DISCRTE MATHEMATICS FOR ENGINEERS - 18MAB302T

Unit - I :

Sets and examples - Operators on sets - Laws of Set theory - Proving set identities using laws of set theory - Partition of a set - examples - Cartesian product of sets - Relations - properties - Equivalence relation and partial order relation - Poset - Graphs of ~~a~~ relation Digraphs - Hasse diagram - Problems - closure of relations - examples. Transitive closure and Warshall's algorithm.

Functions - definitions, domain and range of a function - Egs - Types of functions - one-one and onto bijection - examples - Composition of functions - examples - Associativity of composition of functions - Identity and inverse functions - Necessary and sufficient condition for existence of a function - Uniqueness of identity - Inverse of composition - checking if a function is bijection and if so, finding inverse, domain and range - problems: Applications of sets, relations and functions.

SET THEORY

Set: It is a collection of well-defined objects or elements. A set is represented in two ways: (i) Roaster form
(ii) Set builder form.

Eg: Set of all vowels in English alphabet.

(i) Roaster notation :

$$V = \{a, e, i, o, u\}$$

(ii) Set Builder form :

$$V = \{x / x \text{ is a vowel in English alphabets}\}$$

Operations on set :

(i) $A \cup B = \{x / x \in A \text{ or } x \in B\}$

(ii) $A \cap B = \{x / x \in A \text{ and } x \in B\}$

(iii) $A^c = \{x / x \notin A\}$

(iv) $A - B = \{x / x \in A \text{ but } x \notin B\}$

(v) $A \times B = \{(a, b) / a \in A, b \in B\}$

(vi) $A \oplus B = (A \cup B) - (A \cap B)$ [or] $(A-B) \cup (B-A)$

Let $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$ $B = \{4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

$A \cap B = \{\}$ (or) ϕ

$A - B = \{1, 2, 3\}$

$B - A = \{4, 5\}$

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$

$A \oplus B = \{1, 2, 3, 4, 5\}$.

Laws of set theory:

1. Identity laws: $A \cup \phi = A$
   $A \cap U = A$
2. Domination laws: $A \cup U = U$
   $A \cap \phi = \phi$
3. Idempotent laws: $A \cup A = A$
   $A \cap A = A$
4. Complement or Inverse law:
   $A \cup \bar{A} = U$
   $A \cap \bar{A} = \phi$
5. Double complement law (or) Involution law:
   $\bar{\bar{A}} = A$
6. Commutative law: $A \cup B = B \cup A$
   $A \cap B = B \cap A$
7. Distributive law:
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. Associative law:
   $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$
9. Absorption law:
   $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$
10. Demorgan's law:
   $\overline{A \cup B} = \bar{A} \cap \bar{B}$
   $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Partition of a set:

Let S be any non-empty set. the collection of subsets $A_1, A_2, \dots A_n$ are called partition of S if,

(i) $A_i \neq \phi$ for each i

(ii) $A_1 \cup A_2 \cdots \cup A_n = S$

(iii) $A_i \cap A_j = \phi$, for $i \neq j$

Eg: $S = \{1, 2, 3, \dots 10\}$

$A_1 = \{2, 4, 6, 8\}$, $A_2 = \{1, 3, 5\}$

$A_3 = \{7, 9\}$, $A_4 = \{10\}$

Problems on Set identities:

1. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ analytically.

Proof:

$L.H.S = A \cup (B \cap C)$

$= \{x \mid x \in A \text{ (or) } x \in (B \cap C)\}$

$= \{x \mid x \in A \text{ or } (x \in B \text{ and } x \in C)\}$

$= \{x \mid (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x \mid x \in (A \cup B) \text{ and } x \in (A \cup C)\}$

$= (A \cup B) \cap (A \cup C)$

$= R.H.S$

Hence the proof.

2. Prove analytically and graphically $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(i) Analytical Proof:

$L.H.S = A \cap (B \cup C)$

$= \{x \mid x \in A \text{ and } x \in B \cup C\}$

$= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\}$

$= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$

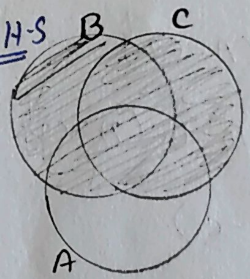
$= \{x \mid x \in (A \cap B) \text{ or } x \in (A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$

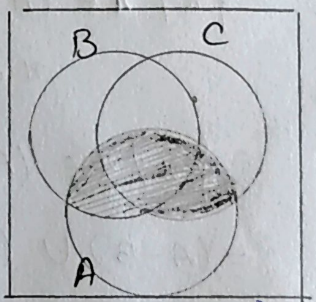
$= R.H.S.$ Hence the proof.

(ii) Graphical Proof by Venn diagram:

L.H.S

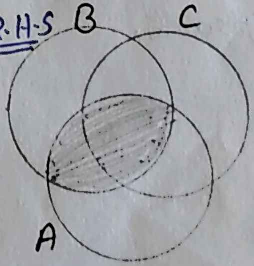


$B \cup C$

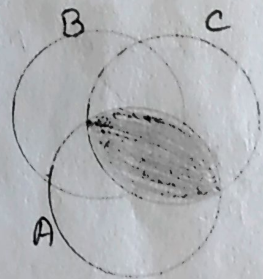


$A \cap (B \cup C)$

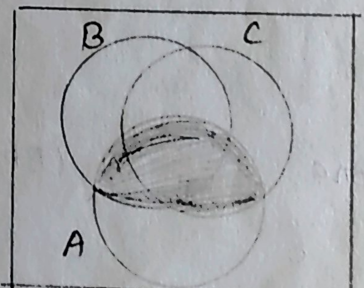
R.H.S



$A \cap B$



$A \cap C$



$(A \cap B) \cup (A \cap C)$

3. Prove both analytically and graphically that,

$A - (B \cap C) = (A - B) \cup (A - C)$

Proof:

$L.H.S = A - (B \cap C)$

$= \{x \mid x \in A \text{ and } x \notin (B \cap C)\}$

$= \{x \mid x \in A \text{ and } (x \notin B \text{ or } x \notin C)\}$

$= \{x \mid (x \in A \text{ and } x \notin B) \text{ or }$
$(x \in A \text{ and } x \notin C)\}$

$= \{x \mid x \in (A - B) \text{ or }$
$x \in (A - C)\}$

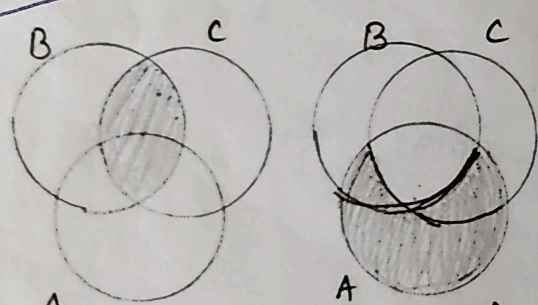
$= \{x \mid x \in (A - B) \cup (A - C)\}$

$= (A - B) \cup (A - C)$

$=$ R.H.S.

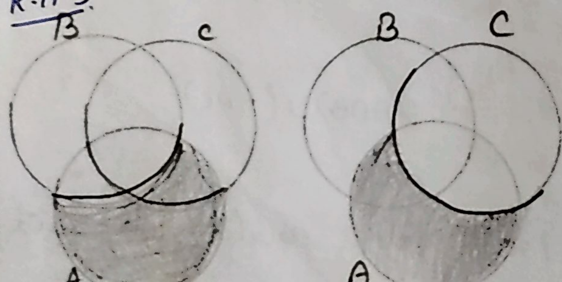
Hence the proof.

L.H.S

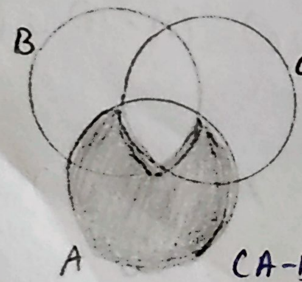


$B \cap C$ $A - (B \cap C)$

R.H.S.



$A - B$ $A - C$



$(A - B) \cup (A - C)$

4. Prove algebraically that
$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof:

L.H.S $= A \times (B \cap C)$

$= \{(x, y) \mid x \in A \text{ and } y \in B \cap C\}$

$= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and }$
$(x \in A \text{ and } y \in C)\}$

$= \{(x, y) \mid (x, y) \in A \times B \text{ and }$
$(x, y) \in A \times C\}$

$= \{(x, y) \mid (x, y) \in (A \times B) \cap$
$(A \times C)\}$

$= (A \times B) \cap (A \times C)$.

$=$ R.H.S

Hence the proof.

Eg:2 $S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{1, 3, 5\}$ $A_2 = \{2, 4\}$

is not a partition, since the union of subsets is not S.

If $A_1 = \{1, 2, 3\}$ $A_2 = \{4, 5\}$ $A_3 = \{6\}$ then the subsets form a partition.

Duality law (or) Principle of Duality:

The dual of an expression is obtained by interchanging $\cup$ and $\cap$, U and ϕ.

Eg:1 $A \cup \phi = A$

$\Rightarrow A \cap U = A$

Eg:2: $A \cap \bar{A} = \phi$

Dual is $A \cup \bar{A} = U$

Minsets:

Let A be a set. Let $\{B_1, B_2, \cdots B_n\}$ be a collection of subsets of A. Then the set of the form $D_1 \cap D_2 \cdots \cap D_n$ where each D_i may be either B_i or B_i^c is called a minset or minterms generated by $B_1, B_2, \cdots B_n$.

Note: For n subsets of A, the number of minterms is 2^n.

Eg:1 Let A be any set and B_1, B_2 be two subsets of A. The minsets of A are:

$B_1 \cap B_2$, $B_1^c \cap B_2$, $B_1 \cap B_2^c$, $B_1^c \cap B_2^c$.

Eg:2 If B_1, B_2, B_3 any subsets of A then minterms are:

$B_1 \cap B_2 \cap B_3$, $B_1^c \cap B_2 \cap B_3$, $B_1 \cap B_2^c \cap B_3$, $B_1 \cap B_2 \cap B_3^c$, $B_1^c \cap B_2^c \cap B_3$, $B_1 \cap B_2^c \cap B_3^c$, $B_1^c \cap B_2 \cap B_3^c$, $B_1^c \cap B_2^c \cap B_3^c$.

Problems:

1. Find the minsets generated by $\{0, 2, 4\}$ and $\{0, 5\}$ of the set $\{0, 1, 2, 3, 4, 5\}$ and also find the partition of A.

Sol: Given, $A = \{0, 1, 2, 3, 4, 5\}$

$B_1 = \{0, 2, 4\}$, $B_2 = \{0, 5\}$

$\overline{B_1} = \{1, 3, 5\}$, $\overline{B_2} = \{1, 2, 3, 4\}$

Min sets are:

$B_1 \cap B_2 = \{0\}$

$\overline{B_1} \cap B_2 = \{5\}$

$B_1 \cap \overline{B_2} = \{2,4\}$

$\overline{B_1} \cap \overline{B_2} = \{1,3\}$

Since each min term is nonempty,

$A_i \cap A_j = \varphi \ (\forall\ i \neq j)$

$\cup A_i = A$

The minterms $\{0\}$, $\{5\}$, $\{2,4\}$, $\{1,3\}$ form a partition of A.

2. Let $A = \{1,2,3,4,5,6,7,8,9\}$ Find the minsets generated by $B_1 = \{5,6,7\}$, $B_2 = \{2,4,5,9\}$ $B_3 = \{3,4,5,6,8,9\}$

<u>Sol:</u> Given, $A = \{1,2,3,4,5,6,7,8,9\}$

$B_1 = \{5,6,7\} \Rightarrow \overline{B_1} = \{1,2,3,4,8,9\}$

$B_2 = \{2,4,5,9\} \Rightarrow \overline{B_2} = \{1,3,6,7,8\}$

$B_3 = \{3,4,5,6,8,9\} \Rightarrow \overline{B_3} = \{1,2,7\}$

$B_1 \cap B_2 \cap B_3 = \{5\}$

$\overline{B_1} \cap B_2 \cap B_3 = \{4,9\}$

$B_1 \cap \overline{B_2} \cap B_3 = \{6\}$

$B_1 \cap B_2 \cap \overline{B_3} = \{\ \}$

$\overline{B_1} \cap \overline{B_2} \cap B_3 = \{3,8\}$

$B_1 \cap \overline{B_2} \cap \overline{B_3} = \{7\}$

$\overline{B_1} \cap B_2 \cap \overline{B_3} = \{2\}$

$\overline{B_1} \cap \overline{B_2} \cap \overline{B_3} = \{1\}$

The minterms $\{5\}$, $\{4,9\}$, $\{6\}$, $\{3,8\}$, $\{7\}$, $\{2\}$, $\{1\}$ forms a partition of A.

<u>Max Set:</u>

Let A be any set and B_1, B_2 be any two subsets of A. Then the max sets of A are $B_1 \cup B_2$, $B_1^c \cup B_2$, $B_1 \cup B_2^c$ and $B_1^c \cup B_2^c$.

<u>Problems:</u>

1. Find the maxterms of $A = \{1,2,3,4,5,6\}$ where $B_1 = \{1,3,5\}$, $B_2 = \{2,4,6\}$.

<u>Sol:</u> $A = \{1,2,3,4,5,6\}$

$B_1 = \{1,3,5\}$ $B_2 = \{2,4,6\}$

$\overline{B_1} = \{2,4,6\}$, $\overline{B_2} = \{1,3,5\}$

$B_1 \cup B_2 = \{1,2,3,4,5,6\}$

$B_1 \cup \overline{B_2} = \{1,3,5\}$

$\overline{B_1} \cup B_2 = \{2,4,6\}$

$\overline{B_1} \cup \overline{B_2} = \{1,2,3,4,5,6\}$

<u>Note:</u> Max terms does not form partition.

~~The~~ ~~max terms~~ ~~{1, 2, 3, 4, 5}~~, ~~{1, 3, 5}~~, ~~{2, 4, 6}~~ ~~forms the partition of A.~~

Relations:

A relation R from A to B is a subset of $A \times B$

$R \subseteq A \times B$.

If R is a relation on a set A, $R \subseteq A \times A$.

Eg: If $A = \{1, 2, 3\}$ and $B = \{1, 4\}$ and aRb if $a < b$; $a \in A$, $b \in B$.

$A \times B = \{(1,1), (1,4), (2,1), (2,4), (3,1) (3,4)\}$

$R = \{(1,1), (1,4), (2,4), (3,4)\}$

ie) $R \subseteq A \times B$.

Basic Operations on Relations:

Let R and S be two relations from a set A to set B. Then,

$R \cup S = \{(a,b) / (a,b) \in R \text{ or } (a,b) \in S\}$

$R \cap S = \{(a,b) / (a,b) \in R \text{ and } (a,b) \in S\}$

$R - S = \{(a,b) / (a,b) \in R \text{ and } (a,b) \notin S\}$

Eg: If $A = \{x, y, z\}$ $B = \{1, 2, 3\}$ $C = \{x, y\}$ $D = \{2, 3\}$. Let R be a relation from A to B defined by $R = \{(x,1), (x,2), (y,3)\}$ and let $S: C \to D$ defined by,

$S = \{(x,2), (y,3)\}$.

Then, $R \cap S = \{(x,2), (y,3)\}$

$R \cup S = \{(x,1), (x,2), (y,3)\}$

$R - S = \{(x,1)\}$

$R' = \{(x,3), (y,1) (y,2) (z,1), (z,2), (z,3)\}$

Types of Relations:

1. **Universal Relation:**

A relation R on a set A is called universal relation if $R = A \times A$.

Eg: $A = \{1, 2, 3\}$

$R = A \times A = \{(1,1), (1,2) (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ is the universal relation on A.

2. **Void Relation:**

A relation R on a set A is called a void relation if $R = \phi$.

Eg: $A = \{3, 4, 5\}$, R is defined

as aRb iff $a+b>10$, then $R=\phi$.

3. Identity Relation:

A relation R on a set A is called identity relation if $R = \{(a,a) / a \in A\}$.

Eg: If $A = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3)\}$ is called the identity relation on A.

4. Inverse Relation:

Let R be a relation from set A to set B. The inverse relation of R is defined by,

$R^{-1} = \{(b,a) / (a,b) \in R\}$.

5. Complementary Relations:

Let A and B be two finite sets and R be a relation from A to B. The complementary relation R is defined as:

$\overline{R} = \{(a,b) \in A \times B / (a,b) \notin R\}$

Problem: Let $A = \{1,2,3\}$, $B = \{1,4\}$. consider the relation R such that "<" and find $R, \overline{R}, R^{-1}$!

Solution:

$A \times B = \{(1,1), (1,4), (2,1), (2,4), (3,1), (3,4)\}$

$R = \{(1,4), (2,4), (3,4)\}$

$\overline{R} = \{(1,1), (2,1), (3,1)\}$

$R^{-1} = \{(4,1), (4,2), (4,3)\}$

Composition of Relations:

If R is a relation from set A to B and S is a relation from set B to C, then the composition of R and S denoted by $R \circ S$ is defined as,

$R \circ S = \{(a,c) / \exists$ some $b \in B$ for which $(a,b) \in R$ and $(b,c) \in S\}$.

1. $R = \{(1,2), (2,4), (3,3)\}$

$S = \{(1,3), (2,4), (4,2)\}$

Find $R \cup S$, $R \cap S$, $R-S$, $S-R$, $R \oplus S$, $R \circ S$, $S \circ R$.

Solution:

$R \cup S = \{(1,2), (1,3), (2,4), (3,3), (4,2)\}$

$R \cap S = \{(2,4)\}$

$R - S = \{(1,2), (3,3)\}$

$S - R = \{(1,3), (4,2)\}$

$R \oplus S = (R \cup S) - (R \cap S)$ [or]

$(R-S) \cup (S-R)$

$= \{(1,2),(1,3),(3,3),(4,2)\}$

$R \circ S = \{(1,4),(2,2)\}$

$S \circ R = \{(1,3)(4,4)\}$

2. $R = \{(1,1),(1,3),(3,2),(3,4),(4,2)\}$

$S = \{(2,1),(3,3),(3,4),(4,1)\}$

find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $(R \circ S) \circ R$, $R \circ (S \circ R)$, R^3

(i) $R \circ S = \{(1,3), \cancel{(3,4)} (1,4), (3,1), (4,1)\}$

(ii) $S \circ R = \{(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$

(iii) $R \circ R = \{(1,1)(1,3)(1,2)(1,4)(3,2)\}$

(iv) $S \circ S = \{(3,3),(3,4),(3,1)\}$

(v) $(R \circ S) \circ R = \{(1,2),(1,4),(3,1),(3,3),(4,1),(4,3)\}$

(vi) $R \circ (S \circ R) = \{(1,2),(1,4)(3,1),(3,3),(4,1),(4,3)\}$

(vii) $R^3 = (R \circ R) \circ R$

$= \{(1,1),(1,3),(1,2),(1,4)\}$

3. If $R_1 = \{(1,2),(2,3),(3,4)\}$ and $R_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}$ from $\{1,2,3\}$ to $\{1,2,3,4\}$. Find

a) $R_1 \cup R_2$ b) $R_1 \cap R_2$

c) $R_1 - R_2$ d) $R_2 - R_1$

e) $R_1 \oplus R_2$

Sol:

a) $R_1 \cup R_2 = \{(1,1)(1,2)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)(3,4)\}$

b) $R_1 \cap R_2 = \{(1,2),(2,3),(3,4)\}$

c) $R_1 - R_2 = \{\}$

d) $R_2 - R_1 = \{(1,1)(2,1)(2,2),(3,1)(3,2)(3,3)\}$

e) $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

$= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

4. $R = \{(x,x^2)\}$, $S = \{(x,2x)\}$ where x is a non-negative integers, find

a) $R \cup S$ b) $R \cap S$ c) $R_1 - R_2$

d) $R_2 - R_1$ e) $R_1 \oplus R_2$

Sol: $R = \{(0,0),(1,1),(2,4),(3,9)(4,16),(5,25)\ldots\}$

$S = \{(0,0),(1,2),(2,4),(3,6),(4,8)\ldots\}$

(a) $R \cup S = \{(0,0), (1,1), (1,2), (2,4), (3,6), (3,9), (4,8), \dots\}$

b) $R \cap S = \{(0,0), (2,4), \dots\}$

c) $R - S = \{(1,1), (3,9), (4,16) \dots\}$

d) $S - R = \{(1,2), (3,6), (4,8) \dots\}$

e) $R \oplus S = \{(1,1), (3,6), (4,8) \dots\}$.

5. If the relation R and S are given by, $R = \{(1,2), (2,2), (3,4)\}$ $S = \{(1,3), (2,5), (3,1), (4,2)\}$ find a) $R \circ S$ b) $S \circ R$ c) $R \circ R$ d) $S \circ S$, e) $R \circ (S \circ R)$, f) $(R \circ S) \circ R$ g) $R \circ R \circ R$

Solution:

a) $R \circ S = \{(1,5), (2,5), (3,2)\}$

b) $S \circ R = \{(1,4), (3,2) (4,2)\}$

c) $R \cdot R = \{(1,2), (2,2)\}$

d) $S \circ S = \{(1,1), (3,3), (4,5)\}$

e) $R \circ (S \circ R) = \{(3,2)\}$

f) $(R \circ S) \circ R = \{(3,2)\}$

g) $R \circ R \circ R = \{(1,2), (2,2)\}$.

Matrices of Relations:

If $A = \{a_1, a_2, \dots a_m\}$ and $B = \{b_1, b_2, \dots b_n\}$ are finite sets containing m & n elements respectively, and R is a relation from A to B, we can represent R by the $m \times n$ matrix:

$$M_R = \{m_{ij}\},$$

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

Note:

1. $M_{R \cup S} = M_R \vee M_S$
2. $M_{R \cap S} = M_R \wedge M_S$
3. $M_{R \circ S} = M_R \cdot M_S$

Eg: If $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

then,

$$M_{R \cup S} = M_R \vee M_S$$

$$= \begin{bmatrix} 1\vee 1 & 0\vee 0 & 1\vee 1 \\ 0\vee 1 & 1\vee 0 & 1\vee 0 \\ 1\vee 0 & 0\vee 1 & 0\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R \cap S} = M_R \wedge M_S$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find Matrix when relation is given:

Eg:1 If $A=\{1,2,3,4\}$, $B=\{x,y,z\}$
$R=\{(1,y),(1,z)\ (3,y),(4,x),(4,z)\}$. Determine the matrix of the relation R.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Eg:2: $A=\{a_1,a_2,a_3\}$, $B=\{b_1,b_2,b_3,b_4\}$
$R=\{(a_1,b_2),(a_2,b_1),(a_2,b_3),(a_2,b_4),(a_3,b_2),(a_3,b_4)\}$. Determine M_R.

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Conversely, to find the relation when matrix is given:

Eg:1 If R is the relation on A, where $A=\{1,3,4\}$ given by
$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find R.

Sol: Given:

$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R=\{(1,1),(1,3),(3,3),(4,4)\}$

Since $m_{ij}=1$ means that the i^{th} element of A is related to the j^{th} element of A.

Eg:2: Let $A=\{1,2,3,4\}$ find the relation R on A determined by the matrix M_R,

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Sol: Given,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$R=\{(1,1),(1,3),(2,3),(3,1),(4,4),(4,1),(4,2)\}$

Problems:

① If R and S be relations on a set A represented by the matrices

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

find a) $R \cup S$ b) $R \cap S$ c) $R \circ S$ d) $S \circ R$ e) $R \oplus S$ by matrix representation.

Sol:

a) $M_{R\cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

b) $M_{R\cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

c) $M_{R\circ S} = M_R \cdot M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 0\vee 0 & 0\vee 1\vee 0 & 0\vee 1\vee 0 \\ 0\vee 0\vee 1 & 1\vee 1\vee 0 & 0\vee 1\vee 0 \\ 0\vee 0\vee 0 & 1\vee 0\vee 0 & 0\vee 0\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

d) $M_{S\circ R} = M_S \cdot M_R$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 1\vee 0 & 0\vee 1\vee 0 & 0\vee 1\vee 0 \\ 0\vee 1\vee 1 & 0\vee 1\vee 0 & 0\vee 1\vee 0 \\ 0\vee 1\vee 1 & 1\vee 1\vee 0 & 0\vee 1\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

e) $M_{R\oplus S} = M_{R\cup S} - M_{R\cap S}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

② If R and S are relations on $A = \{1, 2, 3\}$ represented by the matrices,

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

find the matrices that represent a) $R \cup S$ b) $R \cap S$ c) $R \circ S$ d) $S \circ R$ e) $R \oplus S$.

Sol:

a) $M_{R\cup S} = M_R \vee M_S$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $M_{R\cap S} = M_R \wedge M_S$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) $M_{R\circ S} = M_R \cdot M_S$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) $M_{S\circ R} = M_S \circ M_R$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

e) $M_{R\oplus S} = M_{R\cup S} - M_{R\cap S}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) If R is the relation on $A = \{1,2,3\}$ such that $(a,b) \in R$ if and only if $a+b =$ even, find M_R and also the relational matrices R^{-1}, $\bar{R}$ and R^2.

Sol: $A = \{1,2,3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$\bar{R} = \{(1,2), (2,1), (2,3), (3,2)\}$

Now,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

(i) $M_{R^{-1}} = (M_R)^T$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(ii) $$M_{\bar{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$, since from $\bar{R}$.

(iii) $M_{R^2} = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(4) If R is the relation on $A = \{1, 2, 3\}$ represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find the matrix representing a) R^{-1} b) $\bar{R}$ and c) R^2 and also express them as ordered pairs.

Sol: $A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$\Rightarrow R = \{(1,2), (1,3), (2,1), (2,2), (3,1), (3,3)\}$

(i) $M_{R^{-1}} = [M_R]^T$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} = \{(1,2), (1,3), (2,1), (2,2), (3,1), (3,3)\}$$

(ii) $\bar{R} = \{(1,1), (2,3), (3,2)\}$

$$M_{\bar{R}} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(iii) $M_{R^2} = M_R \cdot M_R$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$R^2 = \{(1,1) (1,2) (1,3) (2,1) (2,2) (2,3) (3,1) (3,2) (3,3)\}$.

Properties of Relations:

1. A relation R on a set A is said to be reflexive if $(a, a) \in R$ for every $a \in A$.
2. A relation R on a set A is said to be symmetric whenever $(a, b) \in R$ then $(b, a) \in R$.
3. A relation R on a set A is said to be transitive if

whenever $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$.

4. A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

5. A relation R on a set A is said to be irreflexive if $(a,a) \notin R$ for every $a \in A$.

6. A relation R is anti-symmetric if whenever $(a,b) \in R$, $(b,a) \in R$ then $a = b$

Note: If $(a,b) \in R$, $(b,a) \in R$ but $a \neq b$ then R is not antisymmetric.

7. A relation R on a set A is called a partial order relation, if R is reflexive, antisymmetric and transitive.

Note: A set A together with partial order relation R is called a partial ordered set or poset.

Equivalence relation:

A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive. ie)

(i) aRa for every $a \in A$

(ii) aRb, then bRa

(iii) aRb, bRc then aRc

Poset: A set A together with partial order relation R is called a poset.

R is a partial order relation if R is reflexive, antisymmetric and transitive. ie,

(i) aRa for every $a \in A$

(ii) aRb and $bRa \Rightarrow a = b$

(iii) aRb and $bRc \Rightarrow aRc$.

Problems:

1. Let $X = \{1, 2, 3, \ldots 25\}$ and $R = \{(x,y) / x-y$ is divisible by $5\}$ be a relation in X. Show that R is an equivalence relation.

Proof: Given, $X = \{1, 2, 3 \ldots 25\}$

$R = \{(x,y) /\ x-y$ is divisible by $5\}$

$X \times X = \{ (1,1)\ (1,2) \dots (1,25)$
$(2,1)\ (2,2) \dots (2,25)$
$\vdots$
$(25,1)\ (25,2) \dots (25,25) \}$

$R = \{ (2,7), (7,2)\ (3,8)\ (8,3)$
$(1,6)\ (6,1)\ (4,24)\ (24,4) \dots \}$

(i) Reflexive:

For any $a \in X$, $a-a$ is divisible by 5 $\Rightarrow (a,a) \in R$, $\forall a \in X$

$\therefore$ R is reflexive.

(ii) Symmetric:

Let $(a,b) \in R$

$\rightarrow a-b$ is divisible by 5

$\Rightarrow -(b-a)$ is also divisible by 5.

$\Rightarrow -(a-b) = b-a$

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

$\therefore$ R is symmetric.

(iii) Transitive:

Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a-b$ is divisible by 5

$b-c$ is divisible by 5.

$\Rightarrow a-b = 5r \; ; \; b-c = 5s$

Now, $a-c = (a-b) + (b-c)$

$= 5r + 5s$

$= 5(r+s)$ is divisible by 5.

$\therefore (a,c) \in R$

$\Rightarrow$ R is transitive

Hence R is an equivalence relation.

2. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x,y) / x-y$ is divisible by $3\}$. Show that R is an equivalence relation.

Sol:

$X = \{1, 2, 3, 4, 5, 6, 7\}$

$X \times X = \{ (1,1)\ (1,2)\ (1,3) \dots (1,7)$
$(2,1)\ (2,2)\ (2,3) \dots (2,7)$
$\vdots$
$(7,1)\ (7,2)\ (7,3) \dots (7,7) \}$

$R = \{ (1,1), (2,2), (3,3), (4,4),$
$(5,5), (6,6), (7,7), (1,4),$
$(1,7)\ (2,5)\ (3,6)\ (4,7)$
$(7,4)\ (7,1)\ (6,3)\ (5,2)$
$(4,1) \}$

(i) For any $a \in X$, $a-a$ is divisible by 3 $\Rightarrow (a,a) \in R, \forall a \in X$

$\therefore$ R is reflexive

(ii) Let $(a,b) \in R$

$\Rightarrow$ $a-b$ is divisible by 3

$\Rightarrow$ $-(b-a)$ is also divisible by 3

$\Rightarrow$ $b-a = -(a-b)$ is also divisible by 3

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

$\therefore R$ is symmetric.

(iii) Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow$ $a-b$ is divisible by 3
$b-c$ is divisible by 3.

$\Rightarrow$ $a-b = 3r$; $b-c = 3r$

Now, $a-c = (a-b)+(b-c)$

$= 3r + 3s$

$= 3(r+s)$

which is divisible by 3

$\Rightarrow (a,c) \in R$

$\therefore R$ is transitive.

Hence R is an equivalence relation.

(3.) Let $S = \{1,2,3, \cdots 9\}$. Define R on S by $R = \{(x,y) / x,y \in S$ and $x+y=10\}$. Is R an equivalence relation?

Solution: $S = \{1,2,3 \cdots 9\}$

$R = \{(1,9),(9,1),(2,8),(8,2),(3,7)(7,3),(4,6),(6,4),(5,5)\}$

(i) For all $a \in S$, $(a,a) \notin R$

$\therefore R$ is not reflexive.

(ii) For $(a,b) \in R$, we can find $(b,a) \in R$ such that $a+b=10$

$\therefore R$ is symmetric.

(iii) For every $(a,b) \in R$, and $(b,c) \in R$ we cannot find $(a,c) \in R$.

$\therefore R$ is not transitive.

(4) Which of the following relations on $\{0,1,2,3\}$ are equivalence relations?

(a) $R_1 = \{(0,0),(1,1),(2,2),(3,3)\}$

(b) $R_2 = \{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2)(3,3)\}$

(c) $R_3 = \{(0,0)(1,1)(1,3),(3,3)(2,2)(2,3)(3,1)(3,2)\}$

Sol:

(a) R_1 is reflexive, symmetric and transitive

$\Rightarrow R_1$ is an equivalence relation.

(b) R_2 is reflexive, Symmetric but not transitive, since $(3,2)$ and $(2,0) \in R_2$ but $(3,0) \notin R_2$

$\Rightarrow R_2$ is not an equivalence relation.

(c) R_3 is reflexive, symmetric but not transitive

$\because (1,3)$ and $(3,2) \in R_3$ but $(1,2) \notin R_3$

$\Rightarrow R_3$ is not an equivalence.

Representation of relations by graphs:

Let R be a relation on a set A. To represent R graphically each element of A is represented by a point, called vertices or nodes.

Whenever a is related to b, an arc is drawn from the point a to b, called edges or arcs. The direction is indicated by an arrow. This diagram is called the directed graph or digraph of R.

(1) Let $A = \{a, b, c, d\}$ and R be the relation on A that has the matrix

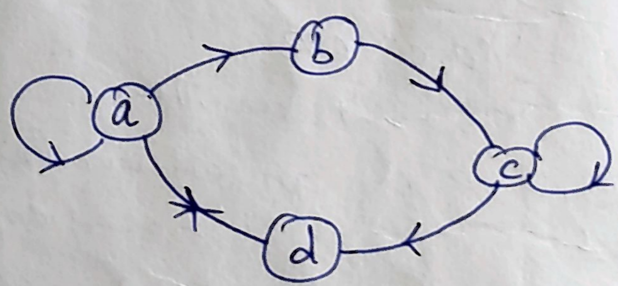
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Construct the digraph of R and list the indegree and out degrees of all vertex.

Sol: Given,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R = \{(a,a), (a,b), (a,d), (b,c), (c,c), (c,d), (d,a)\}$



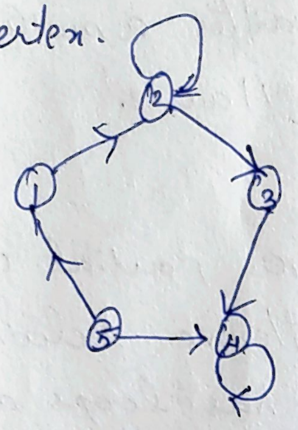
	a	b	c	d
Indegree	2	1	2	2
Out degree	3	1	2	1

Note!

Indegree - 2nd co-ordinate

Out degree - 1st co-ordinate.

2. From the digraph find indegree and outdegree of each vertex.



Sol:

$R = \{(1,2)\ (2,2)\ (2,3)\ (3,4)\ (4,4)\ (5,4)\ (5,1)\}$

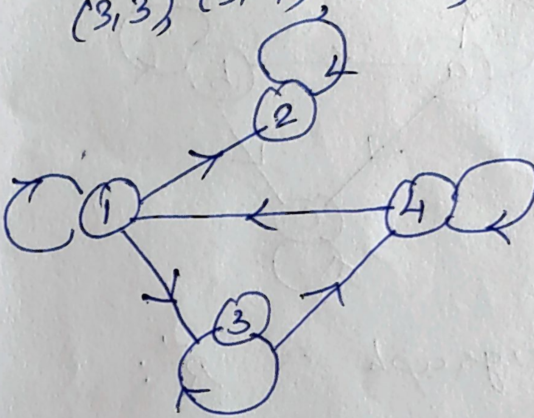
	1	2	3	4	5
Indegree	1	2	1	3	0
Out degree	1	2	1	1	2

3. Let $A = \{1,2,3,4\}$ and $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. Draw the directed graph and use the graph to find if relation is reflexive, symmetric and transitive.

Sol:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3)\ (3,4), (4,1), (4,4)\}$



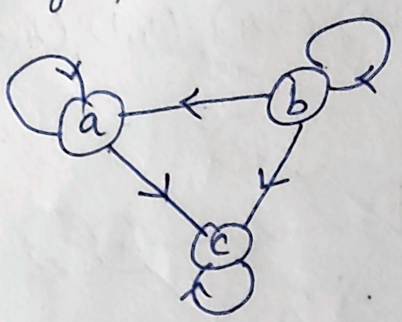
(i) Since there is a loop at every vertex of the digraph, R is reflexive.

(ii) R is not symmetric, since there is an edge from 1 to 2 but no edge from 2 to 1.

(iii) R is not transitive, since there are edges from 1 to 3 and 3 to 4, but there is no edge from (1,4)

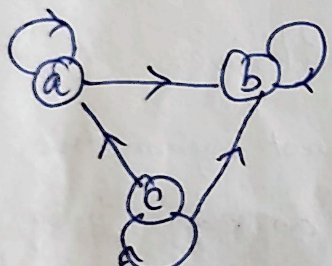
4. Let $A = \{a, b, c\}$, $R = \{(a,a)\ (a,c), (b,a)\ (b,b)\ (b,c)\ (c,c)\}$. Draw the digraphs of R, R^{-1} and $\bar{R}$.

(i) Digraph of R:



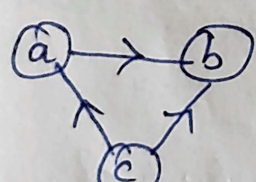
(ii) Digraph of R^{-1}:

$R^{-1} = \{(a,a)\ (c,a)\ (a,b),$
$(b,b)\ (c,b)\ (c,c)\}$



(iii) Digraph of $\bar{R}$:

$\bar{R} = \{(a,b), (c,a), (c,b)\}$



Hasse Diagrams for partial orderings:

The simplified form of the diagraph of a partial ordering on a finite set that contains sufficient information about the partial ordering is called a Hasse diagram, named after a mathematician Helmut Haasse.

Rules:

1. Since the partial ordering is a reflexive relation, its digraph has loops at all vertices. We need not show these loops since they must be present.

2. Since the partial ordering is transitive, we need not show those edges that must be present due to transivity.

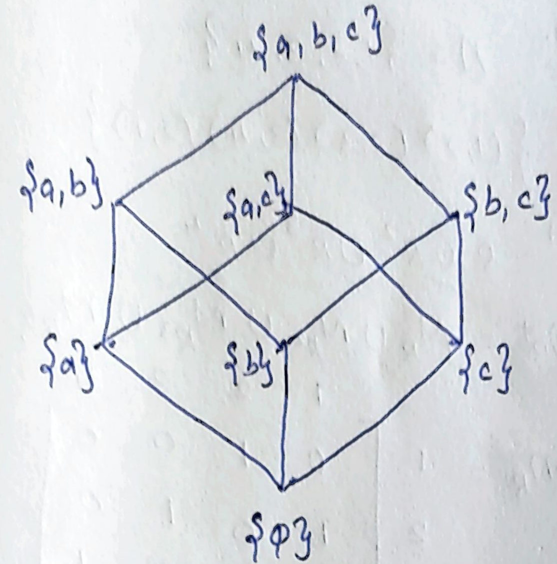
3. We need not show the directions of the edges.

Problems:

① Draw a Hasse diagram representing the partial ordering $\{(A, B) / A \subseteq B\}$ on the power set $P(S)$, where $S = \{a, b, c\}$.

Sol:

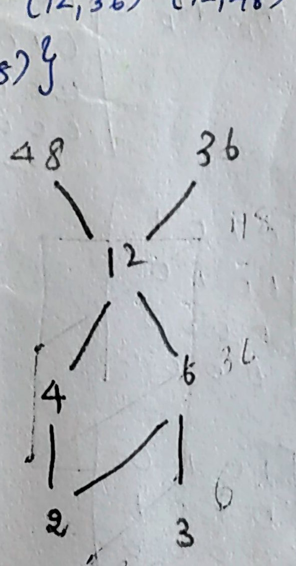
$P(S) = \{\phi, \{a\}, \{b\}, \{c\},$
$\{a,b\}, \{b,c\}, \{c,a\},$
$\{a,b,c\}\}.$



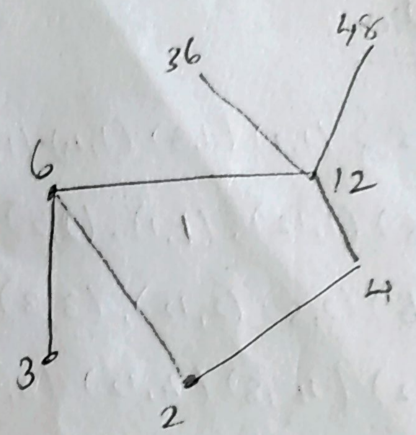
2. Let $B = \{2, 3, 4, 6, 12, 36, 48\}$. Let S be the relation "divides" on B. Draw the Hasse diagram of S.

<u>Sol:</u> $B = \{2, 3, 4, 6, 12, 36, 48\}$

$S = \{(2,2)\ (2,4)\ (2,6)\ (2,12), (2,36), (2,48), (3,3), (3,6), (3,12), (3,36), (3,48), (4,4), (4,12), (4,36), (4,48), (6,6)\ (6,12), (6,36), (6,48), (12,12)\ (12,36)\ (12,48)\ (36,36)\ (48,48)\}$



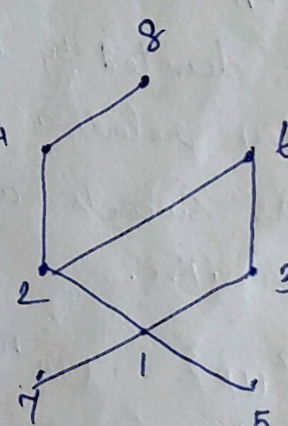
(or)

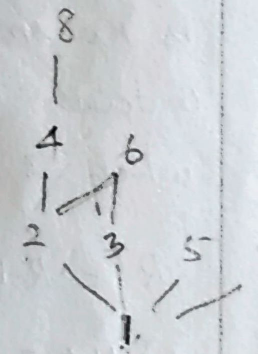


3. Draw a hasse diagram for $A = \{1, 2, 3, \ldots 8\}$ and $R = \{(a,b) \mid a \text{ divides } b\}$.

<u>Sol:</u>

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,4), (2,6), (2,8), (3,6), (4,8)\}$

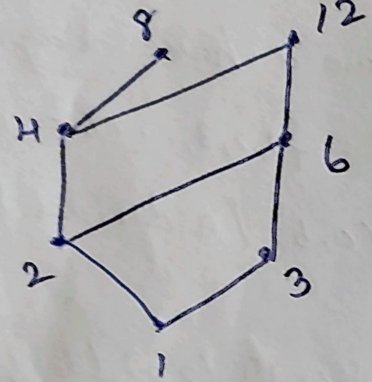




4. Let $A = \{1, 2, 3, 4, 6, 8, 12\}$ $R = \{(a,b) \mid a \text{ divides } b\}$. Draw the hasse diagram of R.

Solution:

$R = \{(1,2)(1,3)(1,4)(1,6),$
$(1,8)(1,12),(1,1),(2,2)(2,4)$
$(2,6)(2,8)(2,12),(3,3)(3,6)$
$(3,12)(4,8)(4,12)(8,8)(12,12)\}$.



Transitive closure:

Let A be a set and R be a relation on A. The transitive closure of R, denoted by R^+ is the smallest relation which contains R as subsets and which is transitive.

$R^+ = R \cup R^2 \cup R^3 \cdots \cup R^n$

ii) $M_{R^+} = M_R + M_{R^2} + \cdots + M_{R^n}$

(1). Let $A = \{1,2,3,4\}$ and let $R = \{(1,2),(2,3)(3,4)(2,1)\}$. Find the transitive closure of R.

Sol: $A = \{1,2,3,4\}$

$R = \{(1,2)(2,3)(3,4)(2,1)\}$

$R^+ = R \cup R^2 \cup R^3 \cup R^4$

$M_{R^+} = M_R \cup M_{R^2} \cup M_{R^3} \cup M_{R^4}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$M_{R^2} = M_R . M_R$

$$= \begin{bmatrix} 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} . M_R = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

$$M_{R^4} = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

$$M_{R^+} = M_R + M_{R^2} + M_{R^3} + M_{R^4}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore R^+ = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$

2. Let $A = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,3)\ (2,3)\ (3,4), (4,1)\ (4,2)\}$. Find the transitive closure of R.

Warshall's Algorithm:

To find the transitive closure of R, we use warshall's algorithm:

Step:1 First transfer to W_k all 1's in W_{k-1}

Step:2 List the locations $P_1, P_2 \ldots$ in column k of W_{k-1} where the entry is 1 and the locations $q_1, q_2 \ldots$ in row k of W_{k-1} where the entry is 1.

Step:3 Put 1 in all positions (P_i, q_j) of W_k.

①. Let $A = \{1,2,3,4,5\}$

$R = \{(1,1)\ (1,3)\ (1,5)\ (2,3), (2,4), (3,3), (3,5), (4,2), (4,4), (5,4)\}$. Find the transitive closure of R.

Sol:

$$W_0 = M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

k	Position of 1's in column k	Position of 1's in row k	W_k has 1's in position	W_k
1	1	1 3 5	(1,1) (1,3) (1,5)	$W_1 = \begin{bmatrix} 1&0&1&0&1\\0&0&1&1&0\\0&0&1&0&1\\0&1&0&1&0\\0&0&0&1&0 \end{bmatrix}$
2	4	3 4	(4,3) (4,4)	$W_2 = \begin{bmatrix} 1&0&1&0&1\\0&0&1&1&0\\0&0&1&0&1\\0&1&1&1&0\\0&0&0&1&0 \end{bmatrix}$
3	1 2 3 4	3 5	(1,3) (1,5) (2,3) (2,5) (3,3) (3,5) (4,3) (4,5)	$W_3 = \begin{bmatrix} 1&0&1&0&1\\0&0&1&1&1\\0&0&1&0&1\\0&1&1&1&1\\0&0&0&1&0 \end{bmatrix}$
4	2 4 5	2 3 4 5	(2,2), (2,3), (2,4), (2,5) (4,2) (4,3) (4,4) (4,5) (5,2) (5,3) (5,4) (5,5)	$W_4 = \begin{bmatrix} 1&\phi&1&\phi&1\\0&1&1&1&1\\0&0&1&0&1\\0&1&1&1&1\\0&1&\phi&1&\phi \end{bmatrix}$
5	1 2 3 4 5	2 3 4 5	(1,2) (1,3) (1,4) (1,5) (2,2) (2,3) (2,4) (2,5) (3,2) (3,3) (3,4) (3,5) (4,2) (4,3) (4,4) (4,5) (5,2) (5,3) (5,4) (5,5)	$W_5 = \begin{bmatrix} 1&1&1&1&1\\0&1&1&1&1\\0&1&1&1&1\\0&1&1&1&1\\0&1&1&1&1 \end{bmatrix}$

$\therefore R^+ = \{(1,1),(1,2),(1,3),(1,4),(1,5),$
$(2,2),(2,3),(2,4),(2,5),$
$(3,2),(3,3),(3,4),(3,5),$
$(4,2),(4,3),(4,4),(4,5),$
$(5,2),(5,3),(5,4),(5,5)\}$

2. Using Warshall's algorithm find the transitive closure of the relation:

$R = \{(1,2)\ (2,3)\ (3,3)\}$ on the set $A = \{1,2,3\}$.

Sol:

$A = \{1,2,3\}$

$R = \{(1,2)\ (2,3)\ (3,3)\}$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

work

Ans: $R^+ = \{(1,2)\ (1,3)\ (2,3)\ (3,3)\}$.

3. Using Warshall's, find the transitive closure of the matrix:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

On $A = \{1,2,3,4\}$.

Ans: $R^+ = \{$

4. Let $A = \{1,2,3,4\}$ and $R = \{(1,2)\ (2,3)\ (3,4)\ (2,1)\}$ Find the transitive closure of R using Warshall's algorithm.

Ans: $R^+ = \{$

Solution of 2:

k	Position of 1's in column k	Position of 1's in row k	W_k has 1's in positions	W_k
1	–	2	–	$W_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
2	1	3	(1,3)	$W_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
3	1 2 3	3	(1,3) (2,3) (3,3)	$W_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore R^+ = \{(1,2), (1,3), (2,3), (3,3)\}$$

Solution of 3:

k	Position of 1's in col k	Position of 1's in row k	W_k has 1's positions	W_k
1	1	1 4	(1,1) (1,4)	$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
2	2	2 4	(2,2) (2,4)	$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
3	–	4	–	$W_3 =$ "

4	1 2 3	1	(1,1) ~~(1,2)~~ (2,1) (3,1)	$W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\therefore R^+ = \{(1,1)\ (1,4)\ (2,1)\ (2,2)\ (2,4)\ (3,1)\ (3,4)\ (4,1)\}$

Solution of (4)!

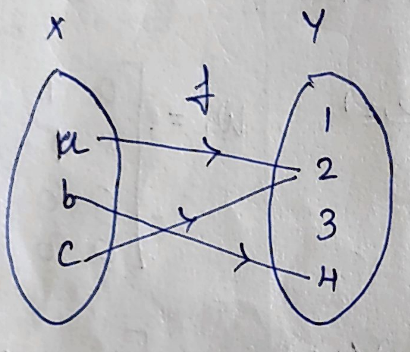
k	Position of 1's in column k	Position of 1's in row k	W_k has 1's position	W_k
1	2	2	(2,2)	$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \phi & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
2	1 2	1 2 3	(1,1) (1,2) (1,3) (2,1) (2,2) (2,3)	$W_2 = \begin{bmatrix} 1 & 1 & \phi & 0 \\ 1 & \phi & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
3	1 2	4	(1,4) (2,4)	$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \phi & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
4	1 2 3	–	–	$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore R^+ = \{(1,1), (1,2)\ (1,3)\ (1,4), (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,4)\}$

Functions:

Def: A relation 'f' from a set X to another set Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$.

Eg:



Domain, $D_f = \{a, b, c\}$

$f(a) = 2, \quad f(b) = 4, \quad f(c) = 2$

Range $R_f = \{2, 4\}$ is a subset of the co-domain $\{1, 2, 3, 4\}$.

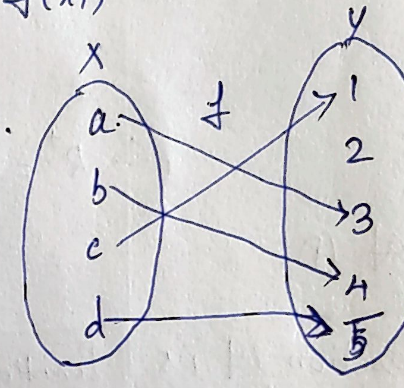
Types of functions:

One to one: A function $f : X \to Y$ is called one-to-one (or) injective if distinct elements of X are mapped into distinct elements on Y. ie).

$f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

[or] $f(x_1) = f(x_2)$ if $x_1 = x_2$.

Eg:

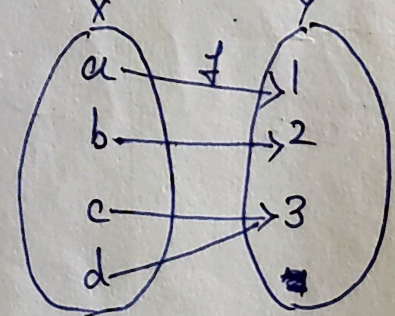


$f : X \to Y$ with, $f(a) = 3$, $f(b) = 4$, $f(c) = 1$ and $f(d) = 5$.

distinct elts in X mapped to distinct elts in Y, so f is one-one.

Onto: A function $f : X \to Y$ is called onto or surjective if the range $R_f = Y$. ie), for every element of $y \in Y$, there is an element $x \in X$ such that $f(x) = y$.

Eg:



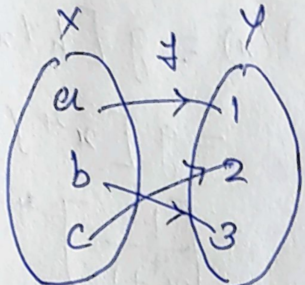
Here $R_f = \{1, 2, 3\} = Y$

So f is onto.

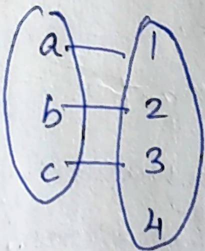
One to one Onto:

A function $f: X \to Y$ is called one to one onto or bijective if it is both one to one and onto.

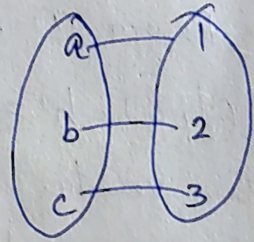
Eg:



Into function: A function $f: X \to Y$ is said to be into if $f(X) \subseteq Y$ i.e) if there exists atleast one $y \in Y$ does not have any pre-image in X.

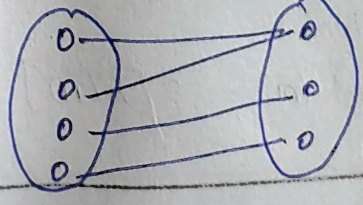


Into



Not into but onto

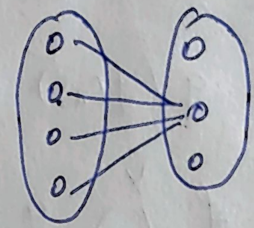
Many to One: A function $f: X \to Y$ is said to be many-one iff two or more elements of X have the same image in Y.



Many-one.

Constant function:

Let $f: X \to Y$ is said to be constant if there exists $y_0 \in Y$ such that $f(x) = y_0$, $\forall x \in X$.

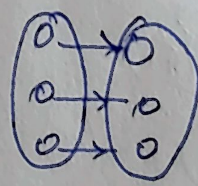


Inverse function:

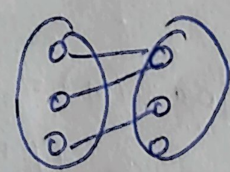
Let $f: X \to Y$ be a bijective function, then the inverse of the function f, denoted as f^{-1} is the function from Y to X defined as $f^{-1}(y) = x$, whenever $f(x) = y$.

If f^{-1} exists, then f is said to be invertible.

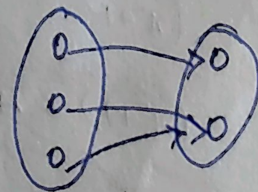
If f is bijective, then f^{-1} is also bijective.



inversible



not inversible

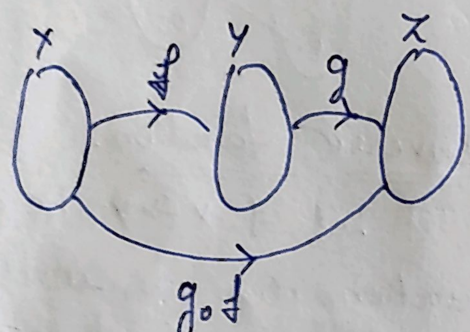


not inversible

Composition of functions:

Let $f: X \to Y$ and $g: Y \to Z$ be two functions. The composite function $g \circ f$ is the function from X to Z such that,

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in X.$$



Problems:

1. If $f, g, h : R \to R$ are defined by $f(x) = x^3 - 4x$, $g(x) = 1/x^2+1$, $h(x) = x^4$, find $(f \circ g) \circ h$ & $f \circ (g \circ h)$.

Sol:

(i) $(f \circ g)(x) = f(g(x))$

$= f\left(1/x^2+1\right) = f(y) = y^3 - 4y$

$y = \frac{1}{x^2+1}$

$= \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right)$

$= (x^2+1)^{-3} - 4(x^2+1)^{-1}$

Now, $h(x) = x^4$.

$(f \circ g) \circ h(x) = f \circ g(\cancel{h(x)})$

$(f \circ g)(h(x))$, $x^4 = z$ $= f \circ g[x^4]$

$= (f \circ g)(x^4)$,

$= (f \circ g)(z)$

$= (z^2+1)^{-3} - 4(z^2+1)^{-1}$

$= (x^8+1)^{-3} - 4(x^8+1)^{-1} \longrightarrow (1)$

(ii) $(g \circ h)(x) = g[h(x)]$

$y = x^4$

$g(y) = \frac{1}{y^2+1}$

$= \frac{1}{x^8+1}$

$= g[x^4] = g(y)$

$= \left[\frac{1}{x^8+1}\right]$

$= (x^8+1)^{-1}$

$f \circ (g \circ h)(x) = f[(x^8+1)^{-1}]$

$f(x^8+1)^{-1}$

$= f(z)$

$= z^3 - 4z$

$= (x^8+1)^{-3} - 4(x^8+1)^{-1}$

$= [(x^8+1)^{-1}]^3 - 4(x^8+1)^{-1} \longrightarrow (2)$

From (1) and (2),

$(f \circ g) \circ h = f \circ (g \circ h)$.

This verifies the associativity of composition functions.

2. $f, g, h : Z \to Z$ defined by $f(n) = n^2$, $g(n) = (n+1)$, $h(n) = (n-1)$. Verify: $f \circ (g \circ h) = (f \circ g) \circ h$.

Sol: L.H.S:

$g \circ h(n) = g[h(n)]$

$= g[n-1]$

$= n-1+1 = n$

$f\circ(g\circ h) = f[(g\circ h)(n)]$

$= f[n]$

$= n^2 \longrightarrow (1)$

R.H.S:

$(f\circ g)(n) = f[g(n)]$

$= f[n+1]$

$= (n+1)^2$

$(f\circ g)\circ h(n) = (f\circ g)[h(n)]$

$= f\circ g[n-1]$

$= (n+1-1)^2 = n^2 \longrightarrow (2)$

From (1) and (2)

$f\circ(g\circ h) = (f\circ g)\circ h.$

3. If $f: Z \to N$ is defined by

$$f(n) = \begin{cases} 2n-1, & \text{if } n > 0 \\ -2n, & \text{if } n \le 0. \end{cases}$$

a) Prove that f is one to one and onto

b) Determine f^{-1}.

Solution:

Let $x_1, x_2 \in Z$ and

$f(x_1) = f(x_2)$

Then $f(x_1)$ and $f(x_2)$ are both odd or even ($\because$ odd $\neq$ even)

If they are both odd, then

$2x_1 - 1 = 2x_2 - 1$

ie) $x_1 = x_2$

If they are both even,

$-2x_1 = -2x_2$

$x_1 = x_2$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence $f(x)$ is one to one.

Let $y \in N$,

If y is odd, its preimage is $y+1/2$, since

$f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1$

$= y$

If y is even, its preimage is $-y/2$, since

$f(-y/2) = -2(-y/2)$

$= y$

Thus, for any $y \in N$, the preimage is $\frac{y+1}{2} \in Z$ or $\frac{-y}{2} \in Z$

Hence $f(x)$ is onto.

(b) Let $y = f(x) = \begin{cases} 2x-1, & \text{if } x > 0 \\ -2x, & \text{if } x \le 0 \end{cases}$

$\therefore f^{-1}(y) = x = \begin{cases} \frac{y+1}{2}, & y = 1, 3, 5 \ldots \\ -y/2, & y = 0, 2, 4 \ldots \end{cases}$

(or)

$$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & \text{if } x = 1, 3, 5 \ldots \\ -x/2, & \text{if } x = 0, 2, 4, \ldots \end{cases}$$

Theorem: 1

If $f: A \to B$ and $g: B \to C$ are bijection then P.T $g \circ f: A \to C$ is also bijection.

Proof: If $f: A \to B$ and $g: B \to C$ are bijection then f and g are 1-1 and onto.

To prove: $g \circ f: A \to C$ is 1-1

Let $a_1, a_2 \in A$

$g \circ f(a_1) = g \circ f(a_2)$

$g[f(a_1)] = g[f(a_2)]$

$f(a_1) = f(a_2)$

$a_1 = a_2$

$\therefore g \circ f$ is 1-1

To prove: $g \circ f: A \to C$ is onto.

Since f is onto, for every $b \in B$, $\exists\, a \in A \ni f(a) = b$

Since g is onto, for every $c \in C$ $\exists\, b \in B \ni g(b) = c$

For every $c \in C$, $\exists\, a \in A$

$\ni g \circ f(a) = c$

$g[f(a)] = c$

$g[b] = c$

$\therefore g \circ f$ is onto.

Hence $g \circ f$ is a bijection.

Theorem: 2

If $f: A \to B$, then f^{-1} exists iff f is one to one and onto.

Proof:

Suppose f^{-1} exists, then $f^{-1}: B \to A$ defined by $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$

$\therefore f: A \to B$ is 1-1

$f(a_1) = f(a_2)$

$f^{-1}[f(a_1)] = f^{-1}[f(a_2)]$

$f^{-1} \cdot f(a_1) = f^{-1} \cdot f(a_2)$

$I_A(a_1) = I_A(a_2)$

$a_1 = a_2$

$\therefore f$ is 1-1

To prove $f: A \to B$ is onto for every $b \in B$ $\exists$ $a = f^{-1}(b) \in A$.

$\Rightarrow f[f^{-1}(b)] = f \circ f^{-1}(b)$

$= I_B(b) = b$

for every $b \in B$, $\exists\, f^{-1}(b) \in A$

$\therefore f : A \to B$ is bijection.

Suppose f is 1-1 and onto.

Let $f : A \to B$ be onto $\forall\, b \in B$

$\exists\, a \in A \ni f(a) = b$

$a = f^{-1}(b)$

$f^{-1} : B \to A \ni f^{-1}(b) = a \;\forall\, b \in B$

Let $f^{-1}(b) = a_1$ & $f^{-1}(b) = a_2$,

where $a_1 \neq a_2$

$\Rightarrow f(a_1) = f(a_2) = b$

which is not possible,

since f is 1-1.

$\therefore f^{-1} : B \to A$ is a unique function.

ie) $\forall\, b \in B$, $\exists$ a unique $a \in A$

$\Rightarrow f(a) = b$, $f^{-1}(b) = a$

$f[f^{-1}(b)] = b$

$f \circ f^{-1}(b) = b$

$I_B(b) = b$

$f \circ f^{-1} = I_B$

|||ly $f^{-1} \circ f = I_A$

Hence the proof.

Theorem: 3:

Let $f : A \to B$ & $g : B \to C$ be any 2 invertible function then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Given, $f : A \to B$ is 1-1 and onto.

$g : B \to C$ is 1-1 and onto.

$\therefore g \circ f : A \to C$ is 1-1 and onto.

Let $a \in A$, $\exists\, b \in B \ni$

$f(a) = b$, $f^{-1}(b) = a$

Let $b \in B$ $\exists\, c \in C \ni$

$g(b) = c$, $g^{-1}(c) = b$

Now, $(g \circ f)(a) = g(f(a))$

$= g(b) = c$

$\therefore (g \circ f)(a) = c \to (1)$

$(f^{-1} \circ g^{-1})(c) = f^{-1} g^{-1}(c)$

$= f^{-1}(b) = a \to (2)$

$(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$

$\Rightarrow (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Hence the proof.