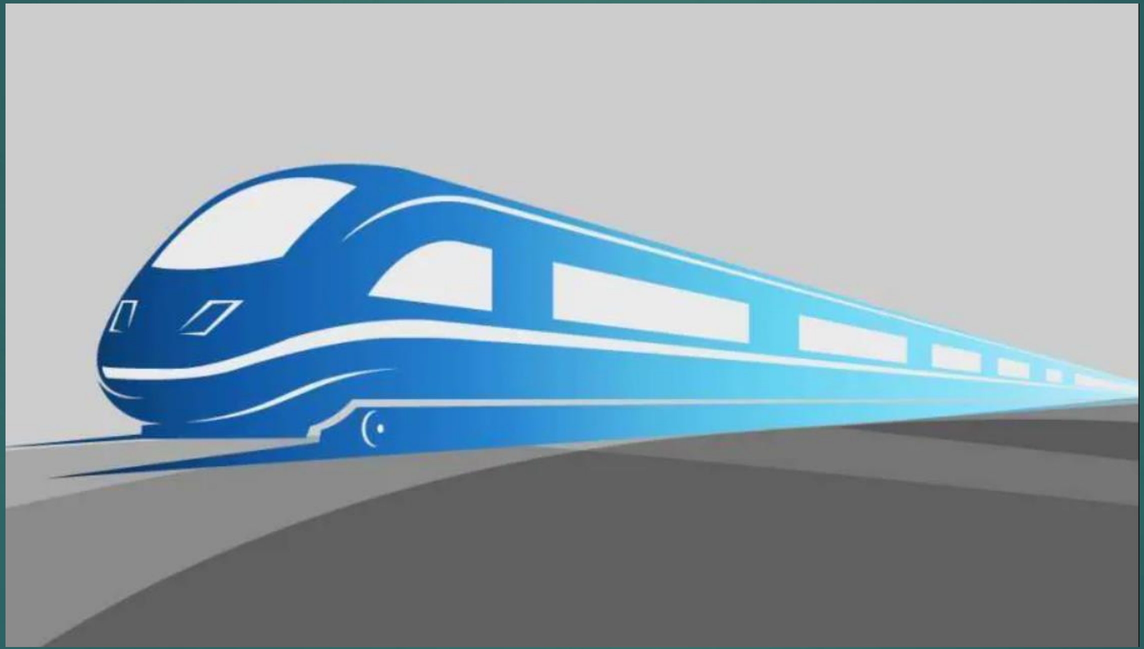
PROJECT REPORT ON

“The fastest stop of a train at a station”



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Content

Theory

- **Pontyragin's Principal:** For deterministic dynamics $\dot{x} = f(x,u)$ we can compute extremal open-loop trajectories (i.e. local minima) by solving a boundary-value ODE problem with given x (0) and $\lambda(T) = \frac{\partial q_T(x)}{\partial x}$ where λ (t) is the gradient of the optimal cost-to-go function (called costate).
- **Theorem (continuous-time maximum principle):** If x (t), u (t), 0 ≤ t ≤ T is the optimal state-control trajectory starting at x (0), then there exists a costate trajectory λ (t) with $\lambda(T) = \frac{\partial q_T(x)}{\partial x}$ satisfying

$$\dot{x} = \overline{H_\lambda}(x,u,\lambda) = f(x,u)$$
$$-\dot{\lambda} = \overline{H_x}(x,u,\lambda) = l_x(x,u) + f_x(x,u)^T \lambda$$
$$u = \arg\min_{\tilde{u}} \overline{H}(x,\tilde{u},\lambda)$$

- **Theorem (discrete-time maximum principle):** If $x_k, u_k, 0 \le k \le N$ is the optimal state-control trajectory starting at x_0, then there exists a costate trajectory λ_k with $\lambda_N = \frac{\partial}{\partial x} q_T(x_N)$ satisfying

$$x_{k+1} = \overline{H_\lambda}(x_k,u_k,\lambda_{k+1}) = f(x_k,u_k)$$
$$\lambda_k = \overline{H_x}(x_k,u_k,\lambda_{K+1}) = l_x(x_k,u_k) + f_x(x_k,u_k)^T \lambda_{k+1}$$
$$u_k = \arg\min_{\tilde{u}} \overline{H}(x_k,\tilde{u},\lambda_{k+1})$$

Theory

- **Filippov's Theorem:** Let the space of control parameters $U \in \mathbb{R}^m$ be compact. Let there exist a compact $K \in M$ such that $f_u(q) = 0 \; for \; q \notin K, u \in U$. Moreover, let the velocity sets

$$f_u(q) = \{f_u(q) | u \in U\} \subset T_q M, \;\; q \in M$$

be convex. Then the attainable sets $A_{q_0}(t)$ & $A_{q_0}^t$ are compact for all $q_0 \in M, t > 0$.

- **Bang-Bang** : Bang-bang control is a type of control system that mechanically or electronically turns something on or off when a desired target (setpoint) has been reached. Bang–bang controllers, which are also known as two-step controllers, on-off controllers or hysteresis controllers, are used in many types of home and industrial control systems (ICS).

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Objective

The problem is to drive the train to a station and stop it there in a minimal time.

- Solving our problem rigorously by using the Pontryagin maximum principle.
- Explaining the problem Analytically & Mathematically using Bang -Bang Control.
- Implementation of our problem in MATLAB & Simulink.

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Introduction

- Consider a train moving on a railway. The problem is to drive the train to a station and stop it there in a minimal time.

Describe position of the train by a coordinate X_1 on the real line the origin $0 \in \mathbb{R}$ corresponds to the station.

Assume that the train moves without friction , and we can control acceleration of the train by applying a force bounded by absolute value.

Using rescaling if necessary, we can assume that absolute value of acceleration is bounded by 1 We obtain the control system

$$\ddot{x} = u, \quad x_1 \in \mathbb{R}, \quad |u| \leq 1$$

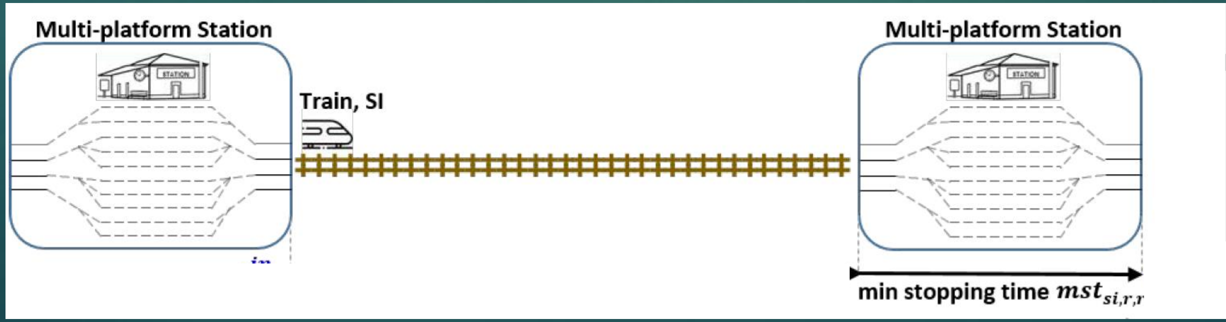
or, in standard form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \ |u| \leq 1 \qquad (1)$$

The time-optimal control problem is

$$x(0) = x^0, \ x(t_1) = 0 \qquad (2)$$

$$t_1 \longrightarrow min. \qquad (3)$$



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Mathematical Modelling

1. Verify existence of optimal controls by Filippov's theorem:

The set of control parameter U = [-1,1] is compact, the vector fields in the right-hand side

$$f(x,u) = \begin{pmatrix} x_2 \\ u \end{pmatrix}, |u| \le 1,$$

are linear and the set of admissible velocities at a point is convex

$$f(x,U) = \{f(x,u) \mid |u| \le 1,$$

The time-optimal control problem has a solution if the origin $0 \in \mathbb{R}^2$ is attainable from the initial point x^0.

We will show that any point $x \in \mathbb{R}^2$ can be connected with the origin by an extremal curve.

Mathematical Modelling

2. Applying Pontryagin Maximum Principle:

Introduce canonical coordinates on the cotangent bundle:

$$M = \mathbb{R}^2,$$

$$T^*M = T^*\mathbb{R}^2 = \mathbb{R}^{2*} \times \mathbb{R}^2 = \{\lambda = (\xi, x) \mid x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \xi = (\xi_1, \xi_2)\}$$

The control dependent Hamiltonian function of PMP is

$$h_u(\xi, x) = (\xi_1, \xi_2)\begin{pmatrix} x_2 \\ u \end{pmatrix} = \xi_1 x_2 + \xi_2 u$$

and the corresponding Hamiltonian system has the form

$$\dot{x} = \frac{\delta h_u}{\delta \xi}, \qquad \dot{\xi} = -\frac{\delta h_u}{\delta x}$$

In coordinates this system splits into two independent subsystems:

$$\begin{Bmatrix} \dot{x_1} = x_2 \\ \dot{x_2} = u \end{Bmatrix}, \qquad \begin{Bmatrix} \dot{\xi_1} = 0 \\ \dot{\xi_2} = -\xi_1 \end{Bmatrix} \qquad (4)$$

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Mathematical Modelling

By PMP, if a control $\tilde{u}(.)$ is time optimal, then the Hamiltonian system has a nontrivial solution

$$\big(\xi(t), x(t)\big), \xi(t) \not\equiv 0$$

$$h_{\tilde{u}(t)}\big(\xi(t), x(t)\big) = \max_{|u|\le 1} h_u(\xi(t), x(t)) \ge 0$$

From this maximality condition if $\xi_2(t) \neq 0$, then $\tilde{u}(t) = sgn\xi_2(t)$.

Since $\ddot{\xi}_2 = 0$,

Then ξ_2 is linear

$$\xi_2(t) = \alpha + \beta(t) , \qquad \alpha,\ \beta = const.$$

Hence, the optimal control has the form

$$\tilde{u}(t) = \mathrm{sgn}(\alpha + \beta t)$$

So $\tilde{u}(t)$is piecewise constant takes only the extremal values ∓ 1 and has not more than one switching (discontinuity point).

Mathematical Modelling

3. Finding all trajectories x(t) that correspond to such controls and come to the origin.

For controls $u = \mp 1$ the first of subsystems (4) reads

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \pm 1 \end{cases}$$

Trajectories of this system satisfy the equation

$$\frac{dx_1}{dx_2} = \pm x_2,$$

thus are parabolas of the form

$$x_1 = \pm \frac{x_2^2}{2} + c, \qquad c = const.$$

i) Finding trajectories from this family that come to the origin without switching's : these are two semi parabolas

$$x_1 = \frac{x_2^2}{2}\,, \quad x_2 < 0, \quad \dot{x}_2 > 0, \qquad (5)$$

And

$$x_1 = -\frac{x_2^2}{2}\,, \quad x_2 > 0, \quad \dot{x}_2 < 0, \qquad (6)$$

for u =+1 and -1 respectively.

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Mathematical Modelling

ii) Find all extremal trajectories with one switching. Let $(x_{1s}, x_{2s}) \in \mathbb{R}^2$ be a switching point for anyone of curves (5),(6). Then extremal

Trajectories with one switching coming to the origin have the form

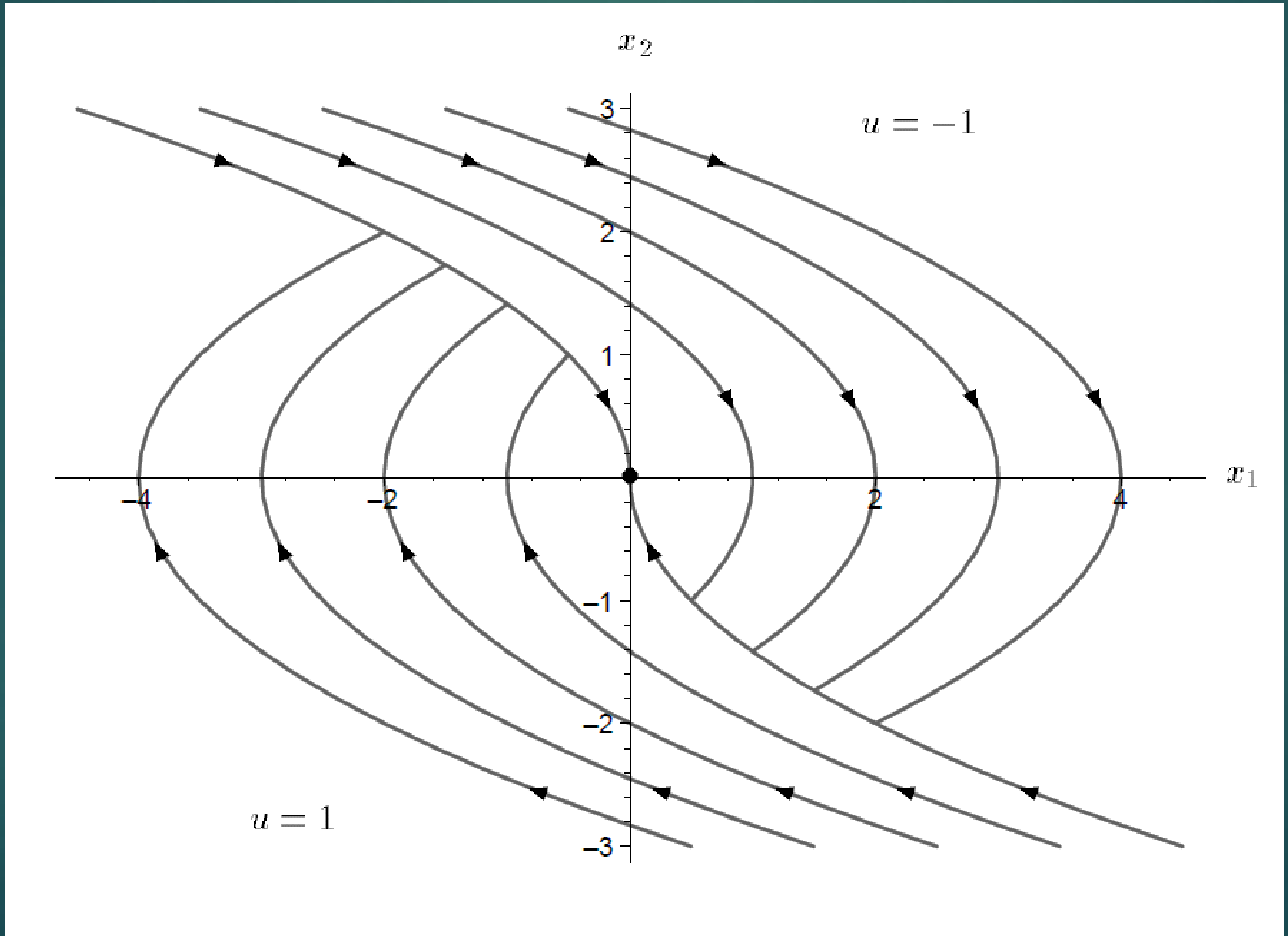
$$x_1 = \begin{cases} -\frac{x_2^2}{2} + \frac{x_{2s}^2}{2} + x_{1s}, & x_2 > x_{2s}, \quad \dot{x}_2 < 0, \\ \frac{x_2^2}{2} & 0 > x_2 > x_{2s}, \quad \dot{x}_2 > 0, \end{cases} \tag{7}$$

And

$$x_1 = \begin{cases} \frac{x_2^2}{2} - \frac{x_{2s}^2}{2} + x_{1s}, & x_2 < x_{2s}, \quad \dot{x}_2 > 0, \\ -\frac{x_2^2}{2} & 0 < x_2 < x_{2s}, \quad \dot{x}_2 < 0, \end{cases} \tag{8}$$

Mathematical Modelling

For any point of the plane there exists exactly one extremal trajectory steering this point to the origin. Since optimal trajectories exist, then the solutions found are optimal. The general view of the optimal synthesis is shown at figure below.



Analytical Explanation

- Bang-Bang Control: A control Signal which always takes it's maximum or minimum value with finitely many switching between than is called a bang-bang control.
- Example: Given the system;

$$\ddot{x}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

A x(t) B u(t)

With the performance index $J = \int_0^{t_f} dt = t_f$ is minimized with $|u(t)| \leq 1$

This constraint means that the control input magnitude must be with one or not greater than 1

Our problem is to transfer the initial state x(t)=x_0 to the final state $x(t_f) = 0$ (origin) in minimum time

Sol. $H(.) = 1 + \lambda^T(t)[Ax(t) + Bu(t)]$

$$H(.) = 1 + [\lambda_1(t):\lambda_2(t)]\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)\right]$$

$$H(.) = 1 + [\lambda_1(t) + \lambda_2(t)]\begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

$$H(.) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \qquad (1)$$

From Eqn.(1) we can say that our H(.) is the linear function of H by using Pontryagin maximum/minimum principle we can minimize H(.), only when u(t) is minimum and it is only possible with the knowledge of $\lambda_2(t)$.

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Analytical Explanation

- Co-state Equations are

$$\dot{\lambda}(t) = -\frac{\partial H(.)}{\partial x(t)} = -\begin{bmatrix} 0 \\ \lambda_1(t) \end{bmatrix}$$

$$\dot{\lambda_1}(t) = 0 \qquad (2)$$

$$\dot{\lambda_2}(t) = -\lambda_1(t) \qquad (3)$$

From (2), $\frac{d\lambda_1}{dt} = 0 \longrightarrow \lambda_1(T) = \lambda_1(0)$ (4)

From (3) $\dot{\lambda_2}(t) = -\lambda_1(t) = -\lambda_1(0)$

$$\frac{d\lambda_2(t)}{dt} = -\lambda_1(0), \quad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0) \qquad (5)$$

From (1) $H(.) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t).u(t)$

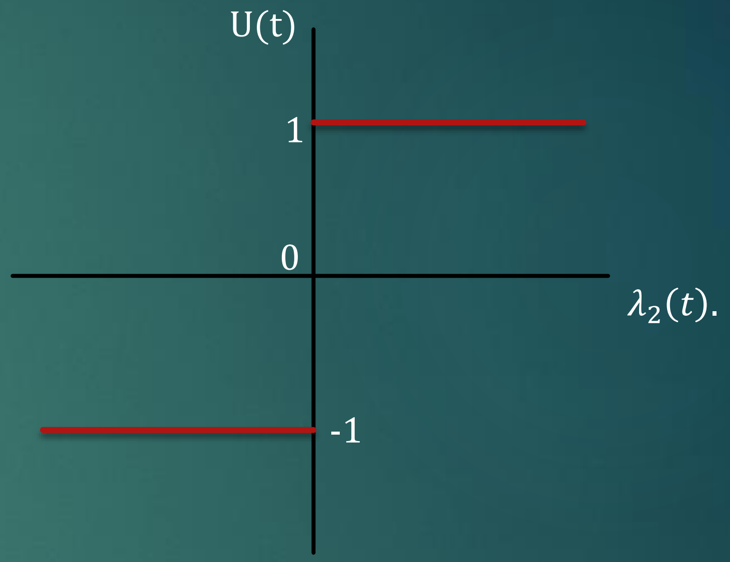
$$u^*(t) = 1 \quad if \quad \lambda_2(t) < 0$$

$$= -1 \quad if \quad \lambda_2(t) > 0$$

$$u^*(t) = -sign(\lambda_2(t)) = -sign(\lambda_2(0) - \lambda_1(0)t)$$

$$Sign = 1 \quad if\ z > 0$$

$$-1 \quad if\ z < 0$$



Analytical Explanation

Now, if u(t) =1

$\dot{x_1}(t) = x_2(t),\ \dot{x_2}(t) = 1,\ \ x_2(t) = t + x_2(0)$

$\dot{x_1}(t) = x_2(0) + t$

$x_1(t) = \frac{t^2}{2} + x_2(0) + x_1(0)$

$x_1(t) = x_1(0) + x_2(0).t + \frac{t^2}{2}$

$x_1(t) = x_1(0) + x_2(0)\big(x_2(t) - x_2(0)\big) + \frac{(x_2(t)-x_2(0))^2}{2}$

$x_1(t) = \frac{x_2^2(t)}{2} + \frac{x_2^2(0)}{2} + x_1(0) \rightarrow for\ u = +1 \qquad (6)$

for $u^*(t) = -1$, the state equation is

$\dot{x_1}(t) = x_2(t);\ \dot{x_2}(t) = -1$

$x_2(t) = x_2(0) - t$

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Analytical Explanation

- $x_1(t) = x_1(0) + x_2(0)t - \frac{t^2}{2}$

$$x_1(t) = -\frac{x_2^2(t)}{2} + \left(x_1(0) + \frac{x_2^2(0)}{2}\right) \rightarrow for\ u = -1 \qquad (7)$$

Depending on the initial condition our parabolas will defined

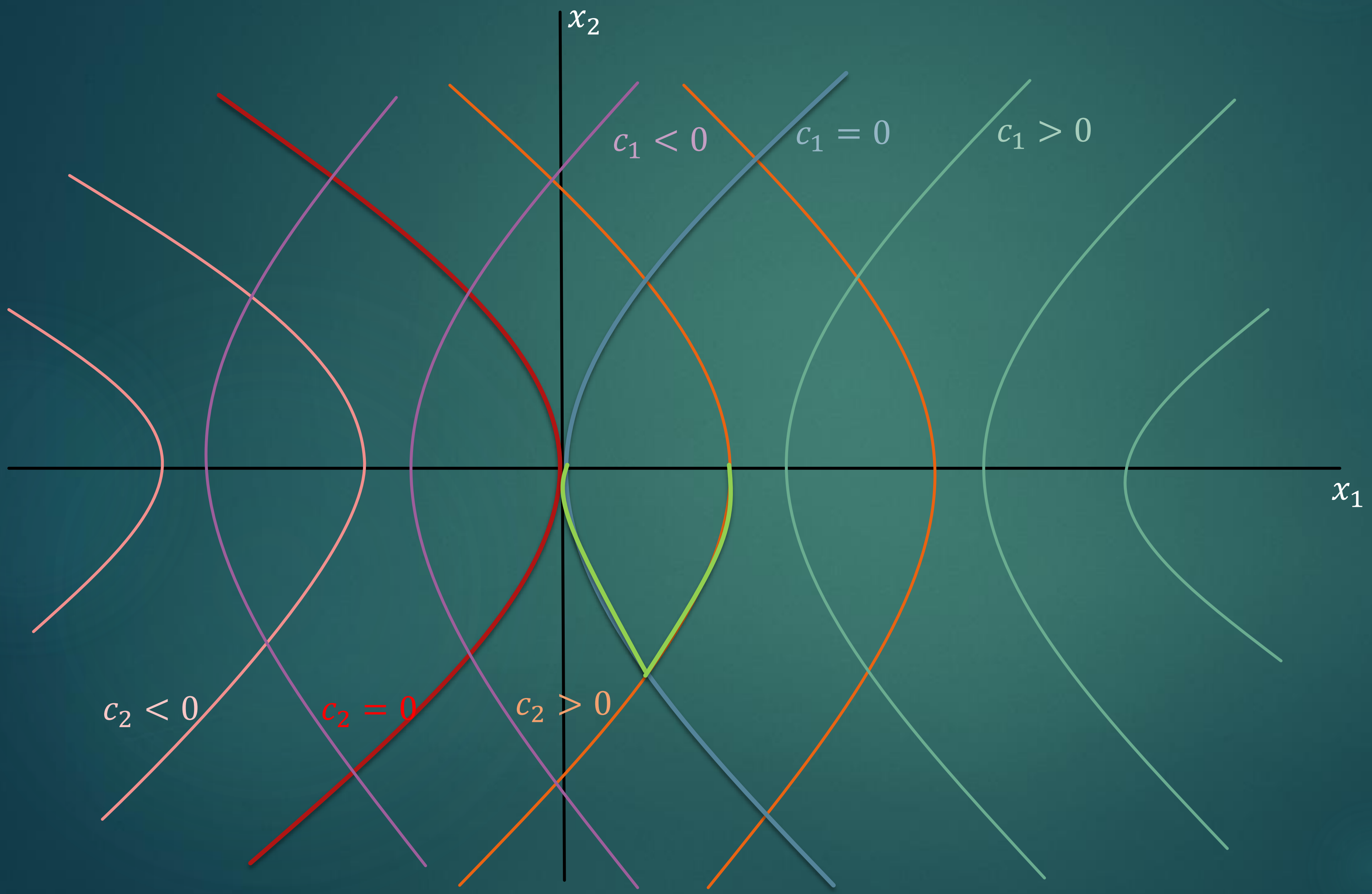
From (6) & (7), we have,

$$\frac{x_2^2(t)}{2} = x_1(t) - c_1, \qquad \text{where } c_1 = x_1(0) - \frac{x_2^2(0)}{2}$$

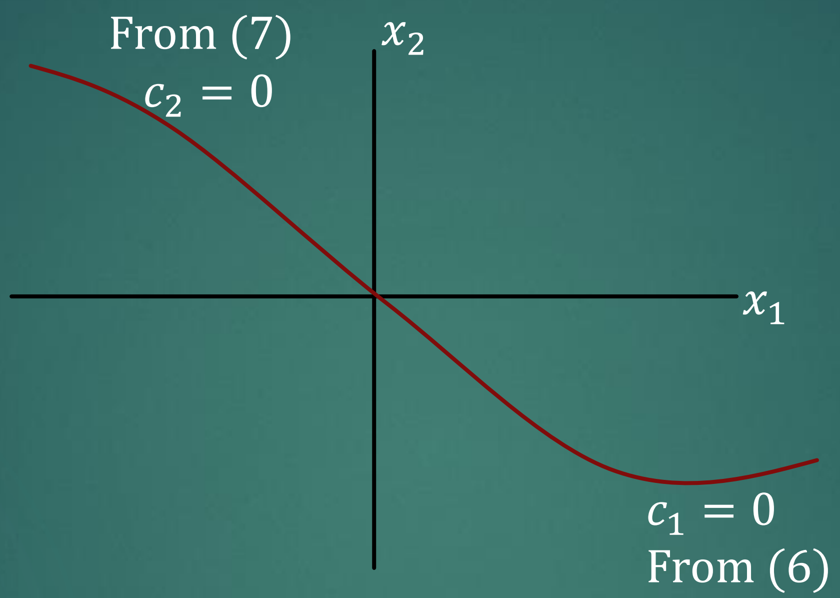
From (7)

$$\frac{x_2^2(t)}{2} = -x_1(t) + c_2, \qquad \text{where } c_2 = x_1(0) + \frac{x_2^2(0)}{2}$$

Phase Portrait of Hamiltonian function



Switching Curve

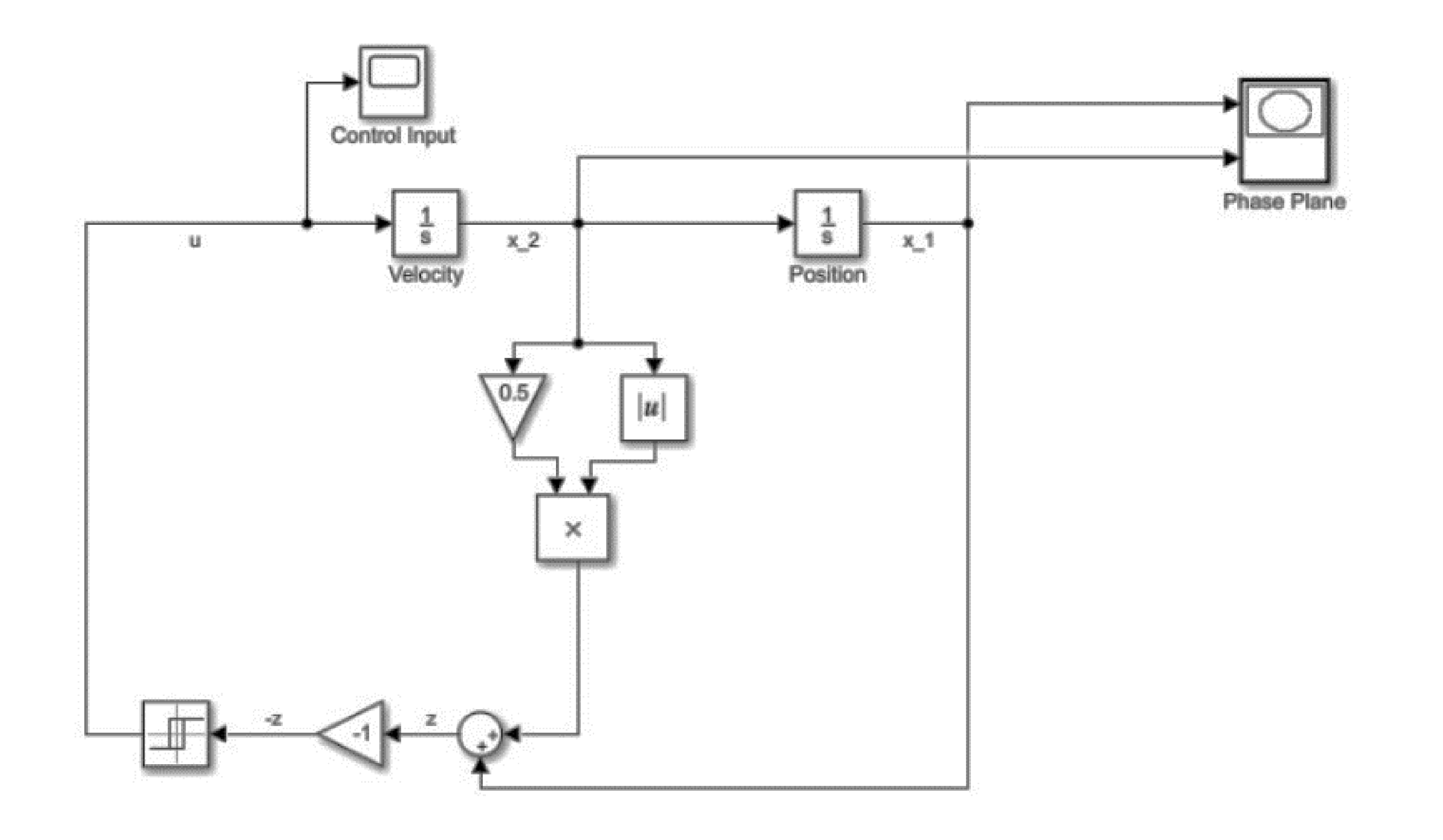


- Control Law

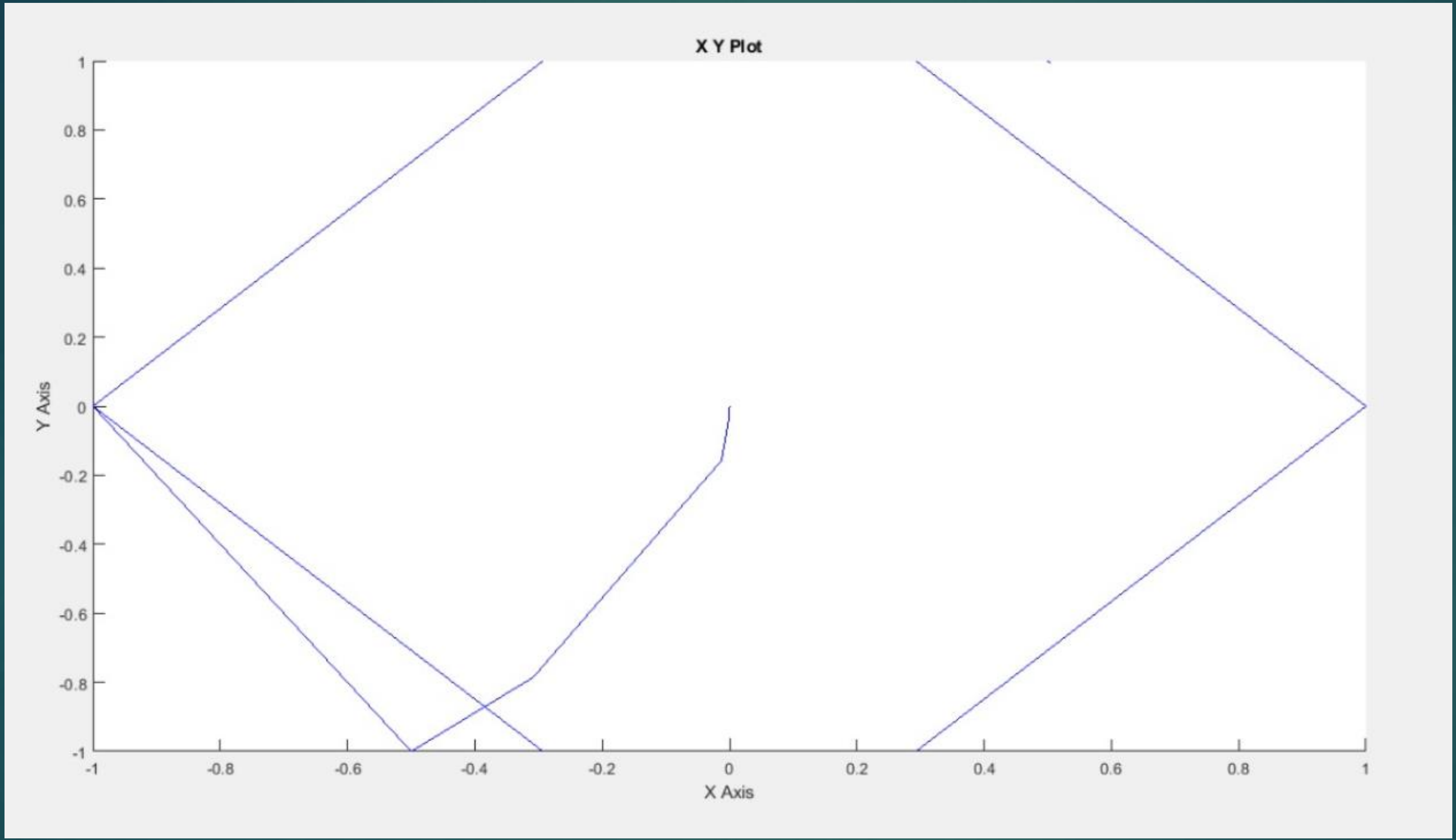
$u^*(t) = -Sign(\lambda_2(0) - \lambda_1(0)t)$ from (6) & (7) when $c_1 = c_2 = 0$

- Note: Only one family from parabola passes through $x = [0,0]^T$

Simulink Model



Simulink Result (Phase Plane Plot)



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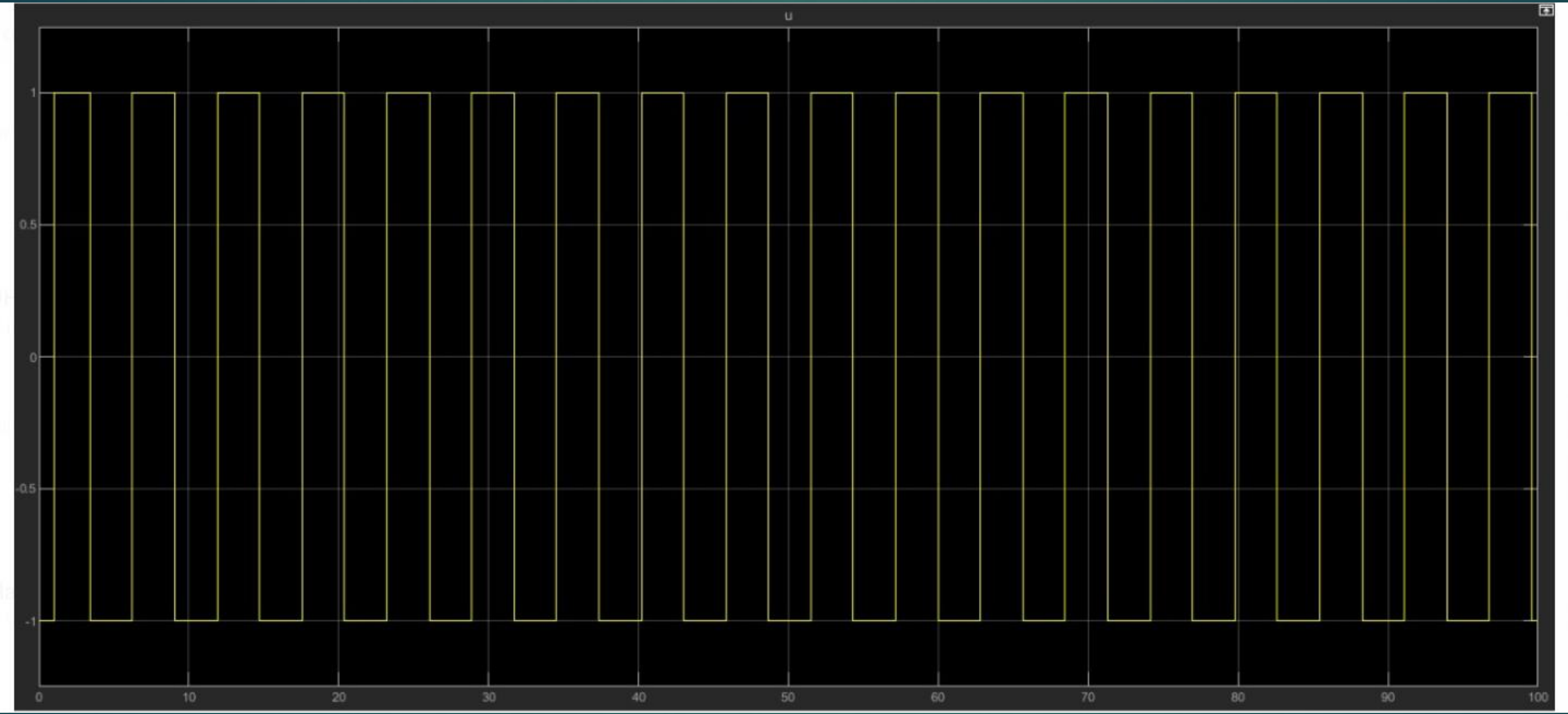
Control Input



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Implementation in matlab

- System

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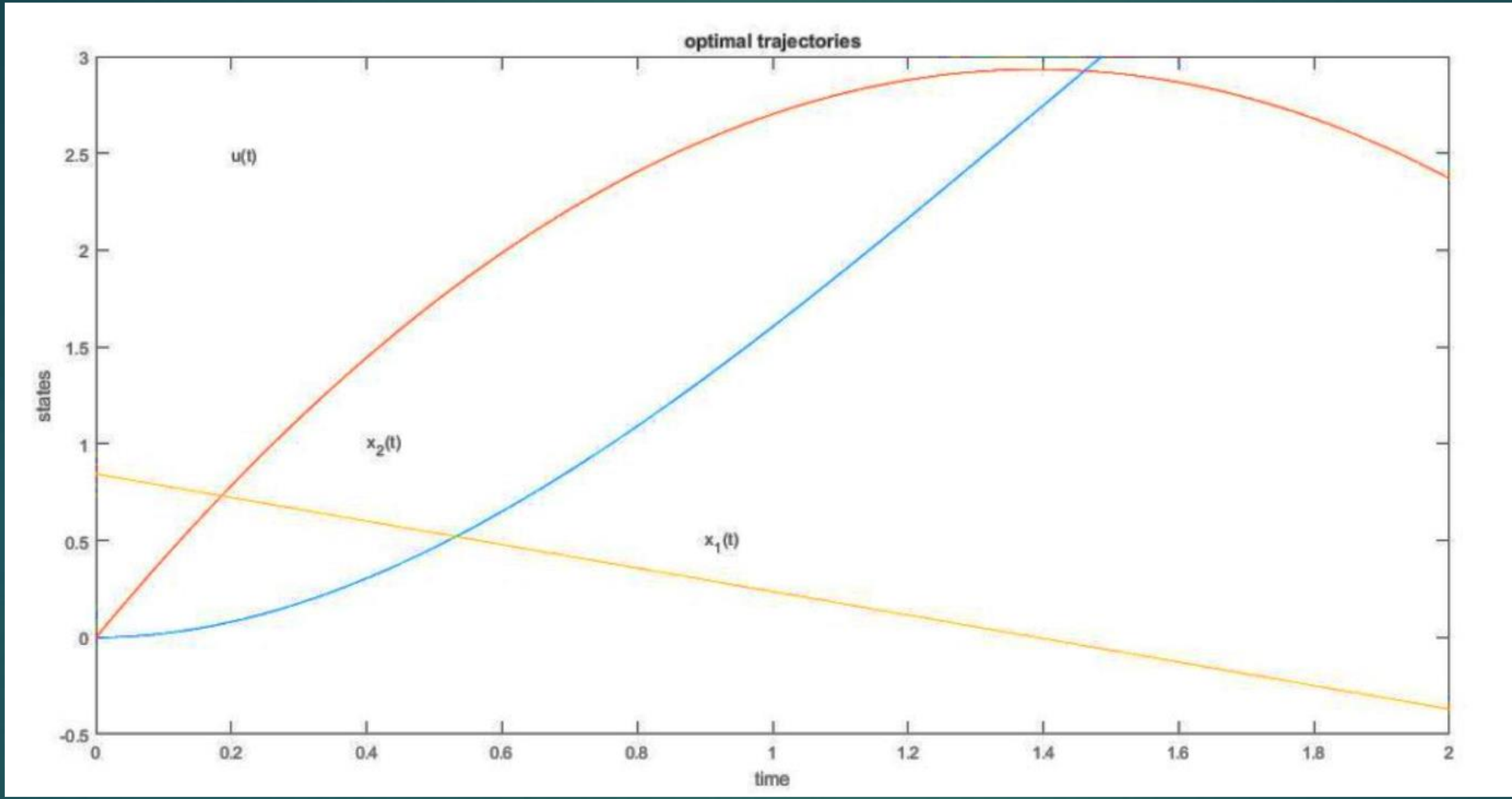
MATLAB Plot (Control Input)



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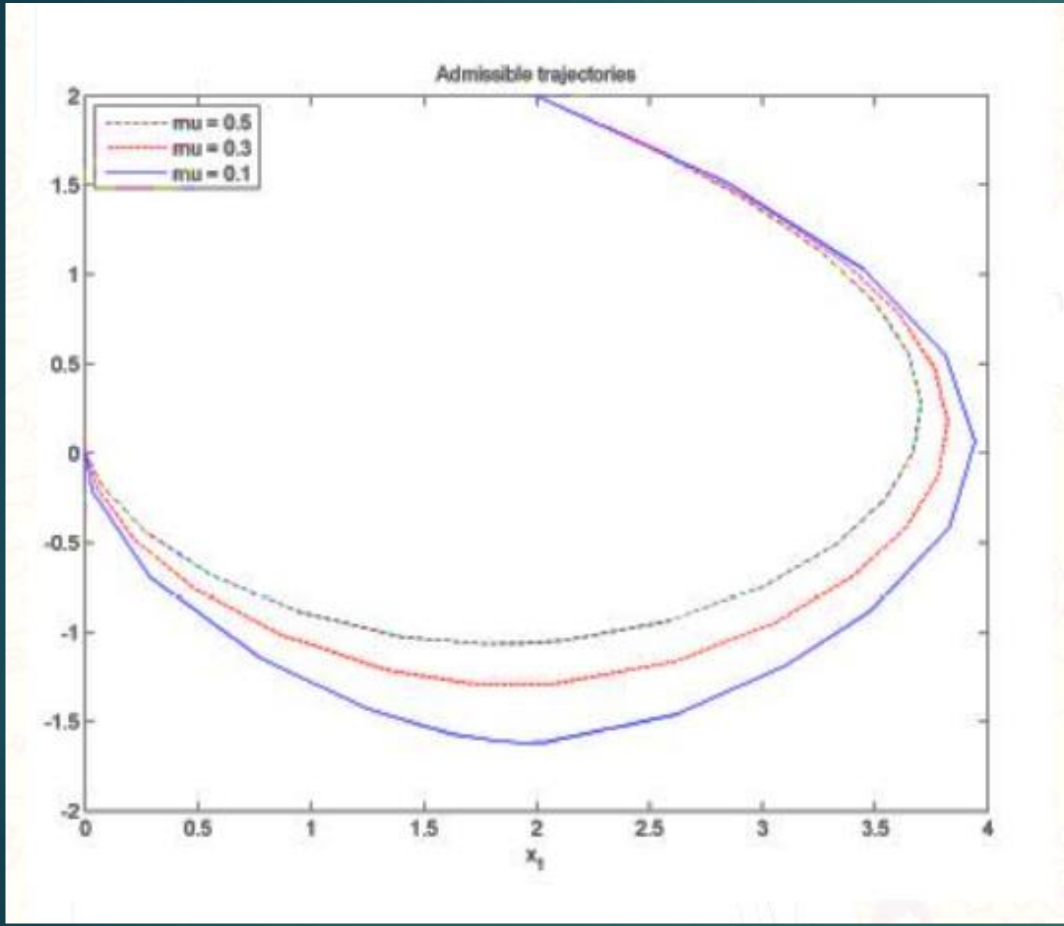
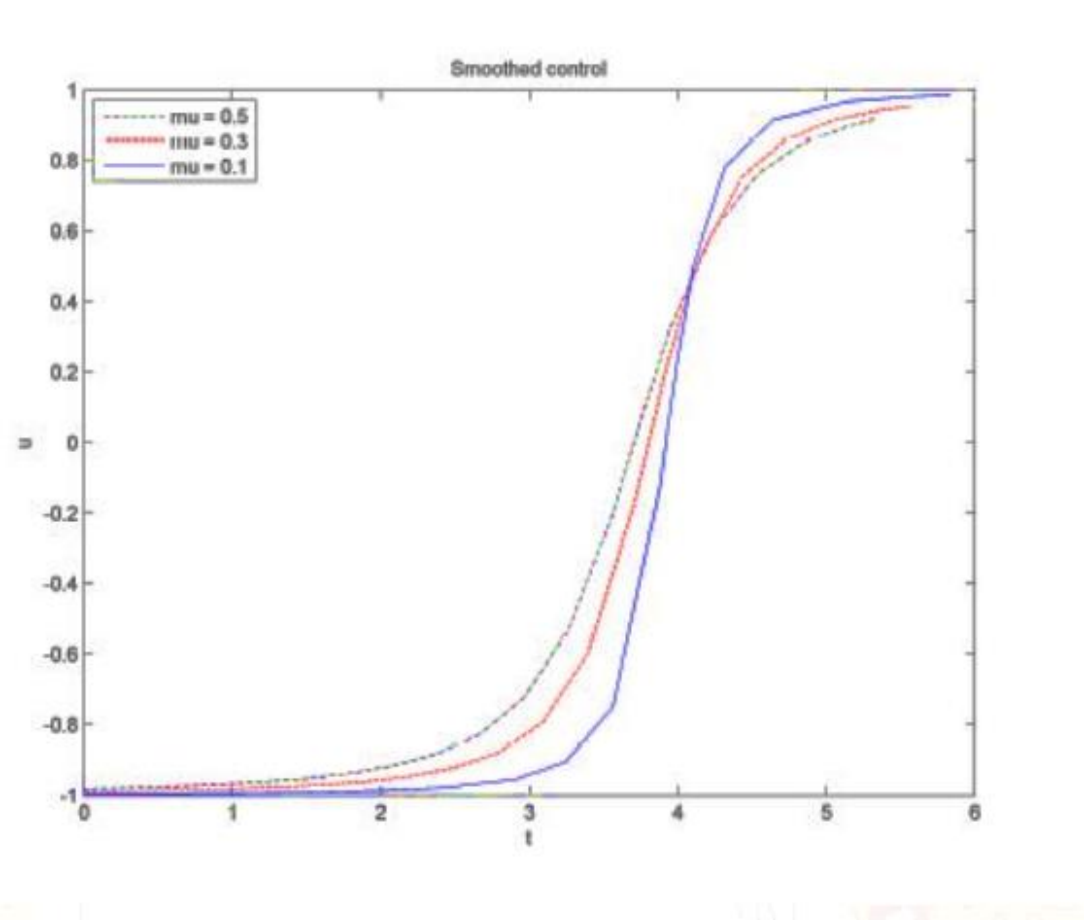
Implementation in matlab

- System
- Condition

Results



Admissible trajectories
mu = 0.5
mu = 0.3
mu = 0.1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
x_1



Smoothed control
mu = 0.5
mu = 0.3
mu = 0.1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
u
0
1
2
3
4
5
6
t

Application:

1.We can apply our Mathematical model also in the automobile sector.

2.This algorithm with bang-bang control can also be used for collision avoidance.

3.This problem have a lot of scope in robotics and aerospace engineering.

4.Pontryagin maximum along with three different strategies can also be used for the modelling and optimal control of typhoid fever disease.

5.This algorithm also proved to be efficient to land prob on mars.

Product Quality:

1.The algorithm that we used is more computationally efficient.

2.We can stop or start our algorithm when desired state is reached in minimum time.

3. There is a hope to minimize the energy in order to stop the train in the future through our problem.

Conclusion

In this project we presented one of the examples of optimal control which is the application of Pontryagin maximum principle and bang-bang control.

We provided the theoretical and mathematical explanation of the problem.

In the first stage we minimized the time which is also our cost function using Pontryagin maximum principle then we derive the equation of optimal trajectories. In second stage we gave the more detailed mathematical explanation, lastly we implemented this mathematical model in Simulink and MATLAB in order to validate our algorithm and to get the more practical exposure.

We can further improve our problem by taking one more factor i.e. energy into the consideration for minimization which will impact the dynamical system quite heavily.

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