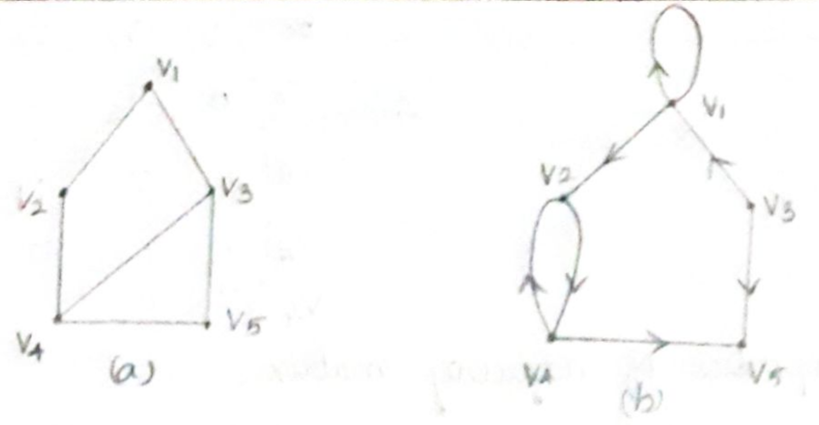
Definition: Complement

The complement of $\bar{G}$ and G is defined as a simple graph with the same vertex set as G and value two vertices u and v are adjacent only when then are not adjacent in G.

Representing graphs and graph isomorphism

Def: Matrix representation of graphs and Digraphs:

We can represent a simple graph in the form of edge list or in the form of adjacency lists which are may be useful in computer programming.



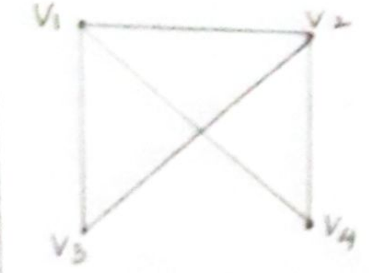
An Edge list of (a)		An Edge list of (b)	
Vertex	Adjacency vertices	Vertex	Adjacency Vertices.
V_1	V_2, V_3	V_1	V_1, V_2
V_2	V_1, V_4	V_2	V_4
V_3	V_1, V_4, V_5	V_3	V_1, V_5
V_4	V_2, V_3, V_5	V_4	V_2, V_5
V_5	V_3, V_4	V_5	—

Def: Adjacency Matrix:

Let $G(V, E)$ be a simple graph with n. Vertices ordered from V_1 to V_n, then the adjacency matrix $A = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix defined by the elements.

$$a_{ij} = \begin{cases} 1 & \text{when } V_i \text{ is adjacent to } V_j \\ 0 & \text{Otherwise.} \end{cases}$$

It is denoted by $A(G)$ or A_G.



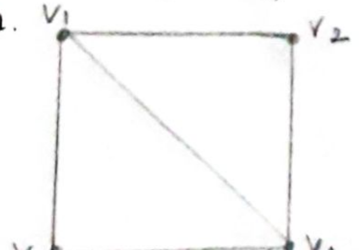
$$A_G = \begin{array}{c c} & \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Properties of adjacency matrix:

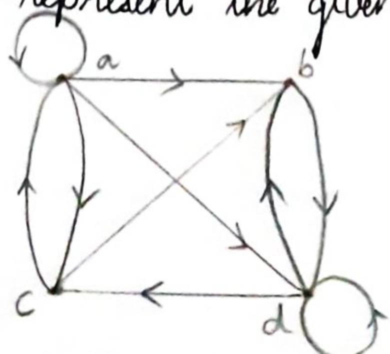
1. An adjacency matrix completely defines a simple graph.
2. The adjacency matrix is symmetric.
3. Any element of the adjacent matrix is either 0 or 1, therefore it is also called as, bit matrix or boolean matrix.
4. The ith row in the adjacency matrix is determined by the edges which originate in the node V_i.
5. If the graph G is simple, the degree of the vertex V_i equals the number of 1's in the ith row (or ith column) of A_G.
6. Given an $n \times n$ symmetric boolean matrix A, we can find a simple graph G s.t A is the adjacency matrix of G.
7. G is null $\Leftrightarrow$ $A(G)$ is the zero matrix of order n.

① use an adjacency list to represent the given graph.

ⓐ



ⓑ



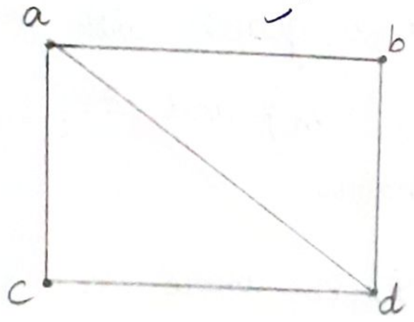
(a)

Vertex	Adjacent vertices
V_1	V_2, V_3, V_4
V_2	V_1, V_4
V_3	V_1, V_4
V_4	V_1, V_2, V_3.

(b)

Vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d.

② Represent the following graph with an adjacency matrix.



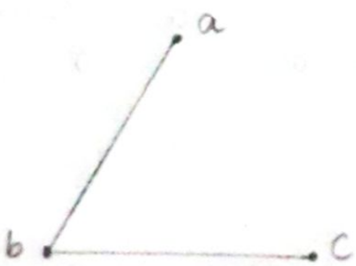
Soln:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

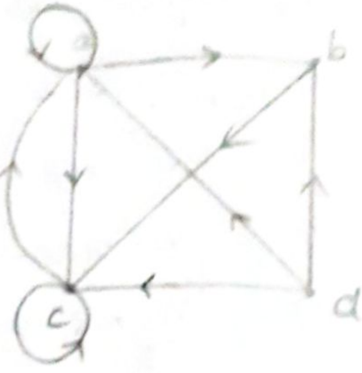
③ Draw a graph of the given adjacency matrix.

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Let
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 0 \end{array}$$



(b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



Let $\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$

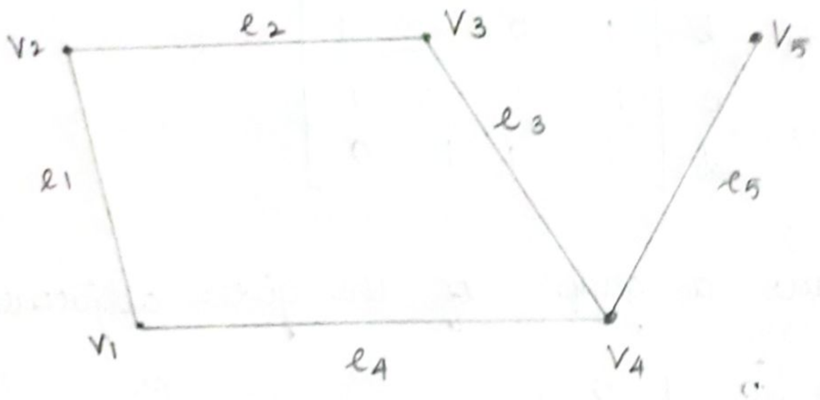
Definition : Incidence matrix

Let G be a graph with n vertices, Let $V = \{v_1, v_2, \dots v_n\}$ and $E = \{e_1, e_2, \dots e_m\}$

Define $n \times m$ matrix

$I_G = [m_{ij}]_{n \times m}$ where

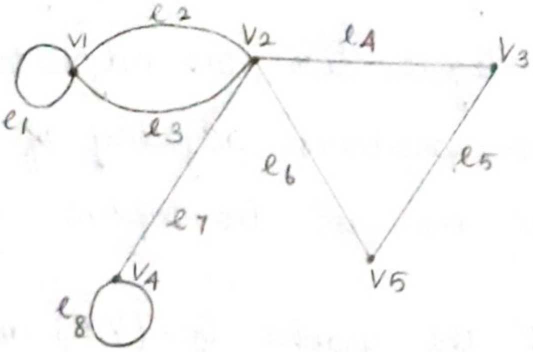
$m_{ij} = \begin{cases} 1 & \text{when } v_i \text{ is incident with } e_j \\ 0 & \text{Otherwise} \end{cases}$



$I_G =$

V_1	e_1	e_2	e_3	e_4	e_5
V_2	1	0 1	0	1	0
V_3	1	1	0	0	0
V_4	0	0	1 1	0	0
V_5	0	0	0	1 0	1 1

(1) Represent pseudograph shown in figure using an incident matrix.



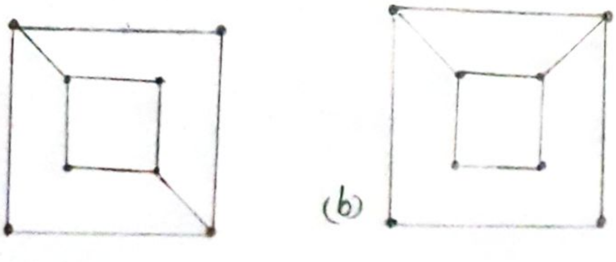
The incident matrix for this graph is

$$
\begin{array}{c|cccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
\hline
V_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
V_2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
V_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
V_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
V_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
$$

Definition: Isomorphic Graphs:

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function F is called an isomorphism.

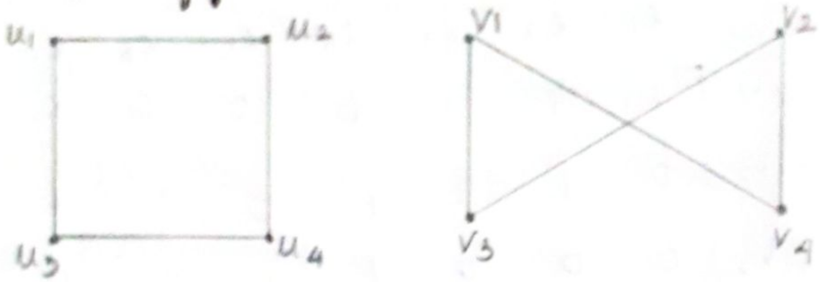
1) Explain the two graphs given below are not isomorphic.



(a) (b)

In the graph (a), no vertices of degree two are adjacent while in the graph (b) vertices of degree two are adjacent. Because isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

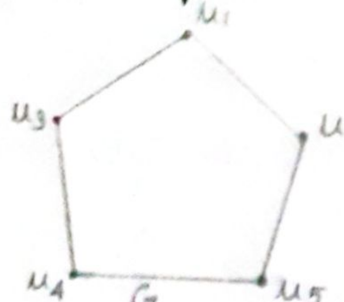
2) Show that the graphs $G = (V, E)$ and $H = (W, F)$, shown in figure are isomorphic.



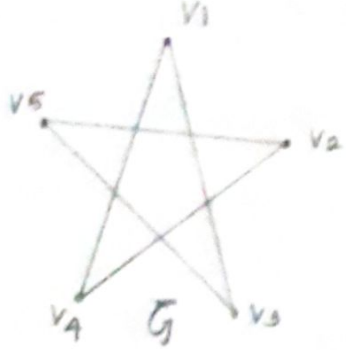
The function f with $f(u_1) = 1$, $f(u_2) = 4$, $f(u_3) = 3$ and $f(u_4) = 2$ is a one-to-one corresponding between V and W.

Here the correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4 and each of the pairs $f(u_1) = 1$ and $f(u_2) = 4$. $f(u_1) = 1$ and $f(u_3) = 3$, $f(u_2) = 4$ and $f(u_4) = 2$, and $f(u_3) = 3$ and $f(u_4) = 2$ are adjacent in H.

3) Prove that the graphs G and $\bar{G}$ given below are isomorphic.



G



$\bar{G}$

The two graphs have the same number of vertices same number of edges and same degree sequence consider the function f.

$f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_4$,
$f(u_4) = v_2$, $f(u_5) = v_5$.

Then the adjacency matrices of two graphs corresponding to f' are

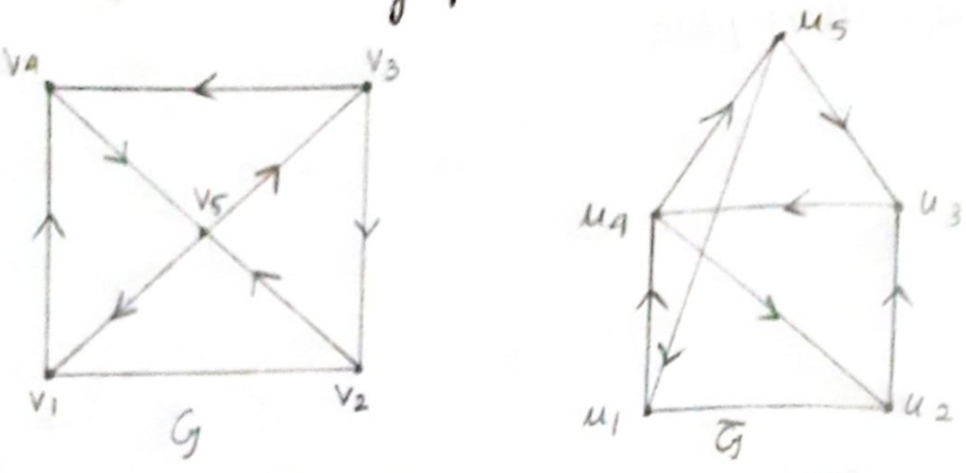
$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A(\bar{G}) = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_2 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore A(G) = A(\bar{G})$

G and $\bar{G}$ are isomorphic to each other.

4) Show that the Digraphs are isomorphic



G and $\bar{G}$ are having 5 vertices and 8 edges. Consider indegree and out degree of the vertices of G and $\bar{G}$.

G	deg + in degree	deg - out degree
V_1	1	2
V_2	2	1
V_3	1	2
V_4	2	1
V_5	2	2

$\bar{G}$	deg + in degree	deg - out degree
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

Now $f(V_1) = u_5$, $f(V_2) = u_1$, $f(V_3) = u_2$

$f(V_4) = u_3$, $f(V_5) = u_4$.

Clearly f is one to one and onto

$\Rightarrow A_G = A_{\bar{G}}$ under this mapping f.

$\therefore G$ and $\bar{G}$ are isomorphic.

Connectivity:

Definition: Path

A path in a multigraph G consists of an alternating sequence of vertices and edges of the form.

$$V_0, e_1 V_1, e_2, V_2, \dots e_{n-1}, V_{n-1}, e_n, V_n$$

where each edge e_i contains the vertices V_{i-1} and V_i.

The number n of edges is called the length of the path.

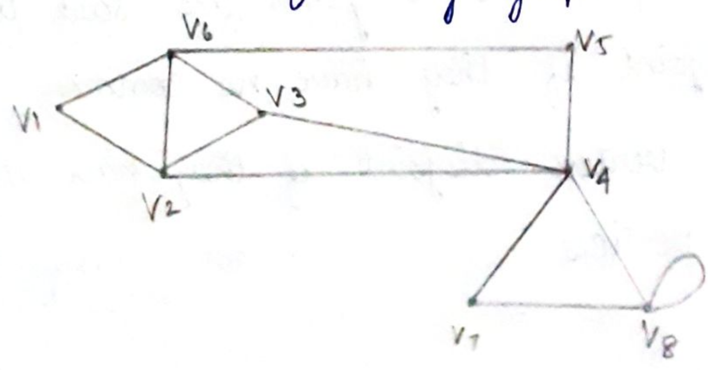
The path is said to be closed if $V_0 = V_n$ we say the path is from V_0 to V_n or between V_0 and V_n or Connects V_0 to V_n.

Note:

① A simple path is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a trial)

② If $V_0 = V_n$, then P is called a closed path. On the other hand, if $V_0 \neq V_n$, then P is an open path.

③ For the following graph



Path	Length	Simple Path	Closed Path	Circuit	Cycle
$V_1 - V_2 - V_6 - V_1$	3	Yes	Yes	Yes	Yes
$V_6 - V_2 - V_3 - V_6$	3	Yes	Yes	Yes	Yes
V_1	0	Yes	Yes	No	No
$V_8 - V_8$	1	Yes	Yes	Yes	Yes
$V_1 - V_2 - V_1$	2	No	Yes	No	No
$V_5 - V_4 - V_7 - V_8 - V_4 - V_3$	5	No	Yes	No	No.

Definition : Circuit :

A path of length ≥ 1 with no repeated edges and whose end vertices are same is called a circuit.

Note :

(4) A cycle is a circuit with no other repeated vertices except its end vertices..

(5) A cycle is a simple circuit, a loop is a cycle of length 1.

(6) In a graph a cycle that is not a loop must have length atleast 3, but there may be cycles of length 2 in a multigraph.

(7) Two paths in a graph are said to be edge disjoint if they have no common edges, they are vertex-disjoint if they have no common vertices.

Definition: Path graph:

A path graph of order 'n' is obtained by removing an edge from a C_n graph, denoted by P_n.

Definition: Trail:

A trail from v to w is a path from v to w that does not contain a repeated edge.

Thus a trail from v to w is a path of the form.

$V = V_0, e_1, V_1, e_2, \dots V_{K-1}, e_K, V_K = W$ where all the e_i are distinct.

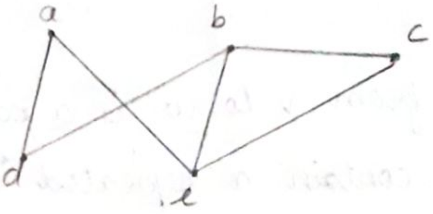
Note: Every simple path is also a trail, and every trail is also a path, but these inclusion do not reverse.

	Repeated edge	Repeated vertex.	Starts and Ends at same point?
Path	allowed	allowed	allowed
Trail	no	allowed	allowed
Simple Path	no	no	no
Closed Path	allowed	allowed	yes
Cycle	no	allowed	yes
Simple Cycle	no	first and last only	yes

Ex1: Does each of these list of vertices from a path is the following graph? Which paths are simple? Which are circuited? What are the lengths of those that are paths?

(a) a, e, b, c, b (b) e, b, a, d, b, e

(c) a, e, a, d, b, c, a (d) c, b, d, a, e, c



(a) path of length 4, not a circuit, not simple.

(b) not a path

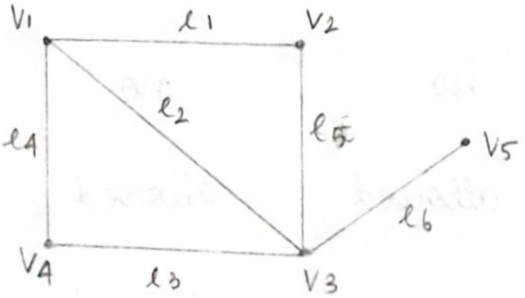
(c) not a path.

(d) simple circuit of length 5.

Definition: Connected and disconnected graphs.

A graph G is a connected graph if there is atleast one path between every pair of vertices in G. Otherwise G is a disconnected graph.

Ex:



Note: A disconnected graph consists of two or more connected graphs. Each of these connected subgroups is called a component (or a block)

Ex:2 Determine whether the given graph is connected.

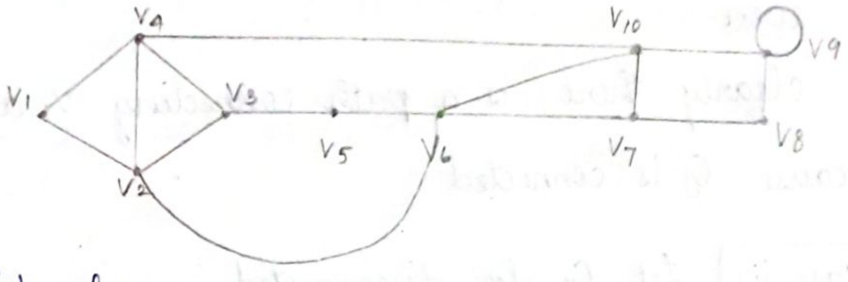
(a) 

(b) 

a) ~~No~~ Yes

b) No.

Ex:3 For the following graph



We have

Path	Length	Simple Path	closed path	Circuit	Cycle
$V_1 - V_4 - V_3 - V_5 - V_6 - V_{10} - V_4 - V_1$	7	No	Yes	No	No
$V_2 - V_3 - V_5 - V_6 - V_7 - V_{10} - V_6 - V_2$	7	No	Yes	Yes	No
$V_1 - V_2 - V_1$	2	No	Yes	No	No
$V_1 - V_4 - V_3 - V_2 - V_1$	4	Yes	Yes	Yes	Yes
$V_9 - V_9$	1	Yes	Yes	Yes	Yes
V_1	0	Yes	Yes	No	No
$V_5 - V_6 - V_7 - V_{10} - V_6 - V_2$	5	No	No	No	No
$V_4 - V_2 - V_3 - V_4$	3	Yes	Yes	Yes	Yes

Theorem: If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof: $\boxed{\text{Case (i)}}$ Let G be connected

Let v_1 and v_2 be the only vertices of G with are of odd degree

But we know that number of odd vertices is even.

Clearly there is a path connecting v_1 and v_2, because G is connected.

$\boxed{\text{Case (ii)}}$ Let G be disconnected.

Then the components of G are connected.

Hence v_1 and v_2 should belong to the same component of G.

Hence, there is a path between v_1 and v_2.

Theorem:

The maximum no. of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$

Proof:

Let the number of vertices in the ith component of G is n_1 $(n_i \geq 1)$

Then $n_1 + n_2 + \dots n_k = n$ or $\sum_{i=1}^{k} n_i = n$

Hence $\sum_{i=1}^{k} (n_i - 1) = n - k$

$$\therefore \left\{ \sum_{i=1}^{k} (n_i - 1) \right\}^2 = n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i=1j} (n_i - 1)(n_j - 1) = n^2 - 2nk + k^2 \rightarrow (3)$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2 \quad [\because (3) \geq 0, \text{ as each } n_i \geq 1]$$

$$\Rightarrow \sum_{i=1} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k.$$

Now, the maximum number of edges in the ith component of

$$G = \frac{1}{2} n_i (n_i - 1)$$

Therefore, maximum number of edges of G

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n, \text{ by } (1)$$

$$\leq \frac{1}{2} (n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2} n, \text{ by } (4)$$

$$\Rightarrow \leq \frac{1}{2} (n^2 - 2nk + k^2 + n - k)$$

$$\Rightarrow \leq \frac{1}{2} [(n-k)^2 + (n-k)]$$

$$\Rightarrow \leq \frac{1}{2} (n-k)(n-k+1)$$

Case (i):

If h covers all edges of G, then h becomes an Euler line, and hence G is an Euler graph.

Case (ii):

If h does not cover all edges of G then remove all edges of h from G and obtain the remaining graph G'. Because both G and G' have all their vertex of even degree.

$\Rightarrow$ Every vertex in G' is also of even degree.

Since G is connected, h will touch G' atleast one vertex v'. Starting from v', we can again construct a new walk h' in G'. This will terminate only at v'; because every vertex in G' is also of even degree.

Now, this walk h' combined with h forms a closed walk starts and ends at v and has more edges than h. This process is repeated until we obtain a closed walk covering all edges of G. Thus G is an Euler graph.

Theorem: A connected multigraph with atleast two vertices has an Euler circuit if and only if each of its vertices has even degree

Theorem:

There is a simple path between every pair of distinct vertices of a connected undirected graph

Proof: Let u and v be two distinct vertices of the connected undirected graph $G = (V, E)$

Since G is connected, there is atleast one path between u and v. Let $x_0, x_1, \dots x_n$, where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length.

This path of least length is simple.

Suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \leq i < j$.

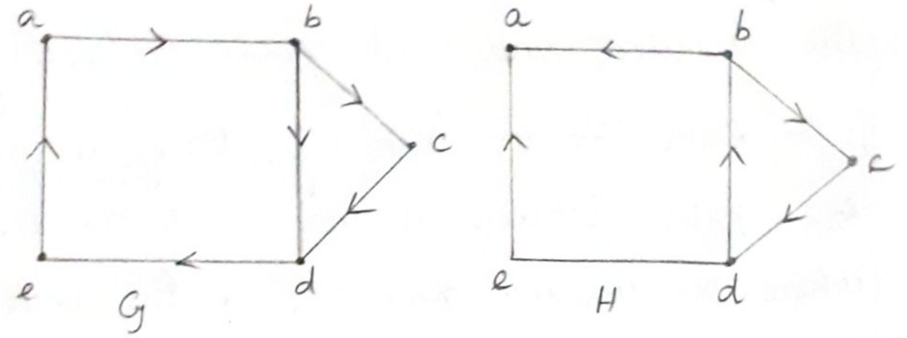
This means that there is a path from u to v of shorter length with vertex sequence $x_0, x_1, \dots x_{i-1}, x_j, \dots x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, x_{i+1}, \dots x_{j-1}$

Definition:

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Definition: A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

1) Are the directed graphs G and H shown in figure strongly connected? Are they weakly connected?



G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.

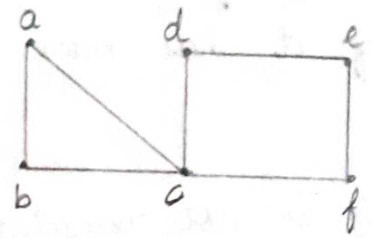
The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graphs of H.

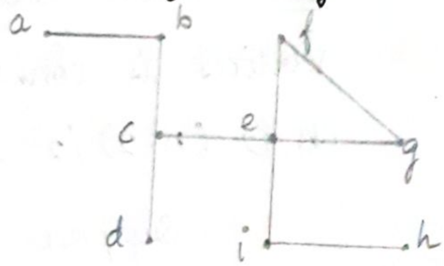
Cut vertex, Cut set and Bridge

A cut vertex of a connected graph G is a vertex whose removal increases the number of components. Clearly if v is a cut vertex of a connected graph G, $G - v$ is disconnected. A cut vertex is also called a cut point.

Bridge: If a graph G is connected and e is an edge such that $G - e$ is not connected, then e is said to be a bridge or a cut edge.

Ex:6. Find all the cut vertices of the given graph.

ⓐ 

ⓑ 

ⓐ e ⓑ b, c, e, i.

Ex:7. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

If a vertex is pendant it is clearly not a cut vertex. A endpoint of a cut edge that is a cut vertex is not pendant.

Remove of a cut edge produces a graph with more connected components than in the original graph.

If an endpoint of a cut edge is not pendant, the connected component it is in after the remove cut edge contains more than just this vertex.

From this, removal of that vertex and all edges incident to it, including the original cut edge, produces a graph with more connected components than were in the original graph.

Hence, an endpoint of a cut edge that is not pendant is a cut vertex.

Ex:8 Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

Suppose that G is not connected. Then it has a component of k vertices for some k, $1 \le k \le n-1$.

The most edges G could have is

$$C(k,2) + C(n-k,2) = [k(k-1) + (n-k)(n-k-1)]/2$$

$c(n,n) = \frac{n}{(n-r)!\,r!}$

$$= k^2 - nk + (n^2-n)/2$$

This quadratic function of f is minimised at $k = n/2$ and not maximised at $k=1$ ~~or~~ or $k = n-1$.

Hence, if G is not connected, then the no. of edges does not exceed the value of this function at 1 and at $n-1$, namely, $(n-1)(n-2)/2$.

Euler and Hamilton paths

Def: Euler circuit:

An Euler circuit in a graph G is a simple circuit containing every edge of G.

Def: Euler path:

An Euler path in G is a simple path containing every degree of G.

Def: Eulerian trail:

A trail in G is called an Eulerian trail if it includes each edge of G exactly once.

Def: Euler line, Euler graph:

A closed walk which contains all edges of the graph G is called an Euler line, and the graph containing atleast one Euler line is said to be an Euler graph.

NOTE: Euler line is also sometimes called Euler circuit.

Theorem:

A given connected graph G is an Euler graph iff all vertices of G is of even degree.

Proof: Suppose, G is and Euler graph.

$\Rightarrow$ G contains an Euler line

$\Rightarrow$ G contains a closed walk covering all edges.

To prove:

All vertices of G is of even degree

In tracing the closed walk, every time the walk meets a vertex v, it goes through two new edges incident on V with one we 'entered' and other 'excited'. This is true, for all vertices, because it is a closed walk. Thus the degree of every vertex is even.

Conversely, suppose that all vertices of G are of even degree.

To prove:

G is an Euler graph.

(i.e) to prove: G contains an Euler line.

Construct a closed walk starting at an arbitrary vertex v and going through the edge of G such that no edge is repeated. Because, each vertex is of even degree, we can exit from each end, every vertex where we enter, the tracing can stop only at the vertex V. Name the closed walk as h.

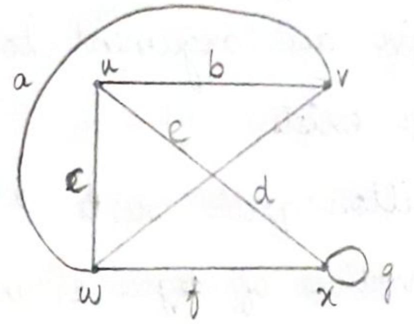
Theorem:

A connected multigraph with atleast two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem:

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

① Show that the graph, has no Eulerian circuit but has a Eulerian trail.



Here $\deg(u) = \deg(v) = 3$ and $\deg(w) = \deg(x) = 4$ because u and v have only two vertices of odd degree, the graph does not contain Eulerian circuit, but the path $b-a-c-d-g-f-e$ is an Eulerian trail.

Def: Hamilton path

A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path. That is, the simple path $x_0, x_1, x_2, \ldots x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots x_{n-1}, x\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$.

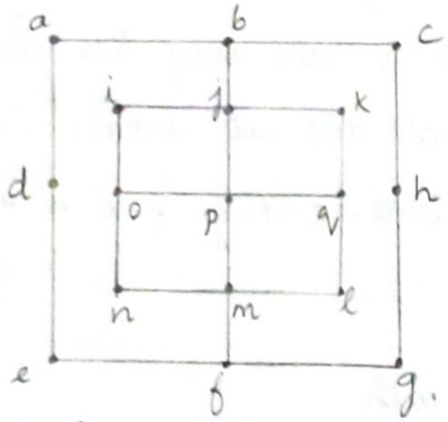
Def: Hamilton circuit:

A simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit. And the simple circuit $x_0, x_1, \ldots x_{n-1}, x_n$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots x_{n-1}, x_n$ is a Hamilton path.

Ex 3: Does the graph given below have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

No Hamilton path exists.

There are eight vertices of degree 2, and only two of them can be end vertices of a path.



For each of the other six, their two incident edges must be in the path.

It is not hard to see that if there is to be a Hamilton path, exactly one of the inside corner vertices must be an end, and that this is impossible.

Ex 5: Show that K_n has a Hamilton circuit whenever $n \geq 3$.

Now we can form a Hamilton circuit in K_n beginning at any vertex.

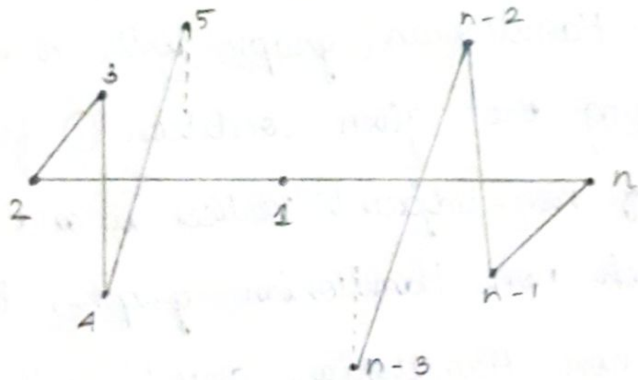
Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once.

It is possible since there are edges in K_n between any two vertices.

Ex: 14 In a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian circuit, if n is an odd number ≥ 3.

Proof: A complete graph G of n vertices has $n(n-1)/2$ edges, and a Hamiltonian circuit in G consists of n edges.

Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed $(n-1)/2$. That there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, when n is odd.



The subgraph (of a complete graph of n vertices) in figure is a Hamiltonian circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by $360/(n-1)$, $2 \cdot 360/(n-1)$, $3 \cdot 360/(n-1)$, ... $(n-3)/2 \cdot 360/(n-1)$

degrees. Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have $(n-3)/2$ new Hamiltonian circuits, all edge disjoint from the one in figure and also edge disjoint among themselves. Hence the theorem.

ORE'S THEOREM:

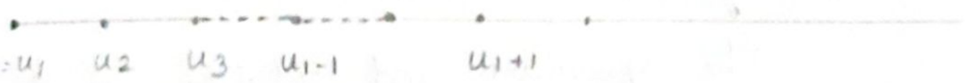
If G is a simple graph with number of vertices $n (\geq 3)$ and if

$$deg(u) + deg(v) \geq n \longrightarrow (1)$$

for every pair of non-adjacent vertices u and v, then G is Hamiltonian.

Proof: We shall prove the theorem by contradiction. We assume that there exists a non-Hamiltonian graph with n vertices satisfying the given condition (1) for every pair of non-adjacent vertices u and v. Among all such non-Hamiltonian graphs, let G be a non-Hamiltonian graph with maximum number of edges. Because G is maximal non-Hamiltonian, it follows that there exist two non-adjacent vertices u and v in G such that addition of an edge joining u and v will result in a Hamiltonian graph. Thus in G, there is a Hamiltonian path.

$u = u_1, u_2, u_3, \dots u_n = v$ with u and v as the end vertices as shown in figure



Dirac's theorem:

If G is a simple graph with n, vertices with $n \geq 3$ such that the degree of every vertex in G is atleast $n/2$, then G has a Hamiltonian circuit.

A knight is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically.

That is, a knight on square (x, y) can move to any of the eight squares $(x \pm 2, y \pm 1)$, $(x \pm 1, y \pm 2)$ if these squares are on the chessboard, as illustrated here.