SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

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Department of Mathematics

Sub Title: DISCRETE MATHEMATICS FOR ENGINEERS

Sub Code: 18MAB 302 T

Unit -V - GRAPH THEORY

1. How many edges are there in a group with 10 vertices each of degree 6 ?

   a.) 30 b)60 c) 15 d) 16 **Ans : a**

2. The maximum number of edges in a simple graph with n vertices is

   a.) n(n-1)/2 b) n(n+1)/2 c) (n-1)(n+1)/2 d) n/2 **Ans : a**

3. A simple graph with n vertices and k components can have atmost ______ edges.

   a.) (n-k) (n-k-1)/2 b) (n-k) (n-k+1)/2 c) (n+k) (n+k-1)/2 d)(n+k) (n-k+1)/2 **Ans : b**

4. The complete graph on a vertices k_n where n>=3 is **Ans : a**

   a.) Hamiltonian b) Eulerian c) Both Hamiltonian and Eulerian d) Neither Hamiltonian and Eulerian

5. The maximum number of edges in a simple graph with 8 vertices is

   a.) 40 b) 32 c) 28 d) 8! **Ans : c**

6. A regular graph G has 10 edges and degree of any vertex is 5, then the number of vertices is

   a.) 4 b) 5 c) 6 d) 25 **Ans : a**

7. A closed directed path containing all the edges in a diagraph G is called an

   a.) Closed circuit b) Hamiltonian circuit c) Eulerian circuit d) Isomorphic circuit **Ans : c**

8. A free graph with n vertices has

   a.) n-1 edges b) atleast one loop c) n edges d) no root **Ans : a**

9. Sum of the degrees of all vertices of a group G is equal to

   a)Thrice the number of edges b) Twice the number of edges

   c) Number of edges d) Five times the number of the edges **Ans : b**

10. A connected graph without any circuit is called

a) Loop b) Bipartite graph c) Tree d) Directed graph **Ans : c**

11. Number of edges in k_6 graph is

a.) 16 b) 17 c) 15 d) 20 **Ans : c**

12. In a graph G, a path which includes each edge of G exactly once is called

a.) Eulerian path b) Hamiltonian path c) Eulerian circuit d) Hamiltonian circuit **Ans : a**

13. The maximum number of edges in a simple graph with 9 vertices is

a.) 36 b) 40 c) 32 d) 45 **Ans : a**

14. A regular graph G has 20 edges and degrees of any vertex is 10, then the number of vertices is

a.) 6 b) 4 c) 5 d) 8 **Ans : b**

15. Any connected graph with n vertices and n-1 edges is

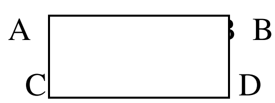
a.) Graph b) Closed graph c) Tree d) Spanning tree **Ans : c**

16. A path of a graph G is called _______ if it includes each vertices of G exactly once

a.) Tree b) Spanning tree c) Directed graph d) Hamiltonian path **Ans : a**

a.) If all the vertices of an undirected graph are each of odd degree 5, then the number of edges of the graph is a multiple of a) 3 b) 2 c) 5 d) 7 **Ans : a**

17. The graph G is



a)Eulerian and Hamiltonian b) Eulerian but not Hamiltonian

c)Hamiltonian but not Eulerian d) Neither Hamiltonian but not Eulerian **Ans : a**

18. A tree with 9 vertices has

a.) 7 edges b) 6 edges c) 10 edges d) 8 edges **Ans : d**

19. A connected graph is a Euler graph if and only if each of its vertices is of

a.) Odd degree b) Even degree c) Equal degree d) Increasing degree **Ans : b**

20. The number of vertices of odd degree in an undirected graph is

a.) Even b) Odd c) 4 d) 3 **Ans : a**

21. A simple graph is which there is exactly one edge between each pair of distinct vertices is

a.) Connected graph b) Bipartite graph c) Euler graph d) Complete graph **Ans : d**

22. Shortest path between two vertices in a weighted graph is a path of least

a.) Vertices b) Edges c) Weight d) Vertices and Edges. **Ans : c**

24. A graph in which all nodes are of equal degree is called

(a) Multi graph (b) non regular graph

(c) Regular graph (d) complete graph **Ans: c**

25. Two isomorphic graphs must have

(a) Same number of vertices (b) Same number of edges

(c) Equal number of vertices (d) all of these **Ans: d**

26.Total number of edges in a complete graph of vertices is

(a) n (b) $\frac{(n-1)}{2}$ (c) (d) $\frac{(n+1)}{2}$ **Ans: b**

27.Number of different rooted labelled trees with *n* vertices is

(a) 2^{n-1} (b) 2^n (c) n^{n-1} (d) n^n **Ans: c**

28.Maximum number of edges in a *n* node undirected graph without self-loops is

(a) n^2 (b) $\frac{(n-1)}{2}$ (c) -1 (d) $\frac{(n+1)}{2}$ **Ans: b**

29.The minimum number of spanning trees in a connected graph with *n* nodes is

(a) 1 (b) n-1 (c) (d) 2 **Ans: d**

30.The length of a Hamiltonian path(if exists) in a connected graph of n vertices is

(a) n-1 (b) n (c) (d) n+1 **Ans: a**

31. A given connected graph G is a Euler graph if and only if all vertices of G are of

(a) Same degree (b) even degree

(c) Odd degree (d) different degrees **Ans: b**

32. A graph is a tree if and only if

(a) Is completely connected (b) is minimally connected

(c) Contains a circuit (d) is planar **Ans: b**

33. The degree of each vertex in K_n is

a) n-1 (b) n (c) n-2 (d) 2n-1 **Ans: a**

34. Number of vertices of ODD degree in a graph is

(a) Always EVEN (b) Always ODD

(c) Either EVEN or ODD (d) Always ZERO **Ans: a**

35. A graph in which all nodes are of equal degree is called

(a) Multi graph (b) non regular graph

(c) Regular graph (d) complete graph **Ans: c**

36. K_n denotes _______graph.

a) Regular (b) Simple (c) Complete (d) Null **Ans: C**

37. Maximum number of edges in an n-node undirected graph without self loops is ____.

a) $\frac{n(n-1)}{2}$ (b) n - 1 (c) n (d) $\frac{n(n+1)}{2}$ **Ans: a**

38. A graph is bipartite if and only if its chromatic number is ________.

a) 1 (b) 2 (c) Odd (d) Even **Ans: b**

39. For a symmetric digraph, the adjacency matrix is _________.

a) Symmetric (b) Anti symmetric (c) asymmetric d) Symmetric & asymmetric **Ans: C**

40. The chromatic number of the chess board is ______.

a) 1 (b) 2 (c) 3 (d) 4 **Ans: b**

41. The total number of degrees of an isolated node is _______.

a) 0 (b) 2 (c) 3 (d) 1 **Ans: a**

42. Every non-trivial tree has at least _____ vertices of degree one.

a) 4 (b) 2 (c) 3 (d) 1 **Ans: b**

43. Every connected graph contains a ________.

a)Tree (b) Sub Tree (c) Spanning tree d) Spanning sub tree **Ans: C**

44. Hamilton cycle is a cycle that contains every ________of G

a) Path (b) Cycle (c) Vertex d) Edge **Ans: C**

45. Edges intersect only at their ends are called ________.

a) Planar (b) Loop (c) Link d) Non-Planar **Ans: a**

46. Two vertices which are incident with the common edge are called ______________vertices.

a) Distinct (b) Loop (c) Direct d) Adjacent **Ans: d**

47. An edge with identical ends is called _________.

a) Distinct (b) Loop (c) Direct d) Adjacent **Ans: b**

48. Each edge has one end in set X and one end in set Y then the graph (X, Y) is called _____graph.

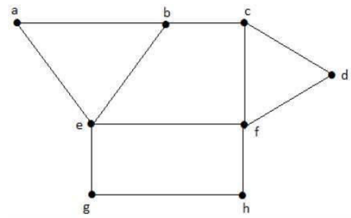
a) Bipartite (b) Simple (c) Complete (d) Trivial **Ans: a**

49. The graph defined by the vertices and edges of a __________ is bipartite.

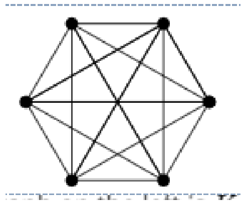
a) Square (b) Cube (c) Rectangle (d) Square and Rectangle **Ans: b**

50.The chromatic number of the null graph is

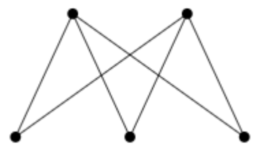
a) 4 (b) 2 (c) 3 (d) 1 **Ans: d**

51.The chromatic number of the region  is

a) 4 (b) 2 (c) 3 (d) 1 **Ans: b**

52. The chromatic number of the graph  is

a) 4 (b) 2 (c) 3 (d) 6 **Ans: d**

53. The chtomatic number of the graph  is

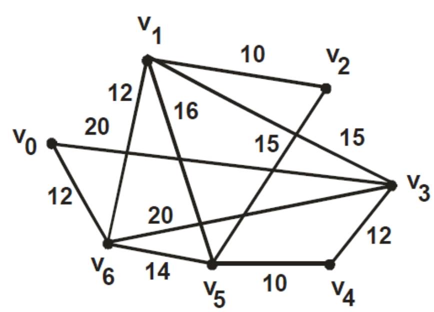
a) 4 (b) 2 (c) 3 (d) 6 **Ans: b**

54. Graph G is 2-colourable iff G is

a) Bipartite (b) Simple (c) Complete (d) Trivial **Ans: a**

55**.** The chromatic number of the graph  is

a) 4 (b) 2 (c) 3 (d) 6 **Ans: b**

56. The minimum weight of the spanning treevfor the graph  is

a) 60 (b) 70 (c) 50 (d) 80 **Ans: b**

PART-B

1. If $G = (V, E)$ is an undirected graph with e edges prove that $\sum_i \deg(v_i) = 2e$. (Hand Shaking theorem).

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertex.

$\therefore$ All the e edges contributes $(2e)$ to the sum of the degree of the vertices.

ie) $\sum_i \deg(v_i) = 2e$.

2. Prove that the number of vertices of odd degree in an undirected graph is even.

Let $G = (V, E)$ be the undirected graph.

Let V_1 and V_2 be the set of vertices of G of even and odd degrees respectively.

Thus, $\sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg v_j = 2e$. —(1)

Since each $\deg(v_i)$ is even, $\sum_{v \in v_j} \deg v_j$ is even.

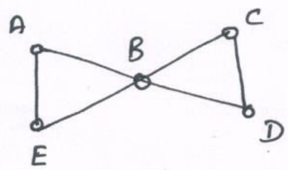
As the R.H.S of (1) is even, we get

$\sum_{v \in v_j} \deg(v_j)$ is even.

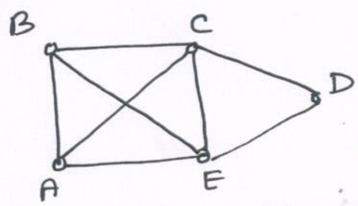
Since each $\deg(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ or in V_2 is even.

∴ ie) the number of vertices of odd degree is even.

(3) Give an example of a graph which contains
(i) An Eulerian circuit but not a Hamiltonian circuit
(ii) A Hamiltonian circuit but not a Eulerian circuit.

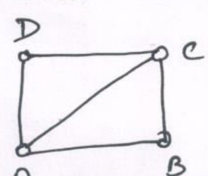
(i)


This graph contains the Eulerian circuit A–B–C–D–B–E–A but this circuit is not Hamiltonian.

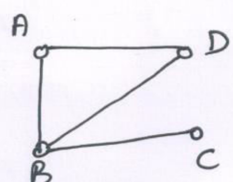
(ii)


This graph contains the Hamiltonian circuit A–B–C–D–E–A. However it does not contain Eulerian circuit as there are 4 vertices each of degree 3.

(4) which of the following graph is a Hamiltonian graph? Give reason.



G_1



G_2

In G_1, degree of A and C are each equal to 3. Hence there is no Euler's circuit in it. Also no circuit passes through each of the vertices exactly once.

∴ Hence it is not Hamiltonian.

In G_2, there is an Hamiltonian path between C and D.

∴ it is Hamiltonian.

(5) Prove that a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof: Case (i): Let G be connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree.

But already we have number of odd vertices is even.

Clearly there is a path connecting v_1 and v_2

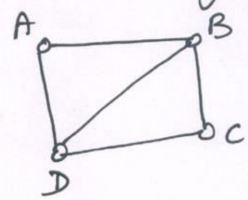
Since G is connected.

Case (ii): Let G be disconnected.

Then the components of G are connected. Hence v_1 and v_2 should belong to the same component of G.

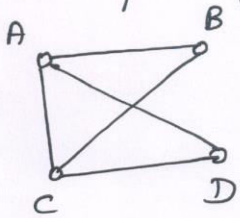
Hence there is a path between v_1 and v_2.

(6) Check whether the following graph is Eulerian and/or Hamiltonian graph.



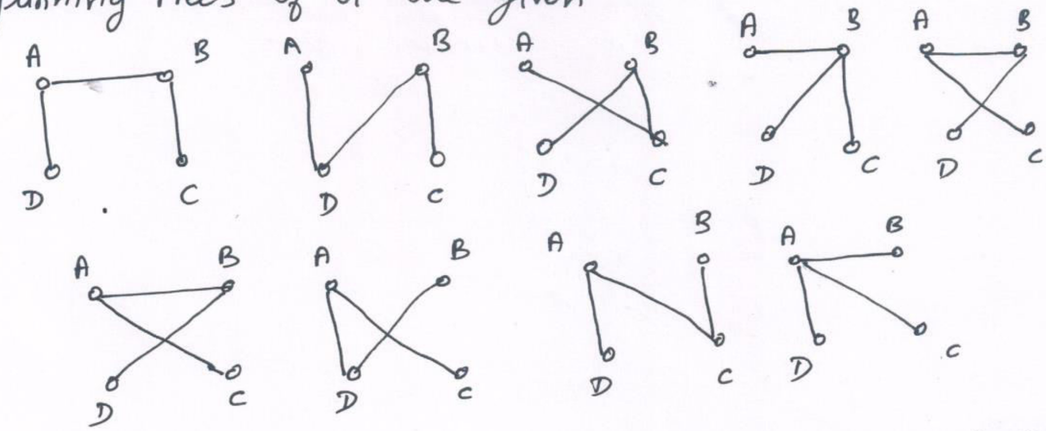
The degree of B and D are each equal to 3. Hence there is no Euler circuit in it. Also no circuit passes through each of the vertices exactly once. So it is not Hamiltonian.

(7) Draw all the spanning trees for the following graph:



The given graph G has 4 vertices. Hence any spanning tree of G will also have 4 vertices and so 3 edges.

Since G has 5 edges we have to delete 2 of the edges of G to get a spanning tree. This deletion can be done in $5C_2 = 10$ ways, but 2 of these 10 ways (namely removal of BC, AC and AD, CD) result in disconnected graph. All the 8 spanning trees of G are given



⑧ If all the vertices of an undirected graph are each of odd degree k. Show that the number of edges of the graph is a multiple of k.

Since the number of vertices of odd degree in an undirected graph is ~~given~~ even.

Let it be $2n$.

Let the number of edges be n_k.

Then by hand shaking thm,

$$\sum_{i=1}^{2n} \deg(v_i) = 2n_k$$

$$\sum_{i=1}^{2n} k = 2n_k \quad \text{or} \quad 2nk = 2n_k$$

$$n_k = nk$$

ie) the number of edges is a multiple of k.

(9) If a graph G contains 21 edges, 3 vertices of degree 4 and other each of degree 3, how many vertices do the graph has?

Let the graph has x vertices (3 vertices of degree 4 and $x-3$ vertices of degree 3).

By hand shaking theorem,

$$\sum \deg(v) = 2e$$

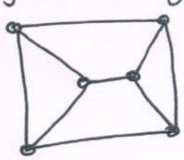
$$(3*4) + ((x-3)*3) = 21.$$

$$12 + 3x - 9 = 21$$

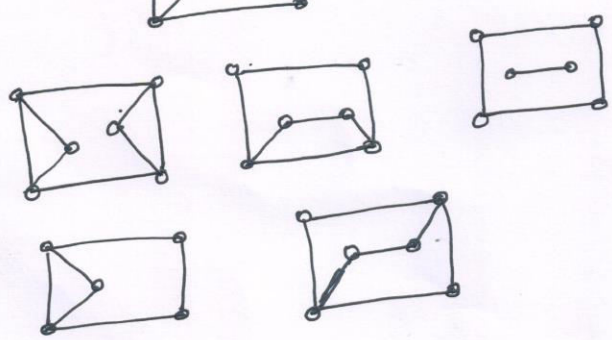
$$3x = 18$$

$$\boxed{x = 6}$$

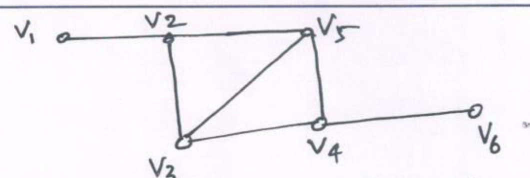
(10) Draw any five subgraphs for the following graph.



Soln:

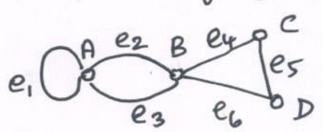


(11) obtain the adjacency matrix of the graph G.



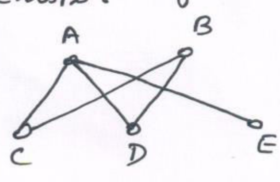
$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 1 \\ v_5 & 0 & 1 & 1 & 1 & 0 & 0 \\ v_6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

(12) Represent the following graph by incidence matrix.



$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline A & 1 & 1 & 1 & 0 & 0 & 0 \\ B & 0 & 1 & 1 & 1 & 0 & 1 \\ C & 0 & 0 & 0 & 1 & 1 & 0 \\ D & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

(13) Find a Hamiltonian path or a Hamiltonian circuit, if it exists. If it does not exist, explain why?

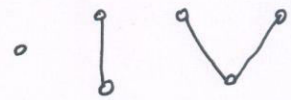


The graph contains a Hamiltonian paths from C to E and from D to E. (namely) C-B-D-A-E and D-B-C-A-E but no Hamiltonian circuit.

PART-C

① Prove that "A tree with n vertices has n-1 edges.

This property is true for $n=1,2,3$ as seen from the graph.



Let us now use mathematical induction to prove the property completely.

Let us now consider a tree T with n vertices.
Let e_k be the edge connecting the vertices v_i and v_j of T.

Then e_k is only the path between v_i & v_j.
If we delete the edge e_k from T, T becomes disconnected and $(T-e_k)$ consists of exactly two components (say) T_1 & T_2 which are connected.

Since T did not contain any circuit T_1 and T_2 also will not have circuits.

Hence both T_1 and T_2 are trees, each having less than n vertices (say) r and n-r respectively.

∴ By induction assumption T_1 has $r-1$ edges and T_2 has $(n-r-1)$ edges.

∴ T has $(r-1)+(n-r-1)+1 = n-1$ edges.

Thus a tree with n vertices has n-1 edges.

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(9)

(2) Prove that the number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Let the vertex set be partitioned into the subsets V_1 and V_2.

Let V_1 contain x vertices. Then V_2 contains $(n-x)$ vertices.

The largest number of ~~the~~ edges of the graph can be obtained, when each of the x vertices in V_1 is connected to each of the $(n-1)$ vertices of V_2.

$\therefore$ Largest number of edges

$f(x) = x(n-x)$, is a function of x.

Now we find the value of x when $f(x)$ is maximum.

$$\Rightarrow f'(x) = n - 2x$$

$$f''(x) = -2 < 0$$

$$f'(x) = 0 \Rightarrow n - 2x = 0$$

$$x = \frac{n}{2}$$

Hence $f(x)$ is maximum, when $x = \frac{n}{2}$

$\therefore$ Maximum number of edges required

$$f\left(\frac{n}{2}\right) = \frac{n}{2}\left(n - \frac{n}{2}\right)$$

$$= \frac{n}{2}\left(\frac{n}{2}\right)$$

$$= \frac{n^2}{4}$$

(3) Prove that the maximum number of edges in a simple disconnected graph G with n vertices and K components is $\frac{(n-K)(n-K+1)}{2}$.

Let the number of vertices in i^{th} component of G be n_i $(n_i \geq 1)$.

Then $n_1 + n_2 + \cdots + n_K = n$ or $\sum_{i=1}^{K} n_i = n$. —(1)

$$\Rightarrow \sum_{i=1}^{K} (n_i - 1) = n - K$$

Squaring on both sides,

$$\left\{ \sum_{i=1}^{K} (n_i - 1) \right\}^2 = (n-K)^2$$
$$= n^2 - 2nK + K^2$$

ie) $$\sum_{i=1}^{K} (n_i - 1)^2 + 2 \sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 - 2nK + K^2 \quad \text{—(2)}$$

ie) $$\sum_{i=1}^{K} (n_i - 1)^2 \leq n^2 - 2nK + K^2$$

($\because$ the second member in the L.S of (2) is ≥ 0, as each $n_i \geq 1$)

ie) $$\sum_{i=1}^{K} (n_i^2 - 2n_i + 1) \leq n^2 - 2nK + K^2$$

$$\sum_{i=1}^{K} n_i^2 \leq n^2 - 2nK + K^2 + 2n - K \quad \text{—(3)}$$

Now the maximum number of edges in the i^{th} component

(11)

of $G = \frac{1}{2} n_i (n_i - 1)$

$\therefore$ Maximum number of edges of G

$$= \frac{1}{2} \sum_{i=1}^{K} n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^{K} n_i^2 - \frac{1}{2} \sum_{i=1}^{K} n_i$$

$$= \frac{1}{2} \sum_{i=1}^{K} n_i^2 - \frac{1}{2} n \quad (\text{by } ①)$$

$$\leq \frac{1}{2} (n^2 - 2nK + K^2 + 2n - K) - \frac{n}{2} \quad \text{by } ③$$

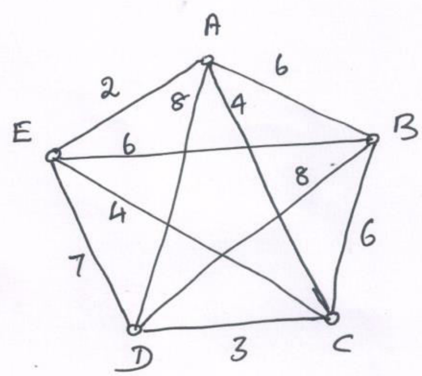
$$\leq \frac{1}{2} (n^2 - 2nK + K^2 + n - K)$$

$$\leq \frac{1}{2} \left((n-K)^2 + (n-K) \right)$$

$$\leq \frac{1}{2} (n-K)(n-K+1)$$

$\therefore$ Maximum number of edges $= \frac{(n-K)(n-K+1)}{2}$.

④ Find the minimum spanning tree for the weighted graph by using Kruskal's algorithm.



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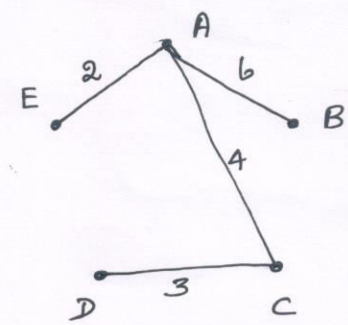
We first arrange the edges in the increasing order of the edges and proceed as per Kruskal's algorithm.

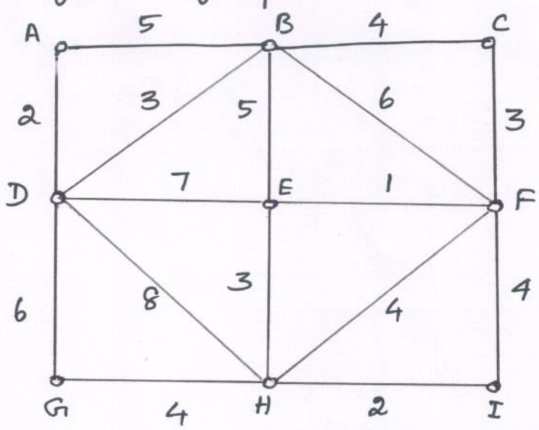
Edge	weight	Included in the Spanning tree or not	If not included, circuit formed
AE	2	YES	–
CD	3	YES	–
AC	4	YES	–
CE	4	NO	A–E–C–A
AB	6	YES	–
BC	6	NO	A–B–C–A
BE	6	–	–
DE	7	–	–
AD	8	–	–
BD	8	–	–

Since there are 5 vertices in the graph, we should stop the procedure for finding the edges of the minimum spanning tree, when 4 edges have been find out.



The edges of the minimum spanning tree are AE, CD, AC and AB whose total length is 15.

⑤ Use Kruskal's algorithm, find the minimum spanning tree for the weighted graph

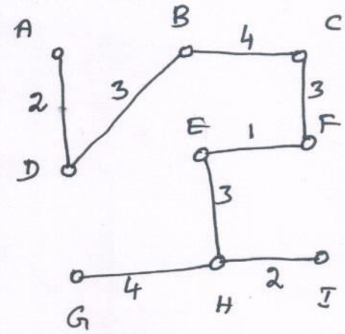


Solution:

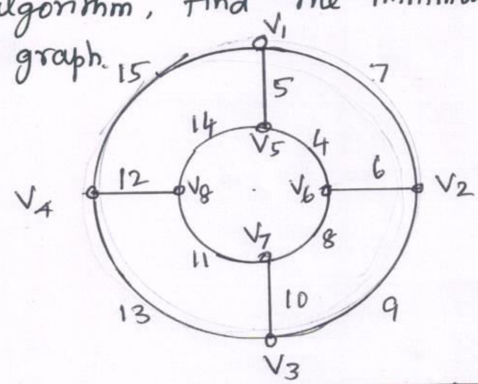
Edge	Weight	Included in the Spanning tree or not	If not included, circuit formed.
EF	1	YES	—
AD	2	YES	—
HI	2	YES	—
BD	3	YES	—
CF	3	YES	—
EH	3	YES	—
BC	4	YES	—
FH	4	NO	E–F–H–E
FI	4	NO	E–F–I–H–E
GH	4	YES	—
AB	5	—	—
BE	5	—	—
BF	6	—	—
DG	6	—	—
DE	7	—	—
DH	8	—	—

The required minimum spanning tree consists of the 8 edges EF, AD, HI, BD, CF, EH, BC and GH.

The total length of the spanning tree = 22.

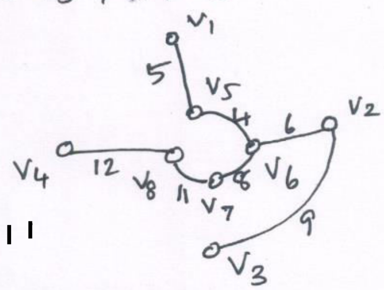


⑥ Use Kruskal's algorithm, find the minimum spanning tree for the weighted graph.



Edge	weight	Included in the spanning tree or not	If not included, circuit formed.
V_5V_6	4	YES	—
V_1V_5	5	YES.	—
V_2V_6	6	YES	—
V_1V_2	7	NO	$V_1-V_5-V_6-V_2-V_1$
V_6V_7	8	YES	—
V_2V_3	9	YES	—
V_3V_7	10	NO	$V_3-V_7-V_6-V_2-V_3$
V_7V_8	11	YES	—
V_4V_8	12	YES	—
V_3V_4	13	—	—
V_5V_8	14	—	—
V_1V_4	15	—	—

The required spanning tree consists of the 7 edges V_5V_6, V_1V_5, V_2V_6, V_6V_7, V_2V_3, V_7V_8 and V_4V_8.



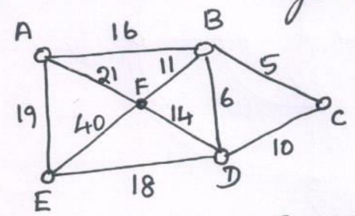
The total length of the minimum spanning tree

$= 5 + 4 + 6 + 9 + 8 + 12 + 11$

$= 55$

⑨ Find the minimum spanning tree for the weighted graph, using Kruskal's algorithm.

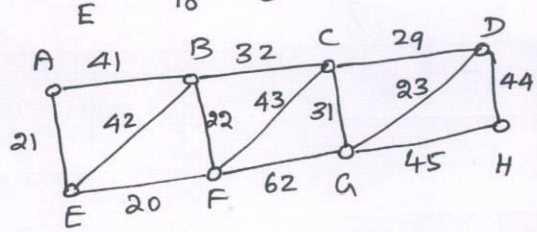
(i)



Ans: The edges of MST are AB, BC, BD, BF, DE

Minimum total weight = 56

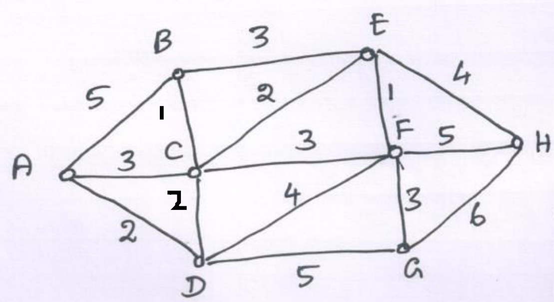
(ii)



Ans: The edges of MST are AE, BC, BF, CD, DG, DH and EF

Minimum total weight = 191

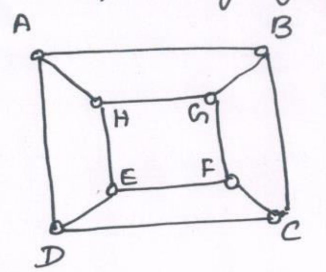
(iii)



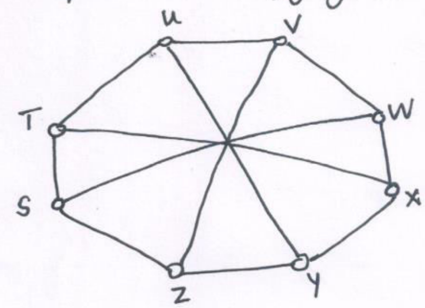
Ans: The edges of MST are AC, AD, BE, CF, EF, EH and FG.

Minimum total weight = 18.

(8) Are the following graphs isomorphic? Justify your answer.



G_1



G_2.

Solution:

Graph G_1

(i) no. of vertices = 8

(ii) no. of Edges = 12

(iii) Deg(A) = 3, Deg(B) = 3
Deg(C) = 3, Deg(D) = 3
Deg(E) = 3, deg(F) = 3
deg(G) = 3, deg(H) = 3.

Graph G_2

(i) no. of vertices = 8

(ii) no. of Eges = 12

(iii) deg(u) = 3, deg(V) = 3
deg(W) = 3, deg(X) = 3
deg(Y) = 3, deg(z) = 3
deg(S) = 3, deg(T) = 3

$\Rightarrow$ All the vertices have same degree.

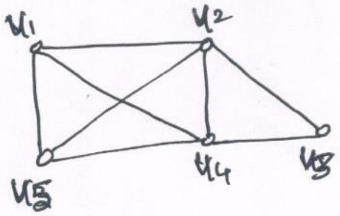
$\therefore$ Basic condition for isomorphism is satisfied.

Now, A – D – C – F – E – H – G – B – A is a circuit of length 8 in G_1.

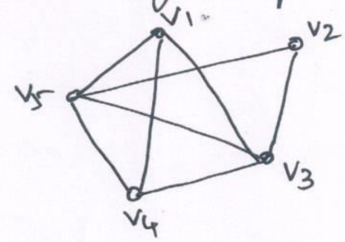
Also, u – v – w – x – y – z – S – T – u is a circuit of length 8 in G_2.

$\therefore$ Two graphs are isomorphic.

(9) Determine whether the following graphs are isomorphic. Exhibit the isomorphism explicitly or prove that it does not exist.



G_1



G_2

The graphs G_1 and G_2 have both 5 vertices and 6 edges. Also they both have 2 vertices each of degree 4 and 2 vertices is of degree 3 and 1 vertex is of degree 2.

$\therefore$ The two graphs G_1 and G_2 agree with respect to 3 invariants.

The adjacency matrices of G_1 and G_2 are

$$A_{G_1} = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \; ; \; A_{G_2} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Since the two adjacency matrices are same, the two graphs are isomorphic.

Graph Theory - Coloring

Graph coloring is the procedure of assignment of colors to each vertex of a graph G such that no adjacent vertices get same color. The objective is to minimize the number of colors while coloring a graph. The smallest number of colors required to color a graph G is called its chromatic number of that graph. Graph coloring problem is a NP Complete problem.

Method to Color a Graph

The steps required to color a graph G with n number of vertices are as follows −

Step 1 − Arrange the vertices of the graph in some order.

Step 2 − Choose the first vertex and color it with the first color.

Step 3 − Choose the next vertex and color it with the lowest numbered color that has not been colored on any vertices adjacent to it. If all the adjacent vertices are colored with this color, assign a new color to it. Repeat this step until all the vertices are colored.

Vertex Coloring

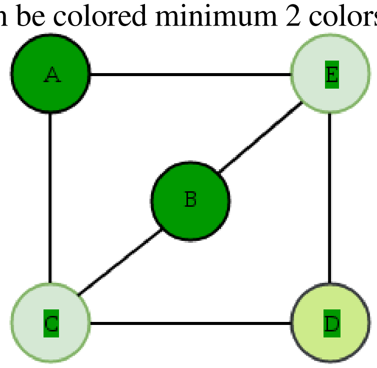
Vertex coloring is an assignment of colors to the vertices of a graph 'G' such that no two adjacent vertices have the same color. Simply put, no two vertices of an edge should be of the same color.

Chromatic Number

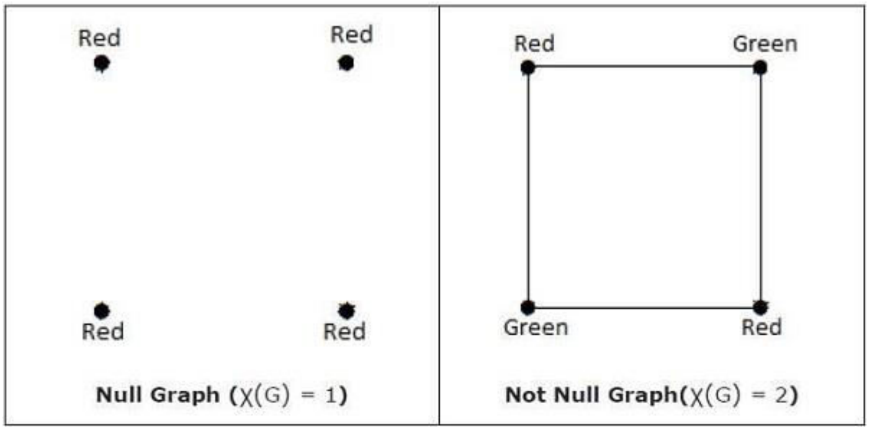
The minimum number of colors required for vertex coloring of graph 'G' is called as the chromatic number of G, denoted by X(G).

$\chi(G) = 1$ if and only if 'G' is a null graph. If 'G' is not a null graph, then $\chi(G) \geq 2$.

. For example, the following can be colored minimum 2 colors.



Example



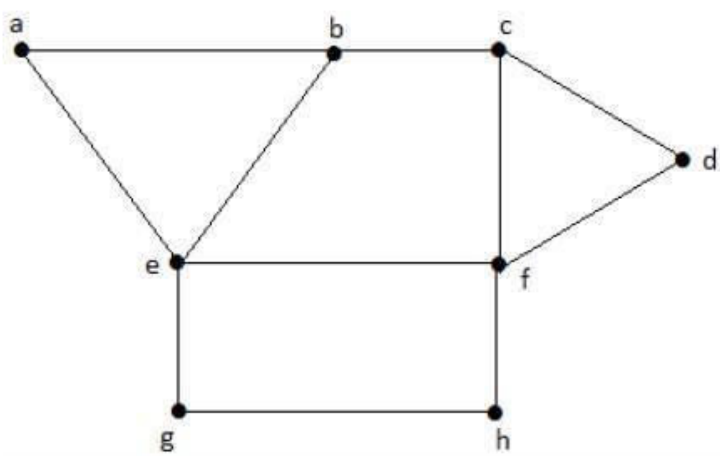
Note − A graph 'G' is said to be n-coverable if there is a vertex coloring that uses at most n colors, i.e., X(G) ≤ n.

Region Coloring

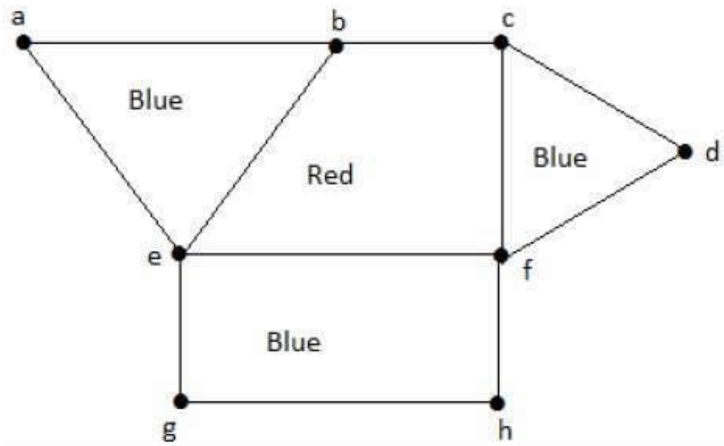
Region coloring is an assignment of colors to the regions of a planar graph such that no two adjacent regions have the same color. Two regions are said to be adjacent if they have a common edge.

Example

Take a look at the following graph. The regions 'aeb' and 'befc' are adjacent, as there is a common edge 'be' between those two regions.



Similarly, the other regions are also coloured based on the adjacency. This graph is coloured as follows −

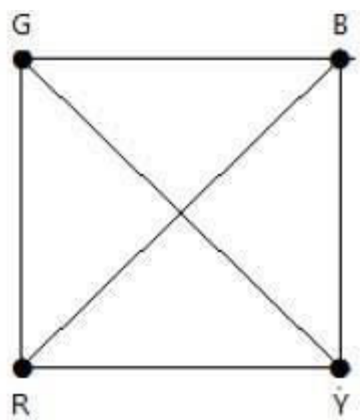


Example

The chromatic number of Kn is

- n
- n–1
- [n/2]
- [n/2]

Consider this example with K_4.



In the complete graph, each vertex is adjacent to remaining (n – 1) vertices. Hence, each vertex requires a new color. Hence the chromatic number of K_n = n.

Applications of Graph Coloring

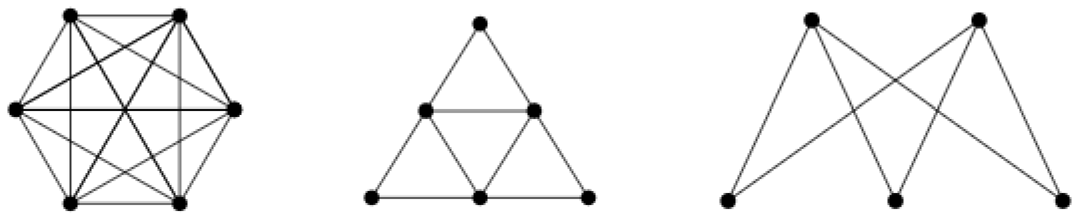
Graph coloring is one of the most important concepts in graph theory. It is used in many real-time applications of computer science such as −

- Clustering
- Data mining
- Image capturing
- Image segmentation

- Networking
- Resource allocation
- Processes scheduling

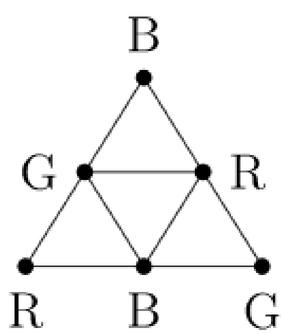
Problem : 1

Find the chromatic number of the graphs below.



The graph on the left is K_6. The only way to properly color the graph is to give every vertex a different color (since every vertex is adjacent to every other vertex). Thus the chromatic number is 6.

The middle graph can be properly colored with just 3 colors (Red, Blue, and Green). For example:



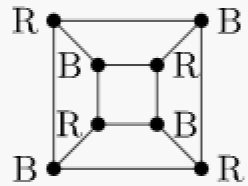
There is no way to color it with just two colors, since there are three vertices mutually adjacent (i.e., a triangle). Thus the chromatic number is 3.

The graph on the right is just $K_{2,3}$. As with all bipartite graphs, this graph has chromatic number 2: color the vertices on the top row red and the vertices on the bottom row blue.

<u>Problem:2</u>

What is the smallest number of colors that can be used to color the vertices of a cube so that no two adjacent vertices are colored identically?

The cube can be represented as a planar graph and colored with two colors as follows:

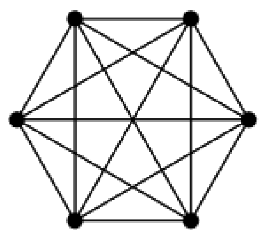


Since it would be impossible to color the vertices with a single color, we see that the cube has chromatic number 2 (it is bipartite).

Problem : 3

Six friends decide to spend the afternoon playing chess. Everyone will play everyone else once. They have plenty of chess sets but nobody wants to play more than one game at a time. Games will last an hour (thanks to their handy chess clocks). How many hours will the tournament last?

Represent each player with a vertex and put an edge between two players if they will play each other. In this case, we get the graph K_6:



We must color the edges; each color represents a different hour. Since different edges incident to the same vertex will be colored differently, no player will be playing two different games (edges) at the same time. Thus we need to know the chromatic index of K_6.

Notice that for sure $\chi'(K_6) \geq 5$, since there is a vertex of degree 5. It turns out 5 colors is enough (go find such a coloring). Therefore the friends will play for 5 hours.