UNIT-II

COMBINATORICS

Principle of Mathematical induction:

Let $P(x)$ be a statement or proposition involving all positive integer n, then we complete two steps.

Basic step: If $P(1)$ is true.

Inductive step: If $P(k+1)$ is true on the assumption that $P(k)$ is true

① Show that $1^2+2^2+3^2+\dots n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \geq 1$ by mathematical induction.

Let $P(n) : 1^2+2^2+3^2+\dots n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: To prove $P(1)$ is true.

$$(i.e)\ P(1) = \frac{(1)(1+1)(2(1)+1)}{6}$$

$$= \frac{1\cdot 2\cdot 3}{6}$$

$$P(1) = 1$$

$\therefore P(1)$ is true.

Step 2: Assume that $P(k)$ is true

$$(i.e)\ 1^2+2^2+3^2+\dots k^2 = \frac{k(k+1)(2k+1)}{6}$$

Step 3: To prove $P(k+1)$ is true.

(i.e) To prove : $\frac{(k+1)(k+2)(2k+3)}{6}$

$\therefore 1^2+2^2+3^2+\dots k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \frac{(k+1)(2k^2+k+6k+6)}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6}$

$1^2+2^2+3^2+\dots k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Hence $P(k+1)$ is true whenever $P(k)$ is true.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all integers.

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② Show that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let $P(n) : \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Step 1: To prove $P(1)$ is true.

(i.e) $P(1) = \frac{1}{1(1+1)} = \frac{1}{(1+1)}$

$\frac{1}{2} = \frac{1}{2}$

$\therefore P(1)$ is true.

Step 2: Assume that $P(k)$ is true.

(i.e) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots \frac{1}{k(k+1)} = \frac{k}{k+1}$

Step 3: To prove P(k+1) is true.

(i.e) To prove : $\frac{k+1}{k+2}$.

$$\therefore \underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)\cancel{(k+1)}}{\cancel{(k+1)}(k+2)}$$

$$=\frac{k+1}{k+2}$$

$\therefore$ P(k+1) is true whenever P(k) is true.

$\therefore$ By Mathematical induction, P(n) is true

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③ Prove by Mathematical induction that $2^n > n$ for $n \in N$ (or) $n < 2^n$ for $n \in N$ all positive integer n.

Let $P(n): n < 2^n$.

Step 1: To prove P(1) is true

(i.e) $P(1) = 1 < 2^1$

$= 1 < 2$

$\therefore$ P(1) is true.

Step 2: ~~To pro~~ Assume that $P(K)$ is true

(i.e) $k < 2^k$

Step 3: To prove $P(k+1)$ is true

i.e) To prove: $(k+1) < 2^{k+1}$

$$K < 2^k$$

$$K+1 < 2^k + 1$$

from $P(1)$

$$K+1 < \cancel{2k+2k}\ 2^K + 2^K \quad [\because 1 \leq 2^K].$$

$a + a = 2a$

$$K+1 < 2(2^K)$$

$a^x a^y = a^{x+y}$

$$K+1 < 2^{K+1}$$

$\therefore P(K+1)$ is true whenever $P(K)$ is true.

By Mathematical induction, $P(n)$ is true.

Sums of Geometric Progression:

Prove that $a + ar + ar^2 + \dots + ar^n = \dfrac{ar^{n+1} - a}{r-1}$ $r \neq$

by the mathematical induction where n is a non-negative integers.

Let $P(n)$: $a + ar + ar^2 + \dots ar^n = \dfrac{ar^{n+1} - a}{r-1}$

Step 1: To prove $P(0)$ is true.

$$ar^0 = \frac{ar^{0+1} - a}{r-1} \qquad \frac{ar^1 - a}{r-1}$$

$$a = \frac{a\cancel{(r-1)}}{\cancel{r-1}}$$

$$a = a$$

$\therefore P(0)$ is true.

Step 2: Assume that $P(k)$ is true.

(i.e) $a + ar + ar^2 + \dots ar^k = \dfrac{ar^{k+1} - a}{r-1}$

Step 3: To prove $P(k+1)$ is true

(i.e) To prove: $\dfrac{ar^{k+2} - a}{r-1}$

$$a + ar + ar^2 + \dots ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + (r-1)(ar^{k+1})}{r-1}$$

$$= \frac{ar^{k+1} - a + rar^{k+1} - ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true. So By Mathematical induction, $P(n)$ is true for all non negative integers.

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⑤ Use the mathematical induction show that $2^n < n!$ for every positive integer n with $n \geq 4$.

Let $P(n)$: $2^n < n!$

Step 1: To prove $P(4)$ is true

(i.e) $2^4 < 4!$

$16 < 24$

$\therefore P(4)$ is true.

Step 2: Assume that $P(k)$ is true

(i.e) $2^k < k!$ ~~is true~~ ; $k \leq 4$

Step 3: To prove $P(k+1)$ is true

(i.e) To prove: $2^{k+1} < (k+1)!$

$n! = n.(n-1)!$

$(k+1)! = (k+1)(k+1-1)!$

$(k+1)! = (k+1)(k)!$

$$2^k < k!$$

$$2 \cdot 2^k < 2 \cdot k!$$

$$2^{k+1} < \underline{(k+1)(k!)} \quad [\because 2 < k+1].$$

$$2^{k+1} < (k+1)!$$

$\therefore$ $P(k+1)$ is true whenever $P(k)$ is true, By Mathematical induction, $P(n)$ is true for all positive integer.

Strong induction and Well Ordering:

Principle of Strong induction:

It is sometimes convenient to replace the induction hypothesis $P(k)$ by the stronger assumption $P(1), P(2), P(3) \ldots P(k)$ are true. The resulting principle known as the principle of strong mathematical induction.

Step 1: Inductive base: To prove $P(1)$ is true.

Step 2: Strong inductive hypothesis: Assume that $P(n)$ is true for all integers $1 \leq n \leq k$.

Step 3: Inductive step: To prove that $P(k+1)$ is true on basis of the strong inductive hypothesis.

① Any positiver integer $n \geq 2$ is either a prime or a product of primes.

To prove this, we use the principle of strong mathematical induction.

Soln:

Let $P(n)$: $n \geq 2$ is either a prime or a product of primes.

Step 1: To prove $P(2)$ is true.

$2 \geq 2$

$2 = 2$ is prime.

$\therefore P(2)$ is true.

Step 2: Assume that the stmt is true for $2 \leq n \leq k$.

Step 3: To prove $P(k+1)$ is true. For the integer $(k+1)$ is ~~k~~ if $k+1$ is prime, the stmt is true. if $k+1$ is not a prime, then $k+1$ can be written as pq for some $2 \leq p \leq k$. and $2 \leq q \leq k$. product of primes

According to the induction hypothesis, p is either a prime or product of primes. Consequently, q is a prime or product of prime. $\therefore pq$ is a product of primes.

02/07/19 Well Ordering property:

Every non empty set of non-negative integers has a least element. The well ordering property can be used directly in proofs.

The Basics of counting:

Basics Counting principles:

The Two basic counting principles are
(i) Product Rule
(ii) Sum Rule.

The Product Rule:

If one job can be done in m ways and ~~the~~ following this another job can be done in n ways. Then, the total no. of ways in which both the jobs can be done in the stated order is mn.

The Sum Rule:

If one job can be done in m ways and another job can be done in n ways and if there is no way common to both jobs, Then the total no. of ways in which either of the two jobs can be done is equal to m+n.

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1) How many different 8 bit strings are there that begin and end with 1?

Soln:

A 8 bit string that begins and end with 1 can be constructed in 6 steps.

By selecting II bit, III, IV, V, VI and VII bit and each bit can be selected in 2 ways. Hence the total no. of 8 bit strings that begins and end with 1 $= 2^6 = 64$.

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② How many different 8 bit strings are there that end with 0111?

Soln: A 8-bit strings that end with 0111 can be constructed in 4 steps. By selecting I, II, III and IV bits and each bit selected in 2 ways. Hence the total no. of 8 bit strings that end with 0111 $= 2^4 = 16$.

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③ ~~In how~~ Suppose that many ways either a member of the chemistry faculty or a student who is a chemistry Major is chosen as a representative to a University Community. How many different choices are there for this representative if there are 23 members of the Chemistry and 80 chemistry Major and no one is both a faculty member and a student?

Soln:

There are 23 ways to choose a member of the Chemistry faculty and there are 80 ways to choose a student who is a Chemistry major.

To choose a member of the Chemistry faculty is never the same as choose a student who is a Chemistry Major. Because no one is both a faculty and a student. By the sum rule, it follows that ~~23~~ there are $23 + 80 = 103$ ways to pick the representative

Inclusion - Exclusion principle in general:

Let $P(1), P(2), \ldots P(n)$ are finite sets

then $|P_1 \cup P_2 \ldots \cup P_n| = \sum_{1 \le i \le n} |P_i| - \sum_{1 \le i \le j \le n} |P_i \cap P_j| +$

$\sum_{1 \le i < j < k \le n} |P_i \cap P_j \cap P_k| \ldots (-1)^{n-1} |P_1 \cap P_2 \ldots \cap P_n|$

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① In a survey of 200 musicians, it was found that 40 wore gloves on the left hand and 39 wore gloves on the right hand. If 160 wore no gloves at all. How many wore a gloves on only the right hand? only the left hand? both the hands?

Soln: The Total no. of musicians wore gloves on the left, right or both hands $|L \cup R| = 200 - 160 = 40$.

Musicians wore gloves on the left hand $= 40$.

Musicians wore gloves on the right hand $= 39$.

$|L \cup R| = |L| + |R| - |L \cap R|$

$40 = 40 + 39 - |L \cap R|$

$|L \cap R| = 39$

Musicians ~~who wore gloves or~~

Musicians who wore gloves only on the right hand $= 39 - 39 = 0$

Musicians who wore gloves only on the left hand $= 40 - 39 = 1$

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③ 40 Computer Programmers interviewed for a job. 25 knew Java, 28 knew Oracle, and 7 knew neither language. How many knew both languages?

$|J| = 25$ [J → Java]

$|O| = 28$ [O → Oracle]

$|J \cup O| = 40 - 7 = 33$

Computer pgms who know both language

$|J \cap O| = 25 + 28 - 33$

$= 20$.

$[\because |J \cap O| = |J| + |O| - |J \cup O|]$

The Pigeonhole Principle:

It states that if there are more pigeons [objects] than the pigeonholes (boxes), then some pigeohole (box) must contain two or more pigeon (objects). The Pigeonhole Principle is called the Dirichlet Drawer Principle (or) Shoe Box Principle.

Theorem: The Pigeonhole Principle:

If k is a positive integer and $k+1$ are more objects are placed into k boxes, then there is atleast one box containing two or more of the objects.

Proof:

We prove this principle by the method of contradiction. Suppose that none of the k boxes contains more than one object. Hence, the total number of objects could be atmost k. This is a contradiction since there are atleast $k+1$ objects.

The Generalisation / Extension of the Pigeonhole principle:

If k pigeons are assigned to n pigeon holes, then one of the pigeonholes must contain atleast $\left(\frac{k-1}{n}\right)+1$ pigeons. $\left(\frac{25-1}{12}\right)+1 = \left(\frac{24}{12}\right)+1 = 3$

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① Give two examples based on Pigeonhole Principle:

(i) Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 maximum possible birthdays.

(ii) In any group of 27 English words, there must be atleast two ~~sto~~ that starts with same letter since there are only 26 letters in alphabets.

② Show that among 13 children, there are atleast 2 children who were born in same month.

Soln:

Let us assume that 13 children as pigeons and 12 months (Jan, Feb, ... Dec) as the pigeonholes. Then by the Pigeonhole Principle, there will be atleast 2 children who were born in the same month.

~~7 members of a 2 fan~~

③ If 9 books are to be kept in 4 selves, there must be atleast 1 self which contains atleast 3 books.

Soln:

Let us assume books - pigeons (objects), selves - pigeonholes (boxes). Now 9 pigeons are to be assigned to 4 pigeonholes. Using the Extended Pigeonhole Principle,

$\left(\frac{k-1}{n}\right)+1$ where $k=9$, $n=4$ $\therefore \left(\frac{9-1}{4}\right)+1=3$

Hence there are 3 books in atleast one self.

Permutation and Combinations:

Permutation:

A permutation of a set of distinct objects is an ordered arrangement of these objects.

Note: Permutation means selection and arrangement of factor.

Notation : ${}_nP_r$ (or) $P(n, r)$ (or) $P_{n,r}$ (or) P_n^r (or) $(n)_r$

r-Permutation:

An r-Permutation of n (distinct) elements $x_1, x_2, x_3 \ldots x$ is an ordering of r elements subset $\{x_1, x_2, x_3 \ldots x_n\}$, the no. of r permutations of a set of n distinct elements is denoted by $P(n, r)$

Example:

2-permutations of $\{x, y, z\}$ are xy, yx, zx, xz, yz, zy, there are six 2-permutations of this set with three elements, note that there are three ways to choose the first element of the arrangment and two ways to choose the second element of the arrangement. By the product rule, it follows that $P(3, 2) = (3)(2) = 6$.

Note: The Product rule to find a formula for $P(n, r)$ whenever n and r are positive integers with $1 \leq r \leq n$

Theorem:

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$ r permutations of a set with n distinct elements.

Proof:

We apply the product rule to prove that this formula is correct. The first element of the permutation can be chosen in n ways since there are n elements in the set. The second element of the permutation can be chosen in $(n-1)$ ways since there are $(n-1)$ elements left in the set, After ~~choo~~ choosing the element picked for first position

Similarly, there are $(n-2)$ ways to choose the element and so on until there are exactly $(n-(r-1)) = (n-r+1)$ ways to chose the r^{th} element.

Formula:

① $nP_r = n(n-1)(n-2)\ldots\overset{n-(r-1)}{(n-r+1)}$ $\quad nP_1 = n$

$= \frac{n!}{(n-r)!}$ $\quad nP_2 = n(n-1)$

$nP_3 = n(n-1)(n-2)$

② $P(n,n) = n!$ $\quad P(n,r) = nP_r$

③ $P(n,r) = 0$ if $r > n$ $\quad P(n,r) = \frac{n!}{(n-r)!}$ (Since $r > n$ $\therefore P(n,r) = 0$)

④ $P(n,0) = 1$ $\quad P(n,0) = \frac{n!}{(n-r)!} = \frac{n!}{(n-0)!} = 1$

(1) Find the value of these quantities P(6, 3) and C(5, 3).

Soln:

$$\boxed{P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}}$$

$$P(6,3) = \frac{6!}{(6-3)!}$$

$$= \frac{6!}{3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1}$$

$$P(6,3) = 120$$

$$\boxed{{}^nC_r = C(n,r) = (n/r) = \frac{n!}{r!(n-r)!}}$$

$$C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5!}{(3!)(2!)}$$

$$= \frac{5\times4\times3\times2\times1}{3\times2\times1\times1\times2} = \frac{20}{2} = 10.$$

(2) Determine the value of n ~~(4)~~ $(4)\,{}^nP_3 = {}^{(n+1)}P_3$.

Soln:

$$(4)\ {}^nP_3 = {}^{(n+1)}P_3.$$

$$(4)\ \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!}$$

$$(4)\ \frac{n!}{(n-3)!} = \frac{(n+1)n!}{(n-2)!}$$

$$(4)\ \frac{\cancel{n!}}{\cancel{(n-3)!}} = \frac{(n+1)\cancel{n!}}{(n-2)\cancel{(n-3)!}}$$

$$(4) = \frac{n+1}{(n-2)}$$

$(4)(n-2) = n+1$

$4n-8 = n+1$

$4n-8-1 = n$

$n = 4n-9$

$3n = 9 \Rightarrow \boxed{n=3}$

③ Determine the value of if $^{20}C_{n+2} = {}^{20}C_{2n-1}$

Soln:

By Formula $nC_x = nC_y$

$n = x+y$ (or) $x=y$

$x=y \Rightarrow n+2 = 2n-1$

$n+2-2n+1 = 0$

$-n+3 = 0$

$\boxed{n=3}$

④ How many bit strings of length ~~to~~ 10 contain

(a) exactly four 1's?

(a) A bit string of length 10 can be considered to have 10 pos; should be four 1's and six 0's

$10C_4$

Therefore the no. of required bit strings $= \frac{10!}{4!6!}$

$\frac{n!}{(n-r)!r!}$

$= 210.$

(b) atmost four 1's.

$+10C_3$ $10C_2 + 10C_1 + 10C_0$

$= \frac{10!}{0!10!} + \frac{10!}{1!9!} + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10}{4!6!}$

210

$= 386.$

120

(c) atleast four 1's : $^{10}C_4 + {}^{10}C_5 + \dots {}^{10}C_{10}$

$$\frac{10!}{4!6!} + \frac{10!}{5!5!} + \frac{10!}{6!4!} + \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$$

$= 848$

(d) an equal no of. 1's and 0's. $^{10}C_5$

$$\frac{10!}{5!5!} = 252.$$

Recurrence ~~Notations~~ Relations:

Sometimes recurrence relations is called difference equation.

1) Let sequence $\{A_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-2} + a_{n-1}$ for $n = 2, 3, 4, 5, \dots$ and suppose that $a_0 = 3, a_1 = 5$, what are a_2 and a_3 ?

Soln: Given $a_n = a_{n-2} + a_{n-1}$

$$a_2 = a_{2-2} + a_{2-1}$$
$$= a_0 + a_1$$
$$= 3+5$$
$$\boxed{a_2 = 8}$$

$$a_3 = a_{3-2} + a_{3-1}$$
$$= a_1 + a_2$$
$$= 5+8$$
$$\boxed{a_3 = 13}$$

2) Let $a_n = 2^n + 5(3^n)$ $n = 0, 1, 2, \ldots$

(a) find a_0, a_1, a_2

(b) Show that $a_4 = 5a_3 - 6a_2$.

Soln:

(a) $a_0 = 2^0 + 5(3^0)$

$= 1 + 5$

$\boxed{a_0 = 6}$

$a_1 = 2^1 + 5(3^1)$

$= 2 + 15$

$\boxed{a_1 = 17}$

$a_2 = 2^2 + 5(3^2)$

$= 4 + 5(9) = 4 + 45$

$\boxed{a_2 = 49}$

$a_3 = 2^3 + 5(3^3)$

$= 8 + 5(27)$

$= 8 + 135$

$\boxed{a_3 = 143}$

81)
405

(b) $a_4 = 5a_3 - 6a_2$

$= 5a_3 - 6(49)$

$= 5(143) - 294$

$= 715 - 294$

$\boxed{a_4 = 421}$

143×5
$\cancel{67}\cancel{1}5$
294
421

3) Show that recurrence $\{a_n\}$ is solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 2n - 9 \text{ if}$$

$a_n = 3(-1)^n + 2^n - n + 2$

$a_n = 72^n - n + 2$

<u>Given</u>: $a_n = 3(-1)^n + 2^n - n + 2$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} + (n-1) + 2$

$= 3(-1)^n \cdot (-1)^{-1} + 2^n \cdot 2^{-1} - (n-1) + 2$

$= -3(-1)^n + 2^n/2 - n + 3 \longrightarrow ①$.

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$= 3(-1)^n (-1)^{-2} + 2^n \cdot 2^{-2} - n + 2 + 2$

$= 3(-1)^n + 2^n/4 - n + 4 \longrightarrow ②$

We have to show that,

$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

$= -3(-1)^n + 2^n/2 - n + 3 + 6(-1)^n + \frac{2^n}{2} - 2n \cancel{+8}$

$+ 2n - 9$

$\frac{2^n \times \cancel{6}^{3}}{\cancel{4}_{2}}$

$= 3(-1)^n + 2^n - n + 2$

$= a_n$. Hence it is proved.

4) How many permutations of $\{a, b, c, d, e, f, g\}$.

i) end with a ?

The last position can be filled in only one way. The remaining 6 letters can be arranged in 6! ways.

The total no. of ~~letter~~ permutations ending with a are $= (6!)(1)$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$.

(ii) begin with c

The first position can be filled in only one way. The remaining 6 letters can be arranged in 6! ways.

∴ Total no. of. permutations ~~can be~~ ~~can~~ starting with 'c' are

$= 1 \times 6! = 720.$

(iii) begin with c and end with a.

The first position can be filled in only one way and the last position can be filled only one way.

The remaining 5 letters can be arranged in 5! ways.

∴ Total no. of. permutations begin with c and with a is $= (1)(5!)(1) = 120.$

(iv) c and a occupy end places:

'c' and 'a' occupy end position in 2! ways and the remaining 5 letters can be arranged in 5! ways.

∴ Total no. of. permutations $= (5!)(2!)$

$= 120 \times 2$

$= 240.$

5) How many permutations of the letters A, B C D E F G contains

(a) the string BCD.

Taking BCD as one object, we have the following 5 objects: A, (BCD), E, F, G.

These 5 objects can be permuted in $P(5,5) = 5!$

$= 120$ ways.

(b) the string CFGA.

CFGA as one object, we have the following 4 objects B, D, E, (CFGA)

$\therefore$ The no. of ways of permuting these 4 objects

$= 4! = 24$ ways.

(C) the string BA and GF

The objects (BA), C, D, E, (GF) can be permutted in $5! = 120$ ways.

(d) the strings ABC and DE

The objects (ABC), (DE), F and G can be permuted in $4! = 24$ ways.

(e) the strings ABC and CDE?

Even though (ABC) and (CDE) are two strings, they contain common letter C.

If we include the strings (ABCDE) in the permutations, it includes both the strings (ABC) and (CDE). We cannot use the letter C twice

Hence, we have to permute the 3 objects (ABCDE), F and G. This can be done in 3! ways = 6 ways.

⑥ How many possibilities are there for the win, place and show (first, second and third) positions in a horse race with 12 horses if all orders of finish are possible?

The no. of ways to pick the three winners in the number of ordered selections of three elements from 12,

i.e) $P(12,3) = (12)(11)(10) = 1320.$

⑦ Find the no. of 5 permutations of a set with nine elements.

The given is nothing but

$P(9,5) = 9P_5 = \frac{9!}{(9-5)!} = \frac{9!}{4!}$

$\frac{24 \times 23 \times 22}{\times 3 \times 4 \times 5}$

$= 15120.$

⑧ In how many ways can a set of five letters to be selected from the English alphabet?

$n!$

The no. of ways to pick 5 of letters from 26 is

$= 26C_5 = \frac{26!}{(26-5)!\,5!} = 65,780$

$5!$

(9) Suppose that there are eight runners in a race. The winner receives 1st prize, the second-place finisher receives second prize, and the third place finisher receives third prize. How many different ways are there to receive these prizes, if all possible outcomes of the race can occur and there are no ties?

The no. of ways to pick the three prize winners in the number of ordered selections of three elements from 8

$\therefore P(8,3) = (8)(7)(6) = 336$ possible ways.

(10) Suppose that there are 9 faculty members in the maths department and 11 in the CS department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the CS department?

By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By theorem, the number of ways to select the committee is $C(9,3) \cdot C(11,4) = \frac{9!}{3!\,6!} \cdot \frac{11!}{4!\,7!} = (84)(330)$

$= 27,720.$

Solving Linear Recurrence Relations:

A Linear Recurrence Relation with constant coefficient is of the form $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots C_k a_{n-k} = f(n)$ where C is the constant.

A linear homogeneous recurrence relation with constant coefficients of degree k is of the form $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots C_k a_{n-k}$ where $C_1, C_2, C_3, \dots C_k$ are real numbers and $C_k \neq 0$.

2m The three methods of solving recurrence relations are (i) Iteration

(ii) characteristic roots

(iii) Generating function.

Theorem:

Let C_1 and C_2 be real numbers, Suppose that $r^2 - C_1 r - C_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is the solution of the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ iff (if and only if) $a_n = \alpha_1 r^n + \alpha_2 r^n$ for $n = 0, 1, 2, \dots$ where α_1, α_2 are constant

Find an explicit formula for the Fibonacci relation?

<u>Soln:</u> The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2} \rightarrow ①$ and satisfies the initial conditions $f_0 = 0, f_1 = 1$.

① implies $f_n - f_{n-1} - f_{n-2} = 0 \rightarrow ②$

Let $\boxed{f_n = r^n}$ be the solution of the given eqn ②.

So $r^n - r^{n-1} - r^{n-2} = 0.$

$r^n[1 - r^{-1} - r^{-2}] = 0.$ $\qquad r^{n-1} = r^n . r^{-1}$

$r^n\left[1 - \frac{1}{r} - \frac{1}{r^2}\right] = 0.$ $\qquad r^{-1} = 1/r$

$r^n\left[\frac{r^2 - r - 1}{r^2}\right] = 0.$

Therefore the characteristic eqn is $r^2 - r - 1 = 0.$

$a = 1, b = -1, c = -1.$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore r_1 = \frac{1+\sqrt{5}}{2}$ $\quad r_2 = \frac{1-\sqrt{5}}{2}$. We know that by the theorem $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

$\therefore f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$f_n = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right] \longrightarrow (3)$$

By the given condition, $f_0 = 0$, $f_1 = 1$

$$f_0 = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^0 + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^0 = 0$$

$$f_0 = \alpha_1 + \alpha_2 = 0.$$

$$\Rightarrow \boxed{\alpha_1 + \alpha_2 = 0} \longrightarrow (4).$$

$$f_1 = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^1 + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^1 = 1$$

$$= \frac{(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2}{2} = 1$$

$$\Rightarrow \boxed{(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2} \longrightarrow (5).$$

Solving (4) and (5)

$$(1+\sqrt{5}) \times (4) \Rightarrow (1+\sqrt{5})\alpha_1 + (1+\sqrt{5})\alpha_2 = 0$$

$$\underset{(-)}{(1+\sqrt{5})\alpha_1} + \underset{(-)}{(1-\sqrt{5})\alpha_2} = \underset{(-)}{2}$$

$(1+\sqrt{5})-(1-\sqrt{5})$

$1+\sqrt{5}-1+\sqrt{5}$

$2\sqrt{5}$

$$2\sqrt{5}\,\alpha_2 = -2.$$

$$\alpha_2 = -1/\sqrt{5}.$$

$$(4) \Rightarrow \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + (-1/\sqrt{5}) = 0$$

$$\alpha_1 = +1/\sqrt{5}$$

$$③ \Rightarrow \alpha_1\left(\frac{1+\sqrt5}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt5}{2}\right)^n$$

$$= (+1/\sqrt5)\left(\frac{1+\sqrt5}{2}\right)^n + (-1/\sqrt5)\left(\frac{1-\sqrt5}{2}\right)^n$$

$$f_n = (1/\sqrt5)\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1}{\sqrt5}\right)\left(\frac{1-\sqrt5}{2}\right)^n.$$

② What is the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

Given: $a_n = 5a_{n-1} - 6a_{n-2} \longrightarrow ①$.

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \longrightarrow ②$$

Let $\boxed{a_n = r^n}$ be the solution of given eqn ②

So $r^n - 5r^{n-1} + 6r^{n-2} = 0.$

$$r^n[1 - 5r^{-1} + 6r^{-2}] = 0$$

$$r^n\left[1 - \frac{5}{r} + \frac{6}{r^2}\right] = 0$$

$$r^n\left[\frac{r^2 - 5r + 6}{r^2}\right] = 0.$$

Therefore the characteristics eqn is $r^2 - 5r + 6 = 0$.

$$(r-3)(r-2) = 0 \quad \cancel{(r-6)(r+1) = 0.}$$

$\boxed{r_1 = 3}$, $\boxed{r_2 = 2.}$ $\cancel{\boxed{r_1 = 6,}\ \boxed{r_2 = -1.}}$

6
$-3 \quad -2$
-5

∴ We know that by the theorem,

$$f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = \alpha_1(3^n) + \alpha_2(2)^n \longrightarrow ③$$

By the given condition $a_0 = 1, a_1 = 0$.

$$a_0 = \alpha_1 3^0 + \alpha_2 2^0 = 1$$

$$a_0 = \alpha_1 + \alpha_2 = 1$$

$$\Rightarrow \boxed{\alpha_1 + \alpha_2 = 1} \rightarrow (4)$$

$$a_1 = \alpha_1 3^1 + \alpha_2 2^1 = 0$$

$$\Rightarrow \alpha_1(3) + \alpha_2(2) = 0$$

$$\Rightarrow \boxed{3\alpha_1 + 2\alpha_2 = 0} \rightarrow (5)$$

Solving (4) and (5)

$3 \times (4) \Rightarrow$

$$3\alpha_1 + 3\alpha_2 = 3$$
$$3\alpha_1 + 2\alpha_2 = 0$$
$$(-) \quad (-) \quad\quad (-)$$

$$\boxed{\alpha_2 = 3}$$

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 + 3 = 1$$

$$\alpha_1 = -3 + 1$$

$$\boxed{\alpha_1 = -2}$$

$$(3) \Rightarrow \boxed{a_n = (-2)\,3^n + (3)\,2^n}$$

Generating functions:

The generating function for the sequence $a_0, a_1, \ldots a_k$ of real numbers is the infinite series.

$$G(x) = a_0 + a_1 x + \ldots a_k x^k + \ldots$$
$$= \sum_{k=0}^{\infty} a_k x^k$$

1. $f(x)+g(x) = \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k$

$$f(x)+g(x) = \sum_{k=0}^{\infty} (a_k+b_k)\, x^k \text{ and}$$

$$f(x)\, g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

① What is the generating function for sequence $(1,1,1,1,1)$?

Soln:

$1, 1, 1, 1, 1$ is

$1x^0 + 1x^1 + 1x^2 + 1x^3 + 1x^4$

(i.e) $1+x+x^2+x^3+x^4$

G.P $a + ar + ar^2 + \dots ar^{(n-1)} \dots \dfrac{ar^n - 1}{r-1}$

$\therefore 1+x^2+x^2+x^3+x^4 = \dfrac{1(x^5)-1}{x-1} = \dfrac{x^5-1}{x-1}$ when $x \neq 1$

$r = \dfrac{x}{1} = \dfrac{x^2}{x} = \dfrac{x^3}{x^2} = \dfrac{x^4}{x^3} = x.$

Consequently, $\boxed{G(x) = \dfrac{x^5-1}{x-1}}$ is the generating function of the sequence $(1,1,1,1,1)$.

② Find the generating fn for the finite sequence $(2,2,2,2,2)$?

Soln: The generating fn of $(2,2,2,2,2)$ is

$2x^0 + 2x^1 + 2x^2 + 2x^3 + 2x^4$

$\Rightarrow 2(1+x+x^2+x^3+x^4)$

$\Rightarrow 2\left(\dfrac{x^5-1}{x-1}\right)$ is the generating fn of the sequence $(2,2,2,2,2)$

$\therefore G(x) = 2\left[\frac{x^5-1}{x-1}\right]$ is the generating fn of the sequence $(2, 2, 2, 2, 2)$.

3) Find the values of the extended binomial Coefficients and $\binom{-2}{4}$ and $\binom{1/3}{0}$.

We know that

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)(u-2)\ldots(u-k+1)}{k!} & \text{if } k>0 \\ 1. & \text{if } k=0 \end{cases}$$

$u-(k-1)$

$$\therefore \binom{1/3}{0} = 1$$

$$\binom{-2}{4} = \frac{(-2)(-2-1)(-2-2)(-2-3)}{4!}$$

stop upto $(k-1)$

$-2-4+1$

$-2-3$

$$= \frac{(-2)(-3)(-4)(-5)}{1\times 2\times 3\times 4}$$

$$= 5$$

4) Find the generating fn for $(1+x)^{-n}$ where n is a positive integer.

Soln:

$(1+x)^{-n} = \sum_{k=0}^{n} \binom{-n}{k} x^k$ by the extended Binomial theorem.

We know that $\binom{-n}{r} = (-1)^r C(n+r-1, r)$

$$\therefore \sum_{k=0}^{n} (-1)^k C(n+k-1, k)\, x^k.$$

5) Find the generating function for the sequence $1, 4, 16, 64, 256, \ldots$ $(1-x)^{-1} = 1+x^2+x^3+\ldots$

The generating function of $1, 4, 16, 64, 256, \ldots$ is

$$1, 4, 4^2, 4^3, 4^4, \ldots$$

$$= 1x^0 + 4x^1 + 4^2x^2 + 4^3x^3 + 4^4x^4 + \ldots$$

$$= 1 + (4x) + (4x)^2 + (4x)^3 + \ldots$$

$$= (1-(4x))^{-1} = \frac{1}{1-4x}.$$

8) (5, 7, 9, 10, 11)

6) Find a closed form for the generating function $3, -3, 3, -3, 3, -3, \ldots$

W.K.T $\frac{3}{1+x} = 3(1+x)^{-1}$

$$= 3(1 - x + x^2 - x^3 \ldots)$$

$$= 3 - 3x + 3x^2 + (-3x^3) + \ldots$$

$$= \sum_{n=0}^{\infty} (-3)^n x^n.$$

$\therefore$ G.F is $\frac{3}{1+x}$.

7) Find a closed form for the generating fn $0, 0, 1, 1, 1, \ldots$

W.K.T $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ generates the sequence $(1, 1, 1, \ldots)$

$\Rightarrow \frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2}$ generates the sequence $(0, 0, 1, 1, \ldots)$

$\therefore \frac{x^2}{1-x}$ is the generating fn of $0, 0, 1, 1, \ldots$

8) Find the generating fn for the sequence $1, a, a^2,$. where a is a fixed constant

The generating fn of $1, a, a^2, \ldots$ is

$$1+ax+a^2x^2+a^3x^3+\ldots \rightarrow ①$$

Let $G(x) = 1+ax+a^2x^2+a^3x^3+\ldots$

$\Rightarrow G(x)-1 = ax+a^2x^2+a^3x^3+\ldots$

$= ax[1+ax+a^2x^2+\ldots]$

$$\frac{G(x)-1}{ax} = 1+ax+a^2x^2+\ldots$$

$= G(x)$ by ①

$G(x)-1 = ax[G(x)]$

$G(x)-ax[G(x)] = 1$

$G(x)[1-ax] = 1$

$G(x) = \dfrac{1}{1-ax}$ which is the generating fn of the sequence $1, a, a^2, \ldots$

9) Find a closed form for the G.F $\binom{7}{0}, \binom{7}{1}, \binom{7}{2} \ldots \binom{7}{7}, 0, 0, 0, 0, \ldots$

We know that

$(1+x)^7 = C(7,0)x^0 + C(7,1)x + \ldots C(7,7)x^7 + 0 + 0 + \ldots$

$$= \sum_{n=0}^{\infty} C(7,n)x^n$$

$\therefore$ The G.F of $C(7,0), C(7,1), \ldots$ (ie) for the sequence $C(7,7,0,0,\ldots$ is $(1+x)^7$.

10) Find a closed form for the G.F $a_n = 5$ for all $n = 0, 1, 2, \dots$

Gn: $a_n = 5$ for all $n = 0, 1, 2, \dots$

(i.e) $a_0 = 5,\ a_1 = 5,\ a_2 = 5,\ a_3 = 5 \dots$

(i.e) $5, 5, 5, 5, \dots$

The G.F of $5, 5, 5, 5 \dots$ is

$$5 + 5x + 5x^2 + \dots = 5(1 + x + x^2 + \dots)$$
$$= 5(1-x)^{-1}$$
$$= \frac{5}{1-x}$$ is the required

G.F of the given sequence.

11) Find a closed form of the G.F for the sequence $\{a_n\}$ when $a_n = 3^n$ for all $n = 0, 1, 2, \dots$

Gn: $a_n = 3^n$ for all $n = 0, 1, 2, \dots$

(i.e) $a_0 = 3^0,\ a_1 = 3^1,\ a_2 = 3^2, \dots$

(i.e) $1, 3, 3^2, 3^3, \dots$

The G.F of $1, 3, 3^2, 3^3 \dots$ is

$$\Rightarrow 1 + 3x + 3^2x^2 + 3^3x^3 + \dots$$
$$\Rightarrow (1-3x)^{-1}$$
$$\Rightarrow \frac{1}{1-3x}$$ which is the required G.F of the

given sequence

12) Use generating f^{ns} to solve recurrence relations $a_{n+2} - 2a_{n+1} + a_n = 2^n$ with initial conditions $a_0 = 2,\ a_1 = 1$.

Soln:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

where $G(x)$ is the generating fn for the sequence $\{a_n\}$. Given that $a_{n+2} - 2a_{n+1} + a_n = 2^n$.

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n \rightarrow (1)$$

(Multiplying each term by x^n and summing from 0 to ∞)

$$\sum_{n=0}^{\infty} a_{n+2} x^n = a_2 x^0 + a_3 x^1 + a_4 x^2 + \dots$$

Sub $n = 0, 1, 2, \dots$

$$= a_2 + a_3 x^1 + a_4 x^2 + \dots$$

Multiply & divide by x^2

$$= \frac{1}{x^2}\left[a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\right]$$

$$= \frac{1}{x^2}\left[\underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots}_{} - a_0 - a_1 x\right]$$

$$= \frac{1}{x^2}\left[G(x) - a_0 - a_1 x\right] \rightarrow (2)$$

$$2\sum_{n=0}^{\infty} a_{n+1} x^n = 2\left[a_1 x^0 + a_2 x^1 + a_3 x^2 + a_4 x^3 + \dots\right]$$

$$= 2 \cdot \frac{1}{x}\left[a_1 x + a_2 x^2 + a_3 x^3 + \dots\right]$$

$$= \frac{2}{x}\left[a_0 + a_1 x + a_2 x^2 + \cdots - a_0\right]$$

$$= \frac{2}{x}\left[G(x) - a_0\right] \longrightarrow ③$$

$$\sum_{n=0}^{\infty} 2^n a^n = 2^0 a^0 + 2^1 a^1 + 2^2 a^2 + \cdots$$

$$= 1 + 2a + 4a^2 + \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= G(x) \longrightarrow ④$$

$$\sum_{n=0}^{\infty} 2^n x^n = 2^0 x^0 + 2^1 x^1 + 2^2 x^2 + 2^3 x^3 + \cdots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \cdots$$

$$= 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

$$= (1-2x)^{-1} \longrightarrow ⑤$$

$$= 1/(1-2x)$$

Sub ②, ③, ④, ⑤ in ①

$$\frac{1}{x^2}\left[G(x) - a_0 - a_1 x\right] - \frac{2}{x}\left[G(x) - a_0\right] + G(x) = \frac{1}{(1-2x)}$$

$$\frac{G(x) - a_0 - a_1 x - 2x\,G(x) + 2x a_0 + x^2 G(x)}{x^2} = \frac{1}{(1-2x)}$$

$$G(x)\left[1 - 2x + x^2\right] - a_1 x + a_0 + 2x a_0 = \frac{x^2}{1-2x}$$

$$G(x)\left\{1 - 2x + x^2\right] = \frac{x^2}{1-2x} + a_1 x + a_0 - 2x a_0$$

Given: $a_0 = 2$, $a_1 = 1$

$$G(x)[x^2-2x+1] = \frac{x^2}{1-2x} + a_1 x + a_0 - 2x a_0$$

$$= \frac{x^2}{1-2x} + 1x + 2 - 2x(2)$$

$$= \frac{x^2}{1-2x} + 1x + 2 - 4x$$

$$G(x)[\underbrace{x^2-2x+1}_{(1-x)^2}] = \frac{x^2}{1-2x} + 2 - 3x$$

$$\therefore G(x) = \frac{x^2}{(1-2x)(1-x)^2} + \frac{2}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$\therefore G(x) = \frac{x^2}{(1-2x)(1-x)^2} + \frac{2}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$(1-x)^2 - (1-2x) = x^2$$

$$x^2 - 1 + 2x$$

$$G(x) \Rightarrow \frac{(1-x)^2 - (1-2x)}{(1-2x)(1-x)^2} + \frac{2}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$\Rightarrow \frac{\cancel{(1-x)^2}}{(1-2x)\cancel{(1-x)^2}} - \frac{\cancel{(1-2x)}}{\cancel{(1-2x)}(1-x)^2} + \frac{2}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$\Rightarrow \frac{1}{1-2x} - \frac{1}{(1-x)^2} + \frac{2}{(1-x)^2} - \frac{3x}{(1-x)^2}$$

$$\Rightarrow (1-2x)^{-1} - (1-x)^{-2} + 2(1-x)^{-2} - 3x(1-x)^{-2}$$

$$\Rightarrow (1-2x)^{-1} + 1(1-x)^{-2} - 3x(1-x)^{-2}$$

$$\Rightarrow 1 + 2x + (2x)^2 + (2x)^3 + \dots + 1 + 2x + 3x^2 + \dots$$

$$- 3x[1 + 2x + 3x^2 + \dots]$$

$$(1-x)^{-1} = 1 + x^2 + x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x^2 + 3x^3 + \dots$$

$G(x) = 1 + 2x + (2x)^2 + \dots + 1 + 2x + 3x^2 + \dots - 3[x + 2x^2 + 3x^3 + \dots]$

$$G(x) = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (n+1)x^n - 3\sum_{n=0}^{\infty} n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n [2^n + (n+1) - 3n]$$

$$a_n = 2^n + n + 1 - 3n$$
$$= 2^n - 2n + 1$$

18/07/19 Inclusion and Exclusion - Application of Inclusion & Exclusion.

Principle of Inclusion and Exclusion:

Let X and Y be two finite subsets of a universal set U if X and Y are disjoint, then

$$|X \cup Y| = |X| + |Y|.$$

If X and Y are not disjoint, then

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

This is called the principle of inclusion and exclusion.

(1) Give a formula for no. of elements in the union of 4 sets.

Soln: By the principle of inclusion and exclusion, we get

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4|$$

$$+|A_2 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_4|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$

(2) In a survey of 200 students, 64 had taken Mathematics course, 94 had taken English course, 58 had taken Computer course, 28 had taken both Mathematics and computer course, 26 had taken both English and Mathematics course, 22 had taken both English and Computer course, 14 had taken all three courses. How many students were surveyed who taken none of the three courses?

$|M| = 64$, $|E| = 94$, $|C| = 58$,

$|M \cap C| = 28$, $|E \cap M| = 26$, $|E \cap C| = 22$

$|M \cap E \cap C| = 14$.

$$|M \cup E \cup C| = |M| + |E| + |C| - |M \cap E| - |M \cap C| - |E \cap C| + |M \cap E \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$

$$= 154$$

$$\begin{array}{r} 300 \\ -154 \\ \hline 146 \end{array}$$

$\therefore$ The students who had taken none of the courses $= 300 - 154$

$= 146$.

(8) Among the first 1000 positive integers, determine the integers which are not divisible by 5, nor by 7, nor by 9.

Soln: Let A be the no. of. integers divisible by 5

B be the no. of. integers divisible by 7

C be the no. of. integers divisible by 9.

$$|A| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$|B| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$$

$$|C| = \left\lfloor \frac{1000}{9} \right\rfloor = 111$$

$$|A \cap B| = \left\lfloor \frac{1000}{LCM(5,7)} \right\rfloor = \left\lfloor \frac{1000}{5 \times 7} \right\rfloor = 28$$

$$|A \cap C| = \left\lfloor \frac{1000}{LCM(5,9)} \right\rfloor = \left\lfloor \frac{1000}{5 \times 9} \right\rfloor = 22$$

$$|B \cap C| = \left\lfloor \frac{1000}{LCM(7,9)} \right\rfloor = \left\lfloor \frac{1000}{7 \times 9} \right\rfloor = 15$$

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{LCM(5,7,9)} \right\rfloor = \left\lfloor \frac{1000}{315} \right\rfloor = 3$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 200 + 142 + 111 - 28 - 22 - 15 + 3$$

$$= 391$$

$\therefore$ The no. of. numbers (integers) which are not divisible by 5, nor by 7, nor by 9 = Total no. of. integers − integers divisible by 5, 7, 9

$= 1000 - 391$

$= 609.$

④ Colomba has two dozen each of n different coloured beads. If she can select 20 beads (with repitition of colour allowed) in 2,30,230 ways, what is the value of n?

$c(n,r)$
$nC_r = \frac{n!}{(n-r)!\,r!}$

Given: $r = 20$.

W.K.T $C(n+r-1, r) = \frac{(n+r-1)!}{(n+r-1-r)!\,r!}$

$= \frac{(n+r-1)!}{(n-1)!\,r!} = 2,30,230.$

Sub $r = 20$

use calculator shift + x^{-1} ⇒ !

$\Rightarrow \frac{(n+19)!}{(n-1)!\,20!} = 230230.$

$\Rightarrow \boxed{n = 7.}$

⑤ Find the no. of Non-negative integers Solution of the eqn. $x_1 + x_2 + x_3 + x_4 + x_5 = 8$

The no. of non-negative integers solns $x_1 + x_2 + \dots x_n = r$ is

$C(n+r-1, r) = \boxed{C(\overset{5+8-1}{n+r-1}, \overset{5-1}{n-1})}$ $r = 8,\ n = 5$

$C(12,8) = C(12,8) = C(12,4) = 495$

$nC_r = \frac{n!}{(n-r)!\,r!}$ $C(12,4) = \frac{12!}{8!\,4!} = 495$

$12C_4 = \frac{12!}{8!\,4!} = 495$

⑥ Find the no. of integer solns of $x_1+x_2+x_3+x_4+x_5=30$ where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$

Given: $x_1+x_2+x_3+x_4+x_5 = 30 \longrightarrow ①$.

$x_1 \geq 2 \Rightarrow y_1 = x_1 - 2 \geq 0$

$\Rightarrow x_1 = y_1 + 2 \geq 0$.

$x_2 \geq 3 \Rightarrow y_2 = x_2 - 3 \geq 0$.

$x_3 \geq 4 \Rightarrow y_3 = x_3 - 4 \geq 0$

$x_4 \geq 2 \Rightarrow y_4 = x_4 - 2 \geq 0$.

$x_5 \geq 0 \Rightarrow y_5 = x_5 \geq 0$.

$\therefore y_1, y_2, y_3, y_4, y_5$ are non-negative integers.

$① \Rightarrow y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$.

$y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11 = 19$

$\therefore n = 5$, $r = 19$.

The no. of non negative integers solns of $y_1 + y_2 + \dots + y_5 = 19$ is

$C(5+19-1, 19) = C(23, 19)$

$= C(23, 4) = \frac{23!}{19!\,4!}$

$\because [C(n+r-1, r) = C(n+r-1, n-1)]$.

$= 8855$

⑦ Find the no. of non negative integer solns of the inequality $x_1 + x_2 + \dots x_6 < 10$.

$x_1 + x_2 + \dots x_6 < 10$.

$x_1 + x_2 + \dots x_6 = 9 - x_7$

$x_1 + x_2 + \dots x_7 = 9$

$\therefore n = 7$, $r = 9$

The no. of. non-negative integer Soln of

negative
↳ strictly positive

$x_1 + x_2 + \dots x_7 = 9$ is $\boxed{C(n+r-1, r)}$

$C(15, 9) = 5005$

⑧ Find the no. of. ~~xxx~~ $\boxed{\text{positive integer}}$ solns of

$x_1 + x_2 + x_3 = 17$

$\therefore n = 3,\ r = 17$

The no. of. positive integer solns of

$x_1 + x_2 + \dots x_n = r$ is $\boxed{C(r-1, r-n)}$

$\therefore$ The no. of. positive integer solns of

$x_1 + x_2 + x_3 = 17$ is

$C(17-1, 17-3) = C(16, 14)$

$= C(16, 2)$

$= 120.$

(2) Solve the recurrence relation.

$$a_{n+2} + 3a_{n+1} + 2a_n = 3^n \text{ for } n \geq 0, \; \underline{a_0 = 0}, \; \underline{a_1 = 1}$$

The characteristic equation is

$$r^2 + 3r + 2 = 0 \qquad \left[\because r^{n+2} + 3r^{n+1} + 2r^n\right] = 0$$

$$(r+2)(r+1) = 0 \qquad \Rightarrow r^n[r^2+3r+2] = 0$$

$$\Rightarrow r^2 + 3r + 2 = 0$$

$$\therefore r = -2 \quad r = -1$$

$$\therefore a_n(h) = A(-2)^n + B(-1)^n \longrightarrow (1)$$

where A and B are arbitrary constants.

Keeping the RHS of the given relation in mind, we seek $a_n(P)$ in the form

$$a_n(P) = A_0 \times 3^n \longrightarrow (2).$$

Putting this for a_n in the given ~~sols~~ relation, we get

$$A_0 \times 3^{n+2} + 3A_0 \times 3^{n+1} + 2A_0 3^n = 3^n$$

$$(A_0 \times 3^2 \cdot 3^n) + (3A_0 \times 3 \cdot 3^n) + 2A_0 3^n = 3^n.$$

$$(A_0 \times 3^2) + (3A_0 \times 3) + 2A_0 = 1$$

So that $(A_0 \times 9) + (\underbrace{3A_0 \times 3}_{9A_0}) + 2A_0 = 1$

$$9A_0 + 9A_0 + 2A_0 = 1 \qquad \Rightarrow A_0(9+9+2) = 1$$

$$A_0 = 1/20 \qquad A_0 = 1/20$$

Putting this into (2) we get

$$a_n(P) = 1/20 \times 3^n \longrightarrow (3)$$

$\therefore$ The general soln of the given relation is

$$a_n = a_n(h) + a_n(P)$$

$$a_n = A(-2^n) + B(-1)^n + \frac{1}{20} \times 3^n \rightarrow (4)$$

Gn: $a_0 = 0$, $a_1 = 1$

$$a_0 = A(-2^0) + B(-1)^0 + 3^0/20$$

$$0 = A + B + \frac{1}{20}$$

$$a_1 = A(-2) + B(-1) + 3/20$$

$$1 = -2A - B + \frac{3}{20}$$

Solve the above eqns, we get

$$A = \frac{-4}{5}, \quad B = \frac{3}{4}$$

Putting these into (4), we get

$$a_n = \left(\frac{-4}{5}\right) \times (-2^n) + \left(\frac{3}{4}\right)(-1)^n + \frac{1}{20}(3^n)$$

This is the required solution

Particular Solution: (for recurrence relation)

1	c, a constant	A, a constant.
2	n	$An + B$
3	n^2	$An^2 + Bn + C$
4	n^t $t \in Z^+$ $(n^2, n^3, ...)$	$An^t + Bn^{t-1} + \dots E$. some constant
5	r^n, $r \in R$ $(2^n, (-3)^n, (1/3)^n)$	Ar^n.
6	$n^t r^n$	$r^n(An^t + B^{t-1} + \dots Z)$ some constant

7	$\cos \alpha n$	$A \sin \alpha n + B \cos \alpha n$
8	$\sin \alpha n$	$A \sin \alpha n + B \cos \alpha n$
9	$r^n \cos \alpha n$	$r^n (A \sin \alpha n + B \cos \alpha n)$
10	$r^n \sin \alpha n$	$r^n (A \sin \alpha n + B \cos \alpha n)$

1. Use mathematical induction P.T, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}} > n$ for $n \geq 2$

<u>Given:</u> $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}} > \sqrt{n} \rightarrow (1)$

<u>To prove:</u> $P(n)$ is true for all $n \geq 2$

Step 1: To prove $P(2)$ is true.

$$P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}.$$

$\Rightarrow 1.707 > 1.414$ is true.

$\therefore P(2)$ is true

Step 2: Assume $P(K)$ is true

$$P(K) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{K}} > \sqrt{K} \rightarrow (2)$$

Step 3: To prove $P(K+1)$ is true

(i.e) T.P $\Rightarrow$ $P(K+1) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \cancel{+1+} \frac{1}{\sqrt{k}\sqrt{K+1}} > \sqrt{K+1}$

LHS: $\left[\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{K}}\right] + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}}$

$$= \frac{\sqrt{k}\sqrt{K+1} + 1}{\sqrt{K+1}}$$

$$= \frac{\sqrt{K(K+1)} + 1}{\sqrt{K+1}}$$

$$> \frac{\sqrt{K \cdot K} + 1}{\sqrt{K+1}}$$

$$= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} \qquad (\because (k+1) > k)$$

$\therefore P(k+1)$ is true whenever $P(k)$ is true

② Use mathematical induction to show that $H_{2^n} \geq 1+\frac{n}{2}$ whenever n is a non-negative Integer.

$P(n)$: $H_{2^n} \geq 1+\frac{n}{2}$.

$\boxed{\text{Step 1:}}$ To prove $P(0)$ is true

$$H_{2^0} \geq 1+\frac{0}{2} \Rightarrow H_1 \geq 1+\frac{0}{2} \quad \left[\because H_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right]$$

$$\Rightarrow 1 \geq 1$$

$\boxed{\text{Step 2:}}$ ~~To prove~~ Assume $P(k)$ is true

$$H_{2^k} \geq 1+\frac{k}{2} \quad \text{i.e.} \quad 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2^k} \geq 1+\frac{k}{2}.$$

$\boxed{\text{Step 3:}}$ To prove $P(k+1)$ is true.

To prove: $H_{2^{k+1}} \geq 1+\frac{k+1}{2}$.

$$H_{2^{k+1}} = 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2^k}+\frac{1}{2^{k+1}}+\cdots\frac{1}{2^{k+1}}$$

$$= H_{2^k}+\frac{1}{2^{k+1}}+\cdots\frac{1}{2^{k+1}}$$

$$H_{2^{k+1}} \geq \left(1+\frac{k}{2}\right)+(2^k)\left(\frac{1}{2^{k+1}}\right)$$

$$H_{2^{k+1}} \geq 1+\frac{k+1}{2}$$

Hence $P(k)$ is true.