By inductive hypothesis, for each vertex i that is not a leaf, $X_i \overset{*}{\Rightarrow} \alpha_i$, since the subtree with root X_i is not the entire tree. If $X_i = \alpha_i$, then surely $X_i \overset{*}{\Rightarrow} \alpha_i$

$$A \Rightarrow X_1 X_2 \ldots X_n \overset{*}{\Rightarrow} \alpha_1 X_2 \ldots X_n \overset{*}{\Rightarrow} \alpha_1 \alpha_2 \ldots X_n \overset{*}{\Rightarrow} \alpha_1 \alpha_2 \ldots \alpha_n = \alpha$$

Thus $A \overset{*}{\Rightarrow} \alpha$

UNIT III

PUSH DOWN AUTOMATA

In order to accept a context free language we need a machine similar to FA called PDA

FA + stack = PDA < DPDA, NDPDA

7 tuple Notation

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

PDA : $\delta : Q \times \{\Sigma \cup \epsilon\} \times \Gamma \rightarrow Q \times \Gamma^*$

NPDA : $\delta : Q \times \{\Sigma \cup \epsilon\} \times \Gamma \rightarrow 2^{(Q \times \Gamma^*)}$

Q - states

Σ - i/p symbol

Γ - stack alphabet

q_0 - (state) - start state

Z_0 → start symbol (Bottom of stack)

F → Final state

One Move:

Based on 3 elements (state, i/p, stack element (TOS))

$$\delta(q, a, x) = (p, \gamma)$$

q - state (current)

a → i/p

x → stack element (TOS)

p → new state

γ → stack symbols that replaces x at TOS

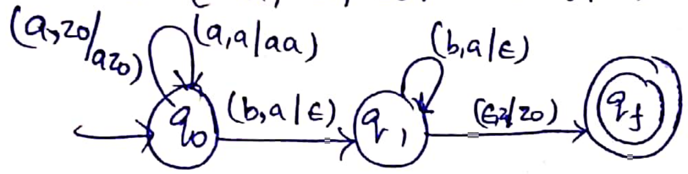
γ <
- ϵ - pop
- x - unchange
- YZ - replaced by YZ

1) $a^n b^n$ ON ACCEPTING STATE

$\delta(q_0, a, z_0) = (q_0, a z_0)$ } PUSH
$\delta(q_0, a, a) = (q_0, aa)$

$\delta(q_0, b, a) = (q_1, \epsilon)$ } POP
$\delta(q_1, b, a) = (q_1, \epsilon)$

$\delta(q_1, \epsilon, z_0) = (q_f, z_0)$ ACCEPT ON FINAL STATE



ON STACK EMPTY

$\delta(q_0, a, z_0) = (q_0, a z_0)$ } PUSH
$\delta(q_0, a, a) = (q_0, aa)$

$\delta(q_0, b, a) = (q_1, \epsilon)$ } POP
$\delta(q_1, b, a) = (q_1, \epsilon)$

$\delta(q_1, \epsilon, z_0) = (q_1, \epsilon)$ ACCEPT ON EMPTY STACK

2. $a^n c b^n$

$\delta(q_0, a, z_0) = (q_0, a z_0)$
$\delta(q_0, a, a) = (q_0, aa)$
$\delta(q_0, c, a) = (q_1, a)$
$\delta(q_1, b, a) = (q_2, \epsilon)$
$\delta(q_2, b, a) = (q_2, \epsilon)$
$\delta(q_2, \epsilon, z_0) = (q_f, z_0)$

$a^n b^m c^n d^m$
$a^n b^n c^m d^m$
$a^n b^n$
$a^n c b^n$
$a^n b^m c^n$
$a^m b^n c^n$
$a^n b^n c^m$
$a^n b^m$
$a^n b^{2n}$
$a^{2n} b^n$
$a^n b^{3n}$

3. $a^n b^{2n}$

$\delta(q_0, a, z_0) = (q_0, a z_0)$
$\delta(q_0, a, a) = (q_0, aa)$
$\delta(q_0, b, a) = (q_1, a)$
$\delta(q_1, b, a) = (q_2, \epsilon)$
$\delta(q_2, b, a) = (q_1, a)$
$\delta(q_2, \epsilon, z_0) = (q_f, z_0)$

4. wcw^R

$\delta(q_0, a, z_0) = (q_0, az_0)$
$\delta(q_0, a, a) = (q_0, aa)$
$\delta(q_0, b, z_0) = (q_0, bz_0)$
$\delta(q_0, b, b) = (q_0, bb)$
$\delta(q_0, a, b) = (q_0, ab)$
$\delta(q_0, b, a) = (q_0, ba)$
} PUSH

$\delta(q_0, c, a) = (q_1, a)$
$\delta(q_0, c, b) = (q_1, b)$
} STATE CHANGE

$\delta(q_0, c, z_0) = (q_f, z_0)$ - i/p is only c

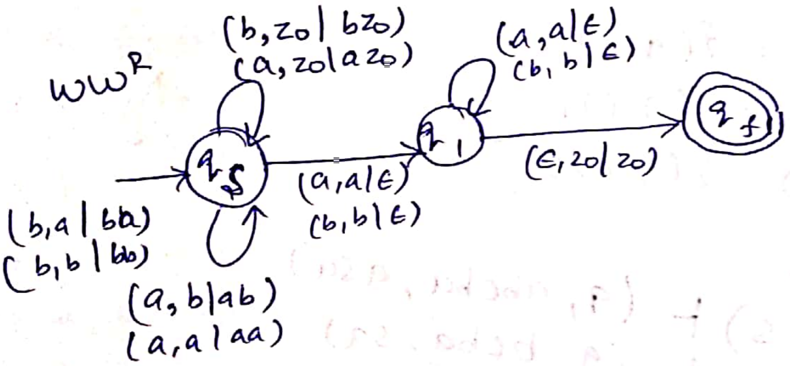
$\delta(q_1, a, a) = (q_2, \epsilon)$
$\delta(q_1, b, b) = (q_2, \epsilon)$
} POP

$\delta(q_2, \epsilon, z_0) = (q_f, z_0)$ - ACC ON FINAL STATE

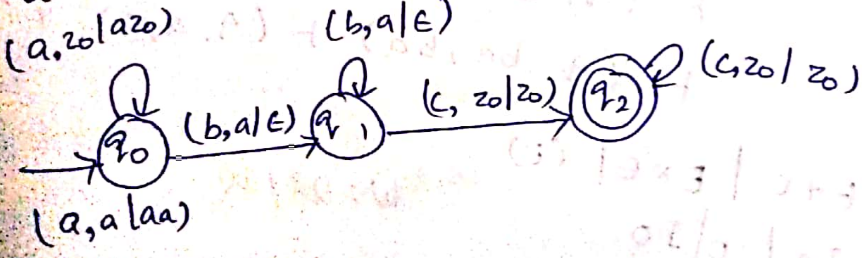
(or)

$\delta(q_2, \epsilon, z_0) = (q_2, \epsilon)$ - ACC ON EMPTY STACK

5. ww^R



6. $a^n b^n c^m \mid n, m \geq 1$



CFG TO PDA CONVERSION

Construct PDA 'P' that accepts $L(G)$ by empty stack as follows

$$P = [\{q\}, T, V \cup T, \delta, q, S]$$

δ is defined by :

1. For each variable A,

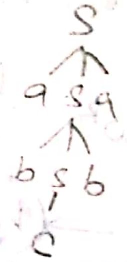
$$\delta(q, \epsilon, A) = \{(q, \beta) \mid A \to \beta \text{ is a production of } P\}$$

2. For each terminal a,

$$\delta(q, a, a) = \{(q, \epsilon)\}$$

1. $S \to aSa \mid bSb \mid c$

$$\delta(q, \epsilon, S) = \{(q, aSa), (q, bSb), (q, c)\}$$



$\delta(q, a, a) = \{(q, \epsilon)\}$

$\delta(q, b, b) = \{(q, \epsilon)\}$

$\delta(q, c, c) = \{(q, \epsilon)\}$

$w = abcba$

$(q, abcba, S) \vdash (q, abcba, aSa)$
$\vdash (q, bcba, Sa)$
$\vdash (q, bcba, bSba)$
$\vdash (q, cba, Sba) \vdash (q, cba, cba)$
$\vdash (q, ba, ba) \vdash (q, a, a) \vdash (q, \epsilon)$

2. $E \to I \mid E+E \mid E*E \mid (E)$ $\qquad W = aa*a0$

$I \to a \mid Ia \mid 0 \mid I0$

$\delta(q, \epsilon, E) = \{(q, I), (q, E+E), (q, E*E), (q, (E))\}$

$\delta(q, \epsilon, I) = \{(q, a), (q, 0), (q, Ia), (q, I0)\}$

$\delta($ q, ϵ, I $)$

$$\left.\begin{array}{ll}\delta(q, a, a), & \delta(q, (, ()\} \\ \delta(q, +, +), & \delta(q,),)) \\ \delta(q, x, x), & \delta(q, 0, 0)\end{array}\right\} \Rightarrow (q, \epsilon)$$

3. $S \to AB$,
$A \to aA \mid \epsilon$, $\quad w = aaab$
$B \to aBb \mid \epsilon$

$\delta(q, \epsilon, S) = \{(q, AB)\}$
$\delta(q, \epsilon, A) = \{(q, aA), (q, \epsilon)\}$
$\delta(q, \epsilon, B) = \{(q, aBb), (q, \epsilon)\}$
$\left.\begin{array}{l}\delta(q, a, a) \\ \delta(q, b, b)\end{array}\right\} = \{(q, \epsilon)\}$

4. $S \to aA$, $\quad B \to b$
$A \to aABC \mid bB \mid a$, $\quad C \to c$

$\delta(q, \epsilon, S) = \{(q, aA)\}$
$\delta(q, \epsilon, A) = \{(q, aABC), (q, bB), (q, a)\}$
$\delta(q, \epsilon, B) = \{(q, b)\}$
$\delta(q, \epsilon, C) = \{(q, c)\}$
$\left.\begin{array}{l}\delta(q, a, a) \\ \delta(q, b, b) \\ \delta(q, c, c\end{array}\right\} = \{(q, \epsilon)\}$

PDA to CFG CONVERSION

Rules:

1. $S \to [q_0, Z_0, P]$ for each P in Q
2. If $\delta(q, x, A)$ contains $(P, B_1, B_2, \ldots B_m)$ when $m \neq 0$
then $[q_1, A, q_{m+1}] \to x [P, B_1, q_2] [q_2, B_2, q_3] \ldots [q_m, B_m, q_m]$
3. If $m = 0$; if $\delta(q, x, A)$ contains (P, ϵ)
$[q, A, P] \to x$

1. $\delta(q_0, 0, z_0) = \{(q_0, xz_0)\}$

$\delta(q_0, 0, x) = \{(q_0, xx)\}$

$\delta(q_0, 1, x) = \{(q_1, \epsilon)\}$

$\delta(q_1, 1, x) = \{(q_1, \epsilon)\}$

$\delta(q_1, \epsilon, x) = \{(q_1, \epsilon)\}$

$\delta(q_1, \epsilon, z_0) = \{(q_1, \epsilon)\}$

Step 1

$S \rightarrow [q_0, z_0, q_0]$

✓ $S \rightarrow [q_0, z_0, q_1]$

Step 2:

✓ $[q_0, x, q_1] \rightarrow 1$

✓ $[q_1, x, q_1] \rightarrow 1$

✓ $[q_1, x, q_1] \rightarrow \epsilon$

✓ $[q_1, z_0, q_1] \rightarrow \epsilon$

Step 3

$\delta(q_0, 0, z_0) = (q_0, xz_0)$

$[q_0, z_0, q_0] \rightarrow 0\,[q_0, x, q_0]\,[q_0, z_0, q_0]$

$[q_0, z_0, q_0] \rightarrow 0\,[q_0, x, q_1]\,[q_1, z_0, q_0]$

$[q_0, z_0, q_1] \rightarrow 0\,[q_0, x, q_0]\,[q_0, z_0, q_1]$

✓ $[q_0, z_0, q_1] \rightarrow 0\,[q_0, x, q_1]\,[q_1, z_0, q_1]$

$\delta(q_0, 0, x) = (q_0, xx)$

$[q_0, x, q_0] \rightarrow 0\,[q_0, x, q_0]\,[q_0, x, q_0]$

$[q_0, x, q_0] \rightarrow 0\,[q_0, x, q_1]\,[q_1, x, q_0]$

$[q_0, x, q_1] \rightarrow 0\,[q_0, x, q_0]\,[q_0, x, q_1]$

✓ $[q_0, x, q_1] \rightarrow 0\,[q_0, x, q_1]\,[q_1, x, q_1]$

Language: $0^n, {}^m, n, m$ ...

Scanned with CamScanner

2. $\delta(q_0, 1, z_0) = (q_0, xz_0)$ ✓
$\delta(q_0, 1, x) = (q_0, xx)$ ✓
$\delta(q_0, 0, x) = (q_1, x)$ ✓
$\delta(q_1, 0, z_0) = (q_0, z_0)$
$\delta(q_0, \epsilon, z_0) = (q_0, \epsilon)$ ✓
$\delta(q_1, 1, x) = (q_1, \epsilon)$

Step 1:

✓ $S \to [q_0, z_0, q_0]$
$S \to [q_0, z_0, q_1]$

Step 2:

✓ $[q_0, z_0, q_0] \to \epsilon$
✓ $[q_1, x, q_1] \to 1$

Step 3:

$(q_0, 1, z_0) = (q_0, xz_0)$
$[q_0, z_0, q_0] \to 1\,[q_0, x, q_0]\,[q_0, z_0, q_0]$
✓ $[q_0, z_0, q_0] \to 1\,[q_0, x, q_1]\,[q_1, z_0, q_0]$
$[q_0, z_0, q_1] \to 1\,[q_0, x, q_0]\,[q_0, z_0, q_1]$
$[q_0, z_0, q_1] \to 1\,[q_0, x, q_1]\,[q_1, z_0, q_1]$

$(q_0, 1, x) = (q_0, xx)$
$[q_0, x, q_0] \to 1\,[q_0, x, q_0]\,[q_0, x, q_0]$
$[q_0, x, q_0] \to 1\,[q_0, x, q_1]\,[q_1, x, q_0]$
$[q_0, x, q_1] \to 1\,[q_0, x, q_0]\,[q_0, x, q_1]$
✓ $[q_0, x, q_1] \to 1\,[q_0, x, q_1]\,[q_1, x, q_1]$

$\delta(q_0, 0, x) = (q_1, x)$
$[q_0, x, q_0] \to 0\,[q_1, x, q_0]$
✓ $[q_0, x, q_1] \to 0\,[q_1, x, q_1]$

$\delta(q_1, 0, z_0) = (q_0, z_0)$
✓ $[q_1, z_0, q_0] \to 0\,[q_0, z_0, q_0]$
$[q_1, z_0, q_1] \to 0\,[q_0, z_0, q_1]$

3) $\delta(q_0, b, z_0) = \{(q_0, zz_0)\}$
$\delta(q_0, \epsilon, z_0) = \{(q_0, \epsilon)\}$
$\delta(q_0, b, z) = \{(q_0, zz)\}$
$\delta(q_0, a, z) = \{(q_1, z)\}$
$\delta(q_1, b, z) = \{(q_1, \epsilon)\}$
$\delta(q_1, a, z_0) = \{(q_0, z_0)\}$

4) $\delta(q, 1, z) = (q, Xz)$
$\delta(q, 1, X) = (q, XX)$
$\delta(q, \epsilon, X) = (q, \epsilon)$
$\delta(q, 0, z) = (p, X)$
$\delta(p, 1, X) = (p, \epsilon)$
$\delta(p, 0, z) = (q, z)$

$S \to [q, z, q]$
$S \to [q, z, p]$

5) $\delta(q_0, a, z) = (q_0, Xz)$
$\delta(q_0, a, X) = (q_0, XX)$
$\delta(q_0, b, X) = (q_1, \epsilon)$
~~$\delta(q_0, b, x) = (q_1, \epsilon)$~~
$\delta(q_1, b, X) = (q_1, \epsilon)$
$\delta(q_1, \epsilon, z) = (q_1, \epsilon)$

$S \to [q_0, z, q_0]$
$S \to [q_0, z, q_1]$

LANGUAGE OF A PDA

Acceptance by final state

$L(P) = \{ w \mid (q_0, w, Z_0) \vdash_P^* (q, \epsilon, \alpha) \}$

for some state q in F & any stack string α

Acceptance by Empty stack

$N(P) = \{ w \mid (q_0, w, Z_0) \vdash_P^* (q, \epsilon, \epsilon) \}$

for any state q i.e $N(P)$ is the set of inputs 'w' that P can consume & at the same time empty its stack.

Instantaneous description of PDA

Triple (q, w, γ) where q - state, $w \to$ Remaining input and $\gamma \to$ stack contents

$(q, w, \gamma) \vdash_P (P, \gamma)$

Final state V/s Empty stack

Theorem: If L is $L(M_2)$ for some PDA M_2, then L is $N(M_1)$ for some PDA M_1.

PROOF: M_1 simulates M_2, M_1 will erase its stack whenever M_2 enters a final state.

Let $M_2 = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ $L \in L(M_2)$

Let $M_1 = (Q \cup \{q_e, q_0'\}, \Sigma, \Gamma \cup \{X_0\}, \delta', q_0', X_0, \phi)$

new initial state

where δ' is defined as follows

1) $\delta'(q_0', \epsilon, X_0) = \{(q_0, Z_0, X_0)\}$ make M_1 to enter into initial ID of M_2

2) $\delta'(q, a, z)$ includes the elements of $\delta(q, a, z)$ simulate M_2 for all q in Q, a in Σ or $a = \epsilon$, and z in Γ

3) For all q in F, and z in $\Gamma \cup \{X_0\}$, $\delta'(q, \epsilon, z)$ contains (q_e, ϵ)

4) For all z in $\Gamma \cup \{X_0\}$, $\delta'(q_e, \epsilon, z)$ contains (q_e, ϵ)

erase stack

Let x be in $L(M_2)$.

Then $(q_0, x, Z_0) \overset{*}{\underset{M_2}{\vdash}} (q, \epsilon, \gamma)$ for some q in F

Now consider M_1 with input x.

By rule (1) $(q_0', x, X_0) \overset{*}{\underset{M_1}{\vdash}} (q_0, x, Z_0X_0)$

By rule (2), every move of M_2 is a legal move of M_1

$$(q_0, x, Z_0) \overset{*}{\underset{M_1}{\vdash}} (q, \epsilon, \gamma)$$

$$(q_0', x, X_0) \underset{M_1}{\vdash} (q_0, x, Z_0X_0) \overset{*}{\underset{M_1}{\vdash}} (q, \epsilon, \gamma X_0)$$

By rules (3) and (4),

$$(q, \epsilon, \gamma X_0) \overset{*}{\underset{M_1}{\vdash}} (q_e, \epsilon, \epsilon)$$

Therefore $(q_0', x, X_0) \overset{*}{\underset{M_1}{\vdash}} (q_e, \epsilon, \epsilon)$ and M_1 accepts x by empty stack.

Thus x accepts by empty stack. in $L(M_2)$

PUMPING LEMMA for CFL

Every sufficiently long string in a regular ~~set~~ contains a short substring that can be pumped.

LEMMA

Let L be any CFL. Then there is a constant n, depending only on L, such that if z is in L and $|z| \geq n$, then we may write $z = uvwxy$ such that

1) $|vx| \geq 1$

2) $|vwx| \leq n$ and

3) for all $i \geq 0$ uv^iwx^iy is in L

1) $L = \{a^i b^i c^i \mid i \geq 1\}$

Consider $z = a^n b^n c^n$ $z = uvwxy$

v, x ? in strips can be pumped? lie in a^i

since $|vwx| \leq n$ v & x can't contain

instances of a's and c's

L is not context-free language

aabbc

2) $\{a^i b^i c^j \mid j \geq 1\}$ ~~and~~ - Not CFL

3) $\{a^i b^j c^k \mid i \leq j \leq k\}$ Not CFL

4) $\{a^i b^j c^i d^j \mid i \geq 1$ and $j \geq 1\}$ - Not ~~CFG~~, CFL

5) $\{a^i b^j c^k \mid i < j < k\}$ - Not CFL

6) $\{a^i b^j \mid j = i^2\}$ - Not CFL

7) $\{a^i \mid i$ is a prime$\}$ - not CFL

8) $\{a^n b^n c^m \mid n \leq m \leq 2n\}$ - not CFL

9) $\{a^i b^{2i} c^j \mid i, j \geq 0\}$ - CFL

$S \to AB$

$A \to aAbb \mid \epsilon$

$B \to cB \mid \epsilon$

11) $\{a^n b^n c^m \mid n, m \geq 0\}$ - CF

$S \to Sc \mid T$

$T \to aTb \mid \epsilon$

10) $\{a^n b^m c^{n+m} \mid n, m \geq 0\}$ - CFL

$S \to aSc \mid T$

$T \to bTc \mid \epsilon$

Scanned with CamScanner