Unit-V

Basic concepts - Definitions - degree and Hand shaking theorem - Some special graphs - Complete, regular and bipartite graphs. Isomorphic graphs - necessary conditions - simple ~~exercises~~ examples. Paths, cycles and circuits - connectivity in undirected graphs - connected graphs and odd degree vertices. Eulerian and Hamiltonian graphs - Necessary and sufficient condition for a graph to be Eulerian. Matrix representation of graphs - adjacent and incidence matrices and examples. Isomorphism using adjacency. Digraphs - indegree and out degree - Hand shaking theorem - verification of hand shaking theorem in digraphs.

Graph colouring - Chromatic number - examples. Four colour theorem (statement only) - problems. Trees - definitions and examples - properties. Spanning trees - examples. Krushkal's algorithm for minimum spanning trees. Application of graph theory in Engineering.

Graph Theory

Introduction:

A graph $G = (V, E, \phi)$ consists of a non-empty set V called set of vertices, E called set of edges and ϕ is a mapping from the set E to a set of ordered pair of elements of V;

Adjacent vertex:

The edge e that connects the vertices u and v is said to be incident on each of the vertex. The pair of vertices that are connected by an edge are called adjacent vertices.

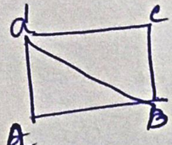
Isolated node:

A vertex (OR) node which is not adjacent to any other node (which is not connected by an edge) is called an isolated node.

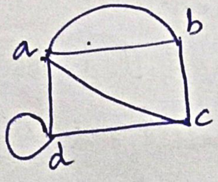
Simple graph:

A graph with no loops and parallel edges is a simple graph:

Eg:



Simplegraph

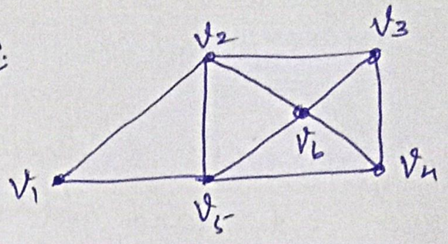


Not a simple graph but a graph

Degree of a Vertex:

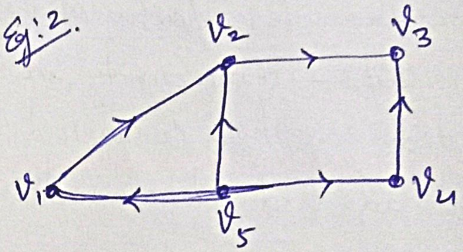
The degree of a vertex in an undirected graph is the number of edges incident with it, with the exception that a loop contributes twice to the degree of that vertex.

Eg:



$d(v_1)=2,\quad d(v_2)=4$
$d(v_3)=3,\quad d(v_4)=3$
$d(v_5)=4\quad d(v_6)=4$

Eg: 2.



In degree of v is $deg^-(v)$
Out degree of v is $deg^+(v)$

$deg^+(v_1)=1$	$deg^-(v_1)=1$
$deg^+(v_2)=1$	$deg^-(v_2)=2$
$deg^+(v_3)=0$	$deg^-(v_3)=2$
$deg^+(v_4)=1$	$deg^-(v_4)=1$
$deg^+(v_5)=2$	$deg^-(v_5)=0$

Theorem: [The Handshaking Theorem]

If $G=(V,E)$ is an undirected graph with 'e' edges, then
$\sum_i deg(v_i) = 2e$

ie), the sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

<u>Proof:</u>

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of degree of the vertices.

$\therefore$ All the e edges contribute (2e) to the sum of the degree of the vertices.

viz, $\sum_i \deg(v_i) = 2e$

<u>Theorem: 2:</u> The number of vertices of odd degree in an undirected graph is even.

<u>Proof:</u>

Let $G = (V, E)$ be the undirected graph. Let V_1 and V_2 be the sets of vertices of G of even and odd degrees respectively.

Then, by handshaking theorem,

$$2e = \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) \longrightarrow (1)$$

Since each $\deg(v_i)$ is even, we get $\sum_{v_i \in V_1} \deg(v_i)$ is even.

As the L.H.S of (1) is even, we get $\sum_{v_j \in V_2} \deg(v_j)$ is even.

Since each $\deg(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$ or in V_2 is even. ie) the number of vertices of odd degree is even.

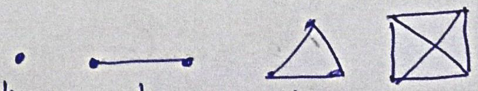
<u>Some special Simple Graphs:</u>

1. <u>Complete graph:</u>

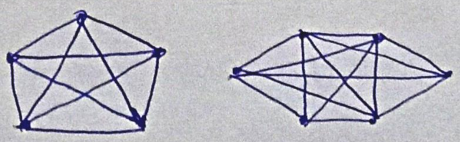
A simple graph, in which there is exactly one edge between each pair of distinct vertices is called a complete graph.

The complete graph on 'n' vertices is denoted by K_n.

Eg:



K_1 K_2 K_3 K_4



K_5 K_6.

Note:

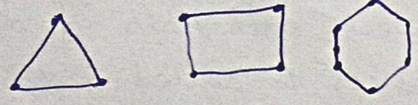
The number of edges in K_n is nC_2 (or) $\frac{n(n-1)}{2}$. Hence the maximum number of edges in a simple graph with n-vertices is $\frac{n(n-1)}{2}$.

Regular graph:

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree n, then the graph is called n-regular.

Eg:



2-regular

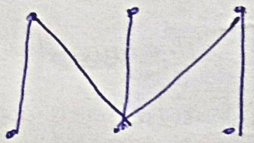
3-regular.

Bipartite graphs:

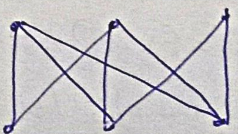
If the vertex set V of a simple graph $G=(V,E)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge connects either two vertices in V_1 or two vertices in V_2), then G is called a bipartite graph.

If each vertex V_1 is connected with every vertex of V_2 by an edge, then G is called a completely bipartite graph. It is denoted by $K_{m,n}$.

Eg:



Bipartite graph



Completely Bipartite, $K_{3,3}$

Subgraphs:

A graph $H=(V',E')$ is called a subgraph of $G=(V,E)$, if $V' \subseteq V$ and $E' \subseteq E$.

If $V' \subset V$ and $E' \subset E$, then H is a proper subgraph of G.

If $V' = V$, and ~~$E'=E$~~ then H is a spanning subgraph of G, which need not contain all its edges.

Isomorphic edges:

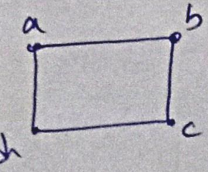
Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a one-to-one correspondance between the vertex sets which preserves adjacency of the vertices.

Two given graphs to be iso-morphic:

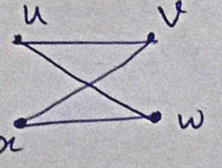
1. must have same number of vertices
2. same number of edges.
3. the corresponding vertices with the same degree.

This is the necessary condition for graph isomorphism.

Eg:



G



G'

Both G and G' have

(i) Number of vertices $= 4$

(ii) Number of edges $= 4$

(iii) $\deg(a) = 2 \rightarrow \deg(u) = 2$

$\deg(b) = 2 — \deg(v) = 2$

$\deg(c) = 2 — \deg(w) = 2$

$\deg(d) = 2 — \deg(x) = 2$

$\Rightarrow$ the corresponding vertices with the same degree.

$\therefore$ G and G' are isomorphic.

Matrix Representation of graphs:

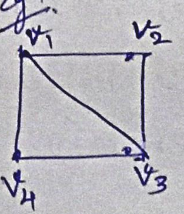
Adjacency Matrix:

When G is a simple graph with n-vertices $v_1, v_2, \ldots v_n$ the matrix $A \equiv [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

is called the adjacency matrix of G.

Eg:



G.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix

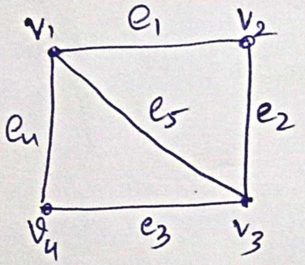
Incidence Matrix:

If $G=(V,E)$ is an undirected graph with n-vertices $v_1, v_2, \dots v_n$ and m edges $e_1, e_2, \dots e_m$ then the $(n \times m)$ matrix $B=[b_{ij}]$, where

$$b_{ij} = \begin{cases} 1, & \text{when } e_j \text{ incident on } v_i \\ 0, & \text{otherwise} \end{cases}$$

is called the incidence matrix of G.

Eg:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

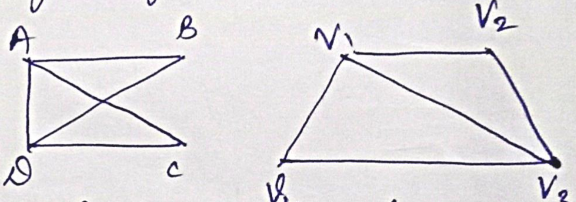
Note:

1. $\deg(v_i)$ is equal to the number of 1's in the ith row.
2. Each column of the incidence matrix contains exactly two unit entries.

Isomorphism and Adjacency matrices:

Thm: Two graphs are isomorphic if and only if their vertices can be labeled in such a way that the corresponding adjacency matrices are equal.

Eg:1 Establish the isomorphism of the two graphs given in figure by considering their adjacency matrices.



G_1 $\qquad$ G_2

Sol:

G_1	G_2
(i) No of vertices = 4	(i) No of vertices = 4
(ii) No of edges = 5	(ii) No of edges = 5
(iii) $d(A)=3$	(iii) $d(v_1)=3$
$d(B)=2$	$d(v_2)=2$
$d(C)=2$	$d(v_3)=3$
$d(D)=3$	$d(v_4)=2$

(iv) Adjacency of G_1 and G_2

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

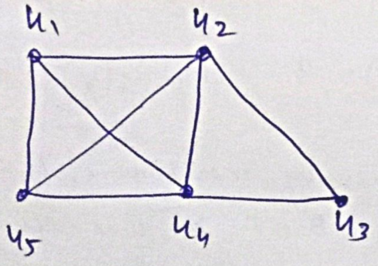
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_4 \\ v_3 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_4 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\Rightarrow$ Adjacency of G_1

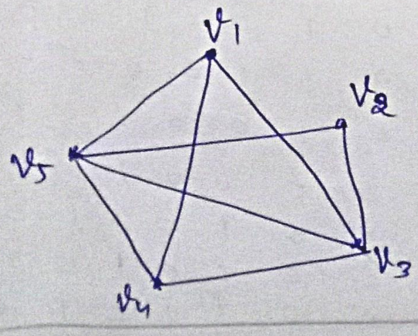
= Adjacency of G_2.

$\therefore$ G_1 and G_2 are isomorphic.

(2).



(G_1)



(G_2)

Solution:

	G_1	G_2
(i)	vertices = 5	vertices = 5
(ii)	edges = 8	edges = 8
(iii)	$d(u_1) = 3$	$d(v_1) = 3$
	$d(u_2) = 4$	$d(v_2) = 2$
	$d(u_3) = 2$	$d(v_3) = 4$
	$d(u_4) = 4$	$d(v_4) = 3$
	$d(u_5) = 3$	$d(v_5) = 4$

(iv) Adjacency:

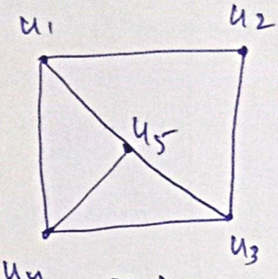
$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ v_1 \\ v_3 \\ v_2 \\ v_5 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_2 & v_5 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

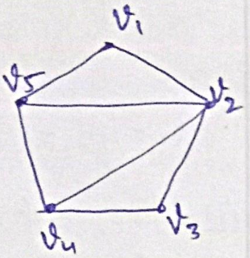
Adjacency of G_1 and G_2 are same

$\therefore$ G_1 and G_2 are isomorphic.

(3)



(G_1)



(G_2)

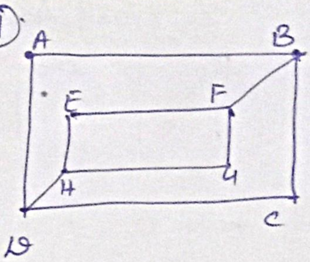
Sol:

G_1	G_2
(i) Vertices = 5	Vertices = 5
(ii) edges = 7	edges = 7
(iii) $deg(u_1) = 2$	$deg(v_1) = 2$
$deg(u_2) = 2$	$deg(v_2) = 4$
$deg(u_3) = 3$	$deg(v_3) = 2$
$deg(u_4) = 3$	$deg(v_4) = 3$
$deg(u_5) = 3$	$deg(v_5) = 3$

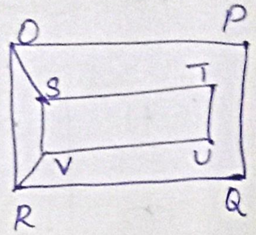
Number of vertices and edges in G_1 and G_2 are equal but not the degrees of the vertices.

$\Rightarrow$ G_1 and G_2 are not isomorphic.

(4)



G_1



G_2

Solution:

G_1	G_2
(i) vertices = 8	Vertices = 8
(ii) edges = 10	edges = 10
(iii) $deg(A) = 2$	$deg(O) = 3$
$deg(B) = 3$	$deg(P) = 2$
$deg(C) = 2$	$deg(Q) = 2$
$deg(D) = 3$	$deg(R) = 3$
$deg(E) = 2$	$deg(S) = 3$
$deg(F) = 3$	$deg(T) = 2$
$deg(G) = 2$	$deg(U) = 2$
$deg(H) = 3$	$deg(V) = 3$

(iv) Adjacency matrix of G_1 and G_2

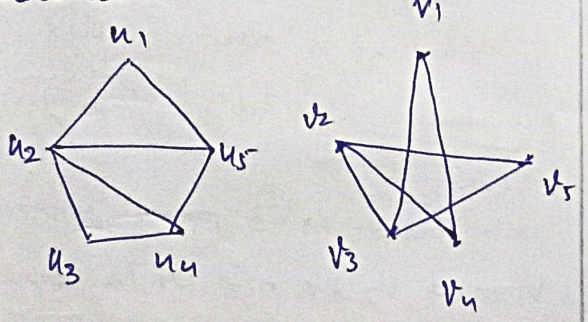
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$
\begin{array}{c|cccccccc}
 & P & O & Q & R & T & S & U & V \\
\hline
P & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
O & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
Q & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
R & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
T & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
S & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
U & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
V & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Since Adjacency of G_1 and G_2 are not same, G_1 and G_2 are not isomorphic.

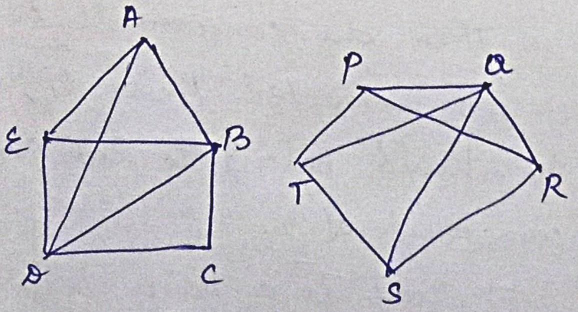
Exercise!

1.



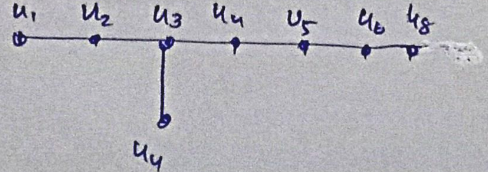
Ans: Not Isomorphic.

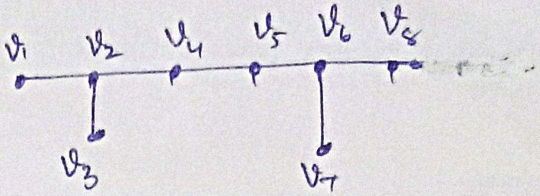
2.



Ans: Not isomorphic.

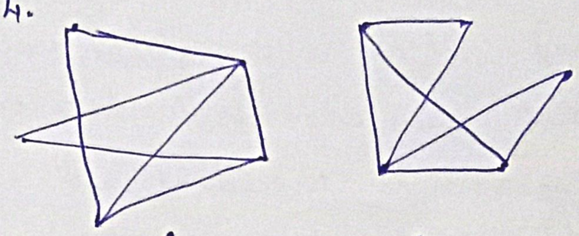
3.





Ans: Not Isomorphic.

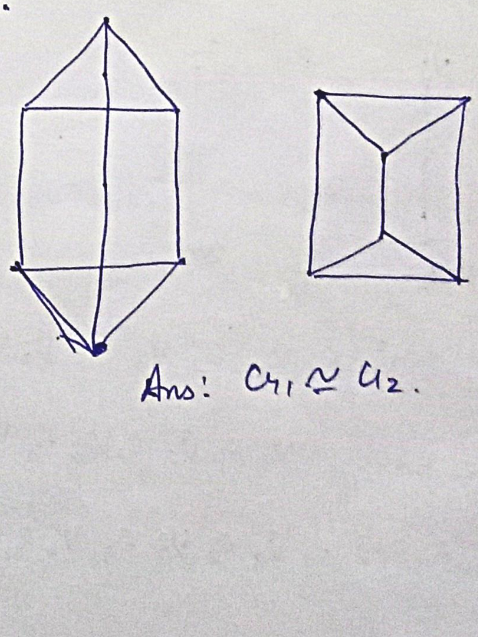
4.



G_1 $\qquad$ G_2

Ans: $G_1 \simeq G_2$.

5.



Ans: $G_1 \simeq G_2$.

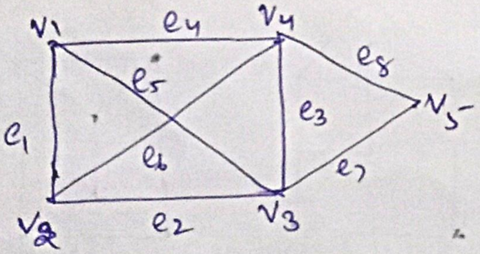
Paths, cycles and Connectivity:

Path:

A path in a graph is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident on the vertices preceding and following it.

If the edges in a path are distinct it is called a simple path.

Eg:



path: $v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_1\ e_1 v_2$

Simple path: $v_1\ e_1\ v_2\ e_2 v_3\ e_3\ v_4$

circuit : $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4 e_4\ v_1$
or
cycle

Connected graphs:

A graph is said to be connected if a path exists between every pair of distinct vertices of the graph.

A graph that is not connected is called disconnected.

Theorem: If a graph G has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof:

Case (i): Let G be connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree.

But it is proved that the number of odd vertices is even.

$\Rightarrow$ There is a path connecting v_1 and v_2, since G is connected.

Case (ii): Let G be disconnected

Then the components of G are connected. Hence v_1 and v_2 should belong to the same component of G.

$\Rightarrow$ There is a path between v_1 and v_2.

Theorem: 2:

The maximum number of edges in a simple disconnected graph G with n-vertices and k-components is:

$$\frac{(n-k)(n-k+1)}{2}.$$

Proof:

Let the number of vertices in the i^{th} component of G be n_i ($n_i \geq 1$).

Then, $n_1 + n_2 + \cdots + n_k = n$

(or)

$$\sum_{i=1}^{k} n_i = n \longrightarrow (1)$$

Hence,

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\therefore \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$= n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2\sum_{i \neq j} (n_i - 1)(n_j - 1)$$

$$= n^2 - 2nk + k^2 \longrightarrow (2)$$

ie) $\sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2$

$$\Rightarrow \sum_{i \neq 1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \longrightarrow (3)$$

Now the max number of edges in the i^{th} component of $G = \frac{1}{2} n_i (n_i - 1)$.

$\therefore$ Maximum number of edges of $G = \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \quad (\because \text{by } (1))$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2} n \quad (\text{by } (3))$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + n - k)$$

$$\leq \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\leq \frac{1}{2}(n-k)(n-k+1).$$

Eularian and Hamiltonian Graphs:

Eulerian graph:

A path of graph G is called an Eulerian path, if it includes each edge of G exactly once.

A circuit of a graph G is called an Eulerian Circuit if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an Eulerian graph.

Necessary and sufficient Condition for the existence of Euler circuits and Euler paths are given in two theorems.

Theorem: 1 A connected graph contains an Euler circuit, if and only if ~~it has exactly two~~ each of its vertices is of even degree.

Theorem: 2 A connected graph contains an Euler path, if and only if it has exactly two vertices of odd degree.

Hamiltonian graph:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.

A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and end vertices which appear twice.

A graph containing a Hamiltonian circuit is called a Hamiltonian graph.

Trees:

A connected graph without any circuits is called a tree.

ie) a tree has to be a simple graph, since loops and parallel edges form circuits.

Properties Of Trees:

Property: 1

An undirected graph is a tree if and only if there is a unique simple path between every pair of vertices.

Proof:

(i) Let the undirected graph T be a tree. Then by definition of a tree, T is connected

$\Rightarrow$ There is a simple path between any pair of vertices say v_i and v_j. If possible let there be two paths between v_i and v_j – one from v_i to v_j and the other from the v_j to v_i. Combination of these two paths form a circuit.

But T cannot have a circuit, be definition. Hence there is a unique simple path between every pair of vertices in T.

(ii) Let a unique path exist between every pair of vertices in the graph T.

$\Rightarrow T$ is connected

If possible let T contain a circuit. This means that there is a pair of vertices v_i and v_j between which two distinct paths exists, which is against the data.

Hence, T cannot have a circuit and so T is a tree.

Property: 2:

A tree with n-vertices has $(n-1)$ edges.

Proof:

Let e_k be the edge connecting the vertices v_i and v_j of T. Then, by property (1), e_k is the only path between v_i and v_j.

If we delete the edge e_k from T, T becomes disconnected and $(T-e_k)$ consists of exactly two components say, T_1 and T_2 which are connected.

Since T do not contain any circuit, T_1 and T_2 will also have no circuits.

$\Rightarrow T_1$ and T_2 are trees, each having less than n-vertices say r and $n-r$ respectively.

$\therefore$ By induction, T_1 has $(r-1)$ edges and T_2 has $(n-r-1)$ edges.

$\therefore T$ has $(r-1)+(n-r-1)+1$
$= n-1$ edges.

Thus, a tree with n-vertices has (n-1) edges.

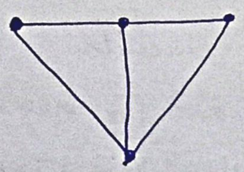
__Property-3:__ Any connected graph with n-vertices and (n-1) edges is a tree.

__Property-4:__ Any circuitless graph with n-vertices and (n-1) edges is a tree.

__Spanning Trees:__

If the subgraph T of a connected graph G is a tree containing all the vertices of G, then T is called a spanning tree of G.

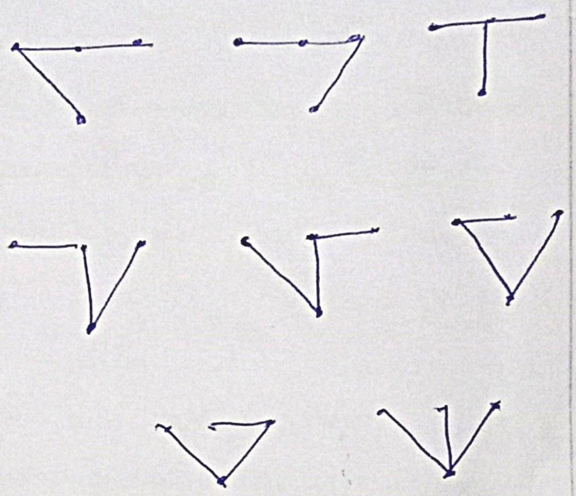
__Eg:__



G

Since G has 4 vertices and 5 edges, removal of 2 edges from G results in spanning tree.

All the possible spanning trees are:

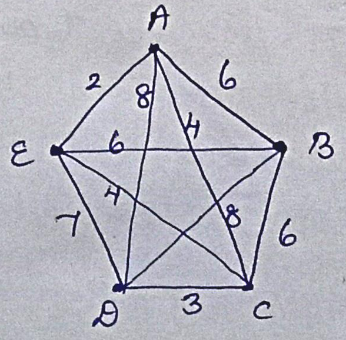


__Minimum spanning tree:__

If G is a connected weighted graph, the spanning tree of G with the smallest total weight is called the minimum spanning tree of G.

__Problems:__

1. Find the minimum spanning tree for the weighted graph shown in figure by Kruskal's algorithm.

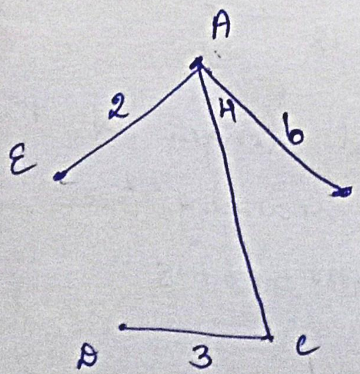


Solution:

The given graph consists of 5 vertices, so the minimum spanning tree must have 4 edges.

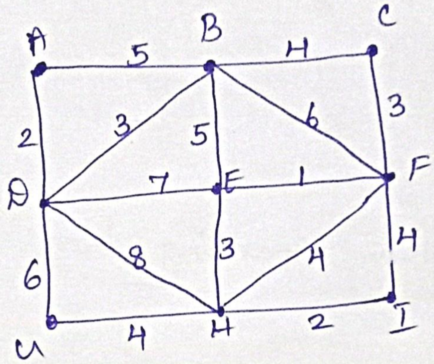
We choose the edges of the given graph with minimum weight, so that no-cycles are formed.

Edge	Weight	Spanning tree
AE	2	Yes
CD	3	Yes
AC	4	Yes
CE	4	NO
AB	6	Yes
BC	6	–
BE	6	–
DE	7	–
AD	8	–
BD	8	–



length = 15

2. Use kruskal's algorithm to find a minimum spanning tree for the given graph:



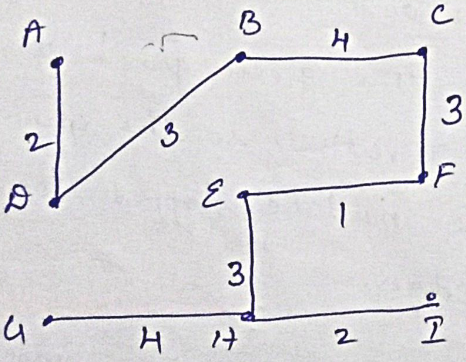
Solution:

The given graph consists of 9 vertices, so the spanning tree must be stopped for 8 edges.

Edge	Weight	Spanning tree
EF	1	Yes
AD	2	Yes
HI	2	Yes
BD	3	Yes
CF	3	Yes
EH	3	Yes
BC	4	Yes
FH	4	NO
FI	4	NO
GH	4	Yes
AB	5	–

BE	5	—
BF	6	—
DG	6	—
DE	7	—
DH	8	—

Minimum Spanning tree:

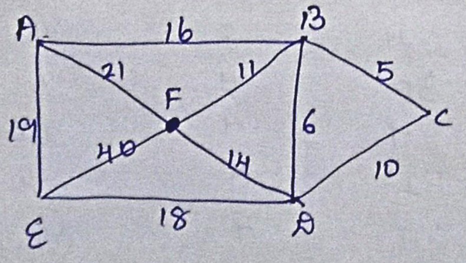


The total length of the minimum spanning tree

$= 2+3+4+3+1+3+4+2$

$= 22$

3.

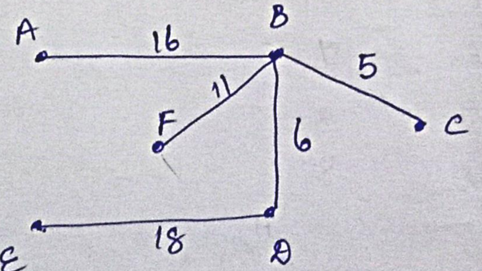


Solution:

The given graph has 6 vertices, so the spanning tree must have 5 edges.

Edge	Weight	Spanning tree
BC	5	Yes
BD	6	Yes
DC	10	NO
BF	11	Yes
FD	14	NO
AB	16	Yes
DE	18	Yes
AE	19	—
AF	21	—
FE	40	

Minimum spanning tree:

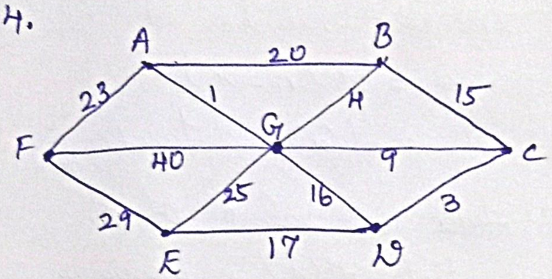


The total length of the minimum spanning tree

$= 16+11+5+6+18$

$= 56$

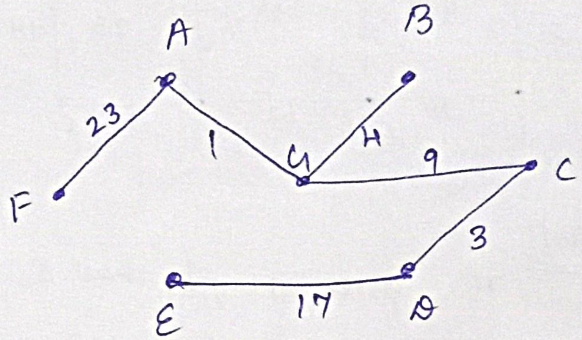
4.



Solution: The given graph has 7 vertices, so the spanning tree will have 6 edges.

Edge	Weight	Spanning tree
AG	1	Yes
DC	3	Yes
GB	4	Yes
GC	9	Yes
BC	15	NO
GD	16	NO
ED	17	Yes
AB	20	NO
AF	23	Yes
GE	25	–
FE	29	–
FG	40	–

Minimum spanning tree:

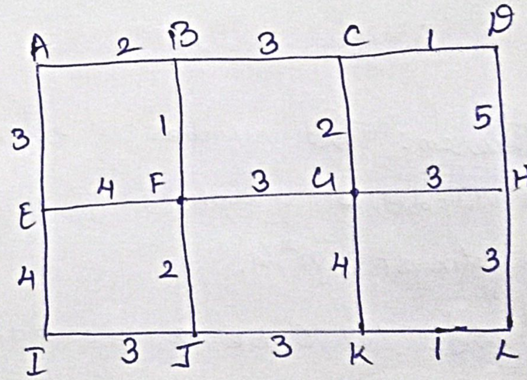


Minimum totatal weight

$= 23 + 1 + 4 + 9 + 3 + 17$

$= 57$

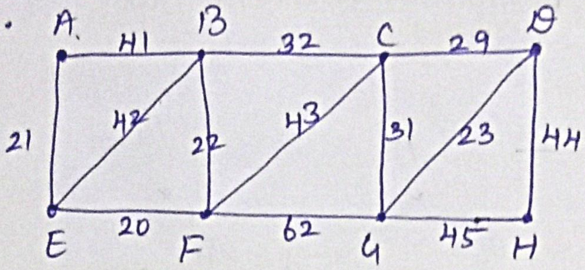
5.



Sol: The given graph has 12 vertices, so the spanning tree will have 11 edges.

The edges are AB, AE, BF, CD, CG, FG, FJ, GH, HL, IJ and KL

Weight = 24.

6. 

Sol:

The given graph has 8 vertices, so the spanning tree will have 7 edges.

The edges are: AE, BC, BF, CD, DG, DH and EF.

Weight = 191.

Theorem: The number of edges in a bipartite graph with n vertices is atmost $n^2/4$.

Proof: Let G be a bipartite graph with n-vertices.

Let the vertex set be partitioned into two subsets V_1 and V_2.

Let V_1 contains x vertices and V_2 contain $n-x$ vertices.

The largest number of edges G can be obtained, when each of the x vertices in V_1 is connected to each of the $n-x$ vertices in V_2.

∴ Largest number of edges $= x(n-x)$

$= f(x)$, a function of x.

To find the value of x, for which $f(x)$ is maximum.

$$f(x) = x(n-x)$$

$$f'(x) = n-2x.$$

$$f''(x) = -2 < 0$$

Also, $f'(x) = 0$

$$\Rightarrow n-2x = 0 \Rightarrow x = n/2$$

∴ Hence $f(x)$ is maximum when $x = n/2$,

∴ Maximum no of edges required $= f(n/2)$.

$$= n(n-n/2) = n^2/4.$$