Basics of Geometric Transformations

CS 650: Computer Vision

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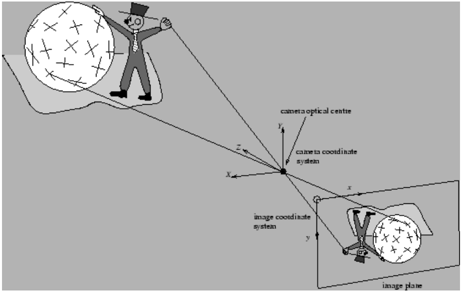
Images of a 3D World

- Cameras take 2D images of a 3D world
- To understand the 3D world, we first need to understand how this happens



Representing Points

2D Points:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Augmented 2D Points:

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Representing Points

Homogeneous 2D Points - Useful for Projection:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

Homogeneous points are equal up to a scaling:

$$a\tilde{\mathbf{x}} = \tilde{\mathbf{x}}$$

Converting “back” to inhomogeneous representation:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} = \tilde{w} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \tilde{w}\bar{\mathbf{x}}$$

Representing Points

3D Points:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Augmented:

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Homogeneous:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \tilde{w} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation

Translating (shifting) points:

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

Using augmented coordinates:

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

or

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

Rotating points around the origin by angle θ:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or for augmented points

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation - More General Form

Rotating into a coordinate system with axes defined by $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or for augmented points

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Any matrix $\mathbf{R}$ *whose row are orthonormal vectors is a rotation matrix:*

$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

Rotation and Translation

Rotating by angle θ *then* translating by $\mathbf{t}$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & t_x \\ -\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or more generally:

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

or in augmented coordinates:

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

Scaling

Scaling $\mathbf{x}' = s\mathbf{x}$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or for augmented points

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaled Rotation and Translation

Scaled rotation and translation (RST or similarity transform):

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t}$$

or

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine

Arbitrary matrix transformation that does not affect the augmented coordinate:

$$\bar{\mathbf{x}}' = \mathbf{A}\bar{\mathbf{x}}$$

or

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective

Arbitrary matrix transformation that operates on homogeneous coordinates:

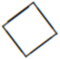
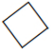
$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}}$$

or

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

- Unique only up to a scaling
- $\tilde{\mathbf{H}}$ is called a *homography*
- Keeps straight lines straight

Hierarchy of 2D Transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times3}$	8	straight lines	

Transformations in 3D

Translation:

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

Rotation:

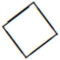
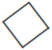
$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

Rotation and translation ("rigid body":

$$\bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \bar{\mathbf{x}}$$

Also equivalents of affine and projective transformations.

Hierarchy of 3D Transformations

Transformation	**Matrix**	**# DoF**	**Preserves**	**Icon**
translation	$\left[\ \boldsymbol{I} \mid \boldsymbol{t}\ \right]_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\left[\ \boldsymbol{R} \mid \boldsymbol{t}\ \right]_{3\times 4}$	6	lengths	
similarity	$\left[\ s\boldsymbol{R} \mid \boldsymbol{t}\ \right]_{3\times 4}$	7	angles	
affine	$\left[\ \boldsymbol{A}\ \right]_{3\times 4}$	12	parallelism	
projective	$\left[\ \tilde{\boldsymbol{H}}\ \right]_{4\times 4}$	15	straight lines