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CHENNAI

COMPUTATIONAL LOGIC

UNIT I

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SRMIST/KTR

Logical thinking keeps you from wasting time worrying, or hoping. It prevents disappointment. Imagination, on the other hand, only gets you hyped up over things that will never realistically happen.

Jodi Picoult

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OVERVIEW OF THE COURSE

- Propositional Logic (PL)
  - Introduction: Syntax, Parsing, Rules – Formation, Production, & Precedence, Theorems and Lemmas, Replacement Laws
  - Natural Deduction, Derived Rules, Soundness, Completeness
- First Order Logic (FL)
  - Introduction- Syntax, Scope, Binding, Substitution, Semantics, Translation, Satisfiability and Validity Meta- theorems- Deduction, Substitution, Chaining
  - First order Calculus (FC)- Introduction, Theorems, Adequacy of FC to FL, Compactness of, and Laws in FL, Natural Deduction, Analytic Tableaux
- Modal Logic
  - System K & KC, Adequacy of KC to K, Natural deduction , Analytic Tableau for K, Modalities and Computation Tree Logic

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LEARNING RESOURCES

- Arindama Singh, "Logics for Computer Science", PHI Learning Private Ltd,2nd Edition, 2018
- Huth M and Ryan M ," Logic in Computer Science : Modelling and Reasoning about systems", Cambridge University Press, 2005
- Wasilewska & Anita,"Logics for computer science: classical and non-classical",Springer ,2018
- Dana Richards & Henry Hamburger,"Logic And Language Models For Computer Science",Third Edition,World Scientific Publishing Co. Pte. Ltd,2018.
- https://www.cs.cornell.edu/courses/cs3110/2012sp/lectures/lec15-logic-contd/lec15.html

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Course Outcome

WHAT YOU WILL BE ABLE TO DO

CO	At the end of the course you will be able to:
1	Apply the skills acquired on propositional logic to solve examples at hand
2	Apply the rules learnt towards problem solving
3	Acquire mastery over FOL and Meta theorems and apply the same with confidence
4	Analyse and use appropriate logics for problem solving
5	Demonstrate ability to use higher model and solve problems

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GRADUATE OUTCOMES

- Engineering Knowledge
- Problem Analysis
- Design & Development
- Analysis, Design, Research
- Modern Tool Usage
- Society & Culture
- Environment & Sustainability
- Ethics
- Individual & Team Work
- Communication
- Project Mgt. & Finance
- Life Long Learning

Program Outcomes (PO)														
1	2	3	4	5	6	7	8	9	10	11	12	PSO		
Engineering Knowledge	Problem Analysis	Design & Development	Analysis, Design,	Modern Tool Usage	Society & Culture	Environment &	Ethics	Individual & Team	Communication	Project Mgt. & Finance	Life Long Learning	PSO - 1	PSO - 2	PSO – 3
3	-	-	-	-	-	-	-	-	-	-	-	-	-	3
3	3	-	-	-	-	-	-	-	-	-	-	-	-	3
3	2	-	-	-	-	-	-	-	-	-	-	-	-	3
3	3	3	-	-	-	-	-	-	-	-	-	-	-	3
3	2	3	-	-	-	-	-	2	-	-	3	-	-	3

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TOPICS

- Logic
- Propositional Logic
- Syntax
- Unique Parsing
- PropDet
- Sub Propositions
- Precedence rules
- Truth Table
- Models – Terminologies
- Replacement Laws
- Interpretations
  - Conventions
  - Lemma
  - Examples
- Theorems
  - Deduction
  - Monotonicity
  - RAA
  - Fitness
  - Paradox of material implication

INTRODUCTION

Propositional Logic

LOGIC

- A proper or reasonable way of thinking about or understanding something.
- The science that studies the formal processes used in thinking and reasoning.
- **Computer logic** is an aspect of **computer** design concerning the fundamental operations and structures upon which all **computer systems are built**.
  - Boolean Logic
  - Logic Programming (Prolog)
  - Digital Logic
  - Fuzzy Logic

APPLICATIONS FOR COMPUTERS

- Programming Languages - logical symbolism (Fortran, C++, Lisp, Prolog)
- Artificial Intelligence/ Machine Intelligence
  - Expert Systems – Decision making
  - Rule-based Systems – Interpret information
- Software Engineering
  - Knowledge Based Software Assistant – Correctness of specifications
- Very Large Scale Integrated design – Implementation corresponds to specifications
- Frame Languages and Classifiers (KL-One)
- Concurrent Systems - Temporal Logic

COMPUTATIONAL LOGIC

- Use of **logic** to perform or reason about **computation**.
- Use of computers to establish facts in a logical formalism.
- Process of designing and analysing logic in computer applications.
- Types of logic:
  - *Informal logic* is the study of natural language arguments.
  - *Formal logic* is the study of inference with purely formal content.
    - **Syllogistic logic:** Deductive reasoning
    - **Symbolic logic:** Symbolic abstractions
      - Propositional (Zeroth order)
      - Predicate (First order)
    - **Mathematical Logic**: Extension of symbolic logic (model, proof, set and computability theories)

LOGICS OF INTEREST

- Propositional Logic: Formed by combining atomic propositions using logical connectors
- Predicate Logic: Provides an account of quantifiers (Universal, Existential) to express arguments occurring in natural language
- Modal Logic: Logic of □necessity and ◊ possibility

ARGUMENTS

TERMINOLOGIES

- **Logic**: Science that studies the principles of correct reasoning- Evaluates arguments
- **Rules:** Instructions indicating what is allowed to be done or not done
- **Arguments:** A reason or set of reasons given in support of an idea, action or theory.
- **Premise:** Statement in an argument that provides reason or support for the conclusion
- **Conclusions**: Decision reached by reasoning
- **Deductions:** Inference of particular instances by reference to a general law or principle

ARGUMENTS

If the train arrives late and there are no taxis at the station, then Singh is late for his meeting. Singh is not late for his meeting. The train did arrive late. Therefore, there were taxis at the station.

If it is raining and Nimmy does not have her umbrella with her, then she will get wet. It is raining, Therefore, Nimmy has her umbrella with her.

If p and Not q then r, Not r, p, Therefore q

If p and Not q then r, p, Therefore q

ARGUMENTS - EXAMPLES

- The bank safe was robbed last night . Whoever robbed the safe knew the safe's combination. Only two people know the safe's combination: Amar and Akbar. Amar needed money to pay his rentals. Amar was seen sneaking around outside the bank last night. Therefore Amar robbed the bank.
- The person who broke open my locker has left finger prints on the door nob. Shanthi is the only person in the world who has these fingerprints. Therefore Shanthi broke open my locker.

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IDENTIFYING ARGUMENTS

- An argument will have premise and conclusion.
- Try to identify what is that you are expected to be convinced (conclusion) about, then try to determine why you ought to be convinced (premise).
- If the above method doesn't work, then you can look for indicative words like since & because to identify the premise and words like hence & therefore that precedes conclusion.

Flaws:

- **False Premise:** All reptiles are green. Bernie Madoff is a reptile. So, Bernie Madoff is green.
- **Bad Form:** Iraq borders Syria. So, The Hague is in Netherlands.

ARE THESE ARGUMENTS?

Ajay would not reveal a third of tax information. He must be hiding something shady. Yes

Price has fallen since the company insiders sold their stake No

Anjali is not American citizen since she has not revealed a third birth certificate. Yes

Chaitanya believes that the state should intervene in cases of market failure. Thus, since all adherents of government intervention are socialists, Chaitanya is socialist. Yes

He couldn't get a loan since he'd had a short sale within the last seven years. No

TERMINOLOGIES

- **Proposition:** Assertion that expresses an opinion
- **Compound Proposition:** Assembly of multiple assertions that expresses a opinion
- **Tautology:** Logical statement in which the conclusion is equivalent to the premise.
- **Parsing:** Analyse a string of symbols into logical syntactic components.
- **Conjectures :** An opinion or conclusion formed on the basis of incomplete information.

PROPOSITIONS

- Proposition
  - Crow is black in Color (T)
  - The Person who committed murder didn't have three hands (T)
  - All cows are white in color (F)
- Not a proposition
  - What is your name? (Question)
  - Come home immediately (Command)
  - Wow what a lovely song (Exclamation)
- Open problems(Conjecture)
  - Every even number bigger than 2 is a sum of two prime numbers

PROPOSITION LOGIC

SYNTAX

COMPONENTS

- **Variables:** These are used to represent the proposition.
- **Connectives:** These are used to connect two or more sentences in a grammatically valid way.
- **Constants:** $\top$ and $\bot$
- **Punctuations:** These are used to avoid the ambiguity in the order of processing
- **Alphabet set:**

  $\{\ ), (, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \top, \bot, p_0, p_1, p_2, \dots \}$

- **Expression:** $(\neg p_0 \rightarrow ()\ \wedge\ p_1 \vee,\ \neg p_{100})(\rightarrow\ \vee,\ (\neg p_0 \rightarrow p_1).$

Proposition

S. No.	connective	Description	Name
1	$\neg$	not	Negation
2	$\wedge$	and	Conjunction
3	$\vee$	or	Disjunction
4	$\rightarrow$	if … then $(\neg p \vee q)$	Conditional
5	$\leftrightarrow$	if and only if	Biconditional

RULES

FORMATION

- $\top$ and $\bot$ are propositions.
- Each p_i is a proposition, where $i \in N$.
- If x is a proposition, then $\neg x$ is a proposition.
- If x, y are propositions, then $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$, $(x \leftrightarrow y)$ are propositions.
- Nothing else is a proposition unless it satisfies some or all of the rules above.

Closure rule

PRODUCTION

Can be

- $\langle \text{proposition} \rangle \mapsto \langle \top \rangle$
- $\langle \text{proposition} \rangle \mapsto \langle \bot \rangle$
- $\langle \text{proposition} \rangle \mapsto \langle \text{propositional variable} \rangle$
- $\langle \text{proposition} \rangle \mapsto \neg \langle \text{proposition} \rangle$
- $\langle \text{proposition} \rangle \mapsto (\langle \text{proposition} \rangle \wedge \langle \text{proposition} \rangle)$
- $\langle \text{proposition} \rangle \mapsto (\langle \text{proposition} \rangle \vee \langle \text{proposition} \rangle)$
- $\langle \text{proposition} \rangle \mapsto (\langle \text{proposition} \rangle \rightarrow \langle \text{proposition} \rangle)$
- $\langle \text{proposition} \rangle \mapsto (\langle \text{proposition} \rangle \leftrightarrow \langle \text{proposition} \rangle)$

DEFINING PROPOSITIONS

- PROP : Set of all propositions
  - Smallest set of expressions such that $\{\top, \bot, p_0, p_1, \ldots\} \subseteq$ PROP
  - If $x, y \in$ PROP, the expressions $\neg x, (x \wedge y), (x \vee y), (x \rightarrow y), (x \leftrightarrow y)$ are also in PROP.
- Recursive Definition

$$\mathrm{PROP}(0) = \{p_0, p_1, \ldots\} \cup \{\top, \bot\}$$
$$\mathrm{PROP}(i+1) = \mathrm{PROP}(i) \cup \{\neg x, (x \wedge y), (x \vee y), (x \rightarrow y), (x \leftrightarrow y) : x, y \in \mathrm{PROP}(i)\}$$
$$\mathrm{PROP} = \cup_{i \in N} \mathrm{PROP}(i)$$

Grammatical rules

$w ::= \top \mid \bot \mid p \mid \neg w \mid (w \wedge w) \mid (w \vee w) \mid (w \rightarrow w) \mid (w \leftrightarrow w)$ **(Backus-Naur Form)**

Where p represent any generic proposition variable, w represent any generic proposition and | describes alternate possibilities.

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CHECK FOR WELL FORMEDNESS

- $p_1 \rightarrow p_2$ is not a proposition

  $(p_1 \rightarrow p_2)$ is a proposition.

- $(p_0 \wedge p_1 \rightarrow p_2)$ is not a proposition

  $((p_0 \wedge p_1) \rightarrow p_2)$ or $(p_0 \wedge (p_1 \rightarrow p_2))$ is a proposition

- $\neg((p_0 \wedge \neg p_1) \rightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$ is a proposition

- $(\vee (p_0 \wedge \neg p_1) \rightarrow (p_2 \leftrightarrow \neg p_3)$ is not a proposition

  $((p_0 \wedge \neg p_1) \rightarrow (p_2 \leftrightarrow \neg p_3))$

PROPOSITION FORMULA

- $(\neg p_0 \rightarrow () \wedge p_1 \vee$ No
- $(\neg p_0 \rightarrow p_1)$ Yes
- $((p_0 \vee (p_1 \wedge p_2)) \rightarrow (\neg p_1 \leftrightarrow p_2))$ Yes
- $(\vee \rightarrow \neg p_1 \leftrightarrow p_2)$ No
- $((p_1 \wedge p_2) \rightarrow (\neg p_3 \vee p_2))$ Yes

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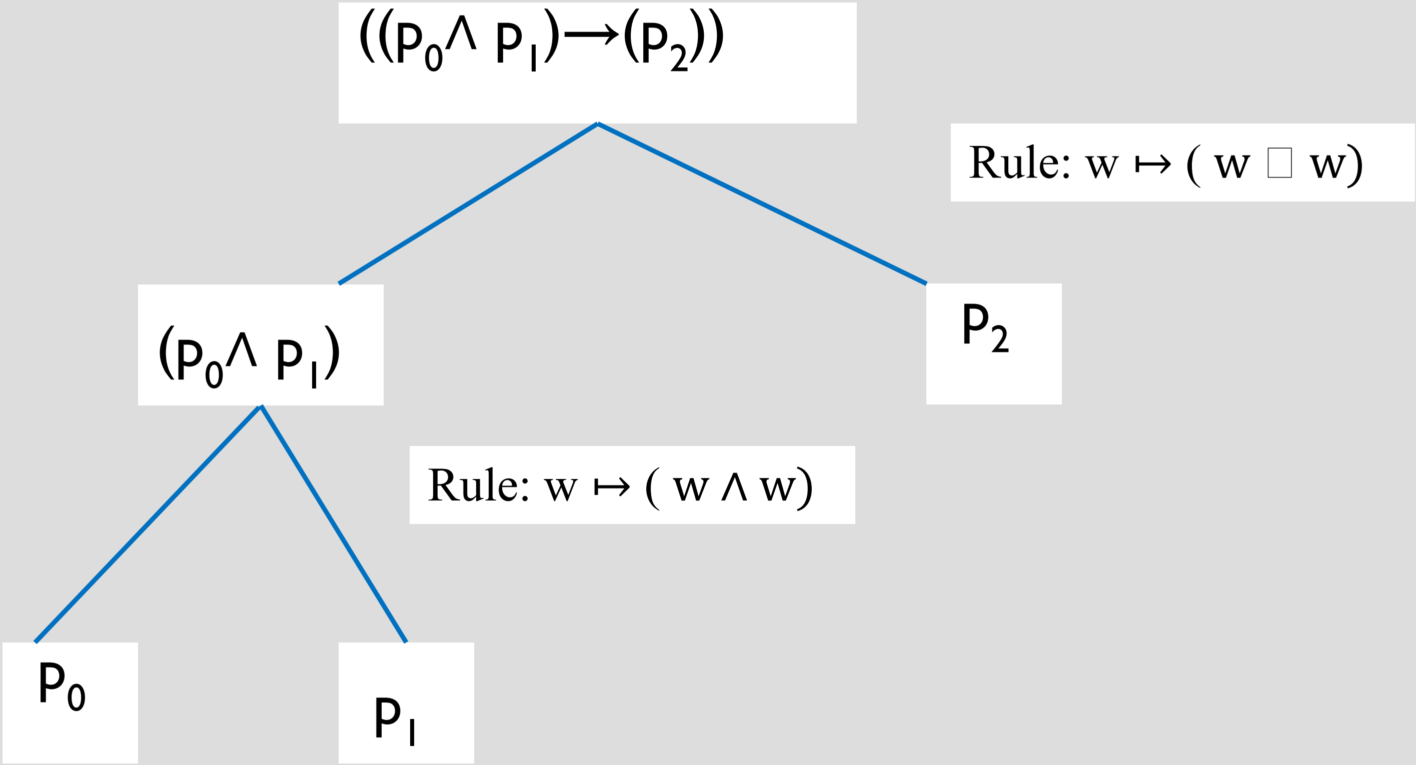
PARSE TREE



$\neg((p_0 \wedge \neg p_1) \rightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$

Rule: $w \mapsto \neg w$

$((p_0 \wedge \neg p_1) \rightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$

Rule: $w \mapsto (w \;\square\; w)$

$(p_0 \wedge \neg p_1)$

Rule: $w \mapsto (w \wedge w)$

p_0

$\neg p_1$

Rule: $w \mapsto \neg w$

p_1

$(p_2 \vee (p_3 \leftrightarrow \neg p_4))$

Rule: $w \mapsto (w \vee w)$

p_2

$(p_3 \leftrightarrow \neg p_4)$

Rule: $w \mapsto (w \leftrightarrow w)$

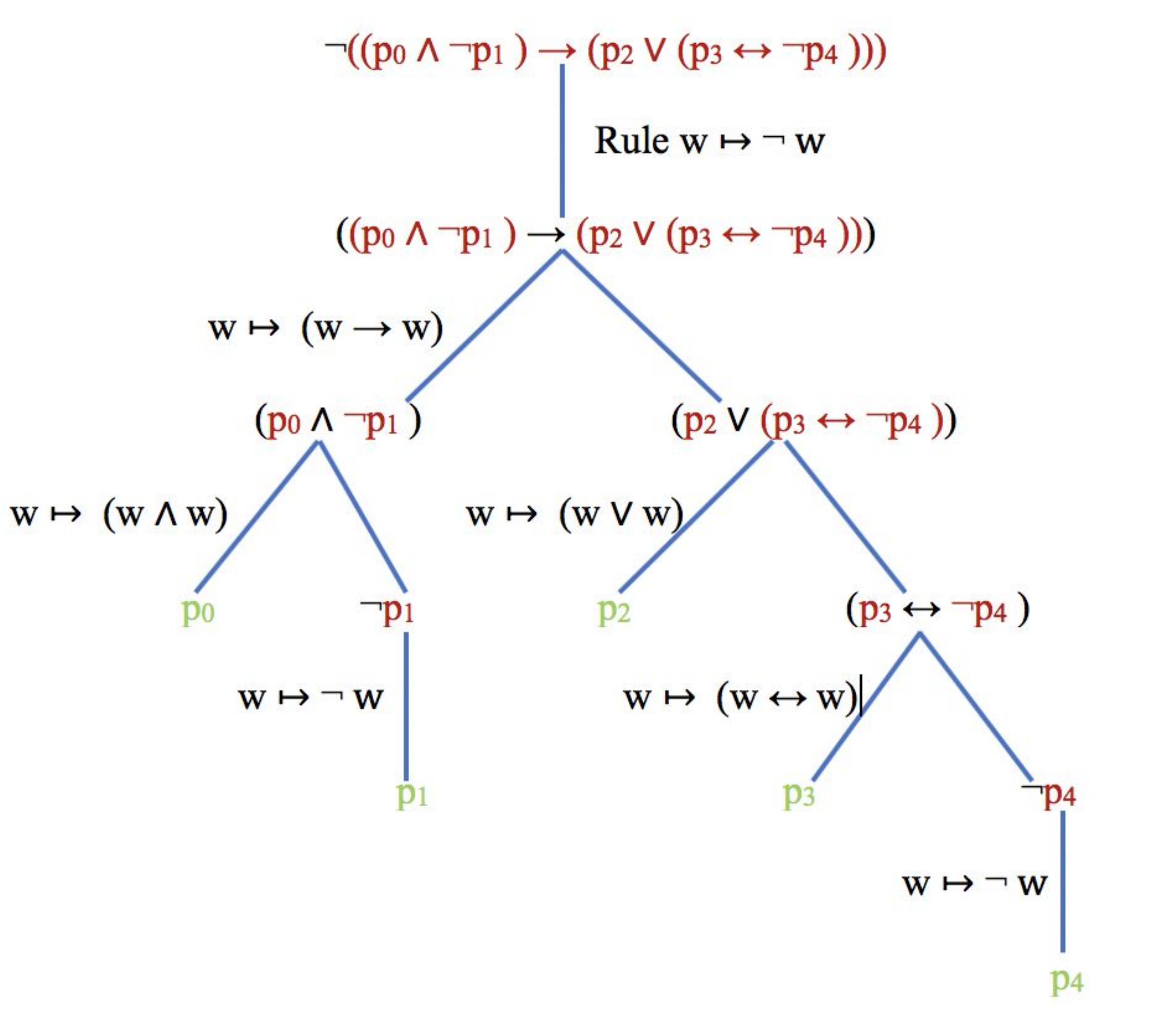
p_3

$\neg p_4$

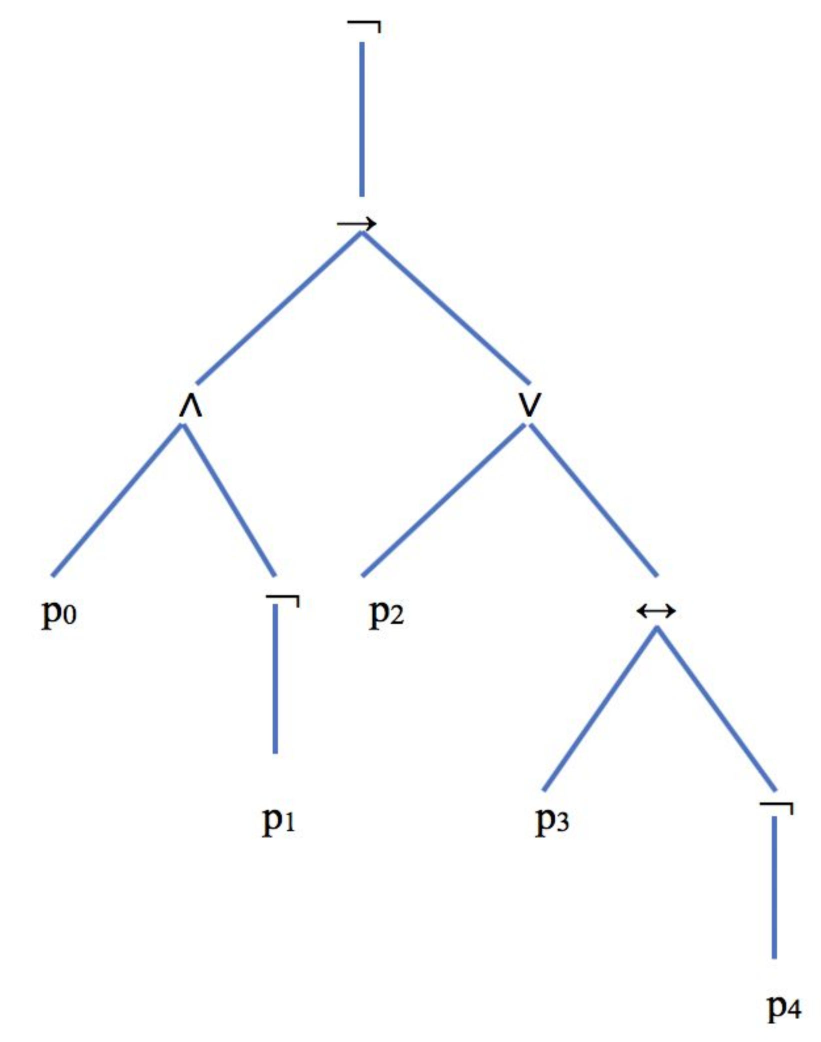
Rule: $w \mapsto \neg w$

p_4

PARSE TREE



¬((p0 ∧ ¬p1) → (p2 ∨ (p3 ↔ ¬p4)))
Rule w ↦ ¬ w
((p0 ∧ ¬p1) → (p2 ∨ (p3 ↔ ¬p4)))
w ↦ (w → w)
(p0 ∧ ¬p1)
(p2 ∨ (p3 ↔ ¬p4))
w ↦ (w ∧ w)
w ↦ (w ∨ w)
p0
¬p1
p2
(p3 ↔ ¬p4)
w ↦ ¬ w
w ↦ (w ↔ w)
p1
p3
¬p4
w ↦ ¬ w
p4



¬
→
∧
∨
p0
¬
p2
↔
p1
p3
¬
p4

PROPDET

1. If e is an atomic proposition, then print yes and stop
2. Scan e from left to get a subexpression w of one of the forms $\neg x$, $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$, $(x \leftrightarrow y)$, where x and y are atomic propositions.
3. If subexpression not found, then print no and stop
4. If sub expression found, then replace w by x
5. Go to step 1

Sub proposition

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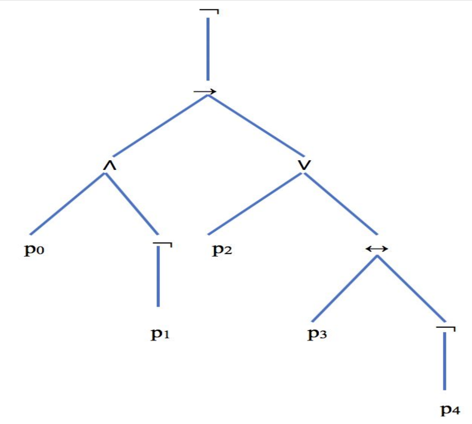
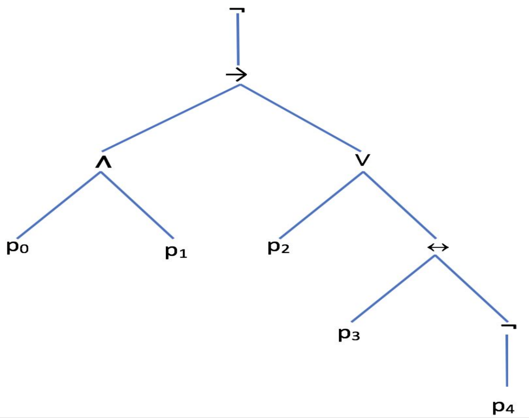
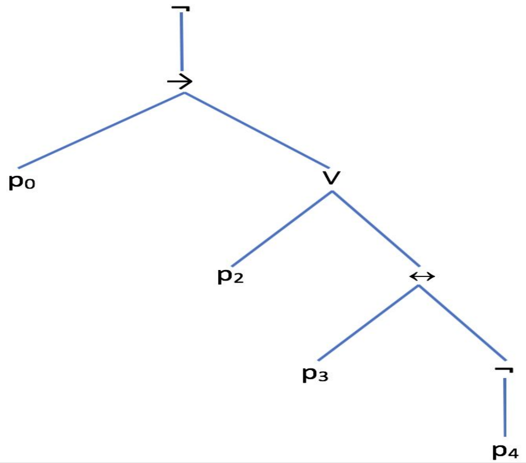
T_0



T_1



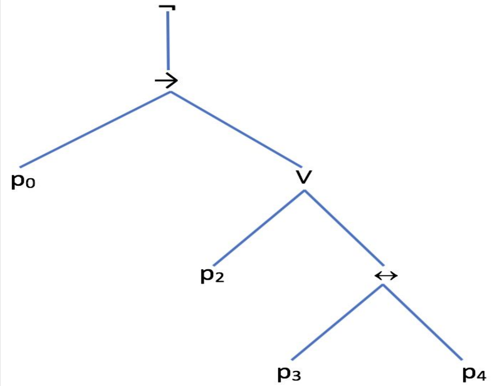
T_2



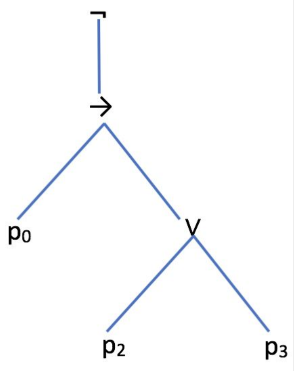
T_7

p_0

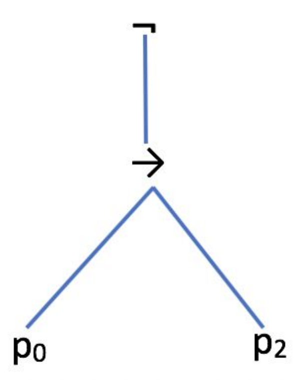
T_3



T_4



T_5



T_6

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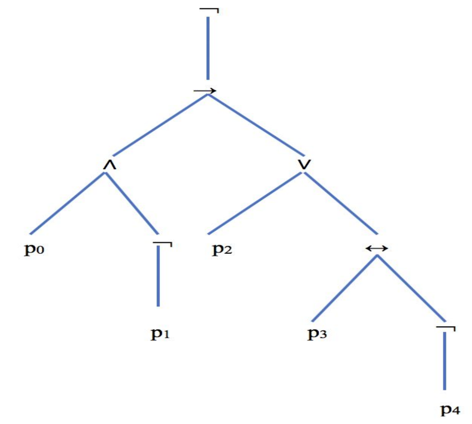
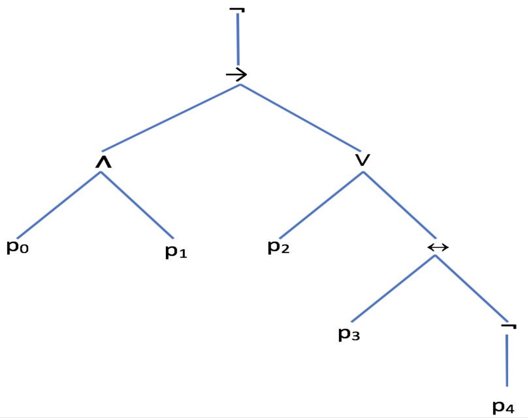
T_0



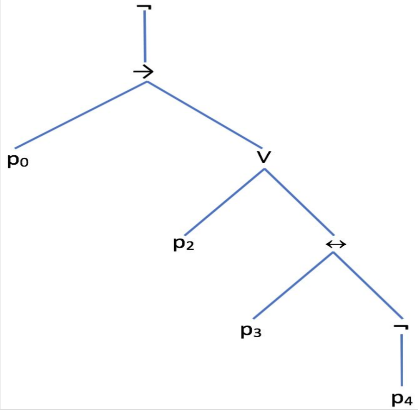
$\neg((p_0 \wedge \neg p_1) \longrightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$

T_1



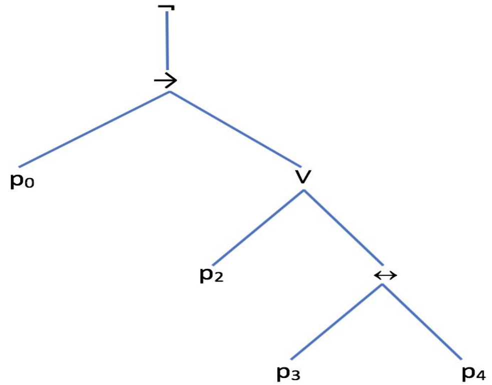
$\neg((p_0 \wedge p_1) \longrightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$

T_2



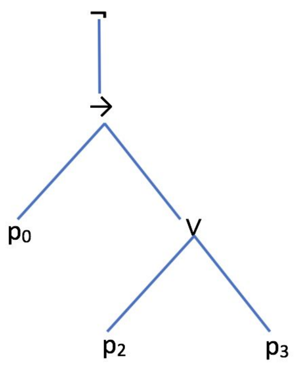
$\neg(p_0 \longrightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$

T_3



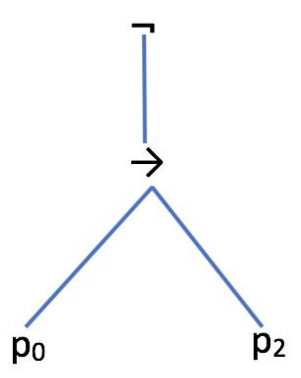
$\neg(p_0 \longrightarrow (p_2 \vee (p_3 \leftrightarrow p_4)))$

T_4



$\neg(p_0 \longrightarrow (p_2 \vee$

T_5



$\neg(p_0 \longrightarrow$

T_6

$\neg p_0$

T_7

p_0

p_0

EXERCISE

- $((p_0 \rightarrow p_1) \vee (p_2 \rightarrow (\neg p_1 \wedge (p_0 \leftrightarrow \neg p_2))))$
- $(((p_5 \rightarrow (p_6 \vee p_8)) \leftrightarrow (p_3 \wedge \neg p_2)) \vee (\neg(p_1 \leftrightarrow p_3) \rightarrow p_{10}))$

UNIQUE PARSING

- Each proposition is parsed in exactly one way; it has a unique parse tree.
- Properties of Proposition
  - Each proposition has the same number of left and right parentheses.
  - If u is a prefix of a proposition w , then u = ∈, or u =w or u is a sequence of ¬'s, or the number of left parentheses is more than the number of right parentheses in u.
  - If u and w are propositions and u is a prefix of w, then u=w

$$\neg((p_0 \wedge \neg p_1) \rightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$$

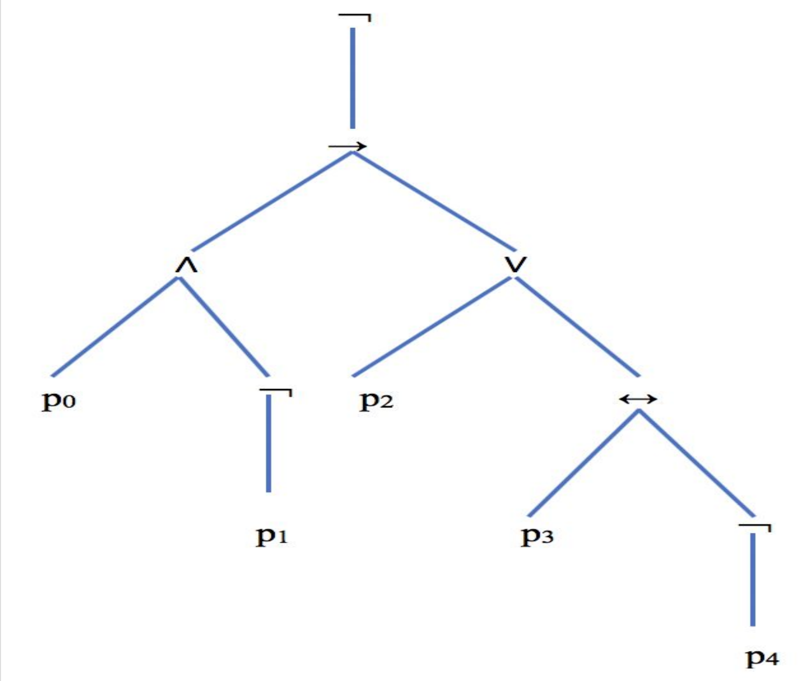
SUB-PROPOSITION

- A sub-proposition of w, is a proposition corresponding to any subtree of the parse tree of w.
- Immediate sub-proposition of a proposition w is any proposition corresponding to a subtree of the parse tree T_w of w whose depth is one less than T_w

Subproposition of $\neg x$ is x

Subproposition of $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$ and $(x \leftrightarrow y)$ are x and y

$$\neg((p_0 \wedge \neg p_1) \rightarrow (p_2 \vee (p_3 \leftrightarrow \neg p_4)))$$



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PRECEDENCE RULES

- $\neg$ has the highest precedence
- $\wedge$ and $\vee$ have the next precedence, both being equal
- $\rightarrow$ and $\leftrightarrow$ have the least precedence, both being equal

$((p \vee (q \wedge s)) \rightarrow (t \leftrightarrow \neg p))$ $\quad$ $p \vee (q \wedge s) \rightarrow (t \leftrightarrow \neg p)$

Remove redundant parentheses without affecting the uniqueness of expression

$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$

$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$

$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$

$((a \vee b) \wedge (\neg(a \wedge b)))$

$a \vee b \wedge \neg(a \wedge b)$

$((a \vee b) \wedge (\neg(a \wedge b)))$

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PRECEDENCE RULES

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$((p \vee (q \wedge s)) \rightarrow (t \leftrightarrow \neg p))$ $\qquad$ $p \vee (q \wedge s) \rightarrow (t \leftrightarrow \neg p)$

Insert parentheses in appropriate p

$(p \rightarrow q) \wedge \neg(r \vee q \leftrightarrow p) \leftrightarrow (\neg p \vee$

$(((p \rightarrow q) \wedge \neg((r \vee q) \leftrightarrow p)) \leftrightarrow ((\neg p \vee q) \rightarrow r))$

$(p \rightarrow q) \leftrightarrow (r \rightarrow t \vee p) \wedge (p \vee q \rightarrow \neg p \wedge t)$

$((p \rightarrow q) \leftrightarrow ((r \rightarrow (t \vee p)) \wedge ((p \vee q) \rightarrow (\neg p \wedge t))))$

$p \vee (\neg q \leftrightarrow r \wedge p) \leftrightarrow (p \vee p \rightarrow \neg q)$

$((p \vee (\neg q \leftrightarrow (r \wedge p))) \leftrightarrow ((p \vee p) \rightarrow \neg q))$

→

TRUTH TABLES FOR CONNECTORS

Conjunction

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Conditional

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

$(p \land q) \lor \neg p$

Bi-Conditional

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

$(p \land q) \lor (\neg p \land \neg q)$

TAUTOLOGY

- Definition: A compound statement, that is always true regardless of the truth value of the individual statements, is defined to be a **tautology**

$(p \wedge q) \rightarrow p$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

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CHECK FOR TAUTOLOGY

$(p \land q) \rightarrow (p \lor q)$

p	q	p ∧ q	p ∨ q	(p ∧ q) → (p ∨ q)
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

$(p \rightarrow q) \land p$

p	q	p → q	(p → q) ∧ p	[(p → q) ∧ p] → p
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$[(p \rightarrow q) \land p] \rightarrow q$

PROPOSITION LOGIC

SEMANTICS

TERMINOLOGIES

- **Interpretation:** Assignment of meaning to the symbols of a formal language.
- **Valuation**: Assignment of truth values to propositional variables with a corresponding assignment of truth values to all propositional formulas with those variables.
- **Relevance Lemma:** Let w be a proposition and A_w be the set of all atomic subpropositions of w. Let s, t be two interpretations of w. If s(p)=t(p) for every p $\in A_w$, then s(w) = t(w).

SEMANTICS

- Meaning – {0, 1}
- Truth Values

  $\top$ is 1 and that of $\perp$ is 0

p	¬p
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

TRUTH ASSIGNMENT & VALUATION

- A truth assignment is any partial function t: AT $\rightarrow\{0,1\}$, where AT= {the set of all atomic propositions including $\top$ and $\bot$ } with $t(\top)=1$ and $t(\bot)=0$.
- A truth valuation is any partial function which is an extension of a truth assignment to the set PROP of all propositions satisfying the following properties:
- $v(\neg x) = 1$ if $v(x) = 0$, else, $v(\neg x) = 0$
- $v(x \wedge y) = 1$ if $v(x) = v(y) = 1$, else, $v(x \wedge y) = 0$
- $v(x \vee y) = 0$ if $v(x) = v(y) = 0$, else, $v(x \vee y) = 1$
- $v(x \rightarrow y) = 0$ if $v(x) = 1, v(y) = 0$, else, $v(x \rightarrow y) = 1$
- $v(x \leftrightarrow y) = 1$ if $v(x) = v(y)$, else, $v(x \leftrightarrow y) = 0$

  for any propositions $x, y \in$ PROP.

VALUATION – TRUTH TABLE

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

$u = \neg(p \wedge q) \rightarrow (p \vee (r \leftrightarrow \neg q))$

p	q	r	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$r \leftrightarrow \neg q$	$p \vee (r \leftrightarrow \neg q)$	u
0	0	0	1	0	1	0	0	0
1	0	0	1	0	1	0	1	1
0	1	0	0	0	1	1	1	1
1	1	0	0	1	0	1	1	1
0	0	1	1	0	1	1	1	1
1	0	1	1	0	1	1	1	1
0	1	1	0	0	1	0	0	0
1	1	1	0	1	0	0	1	1

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

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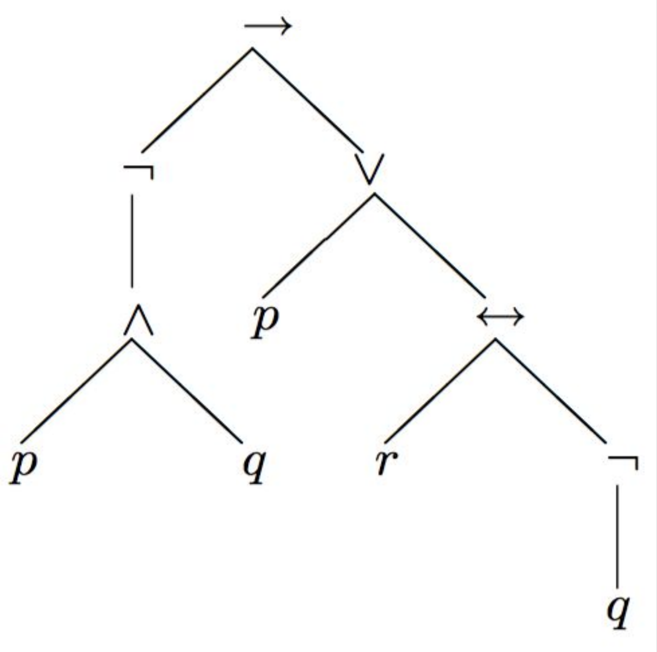
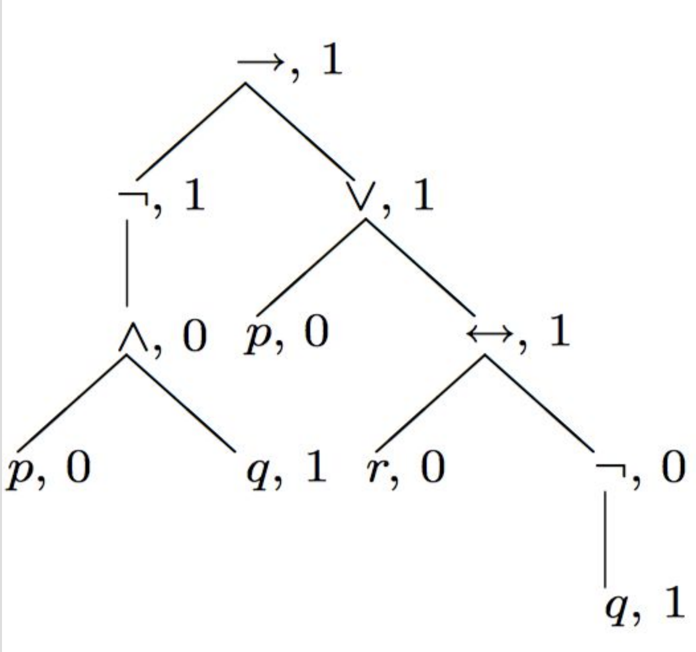
VALUATION – PARSE TREE





EXERCISE

- $v = \neg(p \wedge q) \vee r \rightarrow \neg p \vee (\neg q \vee r)$
- $w = (p \wedge q \rightarrow r) \leftrightarrow p \wedge (q \wedge \neg r)$

p	q	r	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \vee r$ (1)	$\neg q \vee r$	$\neg p \vee (\neg q \vee r)$ (2)	(1) $\rightarrow$ (2)
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	1	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	1	1	1
1	0	0	0	1	1	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	0	1	1	1	1

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$ (1)	$q \wedge \neg r$	$p \wedge (q \wedge \neg r)$ (2)	(1) $\leftrightarrow$ (2)
0	0	0	0	1	0	0	0
0	0	1	0	1	0	0	0
0	1	0	0	1	1	0	0
0	1	1	0	1	0	0	0
1	0	0	0	1	0	0	0
1	0	1	0	1	0	0	0
1	1	0	1	0	1	1	0
1	1	1	1	1	0	0	0

EXERCISE

- $e = (p \rightarrow (\neg p \rightarrow p)) \rightarrow (p \rightarrow (p \rightarrow \neg p))$
- $s = (p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$

p	$\neg p$	$\neg p \rightarrow p$	$p \rightarrow \neg p$	$p \rightarrow (\neg p \rightarrow p)$	$p \rightarrow (p \rightarrow \neg p)$	e
0	1	0	1	1	1	1
1	0	1	0	1	0	0

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$p \rightarrow \neg q$	$(p \rightarrow \neg q) \rightarrow \neg p$	s
0	0	1	1	1	1	1	1
0	1	1	0	1	0	1	1
1	0	0	1	0	1	0	1
1	1	0	0	1	0	1	1

MODEL

p	q	r
0	0	0
1	0	0
0	1	0
1	1	0
0	0	1
1	0	1
0	1	1
1	1	1

- A model of proposition w is an interpretation i of w such that i(w) = 1. The fact that i is a model of w is written as i ⊨w (satisfies). The fact that i is not a model of w is written as i ⊭ w (falsifies)
- Let us consider i to be the interpretation as represented in row 1 of the truth tables of v, w. That is i(p) = i(q) = i(r)=0. From the tables we can derive that i ⊨ v and i ⊭ w as i(v)=1 and i(w)=0
- Consider j to be the interpretation of j(p) = j(q) = 0, j(r)= 1. Indicate whether j satisfies/falsifies the propositions v or w

j ⊨ v and j ⊭ w

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REPRESENTATIONS

- **Connectives as truth functions:** Formally it is represented as p(i)=i(p), $\neg x(i) = \neg(x(i))$, $(x*y)(i) = *(x(i), y(i))$ for $* \in \{\wedge, \vee, \leftrightarrow, \quad\}$.
- **Set of truth assignments:**
  - T - The set of all possible truth assignments,
  - i - any truth assignment in T
  - p - any propositional variable,
  - x, y any proposition(s).
  - Properties
    - $M(\top) = T$, $M(\bot) = \varnothing$
    - $M(p) = \{i : i(p) = 1\}$

REPRESENTATIONS

- $M(\neg x) = T - M(x)$
- $M(x \land y) = M(x) \cap M(y)$
- $M(x \lor y) = M(x) \cup M(y)$
- $M(x \rightarrow y) = (T - M(x)) \cup M(y)$
- $M(x \leftrightarrow y) = (M(x) \cap M(y)) \cup ((T - M(x)) \cup (T - M(y)))$

- **Set of Literals**: Each model is a set of atomic propositions

Formal representation is $i: AT \rightarrow \{0, 1\}$.

Rule: For a truth assignment i, $M_i = \{p : i(p) = 1\} \cup \{\neg p : i(p) = 0\}$.

Models of $p \lor q$ represented: $\{p, q\}$ $\{\neg p, q\}$ $\{p, \neg q\}$

TERMINOLOGIES

- **Valid (Tautologies):** If each interpretation is its model {all i} ⊨ w
- **Satisfiable:** If there is a model i ⊨ w
- **Invalid:** If there are some interpretations that are not its model {some i ⊭ w }
- **Unsatisfiable (Contradictions):** If there is no model i ⊭ w
- **Contingent:** Invalid and satisfiable
- **Equivalence:** u and w are equivalent if every model of u is a model of v and every model of v is a model of u u ≡ v

TERMINOLOGIES

- **Consequence:** Let ∑ be a set of propositions. The set ∑ is called satisfiable if ∑ has a model. **Σ semantically entails w** (Σ ⊨ w) if each model of Σ is a model of w.

For a consequence Σ ⊨ w,

The propositions in Σ are called the **premises or hypotheses**

w is called the **conclusion.**

EXAMPLE

Show that the following argument is valid: If the band performs, then the hall will be full provided that the tickets are not too costly. However, if the band performs, the tickets will not be too costly. Therefore, if the band performs, then the hall will be full.

Declarative sentences

p: the band performs, q: the hall is (will be) full, r: tickets are not too costly

Hypotheses:

$p \rightarrow (r \rightarrow q)$, $p \rightarrow r$

Conclusion: $p \rightarrow q$

p	q	r	$p \rightarrow r$	$r \rightarrow q$	$p \rightarrow (r \rightarrow q)$	$p \rightarrow q$
0	0	0	1	1	1	1
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	1	0	0	0
1	1	0	0	1	1	1
1	1	1	1	1	1	1

TRUTH TABLE

Alternative way to show Σ ⊨ w.

- Common models are to be checked for, if p□ q evaluates to true in these rows, then the argument is correct.
- Whenever the conclusion is falsified, at-least one of the premises is also falsified. Hence the consequence is valid.
- This method lead to RAA theorem

p	q	r	$p \rightarrow r$	$r \rightarrow q$	$p \rightarrow (r \rightarrow q)$	$p \rightarrow q$
0	0	0	1	1	1	1
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	1	0	0	0
1	1	0	0	1	1	1
1	1	1	1	1	1	1

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DEFINITIONS

- **Axiom:** Statement that is accepted without proof and regarded as fundamental to a subject.

Starting point for reasoning

- **Theorem:** Statement which can be shown to be the conclusion of a logical argument which depends on no premises except axioms.
- **Lemma:** Minor result used to help prove a theorem
- **Rule:** Is a theorem that establishes a useful formula
- **Law/Principle:** A theorem that applies in a wide range of circumstances.
- **Conjecture**: An unproved statement that is believe

Used to show the existence of proof

- **Meta theorem:** Statement about a formal system proven in a metalanguage and proved within a metatheory by referencing to the concepts present in the metatheory.
- **Paradox**: Statement that can be shown, to be both true and false.

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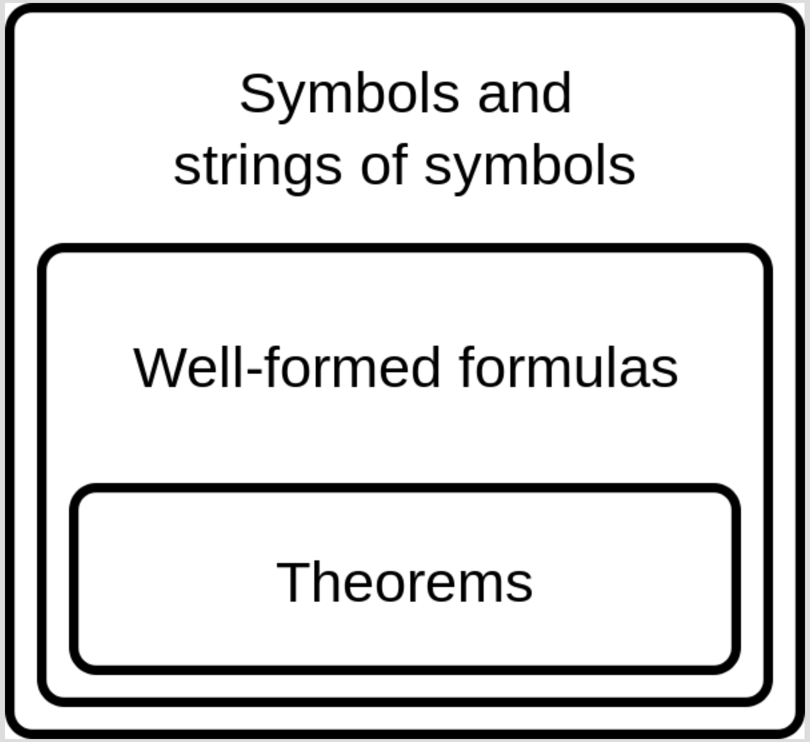
THEOREMS

- Let u and v be propositions. Then
  - $\models u$ iff $u \equiv \top$ iff $\top \models u$ iff $\varnothing \models u$.
  - $u \models v$ iff $\models u \rightarrow v$.
  - $u \equiv v$ iff $\models u \leftrightarrow v$ iff ($u \models v$ and $v \models u$).
  - u is unsatisfiable iff $u \equiv \bot$ iff $u \models \bot$.
- Σ is unsatisfiable iff $\Sigma \models w$ for each w.



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$p \to (r \to q), p \to r$

$p \to q$

META THEOREMS

- **Reduction ad absurdum (RAA):** Let Σ be a set of propositions and w be any proposition. Then
  - $\Sigma \vDash w$ iff $\Sigma \cup \{\neg w\}$ is unsatisfiable.
  - $\Sigma \vDash \neg w$ iff $\Sigma \cup \{w\}$ is unsatisfiable

show that $\{p \to r, p \to (r \to q), p, \neg q\}$ is unsatisfiable

p	q	r	$\neg q$	$p \to r$	$p \to (r \to q)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	0	0	1
1	1	1	0	1	1

- **Deduction**: Let Σ be a set of propositions and x, y be propositions. Then $\Sigma \vDash x \to y$ iff $\Sigma \cup \{x\} \vDash y$.

show that $\{p \to r, p \to (r \to q), p\} \vDash q$

p	q	r	$p \to r$	$r \to q$	$p \to (r \to q)$	$p \to q$
1	0	0	0	1	1	0
1	0	1	1	0	0	0
1	1	0	0	1	1	1
1	1	1	1	1	1	1

META THEOREMS

- **Monotonicity:** Let w be a proposition and Σ, Γ be sets of propositions with $\Sigma \subseteq \Gamma$.
  - If $\Sigma \vDash w$, then $\Gamma \vDash w$.
  - If Σ is unsatisfiable, then Γ is unsatisfiable.

EQUIVALENCE SUBSTITUTION

- Let x, y, w be propositions
- Let w[x := y] denote any proposition obtained from w by substituting some or all or no occurrences of x by y in w.
- If $x \equiv y$, then $w \equiv w[x := y]$.

$$p \rightarrow (r \rightarrow q)$$

PARADOXES OF MATERIAL IMPLICATION

- When we translate if…then or implies of English with material implication, nothing of logical importance is lost.
- Two paradoxes
  - Whenever the antecedent is false, the whole conditional is true
  - Whenever the consequent is true, the conditional is true
- These are paradoxes in the ancient sense, violations of intuition.
- They are not contradictions

→

p	q	p ⊃ q
T	T	T
T	F	F
F	T	T
F	F	T

LAWS OF REPLACEMENT

- Law of constants: $x \wedge \top \equiv x, x \wedge \bot \equiv \bot, x \vee \top \equiv \top, x \vee \bot \equiv x, x \rightarrow \top \equiv \top, x \rightarrow \bot \equiv \neg x, \top \rightarrow x \equiv x, \bot \rightarrow x \equiv \top, x \leftrightarrow \top \equiv x, x \leftrightarrow \bot \equiv \bot, \neg\top \equiv \bot, \neg\bot \equiv \top$
- Law of excluded middle: $x \vee \neg x \equiv \top$
- Law of contradiction: $x \wedge \neg x \equiv \bot, x \leftrightarrow \neg x \equiv \bot$
- Law of double negation: $\neg\neg x \equiv x$
- Law of identity: $x \equiv x, x \leftrightarrow x \equiv \top$
- Law of idempotency: $x \wedge x \equiv x, x \vee x \equiv x$

LAWS OF REPLACEMENT

- **Law of absorption:** $x \wedge (x \vee y) \equiv x, x \vee (x \wedge y) \equiv x$
- **Law of commutativity:** $x \wedge y \equiv y \wedge x, x \vee y \equiv y \vee x, x \leftrightarrow y \equiv y \leftrightarrow x$
- **Law of associativity**: $x \wedge (y \wedge z) \equiv (x \wedge y) \wedge z, x \vee (y \vee z) \equiv (x \vee y) \vee z, x \leftrightarrow (y \leftrightarrow z) \equiv (x \leftrightarrow y) \leftrightarrow z$
- **Law of distributivity:** $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$,

$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z), x \vee (y \rightarrow z) \equiv (x \vee y) \rightarrow (x \vee z)$,

$x \vee (y \leftrightarrow z) \equiv (x \vee y) \leftrightarrow (x \vee z), x \rightarrow (y \wedge z) \equiv (x \rightarrow y) \wedge (x \rightarrow z)$,

$x \rightarrow (y \vee z) \equiv (x \rightarrow y) \vee (x \rightarrow z), x \rightarrow (y \rightarrow z) \equiv (x \rightarrow y) \rightarrow (x \rightarrow z), x \rightarrow (y \leftrightarrow z) \equiv (x \rightarrow y) \leftrightarrow (x \rightarrow z)$

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LAWS OF REPLACEMENT

- **Law of de morgan :** $\neg(x \land y) \equiv \neg x \lor \neg y$, $\neg(x \lor y) \equiv \neg x \land \neg y$
- **Law of implication**: $x \to x \equiv T$, $x \to y \equiv \neg x \lor y$, $\neg(x \to y) \equiv x \land \neg y$, $x \to y \equiv x \leftrightarrow x \land y$, $x \to y \equiv x \lor y \leftrightarrow y$,
- **Law of contraposition**: $x \to y \equiv \neg y \to \neg x$
- **Law of hypothesis invariance**: $x \to (y \to x) \equiv \top$
- **Law of hypothetical syllogism**: $\{x \to y, y \to z\} \models x \to z$

LAWS OF REPLACEMENT

- **Law of exportation:** $x \rightarrow (y \rightarrow z) \equiv (x \wedge y) \rightarrow z \equiv y \rightarrow (x \rightarrow z)$
- **Law of biconditional:** $x \leftrightarrow y \equiv (x \rightarrow y) \wedge (y \rightarrow x)$, $x \leftrightarrow y \equiv (x \wedge y) \vee (\neg x \wedge \neg y)$, $\neg(x \leftrightarrow y) \equiv (x \wedge \neg y) \vee (\neg x \wedge y)$, $\neg(x \leftrightarrow y) \equiv \neg x \leftrightarrow y \equiv x \leftrightarrow \neg y$
- **Law of introduction**: $\{x, y\} \vDash x \wedge y, x \vDash x \vee y, y \vDash x \vee y$
- **Law of elimination** : $x \wedge y \vDash x$,
- **Law of modus ponens**: $\{x, x \rightarrow y\} \vDash y$

LAWS OF REPLACEMENT

- **Law of modus tollens:** $\{x \rightarrow y, \neg y\} \models \neg x$
- **Law of clavius:** $(\neg x \rightarrow x) \models x$
- **Law of the cases:** If $y \models z$, then $x \wedge y \models x \wedge z$. If $x \models z$ and $y \models z$, then $x \vee y \models z$.
- **Law of disjunctive syllogism:** $((x \vee y) \wedge \neg x) \models y$
- **Law of uniform substitution:** For any propositions x, y and any propositional variable p occurring in x, let x[p/y] denote the proposition obtained from x by substituting every occurrence of p by y.

OFTEN USED LAWS

- **Law of double negation:** $\neg\neg x \equiv x$
- **Law of de morgan :** $\neg(x \land y) \equiv \neg x \lor \neg y, \neg(x \lor y) \equiv \neg x \land \neg y$
- **Law of modus ponens**: $\{x, x \rightarrow y\} \models y$
- **Law of modus tollens:** $\{x \rightarrow y, \neg y\} \models \neg x$
- **Law of hypothetical syllogism**: $\{x \rightarrow y, y \rightarrow z\} \models x \rightarrow z$
- **Law of contraposition**: $x \rightarrow y \equiv \neg y \rightarrow \neg x$

$$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

Suppose $\mathbf{i((p \to (q \to r))} \to ((p \to q) \to (p \to r))) = 0$ for an interpretation i.

$i((p \to (q \to r)) = 1$ and $\mathbf{i((p \to q) \to (p \to r)) = 0.}$

$i(p \to q) = 1$ and $\mathbf{i(p \to r) = 0}$.

$i(p \to r) = 0$ gives $\mathbf{i(p) = 1}$ and $i(r) = 0$.

$i(p) = 1 = i(p \to q)$ implies that $i(q) = 1$.

But then, $i(q \to r) = 0$ gives $i((p \to (q \to r)) = 0$ contradicting the fact that $i((p \to (q \to r)) = 1$.

Hence, $i((p \to (q \to r)) \to ((p \to q) \to (p \to r))) = 1$, whatever be the interpretation.

Thus, $\models (p \to (q \to r)) \to ((p \to q) \to (p \to r))$. It is also satisfiable, since it has a model, e.g., the interpretation that assigns each of p, q, r to 0.

$$((P \to Q) \to (P \to R)) \land \neg(P \to (Q \to R))$$

Suppose $i(((p \to q) \to (p \to r)) \land \neg(p \to (q \to r))) = 1$, for an interpretation i.

$i((p \to q) \to (p \to r)) = 1 = \mathbf{i(\neg(p \to (q \to r)))}$.

$p \to (q \to r) = 0$

The last one says that $i(p) = 1$, $i(q) = 1$, and $i(r) = 0$.

But then, $i(p \to q) = 1$, $i(p \to r) = 0$;

consequently, $i((p \to q) \to (p \to r)) = 0$.

So, $i(((p \to q) \to (p \to r)) \land \neg(p \to (q \to r))) = 0$.

That is, the proposition is unsatisfiable. It is also invalid, e.g., the interpretation that assigns each of p, q, r to 0 falsifies the proposition

EXERCISES

EXPRESS THE FOLLOWING AS NATURAL ENGLISH SENTENCES

Let p stand for the proposition "I bought a lottery ticket" and q for "I won the jackpot".

- $\neg p$ — I did not buy a lottery ticket
- $p \vee q$ — I bought a lottery ticket or I won the jackpot
- $p \wedge q$ — I bought a lottery ticket and I won the jackpot
- $p \Rightarrow q$ — If I bought a lottery ticket then I won the jackpot
- $\neg p \Rightarrow \neg q$ — If I did not buy a lottery ticket then I did not win the jackpot
- $\neg p \vee (p \wedge q)$ — I did not buy a lottery ticket or I bought a lottery ticket and I won the jackpot

FORMALISE THE FOLLOWING IN TERMS OF ATOMIC PROPOSITIONS

Use p, q and r for atomic propositions and represent how they correspond to English sentence

- Berries are ripe along the path, but rabbits have not been seen in the area.
- Rabbits have not been seen in the area, and walking on the path is safe, but berries are ripe along the path.
- If berries are ripe along the path, then walking is safe only if rabbits have not been seen in the area.
- It is not safe to walk along the path, but rabbits have not been seen in the area and the berries along the path are ripe.
- For walking on the path to be safe, it is necessary but not sufficient that berries not be ripe along the path and for rabbits not to have been seen in the area.
- Walking is not safe on the path whenever rabbits have been seen in the area and berries are ripe along the path.

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CHECK IF THE GIVEN FORMULAE ARE TAUTOLOGIES/CONTRADICTIONS/ CONTINGENT

1. $p \to (q \to p)$
2. $(q \to p) \to p$
3. $(\neg p \to \neg q) \to (q \to p)$
4. $(p \to (q \to r)) \to ((p \to q) \to (q \to r))$
5. $(((p \land q) \leftrightarrow p) \to q)$
6. $((p \lor (p \land q)) \to (p \land (p \lor q)))$
7. $(((p \lor q) \land (\neg q \lor r)) \to (p \lor r))$
8. $(((p \land q) \land (\neg q \lor r)) \to (p \land r))$
9. $((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow ((p \leftrightarrow q) \land (q \leftrightarrow r))$
10. $(((p \land q \to r) \land (p \land \neg q \to r)) \leftrightarrow (p \to (q \leftrightarrow r))$

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