Unit - II.

LOGIC

Logic is the discipline that deals with the methods of reasoning.

Propositions.

A declarative sentence which is either true or false, but not both, is called a proposition (statement).

$p, q, r, s, \ldots$ are used to denote propositions.

Eg: 1. New Delhi is the capital of India. (T)

2. ~~xxxxx~~. $2+2=3$. (F)

Example for not propositions.

1. How beautiful Rose is!
2. What time is it?
3. $x+y=z$ (Not)(values are not assigned)

Notation:

*) If a proposition is true, then its truth value is denoted by T or 1.

*) If a proposition is false, then its truth value is denoted by F or 0.

Connectives:

① If p and q are two propositions then the proposition "p and q" denoted by $p \wedge q$ is called the conjunction of p and q and is defined as if it is true when p and q are true, otherwise false.

The truth table of $p \wedge q$ is as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(2) Let p and q be two propositions. Then the proposition "p or q" denoted by $p \vee q$ is called the disjunction of p and q and is defined as if it false when p and q are both false, otherwise true.

The truth table of p or q ($p \vee q$) is as follows.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

③ Let p be any proposition, then its negation is denoted by $\neg p$ or $\sim p$. Its truth table is as follows.

p	$\neg p$
T	F
F	T

Conditional and biconditional propositions.

① If p and q are two propositions, then "if p, then q" is denoted by $p \to q$ is called a conditional proposition, which is false when p is true and q is false, otherwise true.

It's truth table is as follows:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

② If p and q are two proposition then "p if and only if q" is denoted by $p \leftrightarrow q$ which is truth when p and q have same values, Otherwise false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Tautology and Contradiction.

A compound proposition which is always true for every truth values is called a tautology.

A compound proposition which is always false for every truth values is called a contradiction.

Example:

$p \vee \neg p$ is a tautology and $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Contingency :

If a proposition is neither a tautology nor a contradiction is called a contingency.

Problems : Contingency

Eg: Construct a truth table for the following.

1. $(p \vee q) \rightarrow (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

7) Construct the truth table for $p \to (p \vee q)$.

p	q	$p \vee q$	$p \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

8) Construct the truth table for

$s: (q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\neg p$	$q \to \neg p$	$p \leftrightarrow q$	s
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

9) P.T. $(p \wedge q) \to (p \vee q)$ is a tautology.

10) P.T. $(\neg p \wedge p) \wedge q$ is a contradiction.

<u>Defn</u>:

<u>Equivalence of Proposition</u>:

Two compound propositions $A(P_1, P_2, \ldots, P_n)$ and $B(P_1, P_2, \ldots, P_n)$ are said to be logically equivalent or simply equivalent, if they have identical truth table.

It is denoted by $A \Leftrightarrow B$ or

$A \equiv B$.

Eg:

12) P.T. $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$p \vee q$	(1) $\neg(p \vee q)$	$\neg p$	$\neg q$	(2) $\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

From (1) & (2),

$\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Duality Law:

The dual of a compound proposition is obtained by replacing each $\wedge$ by $\vee$, each $\vee$ by $\wedge$, each T by F and each F by T.

The dual of A is denoted by A^*.

Duality Theorem:

If $A(P_1, P_2, \ldots, P_n) \equiv B(P_1, P_2, \ldots, P_n)$, where A and B are compound propositions, then $A^*(P_1, P_2, \ldots, P_n) \equiv B^*(P_1, P_2, \ldots, P_n)$.

Algebra of Propositions.

Laws. $\Leftrightarrow$ or $\equiv$

Sl.No.	Name of the law	Primal Form	Dual form
1.	Idempotent law	$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
2.	Identity law	$p \vee F \Leftrightarrow p$	$p \wedge T \Leftrightarrow p$
3.	Dominant law	$p \vee T \Leftrightarrow T$	$p \wedge F \Leftrightarrow F$
4.	Complement law	$p \vee \neg p \Leftrightarrow T$	$p \wedge \neg p \Leftrightarrow F$
5.	Commutative law	$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
6.	Associative law	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
7.	Distributive law	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
8.	Absorption law	$p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$
9.	Demorgan's law	$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

Equivalences involving conditionals

1. $p \rightarrow q \Leftrightarrow \neg p \vee q$
2. $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$
2. $(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$
3. $(p \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (p \vee q) \rightarrow r$
4. $(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$
5. $(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

Equivalences using biconditionals.

(9) $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

1. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
2. $p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \Leftrightarrow \neg q \; \neg p \leftrightarrow \neg q$
4. $\neg (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$ (1) Define Tauto

Tautological Implication.

A compound proposition A is said to tautologically imply or (simply) imply B if B is true whenever A is true (or) $A \to B$ is a tautology.

It is denoted by $A \Rightarrow B$.

(19) P.T. $p \Rightarrow (p \vee q)$.

To prove $p \to (p \vee q)$ is a tautology

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$\therefore p \to (p \vee q)$ is a tautology.

$\therefore p \Rightarrow (p \vee q)$.

(20) P.T. $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$

(21) P.T. $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$

2. $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

Soln:

$(\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$\Leftrightarrow (\neg p \vee q) \wedge ((p \wedge p) \wedge q)$ Associative la

$\Leftrightarrow (\neg p \vee q) \wedge (p \wedge q)$ Idempotent law

$\Leftrightarrow (p \wedge q) \wedge (\neg p \vee q)$ Commutative

$\Leftrightarrow ((p \wedge q) \wedge \neg p) \vee ((p \wedge q) \wedge q)$ distributiv

$\Leftrightarrow ((p \wedge \neg p) \wedge q) \vee (p \wedge (q \wedge q))$ Associative

$\Leftrightarrow (F \wedge q) \vee (p \wedge q)$ Complement & Idempotent

$\Leftrightarrow F \vee (p \wedge q)$ dominant law

$\Leftrightarrow p \wedge q$, dominant law.

(23) Using Truth table, P.T.

$$(p \to (q \to s)) \wedge (\neg r \vee p) \wedge q \Rightarrow r \to s.$$

(24) P.T. $(p \to q) \Leftrightarrow (\neg p \vee q)$.

(25) Construct the truth table for

$$((p \to q) \to r) \to s.$$

(26) Construct the truth table for

$$\neg (p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \to r)).$$

Normal Forms. (Tautological Implication Refer last)

When the no. of propositions $p_1, p_2, \ldots, p_n$ increases, better method be use to reduce into standard forms is called normal forms. There are two types. They are

1. Disjunctive Normal Forms
2. Conjunctive normal form.

Defn: Elementary product

A product of the variables and their negations is called an elementary product.

Eg: $p, \neg p, p \wedge \neg p, \neg p \wedge q, \ldots$ are some examples.

Elementary Sum:

A sum of the variables and their negations is called as elementary sum.

Eg: $p, \neg p, p \vee q, \neg p \vee \neg q \ldots$

Disjunctive Normal Form (DNF)

Sum of elementary products.

Conjunctive Normal Form (CNF)

Product of elementary sum.

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Procedure to obtain DNF & CNF.

Step : 1

If the connectives $\rightarrow$ and $\leftrightarrow$ are present in the given formula they are replaced by $\wedge$, $\vee$ and $\neg$.

i.e, if $p \rightarrow q$ is replaced by $\neg p \vee q$

and $p \leftrightarrow q$ is replaced by $(p \wedge q) \vee (\neg p \wedge \neg q)$

~~$(p \rightarrow q) \wedge (q \rightarrow p)$ and is replaced by~~

~~$(\neg p \vee q) \wedge (\neg q \vee p)$.~~

(Or)

$(\neg p \vee q) \wedge (\neg q \vee p)$

Step : 2

If the negation is present before the given formula or a part of the gn. formula, apply the DeMorgan's law.

Step : 3

If necessary, distributive law and the complement laws are applied.

Step : 4

If there is an elementary product which is equivalent to the truth value F in the DNF, it is omitted.

III^{ly}, If there is an elementary sum which is equivalent to the truth value T in the CNF, it is omitted.

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(30) Find the DNF of $q \to (q \to p)$

Soln:

$$q \to (q \to p) \equiv q \to (\neg q \vee p)$$

$$\equiv \neg q \vee (\neg q \vee p)$$

$\equiv (\neg q \vee \neg q) \vee p$

$\equiv \neg q \vee p.$

(31) Find the CNF of $\neg(p \vee q) \leftrightarrow (p \wedge q)$.

Soln :

$\neg(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv \left(\neg(p \vee q) \wedge (p \wedge q)\right) \vee \left(\neg(\neg(p \vee q)) \wedge \neg(p \wedge q)\right)$

$\equiv \left[(\neg p \wedge \neg q) \wedge (p \wedge q)\right] \vee \left[(p \vee q) \wedge (\neg p \vee \neg q)\right]$

$\equiv \left[(\neg p \wedge p) \wedge (\neg q \wedge q)\right] \vee (p \vee q) \wedge (\neg p \vee \neg q)$ (Associative)

$\equiv (F \wedge F) \vee (p \vee q) \wedge (\neg p \vee \neg q)$

$\equiv F \vee (p \vee q) \wedge (\neg p \vee \neg q)$

$\equiv (p \vee q) \wedge (\neg p \vee \neg q)$

which is the required CNF.

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Principal Disjunctive and Principal Conjunctive Normal Forms (PDNF) & (PC

Definitions

Given a number of variables, the products in which each of its variable or its negation, but not both, occurs only once are called the **minterms.**

For 2 variables p and q, the possible minterms are $p \wedge q$, $\neg p \wedge q$, $p \wedge \neg q$, $\neg p \wedge \neg q$.

For 3 variables p, q and r, the possible minterms are.

$p \wedge q \wedge r$, $\neg p \wedge q \wedge r$, $p \wedge \neg q \wedge r$, $p \wedge q \wedge \neg r$, $\neg p \wedge \neg q \wedge r$, $\neg p \wedge q \wedge \neg r$, $p \wedge \neg q \wedge \neg r$, $\neg p \wedge \neg q \wedge \neg r$.

Defn:

Given a number of variables, the sums in which each of its variable or its negation, but not both occurs only once are called the maxterms.

For 2 variables p and q, the possible max terms are $p \vee q$, $\neg p \vee q$, $p \vee \neg q$, $\neg p \vee \neg q$.

For 3 variables p, q and r, the possible max terms are $p \vee q \vee r$, $\neg p \vee q \vee r$, $p \vee \neg q \vee r$, $p \vee q \vee \neg r$, $\neg p \vee \neg q \vee r$, $\neg p \vee q \vee \neg r$, $p \vee \neg q \vee \neg r$, $\neg p \vee \neg q \vee \neg r$.

Defns:

A formula consisting of sum of minterms in the variables only is known as PDNF. (sum of Products canonical form)

A formula consisting of product of maxterms in the variables only is known as PCNF. (Product of Sums canonical form).

Working Procedure:

① To find the PDNF, first find the DNF.
PCNF / CNF

② Introduce $\wedge T$ in missing terms.
$\vee F$

③ Apply $T \equiv P \vee \neg P$.
$F \equiv P \wedge \neg P$

④ Apply Distributive law.

⑤ Identical terms are deleted (eg: $P \vee P \Leftrightarrow P$)

(1) Using truth table, find the PDNF of $p \vee \neg q$.

Soln:

p	q	$\neg q$	$p \vee \neg q$	Minterms	Maxt
T	T	F	T ✓	$p \wedge q$	
T	F	T	T ✓	$p \wedge \neg q$	
F	T	F	F		$\neg p \vee$
F	F	T	T ✓	$\neg p \wedge \neg q$	

PDNF $\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge$

(2) Using truth table, find the PCNF of $p \leftrightarrow q$.

Soln:

p	q	$p \leftrightarrow q$	Min term	Max
T	T	T ✓	$p \wedge q$	
T	F	F		$p \vee \neg$
F	T	F		$\neg p \vee q$
F	F	T ✓	$\neg p \wedge \neg q$	

PDNF $(p \wedge q) \vee (\neg p \wedge \neg q)$

PCNF $(p \vee \neg q) \wedge (\neg p \vee q)$

(3) Find the PDNF & PCNF of

$(q \vee (p \wedge r)) \wedge (\neg(p \vee r) \wedge q)$

(4) Find the PDNF and PCNF of

$(p \wedge q) \vee (\neg p \wedge q) \vee (q \wedge r)$.

(5) Without using truth table find the PCNF of $p \leftrightarrow q$.

Soln:

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$

which is the required PCNF.

(6) Without using truth table, find the PDNF and PCNF of

$(p \wedge q) \vee (\neg p \wedge q \wedge r)$.

Soln:

$$(p \wedge q) \vee (\neg p \wedge q \wedge r)$$

$$\Leftrightarrow ((p \wedge q) \wedge T) \vee (\neg p \wedge q \wedge r)$$

$$\Leftrightarrow ((p \wedge q) \wedge (r \vee \neg r)) \vee (\neg p \wedge q \wedge r)$$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$

which is the required PDNF.

Let $S \sim (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$.

$\neg S \sim (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

PCNF is

$\neg\neg S : (\neg p \vee q \vee \neg r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$.

(7) Without using truth table, find the PDNF of $p \vee \neg q$.

Soln:

$p \vee \neg q \Leftrightarrow (p \wedge T) \vee (\neg q \wedge T)$

$\Leftrightarrow (p \wedge (q \vee \neg q)) \vee (\neg q \wedge (p \vee \neg p))$

$$\Rightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg q \wedge p)$$
$$\vee (\neg q \wedge \neg p)$$

which is the required PDNF.