UNIT-IV

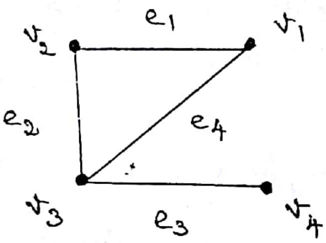
GRAPH THEORY

BASIC DEFINITIONS

① GRAPH

A graph G consists of a pair (V, E), where $V = \{v_1, v_2, \dots\}$ is a set of vertices (or nodes or points) and $E = \{e_1, e_2, \dots\}$ is a set of edges (or lines) such that each edge e_K is associated with a pair of vertices $v_i v_j$.

eg



$G = (V, E)$

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4\}$

such that $e_1 = v_1 v_2$, $e_2 = v_2 v_3$,

$e_3 = v_3 v_4$, $e_4 = v_1 v_3$

Note

$|V|$ = Cardinality of the vertex set of G = Order of G

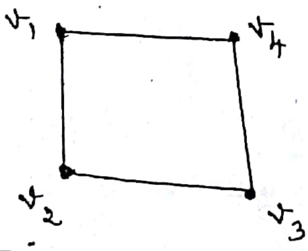
$|E|$ = Cardinality of the edge set of G = Size of G

② UNDIRECTED GRAPH, DIRECTED GRAPH

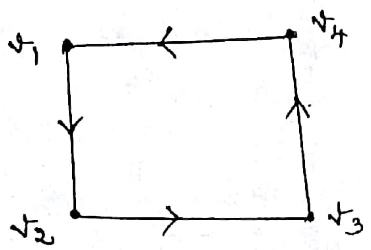
If each edge of a graph G is associated with an unordered pair of vertices, then G is called an undirected graph.

U.Q. If each edge of a graph G is associated with an ordered pair of vertices, then G is called a directed graph or digraph.

eg



UNDIRECTED GRAPH

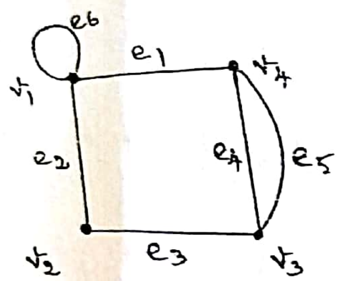


DIRECTED GRAPH

③ LOOP, PARALLEL EDGES

An edge of a graph that joins a vertex to itself is called a loop.

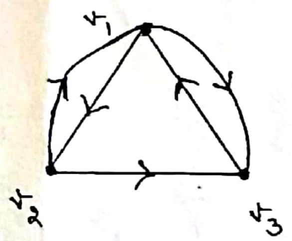
In a graph, certain pair of vertices are joined by more than one edge. Such edges are called parallel edges.



Here e_6 is a loop and e_4, e_5 are parallel edges.

Note

In a directed graph, the two edges between a pair of vertices which are opposite in direction are considered distinct.

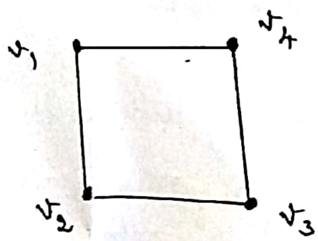


DIRECTED GRAPH WITH DISTINCT EDGES

④ SIMPLE GRAPH

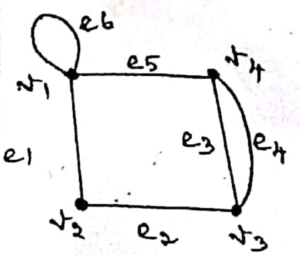
U.Q.

A graph with no loops and parallel edges is called a simple graph.



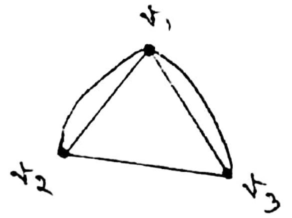
⑤ PSEUDO GRAPH

A graph with loops and parallel edges is called a pseudo graph.



⑥ MULTIGRAPH

A graph with parallel edges is called a multigraph.

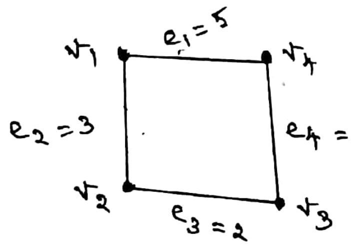


UNDIRECTED MULTIGRAPH

DIRECTED MULTIGRAPH

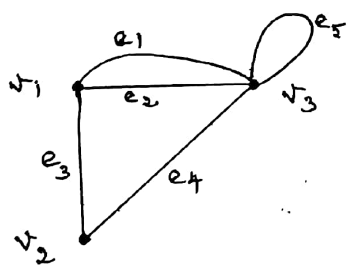
⑦ WEIGHTED GRAPH

A graph in which a number (weight) is assigned to each ed is called a weighted graph.



⑧ DEGREE OF A VERTEX

The degree of a vertex is the number of edges incident with that vertex. A loop contributes 2 to the degree of that vertex.



$deg(v_1) = 3$

$deg(v_2) = 2$

$deg(v_3) = 5$

⑨ ISOLATED VERTEX

A vertex with degree 0 is called an isolated vertex.

⑩ PENDANT VERTEX

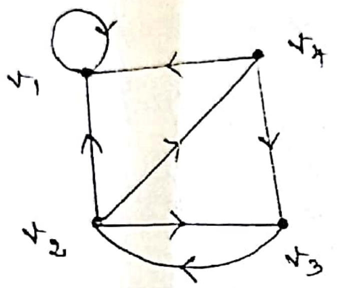
A vertex with degree 1 is called a pendant vertex.

⑪ IN - DEGREE , OUT - DEGREE

In a directed graph, the number of edges that end at v is called the in degree of v. It is denoted by $deg^-(v)$. A vertex with zero in-degree is called a source.

In a directed graph, the number of edges that start from v is called the out-degree of v. It is denoted by $deg^+(v)$. A vertex with zero out-degree is called a sink.

eg



$deg^-(v_1) = 3$ $\quad deg^+(v_1) = 1$

$deg^-(v_2) = 1$ $\quad deg^+(v_2) = 3$

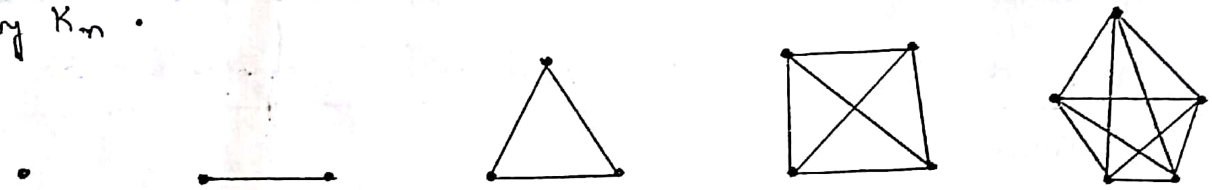
$deg^-(v_3) = 2$ $\quad deg^+(v_3) = 1$

$deg^-(v_4) = 1$ $\quad deg^+(v_4) = 2$

(12) COMPLETE GRAPH

U.Q.

A simple graph, in which there is exactly one edge between each pair of distinct vertices, is called a complete graph. A complete graph on n vertices is denoted by K_n.

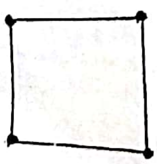


K_1 $\quad K_2$ $\quad K_3$ $\quad K_4$ $\quad K_5$

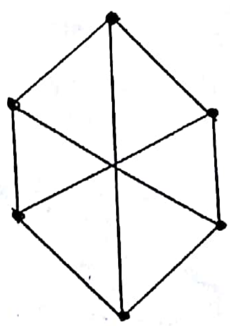
Note: The number of edges in K_n is nC_2 or $\boxed{\frac{n(n-1)}{2}}$.

(13) REGULAR GRAPH

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.



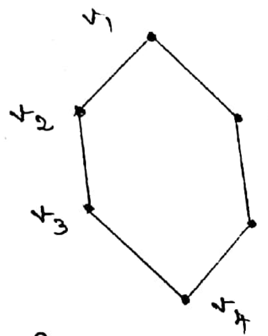
2-regular graph



3-regular graph

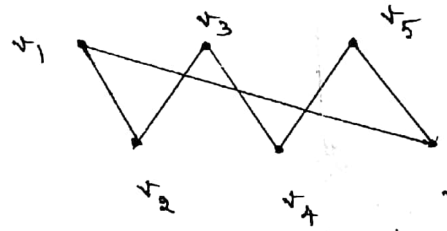
(14) BIPARTITE GRAPH (or) BIGRAPH

A simple graph G is called a bipartite graph if its vertex set V can be partitioned into two disjoint non-empty sets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2.



Graph G

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$V_1 = \{v_1, v_3, v_5\}$$
$$V_2 = \{v_2, v_4, v_6\}$$

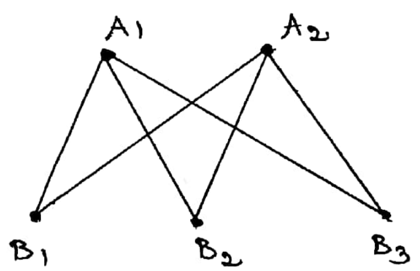


Bipartite graph

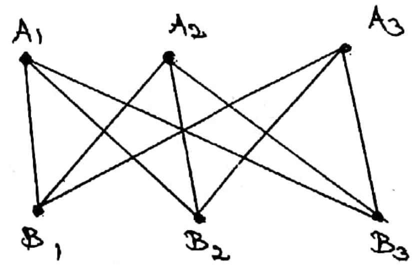
(15) COMPLETE BIPARTITE GRAPH

If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a complete bipartite graph.

If V_1 contains m vertices and V_2 contains n vertices, then the complete bipartite graph is denoted by $K_{m,n}$.



$K_{2,3}$

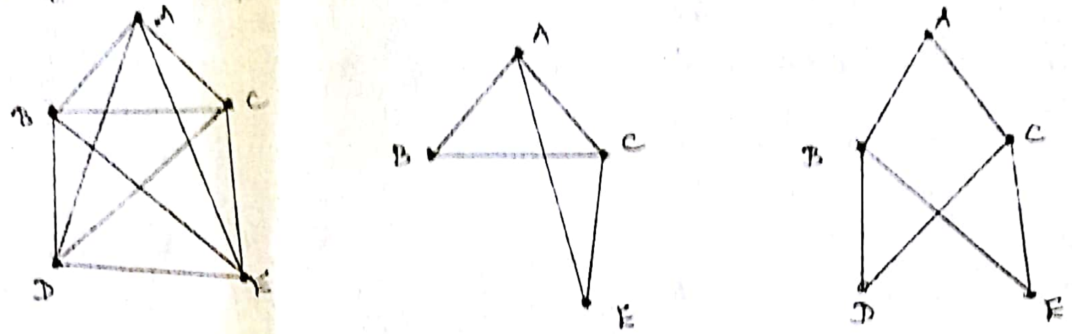


$K_{3,3}$

(16) SUBGRAPH

A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

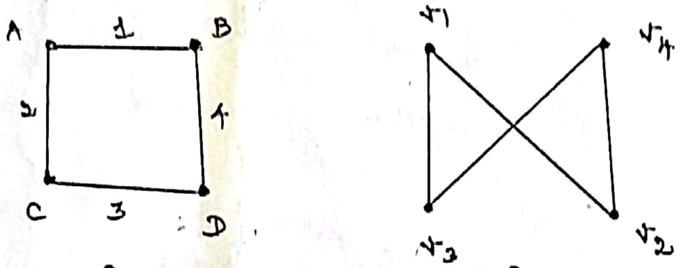
A subgraph H of a graph G is called a spanning subgraph, if $V(H) = V(G)$.



Graph G   Subgraph   Spanning subgraph

(17) **ISOMORPHIC GRAPHS**

Two graphs G_1 and G_2 are said to be isomorphic, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.



G_1 G_2

There are 4 vertices and 4 edges in G_1 and G_2.
Also, degree of each vertex in G_1 is 2 and degree of each vertex in G_2 is 2.

Adjacency matrix of G_1

$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency matrix of G_2

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$\therefore G_1 \cong G_2$

Note

Two graphs are isomorphic if they have the same number of vertices (ii) the same number of edges and (iii) the corresponding vertices with the same degree.

But the converse is not true.

MATRIX REPRESENTATION OF GRAPHS

18 Q. **ADJACENCY MATRIX**

Let $G = (V, E)$ be a graph. Let $V = \{v_1, v_2, \dots, v_n\}$. The $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{Otherwise} \end{cases}$$

is called the adjacency matrix of G.

19 **INCIDENCE MATRIX**

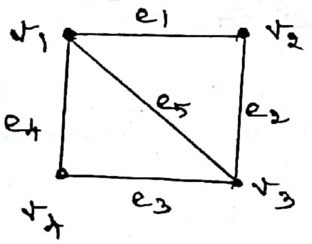
Let $G = (V, E)$ be a graph. Let $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$.

The $n \times m$ matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

is called the incidence matrix of G.

Example



Adjacency matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Incidence matrix

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Theorem-1 : The Handshaking theorem:

U.Q. If $G = (V, E)$ is an undirected graph with e edges, then

$\sum_i \deg(v_i) = 2e$.

(ie) the sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

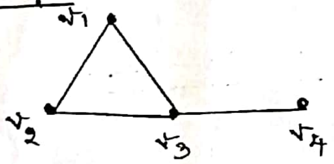
Proof

Every edge of G is incident with two points. Hence each edge contributes 2 to the sum of the degrees of the vertices.

$\therefore$ All the e edges contribute 2e to the sum of the degrees of the vertices.

(ie) $\sum_i \deg(v_i) = 2e$

Example



$$\sum_i \deg(v_i) = 2 + 2 + 3 + 1$$
$$= 8$$
$$= 2 \times \text{No. of edges}$$

Theorem - 2

U.Q. The number of vertices of odd degree in an undirected graph is even.

Proof

Let $G = (V, E)$ be an undirected graph.

Let V_1 be the set of even degree vertices of G and V_2 be the set of odd degree vertices of G.

$$\therefore \sum_i \deg(v_i) = \sum_{\text{Even}} \deg(v_j) + \sum_{\text{Odd}} \deg(v_k)$$

$$\sum_{\text{Odd}} \deg(v_k) = \sum_i \deg(v_i) - \sum_{\text{Even}} \deg(v_j)$$

$$= 2e - \sum_{\text{Even}} \deg(v_j) \quad \text{(By Theorem - 1)}$$

$$= \text{Even}$$

$\therefore \sum_{odd} \deg(V_k) = \text{Even}$

$\therefore$ The number of vertices of odd degree is even.

Theorem - 3

U.Q. The number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Proof

Let G be a bipartite graph with n vertices. Let the vertex set be partitioned into two subsets V_1 and V_2. Let V_1 contain x vertices and V_2 contain $n-x$ vertices.

The largest number of edges of G can be obtained, when each of the x vertices in V_1 is connected to each of the $n-x$ vertices in V_2.

$\therefore$ Largest number of edges $= x(n-x) = f(x)$, a function of x.

To find the value of x, for which $f(x)$ is maximum

$$f(x) = x(n-x)$$

$$f'(x) = n - 2x$$

$$f''(x) = -2$$

$f'(x) = 0$, when $x = \frac{n}{2}$,

and $f''(x) < 0$

Hence $f(x)$ is maximum, when $x = \frac{n}{2}$.

$\therefore$ Maximum number of edges required $= f\left(\frac{n}{2}\right)$

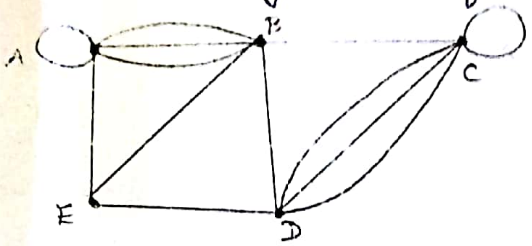
$$= \frac{n}{2}\left(n - \frac{n}{2}\right)$$

$$= \frac{n^2}{4}$$

(ie) The number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Problems

Verify the handshaking theorem for the following graph



Sol

deg (A) = 6, deg (B) = 6, deg (C) = 6, deg (D) = 5,
deg (E) = 3

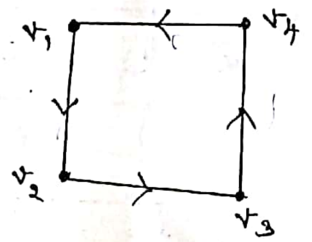
No of edges = 13

$\sum_i \deg(v_i) = 2 \times$ No. of edges

Hence the theorem is verified.

(2) Show that the sum of the indegrees of all the vertices of a simple digraph G is equal to the sum of the outdegrees of all its vertices and that this sum is equal to the number of edges of the graph.

Sol



Indegree	Outdegree
$\deg^-(v_1) = 1$	$\deg^+(v_1) = 1$
$\deg^-(v_2) = 1$	$\deg^+(v_2) = 1$
$\deg^-(v_3) = 1$	$\deg^+(v_3) = 1$
$\deg^-(v_4) = 1$	$\deg^+(v_4) = 1$

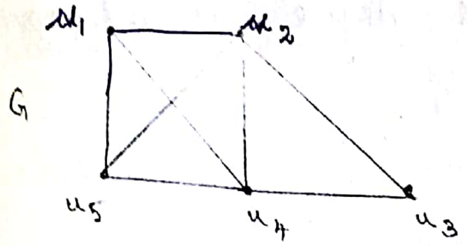
Sum of indegrees of all vertices = Sum of outdegrees of all vertices

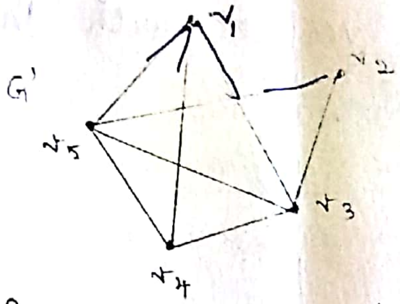
4 = 4

Also No. of edges = 4

Hence
Sum of indegrees = Sum of outdegrees = No. of edges

(3) Determine whether the following pairs of graphs are isomorphic.





Sol There are 5 vertices and 8 edges in G and G'.

$\deg(u_1) = 3$
$\deg(u_2) = 4$
$\deg(u_3) = 2$
$\deg(u_4) = 4$
$\deg(u_5) = 3$

$\deg(v_1) = 3$
$\deg(v_2) = 2$
$\deg(v_3) = 4$
$\deg(v_4) = 3$
$\deg(v_5) = 4$

(i) $|V(G)| = |V(G')|$
(ii) $|E(G)| = |E(G')|$

Adjacency matrix of G

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

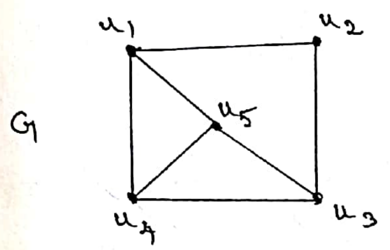
Adjacency matrix of G'

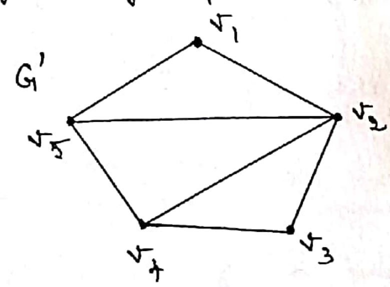
$$\begin{array}{c} \\ v_1 \\ v_5 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_5 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$\therefore$ G and G' are isomorphic.

(7) Determine whether the following pairs of graphs are isomorphic.

U.Q.





Sol

$|V(G)| = |V(G')| = 5$

$|E(G)| = |E(G')| = 7$

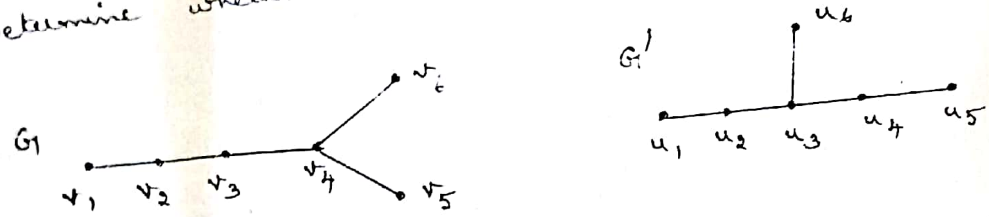
$\deg(u_1) = 3$, $\deg(u_2) = 2$, $\deg(u_3) = 3$, $\deg(u_4) = 3$,
$\deg(u_5) = 3$.
$\deg(v_1) = 2$; $\deg(v_2) = 4$, $\deg(v_3) = 2$, $\deg(v_4) = 3$,
$\deg(v_5) = 3$

No. of vertices with the same degree is not same in G and G'.

∴ G and G' are not isomorphic.

∴ G and G' are isomorphic.

⑤ Determine whether G and G' are isomorphic.



Sol $|V(G)| = 6$, $|V(G')| = 6$

$|E(G)| = 5$, $|E(G')| = 5$

$deg(v_1) = 1$, $deg(v_2) = 2$, $deg(v_3) = 2$, $deg(v_4) = 3$,

$deg(v_5) = 1$, $deg(v_6) = 1$

$deg(u_1) = 1$, $deg(u_2) = 2$, $deg(u_3) = 3$, $deg(u_4) = 2$,

$deg(u_5) = 1$, $deg(u_6) = 1$

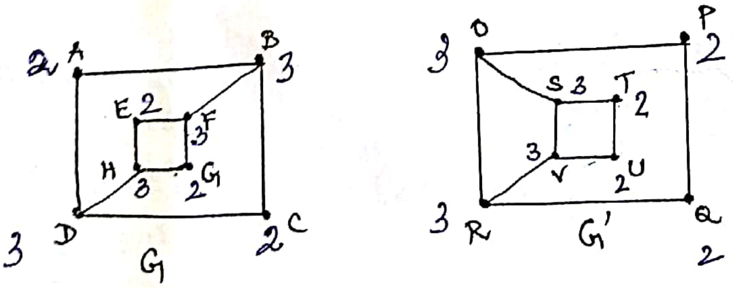
Under isomorphism, v_4 must correspond to u_3.

v_1, v_5, v_6 must correspond to u_1, u_5, u_6 in some order

The vertices v_2 and v_3 are adjacent in G, whereas u_2 and u_4 are not adjacent in G'.

∴ G and G' are not isomorphic.

⑥ Determine whether G and G' are isomorphic.



Ans Not isomorphic

Definitions

PATH

A path is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

(2) SIMPLE PATH

If the edges are distinct in a path, then it is called a simple path.

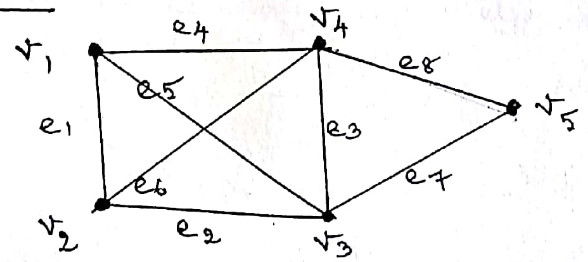
(3) LENGTH OF THE PATH

The number of edges in a path is called the length of the path.

(4) CIRCUIT (or) CYCLE

If the initial and final vertices are same in a path, then the path is called a cycle.

Example



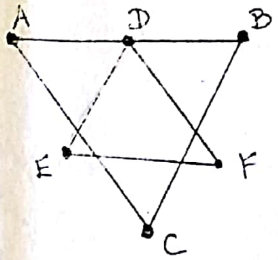
(i) $v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_1\ e_1\ v_2$ is a path of length 4

(ii) $v_1\ e_4\ v_4\ e_6\ v_2\ e_2\ v_3\ e_7\ v_5$ is a simple path of length 4.

(iii) $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_6\ v_2\ e_1\ v_1$ is a circuit of length 5.

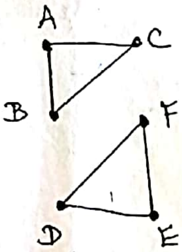
⑤ CONNECTED GRAPH

A graph G is said to be connected if there is a path between every pair of distinct vertices.



⑥ DISCONNECTED GRAPH

A graph that is not connected is called a disconnected graph.



Note

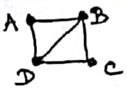
* A disconnected graph is the union of two or more connected subgraphs.
* Each of these connected subgraphs is called a component.

For the above graph, there are 2 components.

⑦ EULERIAN PATH

A path of a graph G is called an Eulerian path, if it includes each edge of G exactly once.

→ eg



B – D – C – B – A – D is an Eulerian path.

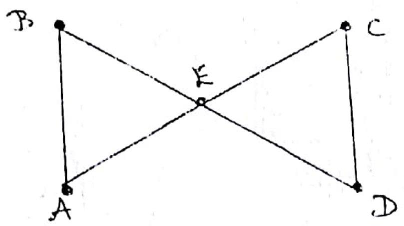
⑧ EULERIAN CIRCUIT

A circuit of a graph G is called an Eulerian circuit, if it includes each edge of G exactly once.

9. EULERIAN GRAPH

A graph containing an Eulerian circuit is called an Eulerian graph.

Graph G

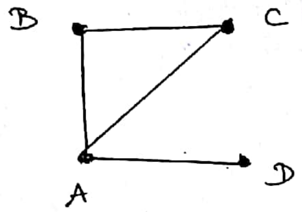


A – E – C – D – E – B – A is an Eulerian circuit.

The graph G is called an Eulerian graph.

10. HAMILTONIAN PATH

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.



D – A – B – C is a Hamiltonian path.

11. HAMILTONIAN CIRCUIT

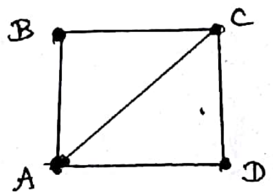
A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and end vertices which appear twice.

U.Q.

12. HAMILTONIAN GRAPH

A graph containing a Hamiltonian circuit is called a Hamiltonian graph.

Graph G



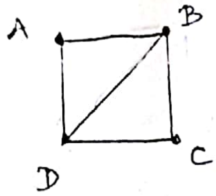
A – B – C – D – A is a Hamiltonian circuit.

The graph G is called a Hamiltonian graph.

Problems

1.Q ① When does a connected graph contain an Euler path? Give an example.

A connected graph contains an Euler path iff it has exactly two vertices of odd degree.

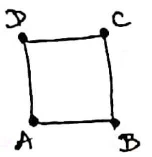


B – D – C – B – A – D is an Euler path

Note The Euler path will have the odd degree vertices as its end points.

Q. ② When does a connected graph contain an Eulerian circuit? Give an example.

A connected graph contains an Eulerian circuit iff each of its vertices is of even degree.



A – B – C – D – A is an Eulerian circuit and each vertex is of degree 2.

1.Q. ③ Give an example of a graph which contains

(i) an Eulerian circuit that is also a Hamiltonian circuit

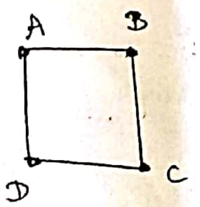
(ii) an Eulerian circuit and a Hamiltonian circuit that are distinct

(iii) an Eulerian circuit, but not a Hamiltonian circuit

(iv) a Hamiltonian circuit, but not an Eulerian circuit

(v) neither an Eulerian circuit nor a Hamiltonian circuit

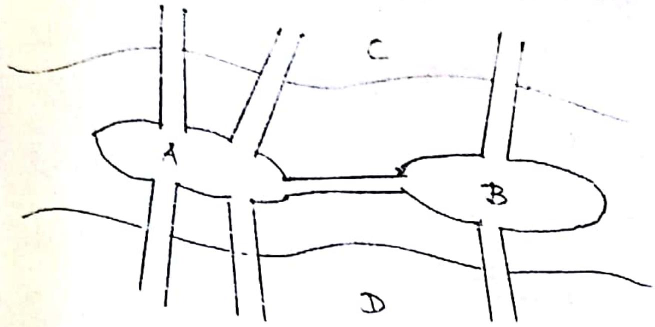
Sol (i)



The circuit A – B – C – D – A consists of all edges and all vertices of G exactly once.

... cond on pg 9.1

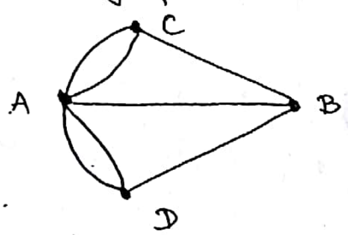
Soln



There are two islands A and B formed by a river. They are connected to each other and to the river banks C and D by means of 7 bridges.

The problem is to start from any one of the four land areas A, B, C, D, walk across each bridge exactly once and return to the same starting point.

When the situation is represented by a graph with vertices representing the land areas and the edges representing the bridges, then the graph will be



The problem is the same as that of drawing the graph without lifting the pen from the paper and without retracing any line.

(ie) The problem is to find an Eulerian circuit in the graph.

W.K.T. "A connected graph has an Eulerian circuit iff each of its vertices is of even degree".

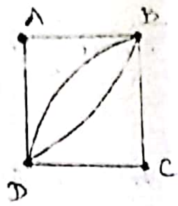
Here, all the vertices are of odd degree.

Hence there is no Eulerian circuit and there is no solution for Konisberg bridge problem.

9.1

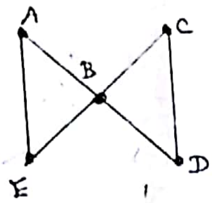
∴ G contains a circuit that is both Eulerian and Hamiltonian.

(ii)



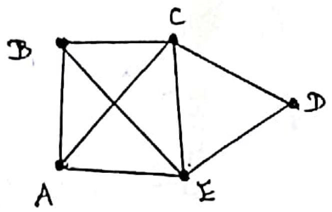
A – B – D – B – C – D – A is an Eulerian circ
A – B – C – D – A is a Hamiltonian circuit.
Both the circuits are different.

(iii)



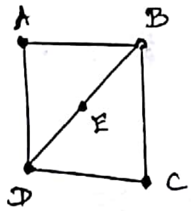
A – B – C – D – B – E – A is an Eulerian circuit
But it is not a Hamiltonian circuit, since the vertex B is repeated twice.

(iv)



A – B – C – D – E – A is a Hamiltonian circ
But it is not an Eulerian circuit, since it does not contain all edges of G

(v)



"A connected graph contains an Eulerian circuit iff each of its vertices is of even degree".

Here degree of B = degree of D = 3

∴ There is no Eulerian circuit in it.

Also, no circuit passes through each of the vertices exactly once. Hence there is no Hamiltonian circuit in G.

S.Q. (4) Explain KONISBERG BRIDGE PROBLEM. Represent the problem by means of graph. Does the problem have a solution?

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Theorem

The maximum number of edges in a simple disconnected graph G with n vertices and k components is

$$\frac{(n-k)(n-k+1)}{2}.$$

Proof:

Let $n_1, n_2, \ldots, n_k$ be the number of vertices in each of k components of the graph G.

Then

$$n_1 + n_2 + \ldots + n_k = n \quad \text{and each } n_i \geq 1. \quad \text{— (1)}$$

$$\text{(ie)} \quad \sum_{i=1}^{k} n_i = n$$

Hence,

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k$$

Now

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} (n_i - 1)^2 + 2\sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk \quad \text{— (2)}$$

$$\text{(ie)} \quad \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

($\because$ The second member in the left side of (2) is ≥ 0, since each $n_i \geq 1$.)

$$\text{(ie)} \quad \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 + k^2 - 2nk$$

$$\text{(ie)} \quad \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq n^2 + k^2 - 2nk$$

$$\text{(ie)} \quad \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k \quad \text{— (3)}$$

WKT "The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$"

Since G is a simple graph, the maximum number of edges in the i th component of $G = \frac{n_i(n_i-1)}{2}$.

$\therefore$ Maximum number of edges of G

$$= \frac{1}{2} \sum_{i=1}^{k} n_i(n_i-1)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \quad \text{By } (1)$$

$$\leq \frac{1}{2}\left[n^2 + k^2 - 2nk + 2n - k - n\right] \quad \text{By } (3)$$

$$\leq \frac{1}{2}\left[(n-k)^2 + n - k\right]$$

$$\leq \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\leq \frac{1}{2}(n-k)(n-k+1)$$

$\therefore$ The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Thm If a graph G (either connected or not) has exactly two vertices of odd degree, then there is a path joining these two vertices.

Pf Case (i) G is connected.

Let v_1, v_2 be the only vertices of G, which are of odd degree.

W.K.T. the number of odd degree vertices is even.

Clearly there is a path connecting v_1 and v_2, since G is connected.

Case (ii) G is disconnected.

Then the components of G are con... Hence v_1 and v_2 should belong to the same component of G.

Hence there is a path between v_1 and v_2.

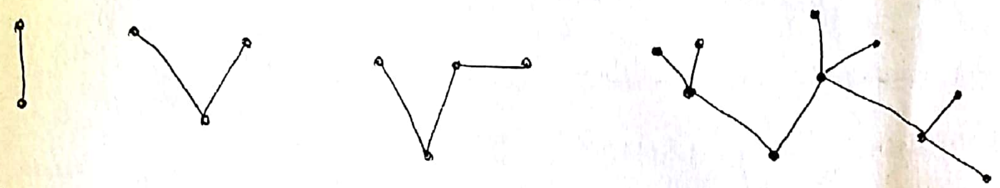
Definitions

TREE

A connected graph without any circuits is called a tree.

(ie) A tree is a simple graph with no loops and parallel edges.

Example

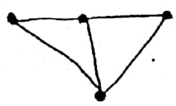


V.Q

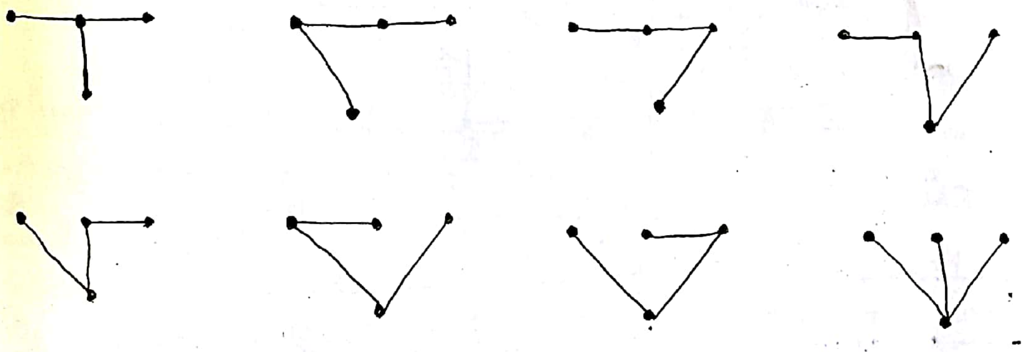
SPANNING TREE

If the subgraph T of a connected graph G is a tree containing all the vertices of G, then T is called a spanning tree of G.

Example Graph G



Spanning trees of G



MINIMUM SPANNING TREE

If G is a connected weighted graph, the spanning tree of G with the smallest total weight is called the minimum spanning tree of G.

KRUSKAL'S ALGORITHM

(i) The edges of the graph G are arranged in the order of increasing weights.

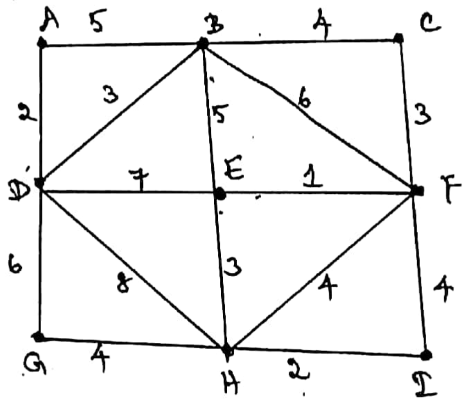
(ii) An edge G with minimum weight is selected as an edge of the required spanning tree.

(iii) Edges with minimum weight that do not form a circuit with the already selected edges are successively added.

(iv) The procedure is stopped after $(n-1)$ edges have been selected.

Problems

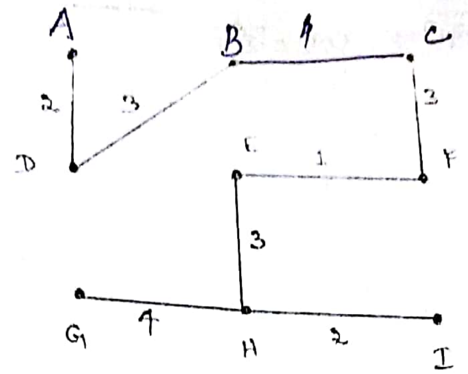
① Use Kruskal's algorithm, to find a minimum spanning tree for the weighted graph.



Sol

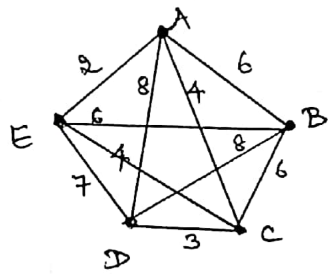
$n-1$ edges
$= 9-1$
$= 8$ edges

Edge	Weight	Selected
FE	1	YES
IH	2	YES
DA	2	YES
CF	3	YES
DB	3	YES
HE	3	YES
BC	4	YES
FI	4	NO
FH	4	NO
GH	4	YES
BE	5	–
AB	5	–
DG	6	–
BF	6	–
DE	7	–
DH	8	–



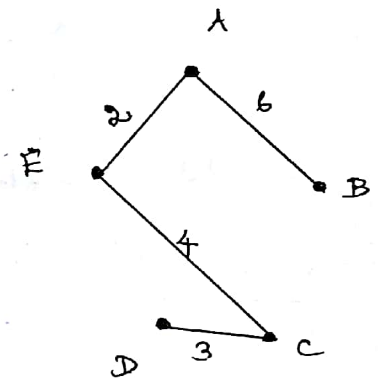
The total length of the minimum spanning tree = 22

Find the minimum spanning tree for the weighted graph by using Kruskal's algorithm.



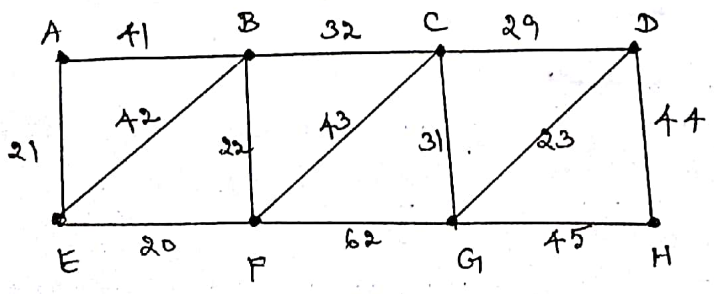
Sol

Edge	Weight	Selected
AE	2	YES
DC	3	YES
EC	4	YES
AC	4	NO
AB	6	YES
EB	6	–
BC	6	–
ED	7	–
BD	8	–
AD	8	–

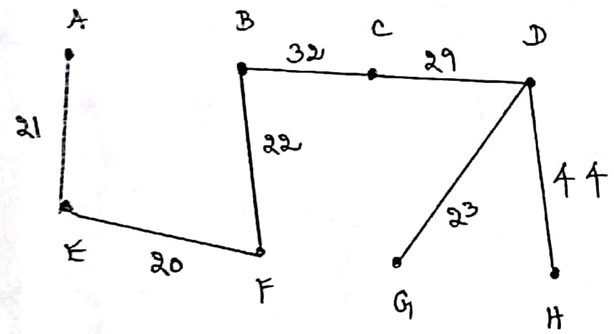


The total length of the minimum spanning tree = 15

Find the minimum spanning tree for the weighted graph by using Kruskal's algorithm.



Edge	weight	selected
EF	20	YES
AE	21	YES
BF	22	YES
GD	23	YES
CD	29	YES
CG	31	NO
BC	32	YES
AB	41	NO
BE	42	NO
CF	43	NO
DH	44	YES
GH	45	–
FG	62	–



The total length of the minimum spanning tree = 191

Theorem – 1

R.U.Q. An undirected graph is a tree iff there is a unique simple path between every pair of vertices.

Proof

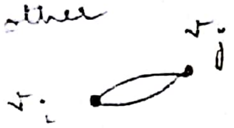
Let T be an undirected tree.

To prove There is a unique simple path between every pair of vertices.

By definition of a tree, T is connected. Hence, there is a simple path between any pair of vertices, say v_i and v_j.

<u>To prove</u> The path is ~~unequal~~ unique

If possible, let there be two paths bet... v_i and v_j – one from v_i to v_j and the other from v_j to v_i.



Then the combination of these two paths forms a circuit.

But by definition of T, a tree T will not have a circuit.

Hence, there is a unique simple path between every pair of vertices in T.

Conversely, there is a unique simple path between every pair of vertices in T.

Then, T is connected.

<u>To prove</u> T is a tree.

If possible, let T contain a circuit.

Then there is a pair of vertices v_i and v_j between which two distinct paths exist, which is a contradiction to our assumption.

Hence T cannot have a circuit.

$\therefore$ T is a tree.

<u>Property – 2</u>

A tree with n vertices has $(n-1)$ edges.

<u>Proof</u>

Let T be a tree with n vertices.

We prove this theorem by the method of mathematical induction on vertices.

Let $n = 1$.

(ie) T is a tree with only one vertex.

(ie) T has no edge.

(ie) T has $(1-1)$ edges.

$\therefore$ The theorem is true for $n = 1$.

Assume that the theorem is true for $n = K-1$ vertices.

(ie) A tree with $K-1$ vertices has $(K-1)-1 = K-2$ edges.

To prove The theorem is true for $n = K$.

Let T be a tree with K vertices.

Let e_K be the edge connecting the vertices v_i and v_j of T.

Then, by Property - 1, e_K is the only path between v_i and v_j.

If we delete the edge e_K from T, T becomes disconnected and $(T - e_K)$ consists of exactly two components, say, T_1 and T_2 which are connected.

Since T did not contain any circuit, T_1 and T_2 also will not have circuits.

Hence, both T_1 and T_2 are trees, each having less than K vertices, say r and $K-r$ respectively.

$\therefore$ By the principle of mathematical induction, T_1 has $(r-1)$ edges and T_2 has $(K-r-1)$ edges.

$\therefore$ T has $(r-1) + (K-r-1) + 1 = K-1$ edges.

Thus, a tree with n vertices has $(n-1)$ edges.

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