MA8351

DISCRETE MATHEMATICS

Unit-I: Logic and Proofs

Propositional logic - Propositional equivalences - Predicates and quantifiers - Nested quantifiers - Rules of inference - Introduction to proofs - Proof methods and strategy.

Unit - II Combinatorics

Mathematical induction - Strong induction and well ordering - The basics of counting - The pigeonhole principle - Permutations and combinations - Recurrence relations - Solving linear recurrence relations - Generating functions - Inclusion and exclusion principle and its applications.

Unit - III Graphs

Graphs and graph models - Graph terminology and special types of graphs - Matrix representation of graphs and graph isomorphism - Connectivity - Euler and Hamilton paths.

unit - IV Algebraic structures

Algebraic systems - Semi groups and monoids - Groups - Subgroups - Homomorphism's - Normal subgroup and coset - Lagrange's theorem - Definition and examples of rings and fields.

Unit - V : Lattices and Boolean Algebra

Partial ordering - Posets - Lattices and posets - Properties of lattices - Lattices as algebraic systems - Sub lattices - Direct product and homomorphism - Some special lattices - Boolean algebra

Unit - I

Logic and Proofs

Defn: Proposition (Statement)

It is a declarative sentences. That is either true or false but not both.

Example:

1. Chennai is the capital of Tamilnadu (True)

2. $2+7=10$ (False)

3. This statement is false (we can't say true or false)

4. Do you speak English (It is a question not a statement).

Notation:

P, Q, R, S are used to denote proposition

T is a True Proposition

F is a False Proposition

Atomic Statements: (Primary or simple):

Declarative sentences which cannot be ~~father~~ further split into simpler sentences are called atomic statements or primitive statements.

Ex: Ramu is a boy.

Five Basic Connectives:

English Language usages	Logical Connectives	Types of Operator	Symbols
AND	Conjuction	Binary	$\wedge$
OR	Disjunction	Binary	$\vee$
NOT	Negation or Denial	Unary	$\neg$ or $\sim$
If ... then	Implication or Conditional	Binary	$\rightarrow$
If and only if	biconditional	binary	$\leftrightarrow$

Molecular (compound or Composite)

New statements can be formed from atomic statements through the use of connectives such as and, but, or etc. The resulting statements are called molecular (or) compound (or composite statement.

Ex: Leema is a girl and Siva is a boy.

Compound Proposition:

Many mathematical statements are constructed by combining one or more proposition new proposition called compound proposition, are formed from existing proposition using logical operator.

Negation : [$\neg$ or $\sim$] [Not]

The negation of a statement is generally formed by introducing a word "NOT" at a proper place in a statement.

Ex: * P: Today is Monday

$\neg p$: Today is not Monday.

* $P: x < y$

$\neg p: x \geq y$ (or) $x \nless y$

* $p: 3 > 2$ (T)

$\neg p: 3 \leq 2$ (F)

Conjuction [$\wedge$] [AND]:

Ex: * P: It is snowing (T)

Q : I am cold (T)

$P \wedge Q$: It is snowing and I am cold (T)

* $P: 4+3>5$ (T)

$Q: -3<-5$ (F)

$P \wedge Q$:

TRUTH TABLE

AND OR

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P lends to Q

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

iff (if and only if)

$P \leftrightarrow Q$ has exactly the same truth values as $(P \to Q) \wedge (Q \to P)$

Note:

(i) The Proposition $Q \to P$ is called the converse of $P \to Q$

(ii) The Proposition $\neg Q \to \neg P$ is called the contrapositive of $P \to Q$

(iii) The Proposition $\neg P \to \neg Q$ the inverse of $P \to Q$.

Contrapositive:

If $P \to Q$ is an implication, then the converse of $P \to Q$ is the implication $Q \to P$. and the contrapositive of $P \to Q$ is the implication $\neg Q \to \neg P$.

① Give the converse and the contrapositive of the implication "If It is raining, then I get wet" $P \to Q$

P : It is raining

Q : I get wet

$\neg P$: It is not raining

$\neg Q$: I do not get wet

$Q \to P$ (converse):

If I get wet, then it is raining.

$\neg Q \to \neg P$ (contrapositive).

If I do not get wet, then it is not raining.

② How can this English sentence be translated to a logical expression?

You can access the internet from campus only if you are a computer Science Major or you are not a freshman.

a $\to$ b $\vee$ $\sim c$

a: you can access the internet from campus

b: you are a Computer Science Major

c: you are a freshman

¬c: you are not a freshman

Ans:

$$a \to b \vee \neg c$$

$$a \to b \vee \sim c$$

③ State the truth value of "If tigers have wings, the earth travels round the sun."

P : Tigers have wings (F).

Q : The earth travels round the sun (T)

($P \to Q$: T (from $P \to Q$ table)

The given statement is $P \to Q$ has the truth value "T."

④ Construct the truth table for the following:

(i) $\neg(\neg P \vee \neg Q)$ (ii) $\neg(\neg P \wedge \neg Q)$.

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T	F.	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	T	F	T	F

⑤ Construct the truth table

(i) $P \vee \neg Q$

(ii) $P \wedge (P \vee Q)$

(iii) $(P \vee Q) \vee \neg P$

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	F	F	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	F
F	F	T	T	T	F	F

P	Q	$\neg P$	$P \vee Q$	$(P \vee Q) \vee \neg P$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Truth Table:

A table giving the truth values of a compound statements in terms of its component parts is called the truth table.

Propositional equivalences - DeMorgan's law.

Tautology:

A statement that is true for all possible values of its propositional variables is called a Tautology or Universally valid formula or a logical truth.

Contradiction:

A statement that is always false is called a contradiction.

① Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

P	Q	$\sim P$	$\sim Q$	$P \wedge \neg Q$	$Q \vee (P \wedge \neg Q)$	$\neg P \wedge \neg Q$	$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

② Show that $(P \wedge Q) \wedge \neg(P \vee Q)$ is a contradiction.

P	Q	$P \wedge Q$	$\cancel{\sim}(P \vee Q)$	$\sim(P \vee Q)$	$(P \wedge Q) \wedge \neg(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Logical Equivalences ~~and~~ - Implication.

The Propositions P and Q are called Logically equivalent if $P \Leftrightarrow Q$ is a tautology. The Notation $P \equiv Q$ denotes that P and Q are logically equivalent or we use $P \Leftrightarrow Q$

3) Show that P is equivalent to the following formula.

(i) $\neg\neg P$ (ii) $P \wedge P$ (iii) $P \vee P$

(iv) $P \vee (P \wedge Q)$ (v) $P \wedge (P \vee Q)$

P	Q	$\neg P$	$\neg\neg P$	$P \wedge P$	$P \vee P$	$P \vee (P \wedge Q)$	$P \wedge (P \vee Q)$
T	T	F	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	F	F	F	F	F
F	F	T	F	F	F	F	F

4) Show that P is equivalent to the following formulas:

(i) $(P \wedge Q) \vee (P \wedge \neg Q)$

(ii) $(P \vee Q) \wedge (P \vee \neg Q)$

(i) $(P \wedge Q) \vee (P \wedge \neg Q)$

P	Q	$\neg Q$	$P \wedge Q$	$P \wedge \neg Q$	$(P \wedge Q) \vee (P \wedge \neg Q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	F	F
F	F	T	F	F	F

P	Q	$\neg Q$	$P \vee Q$	$P \vee \neg Q$	$(P \vee Q) \wedge (P \vee \neg Q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	F

Logically equivalentes:

8 (9) $P \to Q \Leftrightarrow \neg P \vee Q$

9 (10) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $(A \cap B)' = A' \cup B'$

10 (11) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.

(12) $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$

(12). $\neg(P \leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Table Logic Equivalences:

Equivalences	Name
$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow P$	Identity Laws
$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow F$	Domination laws
$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$	Idempotent laws
$\neg(\neg P) \Leftrightarrow P$	Double negation law
$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$	Commutative laws
$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative laws.

Equivalences	Name
$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $(P \vee Q) \wedge R \Leftrightarrow (P \wedge R) \vee (Q \wedge R)$ $(P \wedge Q) \vee R \Leftrightarrow (P \vee R) \wedge (Q \vee R)$	Distributive laws
$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	De Morgan's law
$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$	Absorption laws
$P \vee \neg P \Leftrightarrow T$ (or) $\neg P \vee P \Leftrightarrow T$ $P \wedge \neg P \Leftrightarrow F$ (or) $\neg P \wedge P \Leftrightarrow F$	Negation laws

Table Logic Equivalences involving conditionals:

① $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

② $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

③ $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

④ $P \wedge Q \Leftrightarrow \neg(P \rightarrow \neg Q)$

⑤ $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

⑥ $(P \rightarrow Q) \wedge (P \rightarrow R) \Leftrightarrow P \rightarrow (Q \wedge R)$

⑦ $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$

8. $(P \to Q) \vee (P \to R) \Leftrightarrow P \to (Q \vee R)$

9. $(P \to R) \vee (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

Table Logic Equivalence involving biconditionals:

1. $P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

2. $P \leftrightarrow Q \Leftrightarrow \neg P \leftrightarrow \neg Q$

3. $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

4. $\neg (P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q$

Replacement Process:

Consider the formula $A: P \to (Q \to R)$

Here $Q \to R$ is a part of the formula A.

If we replace $Q \to R$ by an equivalent formula $\neg Q \vee R$ is

$B: P \to (\neg Q \vee R)$. We can easily verify that the formulas A and B are equivalent to each other

This process of obtaining B from A is known as the replacement process.

Tautological Implications:

A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology. In this case, we write $A \Rightarrow B$, read as "A implies B".

Table Implications.

1. $P \wedge Q \Rightarrow P$
2. $P \wedge Q \Rightarrow Q$
3. $P \Rightarrow P \vee Q$
4. $\neg P \Rightarrow P \to Q$
5. $Q \Rightarrow P \to Q$
6. $\neg (P \to Q) \Rightarrow P$
7. $\neg (P \to Q) \Rightarrow \neg Q$
8. $P \wedge (P \to Q) \Rightarrow Q$
9. $\neg Q \wedge (P \to Q) \Rightarrow \neg P$
10. $\neg P \wedge (P \vee Q) \Rightarrow Q$
11. $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$
12. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$

1) Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

	Reasons
(i) $(\neg P \wedge (\neg Q \wedge R))$	
$\Leftrightarrow (\neg P \wedge \neg Q) \wedge R$	Associative law
$\Leftrightarrow \neg (P \vee Q) \wedge R$	De Morgan's law.
(ii) $(Q \wedge R) \vee (P \wedge R)$	
$\Leftrightarrow (Q \vee P) \wedge R$	Distibritive law
$\Leftrightarrow (P \vee Q) \wedge R$	Commutative law.

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$	Given
$\Leftrightarrow (\neg(P \vee Q) \wedge R) \vee ((P \vee Q) \wedge R)$	by (i) & (ii)
$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$	Distributive law.
$\Leftrightarrow T \wedge R$	Negation law $[\neg P \vee P \Leftrightarrow T]$
$\Leftrightarrow R$	Identity law.

② Show that $P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$

	Reasons
(i) $P \to (Q \vee R)$	
$\Leftrightarrow \neg P \vee (Q \vee R)$	$P \to Q \Leftrightarrow \neg P \vee Q$
(ii) $(P \to Q) \vee (P \to R)$	
$\Leftrightarrow (\neg P \vee Q) \vee (\neg P \vee R)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow \neg P \vee (Q \vee \neg P) \vee R$	
$\Leftrightarrow \neg P \vee (\neg P \vee Q) \vee R$	Commutative law.
$\Leftrightarrow (\neg P \vee \neg P) \vee Q \vee R$	
$\Leftrightarrow \neg P \vee (Q \vee R)$	Idempotent law
From (i) and (ii) we get $P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$	

③ Show that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$

	Reasons
(i) $P \to (Q \to P)$	
$P \to (\neg Q \vee P)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\neg P \vee (\neg Q \vee P)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\neg P \vee (P \vee \neg Q)$	Commutative law
$(\neg P \vee P) \vee \neg Q$	Associative law
$T \vee \neg Q$	$\neg P \vee P \Leftrightarrow T$
T	$P \vee T \Leftrightarrow T$

	Reasons
(ii) $\neg P \to (P \to Q)$	
$\neg P \to (\neg P \vee Q)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$(\neg P) \to (\neg P \vee Q)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\neg(\neg P) \vee (\neg P \vee Q)$	~~Negati~~ "
$P \vee (\neg P \vee Q)$	Double Negation law
$(P \vee \neg P) \vee Q$	Associative law
$T \vee Q$	Negation law
T	Domination law

From (i) and (ii) $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$

Principle Disjunctive Normal form (PDNF) (SOP) Minterm

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called Disjunctive Normal Form (DNF) of the given formula.

*Sum of Products Normal Form: (PDNF):

A formula which is equivalent to a given formula and which consists of sum of its min terms is called Principle Disjunctive Normal Form (or) Sum of Product of Canonical form of the given formula.

Principle Conjunctive Normal Form (PCNF) (Product of Sums Canonical form). (POS) (PCNF)

An equivalent formula consisting of conjuction of max terms only is known as its principle Conjuctive Normal Form (or) Product of Sums Canonical form.

Min term:

Let P and Q be two statement variable construct all possible formula ~~with~~ which consists of conjunctions of P (or) its negation and conjunctions of Q (or) its negation. None of the formula should contain both a variable and its negation. Delete a formula if it is the commutative of any one of the remaining formula.

Such a conjunction of P and Q are called the min terms of P and Q.

Max terms:

For a given no. of variables, the max term consists of disjunctions in which each variable or its negation but not both, appears only once.

(without table)

① Obtain PDNF (Principle Disjunctive Normal Form) of (i) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ and also find PCNF.

Let $S = (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$A = (\neg P \wedge R) \vee (Q \wedge R)$

P	Q	R	$\neg P$	$\neg P \wedge R$	$Q \wedge R$	$P \wedge Q$	A	S	Pdnf T (minterm)	Pcnf F (maxterm)
T	T	T	F	F	T	T	T	T	$P \wedge Q \wedge R$	–
T	T	F	F	F	F	T	F	T	$P \wedge Q \wedge \neg R$	–
T_F	F_T	T_F	F	F	F	F	F	F	–	$\neg P \vee Q \vee \neg R$
T_F	F_T	F_T	F	F	F	F	F	F	–	$\neg P \vee Q \vee R$
F	T	T	T	T	T	F	T	T	$\neg P \wedge Q \wedge R$	–
F_T	T_F	F_T	T	F	F	F	F	F	–	$P \vee \neg Q \vee R$
F	F	T	T	T	F	F	T	T	$\neg P \wedge \neg Q \wedge R$	–
F_T	F_T	F_T	T	F	F	F	F	F	–	$P \vee Q \vee R$

$S \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \rightarrow$ PDNF

$S \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R) \rightarrow$ PCNF