Set Theory

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

SET

A **set** is a collection of distinct objects, called **elements** of the set
A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1:

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answers

Some examples of sets defined by listing the elements of the set:

1. {1, 3, 9, 12}
2. {red, orange, yellow, green, blue, indigo, purple}

SUBSET

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.
If B is a subset of A, we write $B \subseteq A$
A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.
If B is a proper subset of A, we write $B \subset A$

Consider these three sets:

A = the set of all even numbers
B = {2, 4, 6}
C = {2, 3, 4, 6}

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A

UNION, INTERSECTION, AND COMPLEMENT

The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.

The **complement** of a set A contains everything that is *not* in the set A. The complement is notated A', or A^c, *or sometimes* $\sim A$.

Consider the sets:

A = {red, green, blue}
B = {red, yellow, orange}
C = {red, orange, yellow, green, blue, purple}

Find the following:

1. Find $A \cup B$
2. Find $A \cap B$
3. Find $A^c \cap C$

Answers

1. The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange} Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \cap B$ = {red}
3. Here we're looking for all the elements that are *not* in set A and are also in C. $A^c \cap C$ = {orange, yellow, purple}

UNIVERSAL SET

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A.

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Set Theory :

Set Operations :

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$A - B = \{x \mid x \in A \;\&\; x \notin B\}$

$A^c = \{x \mid x \notin A\}$

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

$A \oplus B = (A-B) \cup (B-A)$

1. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

: LHS : $\overline{A \cup B}$

Let $x \in \overline{A \cup B}$

$\overline{A \cup B} = \{x \mid x \notin A \cup B\}$

$= \{x \mid x \notin A \text{ and } x \notin B\}$

$= \{x \mid x \notin A\} \text{ and } \{x \mid x \notin B\}$

$= \overline{A} \cap \overline{B}$

$= R.H.S$

$\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$.

*2. Prove that $(A-C) \cap (C-B) = \emptyset$ analytically where A, B, C are the sets. Verify it graphically.

: Consider $(A-C) \cap (C-B) = \{x \mid x \in A-C \text{ and } x \in C-B\}$

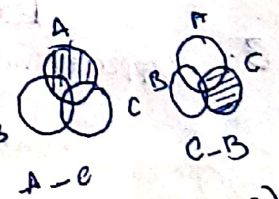
$= \{x \mid (x \in A \text{ and } x \notin C) \text{ and } (x \in C \text{ and } x \notin B)\}$

$= \{x \mid x \in A \text{ and } (x \notin C \text{ and } x \in C) \text{ and } x \notin B\}$

$= A \cap \emptyset \cap \overline{B}$

$= \emptyset$.

$\therefore (A-C) \cap (C-B) = \emptyset$



$(A-C) \cap (C-B) = \phi$.

* Partition of a set :

Let A be any non-empty set.

The collection of subsets of A i.e., $A_1, A_2 \cdots A_n$ is called a partition of A if

i) $\cup A_i = A$

ii) $A_i \cap A_j = \phi$

iii) $A_i \neq \phi \ \forall i$.

* Min Sets or Minterms :

Let A be any non-empty set and B_1, B_2 be subsets of A. Then $B_1 \cap B_2$, $B_1^c \cap B_2$, $B_1 \cap B_2^c$ and $B_1^c \cap B_2^c$ are called minsets or minterms generated by B_1 and B_2.

If B_1, B_2, B_3 are subsets of A then minterms are

$B_1 \cap B_2 \cap B_3$, $B_1 \cap B_2 \cap B_3^c$, $B_1 \cap B_2^c \cap B_3$, $B_1^c \cap B_2 \cap B_3$, $B_1^c \cap B_2^c \cap B_3$, $B_1^c \cap B_2 \cap B_3^c$, $B_1 \cap B_2^c \cap B_3^c$, $B_1^c \cap B_2^c \cap B_3^c$.

{The no. of minterms is 2^n}.

The minterms of A will form a partition of A.

*1. Let $A = \{1,2,3,4,5,6\}$. Find the minsets generated by $B_1 = \{1,3,5\}$ $B_2 = \{1,2,3\}$. Give the partition of A using Minsets.

um :- Minterms are

$B_1 \cap B_2 = \{1,3\} = A_1$

$B_1^c \cap B_2 = \{2\} = A_2$

$B_1 \cap B_2^c = \{5\} = A_3$

$B_1^c \cap B_2^c = \{4,6\} = A_4$

$B_1^c = \{2,4,6\}$

$B_2^c = \{4,5,6\}$

The ~~subs~~ minterms ~~of~~ which form a partition are $\{\{1,3\},\{2\},\{5\},\{4,6\}\}$.

i) Each subset is Non empty

ii) $A_i \cap A_j = \phi \ \forall \ i \neq j$

iii) $A_1 \cup A_2 \cup A_3 \cup A_4 = A$.

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UQ 8M

2. Let $A = \{1,2,3,4,5,6,7,8,9\}$

$B_1 = \{1,5,6,7\}$, $B_2 = \{2,4,5,9\}$, $B_3 = \{3,4,5,6,8,9\}$

Find the minsets generated by B_1, B_2, B_3 & give the partition of A.

: $B_1 = \{1,5,6,7\}$ $B_2 = \{2,4,5,9\}$ $B_3 = \{3,4,5,6,8,9\}$

$B_1^c = \{2,3,4,8,9\}$ $B_2^c = \{1,3,6,7,8\}$ $B_3^c = \{1,2,7\}$

Minsets are :

$B_1 \cap B_2 \cap B_3 = \{5\}$

$B_1 \cap B_2 \cap B_3^c = \{\emptyset\}$

$B_1 \cap B_2^c \cap B_3 = \{6\}$

$B_1^c \cap B_2 \cap B_3 = \{4,9\}$

$B_1 \cap B_2^c \cap B_3^c = \{1,7\}$

$B_1^c \cap B_2 \cap B_3^c = \{2\}$

$B_1^c \cap B_2^c \cap B_3 = \{3,8\}$

$B_1^c \cap B_2^c \cap B_3^c = \{\emptyset\}$

The Minsets which form a partition of A are :

$B_1 \cap B_2 \cap B_3$, $B_1 \cap B_2^c \cap B_3$, $B_1^c \cap B_2 \cap B_3$, $B_1 \cap B_2^c \cap B_3^c$, $B_1^c \cap B_2 \cap B_3^c$,

$B_1^c \cap B_2^c \cap B_3$.

* Max Sets:

The dual of Min sets is called Max sets.

Let A be any set and B_1, B_2 be the subsets of A.

The max sets generated by B_1, B_2 are

$B_1 \cup B_2$, $B_1^C \cup B_2$, $B_1 \cup B_2^C$, $B_1^C \cup B_2^C$.

NOTE: Max sets need not form a partition.

1. Find maxsets for $A = \{1,2,3,4,5,6\}$ generated by $\{1,3,5\}, \{1,2,3\}$.

: $A = \{1,2,3,4,5,6\}$

Let $B_1 = \{1,3,5\}$ $\quad B_1^C = \{2,4,6\}$

$B_2 = \{1,2,3\}$ $\quad B_2^C = \{4,5,6\}$

$B_1 \cup B_2 = \{1,2,3,5\}$

$B_1^C \cup B_2 = \{1,2,3,4,6\}$

$B_1 \cup B_2^C = \{1,3,4,5,6\}$

$B_1^C \cup B_2^C = \{2,4,5,6\}$ are the Maxsets.

Problems on Equivalence Relation:

1. Let, $X = \{1,2,3, \ldots, 25\}$, $R = \{(x,y) / x-y$ is divisible by $5\}$ be a relation. Show that R is an equivalence relation.

: For R to be an equivalence relation, it must be Reflexive, Symmetric and Transitive.

i) **Reflexive:** Let (a,a) belong to R, $(a,a) \in R$.

$a-a=0$ is divisible by 5.

Each element in X is related to itself.

$\Rightarrow$ R is reflexive.

ii) Symmetric:

Let $(a,b) \in R$

If $a-b$ is divisible by 5.

then $b-a$ is also divisible by 5.

$(b,a) \in R$

$\therefore R$ is symmetric.

iii) Transitive:

Let $(a,b) \in R$, $(b,c) \in R$

$\frac{a-b}{5} = n$

$a-b = 5n$

$a-b$ is divisible by 5

$\frac{b-c}{5} = m$

$b-c = 5m$

$b-c$ is divisible by 5

$a-c = a-b+b-c$

$= 5n+5m$

$= 5(n+m)$

$a-c$ is divisible by 5

$\Rightarrow (a,c) \in R$. $\therefore$ It is Transitive.

Hence R is an equivalence relation.

03 August 2016

UQ* 2. If R is a relation on set of ordered pairs of +ve integers such that $(a,b),(c,d) \in R$ whenever $ad = bc$. Show that R is an equivalence relation. $R = \{\{(a,b),(c,d)\} \in R \mid ad = bc\}$.

: To be an equivalence relation, R must be Reflexive, Symmetric and Transitive.

i) Reflexive:

Let $(a,b) \in R$

To prove $(a,b) R (a,b)$

$ab = ba$

$\Rightarrow (a,b) R (a,b)$

$\therefore R$ is reflexive.

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ii) Symmetric:

Let $(a,b) R (c,d)$

Then $ad = bc$

$da = cb$

$\Rightarrow cb = da$

$(c,d) R (a,b)$

$\Rightarrow R$ is symmetric.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

iii) Transitive:

Let $(a,b) R (c,d)$ and $(c,d) R (e,f)$

$(a,b) R (c,d) \Rightarrow ad = bc$

$(c,d) R (e,f) \Rightarrow cf = de$

$(ad)(cf) = (bc)(de)$

$af = be$

$\Rightarrow (a,b) R (e,f)$

$\Rightarrow R$ is Transitive

$\therefore R$ is an equivalence relation.

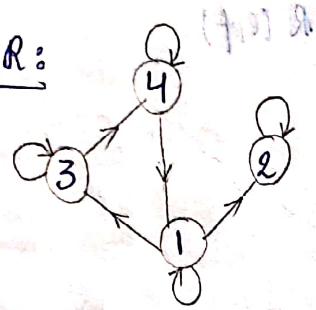
* Graphs of Relations :

1. Let $A = \{1,2,3,4\}$ and $M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$.

Draw the directed graph. Use the graph to find if R is reflexive, symmetric & Transitive. Also find indegree & outdegree of each vertex.

: The relation $R = \{(1,1)(1,2)(1,3)(2,2)(3,3)(3,4)(4,1)(4,4)\}$

Graph of R:



From the graph,

i) R is Reflexive since each vertex has a loop.

ii) R is not Symmetric since there is an edge from 1 to 2 and there is no edge from 2 to 1.

iii) R is not transitive since there are edges from 1 to 3 & 3 to 4, but there is no edge from 1 to 4.

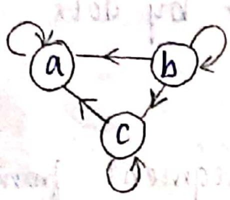
$\therefore$ R is not an Equivalence relation.

	1	2	3	4
In degree	2	2	2	2
Out degree	3	1	2	2

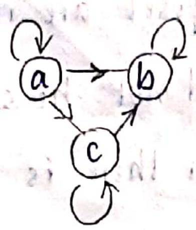
2. Let $A = \{a, b, c\}$ and $R = \{(a,a)(a,c)(b,a)(b,b)(b,c)(c,c)\}$.

Draw the digraph of $R, R^{-1}, \bar{R}$.

: Digraph of R:

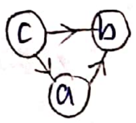


Digraph of R^{-1}:



$\bar{R} = \{(a,b)(c,a)(c,b)\}$

Digraph of $\bar{R}$:



★ Partial Order relation: (POR)

A relation on a set A is called Partial Order relation if it is
i) Reflexive
ii) Antisymmetric
iii) Transitive.

Any set A having the partial order relation is called Partially Ordered set or Poset.

Eg. $\geq$ relation is a PO on the set of integers $\mathbb{Z}$. $(\mathbb{Z}, \geq)$ is a poset.

1. Let $S = \{1, 2, 3, \ldots 9\}$. Define R on S by $R = \{(x,y) / x, y \in S, x+y = 10\}$

Verify R is an equivalence relation or Partial order relation.

: $R = \{(1,9)(2,8)(3,7)(4,6)(5,5)(6,4)(7,3)(8,2)(9,1)\}$

For every $(a,b) \in R$, $(b,a) \in R$

$\therefore$ It is only symmetric

Since $(a,a) \notin R$ it is not reflexive i.e., it is irreflexive.

Since $(1,9), (9,1) \in R$ but $(1,1)$ is not in R, it is not Transitive.

For every $(a,b) \in R$, $(b,a) \in R$ hence it is not asymmetric.

$\therefore$ R is neither equivalence nor Partial order relation.

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Because of its very general or abstract nature, set theory has many applications in other branches of mathematics. In the branch called analysis, of which differential and **integral** calculus are important parts, an understanding of **limit** points and what is meant by the **continuity** of a **function** are based on set theory. The algebraic treatment of set operations leads to **boolean algebra**, in which the operations of intersection, union, and difference are interpreted as corresponding to the logical operations "and," "or," and "not," respectively. Boolean algebra in turn is used extensively in the design of digital electronic circuitry, such as that found in calculators and personal computers. Set theory provides the basis of topology, the study of sets together with the properties of various collections of subsets.

What is the Relation?

It is a subset of the Cartesian product. Or simply, a bunch of points(ordered pairs).

Example: {(-2, 1), (4, 3), (7, -3)}, usually written in set notation form with curly brackets.

Types of Relations

Different types of relations are as follows:

- Empty Relations
- Universal Relations
- Identity Relations
- Inverse Relations
- Reflexive Relations
- Symmetric Relations
- Transitive Relations

Let us discuss all the types one by one.

Empty Relation

When there's no element of set X is related or mapped to any element of X, then the relation R in A is an empty relation also called as void. I.e R= Ø.

For example,

if there are 100 mangoes in the fruit basket. There's no possibility of finding a relation R of getting any apple in the basket. So, R is Void as it has 100 mangoes and no apples.

Universal relation

R is a relation in a set, let's say A is a universal Relation because, in this full relation, every element of A is related to every element of A. i.e $R = A \times A$.

It's a full relation as every element of Set A is in Set B.

Identity Relation

If every element of set A is related to itself only, it is called Identity relation.

I={(A, A), ∈ a}.

For Example,

When we throw a dice, the outcome we get is 36. I.e (1, 1) (1, 2), (1, 3)…..(6, 6). From these, if we consider the relation(1, 1), (2, 2), (3, 3) (4, 4) (5, 5) (6, 6), it is an identity relation.

Inverse Relation

If R is a relation from set A to set B i.e R ∈ A X B. The relation $R-1$ = {(b,a):(a,b) ∈ R}.
For Example,

If you throw two dice if R = {(1, 2) (2, 3)}, $R-1$= {(2, 1) (3, 2)}. Here the domain is the Range $R-1$ and vice versa.

Reflexive Relation

A relation is a reflexive relation If every element of set A maps to itself. I.e for every a ∈ A,(a, a) ∈ R.

Symmetric Relation

A symmetric relation is a relation R on a set A if (a,b) ∈ R then (b, a) ∈ R, for all a &b ∈ A.

Transitive Relation

If (a,b) ∈ R, (b,c) ∈ R, then (a,c) ∈ R, for all a,b,c ∈ A and this relation in set A is transitive.

Equivalence Relation

If and only if a relation is reflexive, symmetric and transitive, it is called an equivalence relation.

How to convert a Relation into a function?

A special kind of relation(a set of ordered pairs) which follows a rule i.e every X-value should be associated with only one y-value is called a Function.

Examples

Example 1: **Is A = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)} a function?**

Solution: If there are any duplicates or repetitions in the X-value, the relation is not a function.

Example 2: **Give an example of an Equivalence relation.**

Solution:

If we note down all the outcomes of throwing two dice, it would include reflexive, symmetry and transitive relations. that will be called an Equivalence relation.

* Relations :

Let A and B be any two sets

A Binary relation R from A to B is a subset of A×B.

Cartesian Product :

$A \times B = \{(a,b) / a \in A, b \in B\}$

Eg: $A = \{1,2,3\}$ $B = \{4,5\}$

$A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

Eg: Let $A = \{1,2,3\}$, $B = \{1,4\}$ and the relation is $\leq$

$R = A \leq B = \{(1,1),(1,4),(2,4),(3,4)\}$

Relation on a set A :

Let A be any set, R be the relation on A, then $R \subseteq A \times A$.

Composition of Relation :

Let A, B, C be any three sets and R is a relation from A to B and S is a relation from B to C, then R.S (R composition S) is a relation from A to C.

Eg: $R = \{(1,2)\}$ $S = \{(2,4)\}$
$R.S = \{(1,4)\}$

Problems on Relations :

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1. $R = \{(1,2),(2,4),(3,3)\}$ and $S = \{(1,3),(2,4),(4,2)\}$ are two relations. Find

i) $R \cup S$
ii) $R \cap S$
iii) $R - S$
iv) $S - R$
v) $R \oplus S$
vi) $R.S$
vii) $S.R$

:

i) $R \cup S = \{(1,2),(2,4),(3,3),(1,3),(4,2)\}$

ii) $R \cap S = \{(2,4)\}$

iii) $R - S = \{(1,2),(3,3)\}$ ~~(1,3)~~ {Remove elements of S from R}

iv) $S - R = \{(1,3),(4,2)\}$

v) $\boxed{R \oplus S = (R \cup S) - (R \cap S)}$ $\oplus \to$ Direct Sum.

$= \{(1,2),(3,3),(1,3),(4,2)\}$

vi) $R.S = \{(1,4),(2,2)\}$

vii) $S.R = \{(1,3),(4,4)\}$.

2. Determine the matrix of the relation

$R = \{(1,y),(1,z),(3,y),(4,x),(4,z)\}$

where $A = \{1,2,3,4\}$ $B = \{x,y,z\}$

: $\boxed{\text{The matrix of relation is denoted by } M_R}$:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}_{4\times 3} \end{array}$$

3. If R and S be relation on a set A represented by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find matrices that represent

i) $R \cup S$ ii) $R \cap S$ iii) $R \circ S$ iv) $S \circ R$ v) $R \oplus S$

: i) $M_{R \cup S} = M_R \cup M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

ii) $M_{R \cap S} = M_R \cap M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

iii) $M_{R \circ S} = M_R \cdot M_S$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

iv) $M_{S \circ R} = M_S \cdot M_R$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

v) $M_{R \oplus S} = M_{R \cup S} - M_{R \cap S} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

27 July 2016

U.Q.

4\. If R is the relation on $A=\{1,2,3\}$ such that $(a,b)\in R$ iff $a+b$ = even. Find the relational matrix and find $M_{R^{-1}}$, $M_{\bar{R}}$ and M_{R^2}.

: $A\times A = \{(1,1), (1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\}$

$R\subseteq A\times A$

$R=\{(1,1)(1,3)(2,2)(3,1)(3,3)\}$

$a+b$ = even.

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\end{array}$$

i) $M_{R^{-1}} = M_{R^T}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

ii) $M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(complement)

iii) $M_{R^2} = M_{R\cdot R} = M_R\cdot M_R$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

5\. If $A=\{1,2,3,4\}$ and $R=\{(1,1)(1,2)(2,3)(2,4)(3,4)(4,1)(4,2)\}$ and $S=\{(3,1)(4,4)(2,3)(2,4)(1,1)(1,4)\}$ on A.

Find i) $R\circ R$

ii) $S\circ R$

: i) $R\circ R = \{(1,2)(1,3)(1,4)(2,4)(2,1)(2,2)(3,1)(3,2)(4,1)(4,2)(4,3)(4,4)\}$

ii) $S\circ R = \{(3,1)(3,2)(4,1)(4,2)(2,4)(2,1)(1,1)(1,2)(2,2)\}$

1) If $R=\{(1,2)(2,4)(3,3)\}$ $S=\{(1,3)(2,4)(4,2)\}$

Verify (i) $\text{dom}(R\cup S) = \text{dom}(R)\cup\text{dom}(S)$

(ii) $\text{range}(R\cap S) \subseteq \text{Range}(R)\cap\text{Range}(S)$

(i) $\text{dom}(R)=\{1,2,3\}$ $\quad\text{dom}(S)=\{1,2,4\}$

$R\cup S = \{(1,2)(2,4)(1,3)(3,3)(4,2)\}$

$\text{dom}(R\cup S) = \{1,2,3,4\} = \text{dom}(R)\cup\text{dom}(S)$

(ii) $\text{Range}(R) = \{2,3,4\}$ $\quad\text{Range}(S)=\{2,3,4\}$

$R\cap S = \{(2,4)\}$ $\quad\text{Range}(R\cap S) = \{4\}$

Types of Relations :

A relation R on a set A is said to be

i) Reflexive if $(a,a) \in R \ \forall \ a \in A$
ii) Irreflexive if $(a,a) \notin R$
iii) Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$
iv) Transitive if $(a,b)(b,c) \in R \Rightarrow (a,c) \in R$.
v) Anti symmetric if $a \cancel{\neq b}$, $(a,b) \in R \cancel{\Rightarrow} (b,a) \notin R$. then $a=b$
vi) Asymmetric if $(a,b) \in R \Rightarrow (b,a) \notin R$ (Antisymmetric & Irreflexive)

★ Equivalence relation :

A relation R on a set A is called Equivalence relation if it is reflexive, symmetric, Transitive.

Eg: Set of parallel lines.

Eg: Let $A = \{1,2,3,4\}$

$R = \{(1,1)(1,2)(2,1)(1,3)(3,1)(2,2)(3,3)\}$

$S = \{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\}$

check R and S are

i) Reflexive, symmetric, Transitive

(or)

Check R and S are equivalence relation.

R:

Reflexive :

Since $(4,4) \notin R$,

R is (not Reflexive) or (Irreflexive)

Symmetric :

R is symmetric $\forall \ (a,b) \in R, (b,a) \in R$

Transitive :

Since $(2,1)(1,3) \in R$, $(2,3) \notin R$

R is not transitive .

Reflexive:

Since $(4,4) \notin S$

S is not Reflexive.

Symmetric:

S is Symmetric $\forall\ (a,b) \in S, (b,a) \in S$.

Transitive:

Since It is transitive

$\therefore$ R and S do not satisfy Equivalence relation.

3 July 2016

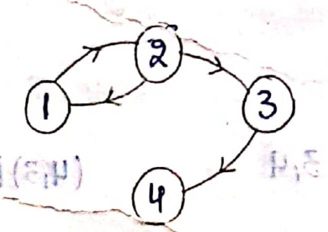
Transitive Closure:

Let R be the relation on a set A
Then the transitive closure of R is the smallest relation which contains R as a subset and it is transitive.

$$R^{\infty} = R \cup R^2 \cup R^3 \cup \dots \cup R^n$$

Eg: Let $A = \{1,2,3,4\}$ and $R = \{(1,2),(2,3),(3,4),(2,1)\}$

Find the transitive closure of R.



$R^{\infty} = \{(1,1)\ (1,3)\ (2,2)\ (2,4)\ (1,4)\ (2,3)\ (1,2)\ (2,1)\ (3,4)\}$.

Reflexive closure:

$S = R \cup \{(x,x) \mid x \in X\}$.

Symmetric closure:

$S = R \cup \{(x,y) : (y,x) \in R\}$

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Warshall's Algorithm : Used to find Transitive closure.

Step 1: First transfer to W_k all is in W_{k-1}

Step 2: Let the location $P_1, P_2, P_3 \dots$ in column of W_{k-1} where the entry is 1 and the location q_1, q_2, q_3 in row k of W_{k-1} where the entry is 1.

Step 3: Put 1 in all positions of (p_i, q_j) of W_k. This procedure is known as Warshall's algorithm.

1. Using Warshall's algorithm, find the Transitive closure of the relation $R = \{(1,1)(1,3)(1,5)(2,3)(2,4)(3,3)(3,5)(4,2)(4,4)(5,4)\}$ where $A = \{1,2,3,4,5\}$.

UQ 12m

: Let

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

K	Position of 1's in column K	Position of 1's in row K	Relation	W_R
1	1	1,3,5	(1,1) (1,3) (1,5)	$\begin{bmatrix} 1&0&1&0&1 \\ 0&0&1&1&0 \\ 0&0&1&0&1 \\ 0&1&0&1&0 \\ 0&0&0&1&0 \end{bmatrix}$
2	4	3,4	(4,3) (4,4)	$\begin{bmatrix} 1&0&1&0&1 \\ 0&0&1&1&0 \\ 0&0&1&0&1 \\ 0&1&1&1&0 \\ 0&0&0&1&0 \end{bmatrix}$
3.	1,2,3,4	3,5	(1,3) (1,5) (2,3) (2,5) (3,3) (3,5) (4,3) (4,5)	$\begin{bmatrix} 1&0&1&0&1 \\ 0&0&1&1&1 \\ 0&0&1&0&1 \\ 0&1&1&1&1 \\ 0&0&0&1&0 \end{bmatrix}$

6M

* Hasse Diagram :

It is a symmetric diagram representing a poset.

Procedure :

1. Elements of poset are represented by dots.
2. Self loops need not be shown
3. If $a \leq b$, then there is a line segment from a to b.
4. If $a \leq b$, $b \leq c$ then there is no line segment from a to c there should be line segments from a to b and b to c.

09 August 2016

1. Let $A = \{1,2,3,4,12\}$. Consider the Partial order relation of divisibility on A. Draw Hasse Diagram for the Poset $(A, /)$

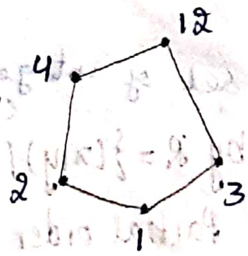
: The relation

$R = \{(1,1)(1,2)(1,3)(1,4)(1,12)(2,4)(2,12)(3,12)(4,12)\}$

The Partial order relation is

$R = \{(1,2),(1,3)(3,12)(2,4)(4,12)\}$

Hasse Diagram :



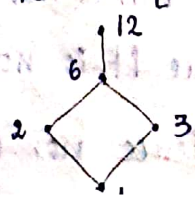
2. $X = \{1,2,3,6,12\}$ $R = \{x \leq y$ iff x divides $y\}$. Draw HD.

: The relation

$R = \{(1,2)(1,3)(1,6)(1,12)(2,6)(2,12)(3,6)(3,12)(6,12)\}$

Partial Order Relation is $R = \{(1,2)(1,3)(2,6)(6,12)(3,6)\}$

Hasse Diagram is



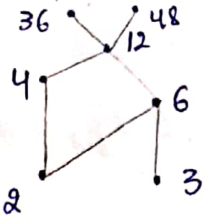
3. Let $B = \{2,3,4,6,12,36,48\}$ and S be the relation divides on B.

Draw HD of S.

: $S = \{(2,4)(2,6)(2,12)(2,36)(2,48)(3,6)(3,12)(3,36)(3,48)$
$(4,12)(4,36)(4,48)(6,12)(6,36)(6,48)(12,36)(12,48)\}$

Partial order Relation

$S = \{(2,4)(2,6)\overset{(3,6)}{}(4,12)(6,12)(12,36)(12,48)\}$



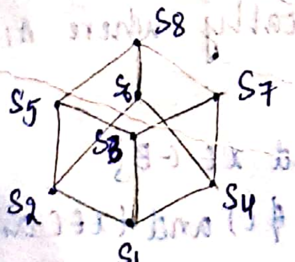
4. Draw HD for the follows $(P(A), \subseteq)$ where $A = \{1,2,3\}$.

: $P(A) = \{\underset{S_1}{\{\phi\}}, \underset{S_2}{\{1\}}, \underset{S_3}{\{2\}}, \underset{S_4}{\{3\}}, \underset{S_5}{\{1,2\}}, \underset{S_6}{\{1,3\}}, \underset{S_7}{\{2,3\}}, \underset{S_8}{\{1,2,3\}}\}$.

Partial Order Relation is

$\{\{\{\phi\},\{1\}\}, \{\{\phi\},\{2\}\}, \{\{\phi\},\{3\}\}, \{\{1\},\{1,2\}\}, \{\{1\},\{1,3\}\},$
$\{\{2\},\{2,3\}\}, \{\{2\},\{1,2\}\}, \{\{3\},\{1,3\}\}, \{\{3\},\{2,3\}\}$
$\{\{1,2\},\{1,2,3\}\}, \{\{2,3\},\{1,2,3\}\}, \{\{1,3\},\{1,2,3\}\}\}$

$R = \{(S_1,S_2)(S_1,S_3)(S_1,S_4)(S_2,S_5)(S_2,S_6)(S_3,S_5)(S_3,S_7)$
$(S_4,S_6)(S_4,S_7)(S_5,S_8)(S_6,S_8)(S_7,S_8)\}$



Least Upper Bound (Supremum)

3. Upper bound for 2,3 is 6 (going up) common max value

Lower bound for 4,6 is 2 (common min value).

Greatest Lower Bound (Infimum).

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Relations are sets of ordered pairs. Usually, the first coordinates come from a set called the domain and are thought of as inputs. The second coordinates are thought of as outputs and come from a set called the range (I actually prefer to call this the co-domain but that's a long story we don't need to go into here).

In order for a relation to be a function, each input must have one and only one output.

So, Five real-world examples:

If you look at a collection of people, you can think of there being a relation between height and age (people generally get taller as they age then remain the same height for a while and then at some point they start getting a bit shorter). This is a relation but not a function because if you input an age people of the same age will have different heights. However, for a *particular person*, height is a function of age. At any given point in their life (age) that person will be exactly one height. It's impossible for one person to be 5' 6" and 6' 2" at the same time.

In one semester at college, there is a relation between a student and their final grades; the same student can have different grades in different courses. In a particular class, there is a function from the students to their grades. The professor assigns a grade to each student and each student is assigned one and only one grade.

Generally there is a relation between time and temperature (as in the weather). If you look at a particular location, temperature is a function of time; at any moment the thermometer at that location can only read one temperature.

Once you pull up to a pump and choose your fuel, the cost of filling up your car with gas is a function of how much gas you put in your car. The cost of taking a (particular) taxi is usually a function of how long the ride is.

What is a Function?

A function is a relation which describes that there should be only one output for each input. OR we can say that, a special kind of relation(a set of ordered pairs) which follows a rule i.e every X-value should be associated to only one y-value is called a Function.

Example:

In the relation, {(-2, 3), {4, 5), (6, -5), (-2, 3)},

The domain is {-2, 4, 6} and Range is {-5, 3, 5}.

Types of Functions

In terms of relations, we can define the types of functions as:

- **One to one function or Injective function:** A function f: P → Q is said to be One to One if for each element of P there is a distinct element of Q.
- **Many to one function:** A function which maps two or more elements of P to the same element of set Q.
- **Onto Function or Surjective function:** A function for which every element of set Q there is pre-image in set P
- **One-one and Onto function or Bijective function:** The function f matches with each element of P with a discrete element of Q and every element of Q has a pre-image in P.

Bijection : (1-1 Correspondance)

A function $f: X \to Y$ is called bijection if f is both 1-1 and Onto.

Composition of function :

If $f: A \to B$ and $g: B \to C$ then the composition $g \circ f : A \to C$ is given by $[g \circ f(x)] = g[f(x)] \; \forall \; x \in A$.

Inverse f^{-1} :

If $f: A \to B$ and $g: B \to A$, then

$g \circ f : A \to A$ and hence $g \circ f = I_A$

and $f \circ g : B \to B$ and hence $f \circ g = I_B$

Here f is called inverse of g.

g is called inverse of f.

Invertible :

The necessary and sufficient condition for a function f to be invertible is f is 1-1 and onto.

Problems :

1. Let $f: R \to R$ and $g: R \to R$ defined by $f(x) = 4x - 1$, $g(x) = \cos x$. Find $g \circ f$ and $f \circ g$.

To find $g \circ f : R \to R$

$g \circ f(x) = g[f(x)] = g[4x-1]$
$= \cos(4x-1)$.

$f \circ g(x) = f[g(x)]$
$= f[\cos x] = 4(\cos x) - 1$

$\therefore g \circ f \neq f \circ g$.

2. Find fog and gof when $f: R \to R$ & $g: R \to R$ defined by

$f(x) = 2x-1$, $g(x) = x^2-2$.

∴ fog: $fog(x)$ $= f(g(x))$ $= f(x^2-2)$ $= 2(x^2-2)-1$ $= 2x^2-4-1$ $= 2x^2-5$

gof: $gof(x)$ $= g(f(x))$ $= g(2x-1)$ $= (2x-1)^2-2$ $= 4x^2-4x+1-2$ $= 4x^2-4x-1$

$fog \neq gof$.

11 August 2016

1. If $f(x) = x^2$, $g(x) = 3x$ $f: R \to R$, $g: R \to R$. Find fog and gof.

∴ $fog(x) = f[g(x)] = f(3x) = (3x)^2 = 9x^2$

$gof(x) = g[f(x)] = g(x^2) = 3(x^2) = 3x^2$

2. Show that the fn. $f: R \to R$ defined by $f(x) = 3x-1$ is a Bijection.

∴ For a fn. to be a bijection, it must be

i) 1-1:

Let $f(x_1) = f(x_2)$

$3x_1 - 1 = 3x_2 - 1$

$3x_1 = 3x_2$

∴ $x_1 = x_2$

∴ f is 1-1.

Onto:

let $y \in R$ ∃ $x = \frac{y+1}{3} \in R$

∋ $f(x) = y$

∴ f is onto

∴ f is Bijection

* Show $f(x) = x^2$ is a Bijection where $f: R \to R$.

Let $f(x_1) = f(x_2)$ ⟹ $x_1^2 = x_2^2$

$x_1 = \pm x_2$

∴ f is not 1-1.

$y = x^2$

$x = \pm y$.

Q. Show that $f: R \to R$ given by $f(x) = \sin x$ is neither 1-1 nor onto.

: 1-1:

$f(x) = f(y)$

$\sin x = \sin y$

$x \neq y$

{ $\sin x$ or $\sin y$ can take value of 0 for 2π or 4π etc. so x & y need not be equal }

f is not 1-1.

$f(x) = \sin x$

$-1 \leq \sin x \leq 1$

Range of f is $[-1,1] \neq R$

$\therefore f$ is not onto.

Part-e

4. If $f: Z \to N$ defined by $f(x) = \begin{cases} 2x-1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}$

i) Prove that f is 1-1 and onto.

ii) Determine f^{-1}.

i)

: 1-1:

$x > 0$	$x \leq 0$
$f(x) = f(y)$	$f(x) = f(y)$
$2x-1 = 2y-1$	$-2x = -2y$
$x = y$	$x = y$
f is 1-1	f is 1-1.

Onto:

$x > 0$	$x \leq 0$
let $y \in N$,	$\forall y \in N$,
$\exists\ x = \frac{y+1}{2} \in Z$	$\exists\ x = -\frac{y}{2} \in Z$
$\ni f(x) = y$	$\ni f(x) = y$

$\therefore f$ is onto.

$\Rightarrow f$ is a Bijection.

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ii) f^{-1}:

$f: Z \to N$

$f^{-1}: N \to Z$

$f(x) = y$

$x = f^{-1}(y)$

$= \begin{cases} \frac{y+1}{2}, & y = 1,3,5\dots \\ -\frac{y}{2}, & y = 0,2,4,\dots \end{cases}$

$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & x = 1,3,5,\dots \\ -\frac{x}{2}, & x = 0,2,4,\dots \end{cases}$

If $A = \{x \in R / x \neq 2\}$, $B = \{x \in R / x \neq 1\}$ and $f(x) = \frac{x}{x-2}$. Prove that f is 1-1 and onto. Also find f^{-1}.

1-1:

$f(x) = f(y)$

$\frac{x}{x-2} = \frac{y}{y-2}$

$x(y-2) = y(x-2)$

$xy - 2x = xy - 2y$

$2x = 2y$

$x = y$

$\therefore f$ is 1-1.

Onto: $f(x) = \frac{x}{x-2}$

$y = \frac{x}{x-2}$

y is defined $x \neq 2$ $\forall$ $y \neq 1 \in B$, $\exists$ $x \neq 2 \in A \ni f(x) = y$

$\therefore f$ is onto.

f^{-1}:

$x = f^{-1}(y)$, $y = \frac{x}{x-2}$

$x = \frac{2y}{y-1}$, $y \neq 1$. $\left\{ \begin{array}{l} x = (x-2)y \\ x = xy - 2y \\ xy - x = 2y \\ x(y-1) = 2y \\ x = \frac{2y}{y-1} \end{array} \right\}$

$\therefore f^{-1}(x) = \frac{2x}{x-1}$, $x \neq 1$.

4. If $A = \{x \in R \mid x \neq 1/2\}$ and $f: A \to R$ defined by

$f(x) = \frac{4x}{2x-1}$

i) Find range of f.

ii) Show that f is invertible

iii) Find domain (f^{-1}), range (f^{-1}) and f^{-1}.

: $f(x) = \frac{4x}{2x-1}$, $f(x) = y \Rightarrow x = \frac{y}{2y-4}$

i) Range of $f = \{y \in R \mid y \neq 2\}$ x is defined for $y \neq 2$.

ii) To prove that f is invertible i.e., f is 1-1 and onto :

f is 1-1 :

$f(x_1) = f(x_2)$

$\frac{4x_1}{2x_1 - 1} = \frac{4x_2}{2x_2 - 1}$

$x_1 = x_2$

f is 1-1

Onto :

$\forall\, y \neq 2 \in R, \exists\, x \neq 1/2 \ni$

$f(x) = y$

$\Rightarrow f$ is onto

$\Rightarrow f$ is invertible.

iii) Domain of f^{-1}:

$f: A \to R$, $f^{-1}: R \to A$.

$\text{dom}(f^{-1}) = \text{Range}(f)$

$= \{y \in R \mid y \neq 2\}$

Range of f^{-1}:

$= \text{dom}(f)$

$= \{x \in R \mid x \neq 1/2\}$

f^{-1}: $x = f^{-1}(y) = \frac{y}{2y-4}$, $y \neq 2$ $\therefore f^{-1}(x) = \frac{x}{2x-4}$, $x \neq 2$.

* Properties of Function:

1. If $f: A \to B$, $g: B \to C$ are functions then $gof: A \to C$ is an injection, Surjection or Bijection according to f and g are injection, Surjection and Bijection.

Proof:

Let $x \in A$, $y \in B$ and $z \in C$ such that $f(x) = y$, $g(y) = z$.

To prove gof is 1-1.

Consider $(gof)(x_1) = (gof)(x_2)$

$g[f(x_1)] = g[f(x_2)]$ since f is 1-1.

$g[x_1] = g[x_2]$ since g is 1-1.

$x_1 = x_2$

Hence gof is 1-1.

To prove gof is Onto:

Let $z \in C$.

Since f is onto, $\forall y \in B$, $\exists x \in A \Rightarrow f(x) = y$

Since g is onto, $\forall z \in C$, $\exists y \in B \Rightarrow g(y) = z$

$(gof)(x) = g[f(x)] = g(y) = z$

$\forall x \in A$, $\exists z \in Z \ni (gof)(x) = z$

$\Rightarrow gof$ is onto

$\therefore gof$ is a Bijection.

2. The inverse of function f is unique if it exists.

Proof:

If possible, let g and h be the inverse of the function f.

Let $f : A \to B$

Then $g \circ f = I_A$ and $f \circ g = I_B$

$h \circ f = I_A$ and $f \circ h = I_B$

$h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$

$\Rightarrow h = g \Rightarrow$ The inverse of f is unique.

3. If $f : A \to B$, $g : B \to C$ are invertible function then $g \circ f : A \to C$ is also invertible. and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof:

Since $f : A \to B$ and $g : B \to C$ are invertible

f and g are 1-1 and onto. Then f and g are Bijections

By property (i), $g \circ f$ is a Bijection

Hence $g \circ f$ is invertible

To prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. $\rightarrow$ (1)

Let $x \in A$, $y \in B$ and $z \in C$ such that $f(x) = y$, $g(y) = z$.

Consider

$(g \circ f)(x) = g[f(x)] = g[y] = z$

$(g \circ f)^{-1}(z) = x \rightarrow$ (1)

Since $f(x) = y$, $g(y) = z$

$x = f^{-1}(y)$, ~~$y = f^{-1}(x)$~~ ; $y = g^{-1}(z)$.

$$(f^{-1} \circ g^{-1})(z) = f^{-1}[g^{-1}(z)]$$

$$= f^{-1}[y] = x \rightarrow (2)$$

From (1) and (2)

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Show that $f: R \rightarrow R$ defined by $f(x) = \frac{x}{x+4}$ is 1-1 and onto and hence find its inverse.

check the foll. relation are function

$A = \{1, 2, 3, 4\}$

(i) $R = \{(1,1)\ (1,3)\ (2,3)\ (2,4)\ (4,4)\}$

$R = \{(1,1)\ (1,3)\ (2,3)\ (2,4)\ (4,4)\}$

(ii) $R = \{(1,1)\ (2,3)\ (3,2)\ (4,4)\}$

(iii) $R = \{(4,1)\ (3,1)\ (2,1)\ (1,1)\}$

A function is merely a “machine” that generates some output in correlation to a given input. So, if $f(x)=2x+1f(x)=2x+1$, then, $f(3)=7f(3)=7$.
Understanding this behavior is essential to recognizing the variety of input\output correlations in the real world. Again, all a function does is provide a mathematical way to model or represent a situation where a certain input will give a certain output.

Driving a Car - When driving a car, your location is a function of time. You see, *Quantum Physics* notwithstanding, you can’t be in two places at once. Therefore, the *vehicle’s position is a function of time.*

Height of a Person - Forensic scientists can determine the height of a person based on the length of their femur. Here is one such function: $h(f)=2.47f+54.10h(f)=2.47f+54.10$, $\pm3.72\pm3.72$cm. Where ff is the length of the femur bone.

A mathematical function can be expressed as $f(x)=yf(x)=y$. we say that it is a function if the value of the variable depends on the value of the other variable. In this case we are presenting xx as the independent variable and yy as the dependent.
A simple example applicable for normal life is the following:

Imagine that your are running at a certain speed and you want to know what is the total distance covered. How do you determine what is the total distance? Well, you can notice that the total distance depends on the time spent running (not considering other factors of course).

f(time)=Distancef(time)=Distance
Another example can be when you are at the gym an you try to bench press. The amount of bench presses that you can do depends on the strength.

f(strength)=benchpress