# SRM UNIVERSITY

RAMAPURAM PART- VADAPALANI CAMPUS, CHENNAI – 600 026

# Department of Mathematics

## Sub Title: DISCRETE MATHEMATICS

**Sub Code: 15MA302**

**Unit -I - MATHEMATIC**

1.Which of the following statement is the negation of the statement " 2 is even and -3 is negative"?

a)2 is even and -3 is not negative b)2 is odd and -3 is not negative

c)2 is not odd and -3 is not negative d)2 is odd or -3 is not negative **Ans (d)**

2.The contra positive of $q \to p$ is a)$p \to q$ b)$\neg p \to \neg q$ c) $\neg q \to \neg p$ d)$p \to \neg q$ **Ans (b)**

3.What is the converse of the assertion I stay only if you go?

a) I stay if you go b)if you don't go then I don't stay

c)if I stay then you go d)if you don't stay then you go **Ans (a)**

4.$PVT \Leftrightarrow T$ is called a)identity law b)complement law c)dominant law d)idempotent law **Ans (c)**

5.The statement PV P is a a)contradiction b)tautology c)contrapositive d)inverse **Ans (b)**

6) Dual of $\neg (p \leftrightarrow Q)=(P \wedge \neg Q)V(\neg P \wedge Q)$

a) $\neg (P \leftrightarrow Q) \equiv (PV\neg Q)V(\neg PVQ)$ b) $(P \leftrightarrow Q) \equiv (\neg PVQ)V(PV\neg Q)$

c) $\neg (P \leftrightarrow Q) \equiv (PV\neg Q) \wedge (\neg PVQ)$ d) $\neg (P \leftrightarrow Q) \equiv (\neg PVQ) \wedge (PV\neg Q)$ **Ans (c)**

7. Symbolized form of the statement "All Roses are red" Where P(x) : x is a Rose , R(x): x is Red

a)$[(\forall x)P(x)] \to R(x)$ b) $\exists x(P(x) \wedge R(x))$ c) $\forall x(P(x) \to R(x))$ d)$\exists xP(x) \wedge R(x)$ **Ans (c)**

8.The rule if a formula S can be derived from another formula R and A set of premises,then the statement

$R \to S$ can be derive from the set of premises is called

a)Rule CP b)Rule T c)Rule P d)Rule US **Ans (a)**

9.The statement $(PVQ) \wedge (P \to R) \wedge (Q \to R)$implies a)R b)P c)Q d)$P \wedge Q$ **Ans (a)**

10.The statement $\neg (P \leftrightarrow Q)$ is equivalent to a)$P \leftrightarrow \neg Q$ b) $\neg P \leftrightarrow \neg Q$ c)$P \to \neg Q$ d) $\neg P \to \neg Q$ **Ans (a)**

11. $\neg P \rightarrow Q$= a)$QV\neg P$ b)$Q \wedge P$ c)PVQ d) $\neg P \wedge Q$ **Ans (c)**

12. $\neg P$,PVQ => a) Q b) $\neg P$ c)PVQ d)$P \wedge Q$ **Ans (a)**

13.$((P \rightarrow Q)V(\neg PV(Q \rightarrow R))VT$= a)$P \rightarrow Q$ b) $\neg P$ c)T d)F **Ans (c)**

14.A compound proposition $P=P(P_1,P_2,\ldots\ldots P_n)$which is true for every truth values for $P_1,P_2,\ldots,P_n$ is called

a)Contradiction b)Tautology c)Negation d)Implication **Ans (b)**

15) $(P \rightarrow \neg P) \rightarrow \neg P$ is equivalent to a)T b)F c)P d) $\neg P$ **Ans (c)**

16)The dual of $\neg P \rightarrow (P \rightarrow Q)$ is

a)$PV(\neg P \wedge Q)$ b) $\neg (\neg P) \wedge (\neg P \wedge Q)$ c)$P \rightarrow \neg (P \rightarrow Q)$ d)$(\neg P \wedge Q) \wedge \neg P$ **Ans (b)**

17.In proving that $P \rightarrow (Q \rightarrow S)$ follows from the premises $P \rightarrow (Q \rightarrow R)$ and $Q \rightarrow (R \rightarrow S)$ using CP rule, the additional premises is a)Q b)$Q \rightarrow R$ c)P d) $\neg P$ **Ans (c)**

18.Let P is sunny this afternoon ,Q is colder than yesterday and R is we will go for swimming. Then the statement if it is not sunny this afternoon and it is colder than yesterday, then we will go for swimming is

a)$(\neg P \wedge Q) \rightarrow R$ b)$(P \wedge \neg Q) \rightarrow \neg R$ c)$(\neg PVQ) \rightarrow R$ d)$(\neg P \wedge Q)VR$ **Ans (a)**

19.Which of the following statement is a contradiction?

a)$(P \rightarrow \neg P) \rightarrow \neg P$ b)$(P \rightarrow (P \vee Q)$ c)$(\neg Q \rightarrow P) \wedge Q$ d) $PV(P \rightarrow Q)$ **Ans (a)**

20.What is the dual of $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P) \equiv T$,

a)$(\neg P \rightarrow Q)$ $(Q$ $\neg P) \equiv F$ b) $\neg (P$ $Q)$ $(Q$ $P) \equiv T$ c)$(\neg P$ $)$ $(Q$ $\neg P) \equiv F$ d) $\neg (\neg PVQ) \wedge (Q$ $\neg P) \equiv F$ **Ans (d)**

21.(PVQ) $\neg (\neg P$ $Q)$ is equivalent to a)P b)Q c)PVQ d)P Q **Ans (a)**

22.Which one is the contra positive of $Q \rightarrow P$?

a)$P \rightarrow Q$ b) $\neg P \rightarrow \neg Q$ c) $\neg Q \rightarrow \neg P$ d) $\neg PVQ$ **Ans (b)**

23.The statement (P Q)=>P is a

a)contradiction b)tautology c)inconsistent d)consistent **Ans (d)**

24.The dual of $\neg$ (P Q)VT is

a)(PVQ) F b)(PVQ) T c)(P Q)VF d) $\neg$ (PVQ) F **Ans (d)**

25.Which of the following is a statement?

(A) Open the door. (B) Do your homework. (C) Switch on the fan **(D)** Two plus two is four. **Ans (D)**

26.Which of the following is a statement in Logic?
(A) Go away (B) How beautiful! (C) $x > 5$ **(D)** $2 = 3$ **Ans (D)**

27. $\sim(p \vee q)$ is (A) $\sim p \vee q$ (B) $p \vee \sim q$ (C) $\sim p \vee \sim q$ **(D)** $\sim p \wedge \sim q$ **Ans (D)**

28.If p: The sun has set, q: The moon has raised, then symbolically the statement 'The sun has not set or the moon has not risen' is written as
(A) $p \wedge \sim q$ (B) $\sim q \vee p$ (C) $\sim p \wedge q$ **(D)** $\sim p \vee \sim q$ **Ans (D)**

29.The inverse of logical statement $p \rightarrow q$ is
**(A)** $\sim p \rightarrow \sim q$ (B) $p \leftrightarrow q$ (C) $q \rightarrow p$ (D) $q \leftrightarrow p$ **Ans (A)**

30.Let p: Mathematics is interesting, q: Mathematics is difficult, then the symbol $p \rightarrow q$ means
**(A)** Mathematics is interesting implies that Mathematics is difficult.
(B) Mathematics is interesting is implied by Mathematics is difficult.
(C) Mathematics is interesting and Mathematics is difficult.
(D) Mathematics is interesting or Mathematics is difficult. **Ans (A)**

31.Which of the following is logically equivalent to $\sim(p \wedge q)$
(A) $p \wedge q$ **(B)** $\sim p \vee \sim q$ (C) $\sim(p \vee q)$ (D) $\sim p \wedge \sim q$ **Ans (B)**

32.$\sim(p \rightarrow q)$ is equivalent to
**(A)**$p \wedge q$ (B) $\sim p \vee q$ (C) $p \vee \sim q$ (D) $\sim p \wedge \sim q$ **Ans (A)**

33.Contrapositive of $p \rightarrow q$ is
(A) $q \rightarrow p$ (B) $\sim q \rightarrow p$ **(C)** $\sim q \rightarrow \sim p$ (D) $q \rightarrow \sim p$ **Ans (C)**

34.A compound statement $p \rightarrow q$ is false only when
**(A)**p is true and q is false. (B) p is false but q is true.
(C) atleast one of p or q is false. (D) both p and q are false. **Ans (A)**

35.Every conditional statement is equivalent to
**(A)** its contrapositive (B) its inverse (C) its converse (D)only itself **Ans (A)**

36.Statement $\sim p \leftrightarrow \sim q \equiv p \leftrightarrow q$ is
**(A)**a tautology (B) a contradiction (C) contingency (D) proposition **Ans (A)**

37.Given that p is 'false' and q is 'true' then the statement which is 'false' is
**(A)** $\sim p \rightarrow \sim q$ (B) $p \rightarrow (q \wedge p)$ (C) $p \rightarrow \sim q$ (D) $q \rightarrow \sim p$ **Ans (A)**

38.Dual of the statement $(p \wedge q) \vee \sim q \equiv p \vee \sim q$ is
(A) $(p \vee q) \vee \sim q \equiv p \vee \sim q$ (B) $(p \wedge q) \wedge \sim q \equiv p \wedge \sim q$
**(C)** $(p \vee q) \wedge \sim q \equiv p \wedge \sim q$ (D) $(\sim p \vee \sim q) \wedge q \equiv \sim p \wedge q$ **Ans (C)**

39.$\sim[p \vee (\sim q)]$ is equal to
(A) $\sim p \vee q$ **(B)** $(\sim p) \wedge q$ (C) $\sim p \vee \sim p$ (D) $\sim p \wedge \sim q$ **Ans (B)**

40.Write Negation of 'For every natural number $x$, $x + 5 > 4$'.
(A) $\forall x \in N, x + 5 < 4$ (B) $\forall x \in N, x - 5 < 4$ (C) For every integer $x$, $x + 5 < 4$
**(D)** There exists a natural number $x$, forwhich $x + 5 \leq 4$ **Ans (D)**

41. If p is false and q is true, then
(A) $p \wedge q$ is true (B) $p \vee q$ is true (C) $q \rightarrow p$ is true **(D)** $p \rightarrow q$ is true **Ans (D)**

42.If p and q have truth value 'F' then $(\sim p \vee q) \leftrightarrow \sim(p \wedge q)$ and $\sim p \leftrightarrow (p \rightarrow \sim q)$ respectively are

**(A)** T, T (B) F, F (C) T, F (D) F, T **Ans (A)**

43.Which of the following is logically equivalent to $\sim[p \rightarrow (p \vee \sim q)]$?

(A) $p \vee (\sim p \wedge q)$ **(B)** $p \wedge (\sim p \wedge q)$ (C) $p \wedge (p \vee \sim q)$ (D) $p \vee (p \wedge \sim q)$ **Ans (B)**

44.If $q \vee p$ is F then which of the following is correct?

(A) $p \leftrightarrow q$ is T **(B)** $p \rightarrow q$ is T (C) $q \rightarrow p$ is T (D) $p \rightarrow q$ is F **Ans (B)**

45.Which of the following is true?

(A) $p \wedge\ p \equiv T$ (B) $p \vee\ p \equiv F$ (C) $p \rightarrow q \equiv q \rightarrow p$ **(D)** $p \rightarrow q \equiv (\sim q) \rightarrow (\ p)$ **Ans (D)**

46.The statement $(p \wedge q) \rightarrow p$ is

(A) a contradiction. **(B)**a tautology .(C) either (A) or (B) (D) a contingency. **Ans (B)**

47.Negation of the statement: "If Dhonilooses the toss then the team wins", is

(A) Dhoni does not lose the toss and theteam does not win.

**(B)**Dhoni loses the toss but the team doesnot win.

(C) Either Dhoni loses the toss or the teamwins. (D) Dhoni loses the toss iff the team wins. **Ans (A)**

48.If $p \Rightarrow (\ p \vee q)$ is false, the truth values of p and q respectively, are

(A) F, T (B) F, F (C) T, T **(D)** T, F **Ans (D)**

49.The logically equivalent statement of $p \leftrightarrow q$ is

(A) $(p \wedge q) \vee (q \rightarrow p)$ (B) $(p \wedge q) \rightarrow (\ p \vee q)$ **(C)** $(p \rightarrow q) \wedge (q \rightarrow p)$ (D) $(p \wedge q) \vee (p \wedge q)$ **Ans (C)**

## UNIT-I - MATHEMATICAL LOGIC

### PART-B

① Show that $(\neg P \wedge (P \to q)) \to \neg P$ is a tautology.

| $P$ | $q$ | $P \to q$ | $\neg P$ | $\neg P \wedge (P \to q)$ | $(\neg P \wedge (P \to q)) \to \neg P$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

Since last column contains only truth value T, it is a tautology.

② Without using truth table, prove the following:

$(P \wedge (P \leftrightarrow Q)) \to Q = T$

$$P \wedge (P \Leftrightarrow Q) \to Q \Rightarrow [P \wedge ((\neg P \vee Q) \wedge (\neg Q \vee P))] \to Q$$

$$\Leftrightarrow (P \wedge (\neg Q \vee P)) \wedge (\neg P \vee Q)) \to Q$$

$$\Leftrightarrow [P \wedge (\neg P \vee Q)] \to Q$$

$$\Leftrightarrow [(P \wedge \neg P) \vee (P \wedge Q)] \to Q \quad \because P \wedge (\neg Q \vee P) = P.$$

$$\Leftrightarrow [F \vee (P \wedge Q)] \to Q$$

$$\Leftrightarrow P \wedge Q \to Q$$

$$\Leftrightarrow \neg(P \wedge Q) \vee Q$$

$$\Leftrightarrow (\neg P \vee \neg Q) \vee Q$$

$$\Leftrightarrow \neg P \vee (\neg Q \vee Q) \Leftrightarrow \neg P \vee T$$

$$\Leftrightarrow T$$

③ Construct the truth table for the following compound proposition: $(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$

| $P$ | $Q$ | $R$ | $\neg P$ | $\neg Q$ | $\neg P \leftrightarrow \neg Q$ | $Q \leftrightarrow R$ | $(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T | T |
| T | T | F | F | F | T | F | F |
| T | F | T | F | T | F | F | T |
| T | F | F | F | T | F | T | F |
| F | T | T | T | F | F | T | F |
| F | T | F | T | F | F | F | T |
| F | F | T | T | T | T | F | F |
| F | F | F | T | T | T | T | T |

④ Prove that $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \to q$ | $\neg q \wedge (p \to q)$ | $(\neg q \wedge (p \to q)) \to \neg p$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | T | T |

Since the last column contains only T, it is a tautology.

⑤ Prove: $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

$$(p \to r) \wedge (q \to r) \Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$$
$$\Leftrightarrow (\neg p \wedge \neg q) \vee r$$
$$\Leftrightarrow \neg (p \vee q) \vee r$$
$$\Leftrightarrow p \vee q \to r.$$

(6) Prove that $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$ without using truth table.

$$P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$$
$$\Leftrightarrow \neg P \vee (\neg Q \vee R)$$
$$\Leftrightarrow \neg P \vee (Q \to R)$$
$$\Leftrightarrow P \to (Q \to R)$$

Hence proved.

(7) Show that $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \to q$ | $\neg q \wedge (p \to q)$ | $(\neg q \wedge (p \to q)) \leftrightarrow \neg p$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | T | T |

(8) Construct the truth table for $(\neg P \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$.

| $P$ | $q$ | $r$ | $\neg P$ | $\neg q$ | $\neg P \leftrightarrow \neg q$ | $q \leftrightarrow r$ | $(\neg P \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T | T |
| T | T | F | F | F | T | F | F |
| T | F | T | F | T | F | F | T |
| T | F | F | F | T | F | T | F |
| F | T | T | T | F | F | T | F |
| F | T | F | T | F | F | F | T |
| F | F | T | T | T | T | F | F |
| F | F | F | T | T | T | T | T |

(9) Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$$
$$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$$
$$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$$
$$\Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$$
$$\Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$$
$$\Leftrightarrow T \wedge R \Leftrightarrow R$$

(10) Show that $(P \to Q) \to Q \Rightarrow P \vee Q$

$$(P \to Q) \to Q \Rightarrow (\neg P \vee Q) \to Q$$
$$\Rightarrow \neg(\neg P \vee Q) \vee Q$$
$$\Rightarrow (P \wedge \neg Q) \vee Q$$
$$\Rightarrow (P \vee Q) \wedge (\neg Q \vee Q)$$
$$\Rightarrow (P \vee Q) \wedge T$$
$$\Rightarrow P \vee Q$$

(11) Derive by using CP rule, $P \to (Q \to S)$ from $P \to (Q \to R)$, $Q \to (R \to S)$.

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $P$ | P (additional) |
| 2. | $P \to (Q \to R)$ | P |
| 3. | $Q \to R$ | T, 1, 2 and modus ponens |
| 4. | $\neg Q \vee R$ | T, 3 |
| 5. | $Q \to (R \to S)$ | P |
| 6. | $\neg Q \vee (R \to S)$ | T, 5 |
| 7. | $\neg Q \vee (R \wedge (R \to S))$ | T, 4,6 |
| 8. | $\neg Q \vee S$ | T, 7 |
| 9. | $Q \to S$ | T, 8 |
| 10. | $P \to (Q \to S)$ | T, 9 and CP rule. |

(12) Show that $R \vee S$ is a valid conclusion from the premises $C \vee D$, $C \vee D \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$

| Step No. | Statement | Reason |
|---|---|---|
| 1. | $C \vee D \to \neg H$ | P |
| 2. | $\neg H \to (A \wedge \neg B)$ | P |
| 3. | $C \vee D \to (A \wedge \neg B)$ | T, 1, 2. |
| 4. | $C \vee D$ | P |
| 5. | $A \wedge \neg B$ | T, 3, 4 |
| 6. | $(A \wedge \neg B) \to (R \vee S)$ | P |
| 7. | $R \vee S$ | T, 5, 6. |

(13) Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$

Answer Same as Qn (12).

In (12) replace
C by p, R by a
D by q, S by b.
H by r
A by s
B by t

(14) Prove $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$ by proving the equivalence of dual.

$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

ie) $\neg (p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$

Dual of the equivalence is

$\neg (p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r)$.

L.H.S $= (\neg p \vee \neg q) \wedge r$, by Demorgan's law.

$\equiv (\neg p \wedge r) \vee (\neg q \wedge r)$, by Distributive law

$\equiv$ R.H.S.

(15) Prove: $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$. by proving the equivalence of dual.

The dual of the given equivalence is

$$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p.$$

L.H.S. $\equiv \neg((\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$. (by distributive law)

$\equiv \neg((\neg p \vee F) \wedge (p \vee q)$, by complement law.

$\equiv \neg(\neg p) \wedge (p \vee q)$, by identity law.

$\equiv p \wedge (p \vee q)$

$\equiv p$ (by absorption law).

(16). Show that $(a \vee b)$ follows logically from the premises $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$.

| Step ① | Statement | Reason. |
|---|---|---|
| (1) | $(p \vee q) \to \neg r$ | P. |
| (2) | $\neg r \to (s \wedge \neg t)$ | P. |
| (3) | $(p \vee q) \to (s \wedge \neg t)$ | T (1,2) |
| (4) | $(s \wedge \neg t) \to a \vee b$ | P. |
| (5) | $p \vee q \to a \vee b$ | T (3,4) |
| (6) | $p \vee q$ | P |
| (7) | $a \vee b$ | T (5,6). |

PART-C

1. without using truth table, prove that

$$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q.$$

$$\begin{aligned}(\neg p \vee q) \wedge (p \wedge (p \wedge q)) &\equiv (\neg p \vee q) \wedge (p \wedge p) \wedge q \quad \text{(by Associative law)}\\ &\equiv (\neg p \vee q) \wedge (p \wedge q) \quad \text{(by Idempotent law)}\\ &\equiv (p \wedge q) \wedge (\neg p \vee q) \quad \text{(commutative law)}\\ &\equiv ((p \wedge q) \wedge \neg p) \vee ((p \wedge q) \wedge q) \quad \text{(Distributive law)}\\ &\equiv (\neg p \wedge (p \wedge q)) \vee (p \wedge (q \wedge q)) \quad \text{(Commutative law)}\\ &\equiv ((\neg p \wedge p) \wedge q) \vee (p \wedge q)\\ &\equiv (F \wedge q) \vee (p \wedge q)\\ &\equiv F \vee (p \wedge q)\\ &\equiv p \wedge q \quad \text{(Dominant law)}.\end{aligned}$$

2. using truth table, show that $((p \vee q) \wedge (p \to r) \wedge (q \to r)) \to r$ is a tautology.

| $p$ | $q$ | $r$ | $p \vee q$ | $p \to r$ | $q \to r$ | $(p \vee q) \wedge (p \to r) \wedge (q \to r)$ (A) | $A \to r$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | T | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | F | F | T |
| F | F | T | F | T | T | F | T |
| F | F | F | F | T | T | F | T |

Since last column contains only T, it is a tautology.

③ Show that $(P\to Q)\wedge(R\to S)$, $(Q\to M)\wedge(S\to N)$, $\neg(M\wedge N)$ and $(P\to R) \Rightarrow \neg P$

| Step | Statement | Reason |
|---|---|---|
| (1) | $(P\to Q)\wedge(R\to S)$ | Rule P |
| (2) | $P\to Q$ | Rule T $(P\wedge Q\Rightarrow P)$ |
| (3) | $R\to S$ | Rule T $(P\wedge Q\Rightarrow Q)$ |
| (4) | $(Q\to M)\to$ | Rule P. |
| (5) | $Q\to M$ | Rule T |
| (6) | $S\to N$ | Rule T |
| (7) | $P\to M$ | Rule T (1, 4) |
| (8) | $R\to N$ | Rule T (1, 4) |
| (9) | $P\to R$ | Rule P |
| (10) | $P\to N$ | Rule T (9, 8) |
| (11) | $\neg N\to\neg P$ | Rule T |
| (12) | $\neg M\to\neg P$ | Rule T (7) |
| (13) | $(\neg M\vee\neg N)\to\neg P$ | Rule T |
| (14) | $\neg(M\wedge N)\to\neg P$ | Rule T |
| (15) | $\neg(M\wedge N)$ | Rule P. |
| (16) | $\neg P$ | Rule T (15, 14) |

④ without using truth table prove that

$\neg p\to(q\to r)\equiv q\to(p\vee r)$.

$$
\begin{aligned}
\neg p\to(q\to r) &\Rightarrow \neg p\to(\neg q\vee r)\\
&\Rightarrow (p\vee\neg q)\vee r\\
&\Rightarrow (\neg q\vee p)\vee r\\
&\Rightarrow \neg q\vee(p\vee r)\\
&\Rightarrow q\to(p\vee r)
\end{aligned}
$$

(5) Show that $(S\to \neg Q,\ S\vee R,\ \neg R,\ \neg R\leftrightarrow Q) \Rightarrow \neg P$. by indirect method.

$r\to\neg q,\ r\vee s,\ s\to\neg q,\ p\to q \Rightarrow \neg p$

To use the indirect method, we shall include $\neg\neg P \equiv P$ as an additional premise and prove a contradiction.

| Step | Statement | Reason |
|---|---|---|
| (1) | $p$ | P.(additional) |
| (2) | $p\to q$ | P |
| (3) | $q$ | T (1,2) |
| (4) | $r\to\neg q$ | P |
| (5) | $s\to\neg q$ | P |
| (6) | $(r\vee s)\to\neg q$ | T (4,5) |
| (7) | $r\vee s$ | P |
| (8) | $\neg q$ | T (6,7) |
| (9) | $q\wedge\neg q$ | T (3,8) |
| (10) | F | T |

(9) Prove the following equivalence by proving the equivalences of dual. $(p\wedge(p\leftrightarrow q))\to q \equiv T$.

ie) $p\wedge((p\to q)\wedge(q\to p))\to q \equiv T$

$p\wedge((\neg p\vee q)\wedge(\neg q\vee p))\to q \equiv T$

ie) $\neg(p\wedge((\neg p\vee q)\wedge(\neg q\vee p)))\vee q \equiv T$

$\therefore$ Dual of equivalences is

$\neg(p\vee((\neg p\wedge q)\vee(\neg q\wedge p)))\wedge q \equiv F$

L.H.S $\equiv \neg[(p\vee(\neg p\wedge q))\vee(\neg q\wedge p)]\wedge q$ (by associative law)

$\equiv \neg[(T\wedge(p\vee q))\vee(\neg q\wedge p)]\wedge q$ (by distributive and complement laws)

$\equiv \neg[(p\vee q)\vee(\neg q\wedge p)]\wedge q$ (identity law)

$\equiv \neg[((p\vee q)\vee\neg q)\wedge((p\vee q)\vee p)]\wedge q$ (Distributive law)

$\equiv \neg[(p\vee T)\wedge(p\vee q)]\wedge q$

$\equiv \neg(T\wedge(p\vee q))\wedge q$ (dominant law)

$\equiv \neg(p\vee q)\wedge q$ (by Demorgan's law)

$\equiv \neg p\wedge F$ (by complement law)

$\equiv F$ (by dominant law).

(7) Show that $(p\to q)\wedge(r\to s)$, $(q\to t)\wedge(s\to u)$, $\neg(t\wedge u)$ and $(p\to r) \Rightarrow \neg p$.

| Step | Statement | Reason |
|---|---|---|
| (1) | $(p\to q)\wedge(r\to s)$ | P |
| (2) | $p\to q$ | T (1, simplification) |
| (3) | $r\to s$ | T (1) |
| (4) | $(q\to t)\wedge(s\to u)$ | P |
| (5) | $q\to t$ | T (4) |
| (6) | $s\to u$ | T (5) |
| (7) | $p\to t$ | T (2, 5) |
| (8) | $r\to u$ | T (3, 6) |
| (9) | $p\to r$ | P |
| (10) | $p\to u$ | T (9, 8) |
| (11) | $\neg t\to\neg p$ | T (7) |
| (12) | $\neg u\to\neg p$ | T (10) |
| (13) | $(\neg t\to\neg p)\to\neg p$ | T (11, 12) |
| (14) | $\neg(t\wedge u)\to\neg p$ | T (13) |
| (15) | $\neg(t\wedge u)$ | P |
| (16) | $\neg p$ | T (14, 15 Modus Ponens) |

(8) Show that $(a \to b) \wedge (a \to c)$, $\neg(b \wedge c)$, $(d \vee a) \Rightarrow d$

| Step | Statement | Reason |
|---|---|---|
| (1) | $(a \to b) \wedge (a \to c)$ | P |
| (2) | $a \to b$ | T (1) |
| (3) | $a \to c$ | T (1) |
| (4) | $\neg b \to \neg a$ | T (2, contrapositive) |
| (5) | $\neg c \to \neg a$ | T (3, contrapositive) |
| (6) | $(\neg b \vee \neg c) \to \neg a$ | T (4, 5) |
| (7) | $\neg(b \wedge c) \to \neg a$ | T (Demorgan's law) |
| (8) | $\neg(b \wedge c)$ | P |
| (9) | $\neg a$ | T (7, 8) |
| (10) | $d \vee a$ | P |
| (11) | $(d \vee a) \wedge \neg a$ | T (9, 10) |
| (12) | $(d \wedge \neg a) \vee (a \wedge \neg a)$ | T (11) |
| (13) | $(d \wedge \neg a) \vee F$ | T (12) |
| (14) | $d \wedge \neg a$ | T (13) |
| (15) | $d$ | T (14). |

(9) Give a direct proof for the implication
$p \to (q \to s)$, $(\neg r \vee p)$, $q \Rightarrow r \to s$.

| Step | Statement | Reason |
|---|---|---|
| (1) | $\neg r \vee p$ | P. |
| (2) | $r \to p$ | T (1) |
| (3) | $p \to (q \to s)$ | P |
| (4) | $r \to (q \to s)$ | T (2,3) |
| (5) | $\neg r \vee (\neg q \vee s)$ | T (4). |

| Step | Statement | Reason |
|---|---|---|
| (6) | $q$ | P |
| (7) | $q \wedge (\neg r \vee \neg q \vee s)$ | T (5,6) |
| (8) | $q \wedge (\neg r \vee s)$ | T (7,8) |
| (9) | $\neg r \vee s$ | T (8) |
| (10) | $r \to s$ | T (9) |

(10) Derive $p \to (q \to s)$ using the CP rule (if necessary) from the premises $p \to (q \to r)$ and $q \to (r \to s)$.

We shall assume $P$ as an additional premise.

| Step | Statement | Reason |
|---|---|---|
| (1) | $P$ | Rule P (Additional) |
| (2) | $p \to (q \to r)$ | P |
| (3) | $q \to r$ | T (1,2) |
| (4) | $\neg q \vee r$ | T (3) |
| (5) | $q \to (r \to s)$ | P |
| (6) | $\neg q \vee (r \to s)$ | T (5) |
| (7) | $\neg q \vee s$ | T, 7 (Modus Ponens) |
| (8) | $q \to s$ | T (8) |
| (9) | $p \to (q \to s)$ | T (9) CP rule. |

(11) Use the indirect method to show that

$r \to \neg p$, $r \vee s$, $s \to \neg q$, $p \to q \Rightarrow \neg p$

To use indirect method, we will include $\neg\neg P \equiv P$ as an additional premise and prove a contradiction.

| Step | Statement | Reason |
|---|---|---|
| (1) | $P$ | Rule P (Additional) |
| (2) | $P \to q$ | P |
| (3) | $q$ | T (1,2) |
| (4) | $r \to \neg q$ | P |
| (5) | $s \to \neg q$ | P |
| (6) | $(r \vee s) \to \neg q$ | T (4,5) |
| (7) | $r \vee s$ | P |
| (8) | $\neg q$ | T (6,7) |
| (9) | $q \wedge \neg q$ | T (3,8) |
| (10) | F | T (9). |

(12) Show that b can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, $d$ by the indirect method.

Let us include $\neg b$ as an additional premise and prove a contradiction.

| Step | Statement | Reason |
|---|---|---|
| (1) | $a \to b$ | P |
| (2) | $c \to b$ | P |
| (3) | $(a \vee c) \to b$ | T (1,2) |
| (4) | $d \to (a \vee c)$ | P |
| (5) | $d \to b$ | T (3,4) |
| (6) | $d$ | P |
| (7) | $b$ | T (5&6) |
| (8) | $\neg b$ | P (additional) |
| (9) | $b \wedge \neg b$ | T (7,8) |
| (10) | F | T (9). |

(13) Using indirect method of proof, derive $p \to \neg s$ from the premise $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, $p$.

Let us include $\neg(p \to \neg s)$ as an additional premise and prove a contradiction.

i.e) $\neg(p \to \neg s) \Rightarrow \neg(\neg p \vee \neg s) = p \wedge s$

Hence the additional premise is $p \wedge s$

| Step | Statement | Reason |
|---|---|---|
| (1) | $p \to (q \vee r)$ | P |
| (2) | $p$ | P |
| (3) | $q \vee r$ | T (1,2) |
| (4) | $p \wedge s$ | P (additional) |
| (5) | $s$ | T (4) |
| (6) | $s \to \neg r$ | P |
| (7) | $\neg r$ | T (5,6) |
| (8) | $q$ | T (3,7) Disjunctive syllogism |
| (9) | $q \to \neg p$ | P |
| (10) | $\neg p$ | T (8,9) |
| (11) | $p \wedge \neg p$ | T (2, 10) |
| (12) | F | T (11) |

(14) Prove that the premises are inconsistent.

$p \to q$, $q \to r$, $s \to \neg r$ & $p \wedge s$

| Step | Statement | Reason |
|---|---|---|
| (1) | $p \to q$ | P |
| (2) | $q \to r$ | P |
| (3) | $p \to r$ | T (1,2) |
| (4) | $s \to \neg r$ | P |

| Step | Statement | Reason |
|---|---|---|
| (5) | $r \to \neg s$ | T (4) contrapositive |
| (6) | $q \to \neg s$ | T (2,5) |
| (7) | $\neg q \vee \neg s$ | T (6) |
| (8) | $\neg(q \wedge s)$ | T (7) |
| (9) | $q \wedge s$ | P |
| (10) | $(q \wedge s) \wedge \neg(q \wedge s)$ | T (8,9) |
| (11) | F | T, (10) |

(15) Prove that the premise $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

| Step | Statement | Reason |
|---|---|---|
| (1) | $a \wedge d$ | P |
| (2) | $a$ | T (1) |
| (3) | $d$ | T (1) |
| (4) | $a \to (b \to c)$ | P |
| (5) | $b \to c$ | T (2,4) |
| (6) | $\neg b \vee c$ | T (5) |
| (7) | $d \to (b \wedge \neg c)$ | P |
| (8) | $\neg(b \wedge \neg c) \to \neg d$ | T (7, contra positive) |
| (9) | $\neg b \vee c \to \neg d$ | (T, 8) |
| (10) | $\neg d$ | (T, 6, 9) |
| (11) | $d \wedge \neg D$ | (T, 3, 10) |
| (12) | F | T, 11. |

Hence the given premises are inconsistent.

(16) Show that the following premises are inconsistent.

(i) If Jack misses many classes through illness, then he fails high school.

(ii) If Jack fails high school, then he is uneducated.

(iii) If Jack reads a lot of books, then he is not uneducated.

(iv) Jack misses many classes through illness and reads a lot of books.

Let $P$ : Jack misses many classes.
$Q$ : Jack fails high school.
$R$ : Jack read a lot of books.
$S$ : Jack is uneducated.

The premises are

$P \to Q$, $Q \to S$, $R \to \neg S$ and $P \wedge R$.

| Step | Statement | Reason. |
|---|---|---|
| (1) | $P \to Q$ | P |
| (2) | $Q \to S$ | P |
| (3) | $P \to S$ | T(1,2) |
| (4) | $R \to \neg S$ | P |
| (5) | $S \to \neg R$ | T(4, contrapositive) |
| (6) | $P \to \neg R$ | T (3,5) |
| (7) | $\neg P \vee \neg R$ | T (6) |
| (8) | $\neg(P \wedge R)$ | T (7) |
| (9) | $P \wedge R$ | P. |
| (10) | $(P \wedge R) \wedge \neg(P \wedge R)$ | T (8,9) |
| (11) | F | T (10) |

$\therefore$ The premises are inconsistent.

(17) Show that the following set of premises is inconsistent.

(i) If Rama gets his degree, he will go for a job.
(ii) If he goes for a job, he will get married soon.
(iii) If he goes for a higher study, he will not get married.
(iv) Rama gets his degree and goes for a higher study.

Let the statements be symbolised as follows:

P : Rama gets his degree.
Q : He will go for a job.
R : He will get married soon.
S : He goes for higher study.

∴ The premises are

$P \to Q, \; Q \to R, \; S \to \neg R, \; P \wedge S$

| Step | Statement | Reason |
|---|---|---|
| (1) | $P \to Q$ | P |
| (2) | $Q \to R$ | P |
| (3) | $P \to R$ | T (1,2). |
| (4) | $P \wedge S$ | P |
| (5) | $P$ | T (4) |
| (6) | $S$ | T (4) |
| (7) | $S \to \neg R$ | P |
| (8) | $\neg R$ | T (6,7) |
| (9) | $R$ | T (5,3) |
| (10) | $R \wedge \neg R$ | T (8,9) |
| (11) | F | T |

∴ The premises are inconsistent.

(18) Show by indirect method of proof, that $\forall x\,(P(x) \vee Q(x)) \Rightarrow (\forall x\, P(x)) \vee (\exists x\, Q(x))$.

Let us assume that $\neg((\forall x\, P(x)) \vee (\exists x\, Q(x)))$ as an additional premise. and prove a contradiction.

| Step | Statement | Reason. |
|---|---|---|
| (1) | $\neg((\forall x\, P(x)) \vee (\exists x\, Q(x)))$ | P. |
| (2) | $\neg(\forall x\, P(x)) \wedge \neg(\exists x\, Q(x))$ | T |
| (3) | $\neg(\forall x\, P(x))$ | T(2) |
| (4) | $\neg(\exists x\, Q(x))$ | T (2) |
| (5) | $\exists x\,(\neg P(x))$ | T (3) |
| (6) | $\forall x\,(\neg Q(x))$ | T (4) |
| (7) | $\neg P(y)$ | T (ES, 5) |
| (8) | $\neg Q(y)$ | T (US, 6) |
| (9) | $\neg P(y) \wedge \neg Q(y)$ | T (7,8) |
| (10) | $\neg(P(y) \vee Q(y))$ | T (9) |
| (11) | $\forall x\,(P(x) \vee Q(x))$ | P |
| (12) | $P(y) \vee Q(y)$ | T (11) |
| (13) | $(P(y) \vee Q(y)) \wedge \neg(P(y) \vee Q(y))$ | T (12,10) |
| (14) | F | T (13) |

Hence Proved.

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(18)

(18) Show by indirect method of proof, that $\forall x\,(P(x) \vee Q(x)) \Rightarrow (\forall x\, P(x)) \vee (\exists x\, Q(x))$.

Let us assume that $\neg((\forall x\, P(x)) \vee (\exists x\, Q(x)))$ as an additional premise and prove a contradiction.

| Step | Statement | Reason |
|---|---|---|
| (1) | $\neg((\forall x\, P(x)) \vee (\exists x\, Q(x)))$ | P. |
| (2) | $\neg(\forall x\, P(x)) \wedge \neg(\exists x\, Q(x))$ | T |
| (3) | $\neg(\forall x\, P(x))$ | T (2) |
| (4) | $\neg(\exists x\, Q(x))$ | T (2) |
| (5) | $\exists x\,(\neg P(x))$ | T (3) |
| (6) | $\forall x\,(\neg Q(x))$ | T (4) |
| (7) | $\neg P(y)$ | T (ES, 5) |
| (8) | $\neg Q(y)$ | T (US, 6) |
| (9) | $\neg P(y) \wedge \neg Q(y)$ | T (7, 8) |
| (10) | $\neg(P(y) \vee Q(y))$ | T (9) |
| (11) | $\forall x\,(P(x) \vee Q(x))$ | P |
| (12) | $P(y) \vee Q(y)$ | T (11) |
| (13) | $(P(y) \vee Q(y)) \wedge \neg(P(y) \vee Q(y))$ | T (12, 10) |
| (14) | F | T (13) |

Hence Proved.

(19) Use the indirect method to prove the conclusion $\exists z\, Q(z)$ follows from the premises $\forall x\,(P(x) \to Q(x))$ and $\exists y\, P(y)$.

Let us assume the additional premise $\neg(\exists z\, Q(z))$ and prove a contradiction.

| Step | Statement | Reason. |
|---|---|---|
| (1) | $\neg(\exists z Q(z))$ | P (additional) |
| (2) | $\forall z\,(\neg Q(z))$ | T (1) |
| (3) | $\neg Q(a)$ | T (US, 2) |
| (4) | $\exists y\, P(y)$ | P |
| (5) | $P(a)$ | T (ES, 4) |
| (6) | $P(a) \wedge \neg Q(a)$ | T (3,5) |
| (7) | $\neg(\neg(P(a)) \vee Q(a))$ | T (6) |
| (8) | $\neg(P(a) \to Q(a))$ | (T, 7) |
| (9) | $\forall x\,(P(x) \to Q(x))$ | P |
| (10) | $P(a) \to Q(a)$ | T (US, 9) |
| (11) | $(P(a) \to Q(a)) \wedge \neg(P(a) \to Q(a))$ | T (8,10) |
| (12) | F | T (11). |

(20) Prove that $\forall x\,(P(x) \vee Q(x)) \Rightarrow \forall x\, P(x) \vee \exists x\, Q(x)$ using indirect method.

Let us assume the additional premise

$$\neg(\forall x\, P(x) \vee \exists x\, Q(x)).$$

$$= \exists x\,(\neg P(x)) \wedge \forall x\,(\neg Q(x))$$

SRM UNIVERSITY

20

| Step | Statement | Reason |
|---|---|---|
| (1) | $\exists x(\neg P(x)) \wedge \forall x(\neg Q(x))$ | P (additional) |
| (2) | $\exists x(\neg P(x))$ | T (1) |
| (3) | $\forall x(\neg Q(x))$ | T (1) |
| (4) | $\neg P(a)$ | T (ES, 2) |
| (5) | $\neg Q(a)$ | T (US, 3) |
| (6) | $\neg P(a) \wedge \neg Q(a)$ | T (4,5) |
| (7) | $\neg(P(a) \vee Q(a))$ | T (8) |
| (8) | $\forall x(P(x) \vee Q(x))$ | P |
| (9) | $P(a) \vee Q(a)$ | T (US, 8) |
| (10) | $(P(a) \vee Q(a)) \wedge \neg(P(a) \vee Q(a))$ | T (9, 7) |
| (11) | F | T (10) |

Hence proved.

(21) construct an argument using rules of inferences to Show that if A work hard then either B or C will enjoy them selves. If B enjoy himself, then A will not work hard. If D enjoy himself then C will not. Therefore, if A works hard, D will not enjoy himself.

Let p : A work hard

q : B enjoy him self

r : C enjoy himself

s : D enjoy himself

The premises are $(p \to \neg s)$

$p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r \Rightarrow p \to \neg s$

we shall assume $P$ as an additional premise (CP rule).

| Step | Statement | Reason |
| --- | --- | --- |
| (1) | $P \to (q \vee r)$ | P |
| (2) | $P$ | P (additional) |
| (3) | $q \vee r$ | T (1,2) |
| (4) | $\neg q \to r$ | T (3, equivalence) |
| (5) | $\neg r \to q$ | T (4, contrapositive) |
| (6) | $s \to \neg r$ | P |
| (7) | $s \to q$ | T (5,6) |
| (8) | $q \to \neg P$ | P |
| (9) | $s \to \neg P$ | T (7,8) |
| (10) | $P \to \neg s$ | T (9, contrapositive) |

—.

(22) Show that the premises "one student in this class knows how to write programs in JAVA" and "Every one who knows how to write program in JAVA can get a high-paying job" imply the conclusion "Some one in this class can get a high paying job".

Let $C(x)$ : $x$ is in this class

$J(x)$ : $x$ knows JAVA Programming.

$H(x)$ : $x$ can get a high paying job.

Then the premises are

$\exists x (C(x) \wedge J(x))$ and $\forall (x) (J(x) \to H(x))$. The Conclusion is $\exists x (C(x) \wedge H(x))$.

| Step | Statement | Reason |
|---|---|---|
| (1) | $\exists x\,(C(x) \wedge J(x))$ | P |
| (2) | $C(a) \wedge J(a)$ | T(ES, 1) |
| (3) | $C(a)$ | T(2) |
| (4) | $J(a)$ | T(2) |
| (5) | $\forall x\,(J(x) \rightarrow H(x))$ | P |
| (6) | $J(a) \rightarrow H(a)$ | T(US, 5) |
| (7) | $H(a)$ | T(4, 6) |
| (8) | $C(a) \wedge H(a)$ | T(3, 7) |
| (9) | $\exists x\,(C(x) \wedge H(x))$ | T(EG, 8) |

23 Prove by mathematical induction that

$$1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$$

Let $P(n)$: $1\cdot2\cdot3 + 2\cdot3\cdot4 \cdots + n(n+1)(n+2) = \frac{n}{4}(n+1)(n+2)(n+3)$

Now, $P(1)$: $1\cdot2\cdot3 = 6$

$$\frac{1}{4}(1+1)(1+2)(1+3) = \frac{1}{\cancel{4}} \times 2 \times 3 \times \cancel{4}$$
$$= 6$$

$\therefore$ L.H.S = R.H.S.

$P(1)$ is true.

Assume $P(n)$ is true for $n = K$.

ie) $P(K)$ is true.

$P(K)$: $1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots + K(K+1)(K+2) = \frac{K}{4}(K+1)(K+2)(K+3)$

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(23)

Now, we prove $P(n)$ is true for $n=k+1$

$$P(k+1) = \underbrace{1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots + k(k+1)(k+2)}+ (k+1)(k+2)(k+3)$$

$$= P(k) + (k+1)(k+2)(k+3)$$

$$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$$

$$= (k+1)(k+2)(k+3)\left[\frac{k}{4}+1\right]$$

$$= (k+1)(k+2)(k+3)\left(\frac{k+4}{4}\right)$$

$$P(k+1) = \frac{1}{4}(k+1)((k+1)+1)((k+1)+2)((k+1)+3).$$

$\therefore$ $P(k+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

(24) use mathematical induction to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, \text{ for } n \geq 2$$

Let $P(n)$ : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

To Prove $P(n)$ is true for $n=2$

$$L.H.S = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707$$

$$R.H.S = \sqrt{2} = 1.414$$

$\therefore$ L.H.S > R.H.S.

$\Rightarrow P(1)$ is true.

Assume $P(n)$ is true for $n=k$.

ie) $P(k)$ is true.

$$P(k) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k} \quad —(1)$$

To Prove $P(n)$ is true for $n=K+1$

$$P(K+1) = \underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}}} + \frac{1}{\sqrt{K+1}}$$

$$= P(K) + \frac{1}{\sqrt{K+1}}$$

$$> \sqrt{K} + \frac{1}{\sqrt{K+1}} \quad \text{by (1)}$$

Now $\sqrt{K} + \dfrac{1}{\sqrt{K+1}} = \dfrac{\sqrt{K(K+1)}+1}{\sqrt{K+1}}$

$$> \frac{\sqrt{K}\,\sqrt{K}+1}{\sqrt{K+1}}$$

$$> \frac{K+1}{\sqrt{K+1}} > \sqrt{K+1}$$

$\therefore$ $P(K+1)$ is true.

$\Rightarrow$ $P(n)$ is true for all $n \in Z^+$

(25) use mathematical induction to prove that $3^n + 7^n - 2$ is divisible by 8, for $n \geq 1$.

Let $P(n)$ : $3^n + 7^n - 2$ is divisible by 8.

To Prove $P(1)$ is true

$$P(1) = 3^1 + 7^1 - 2 = 8 = \text{divisible by } 8.$$

$\therefore$ $P(1)$ is true.

Assume $P(n)$ is true for $n=K$.

ie) $P(K)$ is divisible by 8.

$3^K + 7^K - 2$ is divisible by 8.

$\Rightarrow$ $3^K + 7^K - 2 = 8t$ (say) (multiple of 8) —(1)

$t \rightarrow$ constant

$P(K+1) = 3^{K+1} + 7^{K+1} - 2$

$= 3 \cdot 3^K + 7 \cdot 7^K - 2$

$= 3(8t - 7^K + 2) + 7 \cdot 7^K - 2$

$= 24t - 3 \cdot 7^K + 6 + 7 \cdot 7^K - 2$

$= 24t + 4 \cdot 7^K + 4$

$= 8\left(3t + \frac{7^K + 1}{2}\right)$

= multiple of 8 = divisible by 8.

$\therefore P(K+1)$ is true.

$\therefore P(n)$ is true for all $n \in Z^+$

(26) Use mathematical induction to show that $n! \geq 2^{n-1}$, for $n = 1, 2, 3 \ldots$

Let $P(n) : n! \geq 2^{n-1}$

To Prove $P(1)$ is true:

$P(1) : 1! \geq 2^{1-1}$

$1 \geq 2^0$

$1 \geq 1$ which is true.

Assume $P(n)$ is true for $n = K$.

$P(K) : K! \geq 2^{K-1}$ — (1)

To Prove $P(n)$ is true for $n = K+1$

$P(K+1) : (K+1)! = (K+1) \cdot K!$

$\geq (K+1)\, 2^{K-1}$ by (1)

$\geq 2 \cdot 2^{K-1}$, Since $K+1 \geq 2$

$\geq 2^{(K+1)-1}$

$\therefore P(K+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

27 Prove by mathematical induction, $a^n - b^n$ is divisible by $a-b$.

Let $P(n): a^n - b^n$.

To Prove $P(1)$ is true :

$P(1): a^1 - b^1 = a-b$
$= \text{is divisible by } a-b.$

$\therefore P(1)$ is true.

Assume $P(n)$ is true for $n=k$.

$P(k): a^k - b^k$ is divisible by $a-b$.

$\Rightarrow a^k - b^k = (a-b)t = \text{multiple of } a-b.$

$a^k = b^k + (a-b)t \quad \text{--- (1)}$

To Prove $P(k+1)$ is true

$$
\begin{aligned}
P(k+1) &= a^{k+1} - b^{k+1} \\
&= a \cdot a^k - b^{k+1} \\
&= a\left(b^k + (a-b)t\right) - b^{k+1} \quad \text{by (1)}, \\
&= a \cdot b^k + a(a-b)t - b^{k+1} \\
&= a \cdot b^k + a(a-b)t - b \cdot b^k \\
&= (a-b) \cdot b^k + a(a-b)t \\
&= (a-b)(b^k + at) \\
&= \text{multiple of } a-b \\
&= \text{divisible by } a-b.
\end{aligned}
$$

$\therefore P(k+1)$ is true.

$\Rightarrow P(n)$ is true for all $n \in Z^+$.

— x —.

# SRM UNIVERSITY

RAMAPURAM PART- VADAPALANI CAMPUS, CHENNAI – 600 026

# Department of Mathematics

## Sub Title: DISCRETE MATHEMATICS

**Sub Code:** 15MA302

### Unit -II - SET THEORY

1. From a club consisting of 6 men and 7 women. In how many ways we select a commite of 3 men and 4 womens

   a) 600 ways b) 700 ways c) 900 ways d)800 ways **Ans: b**

2. What is the permutation of the letters A , B, C

   a) 3! b) 2 c) 3 d)2! **Ans: a**

3. In a group of 100 people ; serveral will have birthday in the same month. Atleast how many will have birthdays in the same month ?

   a)9 b)10 c)8 d)4 **Ans: a**

4. From the group containing of 6 men and 7 women the number of ways of selecting a committee of 4 persons which has 2 women exactly is

   a) 405 b) 300 c) 315 d) 786 **Ans: c**

5. How many different outcomes are possible when 5 dice are rolled?

   a) 250 b) 256 c) 252 d) 225

6. Assuming that repetitions are not permitted; How many from digit numbers can be formed from the six digits 1,2,3,4,7,8.

   a) 360 b) 120 c) 180 d) 240 **Ans: a**

7. The relation R={(0,0), (0,2),(2,2), (2,3),(3,2),(3,3)} defined on a set {0,1,2,3} is

   a)Transitive b)Equivalence c)Not equivalence d)Poset **Ans: c**

8.. A Relation R is defined on the set of integers as xRy iff (x+y) is even . Which of the following statements is TRUE?

   (a) R is not an equivalence relation

   (b) R is an equivalence relation having one equivalence class

(c) R is an equivalence relation having two equivalence class

(d) R is an equivalence relation having three equivalence class **Ans: c**

9.The number of binary relations on a set with n elements is

(a) $n^2$ (c) $2^{n^2}$

**(b)** $2^n$ (d) none of these **Ans: c**

10.The number of functions from an m element set to an n element set is

(a) m + n (c) $n^m$

**(b)** $m^n$ (d) m * n **Ans: c**

11.The number of equivalence relations of the set {1,2,3,4} is

(a) 4 (c) 16

(b) 15 (d) 24 **Ans: b**

12. Let R be a symmetric and transitive relation on a set A, if

(a) R is reflexive then R is an equivalence relation

(b) R is reflexive then R is a parital order

(c) R is reflexive then R is not an equivalence relation

(d) R is not reflexive then R is a parital order

**Ans: a**

13.Which of the following sets are null sets?

(a) {0} (c) {a}

**(b)** {ϕ} (d) ϕ **Ans: d**

14.Which of the following set (s) are empty?

(a) {x : x = x} (c) {x: x =$x^2$}

**(b)** {x : x ≠x} (d) { x : x ≠$x^2$} **Ans: b**

15.Relation R defined on the set A ={1,2,3,4},by R={(1,1),(2,2),(3,3) (4,4)} is

(a) irreflexive (c) not transitive

**(b)** anti symmetric (d) equivalence relation **Ans: d**

16.Relation R defined on a set N by R={(a,b) : |a - b| is divisible by 5}, is

(a) reflexive (c) transitive

**(b)** symmetric (d) all of these **Ans: d**

17.Relation R is defined on the set N as {(a,b) : a,b are both odd}, is

(a) Reflexive (c) transitive

(b) Symmetric (d) all of these **Ans: d**

18.If relation R over {a,b,c} is given by R={(a,a),(a,b),(b,a),(b,b),(c,c)},then which of the following properties does R have?

(a) Symmetry (c) transitivity

(b) Reflexivity (d) all of these **Ans: d**

19.The domain and range are same for

(a) constant function (c) absolute value function

(b) identity function (d) greatest integer function **Ans: b**

20.The function f : $Z^+ \rightarrow Z^+$ given by f(x)=$x^2$is

(a) one-one (c) one-one and onto

(b) onto (d) none of these **Ans: a**

21.Which of the following is a partition of the set S={1,2,3,4,5,6} ?

(a) [{(1,3,5),{2,4}{3,6}] (c) [{1,5},{2},{4},{1,5},{3,6}]

**(b)**[{1,5},{2},[3,6}] (d) [{1,2,3,4,5,6}] **Ans: b**

22.A relation over the set S=[x,y,z]is defined by : {(x,x), (y,x),(x,z),(y,y),(z,z)}.

(a) Symmetric (c) Irreflexive

(b) Reflexive (d) Antisymmetric **Ans: b**

23.If sets A and B have 3 and 6 elements each, then minimum number of elements in $A \cup B$ is

(a) 3 (b) 18 (c) 6 (d) 9 **Ans: c**

24.f : R → R is a function defined by f (x) = 10x -7. If g = $f^{-1}$,then g(x)

*(a)* $\frac{1}{10x-7}$ (c) $\frac{x+7}{10}$

(b) $\frac{1}{10x+7}$ (d) $\frac{x-7}{10}$ **Ans: c**

25.The set of all Equivalence classes of a set A of cardinality C

(a) has the same cardinality as A

(b) forms a partition of A

(c) is of cardinality 2C

(d) is of cardinality $C^2$ **Ans: b**

26.Let A={1,2,3........} Define ~ by x ~ y ⇔ x divide y. Then ~ is

(a) reflexive, but not a parital-ordering

(b) symmetric

(c) an equivalence relation

(d) a parital- ordering relation **Ans: d**

27.If A={1,2,3}, then relation S={(1,1),(2,2) }is

(a) symmetric only

(b) anti-symmetric only

(c) both symmetric and anti-symmetric only **Ans: a**

(d) an equivalence relation

28.If A={1,2,3,4}. Let ~ = {(1,2),(2,4),(1,4),(4,2)(4,4)}. Then ~ is

(a) not anti-symmetric (c) reflexive

(b) transitive (d) symmetric **Ans: b**

17.Total number of different partitions of a set having four elements

(a) 16 (b) 8 (c) 15 (d) 14 **Ans : d**

29..Hasse diagrams are drawn for

(a) Partially ordered sets (c) boolean algebra

(b) Lattices (d) all of these **Ans: d**

30.Let X={2,3,6,12,24}, and ≤ be the parital order defined by X ≤ Y if X divides Y. Number of edges in the Hasse diagram of (X, ≤) is

(a) 3 (c) 5

(b) 4 (d) none of these **Ans: c**

31. Partial order relation is

a) Reflexive ,Symmetric, transitive b) Symmetric and transitive

c) Antisymmetric and transitive d) Reflexive , Antisymmetric and transitive **Ans: d**

32. . If A={1,2,3,4,5} , g={(1,2),(3,1),(2,2),(4,3),(5,2)} what is g.g where g:A->A ?

a) {(1,2),(2,2),(3,2),(4,1),(5,2)} b) {(2,1),(2,2),(2,3),(1,4),(5,2)}

c) {(2,1),(2,2),(2,3),(1,4),(2,5)} d) {(1,2),(2,2),(3,1),(4,1),(5,2)} **Ans: a**

33. Which of the following statement is false

a)Every function is a relation b)Every relation is a function

c)Every relation R:X->Y is a subset of the Cartesian product of X & Y d)The Cartesian product of X with Y is a relation **Ans: b**

34. Let R be a relation defined on {a,b,c} and $M_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.Then the reflexive closure of R is

a) R itself b) R U [(1,2)] c)R U [(1,3), (2,1)] d)R U {(3,1)} **Ans: a**

35. Let A={1,2,3} , B={x,y} then the number of function f: A->B is

a)9 b)6 c)2 d)8 **Ans: d**

36. An equivalence relation R on a set A is said to posses

a) Reflexive, antisymmetric and transitive. b) Reflexive, Symmetric and transitive.

c) Reflexive, non-symmetric and antisymmetric. d) Symmetric and antisymmetric only. **Ans: d**

37. A Graph represents the partial order relation

a) Helmut harse b)Poset c) Graph relation d)Hasse diagram **Ans: d**

38. . If 9 colours are used to paint 100 houses, then atleast how many houses will be of the same colours.

a)10 b)11 c)12 d)13 **Ans: c**

39. The minimum number of students required in a class so that atleast six will receive the same grade, If these are 5 possible grade is a) 24 b) 25 c) 26 d) 27 **Ans: c**

40. If 100 pigeons are accommodated in 90 pigeon holes, then one of the pigeon hole must contain

a) $\left\lfloor \frac{100}{90} \right\rfloor + 1$ b) $\left\lfloor \frac{99}{90} \right\rfloor + 1$ c) $\left\lfloor \frac{101}{90} \right\rfloor - 1$ d) $\left\lfloor \frac{99}{90} \right\rfloor - 1$ **Ans: b**

## UNIT - II SET THEORY

### PART-B

(1) Let $A = \{1,2,3,4,5,6\}$ with subsets $B_1 = \{1,3,5\}$, $B_2 = \{1,2,3\}$. Write the minsets and partitions of A.

Given $A = \{1,2,3,4,5,6\}$

$B_1 = \{1,3,5\}$, $B_1' = \{2,4,6\}$

$B_2 = \{1,2,3\}$ $B_2' = \{4,5,6\}$

The minsets generated by $B_1$ & $B_2$ are

$M_1 = B_1 \cap B_2 = \{1, 3\}$

$M_2 = B_1 \cap B_2' = \{5\}$

$M_3 = B_1' \cap B_2 = \{2\}$

$M_4 = B_1' \cap B_2' = \{4, 6\}$

Each min set is non-empty and mutually pairwise disjoint. Also $A = M_1 \cup M_2 \cup M_3 \cup M_4$.

$\therefore$ All the minsets are partitions of A.

$\therefore$ The partitions are $\{\{1,3\}, \{5\}, \{2\}, \{4,6\}\}$

(2). Show that the relation R defined on the set of real numbers such that aRb if and only if a-b is an integer is an equivalence relation.

Given,

aRb ⇒ a−b is an integer $\forall a,b \in R$.

(i) Reflexive:

$a-a=0$ = is a real number.

⇒ aRa $\forall a \in R$.

∴ R is reflexive.

(ii) Symmetry:

Let $a,b \in R$.

aRb ⇒ a−b is an integer.

⇒ b−a is also an integer.

aRb ⇒ bRa

∴ R is symmetric

(iii) Transitive:

Let $a, b, c \in R$.

aRb ⇒ a−b is an integer.

bRc ⇒ b−c is an integer.

aRb, bRc ⇒ $(a-b)+(b-c)$ is also an integer.

(∵ Sum of two integers is also an integer).

⇒ a−c is an integer.

aRb, bRc ⇒ aRc

∴ R is transitive.

⇒ R is an equivalence relation.

③ If $R = \{(1,2), (2,4), (3,3)\}$ and $S = \{(1,3), (2,4), (4,2)\}$

find (i) $R \cup S$ (ii) $R \cap S$ (iii) $R - S$ iv) $R \oplus S$.

Given $R = \{ (1,2), (2,4), (3,3) \}$

$S = \{ (1,3), (2,4), (4,2) \}$

(i) $R \cup S = \{ (1,2), (2,4), (3,3), (1,3), (4,2) \}$

(ii) $R \cap S = \{ (2,4) \}$

(iii) $R - S = \{ (1,2), (3,3) \}$

iv) $R \oplus S = (R-S) \cup (S-R)$

$S - R = \{ (1,3), (4,2) \}$

$R \oplus S = \{ (1,2), (3,3), (1,3), (4,2) \}$

(4) If $R$ is the relation on the set of positive integers such that $(a,b) \in R$ if and only if $ab$ is a perfect square, Show that $R$ is an equivalence relation.

<u>Reflexive:</u>

$(a, a) \in R$, since $a^2$ is a perfect square.

$\therefore R$ is reflexive.

<u>Symmetric</u> :

$aRb \Rightarrow ab$ is a perfect square.

$\Rightarrow ba$ is also a perfect square.

$\Rightarrow bRa$.

$\therefore R$ is symmetric.

<u>Transitive:</u>

$aRb \Rightarrow ab$ is a perfect square

$ab = x^2$ (say) $\quad -(1)$

$bRc \Rightarrow bc$ is a perfect square.

$$bc = y^2 \text{ (say)}. \quad -(2)$$

Multiply ① & ②,

$$(ab)(bc) = x^2y^2$$

$$a(b^2)c = x^2y^2$$

$$ac = \frac{x^2y^2}{b^2} = \left(\frac{xy}{b}\right)^2 = \text{a perfect square.}$$

$\therefore aRc$ i.e) $R$ is transitive.

Hence $R$ is an equivalence relation.

⑤. Examine if the relation $R$ represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of $M_R$.

Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Since all the elements in the main diagonal of $M_R$ are equal to 1 each. $R$ is a reflexive relation.

Since $M_R$ is a symmetric matrix ; $R$ is a symmetric relation.

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$$

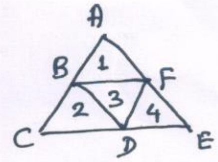
i.e) $R^2 \subseteq R$

$\therefore R$ is a transitive relation.

Hence $R$ is an equivalence relation.

(6) If we select 5 points in the interior of an equilateral triangle of side 1, Show that there must be atleast two points whose distance is apart is less than $\frac{1}{2}$.

Divide the triangle ACE into 4 equilateral triangle of side $\frac{1}{2}$.




$m$ = number of subtriangles = 4 (Pigeon holes)

$n$ = number of points = 5 (Pigeons).

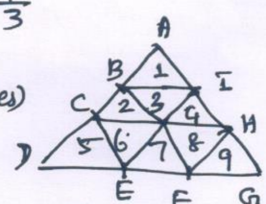
$\therefore$ By Generalised pigeon hole principle, one pigeon hole contains atleast

$$= \left\lfloor \frac{n-1}{m} \right\rfloor + 1 \text{ pigeons.}$$

$$= \left\lfloor \frac{5-1}{4} \right\rfloor + 1 = \left\lfloor \frac{4}{4} \right\rfloor + 1$$

$$= 1 + 1 = 2 \text{ pigeons (number)}$$

(7) If we select 10 points in the interior of an equilateral triangle of side 1, Show that there must be atleast two points whose distance apart is less than $\frac{1}{3}$




$m$ = number of subtriangles = 9 (Pigeon holes)

$n$ = number of pigeons = 10 (pigeons)

By generalised pigeon hole principle, atleast one pigeon hole contains $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

$$= \left\lfloor \frac{10-1}{9} \right\rfloor + 1 = 2 \text{ pigeons (points).}$$

(8) State pigeon hole principle and show that ~~the set~~ in any group of eight people, atleast two have birthdays which fall in the same day of the week in any given year.

"If n pigeons are accommodated in m pigeon holes and $n > m$ then atleast one pigeon hole will contain two or more pigeons"

$m$ = number of days in a week = 7 (pigeon holes)

$n$ = number of people = 8 (pigeons).

$\therefore$ By generalised pigeon hole principle, one pigeon hole contains atleast

$$= \left\lfloor \frac{n-1}{m} \right\rfloor + 1 \text{ pigeons}$$

$$= \left\lfloor \frac{8-1}{7} \right\rfloor + 1 = 2 \text{ pigeons (people).}$$

(9) Prove that in any group of six people, atleast three must be mutual friends or atleast three must be mutual strangers.

Let A be one of the six people. Let the remaining 5 people be accommodated in 2 roms labelled

$$\underbrace{\boxed{\text{A's friends}} \quad \boxed{\text{Strangers to A}}}_{\text{pigeon holes} = 2.}$$

$n =$ number of pigeons (people) $= 5$

$m =$ number of rooms (pigeon hole) $= 2$.

$\therefore$ By generalised pigeon hole principle,

one of the room (pigeon hole) must contain

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3 \text{ people (pigeons)}$$

(10) If seven colours are used to paint 50 bicycles, then show that atleast 8 bicycles will be the same colour.

Number of pigeons = Number of bicycles = 50 = n

Number of pigeon holes = No. of colours = 7 = m

By generalised pigeon hole principle, we have

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{50-1}{7} \right\rfloor + 1$$

$$= 8.$$

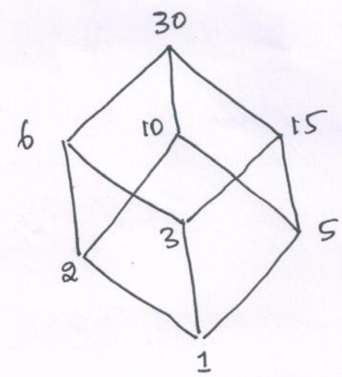
(11). From 7 gentleman and 4 ladies a committee of 6 is to be formed. In how many ways can this be done, when the committee contains atleast 2 ladies.

| Gents 7 | Ladies 4 | No. of ways (6) |
|---|---|---|
| 4 | 2 | $7C_4 \times 4C_2 = 210$ |
| 3 | 3 | $7C_3 \times 4C_3 = 140$ |
| 2 | 4 | $7C_2 \times 4C_4 = 21$ |

$\boxed{371}$ ways.

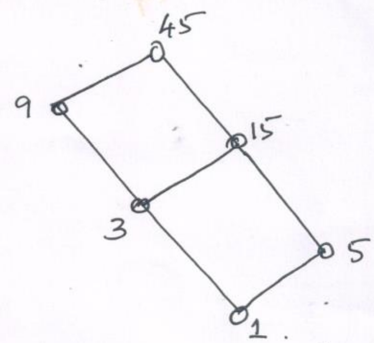
(12) Draw the Hasse diagram for the poset $\{D_{30}, \leq\}$ where $\leq$ is the relation is a divisor of.

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$




(13) Draw the Hasse diagram for the poset $(D_{45}, \leq)$ where $\leq$ is the relation is a divisor of.

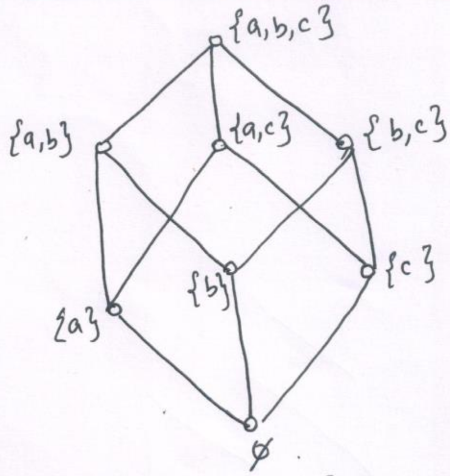
$D_{45} = \{1, 3, 5, 9, 15, 45\}$




(14) Draw the Hasse diagram for $\{P(S), \subseteq\}$ where $S = \{a, b, c\}$

Given $S = \{a, b, c\}$

$P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \emptyset\}$.




(15) If $R = \{(1,2), (2,4), (3,3)\}$ & $S = \{(1,3), (2,4), (4,2)\}$ find (i) $R \cup S$ (ii) $R \cap S$ (iii) $R - S$ iv) $R \oplus S$.

(i) $R \cup S = \{(1,2), (1,3), (2,4), (3,3), (4,2)\}$

(ii) $R \cap S = \{(2,4)\}$

(iii) $R - S = \{(1,2), (3,3)\}$

iv) $R \oplus S = (R-S) \cup (S-R)$ $\qquad S - R = \{(1,3), (4,2)\}$

$= \{(1,2), (1,3), (3,3), (4,2)\}$.

(16) Show that the function $f: Z^+ \to Z^+$ defined by $f(x) = x^2 + 2$ is one to one but not on-to.

Given $f(x) = x^2 + 2$

$$f(x_1) = f(x_2) \Rightarrow x_1^2 + 2 = x_2^2 + 2$$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 - x_2)(x_1 + x_2) = 0$$

$x_1 + x_2 \neq 0, \quad x_1 - x_2 = 0 \qquad (\because x_1 + x_2 \neq 0$ as $x_1, x_2 \in Z^+)$

$$\boxed{x_1 = x_2}$$

$\therefore$ f is one to one.

when $y = f(x)$, ie) $y = x^2 + 2$ we have $x^2 = y - 2$

when $y = 1$, $x$ doesnot exist.

Also when $y = 4$, $x = \pm\sqrt{2} \notin Z^+$

$\therefore f(x)$ is not on-to.

(17) Define closure of a relation. Find reflexive and Symmetric closure of $R = \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4)\}$ define on $A = \{1,2,3,4\}$

The closure of a relation R w.r.to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain the property P.

$A = \{1,2,3,4\}$

Given $R = \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4)\}$

Reflexive closure:

$$\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$$

Reflexive closure $= R \cup \Delta$

$$= \{(1,2), (2,2), (2,3), (3,2), (4,1), (4,4), (1,1), (3,3)\}.$$

Symmetric closure:

$$R^{-1} = \{ (2,1), (2,2), (3,2), (2,3), (1,4), (4,4) \}$$

Symmetric closure $= R \cup R^{-1}$

$$= \{ (1,2), (2,2), (2,3), (3,2), (4,1), (4,4), (2,1), (1,4) \}.$$

PART – C

① A man hiked for 10 hours and covered a total distance of 45 km, it is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked atleast 9 km within a certain period of consecutive hours.

Since the man hiked $6+3 = 9$ km in the first and last hours, he must have hiked $45-9 = 36$ km during the period from second to ninth hour.

| 1st hour | 2 & 3 hour | 4 & 5th hour | 6th & 7th hour | 8 & 9th hour | 10th hour |
|---|---|---|---|---|---|
| | I | II | III | IV | |

So here we have 4 time periods.

No. of Pigeon holes $= 4 = m$

No. of pigeons $= 36$ km $= n$

By generalised pigeon hole principle, one pigeon hole contains $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons (kms)

$$\left\lfloor \frac{36-1}{4} \right\rfloor + 1 = \left\lfloor \frac{35}{4} \right\rfloor + 1 = 8+1 = 9 \text{ pigeons (kms)}$$

(2) Let $A = \{1,2,3,4\}$ find the reflexive closure and symmetric closure of the relation $R = \{(1,3), (1,4), (2,2), (3,4), (4,2)\}$

Given $A = \{1,2,3,4\}$

$$R = \{(1,3), (1,4), (2,2), (3,4), (4,2)\}$$

<u>Reflexive closure:</u> $\Delta = \{(1,1), (2,2), (3,3), (4,4)\}$

$$\text{Reflexive closure} = R \cup \Delta$$

$$= \{(1,3), (1,4), (2,2), (3,4), (4,2), (1,1), (3,3), (4,4)\}$$

<u>Symmetric closure:</u> $R^{-1} = \{(3,1), (4,1), (2,2), (4,3), (2,4)\}$

$$\text{Symmetric closure} = R \cup R^{-1}$$

$$= \{(1,3), (1,4), (2,2), (3,4), (4,2), (3,1), (3,3), (2,4)\}$$

(3) If $R$ is the relation on $A = \{1,2,3\}$ such that $aRb$ if and only if $a+b$ is even then find $M_R$, $M_{R^{-1}}$ and $M_{R^2}$.

$$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \\ 0\vee 0\vee 0 & 0\vee 1\vee 0 & 0\vee 0\vee 0 \\ 1\vee 0\vee 1 & 0\vee 0\vee 0 & 1\vee 0\vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow M_{R^2} = M_R$$

(4) If $R$ is the relation on the set of positive integers such that $(a,b) \in R$ if and only if $a^2+b$ is even. Prove that $R$ is an equivalence relation.

Reflexive:

$a^2+a = a(a+1) =$ even

Since $a$ & $a+1$ are consecutive positive integers

$\therefore (a,a) \in R$

$\Rightarrow R$ is reflexive.

Symmetric:

When $a^2+b$ is even, $a$ & $b$ must be both even or both odd

In either case, $b^2+a$ is even.

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

$\Rightarrow R$ is symmetric.

Transitive:

when $a, b, c$ are even, $a^2+b$ and $b^2+c$ are even.

Also $a^2+c$ is even.

when $a, b, c$ are odd, $a^2+b$ & $b^2+c$ are even

Also $a^2+c$ is even.

Then $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

ie) $R$ is transitive.

$\therefore R$ is an equivalence relation.

(5) Prove that the relation congruence modulo m over the set of positive integers is an equivalence relation.

Reflexive:

$(a-a)$ is a multiple of $m$.

$\therefore a \equiv a \pmod{m}$.

$\therefore R$ is reflexive.

Symmetric:

when $a-b$ is a multiple of $m$, $b-a$ is also a multiple of $m$.

ie) $a \equiv b \pmod{m} \Rightarrow b \equiv a \pmod{m}$

$\therefore R$ is Symmetric

Transitive:

when $a-b = k_1 m$ & $b-c = k_2 m$

we get $a-c = (k_1+k_2) m$ (by addition)

$\therefore$ when $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, $a \equiv c \pmod{m}$

$\therefore R$ is transitive.

$\Rightarrow R$ is an equivalence relation.

⑥ Find the transitive closure of the relation
R = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4), (4,5), (5,4), (5,5) }
defined on a Set A = { 1, 2, 3, 4, 5 } using warshall's algorithm.

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

| K | In $W_{K-1}$ Positions of 1's in column K | In $W_{K-1}$ Positions of 1's in Row K | $W_K$ has 1's in | $W_K$ |
|---|---|---|---|---|
| 1 | 1, 2 | 1, 2 | (1,1), (1,2), (2,1), (2,2) | $W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ |
| 2 | 1, 2 | 1, 2 | (1,1), (1,2), (2,1), (2,2) | $W_2 = W_1$ |
| 3 | 3, 4 | 3, 4 | (3,3), (3,4), (4,3), (4,4) | $W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ |
| 4 | 3, 4, 5 | 3, 4, 5 | (3,3), (3,4), (3,5) (4,3), (4,4), (4,5) (5,3), (5,4), (5,5) | $W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ |
| 5 | 3, 4, 5 | 3, 4, 5 | (3,3), (3,4), (3,5) (4,3), (4,4), (4,5) (5,3), (5,4) (5,5) | $W_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ |

∴ The transitive closure of R is
{ (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (3,5), (4,1) (4,2) (4,3), (4,4) }

(7) Find the transitive closure of the relation $R = \{(1,1), (1,3), (1,5), (2,3), (2,4), (3,3), (3,5), (4,2), (4,4), (5,4)\}$ defined on a set $A = \{1,2,3,4,5\}$ using Warshall's Algorithm.

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

| K | In $W_{K-1}$: Positions of 1's in column K | In $W_{K-1}$: Positions of 1's in Rows | $W_K$ has 1's in | $W_K$ |
|---|---|---|---|---|
| 1 | 1 | 1, 3, 5 | (1,1), (1,3), (1,5) | $W_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ |

(8) If $f: Z \to N$ defined by $f(x) = \begin{cases} 2x-1, & x>0 \\ -2x, & x \le 0 \end{cases}$ prove that $f$ is one to one and onto and hence find $f^{-1}$.

Let $x_1, x_2 \in Z$ and $f(x_1) = f(x_2)$

Then either $f(x_1)$ and $f(x_2)$ are both odd or both even [$\because$ an odd number cannot be equal to an even number]

If they are both odd, then

$$2x_1 - 1 = 2x_2 - 1$$
$$2x_1 = 2x_2$$
$$\text{ie)} \quad x_1 = x_2$$

If they are both even, then

$$-2x_1 = -2x_2$$
$$x_1 = x_2$$

Thus whenever $f(x_1) = f(x_2)$, we get $x_1 = x_2$

Hence $f(x)$ is one to one.

Let $y \in N$. If $y$ is odd, its preimage is $\frac{y+1}{2}$.

Since $f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y$

If $y$ is even its preimage is $-\frac{y}{2}$. $\left(\because y = -2x,\ -\frac{y}{2} = x\right.$

Since $f(-\frac{y}{2}) = -2(-\frac{y}{2}) = y$

Thus for any $y \in W$, the preimage is $\frac{y+1}{2} \in Z$ or $-\frac{y}{2} \in Z$

Hence $f(x)$ is on-to.

$\therefore f$ is invertible.

Let $y = f(x) = \begin{cases} 2x-1, & x > 0 \\ -2x, & \text{if } x \le 0 \end{cases}$

$$\therefore f^{-1}(y) = x = \begin{cases} \frac{y+1}{2}, & \text{if } y = 1,3,5,\ldots \\ -\frac{y}{2}, & \text{if } y = 0,2,4,6,\ldots \end{cases}$$

$$\text{(or)} \quad f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & \text{if } x = 1,3,5,\ldots \\ -\frac{x}{2}, & \text{if } x = 0,2,4,6\ldots \end{cases}$$

(9) State and prove necessary and sufficient condition for a function is invertible.

The necessary and sufficient condition for the function $f: A \to B$ to be invertible is that $f$ is one to one and on-to.

Proof:

Let $f: A \to B$ be invertible.

Then there exist a unique function $g: B \to A$ such that

$$g \circ f = I_A \text{ and } f \circ g = I_B \quad \text{--- (1)}$$

Let $a_1, a_2 \in A$ s.t $f(a_1) = f(a_2)$

where $f(a_1), f(a_2) \in B$ [$\because f: A \to B$ is a function.

Since $g: B \to A$ is a function,

$$g(f(a_1)) = g(f(a_2))$$

ie) $(g \circ f)(a_1) = (g \circ f)(a_2)$

$I_A(a_1) = I_A(a_2)$ by (1),

$\boxed{a_1 = a_2}$

Thus whenever $f(a_1) = f(a_2)$, we have $a_1 = a_2$

Hence $f$ is one to one.

Let $b \in B$. Then $g(b) \in A$, Since $g: B \to A$ is a function

Now $b = I_B(b) = (f \circ g)(b)$

$= f(g(b))$

Thus, corresponding to every $b \in B$ there is an element $g(b) \in A$ s.t $f(g(b)) = b$.

Hence $f$ is on-to.

Thus necessary part of the condition is proved.

<u>Sufficient part:</u>

Let $f: A \to B$ is bijective.

Since $f$ is on-to, for each $b \in B$ there exist an $a \in A$ Such that $f(a) = b$.

Hence we define a function $g: B \to A$ by $g(b) = a$ where $f(a) = b$. — (2)

If possible let $g(b) = a_1$ and $g(b) = a_2$ where $a_1 \neq a_2$

This means that $f(a_1) = f(a_2) = b$

which is not possible since $f$ is one to one.

Thus $g: B \to A$ is a unique function.

Also from (2) we get $g \circ f = I_A$ and $f \circ g = I_B$

ie) $f$ is invertible.

Thus the sufficient condition is proved.

—.

(10) Prove that if $f$ and $g$ are invertible then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let $f : A \to B$ and $g : B \to C$ be invertible functions.

$\Rightarrow f$ & $g$ are bijective.

Hence $(g \circ f) : A \to C$ is also bijective.

$\therefore g \circ f$ is invertible. $(g \circ f)^{-1} : C \to A$ can be formed.

Thus both $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from C to A.

Now for any $a \in A$, let $b = f(a)$ and $c = g(b)$ —(1)

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

$$(g \circ f)^{-1}(c) = a \quad \text{— (2)}$$

By assumption ①, $a = f^{-1}(b)$, $b = g^{-1}(c)$

$$\Rightarrow (f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c))$$

$$= f^{-1}(b) = a \quad \text{— (3)}$$

From (2) & (3) it follows that

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ since $f^{-1}, g^{-1}$ and $(g \circ f)^{-1}$ are bijective.

(11) If $X = \{1,2,3,4,5\}$ and $f, g : X \to X$ given by $f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$, $g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$ then show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ also check $f \circ g = g \circ f$.

(i) $(f\circ g)(1) = f(g(1)) = f(3) = 4$

$(f\circ g)(2) = f(g(2)) = f(5) = 3$ and so on.

$\therefore f\circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$ — (1)

$(g\circ f)(1) = g(f(1)) = g(2) = 5$

$(g\circ f)(2) = g(f(2)) = g(1) = 3$ and so on.

$\therefore g\circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$ — (2)

From (1) & (2), $f\circ g = g\circ f$.

(ii) $f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}$ — (3)

and $g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\}$ — (4)

From (1) $(f\circ g)^{-1} = \{(4,1), (3,2), (2,3), (4,1) (5,1)\}$ — (5)

From (3) & (4),

$g^{-1}\circ f^{-1} = \{(2,3) (1,4) (4,1), (5,5) (3,2)\}$. — (6)

From (1) & (2),

$f^{-1}\circ g^{-1} = \{g^{-1}\circ f^{-1}\}$

Again from (3) & (4),

$f^{-1}\circ g^{-1} = \{(3,2), (5,1), (1,5), (2,3) (4,4)\}$

$\Rightarrow \quad (f\circ g)^{-1} = f^{-1}\circ g^{-1}$