UNIT-I - MATHEMATICAL LOGIC

PART-B

① Show that $(\neg P\wedge(P\to q))\to\neg P$ is a tautology.

P	q	$P\to q$	$\neg P$	$\neg P\wedge(P\to q)$	$(\neg P\wedge(P\to q))\to\neg P$
T	T	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since last column contains only truth value T, it is a tautology.

② without using truth table, prove the following:

$(P\wedge(P\leftrightarrow Q))\to Q = T$

$P\wedge(P\Leftrightarrow Q)\to Q \Rightarrow [P\wedge((\neg P\vee Q)\wedge(\neg Q\vee P))]\to Q$

$\Leftrightarrow (P\wedge(\neg Q\vee P))\wedge(\neg P\vee Q))\to Q$

$\Leftrightarrow [P\wedge(\neg P\vee Q)]\to Q$

$\Leftrightarrow [(P\wedge\neg P)\vee(P\wedge Q)]\to Q \quad \because P\wedge(\neg Q\vee P) = P.$

$\Leftrightarrow [F\vee(P\wedge Q)]\to Q$

$\Leftrightarrow P\wedge Q\to Q$

$\Leftrightarrow \neg(P\wedge Q)\vee Q$

$\Leftrightarrow (\neg P\vee\neg Q)\vee Q$

$\Leftrightarrow \neg P\vee(\neg Q\vee Q)\Leftrightarrow \neg P\vee T$

$\Leftrightarrow T$

③ Construct the truth table for the following compound proposition: $(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$

P	Q	R	$\neg P$	$\neg Q$	$\neg P \leftrightarrow \neg Q$	$Q \leftrightarrow R$	$(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

④ Prove that $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the last column contains only T, it is a tautology.

⑤ Prove: $(p \vee q) \to r \equiv (p \to r) \wedge (q \to R)$

$$(p \to r) \wedge (q \to r) \Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$$
$$\Leftrightarrow (\neg p \wedge \neg q) \vee r$$
$$\Leftrightarrow \neg (p \vee q) \vee r$$
$$\Leftrightarrow p \vee q \to r.$$

(16) Prove that $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$ without using truth table.

$$P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$$
$$\Leftrightarrow \neg P \vee (\neg Q \vee R)$$
$$\Leftrightarrow (\neg P \vee \neg Q) \vee R$$
$$\Leftrightarrow \neg (P \wedge Q) \vee R \Leftrightarrow (P \wedge Q) \to R$$

Hence proved.

(7) Show that $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

(8) Construct the truth table for $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$.

p	q	r	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow r$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

⑨ Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$

$\Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$

$\Leftrightarrow T \wedge R \Leftrightarrow R$

⑩ Show that $(P \to Q) \to Q \Rightarrow P \vee Q$

$(P \to Q) \to Q \Rightarrow (\neg P \vee Q) \to Q$

$\Rightarrow \neg(\neg P \vee Q) \vee Q$

$\Rightarrow (P \wedge \neg Q) \vee Q$

$\Rightarrow (P \vee Q) \wedge (\neg Q \vee Q)$

$\Rightarrow (P \vee Q) \wedge T$

$\Rightarrow P \vee Q$

⑪ Derive by using CP rule, $P \to (Q \to S)$ from $P \to (Q \to R)$, $Q \to (R \to S)$.

Step No.	Statement	Reason
1.	P	P (additional)
2.	$P \to (Q \to R)$	P
3.	$Q \to R$	T, 1, 2 and modus ponens
4.	$\neg Q \vee R$	T, 3
5.	$Q \to (R \to S)$	P
6.	$\neg Q \vee (R \to S)$	T, 5
7.	$\neg Q \vee (R \wedge (R \to S))$	T, 4, 6
8.	$\neg Q \vee S$	T, 7
9.	$Q \to S$	T, 8
10.	$P \to (Q \to S)$	T, 9 and CP rule.

(12) Show that $R \vee S$ is a valid conclusion from the premises $C \vee D$, $C \vee D \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$

Step No.	Statement	Reason
1.	$C \vee D \to \neg H$	P
2.	$\neg H \to (A \wedge \neg B)$	P
3.	$C \vee D \to (A \wedge \neg B)$	T, 1,2.
4.	$C \vee D$	P
5.	$A \wedge \neg B$	T, 3, 4
6.	$(A \wedge \neg B) \to (R \vee S)$	P
7.	$R \vee S$	T, 5,6.

(13) Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$

Answer Same as Qn (12).

In (12) replace C by p   R by a
D by q   S by b.
H by r
A by s
B by t

(14) Prove $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$ by proving the equivalence of dual.

$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

ie) $\neg (p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$

Dual of the equivalence is

$\neg (p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r)$.

S R M UNIVERSITY

L.H.S $= (\neg p \vee \neg q) \wedge r$, by Demorgan's law.

$\equiv (\neg p \wedge r) \vee (\neg q \wedge r)$, by Distributive law

$\equiv$ R.H.S.

(15) Prove: $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$. by proving the equivalence of dual.

The dual of the given equivalence is

$$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p.$$

L.H.S. $\equiv \neg((\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$. (by distributive law)

$\equiv \neg((\neg p \vee F) \wedge (p \vee q)$, by complement law.

$\equiv \neg(\neg p) \wedge (p \vee q)$, by identity law.

$\equiv p \wedge (p \vee q)$

$\equiv p$ (by absorption law).

(16). Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \rightarrow \neg r$, $\neg r \rightarrow (s \wedge \neg t)$ and $(s \wedge \neg t) \rightarrow (a \vee b)$.

Step ①	Statement	Reason.
(1)	$(p \vee q) \rightarrow \neg r$	P.
(2)	$\neg r \rightarrow (s \wedge \neg t)$	P.
(3)	$(p \vee q) \rightarrow (s \wedge \neg t)$	T (1,2)
(4)	$(s \wedge \neg t) \rightarrow a \vee b$	P.
(5)	$p \vee q \rightarrow a \vee b$	T (3,4)
(6)	$p \vee q$	P
(7)	$a \vee b$	T (5,6).

PART-C

1. without using truth table, prove that
$(\neg P \vee q) \wedge (P \wedge (P \wedge q)) \equiv P \wedge q$.

$(\neg P \vee q) \wedge (P \wedge (P \wedge q)) \equiv (\neg P \vee q) \wedge (P \wedge P) \wedge q$ (by Associative law)
$\equiv (\neg P \vee q) \wedge (P \wedge q)$ (by Idempotent law)
$\equiv (P \wedge q) \wedge (\neg P \vee q)$ (commutative law)
$\equiv ((P \wedge q) \wedge \neg P) \vee ((P \wedge q) \wedge q)$ (Distributive law)
$\equiv (\neg P \wedge (P \wedge q)) \vee (P \wedge (q \wedge q))$ (Commutative law)
$\equiv ((\neg P \wedge P) \wedge q) \vee (P \wedge q)$
$\equiv (F \wedge q) \vee (P \wedge q)$
$\equiv F \vee (P \wedge q)$
$\equiv P \wedge q$ (Dominant law).

2. using truth table, show that $((P \vee q) \wedge (P \to r) \wedge (q \to r)) \to r$ is a tautology.

P	q	r	P ∨ q	P → r	q → r	(P ∨ q) ∧ (P → r) ∧ (q → r) (A)	A → r
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Since last column contains only T, it is a tautology.

③ Show that $(P \to Q) \wedge (R \to S)$, $(Q \to M) \wedge (S \to N)$, $\neg(M \wedge N)$ and $(P \to R) \Rightarrow \neg P$

Step	Statement	Reason
(1)	$(P \to Q) \wedge (R \to S)$	Rule P
(2)	$P \to Q$	Rule T $(P \wedge Q \Rightarrow P)$
(3)	$R \to S$	Rule T $(P \wedge Q \Rightarrow Q)$
(4)	$(Q \to M) \to$	Rule P.
(5)	$Q \to M$	Rule T
(6)	$S \to N$	Rule T
(7)	$P \to M$	Rule T (1, 4)
(8)	$R \to N$	Rule T (1, 4)
(9)	$P \to R$	Rule P
(10)	$P \to N$	Rule T (9, 8)
(11)	$\neg N \to \neg P$	Rule T
(12)	$\neg M \to \neg P$	Rule T (7)
(13)	$(\neg M \vee \neg N) \to \neg P$	Rule T
(14)	$\neg(M \wedge N) \to \neg P$	Rule T
(15)	$\neg(M \wedge N)$	Rule P.
(16)	$\neg P$	Rule T (15, 14)

④ without using truth table prove that

$\neg P \to (q \to r) \equiv q \to (p \vee r)$.

$$\begin{aligned}
\neg P \to (q \to r) &\Rightarrow \neg P \to (\neg q \vee r) \\
&\Rightarrow (P \vee \neg q) \vee r \\
&\Rightarrow (\neg q \vee P) \vee r \\
&\Rightarrow \neg q \vee (p \vee r) \\
&\Rightarrow q \to (p \vee r)
\end{aligned}$$

(5) Show that $(S\to \neg Q,\ S\vee R,\ \neg R,\ \neg R \leftrightarrow Q) \Rightarrow \neg P$. by indirect method.
$r\to \neg q,\ r\vee s,\ s\to \neg q,\ p\to q \Rightarrow \neg P$

To use the indirect method, we shall include $\neg\neg P \equiv P$ as an additional premise and prove a contradiction.

Step	Statement	Reason
(1)	P	P. (additional)
(2)	$P\to q$	P
(3)	q	T (1,2)
(4)	$r\to \neg q$	P
(5)	$S\to \neg q$	P
(6)	$(r\vee s)\to \neg q$	T (4,5)
(7)	$r\vee s$	P
(8)	$\neg q$	T (6,7)
(9)	$q\wedge \neg q$	T (3,8)
(10)	F	T

(9) Prove the following equivalence by proving the equivalences of dual. $(P\wedge(P\leftrightarrow q))\to q \equiv T$.

ie) $P\wedge((P\to q)\wedge(q\to p))\to q \equiv T$

$P\wedge((\neg P\vee q)\wedge(\neg q\vee P))\to q \equiv T$

ie) $\neg(P\wedge((\neg P\vee q)\wedge(\neg q\vee p)))\vee q \equiv T$

$\therefore$ Dual of equivalences is

$\neg(P\vee((\neg P\wedge q)\vee(\neg q\wedge p)))\wedge q \equiv F$

L.H.S $\equiv \neg[(P\vee(\neg P\wedge q))\vee(\neg q\wedge p)]\wedge q$ (by associative law)

$\equiv \neg[(T\wedge(P\vee q))\vee(\neg q\wedge p)]\wedge q$ (by distributive and complement laws)

$\equiv \neg[(p\vee q)\vee(\neg q\wedge p)]\wedge q$ (identity law)

$\equiv \neg[((p\vee q)\vee\neg q)\wedge((p\vee q)\vee p)]\wedge q$ (Distributive law)

$\equiv \neg[(p\vee T)\wedge(p\vee q)]\wedge q$

$\equiv \neg(T\wedge(p\vee q))\wedge q$ (dominant law)

$\equiv \neg(p\vee q)\wedge q$ (by Demorgan's law)

$\equiv \neg p\wedge F$ (by complement law)

$\equiv F$ (by dominant law).

(7) Show that $(p\to q)\wedge(r\to s)$, $(q\to t)\wedge(s\to u)$, $\neg(t\wedge u)$ and $(p\to r) \Rightarrow \neg p$.

Step	Statement	Reason
(1)	$(p\to q)\wedge(r\to s)$	P
(2)	$p\to q$	T (1, Simplification)
(3)	$r\to s$	T (1)
(4)	$(q\to t)\wedge(s\to u)$	P
(5)	$q\to t$	T (4)
(6)	$s\to u$	T (5)
(7)	$p\to t$	T (2, 5)
(8)	$r\to u$	T (3, 6)
(9)	$p\to r$	P
(10)	$p\to u$	T (9, 8)
(11)	$\neg t\to\neg p$	T (7)
(12)	$\neg u\to\neg p$	T (10)
(13)	$(\neg t\to\neg p)\to\neg p$	T (11, 12)
(14)	$\neg(t\wedge u)\to\neg p$	T (13)
(15)	$\neg(t\wedge u)$	P
(16)	$\neg p$	T (14, 15 Modus Ponens)

(8) Show that $(a \to b) \wedge (a \to c)$, $\neg(b \wedge c)$, $(d \vee a) \Rightarrow d$

Step	Statement	Reason
(1)	$(a \to b) \wedge (a \to c)$	P
(2)	$a \to b$	T (1)
(3)	$a \to c$	T (1)
(4)	$\neg b \to \neg a$	T (2, Contrapositive)
(5)	$\neg c \to \neg a$	T (3, Contrapositive)
(6)	$(\neg b \vee \neg c) \to \neg a$	T (4,5)
(7)	$\neg(b \wedge c) \to \neg a$	T (Demorgan's law)
(8)	$\neg(b \wedge c)$	P
(9)	$\neg a$	T (7,8)
(10)	$d \vee a$	P
(11)	$(d \vee a) \wedge \neg a$	T (9,10)
(12)	$(d \wedge \neg a) \vee (a \wedge \neg a)$	T (11)
(13)	$(d \wedge \neg a) \vee F$	T (12)
(14)	$d \wedge \neg a$	T (13)
(15)	d	T (14).

(9) Give a direct proof for the implication
$p \to (q \to s)$, $(\neg r \vee p)$, $q \Rightarrow r \to s$.

Step	Statement	Reason
(1)	$\neg r \vee p$	P.
(2)	$r \to p$	T (1)
(3)	$p \to (q \to s)$	P
(4)	$r \to (q \to s)$	T (2,3)
(5)	$\neg r \vee (\neg q \vee s)$	T (4).

Step	Statement	Reason
(6)	q	P
(7)	$q \wedge (\neg r \vee \neg q \vee s)$	T (5,6)
(8)	$q \wedge (\neg r \vee s)$	T (7,8)
(9)	$\neg r \vee s$	T (8)
(10)	$r \to s$	T (9)

(10) Derive $p \to (q \to s)$ using the CP rule (if necessary) from the premises $p \to (q \to r)$ and $q \to (r \to s)$.

We shall assume P as an additional premise.

Step	Statement	Reason
(1)	P	Rule P (Additional)
(2)	$p \to (q \to r)$	P
(3)	$q \to r$	T (1,2)
(4)	$\neg q \vee r$	T (3)
(5)	$q \to (r \to s)$	P
(6)	$\neg q \vee (r \to s)$	T (5)
(7)	$\neg q \vee s$	T, 7 (Modus Ponens)
(8)	$q \to s$	T (8)
(9)	$p \to (q \to s)$	T (9) CP rule.
(10)		

(11) Use the indirect method to show that

$r \to \neg p$, $r \vee s$, $s \to \neg q$, $p \to q \Rightarrow \neg p$

To use indirect method, we will include $\neg\neg p \equiv p$ as an additional premise and prove a contradiction.

Step	Statement	Reason
(1)	p	Rule P (Additional)
(2)	$p \to q$	P
(3)	q	T (1,2)
(4)	$r \to \neg q$	P
(5)	$s \to \neg q$	P
(6)	$(r \vee s) \to \neg q$	T (4,5)
(7)	$r \vee s$	P
(8)	$\neg q$	T (6,7)
(9)	$q \wedge \neg q$	T (3,8)
(10)	F	T (9).

(12) Show that b can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, d by the indirect method.

Let us include $\neg b$ as an additional premise and prove a contradiction.

Step	Statement	Reason
(1)	$a \to b$	P
(2)	$c \to b$	P
(3)	$(a \vee c) \to b$	T (1,2)
(4)	$d \to (a \vee c)$	P
(5)	$d \to b$	T (3,4)
(6)	d	P
(7)	b	T (5&6)
(8)	$\neg b$	P (additional)
(9)	$b \wedge \neg b$	T (7,8)
(10)	F	T (9).

(13) using indirect method of Proof, derive $p \to \neg s$ from the premise $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, p.

Let us include $\neg(p \to \neg s)$ as an additional premise. and prove a contradiction.

ie) $\neg(p \to \neg s) \Rightarrow \neg(\neg p \vee \neg s) = p \wedge s$

Hence the additional premise is $p \wedge s$

Step	Statement	Reason
(1)	$p \to (q \vee r)$	P
(2)	p	P
(3)	$q \vee r$	T (1,2)
(4)	$p \wedge s$	P (additional)
(5)	s	T (4)
(6)	$s \to \neg r$	P
(7)	$\neg r$	T (5,6)
(8)	q	T (3,7) Disjunctive syllogism
(9)	$q \to \neg p$	P
(10)	$\neg p$	T (8,9)
(11)	$p \wedge \neg p$	T (2, 10)
(12)	F	T (11)

(14) Prove that the premises are inconsistent.

$p \to q$, $q \to r$, $s \to \neg r$ & $p \wedge s$

Step	Statement	Reason
(1)	$p \to q$	P
(2)	$q \to r$	P
(3)	$p \to r$	T (1,2)
(4)	$s \to \neg r$	P

Step	Statement	Reason
(5)	$r \to \neg s$	T (4) contrapositive
(6)	$q \to \neg s$	T (2,5)
(7)	$\neg q \vee \neg s$	T (6)
(8)	$\neg(q \wedge s)$	T (7)
(9)	$q \wedge s$	P
(10)	$(q \wedge s) \wedge \neg(q \wedge s)$	T (8,9)
(11)	F	T, (10)

(15) Prove that the premise $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

Step	Statement	Reason
(1)	$a \wedge d$	P
(2)	a	T (1)
(3)	d	T (1)
(4)	$a \to (b \to c)$	P
(5)	$b \to c$	T (2,4)
(6)	$\neg b \vee c$	T (5)
(7)	$d \to (b \wedge \neg c)$	P
(8)	$\neg(b \wedge \neg c) \to \neg d$	T (7, contra positive)
(9)	$\neg b \vee c \to \neg d$	(T, 8)
(10)	$\neg d$	(T, 6, 9)
(11)	$d \wedge \neg D$	(T, 3, 10)
(12)	F	T, 11.

Hence the given premises are inconsistent.

16) Show that the following premises are inconsistent.

(i) If Jack misses many classes through illness, then he fails high School.

(ii) If Jack fails high school, then he is uneducated.

(iii) If Jack reads a lot of books, then he is not uneducated.

iv) Jack misses many classes through illness and reads a lot of books.

Let P : Jack misses many classes.
Q : Jack fails high School.
R : Jack read a lot of books.
S : Jack is uneducated.

The premises are

$P \to Q$, $Q \to S$, $R \to \neg S$ and $P \wedge R$.

Step	Statement	Reason
(1)	$P \to Q$	P
(2)	$Q \to S$	P
(3)	$P \to S$	T(1,2)
(4)	$R \to \neg S$	P
(5)	$S \to \neg R$	T(4, contrapositive)
(6)	$P \to \neg R$	T (3,5)
(7)	$\neg P \vee \neg R$	T (6)
(8)	$\neg(P \wedge R)$	T (7)
(9)	$P \wedge R$	P.
(10)	$(P \wedge R) \wedge \neg(P \wedge R)$	T (8,9)
(11)	F	T (10)

∴ The premises are inconsistent.

(17) Show that the following set of premises is inconsistent.

(i) If Rama gets his degree, he will go for a job.
(ii) If he goes for a job, he will get married soon.
(iii) If he goes for a higher study, he will not get married.
(iv) Rama gets his degree and goes for a higher study.

Let the statements be symbolised as follows:

P : Rama gets his degree.
Q : He will go for a job.
R : He will get married soon.
S : He goes for higher study.

∴ The premises are

$P \to Q$, $Q \to R$, $S \to \neg R$, $P \wedge S$

Step	Statement	Reason
(1)	$P \to Q$	P
(2)	$Q \to R$	P
(3)	$P \to R$	T (1,2).
(4)	$P \wedge S$	P
(5)	P	T (4)
(6)	S	T (4)
(7)	$S \to \neg R$	P
(8)	$\neg R$	T (6,7)
(9)	R	T (5,3)
(10)	$R \wedge \neg R$	T (8,9)
(11)	F	T

∴ The premises are inconsistent.

(21) construct an argument using rules of inferences to Show that if A work hard then either B or C will enjoy them selves. If B enjoy himself, then A will not work hard. If D enjoy himself then C will not. Therefore, if A works hard, D will not enjoy himself.

Let p : A work hard

q : B enjoy him self

r : C enjoy him self

s : D enjoy him self

The premises are $(P \to \neg S)$

$p \to (q \vee r)$, $q \to \neg p$, $S \to \neg r \Rightarrow P \to \neg S$

we shall assume P as an additional premise (CP rule).

Step	Statement	Reason
(1)	$P \to (q \vee r)$	P
(2)	P	P (additional)
(3)	$q \vee r$	T (1,2)
(4)	$\neg q \to r$	T (3, equivalence)
(5)	$\neg r \to q$	T (4, contrapositive)
(6)	$s \to \neg r$	P
(7)	$s \to q$	T (5, 6)
(8)	$q \to \neg P$	P
(9)	$s \to \neg P$	T (7,8)
(10)	$P \to \neg s$	T (9, contrapositive)

(23) Prove by mathematical induction that

$$1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$$

Let $P(n): 1\cdot2\cdot3 + 2\cdot3\cdot4 \cdots + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$

Now, $P(1): 1\cdot2\cdot3 = 6$

$$\frac{1}{4}(1+1)(1+2)(1+3) = \frac{1}{\cancel{4}} \times 2 \times 3 \times \cancel{4}$$
$$= 6$$

$\therefore$ L.H.S = R.H.S.

P(1) is true.

Assume P(n) is true for n=K.

ie) P(K) is true.

$P(K): 1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots + K(K+1)(K+2) = \frac{K}{4}(K+1)(K+2)(K+3)$

Now, we prove P(n) is true for n=K+1

$$P(K+1) = \underbrace{1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots + K(K+1)(K+2)}_{} + (K+1)(K+2)(K+3)$$

$$= P(K) + (K+1)(K+2)(K+3)$$

$$= \frac{1}{4} K(K+1)(K+2)(K+3) + (K+1)(K+2)(K+3)$$

$$= (K+1)(K+2)(K+3)\left[\frac{K}{4}+1\right]$$

$$= (K+1)(K+2)(K+3)\left(\frac{K+4}{4}\right)$$

$$P(K+1) = \frac{1}{4}(K+1)((K+1)+1)((K+1)+2)((K+1)+3).$$

$\therefore$ P(K+1) is true.

$\Rightarrow$ P(n) is true for all $n \in Z^+$.

(24) use mathematical induction to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, \text{ for } n \geq 2$$

Let $P(n) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

~~P(2)~~ To Prove P(n) is true for n=2

$$L.H.S = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707$$

$$R.H.S = \sqrt{2} = 1.414$$

$\therefore$ L.H.S > R.H.S.

$\Rightarrow$ P(1) is true.

Assume P(n) is true for n=K.

ie) P(K) is true.

$$P(K) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}} > \sqrt{K} \quad —(1)$$

To Prove P(n) is true for n=K+1

$$P(K+1) = \underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}}} + \frac{1}{\sqrt{K+1}}$$

$$= P(K) + \frac{1}{\sqrt{K+1}}$$

$$> \sqrt{K} + \frac{1}{\sqrt{K+1}} \quad \text{by (1)}$$

Now $\sqrt{K} + \frac{1}{\sqrt{K+1}} = \frac{\sqrt{K(K+1)} + 1}{\sqrt{K+1}}$

$$> \frac{\sqrt{K}\,\sqrt{K} + 1}{\sqrt{K+1}}$$

$$> \frac{K+1}{\sqrt{K+1}} > \sqrt{K+1}$$

∴ P(K+1) is true.

⇒ P(n) is true for all $n \in Z^+$

(25) Use mathematical induction to prove that $3^n + 7^n - 2$ is divisible by 8, for $n \geq 1$.

Let $P(n) : 3^n + 7^n - 2$ is divisible by 8.

To Prove P(1) is true

$P(1) = 3^1 + 7^1 - 2 = 8$ = divisible by 8.

∴ P(1) is true.

Assume P(n) is true for n=K.

ie) P(K) is divisible by 8.

$3^K + 7^K - 2$ is divisible by 8.

$\Rightarrow 3^K + 7^K - 2 = 8t$ (say) —(1) (multiple of 8)

t → constant

$P(K+1) = 3^{K+1} + 7^{K+1} - 2$

$= 3 \cdot 3^K + 7 \cdot 7^K - 2$

$= 3(8t - 7^K + 2) + 7 \cdot 7^K - 2$

$= 24t - 3 \cdot 7^K + 6 + 7 \cdot 7^K - 2$

$= 24t + 4 \cdot 7^K + 4$

$= 8\left(3t + \frac{7^K + 1}{2}\right)$.

= multiple of 8 = divisible by 8.

$\therefore P(K+1)$ is true.

$\therefore P(n)$ is true for all $n \in Z^+$

(26) Use mathematical induction to show that $n! \geq 2^{n-1}$, for $n = 1, 2, 3 \cdots$

Let $P(n) : n! \geq 2^{n-1}$

<u>To Prove P(1) is true:</u>

$P(1) : 1! \geq 2^{1-1}$

$1 \geq 2^0$

$\therefore 1 \geq 1$ which is true.

Assume $P(n)$ is true for $n = K$.

$P(K) : K! \geq 2^{K-1}$ —— (1)

<u>To Prove $P(n)$ is true for $n = K+1$</u>

$P(K+1) : (K+1)! = (K+1) \cdot K!$

$\geq (K+1) 2^{K-1}$ by (1)

$\geq 2 \cdot 2^{K-1}$, Since $K+1 \geq 2$

$\geq 2^{(K+1)-1}$

$\therefore P(K+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

(27) Prove by mathematical induction, $a^n - b^n$ is divisible by $a-b$.

Let $P(n)$: $a^n - b^n$.

To Prove $P(1)$ is true :

$P(1)$: $a^1 - b^1 = a-b$
= is divisible by $a-b$.

$\therefore P(1)$ is true.

Assume $P(n)$ is true for $n=k$.

$P(k)$: $a^k - b^k$ is divisible by $a-b$.

$\Rightarrow a^k - b^k = (a-b)t$ = multiple of $a-b$.

$$a^k = b^k + (a-b)t \quad \text{--- (1)}$$

To Prove $P(k+1)$ is true

$$P(k+1) = a^{k+1} - b^{k+1}$$
$$= a \cdot a^k - b^{k+1}$$
$$= a(b^k + (a-b)t) - b^{k+1} \quad \text{by (1)},$$
$$= a \cdot b^k + a(a-b)t - b^{k+1}$$
$$= a \cdot b^k + a(a-b)t - b \cdot b^k$$
$$= (a-b) \cdot b^k + a(a-b)t$$
$$= (a-b)(b^k + at)$$

= multiple of $a-b$

= divisible by $a-b$.

$\therefore P(k+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

— x —.