Unit-3:

Propositions and Logical operators – Truth values and truth tables – Propositions generated by a set – Symbolic writing using conditional and biconditional connectives – Writing converse inverse and contra positive of a given conditional – Tautology, contradiction and Contingency – examples. Proving tautology and contradiction using truth table method –

Equivalences – Implications – truth table method to prove equivalences and implications – Laws of logic and some equivalences – Proving equivalences and implications using laws of logic.

Rules of interference – Rule P, Rule T and Rule CP – Direct proofs – Inconsistency and indirect method of proof – Inconsistent premises and proof by contradiction – principle of mathematical induction – problems.

Mathematical Logic

Proposition:

Proposition is a declarative sentence which is either true or false but not both.

Eg:

1. New Delhi is the capital of India – True – Proposition.
2. The Sun rises in the east – True – Proposition.
3. $2+3=7$ – False – Proposition.
4. $x+y=z$ – Not a proposition. Neither true nor false as the values of x, y, z are not assigned.
5. How beautiful the rose is! – Not a proposition.

Note:

(i) If the proposition is true, the truth value of that proposition is true denoted by 'T' or 1.

(ii) If the proposition is false the truth value is said to be false denoted by 'F' or 0.

(iii) Letters like $p, q, r \ldots$ are used to denote propositions.

Atomic Proposition:

Proposition which does not contain any of the logical operators or connectors is called atomic proposition (or) primary (or) primitive.

Compound Proposition:

It is a proportion formed by combining one or more atomic statements using connectives (or Molecular)

Eg: My name is Vijay and I am not a actor.

Logical Operators (or) Connectives:

(i) AND $\wedge$

(ii) OR $\vee$

(iii) NOT $\neg P, \sim P, P', \bar{P}$

(iv) if then $\rightarrow$

(v) iff $\leftrightarrow, \rightleftarrows, \rightleftharpoons$

(vi) Exclusive OR (or) XOR

(vii) NAND

(viii) NOR

(ix) XNOR

(i) **Conjuction:**

When p and q are any two propositions, the conjuction of p and q denoted by $p \wedge q$ is defined as the compound proposition that is true when both p and q are true and is false otherwise.

Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(ii) **Disjunction:**

When p and q are any two propositions, the disjunction of p and q, denoted by $p \vee q$, is defined as the compound proposition which is false when both p and q are false and is true otherwise.

Truth Table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(iii) Negation: Negation of a statement is another statement which has opposite meanings for the statement.

If p is a proposition then negation of p is denoted by $\neg P$, $\sim P$, p' or $\bar{P}$.

Truth Table:

p	$\sim p$
T	F
F	T

(iv) Conditional Proposition:

If p and q are propositions the compound proposition "if p then q" denoted by $p \to q$ is called a conditional proposition, which is false when p is true and q is false and true otherwise.

Truth Table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Eg: If you workhard, then you will get pass marks.

P: You work hard

q: You will get pass marks.

(v) Biconditional Proposition:

When p and q are any two propositions, the compound proposition "p if and only if q" denoted by $p \leftrightarrow q$ is called a biconditional proposition, which is true when p and q here the same truth values and is false otherwise.

Truth Table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(vi) NAND (NOT AND):

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

(vii) Exclusive OR (XOR):

True when exactly one of p and q is true and false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(viii) NOR (NOT OR)

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

(ix) Exclusive NOR (XNOR):

p	q	$p \oplus q$
T	T	T
T	F	F
F	T	F
F	F	T

Precedence of operators:

Symbol	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Tautology or Logic truth or Universally valid formula:

A statement formula that is always true is called a tautology. Eg: $P \vee \sim P$ is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Contradiction:

A statement formula that is always False is called a contradiction.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	T

Contingency:

A proposition which is neither a tautology nor a contradiction is called a Contingency.

Eg: $(p \to q) \vee (p \wedge q)$ is a contingency.

p	q	$p \to q$	$p \wedge q$	$(p \to q) \vee (p \wedge q)$
T	T	T	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

IIIly $p \wedge q$, $p \vee q$, $p \to q$, $p \leftrightarrow q$ are all contingencies.

Equivalent (or) Logically Equivalent:

$\Leftrightarrow$, $\equiv$

Let A, B be two compound propositions A and B are said to be logically equivalent, if A and B have the same set of truth values.

(or)

A and B are said to be logically equivalent if $A \leftrightarrow B$ is a tautology.

Eg: $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$ ①	$\neg p$	$\neg q$	$\neg p \wedge \neg q$ ②	①⇔②
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Here the truth values of $\neg(p \vee q)$ & $\neg p \wedge \neg q$ are same

$\therefore \neg(p \vee q) \equiv \neg p \wedge \neg q$

and $\neg(p \vee q) \to \neg p \wedge \neg q$ is a tautology.

Eg:2 $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p\wedge q)$ ①	$\neg p$	$\neg q$	$\neg p \vee \neg q$ ②	① ↔ ②
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Tautological Implication:

Let A and B be two compound propositions. A is said to be tautologically imply B iff $A \to B$ is a tautology.

Eg: $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Here $p \to p \vee q$ is a tautology.

$\therefore p \Rightarrow (p \vee q)$

Duality Law or Principle of duality:

The dual of a compound proposition is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T.

The dual of a proposition A is denoted by A^*.

Eg: $A : (P \vee F) \wedge (Q \vee F)$

$A^* : (P \wedge T) \vee (Q \wedge F)$

Converse, Inverse, Contrapositive:

If $p \to q$, then the converse is $q \to p$. The inverse is $\neg p \to \neg q$. The contrapositive is $\neg q \to \neg p$.

Eg: p : It rains

q : I will get wet.

$p \to q$: If it rain, then I will get wet.

$q \to p$: If I get wet, then it will rain.
(converse)

$\neg p \to \neg q$: If it does not rain, then I will not get wet.
(inverse)

$\neg q \to \neg p$: If I do not get wet, then it will not rain.
(contrapositive)

Laws of algebra of propositions:

(i) Idempotent law:

$$p \vee p \equiv p, \quad p \wedge p \equiv p$$

(ii) Identity law:

$$p \vee F \equiv p, \quad p \wedge T \equiv p$$

(iii) Dominant law:

$$p \vee T \equiv T, \quad p \wedge F \equiv F$$

(iv) Complement law:

$$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$$

(v) Commutative law:

$$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$$

(vi) Associative law:

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

(vii) Distributive law:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

(viii) Absorption law:

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

(ix) De Morgan's law:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

(x) Double Negation: $\neg(\neg p) \equiv p$.

Equivalence using Conditionals:

(i) $p \to q \equiv \neg p \vee q$

(ii) $p \to q \equiv \neg q \to \neg p$

(iii) $p \to q \to r \equiv (p \wedge q) \to r$

(iv) $p \vee q \equiv \neg p \to q$

(v) $p \wedge q \equiv \neg(p \to \neg q)$

(vi) $\neg(p \to q) \equiv p \wedge \neg q$

(vii) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

(viii) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

(ix) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

(x) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Equivalence involving Biconditionals:

(i) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

(ii) $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

(iii) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(iv) $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Not needed

Implications:

(i) $p \wedge q \Rightarrow p$

(ii) $p \wedge q \Rightarrow q$

Simplification

(iii) $p \Rightarrow p \vee q$

(iv) $q \Rightarrow p \vee q$

Addition.

(v) $q \Rightarrow p \rightarrow q$

(vi) $\neg p \Rightarrow p \rightarrow q$

(vii) $\neg(p \rightarrow q) \Rightarrow p$

(viii) $\neg(p \rightarrow q) \Rightarrow \neg q$

(ix) $p \wedge (p \rightarrow q) \Rightarrow q$ Modus Ponens

(x) $\neg q \wedge (p \rightarrow q) \Rightarrow \neg p$ Modus tollens

(xi) $\neg p \wedge (p \vee q) \equiv q$

(xii) $(p \vee q) \wedge \neg q \equiv p$

Disjunctive Syllogism

(xiii) $(p \rightarrow q) \wedge (q \rightarrow r)$

$\Rightarrow (p \rightarrow r)$ [Hypothetial Syllogism]

(xiv) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$

$\Rightarrow r$ (Dilemmas)

(xv) $p, q \Rightarrow p \wedge q$ [conjunction]

(xvi) $(p \vee q) \wedge (\neg p \vee r) \Rightarrow q \vee r$

(Resolution)

Problems!

Symbolize the following statements!

① If R ! Manasa is rich

H : Manasa is happy.

(i) Manasa is poor but happy

(ii) Manasa is rich and unhappy

(iii) Manasa is neither rich nor happy.

Ans!

(i) $\neg R \wedge H$

(ii) $R \wedge \neg H$

(iii) $\neg R \wedge \neg H$ (or) $\overline{R \vee H}$

② Good food is not cheap

Let p : Food is good

q : Food is cheap.

$p \rightarrow \neg q$

3. Translate the English sentence into logical expression.

You can access the internet from campus only if you are a computer science major or you are not a freshman.

p : You can access the internet from campus.

q : You are computer science major.

r : You are a freshman.

$p \rightarrow q \vee \neg r$

④ p : It is raining

q : The sun is shining

r : There are clouds in the sky.

Write the following statements in symbolic form.

(i) Iff it is raining, then there are clouds in the sky.

(ii) Iff it is not raining, then the sun is not shining and there are clouds in the sky.

(iii) The sun is shining iff it is not raining.

Ans: (i) $p \to q$

(ii) $\neg p \to \neg q \wedge r$

(iii) $q \leftrightarrow \neg p$

Problems:

Construct a truth table for each of the following compound propositions.

1. $(p \vee q) \to p \wedge q$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	F ?

2. $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$q \to \neg p$ (1)	$p \leftrightarrow q$ (2)	(1) $\leftrightarrow$ (2)
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

3. $(p \to q) \leftrightarrow (\neg p \vee q)$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T.	T	T	T

4. $(\neg p \vee \neg q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$ (1)	$p \leftrightarrow q$ (2)	(1) $\leftrightarrow$ (2)
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

5. $(p \leftrightarrow q) \leftrightarrow [(p \wedge q) \vee (\neg p \wedge \neg q)]$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	T

$(p \wedge q) \vee (\neg p \wedge \neg q)$ ②	$p \leftrightarrow q$ ①	① $\leftrightarrow$ ②
T	T	T
F	F	T
F	F	T
T	T	T

6. $R \to (q \to r) \to [(p \to q) \to (p \to r)]$

p	q	r	$q \to r$	$p \to (q \to r)$ ①
~~T~~	~~T~~			
~~T~~	~~F~~			
~~F~~	~~T~~			
~~F~~	~~F~~			

$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$ ②	① $\to$ ②

6. $p \to (q \to r) \to [(p \to q) \to (p \to r)]$

p	q	r	$q \to r$	$p \to (q \to r)$ ①
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$ ②	① $\to$ ②
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T
T	T	T	T
T	T	T	T
T	T	T	T
T	T	T	T

$\therefore p \to (q \to r) \to [(p \to q) \to (p \to r)]$ is a tautology.

Hence, $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$

7. Determine whether the compound proposition $\neg(q \to r) \land r \land (p \to q)$ is a Tautology or Contradiction.

p	q	r	$q \to r$	$\neg(q \to r)$ ①
T	T	T	T	F
T	T	F	F	T
T	F	T	T	F
T	F	F	T	F
F	T	T	T	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F

$p \to q$	$r \land (p \to q)$ ②	① $\land$ ②
T	T	F
T	F	F
F	F	F
F	F	F
T	T	F
T	F	F
T	T	F
T	F	F

Hence, $\neg(q \to r) \land r \land (p \to q)$ is a contradiction.

8. Prove using truth table $\neg q \land (p \to q) \Rightarrow \neg p$.

p	q	$p \to q$	$\neg q \land (p \to q)$
T	T	T	F
T	F	F	F
F	T	T	F
F	F	T	T

$\neg p$	$\neg q \land (p \to q) \to \neg p$
F	T
F	T
T	T
T	T

Hence, $\neg q \land (p \to q) \to \neg p$ is a tautology.

$\therefore \neg q \land (p \to q) \Rightarrow \neg p$.

9. Prove that the relation is equivalance using the truth table.

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

p	q	r	$p\to q$	$p\to r$	$(p\to q)\wedge(p\to r)$ ①
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

$q\wedge r$	$p\to(q\wedge r)$ ②	① $\leftrightarrow$ ②
T	T	T
F	F	T
F	F	T
F	F	T
T	T	T
F	T	T
T	T	T
T	T	T

Since ① $\leftrightarrow$ ② is a tautology

$(p\to q)\wedge(p\to r)\equiv p\to(q\wedge r)$.

Also ① & ② have the same truth values $\Rightarrow$ equivalent.

(10) Prove the implication

$$\underbrace{[(p\vee\neg p)\to q]}_{①}\to[(p\vee\neg p)\to R]\Rightarrow\underbrace{(q\to r)}_{②}$$

If ① $\to$ ② is a Tautology, then ① $\Rightarrow$ ②.

(11) Prove that $\sim(p\wedge q)\equiv\sim p\vee\sim q$

(12) $[p\vee\neg(q\wedge r)]\wedge\neg p\Rightarrow(\neg q\vee\neg r)$

(13) $(p\wedge q)\vee(\neg p\wedge q)\vee(p\wedge\neg q)\vee(\neg p\wedge\neg q)$ $\quad p\to q\Leftrightarrow\neg q\to\neg p$.

(14) $[(p\vee\sim p)\to Q]\to[(p\to\sim p)\to R]\Rightarrow Q\to R$

(15) $(\neg p\vee q)\wedge(p\wedge(p\wedge q))\equiv p\wedge q$.

Problems:

(1) Without using truth tables prove the equivalence $(\neg p\vee q)\wedge(p\wedge(p\wedge q))\equiv p\wedge q$.

L.H.S:

$(\neg p\vee q)\wedge(p\wedge(p\wedge q))$

$\equiv(\neg p\vee q)\wedge[(p\wedge p)\wedge q]$ ($\because$ Associative

$\equiv(\neg p\vee q)\wedge(p\wedge q)$ ($\because$ Idempotent

$\equiv(p\wedge q)\wedge(\neg p\vee q)$ ($\because$ Commutative

$\equiv[(p\wedge q)\wedge\neg p]\vee[(p\wedge q)\wedge q]$ ($\because$ Distributive

$\equiv[\neg p\wedge(p\wedge q)]\vee[p\wedge(q\wedge q)]$ ($\because$ Associative

$\equiv [(\sim P \wedge P) \wedge q] \vee (P \wedge q)$ ($\because$ Idempotent (Associative))

$\equiv (F \wedge q) \vee (P \wedge q)$ ($\because$ Complement law)

$\equiv F \vee (P \wedge q)$ ($\because$ Dominant law)

$\equiv P \wedge q$ ($\because$ Identity law)

$= R.H.S.$ Hence the proof.

2. $P \to (q \to P) \Leftrightarrow \sim P \to (P \to q)$

<u>L.H.S:</u>

$P \to (q \to P) \Leftrightarrow \sim P \vee (q \to P)$ ($\because P \to q \Leftrightarrow \sim P \vee q$)

$\Leftrightarrow \sim P \vee (\sim q \vee P)$ (")

$\Leftrightarrow \sim P \vee (P \vee \sim q)$ (commut)

$\Leftrightarrow (\sim q \vee P) \vee \sim P$ (commutative)

$\Leftrightarrow (\sim q \vee P) \vee \sim P$

$\Leftrightarrow \sim q \vee (P \vee \sim P)$ (Associative)

$\Leftrightarrow \sim q \vee T$ (Complement)

$\Leftrightarrow T$ (Dominant law)

R.H.S $\sim P \to (P \to q) \Leftrightarrow P \vee (P \to q)$

$\Leftrightarrow P \vee (\sim P \vee q)$

$\Leftrightarrow (P \vee \sim P) \vee q$

$\Leftrightarrow T \vee q$ (Complement)

$\Leftrightarrow T$ (Dominant)

$L.H.S \Leftrightarrow R.H.S$

Hence the proof.

3. Show that $\sim P \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (P \wedge r) \Leftrightarrow r$

<u>L.H.S:</u>

$[\sim P \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (P \wedge r)$

$\Leftrightarrow [(\sim P \wedge \sim q) \wedge r] \vee [r \wedge (q \vee P)]$ [$\because$ Associative, distributive]

$\Leftrightarrow [\sim (P \vee q) \wedge r] \vee [r \wedge (P \vee q)]$ [$\because$ Demorgan's, commutative]

$\Leftrightarrow r \wedge (\sim (P \vee q) \vee (P \vee q))$ ($\because$ Distributive]

$\Leftrightarrow r \wedge T$ ($\because$ Complement)

$\Leftrightarrow r$ ($\because$ Identity)

$= R.H.S$

Hence the proof.

4. Show that $(P \wedge q) \to (P \vee q)$ is a tautology.

<u>Proof:</u>

$L.H.S = (P \wedge q) \to (P \vee q)$

$\Leftrightarrow \sim (P \wedge q) \vee (P \vee q)$ ($\because P \to q \Leftrightarrow \sim P \vee q$)

$\Leftrightarrow (\sim P \vee \sim q) \vee (P \vee q)$ ($\because$ Demorgan's)

$\Leftrightarrow (\sim q \vee \sim P) \vee (P \vee q)$ ($\because$ Commutative

$\Leftrightarrow \sim q \vee (\sim P \vee P) \vee q$ ($\because$ Associative)

$\Leftrightarrow \sim q \vee T \vee q$ ($\because$ Complement)

$\Leftrightarrow (\sim q \vee q) \vee T$ (Commutative)

$\Leftrightarrow T \vee T$

$\Leftrightarrow T = R.H.S$ H.T.P

5. Show that, $P \to (q \vee r) \Leftrightarrow (P - q) \vee (P \to r)$

R.H.S:

$P \to (q \vee r) (P \to q) \vee (P \to r)$

$\Leftrightarrow (\neg P \vee q) \vee (\neg P \vee r)$

$\Leftrightarrow (\neg P \vee q \vee \neg P) \vee r$ $\left(\because a \vee (b \vee c) = (a \vee b) \vee c\right)$

$\Leftrightarrow ((\neg P \vee \neg P) \vee q) \vee r$

$\Leftrightarrow (\neg P \vee q) \vee r$

$\Leftrightarrow \neg P \vee (q \vee r)$

$\Leftrightarrow P \to (q \vee r)$

$=$ L.H.S.

6. $(P \vee q) \wedge (P \to r) \wedge (q \to r) \Rightarrow r$

Proof:

$(P \vee q) \wedge (P \to r) \wedge (q \to r) \to r$

$\Leftrightarrow (P \vee q) \wedge [(P \vee q) \to r]$ $(\because 14(vii))$

$\Leftrightarrow (P \vee q) \wedge \sim (P \vee q) \vee r \to r$ $(\because 14(i))$

$\Leftrightarrow (P \vee q) \wedge [r \vee \sim (P \vee q)] \to r$ ($\because$ Commutative)

$\Leftrightarrow [(P \vee q) \wedge r] \vee [P \vee q \wedge \sim (P \vee q)] \to r$ ($\because$ Distributive)

$\Leftrightarrow [(P \vee q) \wedge r] \vee [F] \to r$ ($\because$ Identity)

$\Leftrightarrow (P \vee q) \wedge r \to r$

$\Leftrightarrow \sim [(P \vee q) \wedge r] \vee r$

$\Leftrightarrow \sim [(P \wedge r) \vee (q \wedge r)] \vee r$ ($\because$ Distributive)

$\Leftrightarrow [\sim (P \wedge r) \wedge \sim (q \wedge r)] \vee r$

$\Leftrightarrow \sim (P \wedge r) \vee r \wedge \sim (q \wedge r) \vee r$

$\Leftrightarrow (\sim P \vee \sim r \vee r) \wedge (\sim q \vee \sim r \vee r]$ ($\because$ Distributive)

$\Leftrightarrow \sim P \vee (\sim r \vee r) \wedge (\sim q \vee (\sim r \vee r)]$ ($\because$ Associative)

$\Leftrightarrow (\sim P \vee T) \wedge (\sim q \vee T)$

$\Leftrightarrow T \wedge T$

$\Leftrightarrow T$

Hence tautology.

7. Without using truth table prove that $\neg (P \rightleftharpoons q) \Leftrightarrow (P \wedge \neg q) \vee (\neg P \wedge q)$

Sol:

$\neg (P \leftrightarrow q) \Leftrightarrow (P \wedge \neg q) \vee (\neg P \wedge q)$

L.H.S $\Leftrightarrow \neg [(P \to q) \wedge (q \to P)]$

$\Leftrightarrow \neg [(\neg P \vee q) \wedge (\neg q \vee P)]$

$\Leftrightarrow \neg [\neg P \vee q] \vee \neg [\neg q \vee P]$

$\Leftrightarrow (P \wedge \neg q) \vee (q \wedge \neg P)$

$\Leftrightarrow (P \wedge \sim q) \vee (\sim P \wedge q)$

$=$ R.H.S

Hence the proof.

Premise (or) Hypothesis: It is a statement which is assumed to be true.

Theory of Inference:

Inference theory is concerned with the infering of a conclusion from certain hypothesis or basic assumptions called) premises, by applying certain principles of reasoning called Rules of Inference.

Rules of Inference:

A set of premises $H_1, H_2, \ldots H_n$ and a conclusion C are given. We assume $H_1, H_2, \ldots H_n$ are all true. We have to prove the conclusion is true.

Two rules are used here:

1. Rule P : A premise may be introduced at any step in the derivation.
2. Rule T : A formula S may be introduced in the derivation if S is tautologically implied by one or more preceeding formula's in the derivations.

Direct Proof:

1. Show that R is a valid inference from the premises $P \to q$, $q \to r$ and P.

Solution:

1. $P \to q$ Rule (P)
2. $q \to r$ Rule (P)
3. P Rule (P)
4. $P \to r$ Rule (T) (1,2)
5. r Rule (T) (3,4)

Hence proved.

2. Show that $R \vee S$ logically from the premises $C \vee D$, $C \vee D \to \neg H$, $\neg H \to A \wedge \neg B$ and $A \wedge \neg B \to R \vee S$

1. $C \vee D$ Rule P
2. $C \vee D \to \neg H$ Rule P
3. $\neg H \to A \wedge \neg B$ Rule P
4. $A \wedge \neg B \to R \vee S$ Rule P
5. $C \vee D \to A \wedge \neg B$ Rule T (2,3)
6. $C \vee D \to R \vee S$ Rule T (5,4)
7. $R \vee S$ Rule T (1,6)

Hence the proof.

3. Show that $R \wedge (P \vee q)$ is a valid conclusion from the premises $P \vee q$, $q \rightarrow R$, $P \rightarrow M$, $\neg M$.

1. $P \vee q$ Rule P
2. $q \rightarrow R$ Rule P
3. $P \rightarrow M$ Rule P
4. $\neg M$ Rule P
5. $\neg M \rightarrow \neg P$ Rule (T) [3] ($P \rightarrow q \equiv \neg q \rightarrow \neg P$)
6. $\neg P$ Rule T [4,5] (Modulus ~~Ponens~~ ~~tollens~~)
7. q Rule T (6,1) (Disjunctive Syllogi
8. R Rule T (7,2) (Modulus Ponens)
9. $R \wedge (P \vee q)$ Rule T (8,1)

Hence proved.

4. Show that $S \vee R$ logically follows from $P \vee q$, $P \rightarrow r$ & $q - s$.

1. $P \vee q$ Rule P
2. $P \rightarrow r$ Rule P
3. $q \rightarrow s$ Rule P
4. $\sim P \rightarrow q$ Rule T (1) ($P \vee q \Leftrightarrow \sim P \rightarrow q$)
5. $\sim P \rightarrow S$ Rule T (4,3) Hypo. Syll
6. $\sim S \rightarrow P$ Rule T (5) ($P \rightarrow q \Leftrightarrow \sim q \rightarrow \sim P$)
7. $\sim S \rightarrow R$ Rule T (6,2) (Hypo Syll)
8. $\sim(\sim S) \vee R$ Rule T (7) ($P \rightarrow q \Leftrightarrow \neg P \vee q$)
9. $S \vee R$ Rule T (8) ($\neg(\neg P) \equiv P$)

5. Demonstrate S is a valid inference from the $P \rightarrow \neg q$, $q \vee r$, $\neg S \rightarrow P$ and $\neg r$.

Sol;

1. $P \rightarrow \neg q$ Rule P
2. $q \vee r$ Rule P
3. $\neg S \rightarrow P$ Rule P
4. $\neg r$ Rule P
5. $\neg S \rightarrow \neg q$ Rule (T) (3,1) (Hyp. Syll)
6. $r \vee q$ Rule T (2)
7. q Rule T (4,6) (Dist. Syll.
8. $q \rightarrow s$ Rule T (5) ($\because P \rightarrow q \Leftrightarrow \neg q \rightarrow \neg P$)
9. S Rule T (7,8) (Modulus Ponens)

Direct proof using CP rule:

Definition: This is a rule states that if $R \to S$ is a conclusion obtained from a given set of $H_1, H_2, \dots H_n$ premises, it is enough to conclude S alone from $H_1, H_2, \dots H_n$ & R.

1. Derive $P \to (q \to s)$ using rule CP if necessary from $P \to (q \to r)$, $q \to (r \to s)$.

Sol:

1. P Rule CP
2. $P \to (q \to r)$ Rule P
3. $q \to (r \to s)$ Rule P
4. $q \to r$ Rule T (1,2)
5. $\neg q \vee r$ Rule T (4)
6. $\neg q \to (r \to s)$ Rule T
7. $\neg q \vee (r \to s)$ ~~$\neg r \vee s$~~ Rule T (6)
8. $(\neg q \vee r) \wedge (\neg q \vee (r \to s))$ ~~$(\neg r \vee s)$~~ Rule T (5,7)
9. $\neg q \vee (r \wedge (q \to s))$ ~~$(\neg r \vee s)$~~ Rule T (8) (Modus Ponnes)

10. $\neg q \vee s$

10. ~~$\neg q \vee (r \wedge \neg r) \vee (r \wedge s)$~~ Rule T (9)
11. ~~$\neg q \vee F \vee (r \wedge s)$ Rule T (10)~~
12. ~~$\neg q \vee (r \wedge s)$ Rule T (11)~~
13. ~~$(\neg q \vee r) \wedge (\neg q \vee s)$ Rule T (12), (Dist...)~~
14. ~~$(q \to r) \wedge (q \to s)$~~
15. $q \to s$ Rule T (14) (Equivalence conditional)
12. $P \to (q \to s)$ CP (1,11)

(2) Using CP rule P.T $R \to S$ can be derived from the premises $P \to (q \to s)$, $\neg R \vee P$, q.

Sol:

1. R Rule CP
2. $P \to (q \to s)$ Rule P
3. $\neg R \vee P$ Rule P
4. q Rule P
5. $R \to P$ Rule T (3)
6. $R \to (q \to s)$ Rule T (5,2)
7. $q \to s$ Rule T (1,6)
8. S Rule T (4,7)
9. $R \to S$ Rule CP.

(3) Direct proof. Show that $(t \wedge s)$ can be derived from the premises $P \to q$, $q \to \neg r$, r, $P \vee (t \wedge s)$

1. $p \to q$ Rule P
2. $q \to \neg r$ Rule P
3. r Rule P
4. $p \vee (t \wedge s)$ Rule P
5. $p \to \neg r$ Rule T (1,2)
6. $r \to \neg p$ Rule T (5)
   $(p \to q \Leftrightarrow \neg q \to \neg p)$
7. $\neg p$ Rule T (3,6)
   $r \wedge r \to \neg p \Leftrightarrow \neg p$
8. $t \wedge s$ Rule T (7,4)
   (Disjunctive Syllo

4. S.T $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$, $(s \wedge \neg t) \to a \vee b$.

Sol!

1. $p \vee q$ Rule P
2. $(p \vee q) \to \neg r$ "
3. $\neg r \to (s \wedge \neg t)$ "
4. $s \wedge \neg t \to (a \vee b)$ "
5. $\neg r$ Rule T (1,2)
   (Modulus Ponnes)
6. $s \wedge \neg t$ Rule T (5,3) "
7. $a \vee b$ Rule T (6,4) "

Indirect Method:

1. Using indirect method of proof, derive $p \to \neg s$ from $p \to q \vee r$, $q \to \neg p$, $s \to \neg r$, p.

Sol: Let us include $\neg(p \to \neg s)$ as an additional premise and prove a contradiction.

Now,
$\neg(p \to \neg s) \Rightarrow \neg(\neg p \vee \neg s)$
$\Rightarrow \neg\neg(p \wedge s)$
$\Rightarrow p \wedge s$

1. $p \wedge s$ Rule P (additional premise)
2. $p \to (q \vee r)$ Rule P
3. $q \to \neg p$ Rule P
4. $s \to \neg r$ Rule P
5. p Rule P
6. $q \vee r$ Rule T (5,2)
7. s Rule T (1)
8. $\neg r$ Rule T (7,4)
9. $r \vee q$ Rule T (6)
10. q Rule T (8,9)
11. $\neg p$ Rule T (10,3).
12. F Rule T (5,11).

2. Using Indirect method

prove $p \to q,\ q \to r,\ p \vee r \Rightarrow r$

Sol:

1. $\neg r$ Rule P
2. $p \to q$ Rule P
3. $q \to r$ Rule P
4. $p \to r$ Rule T (2,3)
5. $p \vee r$ Rule P
7. $\neg r \to \neg p$ Rule T (4)
8. $\neg p$ Rule T (1,7)
9. r Rule T (8,6)
10. $r \wedge \neg r$ Rule T
11. F

Contradiction.

$\therefore p \to q,\ q \to r,\ p \vee r \Rightarrow r.$

Practice Questions:

1. Show that $r \to \neg q,\ r \vee s,\ s \to \neg q,\ p \to q \Rightarrow \neg p$ by indirect method.
2. Show by indirect method $p \to q,\ q \to r,\ s \to \neg r,\ q \wedge s$.