UNIT-V

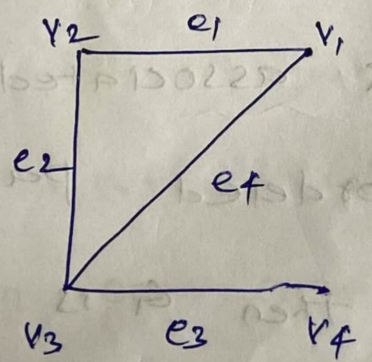
GRAPH THEORY.

BASIC DEFINITIONS:

Graph:

A graph G consists of a pair (V, E), where $V = \{v_1, v_2, \dots\}$ is a set of vertices (or nodes or points) and $E = \{e_1, e_2, \dots\}$ is a set of edges (or lines), such that each edge e_k is associated with a pair of vertices $v_i v_j$.

Eg:



$G = (V, E)$

$V = \{v_1, v_2, v_3, v_4\}$

$e = \{e_1, e_2, e_3, e_4\}$

Note!

$|V|$ = Cardinality of the vertex set of G.

= Order of G.

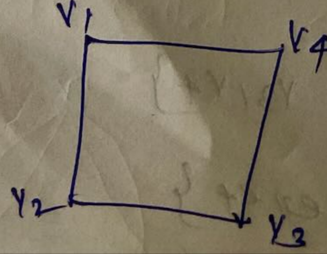
$|E|$ = Cardinality of the edge set of G

= size of G.

Undirected graph, Directed graph

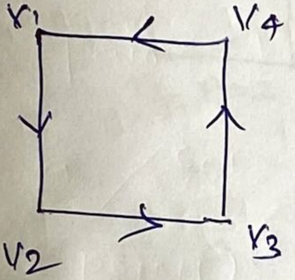
If each edge of a graph G is associated with an unordered pair of vertices, then G is called an undirected graph.

Ex!



If each edge of a graph G is associated with an ordered pair of vertices, Then G is called a directed graph or digraph.

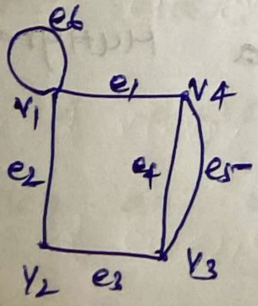
Ex:



Loop, parallel Edges:-

An edge of a graph that joins a vertex to itself is called a loop.

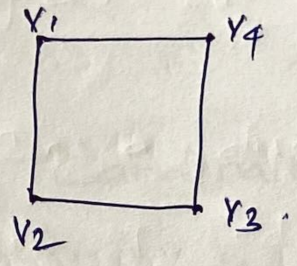
In a graph, certain pair of vertices are joined by more than one edge. Such edges are called parallel edges.



Here e_6 is a loop and e_4, e_5 are ||el edges.

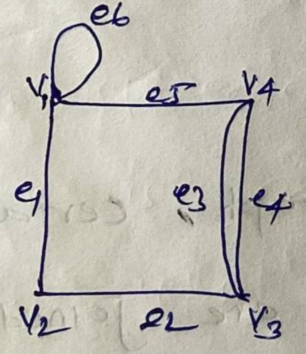
Simple graph:

A graph with no loops and parallel edges is called a simple graph.



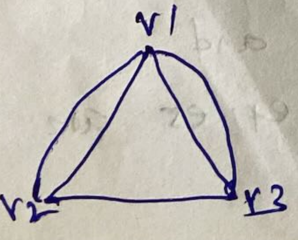
Pseudo graph!

A graph with loops and parallel edges is called a pseudo graph.



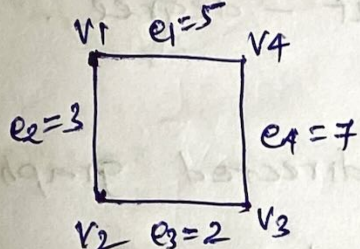
Multi graph!-

A graph with parallel edges is called a Multigraph.



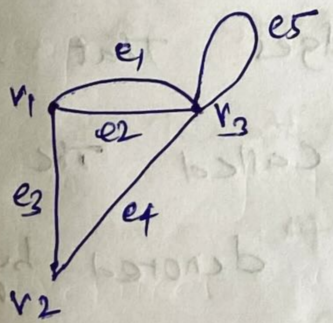
Weighted graph:

A graph in which a number (weight) is assigned to each edge is called a weighted graph.



Degree of a vertex:

The degree of a vertex is the number of edges incident with that vertex. A loop contributes 2 to the degree of that vertex.



$deg(v_1) = 3$, $deg(v_2) = 2$, $deg(v_3) = 5$.

Isolated vertex:

A degree with degree zero is called an isolated vertex.

Pendant vertex:

A vertex with degree 1 is called a pendant vertex.

In-degree, out-degree.

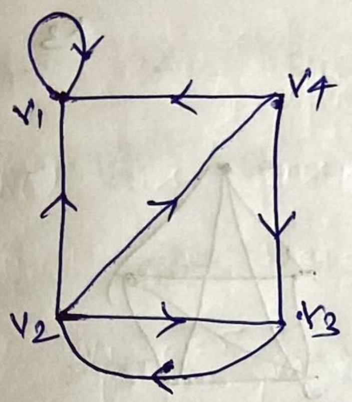
In a directed graph, the number of edges that ends at v is called the in degree of v. It is denoted by $deg^{-}(v)$.

A vertex with zero in degree is called a SOURCE.

In a directed graph, the number of edges that start from v is called the out-degree of v. It's denoted by $deg^{+}(v)$

A vertex with zero out degree is called a SINK.

Eg:



$\deg^-(v_1) = 3$

$\deg^+(v_1) = 1$

$\deg^-(v_2) = 1$

$\deg^+(v_2) = 3$

$\deg^-(v_3) = 2$

$\deg^+(v_3) = 1$

$\deg^-(v_4) = 1$

$\deg^+(v_4) = 2$.

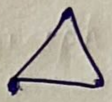
Complete graph:

A simple graph, in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.

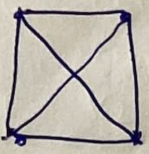
A complete graph on n vertices is denoted by K_n.

K_1

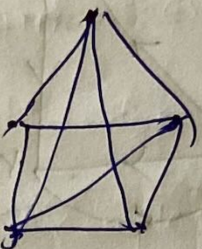
K_2



K_3



K_4



K_5.

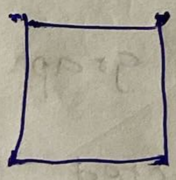
Note:

The number of edges in K_n is

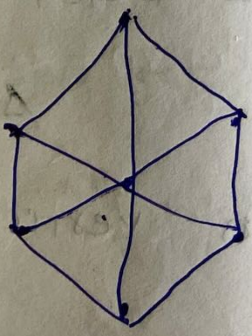
nC_2 or $\frac{n(n-1)}{2}$.

Regular graph:

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.



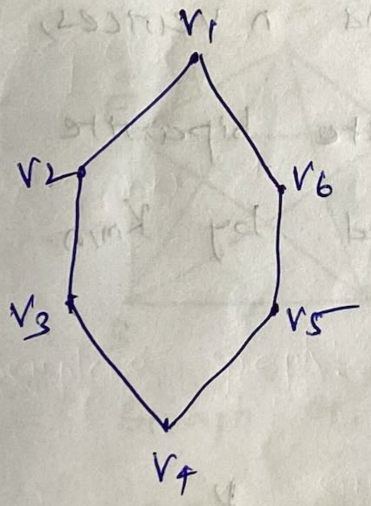
2- regular



3- regular.

Bipartite graph or Bigraph:

A simple graph G is called a bipartite graph if its vertex set V can be partitioned into two disjoint non-empty sets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2.

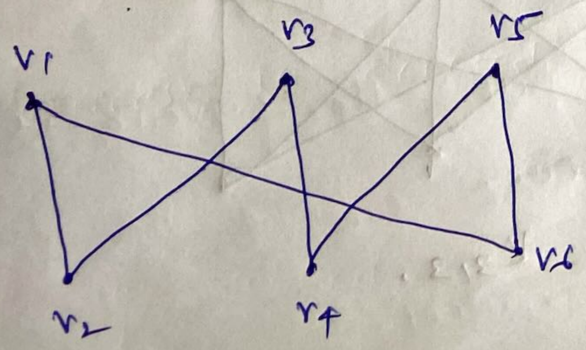


Graph G.

$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$V_1 = \{v_1, v_3, v_5\}$

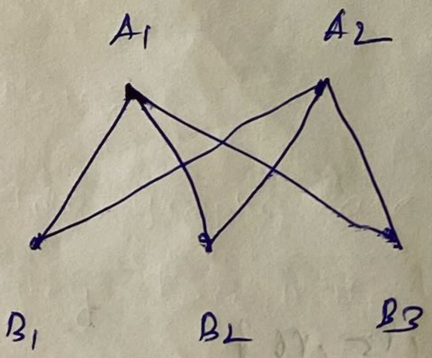
$V_2 = \{v_2, v_4, v_6\}$



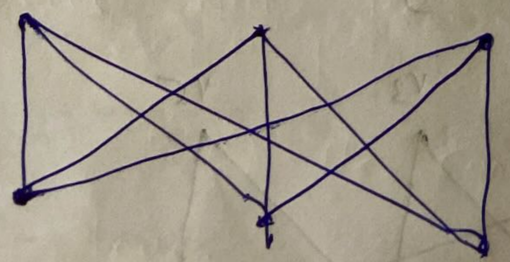
Complete Bipartite graph:

If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a complete bipartite graph.

If V_1 contains m vertices and V_2 contains n vertices, then the complete bipartite graph is denoted by $K_{m,n}$.



$K_{2,3}$

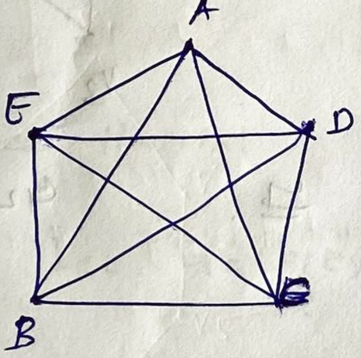


$K_{3,3}$.

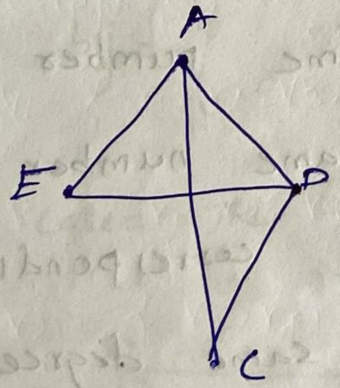
subgraph:

A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

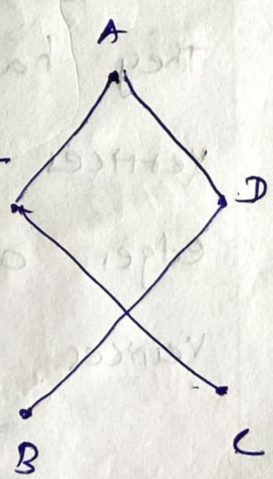
A subgraph H of a graph G is called a spanning subgraph, if $V(H) = V(G)$



Graph G.



Subgraph of G

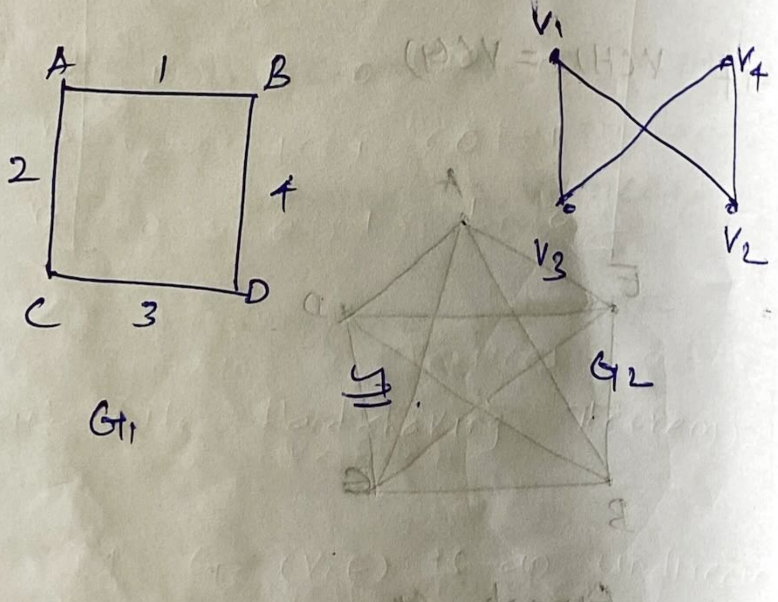


Spanning-subgraph.

ISOMORPHIC GRAPHS!

Two graphs G_1 and G_2 are said to be isomorphic, if there exists a 1-1 correspondence between the vertex sets which preserves adjacency of the vertices.

Ex!



G_1 G_2

Note!

Two graphs are isomorphic if they have (i) the same number of vertices (ii) the same number of edges and (iii) the corresponding vertices with the same degree.

But the converse is not true.

MATRIX REPRESENTATION OF GRAPHS.

Adjacency Matrix:

Let $G = (V, E)$ be a graph.

Let $V = \{v_1, v_2, \ldots, v_n\}$. The $n \times n$ Matrix $A = (a_{ij})$, Where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

is called The adjacency Matrix of G.

Incidence Matrix:

Let $G = (V, E)$ be a graph.

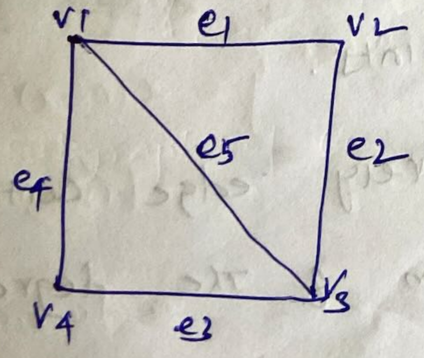
Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$

The $n \times m$ matrix $B = (b_{ij})$, Where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise.} \end{cases}$$

is called The incidence Matrix of G.

Ex:



Adjacency Matrix:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence Matrix:

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem:- (The Handshaking Theorem)

If $G = (V, E)$ is an undirected graph with e edges, Then

$$\sum_i \deg(v_i) = 2e.$$

Proof:-

Every edge of G is incident with two points.

Hence every edge contributes 2 to the sum of the degrees of the vertices.

∴ All The e edges contribute 2e to The sum of the degree of The vertices.

i.e, $\sum_{i} deg(v_i) = 2e$.

Theorem:- 2.

The number of vertices of odd degree in an undirected graph is even.

proof:

Let $G=(V,E)$ be an undirected graph. Let V_1 be The set of even degree vertices of G and V_2 be The set of odd degree vertices of G.

$$\therefore \sum_{i} deg(v_i) = \sum_{even} deg(v_j) + \sum_{odd} deg(v_k).$$

$$\sum_{odd} deg(v_k) = \sum_{i} deg(v_i) - \sum_{even} deg(v_j)$$

$$= 2e - \sum_{even} deg(v_j)$$

$$= \text{Even}.$$

Theorem - 3.

The number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Proof:

Let G be a bipartite graph with n vertices.

Let the vertex set be partitioned into two subsets V_1 and V_2.

Let V_1 contain x vertices and V_2 contain $n-x$ vertices.

The Largest number of edges of G can be obtained, when each of the x vertices in V_1 is connected to each of the $n-x$ vertices in V_2.

$$\therefore \left.\begin{array}{l}\text{Largest number} \\ \text{of edges}\end{array}\right\} = x(n-x)$$

$$= f(x),$$

a function of x.

To find the value of x, for which $f(x)$ is maximum.

$$f(x) = x(n-x)$$

$$f'(x) = n-2x$$

$$f''(x) = -2 < 0.$$

$$f'(x) = 0 \Rightarrow n-2x = 0 \Rightarrow x = n/2.$$

$\therefore$ Hence $f(x)$ is maximum, when $x = n/2$.

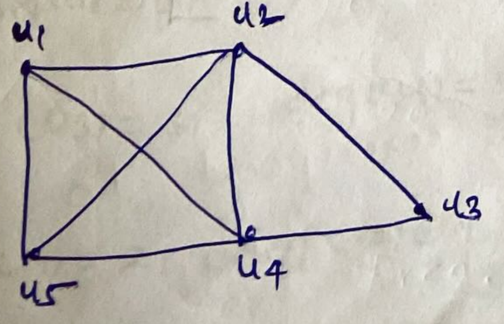
$\therefore$ Max. no. of edges required $= f(n/2)$

$$= \frac{n}{2}\left(n - n/2\right)$$

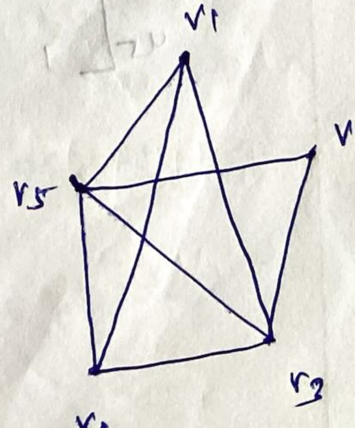
$$= \frac{n^2}{4}.$$

Ex!

Determine whether the following pair of graphs are isomorphic.



G



G'

Solution!

There are 5 vertices and 8 edges in G and G'.

deg(u1) = 3 deg(v1) = 3

deg(u2) = 4 deg(v2) = 2

deg(u3) = 2 deg(v3) = 4

deg(u4) = 4 deg(v4) = 3

deg(u5) = 3 deg(v5) = 4.

Also $|V(G)| = |V(G')|$

$|E(G)| = |E(G')|$

Adjacency Matrix of G:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of G'

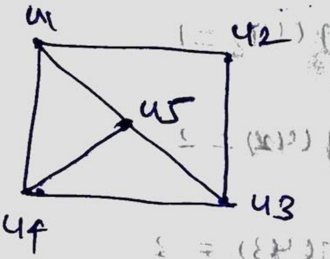
$$
\begin{array}{c|ccccc}
 & v_1 & v_5 & v_2 & v_3 & v_4 \\
\hline
v_1 & 0 & 1 & 0 & 1 & 1 \\
v_5 & 1 & 0 & 1 & 1 & 1 \\
v_2 & 0 & 1 & 0 & 1 & 0 \\
v_3 & 1 & 1 & 1 & 0 & 1 \\
v_4 & 1 & 1 & 0 & 1 & 0
\end{array}
$$

$\therefore$ G'' and G' are isomorphic.

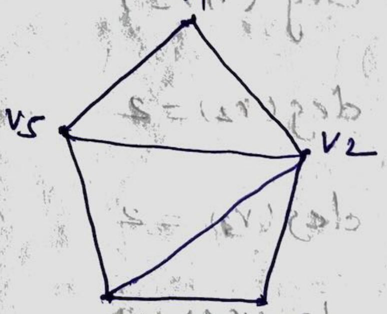
Ex:

Determine whether the following pairs of graphs are isomorphic.

Solution:



G



G'

$|V(G)| = |V(G')| = 5$

$|E(G)| = |E(G')| = 7$,

$\deg(u_1) = 3$, $\deg(u_2) = 2$,

$\deg(u_3) = 3$, $\deg(u_4) = 3$, $\deg(u_5) = 3$.

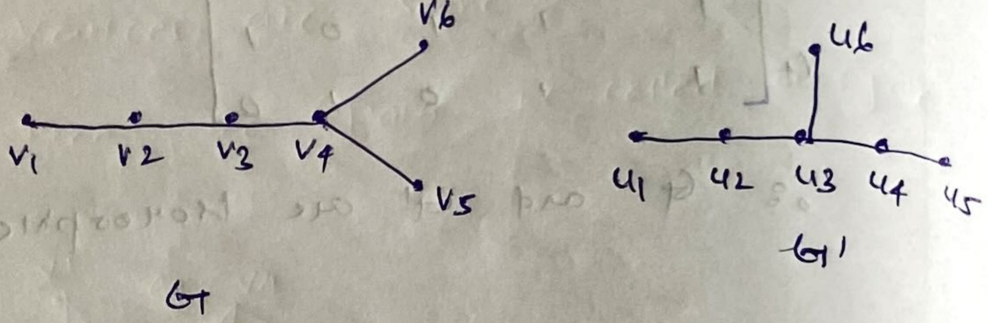
$\deg(v_1) = 2$, $\deg(v_2) = 4$, $\deg(v_3) = 2$,

$\deg(v_4) = 3$, $\deg(v_5) = 3$.

No. of vertices with the same degree is not same in G and G'.

Ex:

Determine whether G and G' are isomorphic.



Solution:-

$|V(G)| = 6$, $|V'(G)| = 6$

$|E(G)| = 5$, $|E(G')| = 5$.

$\deg(v_1) = 1$ $\qquad$ $\deg(u_1) = 1$

$\deg(v_2) = 2$ $\qquad$ $\deg(u_2) = 2$

$\deg(v_3) = 2$ $\qquad$ $\deg(u_3) = 3$

$\deg(v_4) = 3$ $\qquad$ $\deg(u_4) = 2$

$\deg(v_5) = 1$ $\qquad$ $\deg(u_5) = 1$

$\deg(v_6) = 1$ $\qquad$ $\deg(u_6) = 1$.

Under isomorphism, v_4 must correspond to u_3.

v_1, v_5, v_6 must correspond to u_1, u_5, u_6 in some order.

The vertices v_2 and v_3 are adjacent in G, whereas u_2 and u_4 are not adjacent in G'.

$\therefore$ G and G' are not isomorphic.

Path:

A path is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

Simple path:

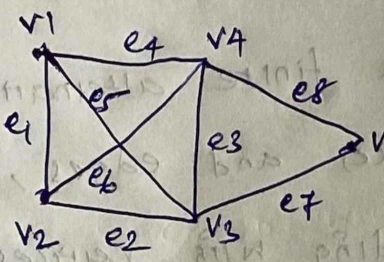
If the edges are distinct in a path, then it is called a simple path.

Length of the path:

The number of edges in a path is called the length of the path.

Circuit or cycle:

If the initial and final vertices are same in a path, then the path is called a cycle.



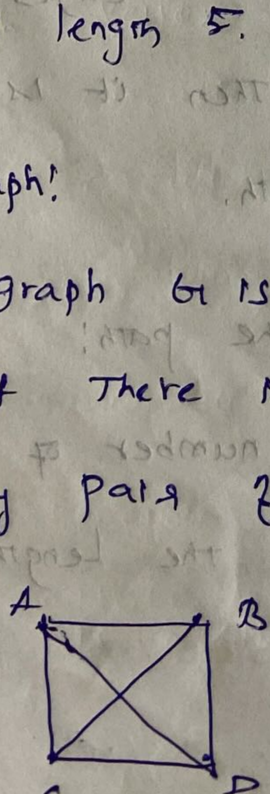
(i) $v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_1\ e_1\ v_2$ is a path of length 4.

(ii) $v_1\ e_4\ v_4\ e_6\ v_2\ e_2\ v_3\ e_7\ v_5$ is a simple path of length 4.

(iii) $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_6\ v_2\ e_1\ v_1$ is a circuit of length 5.

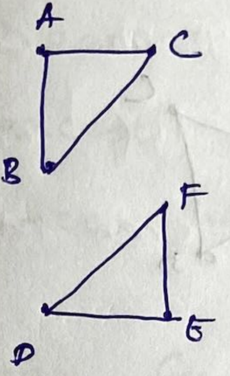
Connected Graph:

A graph G is said to be connected if there is a path between every pair of distinct vertices.



Disconnected graph:

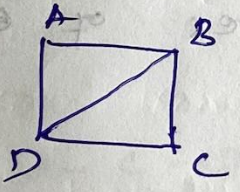
A graph that is not connected is called a disconnected graph.



Eulerian path:

A path of a graph G is called an Eulerian path, if it includes each edge of G exactly once.

Eg:



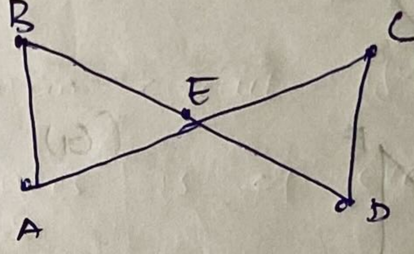
B — D — C — B — A — D is an Eulerian path.

Eulerian circuit:

A circuit of a graph G is called an Eulerian circuit, if it includes each edge of G exactly once.

Eulerian graph:

A graph containing an Eulerian circuit is called an Eulerian graph.

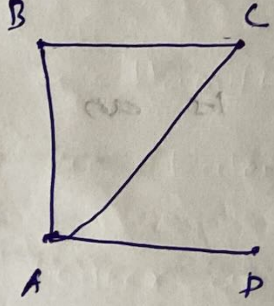


(G)

A – E – C – D – E – B – A is an Eulerian circuit. The graph G is called Eulerian graph.

Hamiltonian path:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.



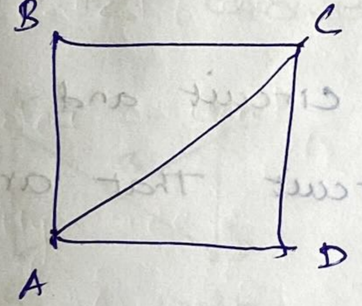
D – A – B – C is a Hamiltonian path.

Hamiltonian Circuit:

A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and end vertices which appear twice.

Hamiltonian Graph:

A graph containing a Hamiltonian circuit, is called a Hamiltonian graph.



A - B - C - D - A is a Hamiltonian Circuit.

The graph G is called Hamiltonian.

Note:

* A connected graph contains an Euler path iff it has exactly two vertices of odd degree.

* The Euler path will have the odd degree vertices as its end points.

* A connected graph contains an Eulerian circuit iff each of its vertices is of even degree.

Ex:

Give an example of a graph which contains

(i) an Eulerian circuit that is also a Hamiltonian circuit.

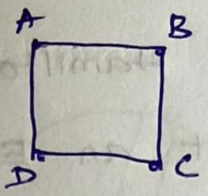
(ii) an Eulerian circuit and a Hamiltonian circuit that are distinct.

(iii) an Eulerian circuit, but not a Hamiltonian circuit.

(iv) a Hamiltonian circuit, but not an Eulerian circuit.

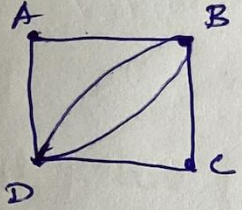
(v) neither an Eulerian circuit nor a Hamiltonian circuit.

Solution:

(i) 

(Eulerian and Hamiltonian)

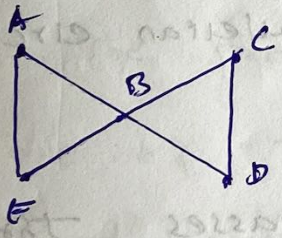
The circuit A – B – C – D – A consists of all edges and all vertices of G exactly once.

(ii) 

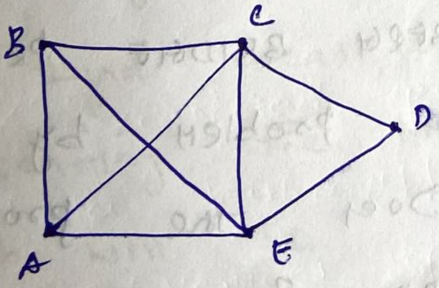
A – B – D – B – C – D – A is an Eulerian circuit.

A – B – C – D – A is a Hamiltonian circuit.

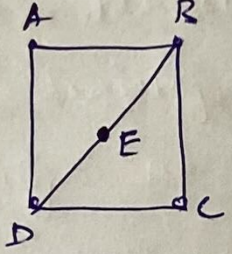
Both the circuits are different.

(iii) 

A – B – C – D – B – E – A is an Eulerian circuit but it is not a Hamiltonian circuit, since the vertex B is repeated twice.

(iv) 

A – B – C – D – E – A is a Hamiltonian circuit. But it is not an Eulerian circuit since it does not contain all edges of G.

(v) 

" A connected graph contains an Eulerian circuit iff each of its vertices is of even degree".

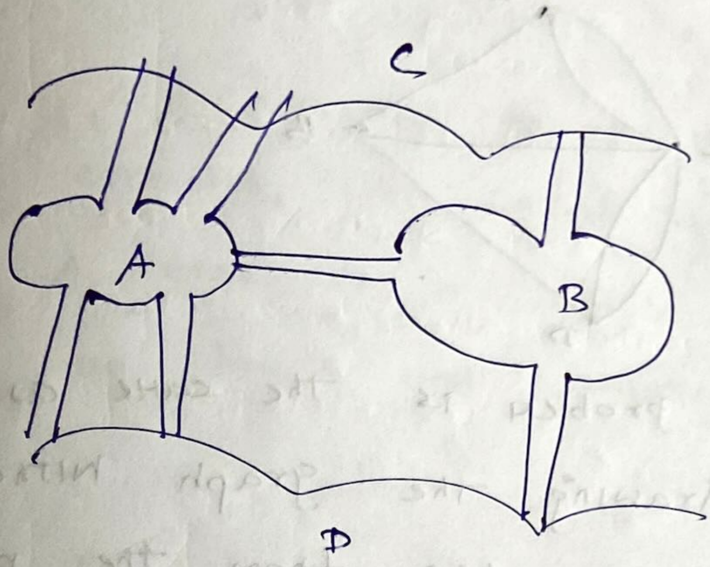
Here degree of B = degree of D = 3

∴ There is no Eulerian circuit in it.

Also, no circuit passes through each of the vertices exactly once. Hence there is no Hamiltonian circuit in G.

Ex!

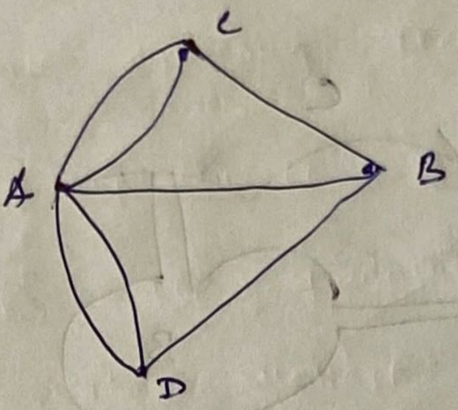
Explain KONISBERG BRIDGE PROBLEM. Represent the problem by means of graph. Does the problem have a solution ?



There are two islands A and B formed by a river. They are connected to each other and to the river banks C and D by means of 7 bridges.

The problem is to start from any one of the four land areas A, B, C, D, walk accross each bridge exactly once and return to the same starting point.

When the situation is represented by a graph with vertices representing the land areas and the edges representing the bridges, then the graph will be



The problem is the same as that of drawing the graph without lifting the pen from the paper and without retracing any time.

i.e, The problem is to find an Eulerian circuit in the graph.

W.K.T, " A connected graph has an Eulerian circuit iff each of its vertices is of even degree".

Here, all the vertices are off odd degree.

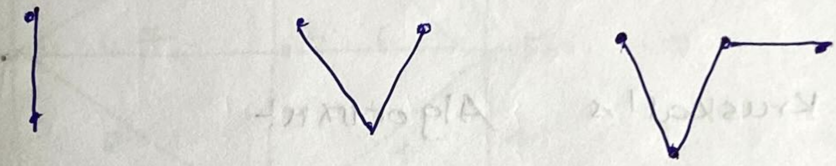
Hence there is no Eulerian circuit and there is no solution for the Konisberg bridge problem.

TREE:

A connected graph without any circuits is called a tree.

i.e., A tree is a simple graph with no loops and parallel edges.

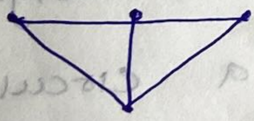
Ex:



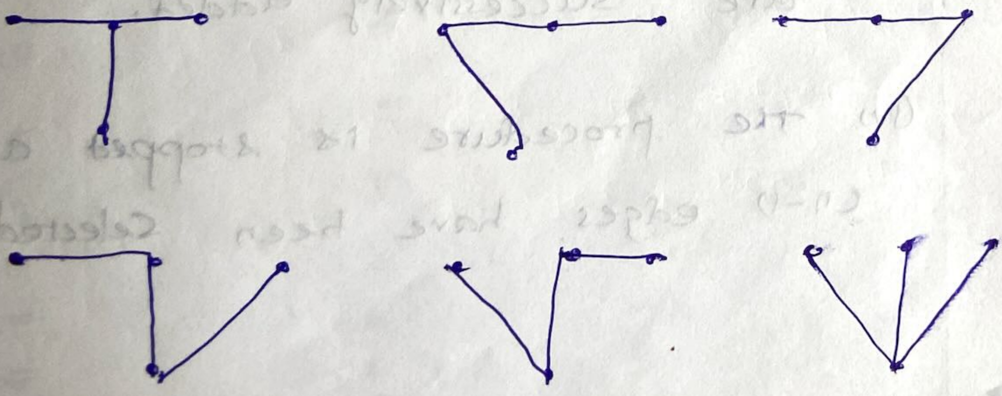
spanning tree:

If the subgraph T of a connected graph G is a tree containing all the vertices of G, then T is called a spanning tree of G.

Ex:



spanning trees of G:



Minimum spanning Tree:

If G is a connected weighted graph, The spanning tree of G with The smallest total weight is called The minimum spanning tree of G.

Kruskal's Algorithm:

(i) The edges of the graph G are arranged in the order of increasing weights.

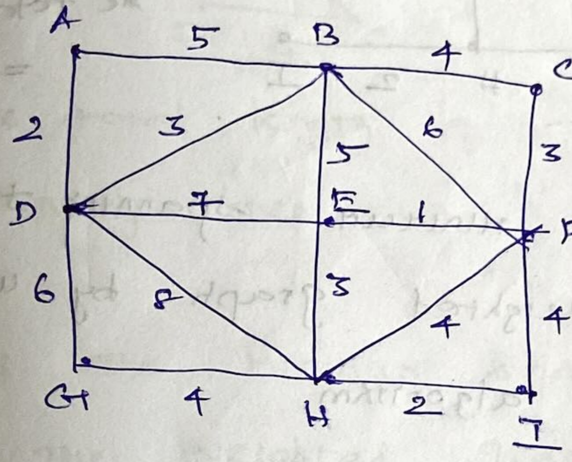
(ii) An edge of G with minimum weight is selected as an edge of The required spanning tree.

(iii) Edges with minimum weight that do not form a circuit with The already selected edges are successively added.

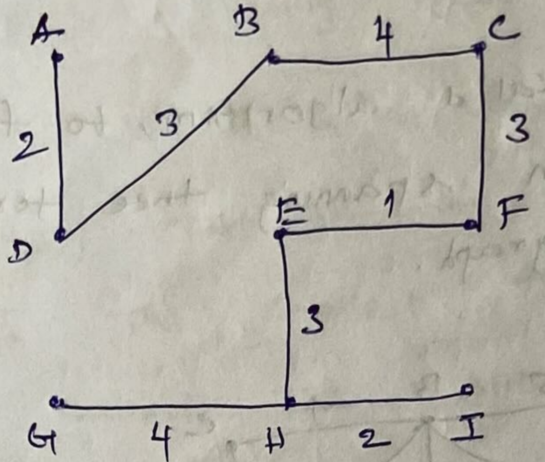
(iv) The procedure is stopped after (n-1) edges have been selected.

problems:-

1 Use Kruskal's algorithm, to find a minimum spanning tree for the weighted graph.

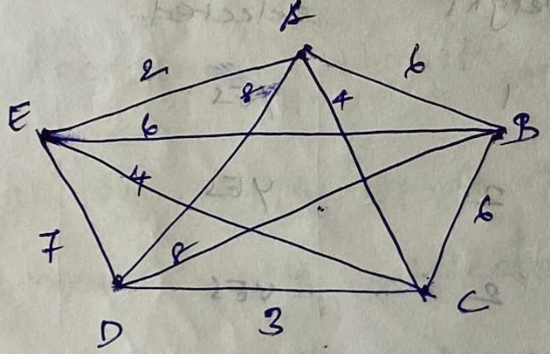


Edge	weight	selected.
FE	1	YES.
IH	2	YES
DA	2	YES
CF	3	YES
DB	3	YES
HE	3	YES
BC	4	YES
FI	4	NO
FH	4	NO
GH	4	Yes.
BE	5	—
AB	5	—
DG	6	—
BF	6	—
DE	7	—
DH	8	—



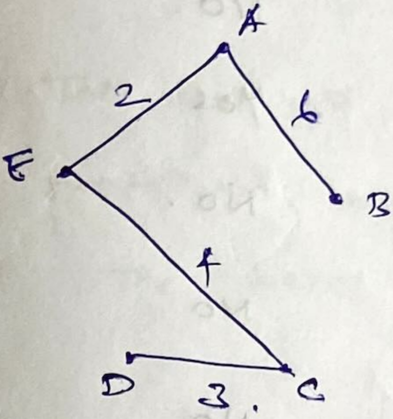
the total length = 22.

2. Find the minimum spanning tree of the weighted graph by using Kruskal's algorithm.



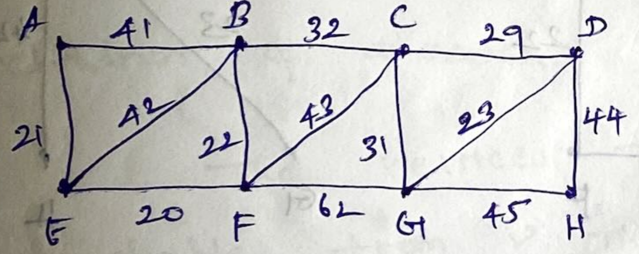
Solution:-

AE	2	yes
DC	3	yes
EC	4	yes
AC	4	NO
AB	6	yes
EB	6	–
BC	6	–
ED	7	–
BD	8	–
AD	8	–



The total length of The Minimum spanning tree = 15.

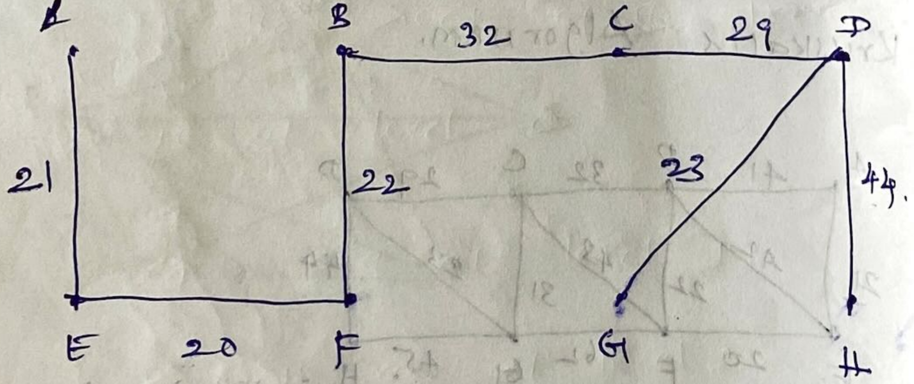
3. Find The Minimum spanning tree for The weighted graph by using Kruskal's Algorithm.



Solution:

Edge	Weight	Selected.
EF	20	Yes.
AE	21	Yes.
BF	22	Yes.
GD	23	Yes.
CD	29	Yes.

CG	31	NO.
BC	32	Yes
AB	41	No.
BF	42	NO
CF	43	No.
DH	44	Yes.
GH	45	—
FG	62	—



The total Length of The Minimum Spanning tree is 191.

Definitions:-

Rooted Tree:

A tree in which a particular vertex is designated as the root of the tree is called a rooted tree.

Level (or) Height of a vertex:

The length of the path from the root to any vertex v is called the Level of v.

Note!

The root is said to be at level zero.

Height of the tree!

The Maximum level of any vertex is called the height of the tree.

Children of a vertex!

The vertices that are reachable from v through a single edge are called the children of v.

Leaf (or) Terminal vertex (or) pendant vertex.

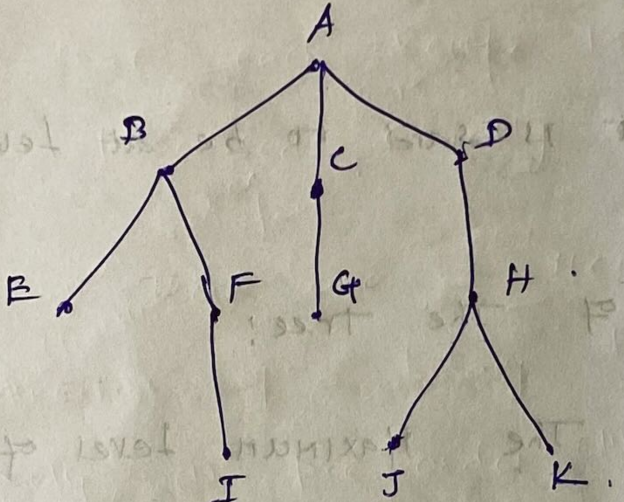
If a vertex v has no children, then v is called a pendant vertex.

i.e., The degree of a leaf is 1.

Internal vertex:

A non-pendant vertex is called an internal vertex.

Ex:



* A is the root of the tree.

* A is at level 0

* B, C, D are at level 1.

E, F, G, H are at level 2.

I, J, K are at level 3.

* The height of the tree is 3.

* E, I, G, J, K are Leaves of the tree.

* A, B, F, C, D and H are Internal vertices of the tree.

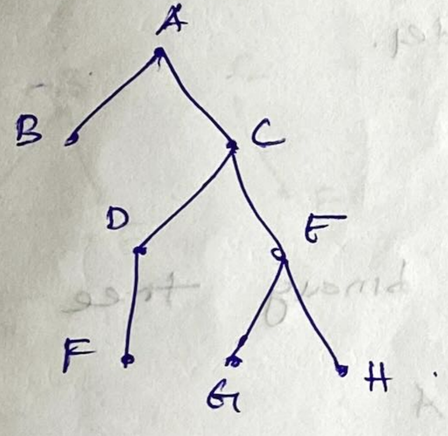
Binary Tree:

If every internal vertex of a rooted tree has atmost 2 children, Then The tree is called a binary.

(or)

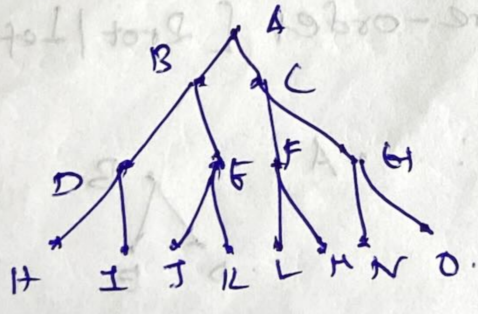
A tree T is called a binary tree, if there is only one vertex with degree 2 and The remaining vertices are of degree 1 or 3.

Eg:



Full Binary Tree:

If every internal vertex of a rooted tree has exactly 2 children then The tree is called a full binary tree.



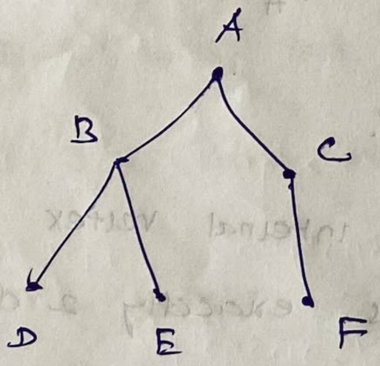
Tree Traversals!

A traversal of a tree is a process to travelle (walk along) a tree in a systematic manner so that each vertex is visited and processed exactly once.

There are 3 methods of traversal of a binary tree, namely, (i) pre-order (ii) In-order. (iii) Post order.

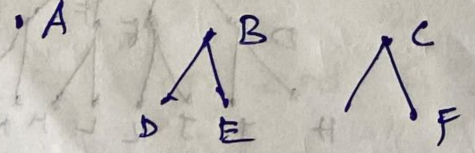
Ex:

Consider a binary tree



(i) pre-order [Root | Left | Right]

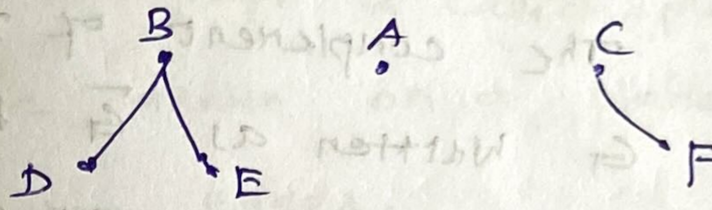
Step 1:



Step 2: A B D E C F

(ii) Inorder (Left | Root | Right)

Step 1:



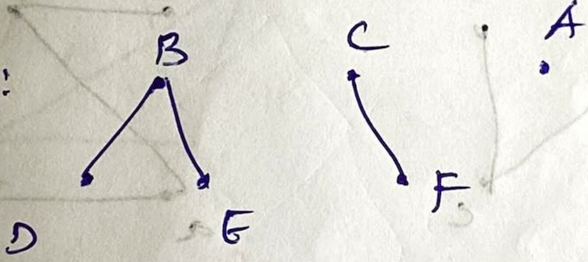
Step 2:

D B E A C F

Inorder is D B E A C F.

(iii) postorder: (Left | Right | Root)

Step 1:



Step 2:

D E B F C A

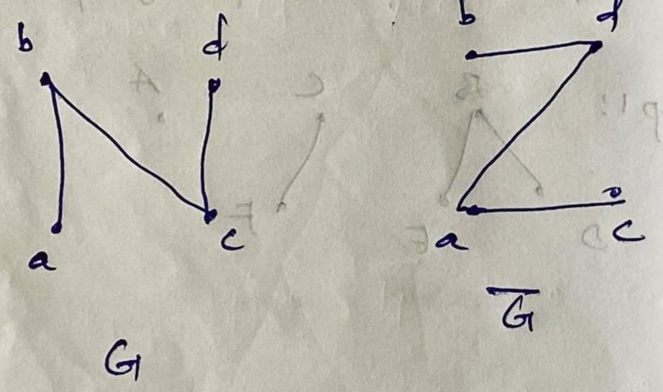
∴ The post order is

D E B F C A.

Complement of a graph:

The complement of the graph G written as $\overline{G}$ has the same vertex set as G and for every pair of distinct vertices u and v in G, uv is an edge of G iff uv is not an edge of $\overline{G}$.

Ex:



Note: Graphs that are isomorphic to their complement are called self complemented graph.

Euler's formula for planar graph:

For any connected planar graph

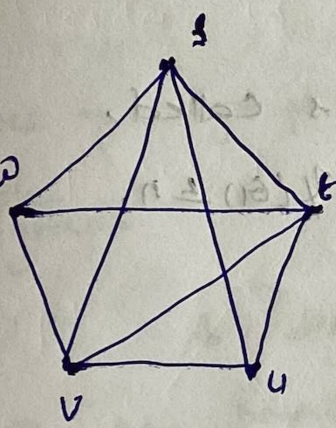
$v = e + r - 2$.

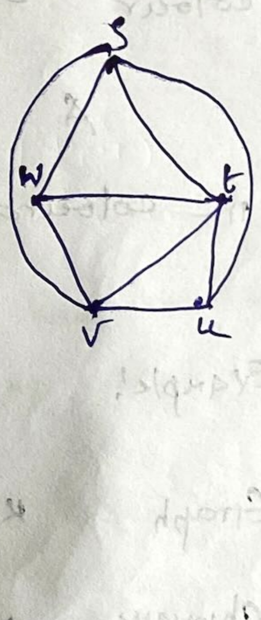
v - vertices, e - edges, r - regions of the graph.

Planar graph:

A graph is called planar if it can be drawn on a plane without intersecting edges.

A graph is called non-planar if it is not planar.





Ex: K_5 is non-planar.

Graph colouring:

An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour is called a colouring of the graph.

For each colour, the set of all points which get the colour is independent and is called a colour class.

A colouring of a graph G using atmost n colours is called an n-colouring.

The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour G.

A graph G is called n-colourable if $\chi(G) \leq n$.

Example:

Graph	K_p	$K_p - x$	$\overline{K_p}$	$K_{m,n}$	C_{2n}	C_{2n+1}
Chromatic number.	p	p-1	1	2	2	3.

When T is a tree with atleast two points, $\chi(T) = 2$.

Four Colour Theorem:

Every planar graph is 4-colourable.

Note:

* The chromatic number of any totally disconnected graph is 1

* The chromatic number of any non-trivial tree is 2.

* Any connected bipartite graph is uniquely 2-colourable.

Types of digraph:

Simple Digraph:

A digraph that has no self loop or parallel edges is called a simple digraph.

Asymmetric digraph:

Digraphs that have atmost one directed edge between a pair of vertices but are allowed to have self loops are called asymmetric or antisymmetric digraphs.

Symmetric Digraph:

Digraphs in which for every edge (a,b) There is also an edge (b,a).

Simple symmetric digraph:

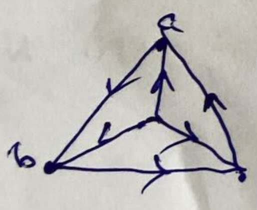
It is both simple and symmetric. A digraph which is both simple and asymmetric is called simple asymmetric digraph.

Complete digraph:

A complete symmetric digraph is a simple digraph in which There is exactly one edge directed from every vertex to every other vertex.

A complete asymmetric digraph:

is an asymmetric graph in which There is exactly one edge between every pair of vertices. It is also called a Tournament.



Note:

* A complete asymmetric digraph with n vertices contains $\frac{n(n-1)}{2}$ edges.

* A complete symmetric digraph of n vertices contains $n(n-1)$ edges.

Recall:

Region:

Suppose we draw a planar graph and take a sharp knife to cut along the edges then the plane will be cut into pieces. That is called regions of the graph.