PROJECT REPORT ON

"The fastest stop of a train at a station"

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Content

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Theory

- **Pontryagin Principle:**

For deterministic dynamics $\dot{x} = f(x, u)$ we can compute extremal open-loop trajectories (i.e. local minima) by solving a boundary-value ODE problem with given x (0) and $\lambda(T) = \frac{\partial q_T(x)}{\partial x}$ where λ (t) is the gradient of the optimal cost-to-go function (called costate).

- **Continuous-time maximum principle:**

If x (t), u (t), 0 ≤ t ≤ T is the optimal state-control trajectory starting at x (0), then there exists a costate trajectory λ (t) with $\lambda(T) = \frac{\partial q_T(x)}{\partial x}$ satisfying

$$\dot{x} = \overline{H_\lambda}(x, u, \lambda) = f(x, u)$$
$$-\dot{\lambda} = \overline{H_x}(x, u, \lambda) = l_x(x, u) + f_x(x, u)^T \lambda$$
$$u = \arg\min_{\tilde{u}} \overline{H}(x, \tilde{u}, \lambda)$$

- **Discrete-time maximum principle:**

If $x_k, u_k, 0 \le k \le N$ is the optimal state-control trajectory starting at x_0, then there exists a costate trajectory λ_k with $\lambda_N = \frac{\partial}{\partial x} q_T(x_N)$ satisfying

$$x_{k+1} = \overline{H_\lambda}(x_k, u_k, \lambda_{k+1}) = f(x_k, u_k)$$
$$\lambda_k = \overline{H_x}(x_k, u_k, \lambda_{K+1}) = l_x(x_k, u_k) + f_x(x_k, u_k)^T \lambda_{k+1}$$
$$u_k = \arg\min_{\tilde{u}} \overline{H}(x_k, \tilde{u}, \lambda_{k+1})$$

Theory

- Filippov's Theorem: Let the space of control parameters $U \in \mathbb{R}^m$ be compact. Let there exist a compact $K \in M$ such that $f_u(q) = 0\ for\ q \notin K, u \in U$. Moreover, let the velocity sets

$$f_u(q) = \{f_u(q) | u \in U\} \subset T_q M,\ \ q \in M$$

be convex. Then the attainable sets $A_{q_0}(t)$ & $A^t_{q_0}$ are compact for all $q_0 \in M, t > 0$.

- Bang-Bang Control : Bang-bang control is a type of control system that mechanically or electronically turns something on or off when a desired target (setpoint) has been reached. Bang–bang controllers, which are also known as two-step controllers, on-off controllers or hysteresis controllers, are used in many types of home and industrial control systems (ICS).

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Objective

The problem is to drive the train to a station and stop it there in a minimal time.

- Solving the problem mathematically
- Explanation of the problem analytically
- Implementation of the problem in MATLAB & Simulink.

Introduction

- The problem is to drive the train to a station and stop it there in a minimal time.

<u>Denote</u>:

- Position of the train→ x_1 , $x_1 \in \mathbb{R}$
- Station where the train will stop→ The origin $0 \in \mathbb{R}$
- Acceleration→ u

<u>Assumption</u>:

The train moves without friction,

Controlled acceleration $|u| \leq 1$

We obtain the control system:

$$\ddot{x}_1 = u, \quad x_1 \in \mathbb{R}, \quad |u| \leq 1$$

Standard form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \ |u| \leq 1 \qquad (1)$$

The time-optimal control problem:

$$x(0) = x^0, \ x(t_1) = 0 \qquad (2)$$

$$t_1 \longrightarrow min. \qquad (3)$$

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Mathematical Modelling

1. *Verify existence of optimal controls by Filippov's theorem:*

The set of control parameter U = [-1,1] is compact, the vector fields in the right-hand side

$$f(x,u) = \binom{x_2}{u}, |u| \le 1,$$

are linear and the set of admissible velocities at a point is convex

$$f(x,U) = \{f(x,u) \mid |u| \le 1,$$

The time-optimal control problem has a solution if the origin $0 \in \mathbb{R}^2$ is attainable from the initial point x^0.

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Mathematical Modelling

2. *Applying Pontryagin Maximum Principle:*

Introduce canonical coordinates on the cotangent bundle:

$$M = \mathbb{R}^2,$$

$$T^*M = T^*\mathbb{R}^2 = \mathbb{R}^{2*} \times \mathbb{R}^2 = \{\lambda = (\xi, x) \mid x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \xi = (\xi_1, \xi_2)\}$$

The control dependent Hamiltonian function of PMP is

$$h_u(\xi, x) = (\xi_1, \xi_2)\begin{pmatrix} x_2 \\ u \end{pmatrix} = \xi_1 x_2 + \xi_2 u$$

and the corresponding Hamiltonian system has the form

$$\dot{x} = \frac{\delta h_u}{\delta \xi}, \qquad \dot{\xi} = -\frac{\delta h_u}{\delta x}$$

In coordinates this system splits into two independent subsystems:

$$\begin{Bmatrix} \dot{x_1} = x_2 \\ \dot{x_2} = u \end{Bmatrix}, \qquad \begin{Bmatrix} \dot{\xi_1} = 0 \\ \dot{\xi_2} = -\xi_1 \end{Bmatrix} \qquad (4)$$

Mathematical Modelling

By PMP, if a control $\tilde{u}(.)$ is time optimal, the Hamiltonian system has a nontrivial solution

$$\big(\xi(t), x(t)\big), \xi(t) \not\equiv 0$$

$$h_{\tilde{u}(t)}\big(\xi(t), x(t)\big) = \max_{|u|\leq 1} h_u(\xi(t), x(t)) \geq 0$$

From this maximality condition if $\xi_2(t) \neq 0$, then

$$\tilde{u}(t) = sgn\xi_2(t).$$

Since $\ddot{\xi}_2 = 0 \ \rightarrow \ \xi_2(t) = \alpha + \beta(t)\,, \quad \alpha,\ \beta = const.$

Hence, the optimal control is:

$$\tilde{u}(t) = \mathrm{sgn}(\alpha + \beta t)$$

- $\tilde{u}(t)$ Piecewise constant(u= ∓ 1)
- $\tilde{u}(t)$ has not more than one switching.

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Mathematical Modelling

3. Finding all trajectories x(t) that correspond to such controls and come to the origin.

For controls u $= \mp 1$ the first of subsystems (4) reads

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \pm 1 \end{cases}$$

Trajectories of this system satisfy the equation

$$\frac{dx_1}{dx_2} = \pm x_2,$$

thus are parabolas of the form

$$x_1 = \pm \frac{x_2^2}{2} + c, \qquad c = const.$$

i) Finding trajectories from this family that come to the origin without switchings :

$$x_1 = \frac{x_2^2}{2}\ , \quad x_2 < 0, \ \ \dot{x}_2 > 0, \qquad (5)$$

And

$$x_1 = -\frac{x_2^2}{2}\ , \quad x_2 > 0, \ \ \dot{x}_2 < 0, \qquad (6)$$

for u =+1 and -1 respectively.

Mathematical Modelling

ii) Find all extremal trajectories with one switching.

Let $(x_{1s}, x_{2s}) \in \mathbb{R}^2$ be a switching point for anyone of curves (5),(6).

Extremal trajectories with one switching coming to the origin have the form:

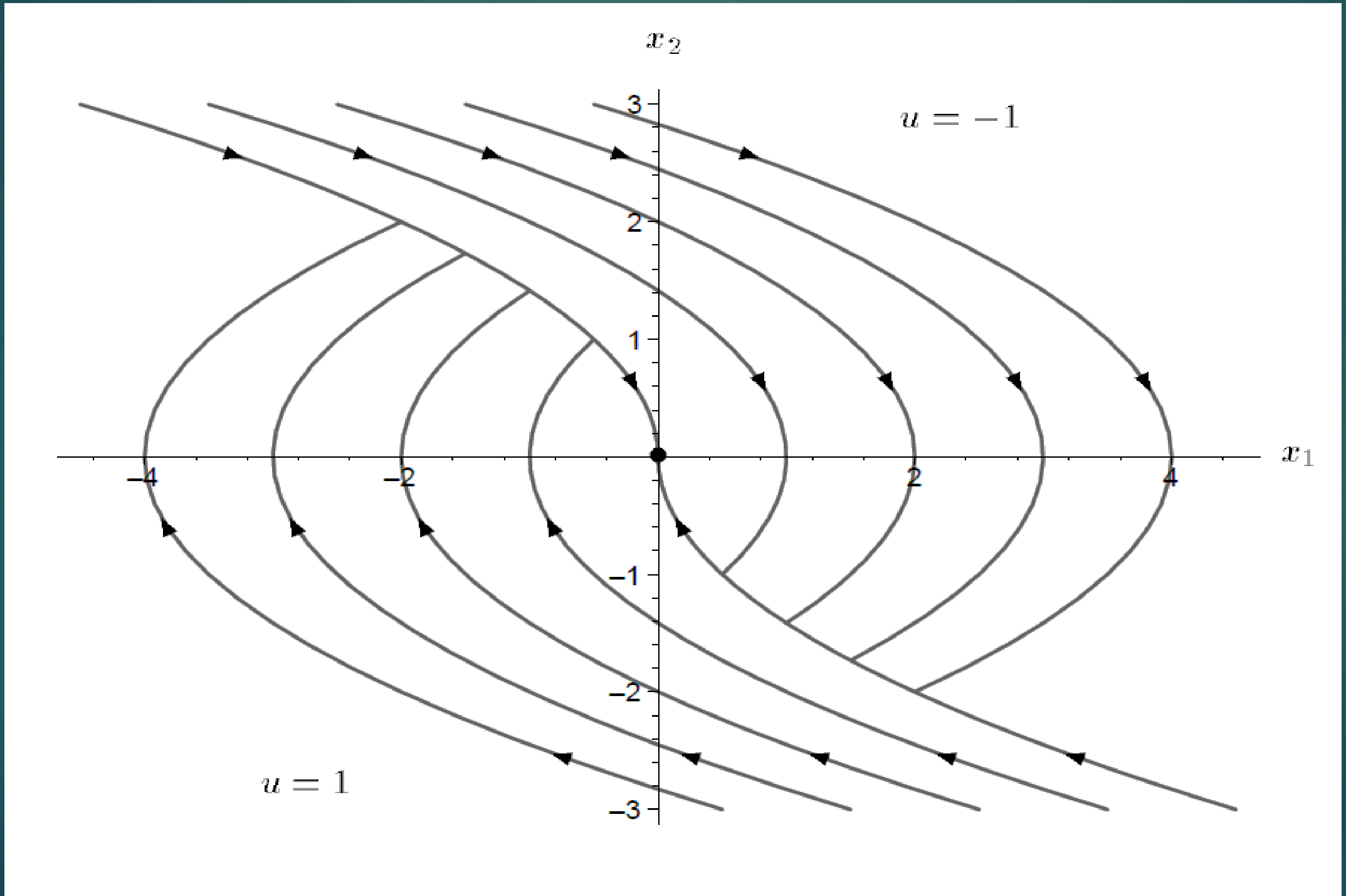
$$x_1 = \begin{cases} -\frac{x_2^2}{2} + \frac{x_{2s}^2}{2} + x_{1s}, & x_2 > x_{2s}, \quad \dot{x}_2 < 0, \\ \frac{x_2^2}{2} & 0 > x_2 > x_{2s}, \quad \dot{x}_2 > 0, \end{cases} \tag{7}$$

And

$$x_1 = \begin{cases} \frac{x_2^2}{2} - \frac{x_{2s}^2}{2} + x_{1s}, & x_2 < x_{2s}, \quad \dot{x}_2 > 0, \\ -\frac{x_2^2}{2} & 0 < x_2 < x_{2s}, \quad \dot{x}_2 < 0, \end{cases} \tag{8}$$

General view of the optimal synthesis

Optimal trajectories exist → Solutions are optimal



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Analytical Explanation

- Given the system;

$$\ddot{x}(t) = \begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t)$$

A x(t) B u(t)

With the performance index $J = \int_0^{t_f} dt = t_f$ is minimized with $|u(t)| \leq 1$

Sol. $H(.) = 1 + \lambda^T(t)[Ax(t) + Bu(t)]$

$$H(.) = 1 + [\lambda_1(t) : \lambda_2(t)]\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t)\right]$$

$$H(.) = 1 + [\lambda_1(t) + \lambda_2(t)]\begin{bmatrix}x_2(t)\\ u(t)\end{bmatrix}$$

$$H(.) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \qquad (1)$$

Analytical Explanation

- Co-state Equations are

$$\dot{\lambda}(t) = -\frac{\partial H(.)}{\partial x(t)} = -\begin{bmatrix} 0 \\ \lambda_1(t) \end{bmatrix}$$

$$\dot{\lambda_1}(t) = 0 \qquad (2)$$

$$\dot{\lambda_2}(t) = -\lambda_1(t) \qquad (3)$$

From (2), $\frac{d\lambda_1}{dt} = 0 \longrightarrow \lambda_1(T) = \lambda_1(0)$ (4)

From (3) $\dot{\lambda_2}(t) = -\lambda_1(t) = -\lambda_1(0)$

$$\frac{d\lambda_2(t)}{dt} = -\lambda_1(0), \quad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0) \qquad (5)$$

From (1) $H(.) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t).u(t)$

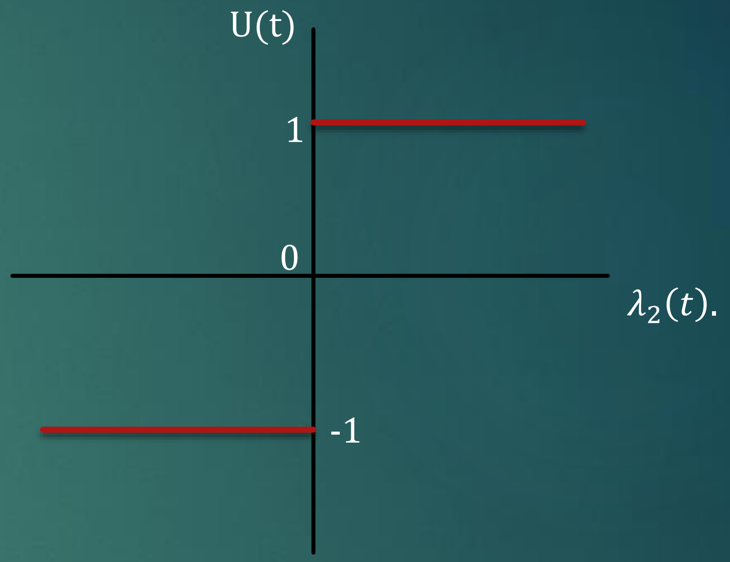
$$u^*(t) = 1 \quad if \quad \lambda_2(t) < 0$$

$$= -1 \quad if \quad \lambda_2(t) > 0$$

$$u^*(t) = -sign\big(\lambda_2(t)\big) = -sign(\lambda_2(0) - \lambda_1(0)t)$$

$$Sign = 1 \quad if\ z > 0$$

$$-1 \quad if\ z < 0$$



Analytical Explanation

$u^*(t) = 1$

$\dot{x_1}(t) = x_2(t),\ \dot{x_2}(t) = 1,\ \ x_2(t) = t + x_2(0)$

$\dot{x_1}(t) = x_2(0) + t$

$x_1(t) = \frac{t^2}{2} + x_2(0) + x_1(0)$

$x_1(t) = x_1(0) + x_2(0).t + \frac{t^2}{2}$

$x_1(t) = x_1(0) + x_2(0)\big(x_2(t) - x_2(0)\big) + \frac{(x_2(t) - x_2(0))^2}{2}$

$x_1(t) = \frac{x_2^2(t)}{2} + \frac{x_2^2(0)}{2} + x_1(0) \rightarrow for\ u = +1 \qquad (6)$

for $u^*(t) = -1$, the state equation is

$\dot{x_1}(t) = x_2(t);\ \dot{x_2}(t) = -1$

$x_2(t) = x_2(0) - t$

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Analytical Explanation

- $x_1(t) = x_1(0) + x_2(0)t - \frac{t^2}{2}$

$$x_1(t) = -\frac{x_2^2(t)}{2} + \left(x_1(0) + \frac{x_2^2(0)}{2}\right) \rightarrow for\ u = -1 \qquad (7)$$

Depending on the initial condition our parabolas will defined

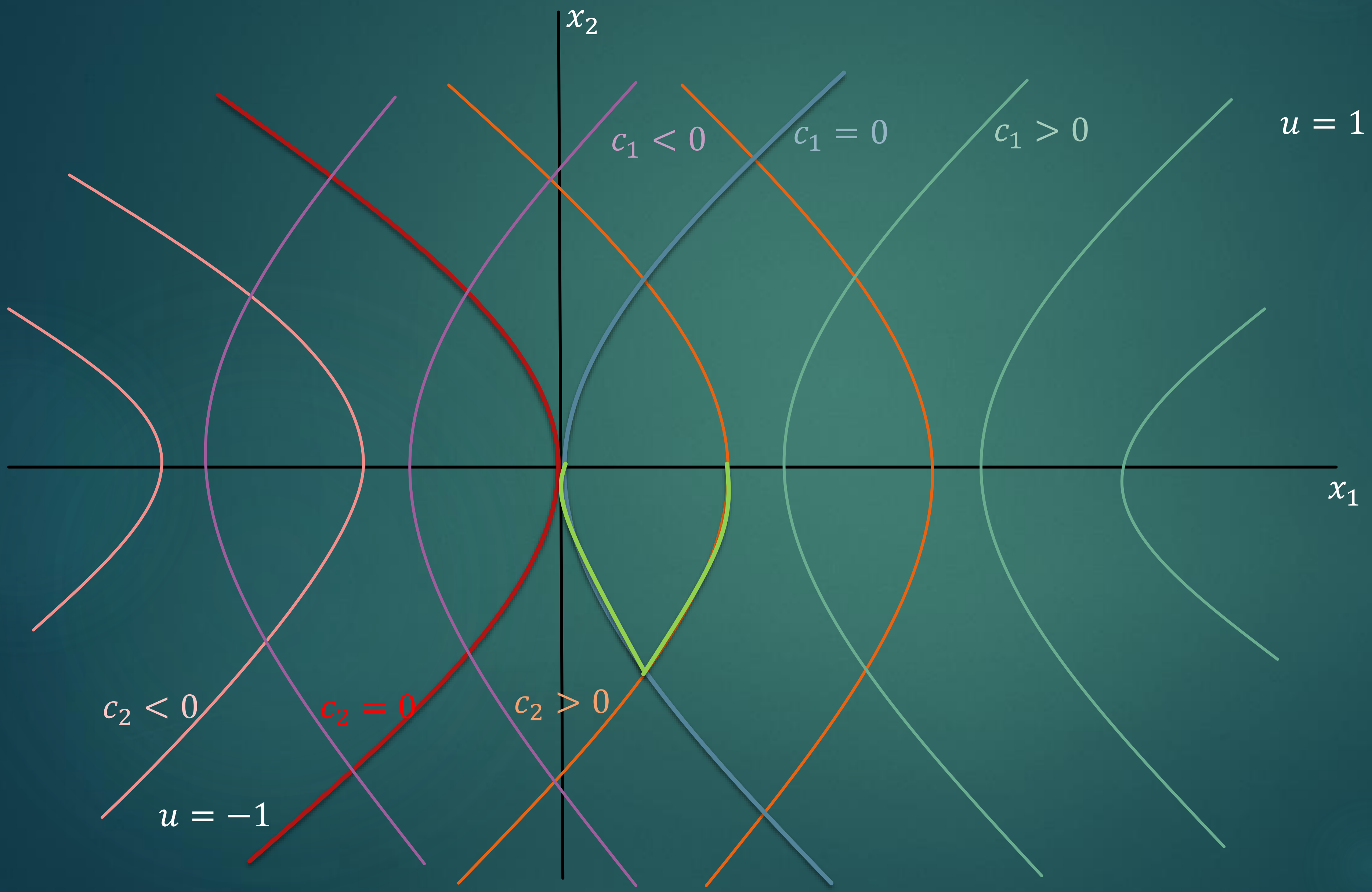
From (6) & (7), we have,

$\frac{x_2^2(t)}{2} = x_1(t) - c_1,$ where $c_1 = x_1(0) - \frac{x_2^2(0)}{2}$

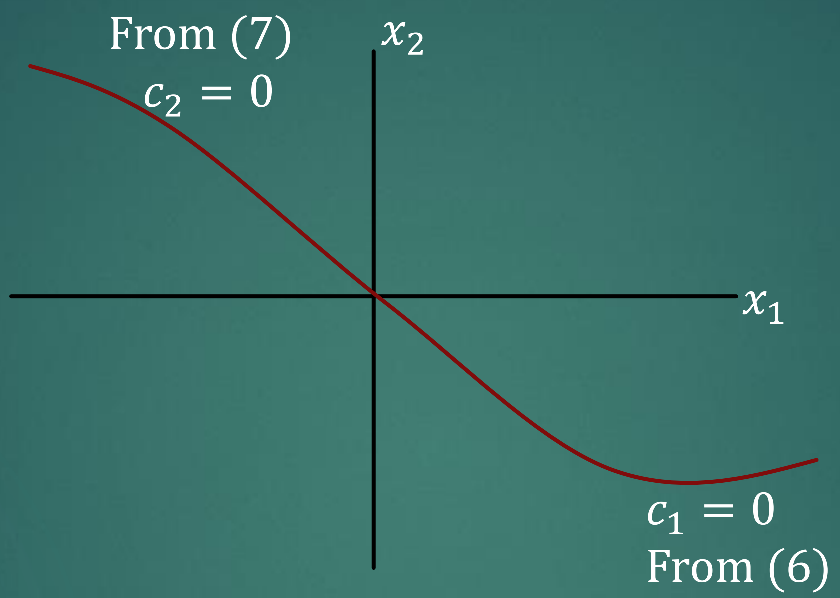
From (7)

$\frac{x_2^2(t)}{2} = -x_1(t) + c_2,$ where $c_2 = x_1(0) + \frac{x_2^2(0)}{2}$

Phase Portrait of Hamiltonian function



Switching Curve

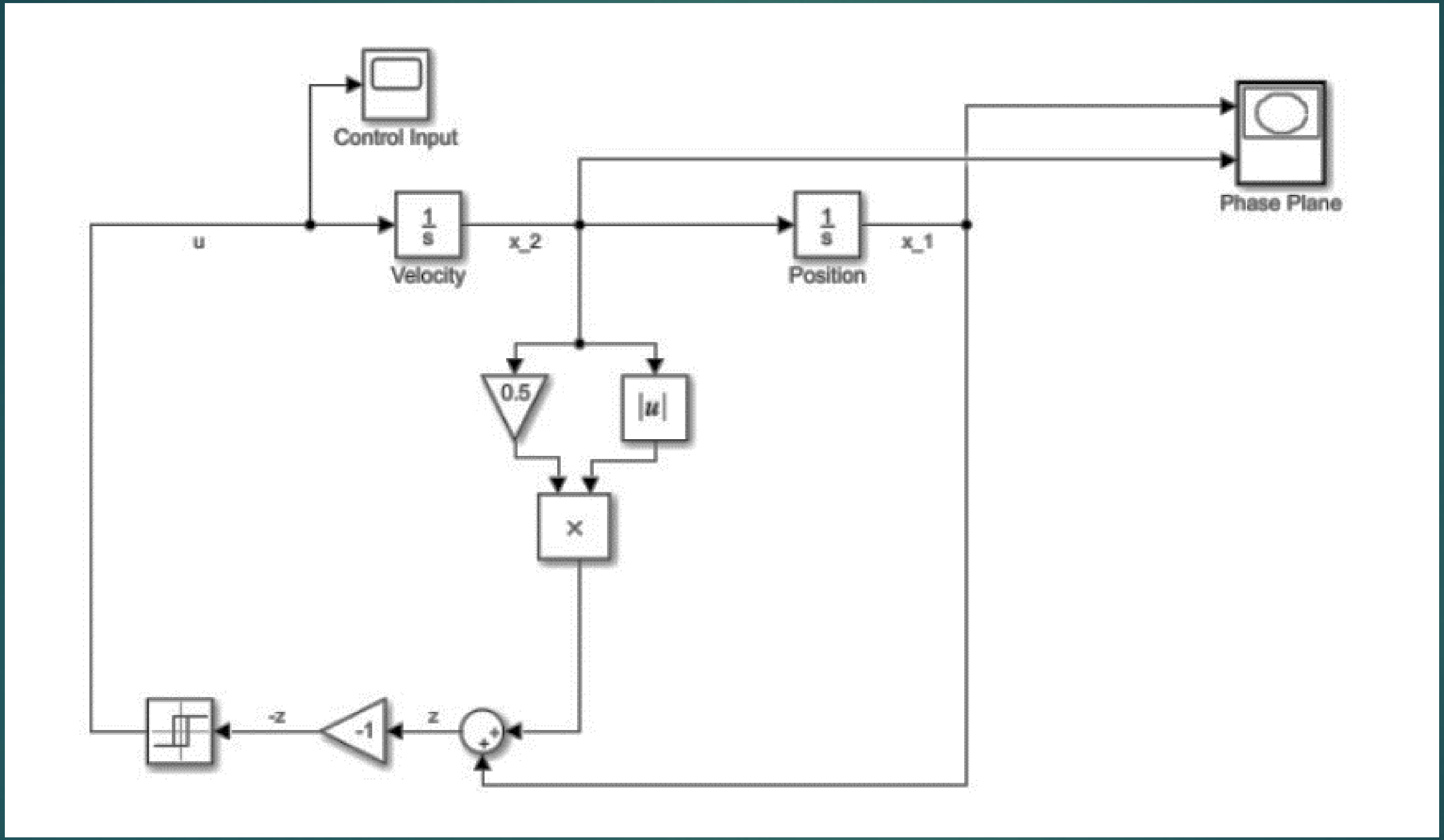


- Control Law

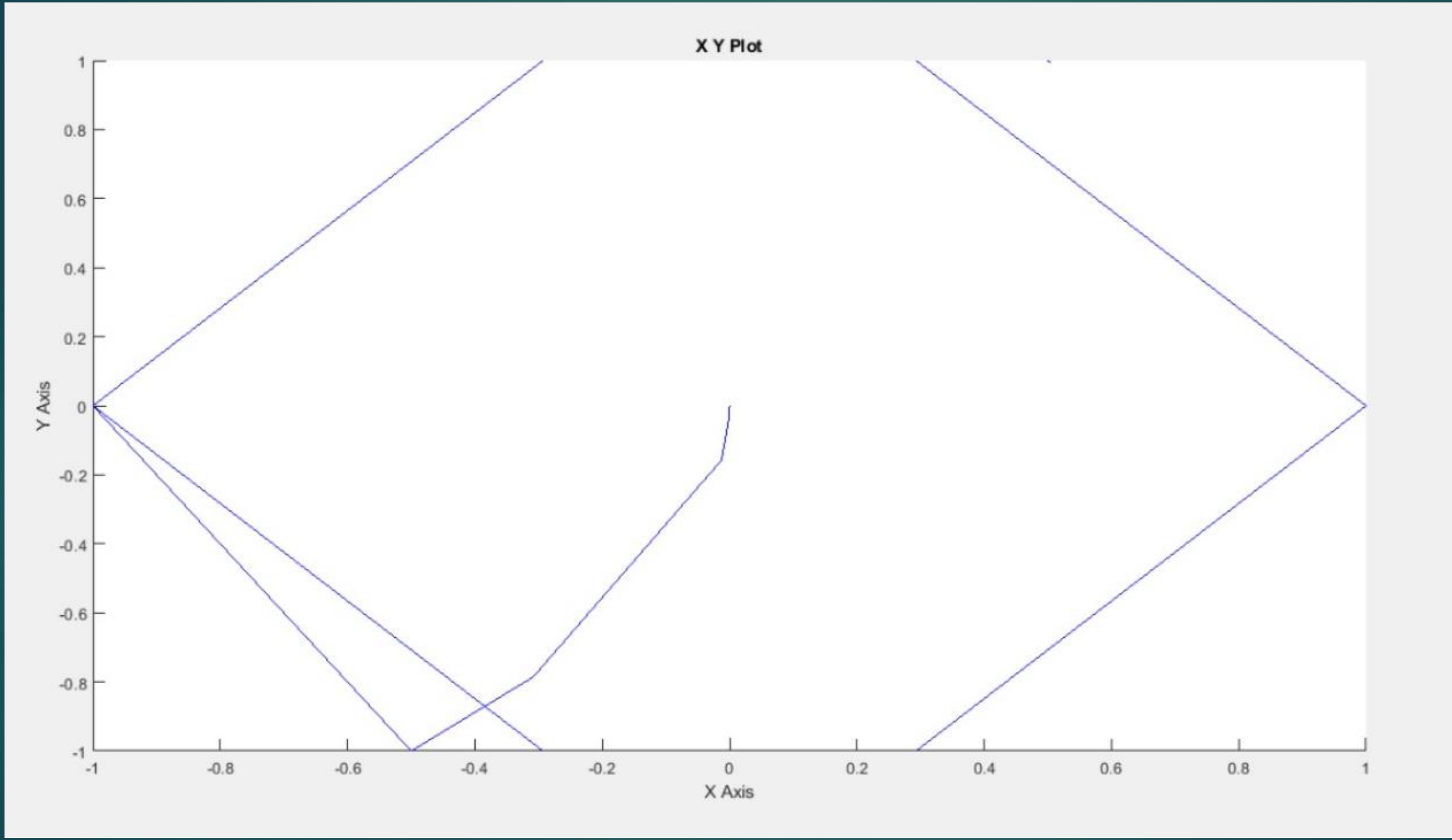
$u^*(t) = -Sign(\lambda_2(0) - \lambda_1(0)t)$ from (6) & (7) when $c_1 = c_2 = 0$

- Note: Only one family from parabola passes through $x = [0,0]^T$

Simulink Model



Simulink Result (Phase Plane Plot)



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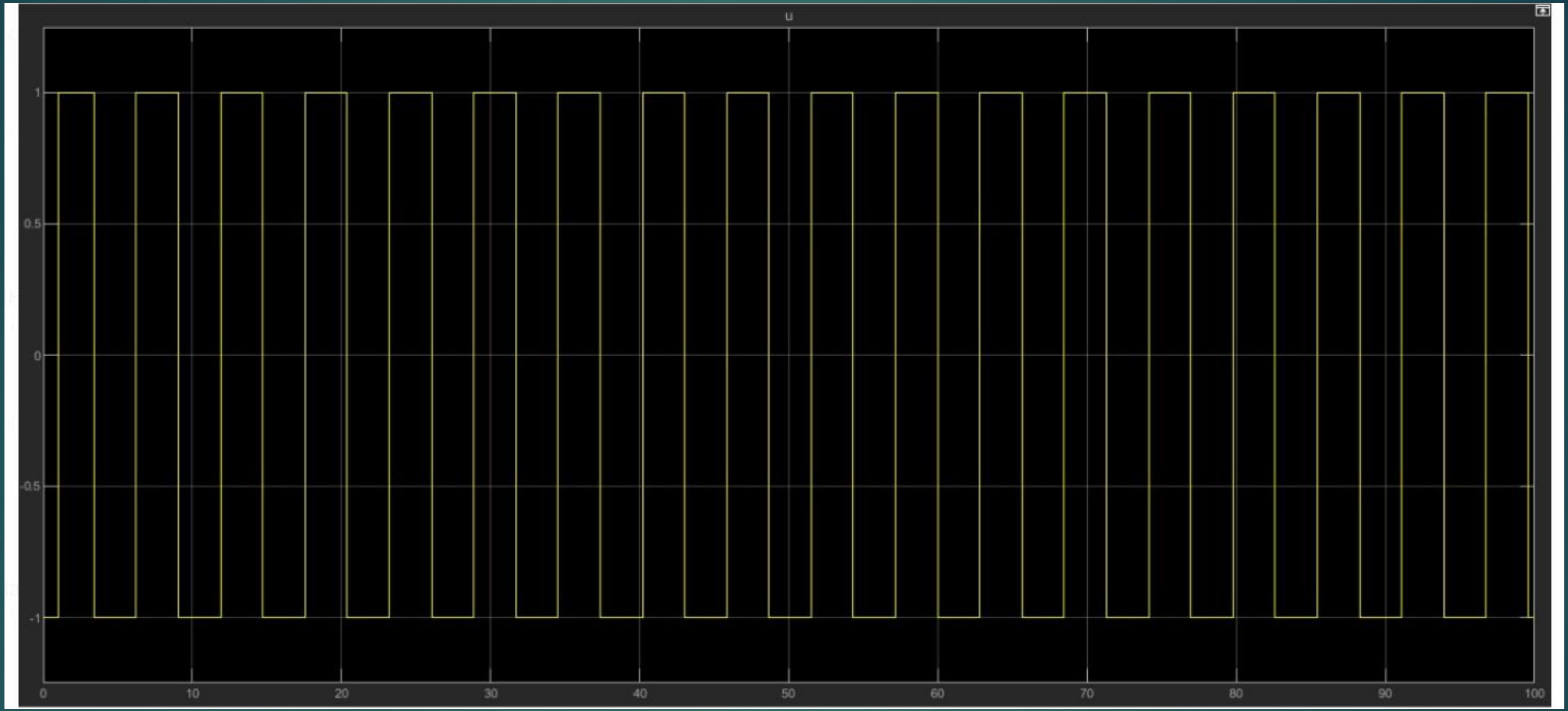
Control Input



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Implementation in MATLAB

- System:

Given a double integral system as:

$$\dot{x_1} = x_2(t)$$

$$\dot{x_2} = u(t)$$

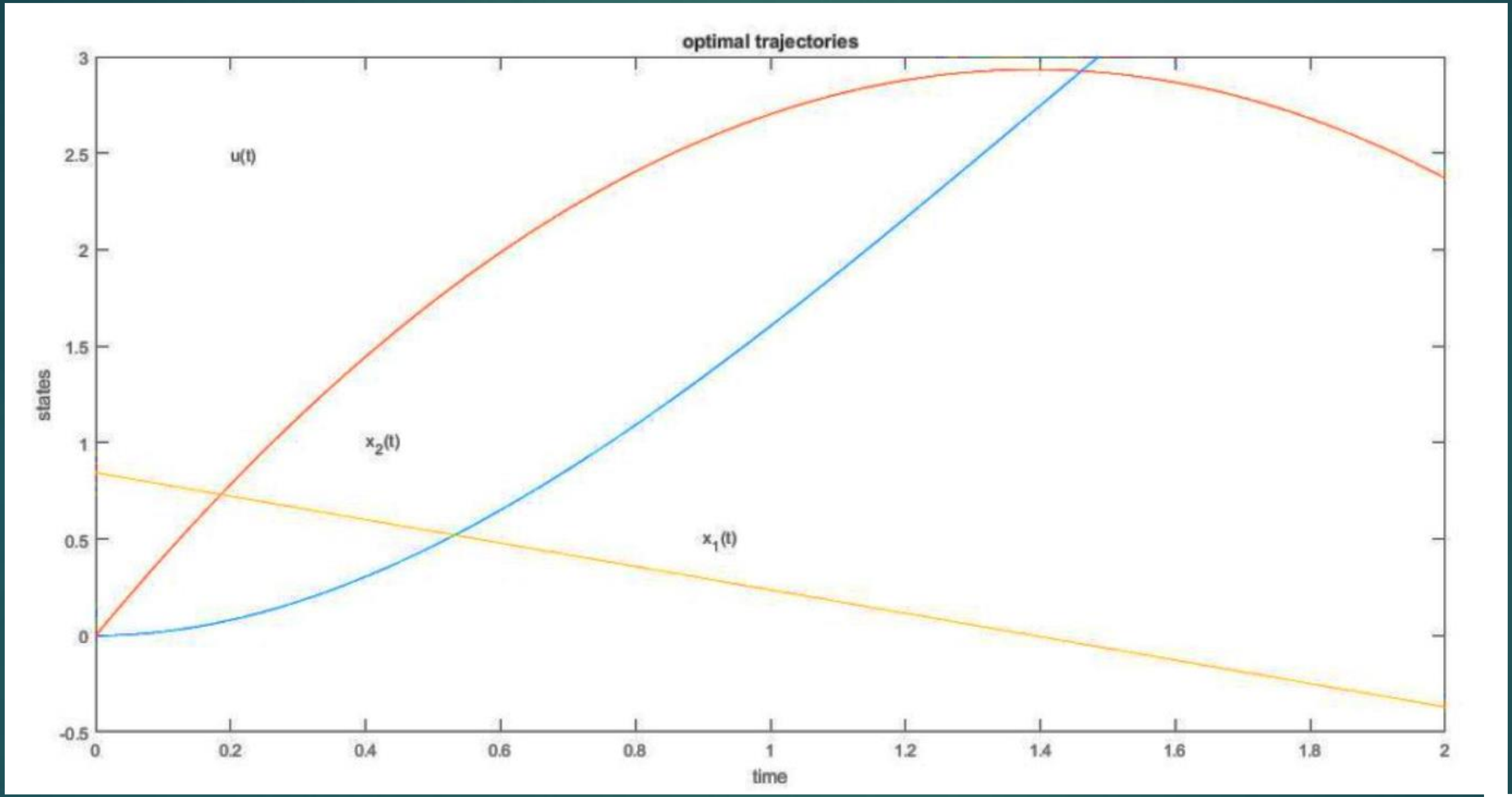
Minimize the final time:

$$T = \int_0^{t_f} 0.1 * u^2$$

- Condition:

$$x_1(0) = 0,\ x_2(0) = 0,\ x_1(2) = 5,\ x_2(2) = 5$$

MATLAB Result (Optimal Trajectories)



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Implementation in MATLAB

- System

  Given a double integral system as:

  $$\dot{x_1} = x_2(t)$$

  $$\dot{x_2} = u(t)$$

  Minimize the final time:

  $$T = \int_0^{t_f} dt$$

- To drive the states to the origin:

  $$x(0) = [2 \quad 2]^T, \quad x(t_f) = [0 \quad 0]^T, \quad |u| \leq 1$$

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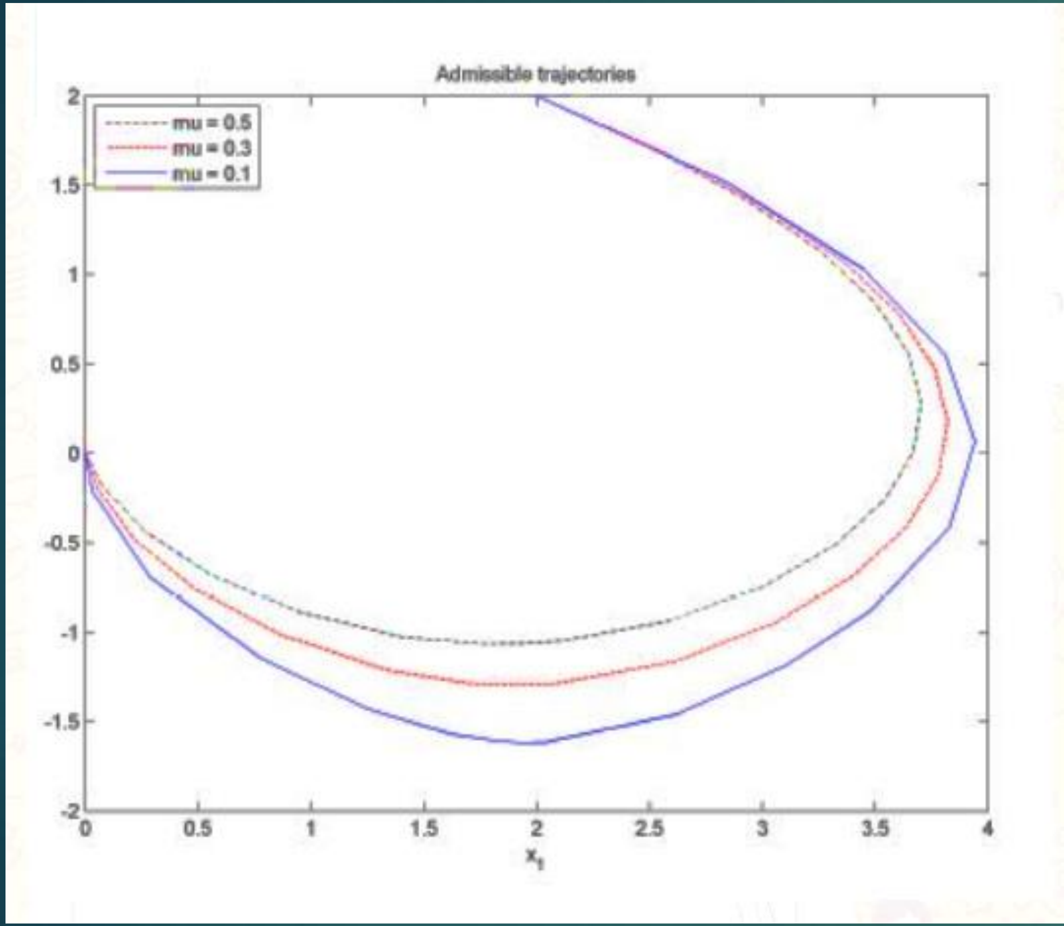
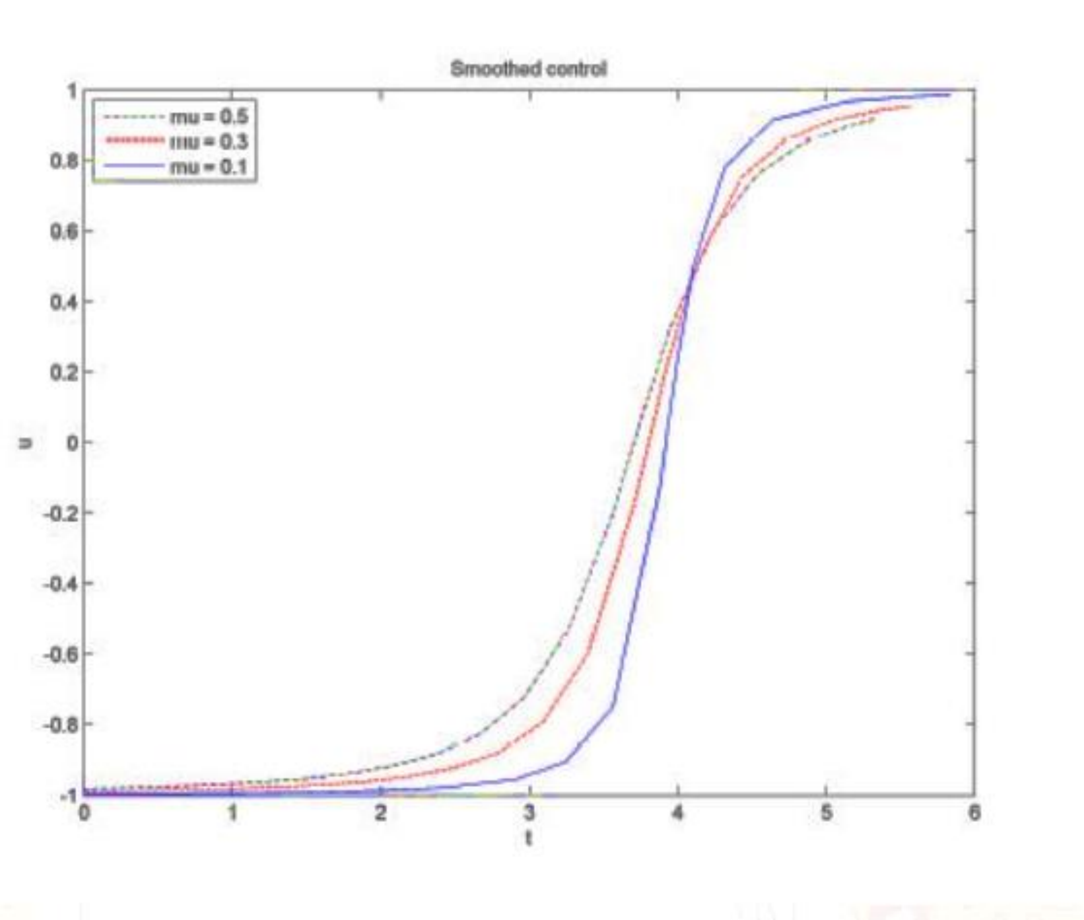
MATLAB Results





Admissible Trajectories

Smoothed Control

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Application:

- We can apply our Mathematical model also in the automobile sector.
- This algorithm with bang-bang control can also be used for collision avoidance.
- This problem have a lot of scope in robotics and aerospace engineering.
- Pontryagin maximum along with three different strategies can also be used for the modelling and optimal control of typhoid fever disease.
- This algorithm also proved to be efficient to land prob on mars.

Product Quality:

- The algorithm that we used is more computationally efficient.
- We can stop our start our algorithm when desired state is reached in minimum time.
- There is a hope to minimize the energy in order to stop the train in the future through our problem.

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Conclusion

- In this project it was presented one of the examples of optimal control which is the application of Pontryagin maximum principle and bang-bang control.
- We provided the theoretical and mathematical explanation of the problem.
- In the first stage we minimized the time which is also our cost function using Pontryagin maximum principle then we derive the equation of optimal trajectories. In second stage we gave the more detailed mathematical explanation, lastly we implemented this mathematical model in Simulink and MATLAB in order to validate our algorithm and to get the more practical exposure.
- We can further improve our problem by taking one more factor i.e. energy into the consideration for minimization which will impact the dynamical system quite heavily.

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