UNIT – II

Logic

Logic

A proposition or statement is a declarative sentence which is either true or false but not both.

Negation $\neg$ or $\sim$

If P is the statement then its negation is denoted by $\neg P$ or $\sim P$ and defined by the truth table

P	$\neg P$ (or) $\sim P$
T	F
F	T

Disjunction : $\vee$ (OR)

If P and Q be two statement then P & Q denoted by $P \vee Q$ (read P or Q) and defined by the truth table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction : $\wedge$ (AND)

P and Q are statements then P & Q denoted by $P \wedge Q$ (read as P & Q) and defined by the truth table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Conditional Statement → (If then)

P & Q are 2 statement, the P Condition is true denoted by $P \to Q$ is called Conditional Statement (read as if P and Q) and defined by the truth table.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: $P \to Q$ is false when P is true and Q is false otherwise in all the cases it is true.

BiConditional ⇔

P and Q are 2 statements, then P ~~Condition is true~~ ~~denoted~~ by if and only Q Biconditional Statement (read as if P iff Q) and defined by truth table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: If both P and Q have the same truth values then P biCondition is true otherwise it is false.

Write Symbolic form

(i) Ram is Rich but happy

(ii) Ram is rich or unhappy

(iii) Ram is neither Rich nor happy.

Soln P : Ram is rich

Q : Ram is happy

(i) $P \vee Q$

(ii) $P \wedge \neg Q$

(iii) $\neg P \vee \neg Q$

(2) If either Ram takes Calculus or Ravi takes sociology then Kumar will take English.

Soln P : Ram takes calculus

Q : Ravi takes sociology

R : Kumar will take English.

$(P \vee Q) \rightarrow R$

(3) Construct a truth table negative $\neg P \vee \neg Q$

Soln

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Tautology :

A statement is always true for the truth values of the components is called tautology

Contradiction :

A statement which is always false for the truth values of the components is called contradiction.

Prove that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

Soln

P	Q	$P \wedge Q$	$P \vee Q$	$S: (P \wedge Q) \rightarrow (P \vee Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

$\therefore$ $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology

Prove that negation $(\neg P \wedge P) \wedge Q$ is a Contradiction.

Soln

P	Q	$\neg P$	$\neg P \wedge P$	$(\neg P \wedge P) \wedge Q$
T	T	F	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

$\therefore$ $(\neg P \wedge P) \wedge Q$ is a Contradiction.

Prove that $(p \rightarrow q) \rightarrow q \Leftrightarrow p \vee q$

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow q$	$p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

$\therefore$ $(p \rightarrow q) \rightarrow q \Leftrightarrow p \vee q$

Logical Equivalence (or) Equivalence Rules:

1. Absorption Laws

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \wedge (p \vee q) \Leftrightarrow p$$

2. Idempotent Laws

$$p \vee p \Leftrightarrow p$$
$$p \wedge p \Leftrightarrow p$$

3. Commutative Laws

$$p \wedge q \Leftrightarrow q \wedge p$$
$$p \vee q \Leftrightarrow q \vee p$$

4. Associative Law

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

5. Distributive Laws

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

6. Demorgan's Law

$$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$$
$$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

Rules: (i) $p \to q \Leftrightarrow \neg p \vee q$

(ii) $p \to q \Leftrightarrow \neg q \to \neg p$

(iii) $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

(iv) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

(v) $\neg(\neg p) = p$

(vi) $p \vee T = T$, $p \wedge T = p$

(vii) $p \vee F = p$, $p \wedge F = F$

(viii) $P \wedge \neg P = F$

$P \vee \neg P = T$

(1) $(P \to q) \wedge (\neg P \to q) \Leftrightarrow q$

$(\neg P \vee q) \wedge (\neg P \to q)$ $\quad (p \to q \Leftrightarrow \neg P \vee q)$

$= (\neg P \vee q) \wedge (\neg(\neg P) \vee q)$ $\quad (\neg(\neg P) \Leftrightarrow P)$

$= (\neg P \wedge P) \vee q$ (Distributive Law)

$= F \vee q$ $\quad (P \wedge \neg P = F)$

$= q$ $\quad (P \vee F = P)$

L.H.S = R.H.S

$(P \to q) \wedge (\neg P \to q) \Leftrightarrow q$

(2) P.T $\neg P \to (q \to r)$ iff $q \to (p \vee r)$

Soln L.H.S $\quad \neg P \to (q \to r)$

$= \neg P \to (\sim q \vee r)$

$= \neg(\neg P) \vee (\sim q \vee r)$

$= P \vee (\sim q \vee r)$

$= (\sim q \vee r) \vee p$

$= \sim q \vee (r \vee P)$

$= \sim q \vee (P \vee r) \longrightarrow (1)$

R.H.S $\quad q \to (p \vee r)$

$= \neg q \vee (P \vee r)$

$= \neg q \vee (P \vee r) \longrightarrow (2)$

L.H.S = R.H.S

$\neg P \to (q \to r) \Leftrightarrow q \to (p \vee r)$

Minterm : Let P & Q be two statements. Let us construct all possible formulas which consists of conjunction of P or its negation : conjunction of Q or its negation

e.g.

$P \wedge Q$

$\neg P \wedge Q$

$P \wedge \neg Q$

$\neg P \wedge \neg Q$

Maxterm : Let P and Q be two statements. Let us construct all possible formulas which consist of disjunction of P or its negation. disjunction of Q & its negation

$P \vee Q$

$\neg P \vee Q$

$P \vee \neg Q$

$\neg P \vee \neg Q$

Note : $P \wedge \neg P$ is not Minterm

$P \vee \neg P$ is not Maxterm

Principal Disjunction Normal form (PDNF)

For a given statement formula consisting of disjunction of minterms only is called PDNF

PDNF: (Minterm) $\vee$ (minterm) $\vee$ (minterm)

Principal Conjunction Normal form (PCNF)

For a given statement formula consisting of conjunction of maxterms only is called PCNF

PCNF: (Maxterm) $\wedge$ (maxterm) $\wedge$ (maxterm)

① obtain PDNF & PCNF of $P \vee (\neg P \wedge Q)$

$P \vee (\neg P \wedge Q)$

$= (P \wedge T) \vee (\neg P \wedge Q)$

$= (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$

$= (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

PDNF of S: $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

PDNF of $\neg S$: $P \wedge Q$
$\neg P \wedge Q$
$P \wedge \neg Q$
$\neg P \wedge \neg Q$

PDNF of $\neg S = \neg P \wedge \neg Q$

PCNF of $S = P \vee Q$

② obtain the PCNF and PDNF of

$S: (q \to p) \wedge (\neg p \wedge q)$

Soln Since $\neg p \wedge q \Leftrightarrow \neg(p \vee \neg q) \Leftrightarrow \neg(p \to q)$.

we have $S \Leftrightarrow (q \to p) \wedge \neg(q \to p) \Leftrightarrow F$

Since the PCNF of a Contradiction Contains all the max terms while its PDNF has none of the min terms, the PCNF of S is given by

$$S \Leftrightarrow (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$$

and the PDNF is the empty sum.

③ Find the PCNF is obtained as follows

Soln The PCNF is obtained as follows

$P \wedge (p \to q) \Leftrightarrow (p \wedge (\neg p \vee q)$

$\Leftrightarrow (p \wedge T) \wedge (\neg p \cancel{\vee} q)$

$$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q)$$

The PDNF is obtained as follows:

$$p \wedge (p \to q) \Leftrightarrow p \wedge (\neg p \vee q)$$

$$\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$$

$$\Leftrightarrow F \vee (p \wedge q)$$

$$\Leftrightarrow p \wedge q.$$

Alternately, the PDNF can be obtained as follows: Let S denote $p \wedge (p \to q)$. The PCNF of $\neg S$ is

$$\neg S \Leftrightarrow \neg p \vee \neg q.$$

and hence

$$S \Leftrightarrow \neg\neg S \Leftrightarrow \neg(\neg p \vee \neg q) \Leftrightarrow \neg\neg p \wedge \neg\neg q \Leftrightarrow p \wedge q.$$

(4) Find the PCNF and PDNF of

$$\neg(p \vee q) \leftrightarrow (p \wedge q)$$

Soln.

$$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q).$$

$$\Leftrightarrow (\neg(p \vee q) \wedge (p \wedge q)) \vee (\neg\neg(p \vee q) \wedge \neg(p \wedge q))$$

$$\Leftrightarrow ((\neg p \wedge \neg q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge (\neg p \vee \neg q))$$

$$\Leftrightarrow (F \wedge F) \vee (p \vee q) \wedge (\neg p \vee \neg q)$$

$$\Leftrightarrow F \wedge \vee ((p \vee q) \wedge (\neg p \vee \neg q))$$

$$\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q).$$

The PDNF is obtained as follows:

$$\neg(p \vee q) \leftrightarrow (p \wedge q)$$

$$\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$$

$$\Leftrightarrow ((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q))$$

$\Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg q)$

$\Leftrightarrow F \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee F$

$\Leftrightarrow (q \wedge \neg p) \vee (p \wedge \neg p)$

$\Leftrightarrow (\neg p \wedge q) \vee (p \wedge \neg q).$

Theory of Inference:

Implication formula:

1. $P \wedge Q \Rightarrow p$
   $P \vee Q \Rightarrow p$
2. $P \wedge Q \Rightarrow Q$
   $P \vee Q \Rightarrow Q$
3. $p, p \to Q \Rightarrow Q$
4. $p \to Q, \sim Q \Rightarrow \sim p$
5. $p \to Q, Q \to R \Rightarrow p \to R$
6. $\neg p, p \vee Q \Rightarrow Q$

Rule P:

A given premises may introduced at any step in the derivation.

Rule T:

A formula S may be introduced if S is tautologically implide by one or more preceding formulas in the derivation.

Show that $\neg P$ follows logically from the premises $\neg(P \wedge Q)$, $\sim Q \vee R$ and $\neg R$.

1. (1) $\neg(P \wedge \neg Q)$ Rule P
   (2) $\neg P \vee Q$ Demorgans
   (3) $P \to Q$ Rule T
   (4) $\sim Q \vee R$ Rule P

⑤ $Q \to R$ Rule T

⑥ $P \to R$ Rule T (3,5)

⑦ $\neg R$ Rule P

⑧ $\neg P$ Rule T

② S.T $\neg(p \wedge \neg q)$, $\neg p \vee r$ and $\neg r$ implies $\neg p$.

Soln

① $\neg r$ Rule P

② $\neg q \vee r$ Rule P

③ $\neg q$ ① & ② Disjunctive

④ $\neg(p \wedge \neg q)$ Rule P

⑤ $\neg p \vee \neg\neg q$ Demorgan's Law

⑥ $\neg p \vee q$ ⑤ Double negation Law

⑦ $\neg p$ ③ & 6 disjunctive Syllogism

we now give an indirect proof of this implication:

① $\neg(p \wedge \neg q)$ Rule P

② $\neg p \vee (\neg\neg q)$ Demorgan's Law

③ $\neg p \vee q$ Double negation Law

④ $p \to q$ Conditional as disjunction

⑤ p negated conclusion

⑥ $\neg(\neg q)$ ④ & ⑤ modus Ponens

⑦ $\neg q \vee r$ Rule P

⑧ r ⑥ & ⑦ Disjunctive Syllogism

⑨

(9) $\neg r$ Rule P

(10) $r \wedge \neg r$ (8), (9) $r \wedge \neg r$ is a Contradiction.

Conditional Rule (CP Rule)

Let $M_1, M_2 \ldots M_n$ be the set of Premises we drive $R \to S$

(1) Add R a new premises
(2) Derive S using equivalent rules
(3) $R \to S$ (CP rule)

Consistency

Let $M_1, M_2 \ldots M_n$ be set of premises the given set of premises R is Consistance, if derive a tautology.

InConsistency

Let $M_1, M_2 \ldots M_n$ be the set of Premises, the given Set of premises are inConsistancy, if we derive a Contradiction.

Indirect method:

Let $M_1, M_2 \ldots M_n$ be the Set of Premises use drive $S \to R$

(i) Add $\sim(R \to S)$ a new premises
(ii) Derive Contradiction (F) using equivalence Rules.

(1) Using Conditional proof (CP rules), Derive $R \to S$ from the premises $P \to (Q \to S)$, $\sim R \vee P$, Q.

Soln

(1) R Add a new premise

(2) $\neg R \vee P$ Rule P

(3) $R \to P$ Rule T (1,2)

(4) P Rule T $P, P \to Q \Rightarrow Q$

(5) $P \to (Q \to S)$ Rule P

(6) $Q \to S$ Rule T $P, P \to Q \Rightarrow Q$

(7) Q Rule P

(8) S Rule T, $P, P \to Q \Rightarrow Q$

(9) $R \to S$ CP Rule.

(2) Use indirect method P.T $P \to R, Q \to S, P \vee Q \Rightarrow S \vee R$

Soln

(1) $\neg(S \vee R)$ add a new Premises

(2) $\neg S \wedge \neg R$ Rule T, Demorgan's law

(3) $\neg S$ Rule T, $P \wedge Q \Rightarrow P$

(4) $\neg R$ Rule T, $P \wedge Q \Rightarrow Q$

(5) $P \vee Q$ Rule P

(6) $\neg P \to Q$ Rule T

(7) $Q \to S$ Rule P

(8) $\neg P \to S$ Rule T

(9) $P \to R$ Rule P

(10) $\neg P$ Rule T

(11) S Rule T. $P, P \to Q \Rightarrow Q$

(12) F Rule T

(3) Show that the following statements constitutes a valid argument: "If A works hard, then either B or C will enjoy themselves, If B enjoy himself, then A will not work hard. If D enjoy himself, then C will not. Therefore, If A works hard, then D will not enjoy himself."

Soln Let

p: A works hard

q: B enjoy himself

r: C enjoy himself

s: D enjoy himself

Then the given statements translate into

$$p \to (q \vee r),\ q \to \neg p,\ s \to \neg r \Rightarrow p \to \neg s$$

The argument is valid as shown below. We use premise of the conclusion as additional hypothesis and obtain the conclusion of the conclusion

$$p \to (q \vee r),\ q \to \neg p,\ s \to \neg r,\ p \Rightarrow \neg s$$

1. p — premise
2. $p \to (q \vee r)$ — premise
3. $q \vee r$ — 1 & 2, Modus pollens.
4. $q \to \neg p$ — premise
5. $p \to \neg q$ — (4) Contrapositive
6. $\neg q$ — 1 & 5 Modus Pollens
7. r — 3 & 6 disjunctive syllogism
8. $s \to \neg r$ — premise
9. $r \to \neg s$ — (8) Contrapositive
10. $\neg s$ — 7 & 9 Modus Pollens.