Unit - II

Permutation and Combination - Simple problems using addition and product rules - Principle of Inclusion and Exclusion - Problems - Pigeon hole and generalized pigeon hole principle - problems - Divisibility and prime numbers - Fundamental theorem of arithmetic - problems - Finding prime factorization of a given number - Some more problems using fundamental theorem of arithmetic - Division algorithm - gcd and properties - problems - Euclid's algorithm examples - problems on LCM - Relation between GCD and LCM. Problems on GCD and LCM using prime factorization and Euclid's lemma - problems for practice.

Introduction:

The counting numbers 1, 2, 3, ... n are considered to be man's first mathematical creation. Area of applications are art, coding theory, cryptography etc.

The well-ordering principle:

Every non-empty set of positive integers has a least number.

Pigeon hole principle: [or] Dirichlet box principle:

If m pigeons are assigned to n-pigeon holes, where $m > n$, then atleast two pigeons must occupy the same hole.

Floor function (or) Greatest integer function:

$[x]$ = the greatest integer $\leq x$

Eg: $[4.1] = 4$

$[4.5] = 4$

$[4.9] = 4$

Generalized pigon hole principle:

If m pigeons are assigned to n pigeon holes, where $m > n$, then atleast one of the pigeon hole will contain $\left[\frac{m-1}{m}\right]+1$ pigeons.

Problems:

1. Show that if seven colours are used to paint 50 bicycles then atleast 8 bicycles will have the same colour.

Sol: $m = 50,\ n = 7$

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{50-1}{7}\right]+1$$

$$= \left[\frac{49}{7}\right]+1$$

$$= 7+1$$

$= 8$ bicycles.

2. Show that of 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionary must have 2045 pages.

Solution: $m = 61327,\ n = 30$

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{61327-1}{30}\right]+1$$

$$= \left[\frac{61326}{30}\right]+1$$

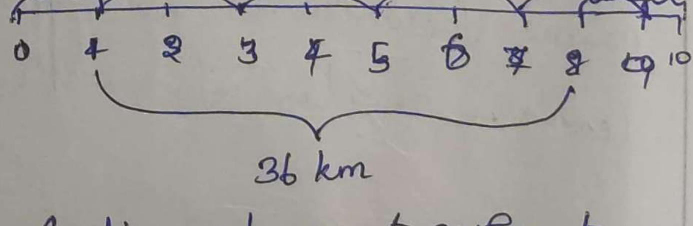
$$= [2044.2]+1$$

$$= 2044+1$$

$= 2045$ pages.

3. A man hiked for 10 hours and covered a total distance of 45 km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked atleast 9 km within a certain period of 2 consecutive hours.

Sol: The man hiked in first and last hours $= 6+3 = 9$ km.
So during the period of second to ninth hour he must have hiked $45-9 = 36$ km.



Combine 2 hours together to get 4 time periods.

Now, 4 time periods = pigeon holes = n

36 km = pigeons. = m

Using generalised pigeonhole principle,

The least number of pigeons accomodated in one pigeon hole

$= \left[\frac{m-1}{n}\right]+1$

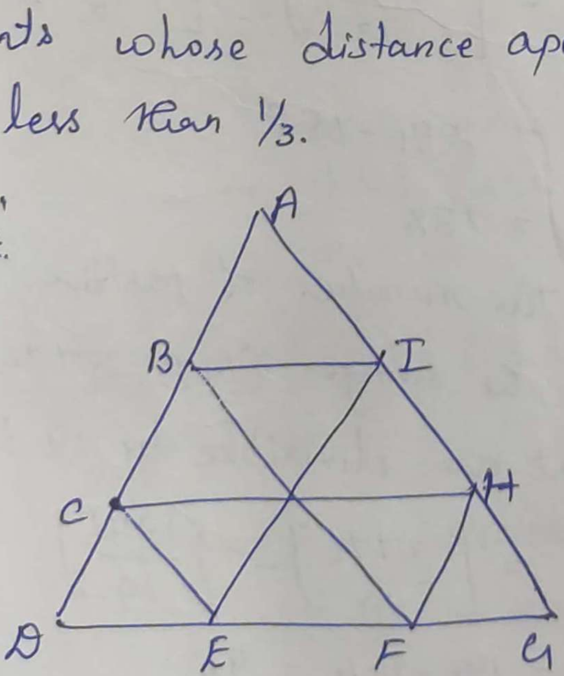
$= \left[\frac{36-1}{4}\right]+1$

$= [8.75]+1 = 8+1 = 9$

∴ The man must have hiked atleast 9 km in one time period of 2 consecutive hours.

4. If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be atleast two points whose distance apart is less than 1/3.

Sol:



Let ADG be the given equilateral triangle. We divide it into 9 equilateral triangles each of side 1/3.

The 9 sub-triangles may be regarded as 9 pigeon holes and 10 interior points as pigeons.

$m = 10, \ n = 9$

By generalised pigeon hole principle,

The least number of pigeons accomodated in one pigeon hole

$= \left[\frac{m-1}{n}\right]+1$

$= \left[\frac{10-9}{9}\right]+1 = 1+1 = 2$

ie) one subtriangle must contain 2 interior points.

The distance between any two interior points of any sub-triangle cannot exceed the length of the side, namely 1/3.

Hence the result.

Principle of Inclusion - Exclusion:

If A and B are finite subsets of a finite universal set U, then

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where $|A|$ denotes the cardinality of the set A.

Note:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|.$$

Problems:

1. Find the number of +ve integers ≤ 3076 and are
(i) divisible by 19
(ii) not divisible by 24.

Solution:

(i) Number of positive integers ≤ 3076 and divisible by 19 is:

$$= \left[\frac{3076}{19}\right] = 161$$

(ii) The number of positive integers ≤ 3076 and divisible by 24 is:

$$= \left[\frac{3076}{24}\right] = 128$$

$\Rightarrow$ The number of integers ≤ 3076 and not divisible by 24 is

$3076 - 128 = \underline{2948}$. Ans.

2. Find the number of positive integers in the range 1976 through 3776 that are (i) divisible by 13 (ii) not divisible by 19.

Solution:

(i) The required number of positive integers is:

$$= \left[\frac{3776}{13}\right] - \left[\frac{1976}{13}\right]$$

$$= 290 - 152$$

$$= 138$$

(ii) The number of positive integers in the range (1976, 3776) that are divisible by 19 is:

$$= \left[\frac{3776}{19}\right] - \left[\frac{1976}{19}\right]$$

$$= 198 - 104 = 94$$

⇒ The number of positive integers in the range that are not divisible by 19 is:

$= (3776 - 1976) - 94$

$= 1706.$

3. Find the number of positive integers ≤ 2076 and divisible by neither 4 nor 5.

Solution:

Let A and B be the set of integers ≤ 2076 that are divisible by 4 or 5 respectively.

By inclusion - exclusion principle:

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = \left[\frac{2076}{4}\right] = 519$

$|B| = \left[\frac{2076}{5}\right] = 415$

$|A \cap B| = \left[\frac{2076}{4 \times 5}\right] = 103$

$|A \cup B| = 519 + 415 - 103$

$= 831$

∴ The set of integers divisible by neither 4 nor 5 is:

$|A' \cap B'| = [|A \cup B|]'$

$= \left[\text{The total number of integers}\right] - |A \cup B|$

$= 2076 - 831$

$= 1245$

4. Find the positive integers ≤ 3000 and divisible by 3, 5 or 7.

Solution: Let A, B, C be the set of numbers ≤ 3000 that are divisible by 3, 5 and 7 respectively.

$|A| = \left[\frac{3000}{3}\right] = 1000$

$|B| = \left[\frac{3000}{5}\right] = 600$

$|C| = \left[\frac{3000}{7}\right] = 428$

$|A \cap B| = \left[\frac{3000}{3 \times 5}\right] = 200$

$|A \cap C| = \left[\frac{3000}{3 \times 7}\right] = 142$

$|B \cap C| = \left[\frac{3000}{5 \times 7}\right] = 85$

$|A \cap B \cap C| = \left[\frac{3000}{3 \times 5 \times 7}\right] = 28.$

By inclusion-exclusion principle, we know,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

$= 1000 + 600 + 428 - 200 - 142 - 85 + 28$

$= 1629$

⑤ Find the number of integers from 1 to 250 that are not divisible by any of the integers 2, 3, 5 and 7.

Sol: Let A, B, C, D be the set of integers divisible by 2, 3, 5, 7 respectively.

By inclusion-exclusion,

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$.

$= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$

$= 193$

Number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5 and 7

= Total no of integers $- |A \cup B \cup C \cup D|$

$= 250 - 193 = \underline{57}$ Ans.

Permutation and Combination:

Permutation is an arrangement of r elements of a set containing n distinct elts. It is denoted by $P(n, r)$ or nP_r.

(i) $nP_r = \frac{n!}{(n-r)!}$

(ii) If repetitions is allowed:

$nP_r = \frac{n!}{p_1!\, p_2!\, p_3!}$

Combination is an unordered selection of r elements of a set containing n distinct elements. It is denoted by $C(n, r)$ or nC_r.

$nC_r = \frac{n!}{(n-r)!\, r!}$

Problems:

1. In how many ways can a coach choose three swimmers from five swimmers?

Solution:

Here $n=5$, $r=3$

$$nC_r = 5C_3 = \frac{5!}{3!\,2!}$$

$$=10 \text{ ways.}$$

2. In a lottary each ticket has 5 one digit numbers 0-9 on it. (a) If your ticket has the digits in any order r, you win. what are your chances of winning? (b) You win only if your ticket has the digits in the required order. What are your chances of winning?

Sol: (i) $nC_r = 10C_5$

$$= \frac{10!}{5!\,5!} = \frac{1}{252}$$

(ii) $nP_r = 10P_5$

$$= \frac{10!}{5!} = 30240$$

3. How many ways can you arrange the letters in the word LOLLIPOP?

$N=8$, $L=3$, $O=2$, $I=1$, $P=2$.

Repeated numbers:

$$nP_r = \frac{n!}{P_1!\,P_2!\,P_3!}$$

$$= \frac{8!}{3!\,2!\,2!}$$

4. 5 balls are to be placed in 3 boxes. Each can hold all the 5 ~~boxes~~ balls. In how many different ways can we place the balls so that no box is left empty if:

a) balls and boxes are different?

b) balls are identical and boxes are different?

c) balls are different and boxes are identical?

d) balls as well as boxes are identical?

Solution:

a) 5 balls can be distributed such that the first, second and third boxes 1, 1 and 3 balls respectively

$$nP_r = \frac{n!}{P_1!\,P_2!\,P_3!}$$

$$= \frac{5!}{1!\,1!\,3!} = 20$$

Similarly the boxes I, II, III may contain 1, 3 and 1 balls respectively or 3, 1 and 1 balls respectively.

Number of ways of distributing in each of these manners = 20.

Again I, II, III may contain 1, 2, 2 balls (or) 2, 1, 2 balls (or) 2, 2, 1 balls respectively.

$\Rightarrow$ Number of ways $= \dfrac{5!}{1!\,2!\,2!}$

$= 30$

$\therefore$ Total no of required ways $= 20+20+20+30+30+30$

$= 150$

b) Total number of ways of distributing r identical balls in n different boxes is same as the no of r-combinations of n items, repetitions allowed.

$$C(n+r-1,\ r) = C(3+2-1,\ 2)$$
$$= C(4, 2)$$
$$= \frac{4!}{2!\,2!}$$
$$= 6$$

c) When the boxes are identical the distributions 1, 1, 3 balls, 1, 3, 1 balls and 3, 1, 1 balls considered will be treated as identical distributions. Thus, there are 20 ways.

Similarly, there are 30 ways of distributing 2 balls in each of any 2 boxes and 1 ball in the third box.

$\therefore$ Number of ways $= 20+30$
$= 50$

d) By an argument similar to that given in (c), we get from the answer in (b) that the required no of ways

$= \dfrac{6}{3} = 2.$

Sum rule:

If r activities can be performed in $n_1, n_2, \dots n_r$ ways and if they are disjoint, viz cannot be perfomed simultaneously, then any one of the r activities can be performed in $(n_1 + n_2 + \dots + n_r)$ ways.

Product Rule:

If an activity can be performed in r successive steps and step 1 can be done in n_1 ways, step 2 can be done in n_2 ways ... step r can be done in n_r ways, then the activity can be done in $(n_1 . n_2 . n_3 \dots . n_r)$ ways.

Division algorithm:

Let a be any integer and b a positive integer. Then there exist unique integers q and r such that

$a = qb + r, \quad 0 \le r < b$

The divisibility relation:

In division algorithm if $r = 0$, then $a = qb$.

We say that b divides a [or] b is a factor of a [or] a is divisible by b (or) a is a multiple of b and we write b/a.

Prime and Composite Number:

A positive integer $p > 1$ is called a prime if its only positive factors are 1 and p. If $p > 1$ is not a prime, then it is called a composite number.

Note: The integer n is composite if and only if there exists an integer a such that a/n and $1 < a < n$.

Eg: 5 is prime because its only positive factors are 1 and 5.

But 6 is composite because it has 2 and 3 as factors.

Note: If n has no prime factors $\le [\sqrt{n}]$, then n is a prime.

1. Determine whether 101 is a prime.

<u>Sol:</u>

All prime numbers $\} \leq [\sqrt{101}]$

$= 10$

The prime are 2, 3, 5 & 7. Since none of these is a factor of 101, we get 101 is a prime number.

2. Find whether 1001 is a prime.

<u>Sol:</u> All primes $\leq [\sqrt{1001}]$

$= 31$

The primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.

We find $7 / 1001$

$\Rightarrow$ 1001 is not a prime.

<u>Fundamental Theorem of Arithmetic:</u>

Every integer $n > 1$ can be written uniquely as a product of prime numbers.

<u>Eg: 1</u> Prime factorisation of

$100 = 2^2 5^2$

<u>Eg: 2</u> $5096 = 2^3 \cdot 7^2 \cdot 13$

$$\begin{array}{r|l} 5 & 100 \\ \hline 5 & 20 \\ \hline 2 & 4 \\ \hline & 2 \end{array} \qquad \begin{array}{r|l} 7 & 5096 \\ \hline 7 & 728 \\ \hline 2 & 104 \\ \hline 2 & 52 \\ \hline 2 & 26 \\ \hline & 13 \end{array}$$

<u>Greatest Common Divisor:</u>

<u>Definition:</u> The gcd of two integers a and b not both zero is the integer (largest positive integer) that divides both a and b.

It is denoted by gcd (a, b) or (a, b).

<u>Eg:</u> $(12, 18) = 6$

$(-15, 20) = 5$

$(3, 6) = 3$

$(-3, 6) = 3$

$(24, 36) = 12$

Common factors of 24 and 36 are 1, 2, 3, 4, 12 of which the largest is 12.

<u>Note:</u> $\gcd(a, -b) = \gcd(-a, b)$

$= g(-a, -b) = \gcd(a, b)$.

Definition:

A positive integer d is the gcd of integers a and b if:

(i) d/a and d/b

(ii) If c/a and c/b then c/d, where c is a positive integer.

Definition:

Two positive integers a and b are relatively prime if their gcd is 1.

ie) $(a, b) = 1$.

Eg: $\gcd(6, 25) = 1$

$\Rightarrow$ 6 and 25 are relatively prime.

Note: 1

The gcd of two positive integers a and b is a linear combination of a and b, ie) if $d = (a, b)$ then $d = la + mb$ for some integers l and m.

Note: 2

Two positive integers a and b are relatively prime iff $\exists$ integers α and β such that $\alpha a + \beta b = 1$.

Note: 3: If a/c and b/c and $(a, b) = 1$ then ab/c.

Note: 4 a/bc does not mean that a/b or a/c.

Eg: $6/24 \Rightarrow 6/3.8$

But this does not mean $6/3$ or $6/8$.

Euclid's lemma:

If p is a prime and p/ab then p/a or p/b.

Least Common Multiple: (LCM)

Definition:

The least common multiple of two positive integers a and b is the smallest positive integer that is divisible by both a and b.

The lcm of a and b is denoted by $[a, b]$ or $\mathrm{lcm}(a, b)$.

Canonical decomposition:

If $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and

$b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_k^{\beta_k}$,

where $p_1, \dots, p_k$ are primes and α_i, β_i are non-negative integers.

$$\gcd(a,b) = p_i^{\min(\alpha_i, \beta_i)}, \quad i = 1, 2 \dots k$$

$$\text{lcm}(a,b) = p_i^{\max(\alpha_i, \beta_i)} \quad i = 1, 2 \dots k.$$

Problems:

1. Find the gcd and lcm of 1050 and 2574 using prime factorization.

Sol:

$$\begin{array}{r|l} 5 & 1050 \\ 5 & 210 \\ 7 & 42 \\ 3 & 6 \\ & 2 \end{array} \qquad \begin{array}{r|l} 2 & 2574 \\ 3 & 1287 \\ 3 & 429 \\ 11 & 143 \\ & 13 \end{array}$$

$1050 = 2 \cdot 3 \cdot 5^2 \cdot 7$

$2574 = 2 \cdot 3^2 \cdot 11 \cdot 13$

$\gcd(1050, 2574)$

$= 2 \cdot 3 \cdot 5^0 \cdot 7^0 \cdot 11^0 \cdot 13^0$

$= 6$

$\text{lcm}(1050, 2574)$

$= 2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$

$= 450450$

2. Find the lcm and gcd of 504 and 540.

Sol:

$$\begin{array}{r|l} 2 & 504 \\ 2 & 252 \\ 2 & 126 \\ 7 & 63 \\ 3 & 9 \\ & 3 \end{array} \qquad \begin{array}{r|l} 2 & 540 \\ 2 & 270 \\ 5 & 135 \\ 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

$504 = 2^3 \cdot 3^2 \cdot 7 = 2^3 \cdot 3^2 \cdot 5^0 \cdot 7^1$

$540 = 2^2 \cdot 3^3 \cdot 5 = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^0$

$\gcd = 2^2 \cdot 3^2 \cdot 5^0 \cdot 7^0 = 4 \cdot 9$
$= 36$

$\text{lcm} = 2^3 \cdot 3^3 \cdot 5^1 \cdot 7^1 = 8 \times 27 \times 5 \times 7$
$= 7560$

3. Find the lcm and gcd of 414 and 662.

Sol:

$$\begin{array}{r|l} 2 & 414 \\ 3 & 207 \\ 3 & 69 \\ & 23 \end{array} \qquad \begin{array}{r|l} 2 & 662 \\ & 331 \end{array}$$

$414 = 2 \cdot 3^2 \cdot 23 = 2 \cdot 3^2 \cdot 23^1 \cdot 331^0$

$662 = 2 \cdot 331 = 2 \cdot 3^0 \cdot 23^0 \cdot 331^1$

$\gcd = 2 \cdot 3^0 \cdot 23^0 \cdot 331^0 = 2$

$\text{lcm} = 2 \cdot 3^2 \cdot 23 \cdot 331 = 137{,}034$

4. Find gcd and lcm of 175 and 192

Sol:

$5 \,|\, 175$, $5 \,|\, 35$, 7

$2 \,|\, 192$, $2 \,|\, 96$, $2 \,|\, 48$, $2 \,|\, 24$, $2 \,|\, 12$, $2 \,|\, 6$, 3

$175 = 5^2 \cdot 7$

$192 = 2^6 \cdot 3$

$\gcd = 1$, since there is no common factor except 1.

$\text{lcm} = 2^6 \cdot 3 \cdot 5^2 \cdot 7$

$= 33600$

5. 168 and 180.

$168 = 2^3 \cdot 3 \cdot 7$

$180 = 2^2 \cdot 3^2 \cdot 5$

$\gcd = 2^2 \cdot 3 = 12$

$\text{lcm} = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 8 \cdot 9 \cdot 5 \cdot 7$

$= 72 \cdot 35$

$= 2520$

Euclidean Algorithm to find the gcd:

Suppose a and b are positive integers with $a \geq b$, $a = qb + r$, $0 \leq r < b$, then $(a, b) = r_n$, where r_n is the last non-zero remainder in the sequence of divisions.

Problems:

1. Find the gcd of $(2076, 1776)$ using Euclidean algorithm.

Solution:

Using Euclidean:

$2076 = 1(1776) + 300$

$1776 = 5(300) + 276$

$300 = 1(276) + 24$

$276 = 11(24) + \boxed{12}$

$24 = 2(12) + 0$

$1776 \overline{)2076}(1$; 1776; $300 \overline{)1776}(5$; 1500; $276 \overline{)300}(1$; 276; $24 \overline{)276}(11$; 24; 36; 24; $12 \overline{)24}(2$; 24; 0

The last non-zero remainder is 12. $\therefore \boxed{(2076, 1776) = 12}$

2. Apply Euclidean algorithm to compute (3076, 1976).

Sol:

Applying Euclidean division algorithm,

$$
\begin{array}{l}
1976)\,3076\,(1 \\
\quad 1976 \\
\quad 1110)\,1976\,(1 \\
\qquad 1110 \\
\qquad 876)\,1110\,(1 \\
\qquad\quad 876 \\
\qquad\quad 224)\,876\,(3 \\
\qquad\qquad 672 \\
\qquad\qquad 204)\,224\,(1 \\
\qquad\qquad\quad 204 \\
\qquad\qquad\quad 20)\,204\,(10 \\
\qquad\qquad\qquad 200 \\
\qquad\qquad\qquad 4)\,20\,(5 \\
\qquad\qquad\qquad\quad 20 \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

$3076 = 1(1976) + 1110$

$1976 = 1(1110) + 876$

$1110 = 1(876) + 224$

$876 = 3(224) + 204$

$224 = 1(204) + 20$

$204 = 10(20) + \boxed{4}$

$20 = 5(4) + 0$

The last non-zero remainder is 4.

$\therefore$ gcd (3076, 1976) = 4.

3. Find gcd (1976, 1776) using Euclidean algorithm and express as a linear combination of them.

Sol:

$$
\begin{array}{l}
1776)\,1976\,(1 \\
\quad 1776 \\
\quad 200)\,1776\,(8 \\
\qquad 1600 \\
\qquad 176)\,200\,(1 \\
\qquad\quad 176 \\
\qquad\quad 24)\,176\,(7 \\
\qquad\qquad 168 \\
\qquad\qquad 8)\,24\,(3 \\
\qquad\qquad\quad 24 \\
\qquad\qquad\quad 0
\end{array}
$$

Applying Euclidean division algorithm,

$1976 = 1(1776) + 200$

$1776 = 8(200) + 176$

$200 = 1(176) + 24$

$176 = 7(24) + \boxed{8}$

$24 = 3(8) + 0$

The last non-zero remainder is 8.

$\therefore \boxed{\text{gcd} = 8}$

Using the above equations in reverse order and subst. for remainder each time, we get the linear combination.

$(1976, 1776) = 8$

$$\begin{aligned}
\Rightarrow 8 &= 176 - 7(24) \\
&= 176 - 7[200 - 1(176)] \\
&= 176 - 7(200) + 7(176) \\
&= 8(176) - 7(200) \\
&= 8[1776 - 8(200)] - 7(200) \\
&= 8(1776) - 64(200) - 7(200) \\
&= 8(1776) - 71(200) \\
&= 8(1776) - 71[1976 - 1(1776)] \\
&= 8(1776) - 71(1976) + 71(1776) \\
&= 79(1776) + (-71)(1976)
\end{aligned}$$

Hence gcd as the linear combination of 1776 and 1976.

4. Use Euclidean algorithm to express the gcd of 4076 and 1024 as a linear combination of them.

<u>Sol:</u>

1024) 4076 (3
3072
1004) 1024 (1
1004
20) 1004 (50
1000
4) 20 (5
20
0

By using Euclidean division algorithm,

$$\begin{aligned}
4076 &= 3(1024) + 1004 \\
1024 &= 1(1004) + 20 \\
1004 &= 50(20) + \boxed{4} \\
20 &= 5(4) + 0
\end{aligned}$$

The last non-zero remainder is 4 $\Rightarrow$ $\boxed{gcd = 4}$

Now, using the above equations in reverse order:

$(4076, 1024) = 4$

$$\begin{aligned}
\Rightarrow 4 &= 1004 - 50(20) \\
&= 1004 - 50[1024 - 1(1004)] \\
&= 1004 - 50(1024) + 50(1004) \\
&= 51(1004) - 50(1024) \\
&= 51[4076 - 3(1024)] - 50(1024) \\
&= 51(4076) - 153(1024) - 50(1024) \\
&= 51(4076) - 203(1024) \\
&= 51(4076) + (-203)(1024)
\end{aligned}$$

$\therefore$ gcd is a linear combination of 1024 and 4076.

5. For any positive integer n, Prove that $8n+3$ and $5n+2$ are relatively prime.

<u>To prove</u>: $(8n+3,\ 5n+2) = 1$

when $n = 1$,

$8n+3 = 11$ and $5n+2 = 7$

$\gcd(11, 7) = 1$

Hence, it is true for $n = 1$.

For $n \geq 2$, we have $8n+3 > 5n+2$

By Euclidean division algorithm,

$$
\begin{array}{l}
5n+2)\ 8n+3\ (1 \\
\quad\ \underline{5n+2} \\
\quad 3n+1)\ 5n+2\ (1 \\
\qquad\quad\ \underline{3n+1} \\
\qquad\ 2n+1)\ 3n+1\ (1 \\
\qquad\qquad\quad \underline{2n+1} \\
\qquad\qquad\quad n)\ 2n+1\ (2 \\
\qquad\qquad\qquad\ \underline{2n} \\
\qquad\qquad\qquad\ 1)\ n\ (n \\
\qquad\qquad\qquad\qquad \underline{n} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

$8n+3 = 1(5n+2) + 3n+1$

$5n+2 = 1(3n+1) + 2n+1$

$3n+1 = 1(2n+1) + n$

$2n+1 = 2(n) + \boxed{1}$

$n = n(1) + 0$

The last non-zero remainder is 1.

$\therefore \gcd(8n+3, 5n+2) = 1,\ \forall n \geq 2$

So they are relatively prime.

<u>Relation between gcd and lcm</u>:

$$\operatorname{lcm}(a,b) = \frac{a \cdot b}{\gcd(a,b)}$$

<u>Note</u>:

If a and b are relatively prime then $(a,b) = 1$.

$\therefore [a,b] = \frac{a \cdot b}{1} = a \cdot b$

The lcm of relatively prime numbers is their products.