UNIT - 2

GENERALISATION OF PIGEONHOLE PRINCIPLE

If n pigeons are accomadated in m pigeonholes and $n > m$, then one of the pigeon holes must contain $\left[\frac{n-1}{m}\right] + 1$ Pigeons, where $[x]$ denotes the greatest integer less than or equal to x.

1. Show that in any group of 8 people at least two will birthday's which fall on the same day of the week in any given here.

$m = 7$, $n = 8$. The least no of b'day's that fall on the same day of week is equal to $\left[\frac{n-1}{m}\right] + 1$ Pigeons $= \left[\frac{8-1}{7}\right] + 1 = 2$

2. In 10 hrs journey, a man covered a total distance of 45 km. It is known that he travelled 6 km in the first hr and only 3 km in the last hr. show that he must be travelled at least 9 km within a certain period of two consecutive hrs.

soln: Since he travelled $6+3=9$ km in the first and last hr, he must have travelled $45-9=36$ km during the period from second to nineth hr.

If we combine the second and third hr together, 4^{th} & 5^{th} etc .. 8^{th} and 9^{th}. We have 4 time period

$n = 36$, $m = 4$

The least no of Pigeons in one of the period of consecutive two hrs.

$$\left[\frac{36-1}{4}\right] + 1 = 8 + 1 = 9$$

MATHEMATICAL INDUCTION

Let $p(n)$ be a mathematical statement defined for every natural number N. Suppose $p(1)$ is true and assume that $p(a)$ is true for some k. If we are able to prove that $p(k+1)$ is true then say that $p(n)$ is true for every natural numbers

Problem.

1. Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$ by PMI

Soln: $p(n) : 1+2+\dots+n = \frac{n(n+1)}{2}$ $\qquad \frac{1(1+1)}{2} = 1$

$p(1)$ is true

Assume $p(k)$ is true.

$$1+2+\dots+k = \frac{k(k+1)}{2} \quad \text{---(1)}$$

Addition $k+1$ on both side of (1)

$$1+2+\dots+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= (k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{(k+1)(k+1)}{2}$$

$p(k+1)$ is also True

$p(n)$ is true $\forall$ $n \in N$

P.T $a^n - b^n$ is divisible by $a-b$ by PMI

$p(n)$: $a^n - b^n$ is divisible by $a-b$

$P(1)$ is true

Assume that $p(k)$ is true

$a^k - b^k$ is divisible by $a-b$

$a^k - b^k = \lambda(a-b)$ — ①

To prove $p(k+1)$ is true.

$a^{k+1} - b^{k+1} = a^k \cdot a - b^{k+1}$

$= a(\lambda(a-b) + b^k) - b^{k+1}$

$= a\lambda(a-b) + ab^k - b^{k+1}$

$= a\lambda(a-b) + b^k(a-b)$

$= (a-b)(a\lambda + b^k)$

$a-b$ divides $a^{k+1} - b^{k+1}$ also

$P(k+1)$ is true

$P(n)$ is true for any n.

P.T $n^3 + 2n$ is divisible by 3.

Soln:

$P(n)$: $n^3 + 2n$ is divisible by 3

$P(1)$ is true

Assume $p(k)$ is true

$k^3 + 2k$ is divisible by 3

$k^3 + 2k = 3\lambda$ — ①

Now $(k+1)^3 + 2(k+1) = k^3 + 1 + 3k^2 + 3k + 2(k+1)$

$= k^3 + 2k + 3k^2 + 3k + 3$

$= 3\lambda + 3(k^2 + k + 1)$

$\Rightarrow (k+1)^3 + 2(k+1)$ is a multiple of 3

$\therefore P(k+1)$ is also true

$\Rightarrow P(n)$ is true for any n.

Q P.T. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$P(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots \frac{1}{n(n+1)} = \frac{n}{n+1}$

$P(1)$ is true

Assume $P(k)$ is true

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

RECURRENCE RELATION

An equation that expresses in terms of one or more of the previous terms of the sequence is called a recurrence relation.

To Solve Recurrence Relation

Consider a recurrence relation.

$C_0 a_{n+2} + C_1 a_{n+1} + C_2 a_n = f(n)$ —(1)

The Characteristic equation is

$C_0 r^2 + C_1 r + C_2 = 0$ — (2)

Solving (2), we get roots

Case i. Roots are real and distinct

i.e., $r_1 \neq r_2$

Scanned by CamScanner

$$a_n^{(C.F)} = k_1(r_1)^n + k_2(r_2)^n$$

C.F - Complementary function

Case (ii) Roots are real and equal

$r = r_1 = r_2$.

$$a_n^{(C.F)} = (k_1 + k_2 n)(r)^n$$

Case (iii) Roots are imaginary

$r = \alpha \pm i\beta$

$$a_n^{(C.F)} = |r|^n (k_1 \cos n\theta + k_2 \sin n\theta)$$

where $|r| = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

Solve

$$a_{n+2} + 5a_{n+1} + 6a_n = 0$$

Soln. The characteristic eqn is

$r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r_1 = -3 \quad r_2 = -2$

$$a_n^{(C.F)} = k_1(-3)^n + k_2(-2)^n$$

To find Particular Solution $(a_n^{(P)})$

Form of f(n)	Form of f(n) to be assumed
$f(n) = k$	$a_n^{(P)} = A$
$f(n) = n^t$	$a_n^{(P)} = A_0 n^t + A_1 n^{t-1} + \dots + A_t$
$f(n) = a^n$	$a_n^{(P)} = Aa^n$ if a is not a root of ② $a_n^{(P)} = An^p a^n$ if a is a root repeated p times
$f(n) = n^t \cdot a^n$	$a_n^{(P)} = a^n (A_0 n^t + A_1 n^{t-1} + \dots + A_t)$ If "a" is not a root of ②.

$\therefore a_n^{(P)} = n^p a^n (A_0 n^t + A_1 n^{t-1} + \dots A_t)$

If "a" is a root of (2) repeated p times

Solve

$a_{n+2} + 3a_{n+1} + 2a_n = 3^n$ given $a_0 = 0$, $a_1 = 1$

The char eqn is $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$r_1 = -1 \quad r_2 = -2$

$a_n^{(c.p)} = k_1(-1)^n + k_2(-2)^n$

Here $a = 3$

$a_n^{(P)} = A \cdot 3^n \quad a_{n+1}^{(P)} = A\, 3^{n+1}$

$a_{n+2}^{(P)} = A\, 3^{n+2}$

$A\, 3^{n+2} + 3 A\, 3^{n+1} + 2 A \cdot 3^n = 3^n$

$3^n [9A + 9A + 2A] = 3^n \cdot 1$

$20 A = 1$

$A = 1/20$

$a_n^{(P)} = \frac{3^n}{20}$

$a_n = a_n^{C.F} + a_n^{(P)}$

$a_n = k_1(-1)^n + k_2(-2)^n + \frac{3^n}{20}$

Given $a_0 = 0$

$k_1 + k_2 + 1/20 = 0$

$a_1 = 1$

$\boxed{-k_1 - 2k_2 + 3/20 = 1}$

$$-k_2 + 4/20 = 1$$

$$-k_2 = 1 - 1/5$$

$$k_2 = -4/5$$

$$k_1 = -1/20 + 4/5$$

$$k_1 = \frac{-1+16}{20} = 3/4$$

$$a_n = \frac{3}{4}(-1)^n - \frac{4}{5}(-2)^n + \frac{3^n}{20}$$

Q Solve $a_{n+2} - 2a_{n+1} + a_n = 4^n$

The char eqn is $r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

$$r = 1, 1$$

$$a_n^{(C.F)} = (k_1 + k_2 n)(1)^n = k_1 + k_2 n$$

To find Particular soln

$$a = 4$$

Let $a_n^{(P)} = A4^n$

$$a_{n+1}^{(P)} = A4^{n+1}$$

$$a_{n+2}^{(P)} = A4^{n+2}$$

$$A4^{n+2} - 2A4^{n+1} + A4^n = 4^n$$

$$4^n(16A - 8A + A) = 4^n \cdot 1$$

$$9A = 1$$

$$A = 1/9$$

$$a_n^{(P)} = 4^n/9$$

$$\boxed{a_n = k_1 + k_2 n + \frac{4^n}{9}}$$

Scanned by CamScanner

Q- Solve $a_{n+2} - 3a_{n+1} + 2a_n = 2$

The char eqn is $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r_1 = 1, r_2 = 2$

$a_n^{(C.F)} = K_1(1)^n + K_2 2^n$. Here $a=2$ which is a root of char eqn

Let $a_n^{(P)} = An2^n$

$a_{n+1}^{(P)} = A(n+1)2^{n+1}$

$a_{n+2}^{(P)} = A(n+2)2^{n+2}$

$A(n+2)2^{n+2} - 3A(n+1)2^{n+1} + 2An2^n = 2^n$

$A\cancel{2^n}[(n+2)4 - 6(n+1) + 2n] = \cancel{2^n}\ 1$

$A[4n + 8 - 6n - 6 + 2n] = 1$

$A(2) = 1$

$A = 1/2$

$a_n^{(P)} = n/2\ 2^n$

$$\boxed{a_n = K_1(1)^n + K_2(2)^n + n/2\ 2^n}$$

Q. Solve $a_{n+2} - 6a_{n+1} + 9a_n = 3 \cdot 2^n + 7 \cdot 3^n$

The char eqn is

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$.

$a_n^{(C.F)} = (K_1 + K_2 n)3^n$

To find Particular integral.

Let $a_n^{(P_1)} = A \cdot 2^n$

$a_{n+1}^{(P_1)} = A \cdot 2^{n+1}$. $a_{n+2}^{(P_1)} = A \cdot 2^{n+2}$

$$A\cdot 2^{n+2} - 6A2^{n+1} + 9A2^{n} = 3\cdot 2^{n}$$

$$A2^{n}[4-12+9] = 3\cdot 2^{n}$$

$$A = 3$$

$$\boxed{a_n^{(P_1)} = 3\cdot 2^{n}}$$

Let $a_n^{(P_2)} = Bn^2 3^n$

$$a_{n+1}^{(P_2)} = B(n+1)^2 3^{n+1}$$

$$a_{n+2}^{(P_2)} = B(n+2)^2 3^{n+2}$$

$$B(n+2)^2 3^{n+2} - 6B(n+1)^2 3^{n+1} + 9Bn^2 3^n = 7\, 3^n$$

$$B3^n[(n+2)^2 9 - 18(n+1)^2 + 9n^2] = 7\cdot 3^n$$

$$B[(n^2+4n+4)9 - 18(n^2+2n+1) + 9n^2] = 7$$

$$B[18] = 7$$

$B = 7/18$ $\qquad a_n^{(P_2)} = \frac{7}{18} n^2 3^n$

$\cancel{A\cdot 2^{n+2}} - 6$ $\qquad \boxed{a_n = a_n^{(C.F)} + a_n^{(P_1)} + a_n^{(P_2)}}$

Solve: $\quad a_n - 2a_{n-1} = n+5, \; a_0 = 4 \qquad n-(n-1) = 1$

Soln: The char eqn is

$$r - 2 = 0$$

$$r = 2$$

$$a_n^{(C.F)} = K_1 2^n$$

$a_{n-1}^{(P)}$ To find Particular soln

Let $a_n^{(P)} = A_0 n + A_1$

$$a_{n-1}^{(P)} = A_0(n-1) + A_1$$

$$A_0 n + A_1 - 2\{A_0(n-1) + A_1\} = n+5$$

$$-A_0 n - A_1 + 2A_0 = n+5$$

Comparing the like terms

$n: \boxed{-A_0 = 1}$

$A_0 = -1$

Const: $-A_1 + 2A_0 = 5$

$A_1 = -7$

$$a_n^{(P)} = -n-7$$

$$a_n = K_1 2^n - n - 7$$

Given $a_0 = 4$

$K_1 - 7 = 4$

$K_1 = 11$

$$\boxed{a_n = 11 \cdot 2^n - n - 7}$$

Solve: $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + 3n$

soln: the Char eqn is

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$$r = 3 \quad r = 2$$

$$a_n^{(C.F)} = K_1 3^n + K_2 2^n$$

To find PS

Let $a_n^{(P_1)} = An2^n$

$a_{n-1}^{(P_1)} = A(n-1)2^{n-1}$

$a_{n-2}^{(P_1)} = A(n-2)2^{n-2}$

$$An2^n - 5A(n-1)2^{n-1} + 6A(n-2)2^{n-2} = 2^n$$

$$A \cdot 2^n \left[n - \frac{5(n-1)}{2} + \frac{6(n-2)}{4}\right] = 2^n$$

$$A\left[\frac{4n-10(n-1)+6(n-2)}{4}\right]=1$$

$$A[-2]=4$$

$$A=2$$

$$a_n^{(P_1)}=-2\cdot n2^n=-n\cdot 2^{n+1}$$

Let $a_n^{(P_2)}=A_0 n+A_1$

$$a_{n-1}^{(P_2)}=A_0(n-1)+A_1$$

$$a_{n-2}^{(P_2)}=A_0(n-2)+A_1$$

$$An2^n-5A(n-1)2^{n-1}+6A(n-2)2^{n-2}=2^n$$

$$A\cdot 2^n\left[n-\frac{5(n-1)}{2}+\frac{6(n-2)}{4}\right]=2^n$$

$$A_0 n+A_1-5(A_0(n-1)+A_1)+6(A_0(n-2)+A_1)$$
$$=3n$$

$$2A_0n-7A_0+2A_1=3n$$

n: $2A_0=3$

$$A_0=3/2$$

Const $-7A_0+2A_0=0$

$$2A_1=7A_0=7\times 3/2=21/2$$

$$A_1=21/4$$

$$a_n^{(P_2)}=3/2\, n+21/4$$

$$a_n=a_n^{(C.P)}+a_n^{(P_1)}+a_n^{(P_2)}$$

$$a_n=k_1 3^n+k_2 2^n-n2^{n+1}$$
$$+\frac{3n}{2}+21/4$$

8) $a_{n+2} - 4a_{n+1} + 3a_n = 2^n \cdot n^2$

$r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r = 1 \quad , r = 3$

3

-1   -3

$$\boxed{a_n^{(C.F)} = k_1 (1)^n + k_2 3^n}$$

$a_n^{(P.I)} = 2^n (A_0 + A_1 n + A_2 n^2)$

$a_{n+1}^{(P.I)} = 2^{n+1} (A_0 + A_1(n+1) + A_2(n+1))$

$a_{n+2}^{(P.I)} = 2^{n+2} (A_0 + A_1(n+2) + A_2(n+2)^2)$

$2^{n+2}(A_0 + A_1(n+2) + A_2(n+2)^2) - 4 \cdot 2^{n+1}$
$(A_0 + A_1(n+1) + A_2(n+1)^2) + 3 \cdot 2^n (A_0 + A_1 n + A_2 n^2)$
$= 2^n \cdot n^2$

$2^n [4(A_0 + A_1(n+2) + A_2(n+2)^2) -$
$8(A_0 + A_1(n+1) + A_2(n+1)^2) + 3(A_0 + A_1 n + A_2 n^2)]$
$= 2^n \cdot n^2$

$n^2 = 4A_2 - 8A_2 + 3A_2 = 1$

$-A_2 = 1$

$A_2 = -1$

$n = 4A_1 + 16A_2 - 8A_1 - 16A_2 + 3A_1 = 0$

$A_1 = 0$

$4A_0 + 8A_1 + 16 A_2 - 8A_0 - 8A_0 - 8A_1 - 8A_2 + 3A_0 = 0$

$-A_0 - 16 + 8 = 0$

$-A_0 - 8 = 0$

$A_0 = -8$

$a_n^{(PI)} = 2^n (-8 - n^2)$

$a_n = a_n^{(C.F)} + a_n^{(P.I)}$

Q)

$$a_n - 4a_{n-1} + 4a_{n-2} + (n+1)2^n = (n+1)2^n$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = +2, 2$$

$$a_n^{(CF)} = (K_1 + K_2 n)2^n$$

$$a_n^{(P)} = 2^n n^2 (A_0 + A_1 n)$$

$$a_{n-1}^{(P)} = 2^{n-1}(n-1)^2(A_0 + A_1(n-1))$$

$$a_{n-2}^{(P)} = 2^{n-2}(n-2)^2(A_0 + A_1(n-2))$$

① becomes

$$= 2^n n^2 (A_0 + A_1 n) - 4(2^{n-1}(n-1)^2(A_0 + A_1(n-1))$$
$$+ 4(2^{n-2}(n-2)^2(A_0 + A_1(n-2))) = 2^n(n+1)$$

$$= 2^n\left[n^2(A_0 + A_1 n) - \tfrac{4}{2}((n-1)^2(A_0 + A_1(n-1)) + \tfrac{4}{2}((n-2)^2(A_0 + A_1(n-2)))\right) = 2^n(n+1)$$

$$= n^2(A_0 + A_1 n) - 2(n-1)^2(A_0 + A_1(n-1)) + (n-2)^2(A_0 + A_1(n-2)) = n+1$$

$$= n^2 A_0 + A_1 n^3 - 2(n^2 - 2n + 1)(A_0 + A_1 n - A_1)$$
$$+ (n^2 - 4n + 4)(A_0 + A_1 n - 2A_1) = n+1$$

$$= n^2 A_0 + A_1 n^3 - 2n^2 A_0 - 2A_1 n^3 + 2A_1 + 4A_0 n + 4A_1 n^2$$
$$- 4A_1 n - 2A_0 - 2A_1 n + 2A_1 + A_0 n^2 + A_1 n^3$$
$$- 2A_1 n^2 - 4A_0 n - 4A_1 n^2 + 8A_1 n + 4A_0 + 4A_1 n$$
$$- 8A_1 = n+1$$

$$= -14A_1 n + 2A_0 - 6A_1 + 4A_0 n = n+1$$

n: $-14A_1 + 4A_0 = 1$ —(1)

const: $2A_0 - 6A_1 = 1$ —(2)

solving (1) & (2)

$A_0 = -3/2$, $A_1 = -1/2$

$$a_n = a_n^{(C.F)} + a_n^{(P)}$$

$$= (K_1 + K_2 n) 2^n + 2^n n^2 (-3/2 - n/2)$$

TO SOLVE RR USING GENERATING FUNCTION

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots \infty = \sum_{n=0}^{\infty} a_n x^n$$

$$a_{n+1} + a_n = 0$$

$$a_{n+1} x^n + a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\frac{1}{x} \sum_{n=0}^{\infty} a_{n+1} x^{n+1} + \sum a_n x^n = 0$$

$$\frac{1}{x} \left[a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - a_0\right]$$

$$\frac{1}{x} (G(x) - a_0) + G(x) = 0$$

$$\frac{1}{x} (G(x)) + G(x) = \frac{a_0}{x}$$

$$G(x) \left(\frac{1}{x} + 1\right) = \frac{a_0}{x}$$

$$G(x) = \frac{a_0}{1+x}$$

NOTE

* $\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$

* $\frac{1}{1+x} = 1 - x + x^2 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$

* $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots (n-1) x^n + \dots \infty$
$= \sum_{n=0}^{\infty} (n+1) x^n$

Scanned by CamScanner

Q $a_n - 3a_{n-1} = 1$ given $a_0 = 1$

$$a_n - 3a_{n-1} = 1$$

$$a_n x^n - 3a_{n-1} x^n = x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} x^n$$

$n-1=0$

$n=1$

$$a_0 + \sum a_n x^n - a_0 - 3x \underbrace{\sum_{n=1}^{\infty} a_{n-1} x^{n-1}}_{G(x)} = x\sum_{n=1}^{\infty} x^{n-1}$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) - 1 - 3x\, G(x) = x\frac{1}{1-x}$$

$$(1-3x)(G(x)) = \frac{x}{1-x} + 1$$

$$(1-3x)\, G(x) = \frac{x+1-x}{1-x}$$

$$G(x) = \frac{1}{(1-x)(1-3x)}$$

Let $\dfrac{1}{(1-x)(1-3x)} = \dfrac{A}{(1-x)} + \dfrac{B}{1-3x}$

$$1 = A(1-3x) + B(1-x)$$

$$A = -1/2, \quad B = 3/2$$

$$G(x) = \frac{(-1/2)}{(1-x)} + \frac{3/2}{(1-3x)}$$

Scanned by CamScanner

$a_n =$ Coefficient of x^n in $G(x)$

$$G(x) = \left(\frac{-1}{2}\right)\left[1+x+x^2+\ldots\ldots+x^n+\ldots\infty\right] + \left(\frac{3}{2}\right)\left[1+(3x)+(3x)^2+\ldots+(3x)^n+\ldots\infty\right]$$

$$\boxed{a_n = \left(\frac{-1}{2}\right)(1) + \frac{3}{2}(3)^n}$$