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UNIT-3.

Mathematematical Logic.

Propositions & Logical Operators - Truth tables - proposition generated by equivalence & implication - Tautologies - Law of Logics - propositional Calculus - Direct ~~proposition~~ Product - Conditional Conclusion - Mathematical induction.

26/9. PROPOSITION:

Proposition is a declarative Sentence that is true or false but not Simultaneously.

Ex: integer 5 is a prime number - True

Today is a rainy day - False.

Logical Operators:

A Systematic study of arguments by extensive use of Symbols is known by Logical Symbols.

Ex: AND, OR, NOT, IF, THEN, etc

Name	Symbol
union	$\vee$, $\cup$, $+$, OR
Intersection	$\wedge$, $\cap$, AND
Negation	$\neg$, $\sim$, Complement
Conditional	$\rightarrow$, $\Rightarrow$, IF . then

Name	Symbol
Biconditional	↔, ⇔, IFF

Truth Table for Logical Operator:

Conjunction (intersection)

Disjunction (union)

Conditional (→)

Biconditional (↔)

Negation (¬)

Conjunction (∧)			DISJUNCTION (∨)	CONDITIONAL (→)
p	q	p ∧ q	p ∨ q	p → q
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

NEGATION	
p	¬p
T	F
F	T

Problems:

1. Construct Truth table for the following proposition where p, q are proposition.

(i) $p \vee q \rightarrow p \wedge q$

(ii) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

(iii) $(\neg p \vee \neg q) \leftrightarrow (p \leftrightarrow q)$

(iv) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$

(i) $\overset{A}{p \vee q} \rightarrow \overset{B}{p \wedge q}$

Both True True value.

p	q	$\overset{A}{p \vee q}$	$\overset{M}{p \wedge q}$	$A \rightarrow B$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(ii) $\overset{A}{(q \rightarrow \neg p)} \leftrightarrow \overset{B}{(p \leftrightarrow q)}$

p	q	$\neg p$	$q \rightarrow \neg p$	$p \rightarrow q$	$A \leftrightarrow B$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	T	T	T

(iii) $(\neg p \vee \neg q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$\neg q$	$\overset{A}{\neg p \vee \neg q}$	$\overset{B}{p \leftrightarrow q}$	$A \leftrightarrow B$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

(iv) $\overset{C}{(p \leftrightarrow q)} \leftrightarrow ((p \wedge q) \vee \overset{D}{(\neg p \wedge \neg q)})$

p	q	$p \leftrightarrow q$	$\overset{A}{p \wedge q}$	$\overset{B}{\neg p \wedge \neg q}$	$A \cup B$	$C \leftrightarrow D$
TF	TF	T	T	F	T	T
TF	FT	F	F	F	T	F
FT	TF	F	F	F	F	F
FT	FT	T	F	T	T	T

TAUTOLOGY: (Only True)

A Compound proposition is called tautology if every proposition has a truth Assignment

CONTRADICTION (Only false)

A compound proposition has false assignment then it is called contradiction.

Problems:

(1) $p \wedge q \rightarrow p \vee q$

(2) $(p \wedge q) \vee (\neg p \wedge q) \vee ((p \vee \neg q) \vee (\neg p \wedge \neg q))$

(i) $p \wedge q \rightarrow p \vee q$

p	q	$p \wedge q$	$p \vee q$	$A \rightarrow B$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	F	

27/9 Problem:

Using Truth table determine whether the Compound proposition is a Tautology or Contradiction. $\neg(p \to r) \wedge r \wedge (p \to q)$

p	q	r	$p \to r$	$\neg(p \to r)$	$\neg(p \to r) \wedge r$ (A)
T	T	T	T	F	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	F	T	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

$p \to q$ (B)	$\neg(p \to r) \wedge r \wedge (p \to q)$ (A) $\wedge$ (B)
T	F
T	F
F	F
F	F
T	F
T	F
T	F
T	F

$\neg(p \to r) \wedge r \wedge (p \to q)$ is a Contradiction

Q. P.T $\neg(p \vee (\neg p \vee q))$ & $\neg p \wedge \neg q$ are logically equivalent.

i.e $\neg(p \vee (\neg p \vee q)) \Leftrightarrow \neg p \wedge \neg q$.

p	q	$\neg p$	$\neg p \vee q$	$p \vee (\neg p \vee q)$ (A)	$\neg A$	$\neg q$	$\neg p \wedge \neg q$ (B)
T	T	F	T	T	F	F	F
T	F	F	F	T	F	T	F
F	T	T	T	T	F	F	F
F	F	T	T	T	F	T	F
~~T~~	~~T~~	~~F~~					
~~T~~	~~F~~	~~F~~					
~~F~~	~~T~~	~~T~~					
~~F~~	~~F~~	~~T~~					

$\therefore \neg(p \vee (\neg p \vee q)) \Leftrightarrow \neg p \wedge \neg q$

Q. Using truthtable Determine whether the compound proposition is a tautology or contradiction.

$(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$.

p	q	r	$p \vee q$	$p \to r$	$(p \vee q) \wedge (p \to r)$	$q \to r$	$A \cap B$	$A \cap B \to r$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Properties in Logics:

1. Commutative law:

$$p \wedge q \Leftrightarrow q \wedge p \; ; \; p \vee q \Leftrightarrow q \vee p$$

2. Associative law:

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \; ; \; p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

3. Distributive law:

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) ;$$

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

4. De Morgan's law:

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \; ; \; \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

5. Negation law:

$$\neg(\neg p) = p$$

6. Idempotent law:

$$p \vee p \Leftrightarrow p$$

7. Identity law

$$p \vee F \Leftrightarrow p.$$

8. Dominant law:

$$p \vee T \Leftrightarrow T$$

9. Complement law:

$$p \vee \neg p \Leftrightarrow T$$

10. Absorption law:

$$p \vee (p \wedge q) \Leftrightarrow p.$$

Q. Without using Truthtable, prove that equivalence.

$$(\neg p \vee q) \wedge (p \wedge p \wedge q) \Leftrightarrow p \wedge q.$$

$(\neg p \vee q) \wedge (p \wedge p \wedge q) \Leftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ (By Associative

$\Leftrightarrow (\neg p \vee q) \wedge (p \wedge q)$

$\Leftrightarrow (p \wedge q) \wedge (\neg p \vee q)$ (By Commutative)

$\Leftrightarrow (p \wedge \neg p \vee q) \wedge (q \wedge \neg p \vee q)$ (By Distributive)

$\Leftrightarrow (p \wedge \neg p) \vee q \wedge (q \wedge \neg p \vee q)$

$p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

F	q
	T
F	
F	F

$\Leftrightarrow F \vee q \wedge (\neg p \wedge (q \vee q))$ (By Associative)

$\Leftrightarrow (\cancel{p} F \vee q) \wedge (\neg p \wedge q)$

$\Leftrightarrow F \wedge (\neg p \wedge q) \vee q \wedge (\neg p \wedge q)$

$\Leftrightarrow p \wedge q$

Tautological implications

(i) $A \Rightarrow B$

(ii) $A \Leftrightarrow B$

A Compound proposition A is said to imply tautologically B, if B is true whenever A is true

(i.e) $A \Rightarrow B$ or $A \to B$ is a Tautology

Two proposition A & B are logically equivalent if & only if $A \leftrightarrow B$ is a tautology or have identical truth values

(i.e) $A \Leftrightarrow B$ iff $A \Leftrightarrow B$ is a Tautology.

Some Equivalence formula:

Equivalent Symbols

$\Leftrightarrow$ / $\equiv$

1. $p \to q \Leftrightarrow \neg p \vee q$ (*)
2. $p \to q \Leftrightarrow \neg q \to \neg p$
3. $p \vee q \Leftrightarrow \neg p \to q$

without using Truthtable.

1) $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

Sol

$P \to (Q \to R) \Leftrightarrow \neg P \vee (Q \to R)$ By (*)

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ By (*).

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ By Associative

$\Leftrightarrow \neg (P \wedge Q) \vee R$ By De Morgan

$\Leftrightarrow (P \wedge Q) \to R$ (By (*))

By ①

$p \to q \Leftrightarrow \neg p \vee q$

p	q	$p \to q$ (A)	$\neg p$	$\neg p \vee q$ (B)	$A \leftrightarrow B$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

2. $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$

Sol $(P \to R) \wedge (Q \to R) \Leftrightarrow (\neg P \vee R) \wedge (Q \to R)$ By ①.

$\Leftrightarrow (\neg P \vee R) \wedge (\neg Q \vee R)$ By ①.

$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$ (By Distributive)

$\Leftrightarrow \neg(P \vee Q) \vee R$ (By DeMorgan's)

$\Leftrightarrow (P \vee Q) \to R$

3. $P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$

Sol $(P \to Q) \vee (P \to R) \Leftrightarrow (\neg P \vee Q) \vee (P \to R)$ By ①.

$\Leftrightarrow (\neg P \vee Q) \vee (\neg P \vee R)$ By ①.

$\Leftrightarrow \neg P \vee (Q \vee R)$

$\Leftrightarrow P \to (Q \vee R)$ By ①.

4 $((P \vee \neg P) \to Q) \to ((P \vee \neg P)) \to R \Rightarrow Q \to R$

$P \vee \neg P \Rightarrow$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(By Complement law)

$(T \to Q) \to (T \to R)$

4. $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R) \Rightarrow Q \to R$

Sol $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R))$

$\Rightarrow (T \to Q) \to (T \to R)$

$\Rightarrow (\neg T \vee Q) \to (\neg T \vee R)$

$\Rightarrow (F \vee Q) \to (F \vee R)$

$\Rightarrow Q \to R$

5. $(p \vee q) \wedge (p \to r) \wedge (q \to s) \Rightarrow s \vee r$

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Equivalence involving Conditionals.

1. $p \to q \Leftrightarrow \neg p \vee q$
2. $p \to q \Leftrightarrow \neg q \to \neg p$
3. $p \vee q \Leftrightarrow \neg p \to q$
4. $p \wedge q \Leftrightarrow \neg(p \to \neg q)$
5. $\neg(p \to q) \Leftrightarrow p \wedge \neg q$
6. $(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \Leftrightarrow p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \Leftrightarrow (p \wedge q) \to r$

Equivalence involving Bi-Conditionals.

1. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

Solve:

1. $(p \wedge q) \to (p \vee q)$ is a Tautology without using Truthtable.

Sol

$(p \wedge q) \to (p \vee q) \Leftrightarrow \neg(p \wedge q) \vee (p \vee q)$ (By ① of conditional)

$\Leftrightarrow (\neg p \vee \neg q) \vee (p \vee q)$ (By DeMorgan's law)

$\Leftrightarrow (\neg q \vee \neg p) \vee (p \vee q)$ (By commutative)

$\Leftrightarrow \neg q \vee (\neg p \vee p) \vee q$ (Associative)

$\Leftrightarrow \neg q \vee T \vee q$ (By Dominant)

$\Leftrightarrow (\neg p \vee T) \vee q \Leftrightarrow T \vee q \Leftrightarrow T$

Q. Show that

② ⇒ $(\neg p \wedge (\neg q \wedge r)) \vee (q \vee r) \vee (p \wedge r) \Leftrightarrow r$

③ ⇒ $p \to (q \to p) \Leftrightarrow \neg p \to (p \to q)$

④ ⇒ $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

② $(\neg p \wedge (\neg q \wedge r)) \wedge (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow (\neg p \wedge \neg q) \wedge r \vee (q \wedge r) \vee (p \wedge r)$ (By Associative

$\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee (q \vee p) \wedge r$ (Distributive)

$\Leftrightarrow ((\neg p \wedge \neg q) \vee (p \vee q)) \wedge r$ (Distributive & commutative)

$\Leftrightarrow \neg(p \vee q) \vee (p \vee q) \wedge r$ (Demorgan's)

$\Leftrightarrow T \wedge r$ [$T p \vee R \Leftrightarrow T$]

$\Leftrightarrow r$ //.

③ $p \to (q \to p) \Leftrightarrow \neg p \to (p \to q)$

~~$p \to (q \to p) \Leftrightarrow \neg p \vee (\neg q \vee p)$~~

L.H.S $p \to (q \to p) \Leftrightarrow \neg p \vee (q \to p)$ (By condition ①)

$\Leftrightarrow \neg p \vee (\neg q \vee p)$ (By ")

$\Leftrightarrow \neg p \vee (p \vee \neg q)$ (By Commutative)

$\Leftrightarrow (\neg p \vee p) \vee \neg q$ (By Associative)

$\Leftrightarrow T \vee \neg q$

$\Leftrightarrow T$

R.H.S $\neg p \to (p \to q)$

$\Leftrightarrow \neg\neg p \vee (p \to q)$ By ①.

$\Leftrightarrow p \vee (\neg p \vee q)$ By ①.

$\Leftrightarrow (p \vee \neg p) \vee q$ (By Associative)

$\Leftrightarrow T \vee q$

$\Leftrightarrow T$

④ $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

$\neg(p \leftrightarrow q) \Leftrightarrow \neg((p \to q) \vee (q \to p))$

$\Leftrightarrow \neg(p \to q) \vee \neg(q \to p)$

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$ (By De Morgan's)

H.W

1. $[(p \vee \neg p) \to q] \to [(p \vee \neg p) \to r] \Leftrightarrow q \to r$.
2. $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$.
3. $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$

TABLE I :

1.	Idempotent law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2.	Identity law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3	Dominant	$p \vee T \equiv T$	$p \wedge F \equiv F$
4.	Complement	$p \vee \neg p \equiv T$	$p \wedge \neg p$
5.	Commutative	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6	Associative	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7.	Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Absorption	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	De Morgan	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

TABLE 2 :

$p \rightarrow q \Leftrightarrow \neg p \vee q$	$(p \rightarrow q) \wedge (q \rightarrow r) \Leftrightarrow (p \vee q) \rightarrow r$
$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	$(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$
$p \vee q \Leftrightarrow \neg p \rightarrow q$	$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$
$p \wedge q \Leftrightarrow \neg(p \rightarrow \neg q)$	
$\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg p$	
$(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$	

TABLE 3:

1. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
2. $p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \leftrightarrow q) \Leftrightarrow \neg q$.

TABLE 4:

1. $p \wedge q \Rightarrow p$
2. $p \wedge q \Rightarrow q$

(Simplification)

3. $p \Rightarrow p \vee q$
4. $q \Rightarrow p \vee q$

(Addition)

5. $\neg p \Rightarrow p \rightarrow q$
6. $q \Rightarrow p \rightarrow q$
7. $\neg(p \rightarrow q) \Rightarrow \neg q$
8. $\neg(p \rightarrow q) \Rightarrow p$
9. $p \wedge (p \rightarrow q) = q$ (Modus Ponens)
10. $\neg q \wedge (p \rightarrow q) \rightarrow \neg q$ (Modus tollens)
11. $\neg p \wedge (p \vee q) = q$ (Disjunctive Syllogism)
12. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$ (Hypothetical Syllogism)
13. $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)$

$$\Rightarrow r$$

(Dynamic)

Theory of Inference:

Inference Theory is concerned with getting a conclusion from certain hypothesis or basic assumptions or premises by applying principle of reason or rule of inference.

Premises:

Premises is a statement which is assumed to be true.

Formal proof:

The process of driving conclusion from a set of premises by the accepted rules: Types: (i) Direct (ii) InDirect

Rule P : A Premises may be introduced at any step in the derivation.

Rule T : A formulae S is introduced in the derivation, if S is tautologically implied by one or more premise in the derivation.

Problem:

(, is always And op)
"$\wedge$"

Q. Show that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.

Sol

Step	Statement	Rule	Reason
1	$P \to Q$	P	given
2	$Q \to R$	P	given
3	P	P	given.
4	$P \to R$	T	Statement (1,2) Hypothecal Syllogism
5	R	T	Statement (3,4) Modus Ponens $P \wedge (P \to R) = R$

Q To show that $P \to S$ follows logically from the premises $\neg P \vee Q$, $\neg Q \vee R$ and $R \to S$

Step	Statement	Rule	Reason
1	$\neg P \vee Q$	P	given
2	$\neg Q \vee R$	P	given
3	$R \to S$	P	given
4	$P \to Q$	T	Table 2 ①.
5	$Q \to R$	T	Table 2 ①

6	$P \to R$	T	(4,5) Hypothesis Syllogism
7	$P \to S$	T	(6.3) Hypothesis Syllogism