UNIT - I

SET THEORY

SET:

It is a well-defined collection of objects called the elements or members of the set.

Set Operations:

* $A \cup B = \{x / x \in A \text{ or } x \in B\}$

* $B \cap A = \{x / x \in A \text{ and } x \in B$

* $A^c = \{x / x \notin A\}$

* $A - B = \{x / x \in A, x \notin B\}$

Symmetric Diff * $A \oplus B = (A - B) \cup (B - A)$

Example:

Let $U = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$ and

$B = \{3, 4\}$.

be the subsets of U.

Find $A \cup B$, $A \cap B$, $\bar{A}$, $A - B$, $B - A$, $A \oplus B$ and $A \times B$.

Sol:-

$A \cup B = \{1, 2, 3, 4\}$

$A \cup B = \{3\}$

$\bar{A} = \{4, 5, 6\}$

$A - B = \{1, 2\}$

$B - A = \{4\}$

$A \oplus B = (A - B) \cup (B - A)$

$= \{1, 2, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

Example:

List all the subsets of $A = \{1, 2, 3\}$

Sol:-

$\{1\}, \{2\}, \{3\},$

$\{1, 2\}, \{2, 3\}, \{3, 1\},$

$\{1, 2, 3\}, \phi.$

Example:

List all the proper non-empty subsets of $A = \{1, 2\}$

Sol:-

$\{1\}, \{2\}.$

The Duality Principle:

The dual of an expression is obtained by interchanging $\cup$ and $\cap$, U and ϕ, $\leq$ and $\geq$, $<$ and $>$.

Example:

$A \cup \phi = A$

Its dual is $A \cap U = A$

Partition of a Set:

Let S be any non-empty set. The collection of subsets $A_1, A_2 \ldots A_n$ is called a partition of S, iff

i) $A_i \neq \phi$ for each i

ii) $A_i \cap A_j = \phi$ for $i \neq j$

iii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

Example:

For the set $S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 3, 5\}$, $A_2 = \{2, 4\}$, $A_3 = \{6\}$ is a partition of S.

Minset:

Let A be a set. Let $\{B_1, B_2, \ldots, B_n\}$ be a collection of subsets of A.

A set of the form $D_1 \cap D_2 \cap \ldots \cap D_N$ where each D_i may be either B_i or B_i^c is called a Minset.

Note:

1. Let A be a set.
   Let the subsets of A be B_1 and B_2.
   Then the minsets are

   $D_1 = B_1 \cap B_2$,

   $D_2 = \overline{B_1} \cap B_2$,

   $D_3 = B_1 \cap \overline{B_2}$,

   $D_4 = \overline{B_1} \cap \overline{B_2}$

2. If B_1, B_2, B_3 are the subsets of A, then the minsets are

   $D_1 = B_1 \cap B_2 \cap B_3$

   $D_2 = \overline{B_1} \cap B_2 \cap B_3$

   $D_3 = B_1 \cap \overline{B_2} \cap B_3$

   $D_4 = B_1 \cap B_2 \cap \overline{B_3}$

   $D_5 = \overline{B_1} \cap \overline{B_2} \cap B_3$

   $D_6 = B_1 \cap \overline{B_2} \cap \overline{B_3}$

   $D_7 = \overline{B_1} \cap B_2 \cap \overline{B_3}$

   $D_8 = \overline{B_1} \cap \overline{B_2} \cap \overline{B_3}$

   D.

3. The collection of minsets form a partition of A.

Maxset:

Let A be a set. Let $\{B_1, B_2, \ldots, B_n\}$ be a collection of subsets of A. A set of the form $D_1 \cup D_2 \cup \ldots \cup D_n$ where each D_i may be either B_i or $\overline{B_i}$ is called a Maxset.

Note:

1. Let A be a set. Let the subsets of A be B_1, B_2. Then the Maxsets are

$D_1 = B_1 \cup B_2$

$D_2 = \overline{B_1} \cup B_2$

$D_3 = B_1 \cup \overline{B_2}$

$D_4 = \overline{B_1} \cup \overline{B_2}$

2. The collection of Maxsets does not form a partition of A.

Laws of Set Theory

1. $A \cup \emptyset = A$.	$A \cap U = A$	Identity Laws
2. $A \cup U = U$	$A \cap \phi = \emptyset$	Domination Laws
3. $A \cup A = A$	$A \cap A = A$	Idempotent Laws.
4. $A \cup \overline{A} = U$	$A \cap \overline{A} = \phi$	Inverse Laws (or) Complement Laws
5. $\overline{\overline{A}} = A$		Double complement (or) Involution Law
6. $A \cup B = B \cup A$	$A \cap B = B \cap A$	Commutative Laws
7. $A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
9. $A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	Absorption Laws
10. $\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's Law

Problems:

1. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S

$$A \cup (B \cap C) = \{x / x \in A \text{ or } x \in (B \cap C)\}$$

$$= \{x / (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$$

$$= \{x / x \in (A \cup B) \text{ and } x \in (A \cup C)\}$$

$$= \{x / x \in (A \cup B) \cap (A \cup C)\}$$

$$= (A \cup B) \cap (A \cup C) \quad \text{R.H.S}$$

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S

$$A \cap (B \cup C) = \{x / x \in A \text{ and } x \in (B \cup C)\}$$

$$= \{x / x \in A \text{ and } (x \in B \text{ or } x \in C)\}$$

$$= \{x / (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$$

$= \{x / x \in (A \cap B) \text{ or } x \in (A \cap C)\}$

$= \{x / x \in (A \cap B) \cup (A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$ R.H.S

3. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$

Proof:

$A - (B \cap C) = \{x / x \in A \text{ and } x \notin (B \cap C)\}$

$= \{x / x \in A \text{ and } (x \notin B \text{ or } x \notin C)\}$

$= \{x / (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\}$

$= \{x / x \in (A - B) \text{ or } x \in (A - C)\}$

$= \{x / x \in (A - B) \cup (A - C)\}$

$= (A - B) \cup (A - C)$

4. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof:

$A \times (B \cup C) = \{(a, b) / a \in A \text{ and } b \in (B \cup C)\}$

$= \{(a,b) / (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C)\}$

$= \{(a,b) / (a,b) \in A \times B \cup A \times C\}$

$= (A \times B) \cup (A \times C)$

Example:
5. Find the Minsets generated by $\{0,2,4\}$ and $\{1,5\}$ of the set $\{0,1,2,3,4,5\}$. Also find the Maxsets.

Sol:-

Let $A = \{0,1,2,3,4,5\}$

Let $B_1 = \{0,2,4\}$

$B_2 = \{1,5\}$

$\therefore \overline{B_1} = \{1,3,5\}$

$\overline{B_2} = \{0,2,3,4\}$

The Minsets are

$D_1 = B_1 \cap B_2 = \{\}$

$D_2 = \overline{B_1} \cap B_2 = \{1,5\}$

$D_3 = B_1 \cap \overline{B_2} = \{0,2,4\}$

$D_4 = \overline{B_1} \cap \overline{B_2} = \{3\}.$

The Maxsets are

$D_1 = B_1 \cup B_2 = \{1, 2, 0, 4, 5\}$

$D_2 = \overline{B_1} \cup B_2 = \{1, 3, 5\}$

$D_3 = \{0, 2, 3, 4\} = B_1 \cup \overline{B_2}$

$D_4 = \overline{B_1} \cup \overline{B_2} = \{0, 1, 2, 3, 4, 5\}$

Example:

6. Find the Minsets and Maxsets generated by $\{(1, 3)\}, \{2, 4\}, \{1, 4, 6\}$ of the set $\{1, 2, 3, 4, 5, 6\}$.

RELATIONS

Definition:

When A and B are sets, a subset R of the cartesian product $A \times B$ is called a binary relation from A to B.

If R is a binary relation from A to B, R is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

When $(a, b) \in R$, we use the notation aRb and read it as "a is related to b" by R.

If $(a, b) \notin R$, it is denoted as $a \not R b$.

Note: 1

If R is a relation from a set A to itself.

(ie) if R is a subset of $A \times A$, then R is called a relation on the set A.

Note: 2

The set $\{a \in A \mid aRb, \text{ for some } b \in B\}$ is called the domain of R and denoted by $D(R)$.

Note : 3

The set $\{ b \in B / aRb$, for some $a \in A \}$ is called the range of R and denoted by $R(R)$.

Example :

1. Let $A = \{0, 1, 2, 3, 4\}$
$B = \{0, 1, 2, 3\}$
and aRb iff $a+b=4$.

then $R = \{(1, 3), (2, 2), (3, 1), (4, 0)\}$
the domain of $R = \{1, 2, 3, 4\}$
the image of $R = \{0, 1, 2, 3\}$

2. Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by aRb if $\boxed{a \leq b}$; $a, b \in A$.

then $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

The domain and range of R are both equal to A.

Types of Relations:

i) Universal Relation:

A relation R on a set A is called a Universal relation if

$$R = A \times A$$

Example:

If $A = \{1, 2, 3\}$, then

$$R = A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

is the Universal relation on A

ii) Void Relation:

A relation R on a set A is called a void relation, if R is the Null Set ϕ.

Example:

If $A = \{3, 4, 5\}$ and R is defined as aRb iff $a+b > 10$.

then R is a null set

$\because$ no element in $A \times A$ satisfies the given condition.

Note:

The entire cartesian product $A \times A$ and the empty set are subsets of $A \times A$.

iii) Identity Relation:

A relation R on a set A is called an Identity relation, $R = \{(a, a) / a \in A\}$ and is denoted by I_A.

Example:

If $A = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on R.

iv) Inverse of Relation

When R is any relation from a set A to a set B, the inverse of R, denoted by R^{-1}, is the relation from B to A which consists of the ordered pairs got by interchanging the elements of the ordered pairs in R.

$$R^{-1} = \{(b, a) / (a, b) \in R\}$$

(ie) If aRb, then $bR^{-1}a$.

Example:

If $A = \{2, 3, 5\}$, $B = \{6, 8, 10\}$ and aRb iff $a \in A \div b \in B$

then $R = \{(2,6), (2,8), (2,10), (3,6), (5,10)\}$

$R^{-1} = \{(6,2), (8,2), (10,2), (6,3), (10,5)\}$.

Note: 1

$bR^{-1}a$, iff $b \in B$ is a multiple of $a \in A$

Note: 2

$D(R) = R(R^{-1}) = \{2, 3, 5\}$ and

$R(R) = D(R^{-1}) = \{6, 8, 10\}$

Properties of Relations

A relation R on a set A is called

i) reflexive if $(a,a) \in R$, $\forall a \in A$

ii) symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$

iii) transitive if $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$

iv) an equivalance relation, if R is reflexive, symmetric and transitive.

v) irreflexive, if $(a, a) \notin R, \forall a \in A$.

vi) anti-symmetric, if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$

vii) a partial order relation, if R is reflexive, anti-symmetric and transitive.

Poset:

A set together with a partial order relation R is called a partially ordered set or POSET.

Composition of Relations:

If R is a relation from A to B and S is a relation from B to C, then the composition of R and S is

$$ROS = \{(a, c) / (a, b) \in R, (b, c) \in S\}$$

Matrices of relations:

If $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets and R is the relation from A to B, then $M_R = \{M_{ij}\}$ where.

$$M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Problems:

Example: 1

Let $R = \{(1,2), (2,1), (2,2), (2,3), (3,1)\}$ and

$S = \{(1,2), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

Find i) $R \cup S$ ii) $R \cap S$ iii) $R - S$ iv) $S - R$ v) $R \oplus S$ vi) $R \circ S$ vii) $S \circ R$

Sol:-

i) $R \cup S = \{(1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

ii) $R \cap S = \{(1,2), (2,2), (2,3), (3,1)\}$

iii) $R - S = \{(2,1)\}$

iv) $S - R = \{(3,2), (3,3)\}$

v) $R \oplus S = (R \cup S) - (R \cap S)$

$= \{(2,1), (3,2), (3,3)\}$

vi) $R \circ S = \{(1,2), (1,3), (2,2), (2,1), (2,3), (3,2)\}$

vii) $S \circ R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,2), (3,1), (3,3)\}$

Example: 2

Let $X = \{1, 2, 3, \ldots, 25\}$

Let $R = \{(x, y) / x-y$ is divisible by $5\}$

is a relation on X. Show that R is an equivalence relation.

Proof:

$R = \{(1,1), (1,6), (1,11), \ldots\}$

i) Let $a \in X$; clearly $(a,a) \in R$

also $a-a = 0$ is divisible by 5

$\therefore R$ is reflexive.

ii) Let $(a,b) \in R$

$\Rightarrow a-b$ is divisible by 5

$\Rightarrow -(b-a)$ is also divisible by 5

$\Rightarrow (b,a) \in R$

$\therefore R$ is symmetric.

iii) Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a-b$ and $b-c$ is divisible by 5.

$\Rightarrow a-c = (a-b)+(b-c)$ is also divisible by 5.

$\therefore (a,c) \in R$

$\therefore R$ is transitive.

Hence, R is an equivalence relation

Example: 3

Let $S = \{1, 2, \ldots, 9\}$. Define R on S by $R = \{(x,y) / x, y \in S \text{ and } x+y=10\}$. What are the properties of R?

Sol:-

$$R = \{(1,9),(2,8),(3,7),(4,6),(5,5),(6,4),(7,3),(8,2),(9,1)\}.$$

i) Clearly $(a,a) \notin R$

$\therefore$ Not reflexive

ii) clearly for every $(a,b) \in R$, we have $(b,a) \in R$

$\therefore R$ is symmetric

iii) Clearly for every $(a,b) \in R$ and $(b,c) \in R$, we have $(a,c) \notin R$

$\therefore$ Not transitive.

iv) R is not an equivalance relation

v) Also, for every $(a,b) \in R$ and $(b,a) \in R$

$\therefore a \neq b$

$\Rightarrow R$ is not anti-symmetric.

vi) R is not a partial order relation.

Example: 4

Let $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$.

Let $R = \{(1,y), (1,z), (3,y), (4,x), (4,z)\}$

Determine the matrix of R.

Sol:-

$$M_R = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Note:

$$M_{R \cup S} = M_R \vee M_S$$

$$M_{R \cap S} = M_R \wedge M_S$$

$$M_{R \circ S} = M_R \circ M_S$$

$$M_{R^{-1}} = (M_R)^T$$

Example : 5

If R and S are two relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find $M_{R\cup S}$, $M_{R\cap S}$, $M_{R\oplus S}$, $M_{R\circ S}$, $M_{S\circ R}$, $M_{R^{-1}}$, M_{R^2}.

Sol:-

i) $M_{R\vee S} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

ii) $M_{R\wedge S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

iii) $M_{R\oplus S} = M_{R\cup S} - M_{R\cap S}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

iv) $M_{R \circ S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

v) $M_{S \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

vi) $M_{R^{-1}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

vii) $M_{R^2} = M_R \cdot R_R$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Closure Operations on Relations:

Let A be a set. Let R be a relation on A. The transitive closure of R is the smallest relation which contains R as a subset and is transitive, denoted by R^+.

Warshall's Algorithm:

1. Find the transitive closure of the relation whose matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

using Warshall's algorithm.

Sol:-

Let $W_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

k	In W_{k-1}: Position of 1's in column	Position of 1's in row	W_k has 1's in	W_k
1	1	1, 4	(1, 1); (1, 4)	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
2	2	2, 4	(2, 2), (2, 4)	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
3	-	4	-	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
4	1, 2, 3, 4	4	(1, 4), (2, 4), (3, 4), (4, 4)	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ The transitive closure

$$R^+ = \{(1,1), (1,4), (2,2), (2,4), (3,4), (4,4)\}$$

2. Using Warshall's algorithm, find the transitive closure of $S = \{1, 2, 3, 4, 5\}$.

The relation $R = \{(1,1), (1,3), (1,5), (2,3), (2,4), (3,3), (3,5), (4,2), (4,4), (5,4)\}$

K	In W_{K-1} Position of 1's in column	In W_{K-1} Position of 1's in row	W_K has 1's in	W_K
1	1	1, 3, 5	(1,1), (1,3), (1,5)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
2	4	3, 4	(4,3) (4,4)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
3	1, 2, 3, 4	3, 5	(1,3), (1,5) (2,3), (2,5) (3,3), (3,5) (4,3), (4,5)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$
4.	2, 4, 5	2, 3, 4, 5	(2,2), (2,3) (2,4), (2,5) (4,2), (4,3) (4,4), (4,5) (5,2), (5,3) (5,4), (5,5)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

5	1, 2, 3, 4, 5	2, 3, 4, 5	(1, 2), (1, 3), (1, 4), (1, 5) (2, 2), (2, 3), (2, 4), (2, 5) . . . . (5, 2), (5, 3), (5, 4), (5, 5)	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

∴ The transitive closure is

$R^+ = \{(1,2), \ldots, (1,5), (2,2), \ldots, (2,5), (3,2)$
$\ldots, (3,5), (4,2), \ldots, (4,5), (5,2)$
$\ldots, (5,5)\}$

Graphs of relations:

1. Let $A = \{1, 2, 3, 4\}$ and

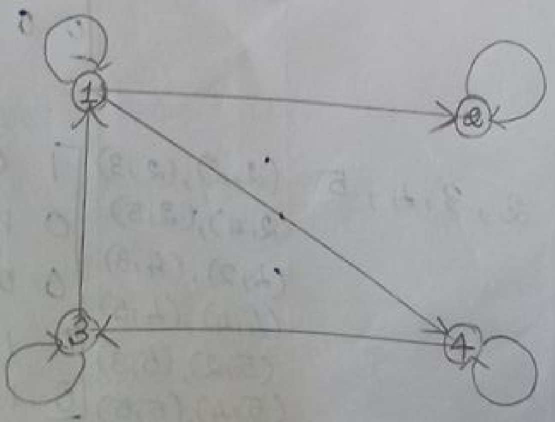
$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

i) Draw the Digraph

ii) Find the indegree & Outdegree

iii) Use the graph to find if the relation is reflexive, symmetric, transitive

Sol:-

i) 

(If a loop, its both indegree and Outdegree)

	1	2	3	4
ii) Indegree	2	2	2	2
Outdegree	3	1	2	2

iii) * Since there is a loop at every vertex, the relation R is Reflexive.

* R is not symmetric, because there is an edge from 1 to 2, but no edge from 2 to 1.

* R is not transitive, since there are edges from 1 to 3 and 3 to 4 but no edge from 1 to 4.

Hasse Diagram:

* It is named after the Mathematician Helmut Hasse
* Diagrammatic representation of the relation.

Rules:

1. Each vertex of A must be related to itself, so that arrows from a vertex to itself isn't necessary.

2. If a vertex appears above vertex "a" and if "a" is connected to "b" by an edge, then aRb. So, direction arrows aren't necessary.

3. If a vertex "c" is above vertex "a" and if "c" is connected to "a" by a sequence of edges, then aRc

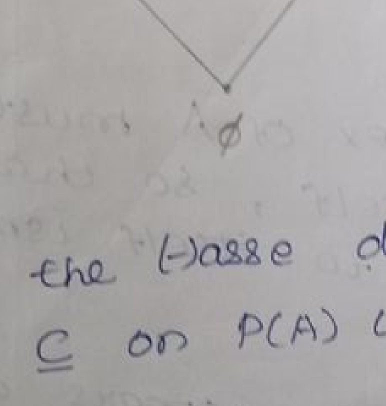
4. The vertices are denoted by points rather than circles.

Problems:

1. Draw the Hasse diagram of the relation $\subseteq$ on $P(A)$ where $A = \{a, b\}$.

Sol:-

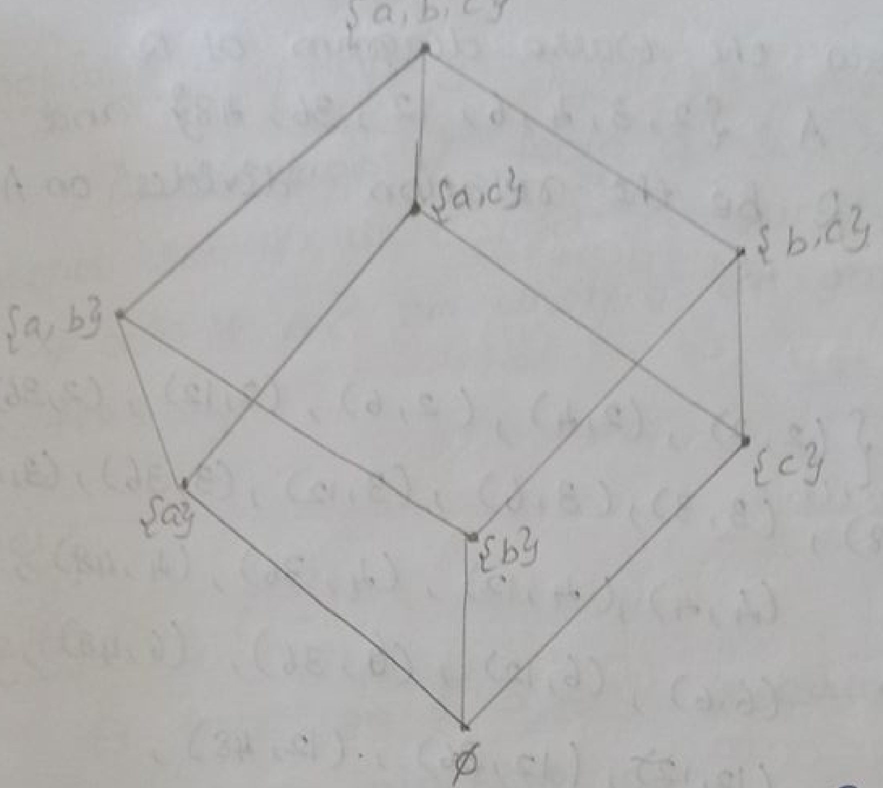
$$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$



2. Draw the Hasse diagram of the relation $\subseteq$ on $P(A)$ where $A = \{a, b, c\}$

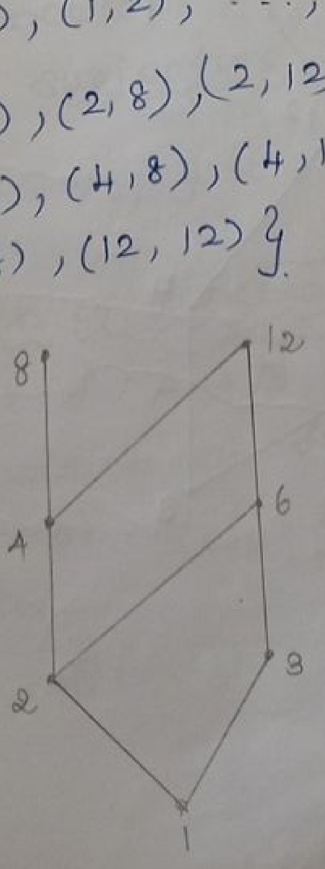
Sol:-

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$



3. Draw the Hasse diagram of R.
$R = \{(a, b) / a \text{ divides } b\}$ where.
$A = \{1, 2, 3, 4, 6, 8, 12\}$

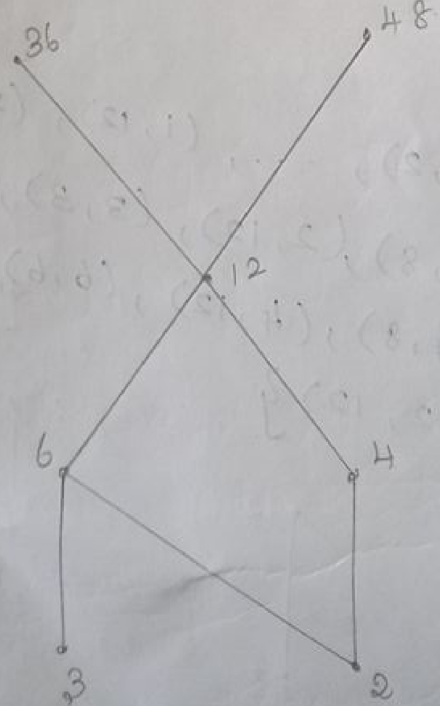
Sol:-
$R = \{(1,1), (1,2), \ldots, (1,12), (2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12), (4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12)\}$.



4. Draw the Hasse diagram of R. Let $A = \{2, 3, 4, 6, 12, 36, 48\}$ and let R be the relation "divides" on A.

Sol:-

$R = \{(2,2), (2,4), (2,6), (2,12), (2,36),$
$(2,48), (3,3), (3,6), (3,12), (3,36), (3,48),$
$(4,4), (4,12), (4,36), (4,48),$
$(6,6), (6,12), (6,36), (6,48),$
$(12,12), (12,36), (12,48),$
$(36,36), (48,48)\}$



Functions :- [Mappings or transformation]

A relation f from a set X to another set Y is called a function if $\forall\, x \in X$, $\exists$ an unique image $y \in Y$ of $f(x) = y$.

Types :

One to One (1-1) :

A function $f : x \to y$ is called one to one or injective if

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

(ie) distinct elements of X have distinct images.

Onto :

A function $f : x \to y$ is called onto or surjective if

$\forall\, y \in Y,\ \exists\, x \in X \,/\, f(x) = y$.

Composition of function :

Let $f : A \to B$ and $g : B \to C$.

The composition of function f and g is a function $g \circ f : A \to C$ and it is defined by

$(g \circ f)x = g(f(x)),\ \forall\, x \in A$.

Inverse function:

A function $f^{-1}: y \to x$ is called the inverse function of $f: x \to y$ if $f^{-1}(y) = x, \forall y \in Y$.

Invertible:

A function f is called as invertible if f is both one to one and onto

Problems:

1. Let $f: R \to R$ and $g: R \to R$ defined by $f(x) = 4x - 1$, $g(x) = \cos x$. Find fog and gof

Sol:-

$(fog)(x) = f(g(x)) = f(\cos x) = 4\cos x - 1$

$(gof)(x) = g(f(x)) = g(4x-1) = \cos(4x-1)$

2. Find fog and gof when $f: R \to R$ and $g: R \to R$ defined by $f(x) = 2x - 1$ and $g(x) = x^2 - 2$

Sol:- $(fog)(x) = f(g(x)) = f(x^2 - 2)$

$= 2(x^2 - 2) - 1$

$= 2x^2 - 5$

$(g\circ f)(x) = g(f(x)) = g(2x-1)$

$= (2x-1)^2 - 2$

$= 4x^2 - 4x - 1.$

Note:

The composition of functions are not commutative.

(ie) $f\circ g \neq g\circ f$.

3. Show that $f: R \to R$ defined by $f(x) = 3x-1$ is bijective

Sol:-

1-1: Let $f(x_1) = f(x_2)$

$3x_1 - 1 = 3x_2 - 1$

$x_1 = x_2$.

$\therefore f$ is 1-1

Onto: Let $y \in R$, $\exists\, x = \frac{y+1}{3} \in R \,/\, f(x) = y$

$\therefore f$ is onto

$\therefore$ It is bijective

4. Determine whether the function $f: Z^+ \to Z^+$ defined $f(x) = x^2 + 2$ is bijective or not?

Sol:-

1-1:- Let $f(x_1) = f(x_2)$

$x_1^2 + 2 = x_2^2 + 2$

$(x_1 + x_2)(x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 = 0 \quad , (\because x_1 + x_2 \neq 0)$

$\Rightarrow x_1 = x_2.$

$\therefore f$ is $1-1$

Onto :- Here $y = x^2 + 2$

$x^2 = y - 2.$

$\forall y \in Y, \exists x \in X$ such that $f(x) = y$

clearly its not possible.

$\therefore$ Non - onto

Hence f is not bijective.

5. If $f : Z \rightarrow N$ is defined by

$$f(x) = \begin{cases} 2x - 1 , & \text{if } x > 0 \\ -2x , & \text{if } x \leq 0. \end{cases}$$

Then prove that f is bijective and determine f^{-1}.

Sol:-

1-1:- Let $x_1, x_2 \in Z$.

$x > 0, \quad f(x_1) = f(x_2)$

$2x_1 - 1 = 2x_2 - 1$

$x_1 = x_2$

$x \leq 0, \quad -2x_1 = -2x_2$

$x_1 = x_2$

$\therefore f$ is $1-1$

Onto:-

$x > 0$, Let $y = 2x - 1$

$$\Rightarrow x = \frac{y+1}{2} \quad (\text{odd})$$

$x \le 0$, Let $y = -2x$

$$\Rightarrow x = -y/2$$

$\therefore$ for any $y \in N$, $\exists \frac{y+1}{2} \in Z$ or

$-y/2 \in Z$

$\therefore$ f is onto

Hence f is bijective.

f^{-1}:-

$$f^{-1}(y) = \begin{cases} \frac{y+1}{2} & ; y = 1, 3, 5, \dots \\ -y/2 & ; y = 0, 2, 4, 6, \dots \end{cases}$$

6. If $A = \{x \in R / x \neq 1/2\}$ and $f: A \to R$ is defined by $f(x) = \frac{4x}{2x-1}$ find

i) Range of f

ii) Show that f is invertible

iii) find the domain of f^{-1}.

iv) find the range of f^{-1}.

v) formula for f^{-1}.

Sol:-

ii) 1-1:- Let $x_1, x_2 \in A$

Let $f(x_1) = f(x_2)$

$$\frac{4x_1}{2x_1 - 1} = \frac{4x_2}{2x_2 - 1}$$

$$\Rightarrow x_1 = x_2$$

Onto:- Let $y = \frac{4x}{2x-1}$

$$\Rightarrow x = \frac{y}{2(y-2)}$$

$\therefore \forall y \in R, \exists x = \frac{y}{2(y-2)} \in A$ such

that $f(x) = y$.

$\therefore f$ is onto

Hence f is bijective.

i) Range of $f = \{y \in R / y \neq 2\}$

iii) Domain of $f^{-1} = \{y \in R / y \neq 2\}$

(iv) Range of $f^{-1} = \{x \in A / x \neq 1/2\}$

v) Formula for f^{-1}:

$$f^{-1}(y) = \frac{y}{2(y-2}$$ where $y \neq 2$.

Theorem 1:

If $f: A \to B$ and $g: B \to C$ are bijective functions, then $g \circ f: A \to C$ is also bijective.

Proof:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

To prove: $g \circ f$ is bijective.

1 - 1: Let $a_1, a_2 \in A$.

$$(g \circ f)(a_1) = (g \circ f)(a_2)$$

$$g(f(a_1)) = g(f(a_2))$$

$$f(a_1) = f(a_2) \quad \text{Since } g \text{ is } 1-1$$

$$a_1 = a_2. \quad \text{,, } f \text{ is } 1-1$$

onto: Let $c \in C$.

Since g is onto, there is an element $b \in B$, such that $g(b) = c$

Since f is onto, there is an element $a \in A$, such that $f(a) = b$

for every $c \in C$, there is an element $a \in A$, such that $(g \circ f)(a) = c$.

$\therefore g \circ f$ is onto.

Hence $g \circ f$ is bijective.

Theorem : 2

If $f : A \to B$ and $g : B \to C$ are invertible functions, then $gof : A \to C$ is also invertible and $(gof)^{-1} = f^{-1} o g^{-1}$.

Proof:

$(gof)^{-1} : C \to A$

$f^{-1} : B \to A$

$g^{-1} : C \to B$

Hence $f^{-1} o g^{-1} : C \to A$

Both $(gof)^{-1}$ and $f^{-1} o g^{-1}$ are functions from C to A.

Let $x \in A$

Then $\exists\, y \in B$ such that $f(x) = y$

$\Rightarrow x = f^{-1}(y)$

Also $\forall\, y \in B$, $\exists\, z \in C \ni g(y) = z$.

$\Rightarrow y = f^{-1}(z)$

Now, $(gof)x = g(f(x)) = g(y) = z$

$\therefore (gof)^{-1}(z) = x \quad \dots (1)$

Now, $(f^{-1} o g^{-1})(z) = f^{-1}(g^{-1}(z))$

$= f^{-1}(y)$

$= x \quad \dots (2)$

From (1) and (2)

$(gof)^{-1} = f^{-1} o g^{-1}$

Theorem 3:

A function $f: A \to B$ is invertible iff f is 1-1 and onto.

Proof:

Let $f: A \to B$ be invertible.

To prove: f is 1-1 and onto.

Since f is invertible, there exists an unique function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ .....(1)

i) Let $a_1, a_2 \in A$.

$$f(a_1) = f(a_2)$$

$$\Rightarrow g(f(a_1)) = g(f(a_2))$$

$$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$$

$$\Rightarrow I_A(a_1) = I_A(a_2)$$

$$\Rightarrow a_1 = a_2.$$

ii) Let $b \in B$

Then $g(b) \in A$.

Now, $b = I_B b$

$= (f \circ g) b$

$= f(g(b))$

$\therefore$ for every $b \in B$, $\exists\, g(b) \in A \ni f(g(b)) = b$

$\therefore f$ is onto.

Conversely, f is 1-1 and onto

<u>To prove</u>: f is invertible

Since f is onto, $\forall\, b \in B$, $\exists\, a \in A$ such that $f(a) = b$

Hence, we define a function $g : B \to A \ni g(b) = a$ where $f(a) = b$

If possible,

let $g(b) = a_1$ and $g(b) = a_2$

where $a_1 \neq a_2$

Then $f(a_1) = b$ and $f(a_2) = b$ which is impossible, since f is 1-1.

$\therefore\, g : B \to A$ is an unique function such that $g \circ f = I_B$ and $f \circ g = I_A$

$\therefore\, f$ is invertible.