CS7015 (Deep Learning) : Lecture 13

Sequence Learning Problems, Recurrent Neural Networks, Backpropagation Through Time (BPTT), Vanishing and Exploding Gradients, Truncated BPTT

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Module : Sequence Learning Problems

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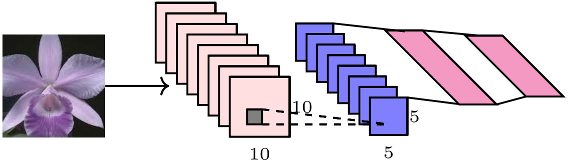

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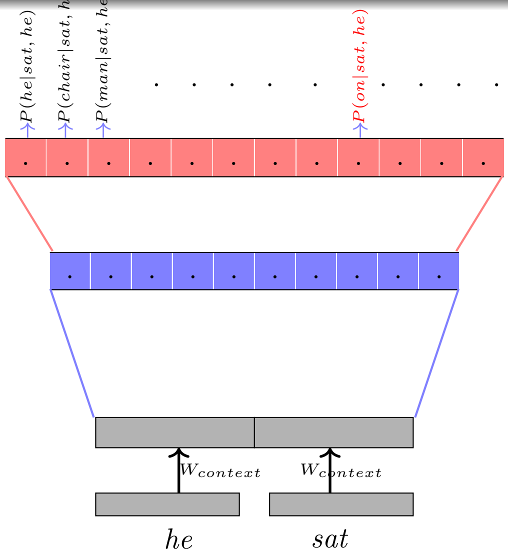

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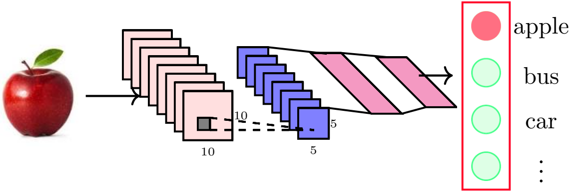

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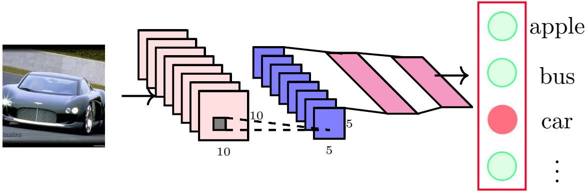

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- For example, the computatations, outputs and decisions for two successive images are completely independent of each other

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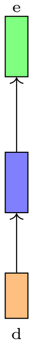

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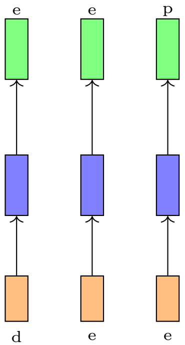

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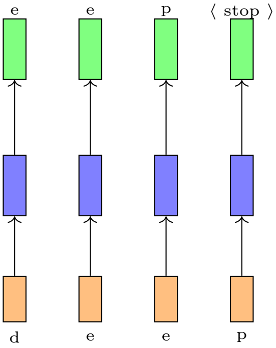

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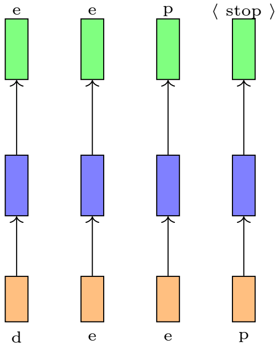

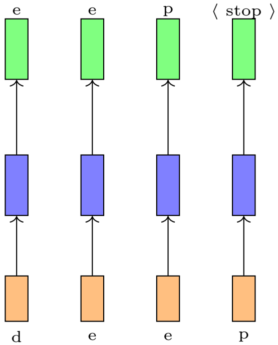

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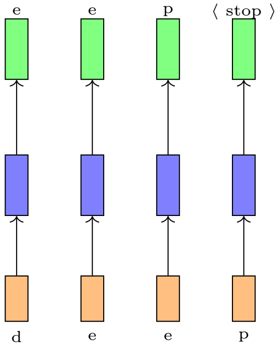

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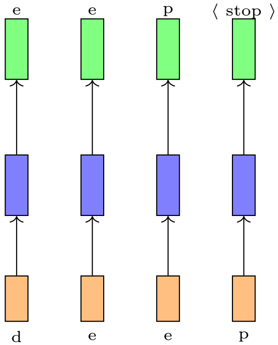

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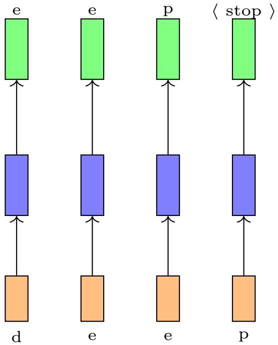
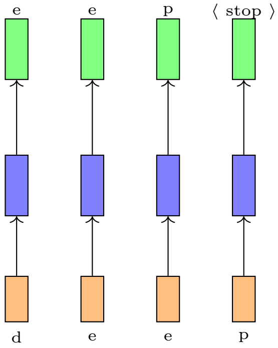
- Second, the length of the inputs and the number of predictions you need to make is not fixed (for example, “learn”, “deep”, “machine” have different number of characters)
- Third, each network (orange-blue-green structure) is performing the same task (**input** : character **output** : character)

- These are known as sequence learning problems





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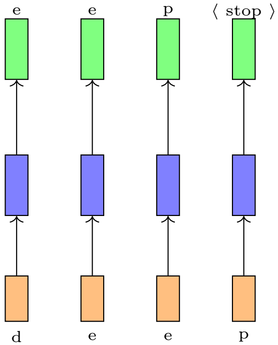

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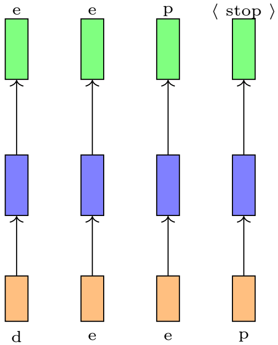

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- Let us look at some more examples of such problems

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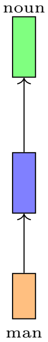

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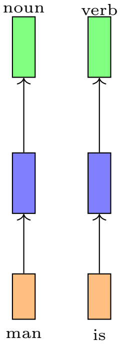

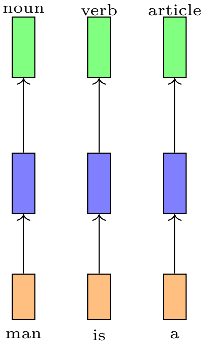

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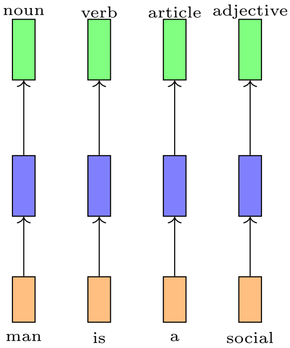

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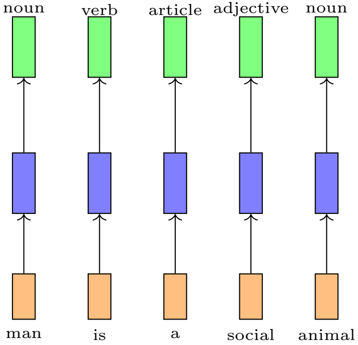

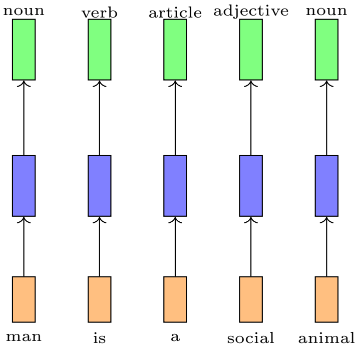

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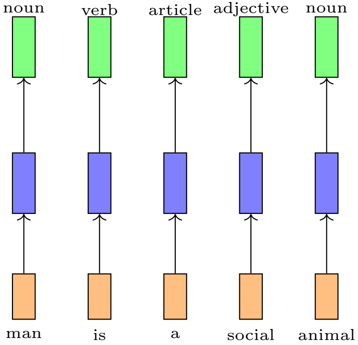

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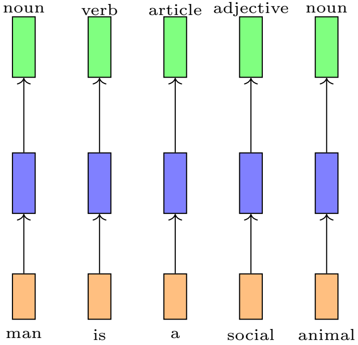

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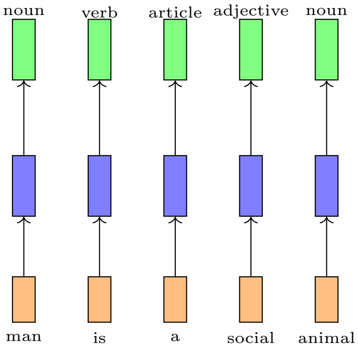

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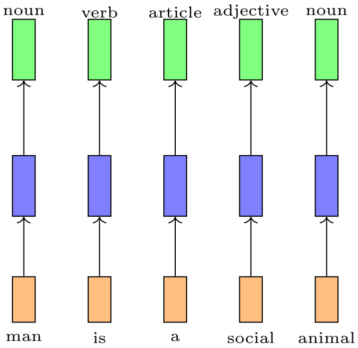

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- Each network is performing the same task (**input** : word, **output** : tag)

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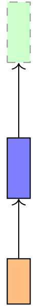

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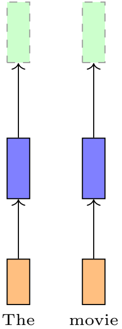

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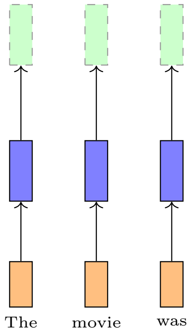

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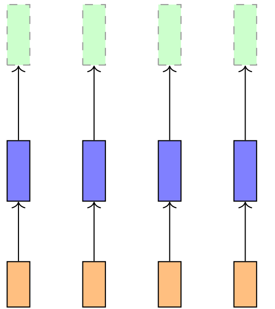

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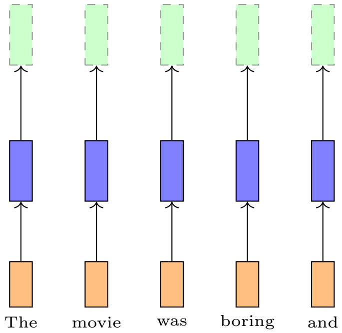

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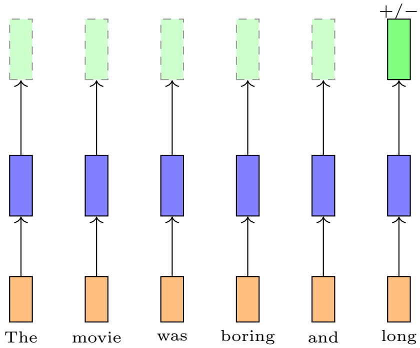

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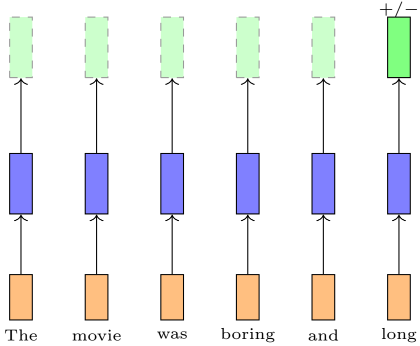

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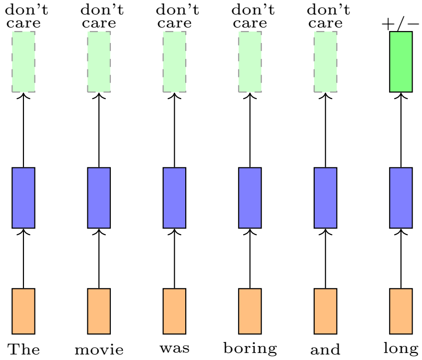

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- Here again we could think that the network is performing the same task at each step (input : word, output : $+/-$) but it's just that we don't care about intermediate outputs

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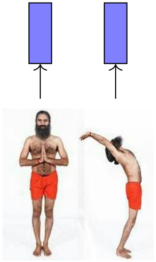

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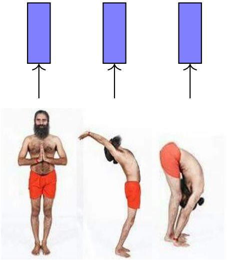

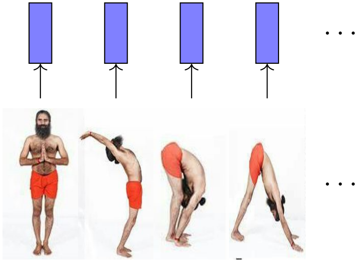

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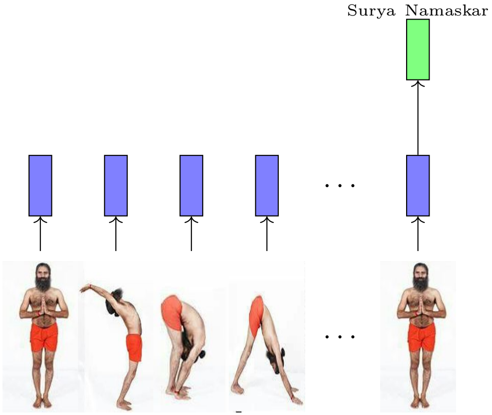

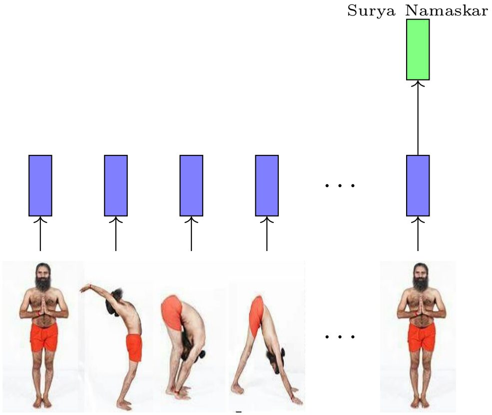

- Sequences could be composed of anything (not just words)
- For example, a video could be treated as a sequence of images
- We may want to look at the entire sequence and detect the activity being performed

Module : Recurrent Neural Networks

How do we model such tasks involving sequences ?

Wishlist

- Account for dependence between inputs

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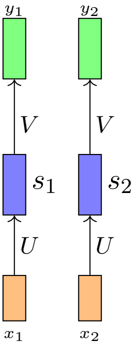
- Make sure that the function executed at each time step is the same
- We will focus on each of these to arrive at a model for dealing with sequences

- What is the function being executed at each time step ?



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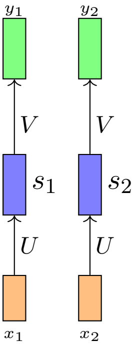

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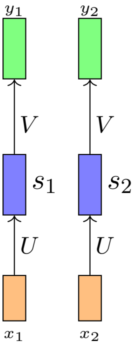

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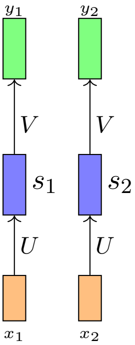

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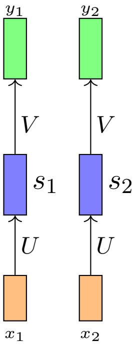
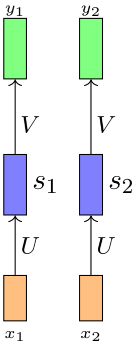
- Since we want the same function to be executed at each timestep we should share the same network (i.e., same parameters at each timestep)

- This parameter sharing also ensures that the network becomes agnostic to the length (size) of the input





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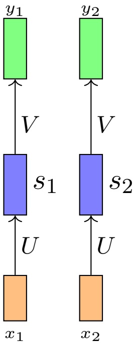

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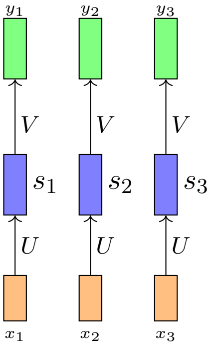

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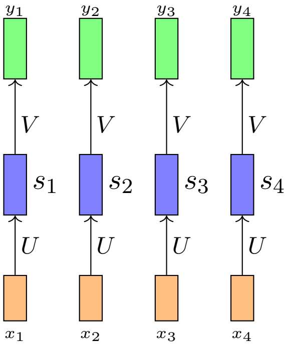

- This parameter sharing also ensures that the network becomes agnostic to the length (size) of the input
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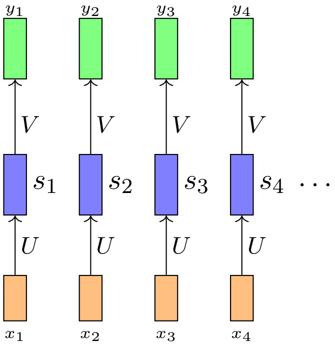

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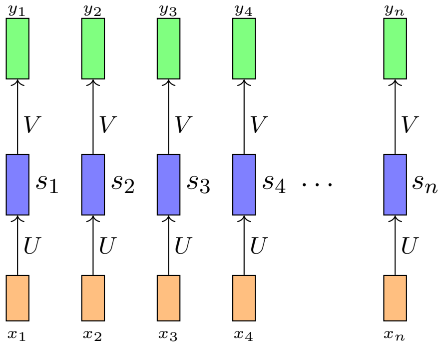

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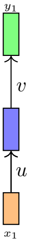
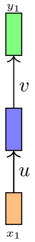
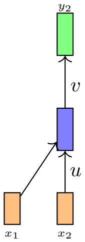
- How do we account for dependence between inputs ?

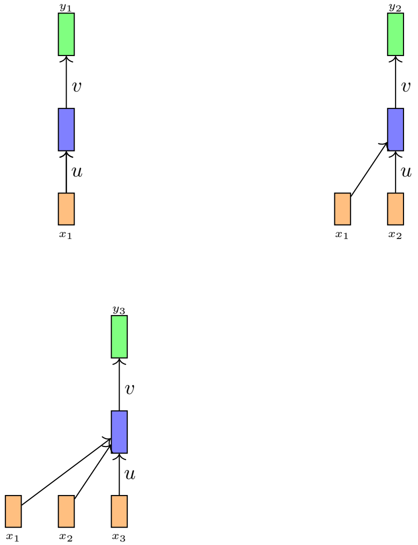


- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this





- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this
- At each timestep we will feed all the previous inputs to the network



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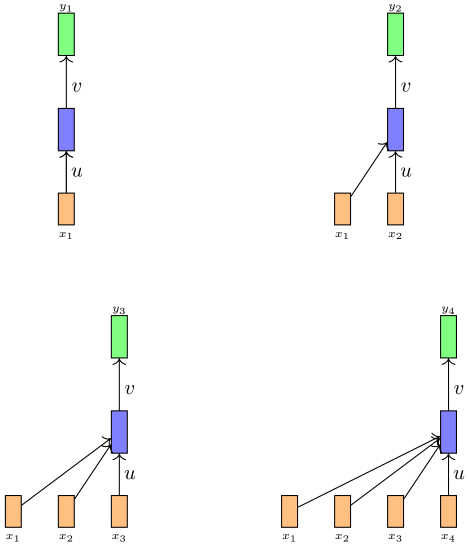

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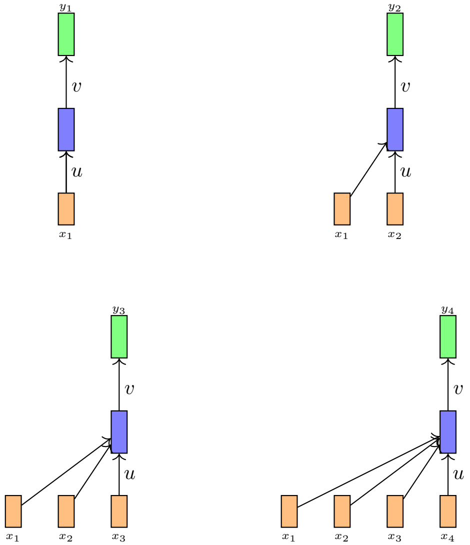

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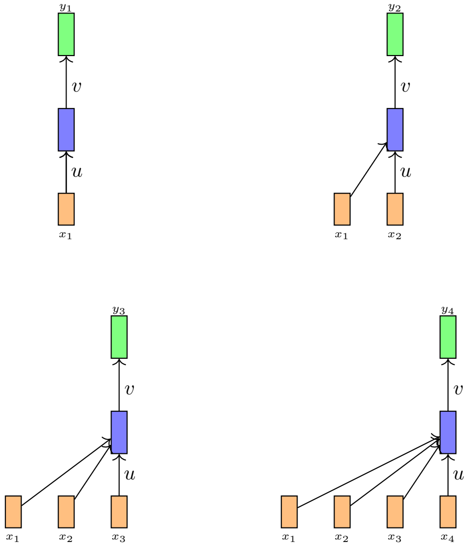
- Is this okay ?



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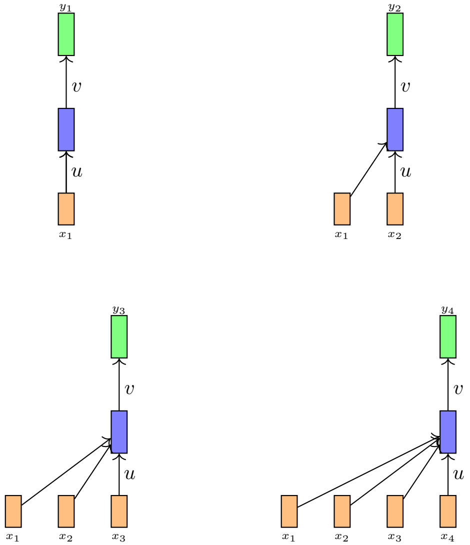
- Is this okay ?
- No, it violates the other two items on our wishlist



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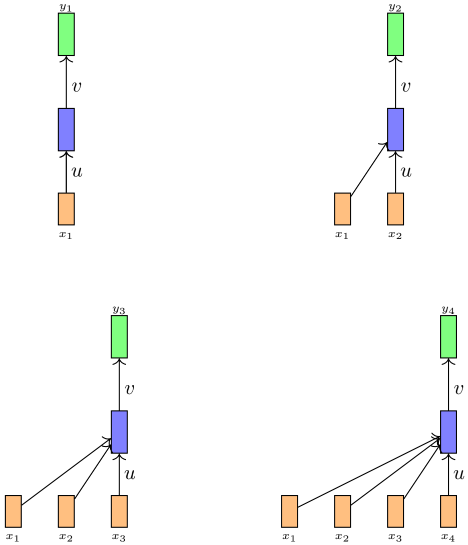

- How do we account for dependence between inputs ?
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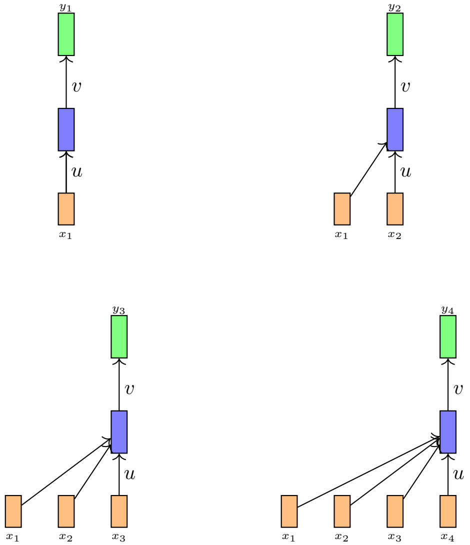
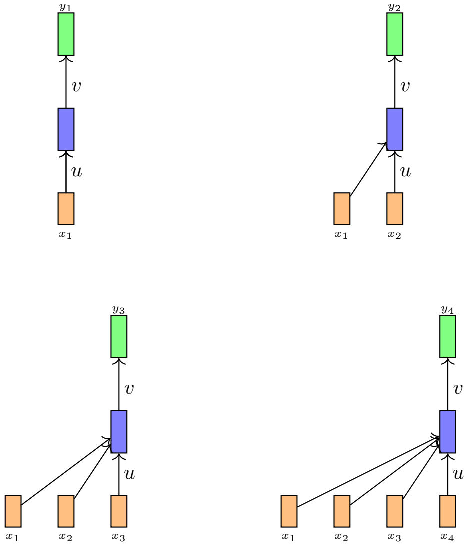
- Is this okay ?
- No, it violates the other two items on our wishlist
- How ? Let us see



- First, the function being computed at each time-step now is different



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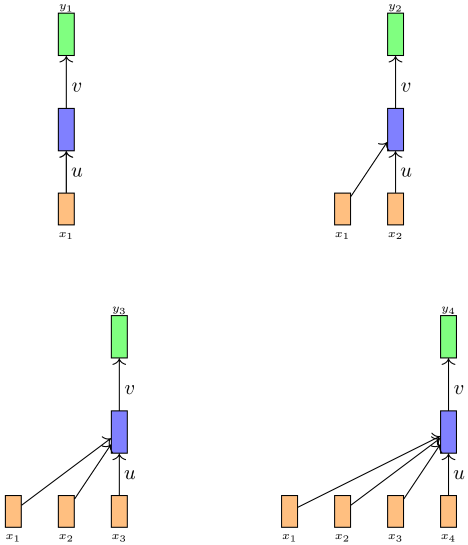
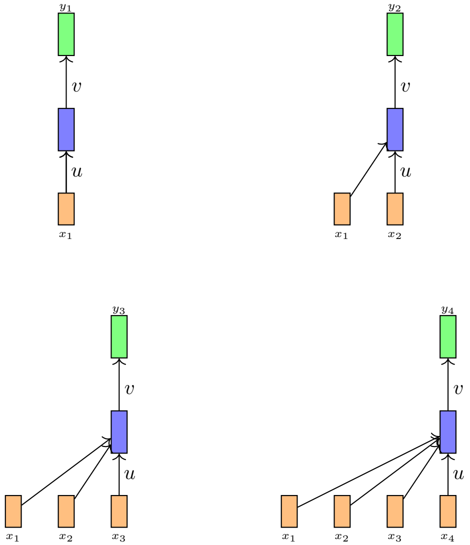
$$y_1 = f_1(x_1)$$



- First, the function being computed at each time-step now is different

$$y_1 = f_1(x_1)$$
$$y_2 = f_2(x_1, x_2)$$



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$$y_1 = f_1(x_1)$$
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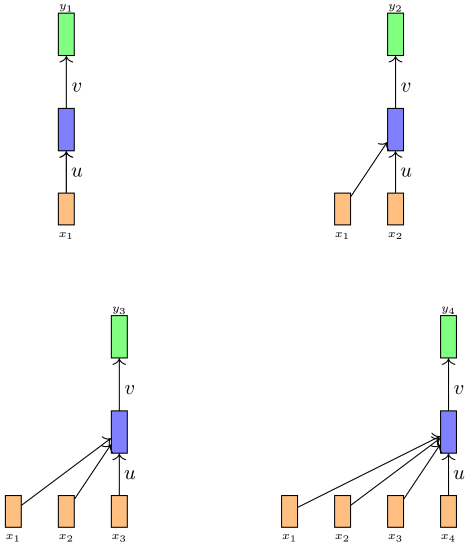

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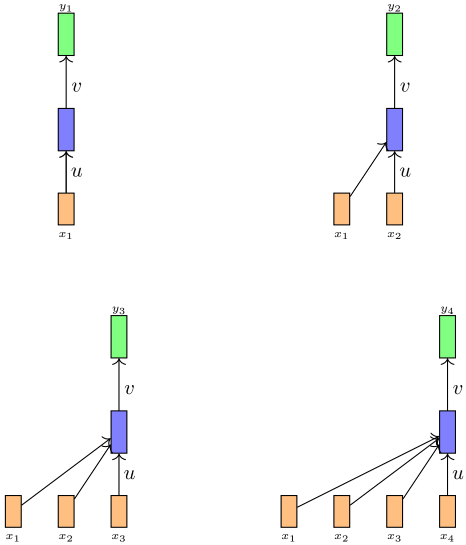
- The network is now sensitive to the length of the sequence



- First, the function being computed at each time-step now is different

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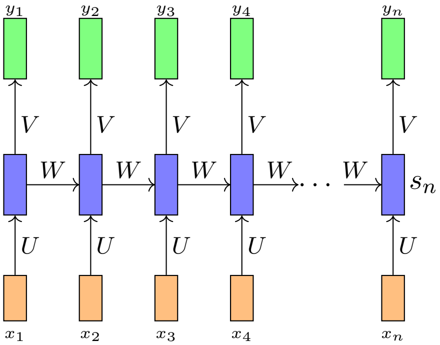
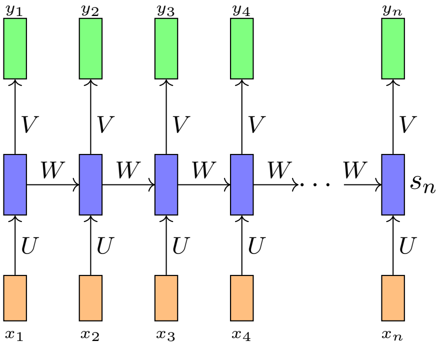
\begin{aligned}
y_1 &= f_1(x_1) \\
y_2 &= f_2(x_1, x_2) \\
y_3 &= f_3(x_1, x_2, x_3)
\end{aligned}
$$

- The network is now sensitive to the length of the sequence
- For example a sequence of length 10 will require $f_1, \ldots, f_{10}$ whereas a sequence of length 100 will require $f_1, \ldots, f_{100}$

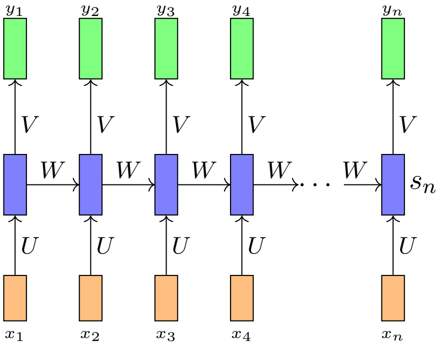
- The solution is to add a recurrent connection in the network,





- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$



- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
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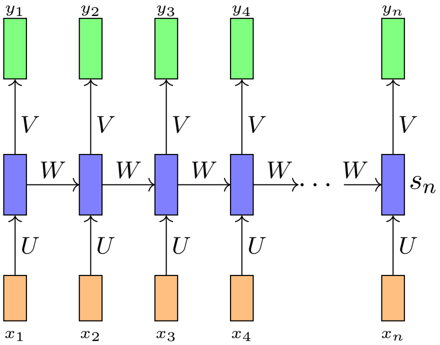

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$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
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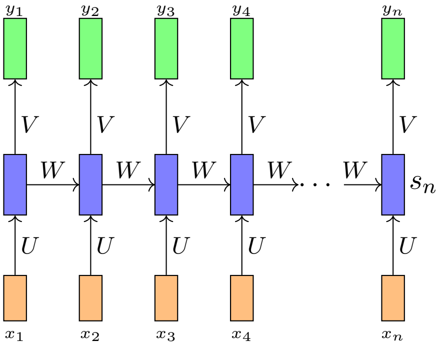
$$or$$



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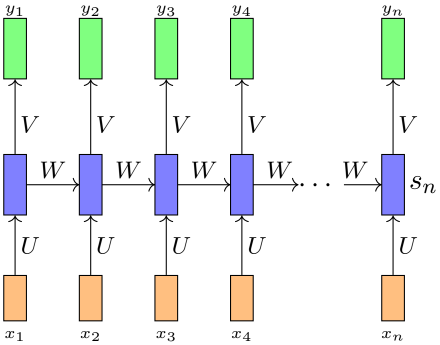
$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
$$y_i = \mathcal{O}(Vs_i + c)$$
$$or$$
$$y_i = f(x_i, s_{i-1}, W, U, V, b, c)$$



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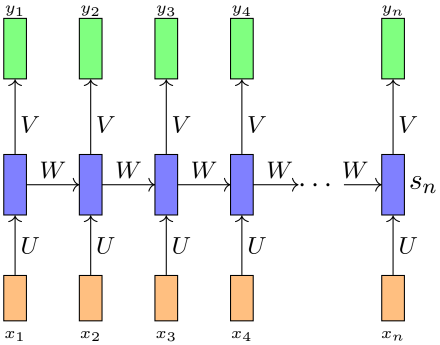
- s_i is the state of the network at timestep i



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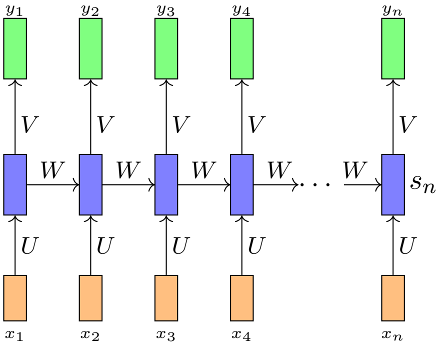
- s_i is the state of the network at timestep i
- The parameters are W, U, V, c, b which are shared across timesteps



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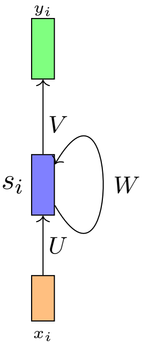
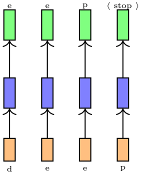
\begin{aligned}
s_i &= \sigma(Ux_i + Ws_{i-1} + b) \\
y_i &= \mathcal{O}(Vs_i + c) \\
& \quad or \\
y_i &= f(x_i, s_{i-1}, W, U, V, b, c)
\end{aligned}
$$

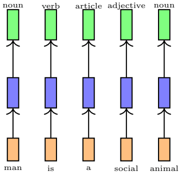
- s_i is the state of the network at timestep i
- The parameters are W, U, V, c, b which are shared across timesteps
- The same network (and parameters) can be used to compute $y_1, y_2, \ldots, y_{10}$ or y_{100}

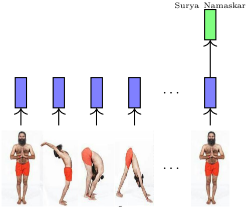


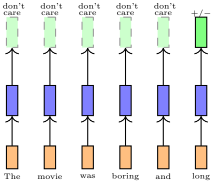
- This can be represented more compactly

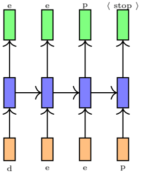
- Let us revisit the sequence learning problems that we saw earlier

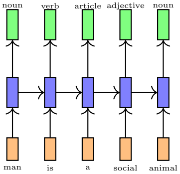


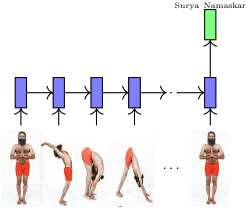


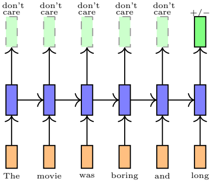






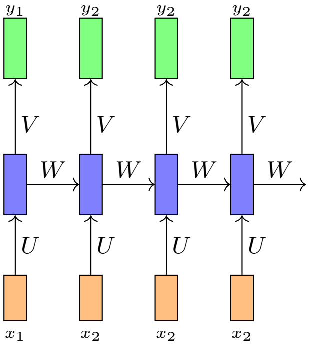




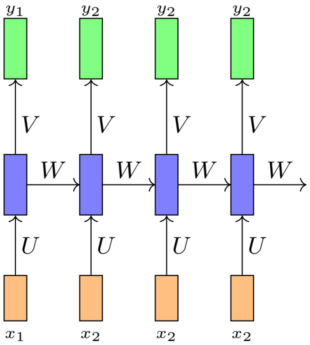


- Let us revisit the sequence learning problems that we saw earlier
- We now have recurrent connections between time steps which account for dependence between inputs

Module : Backpropagation through time



- Before proceeding let us look at the dimensions of the parameters carefully



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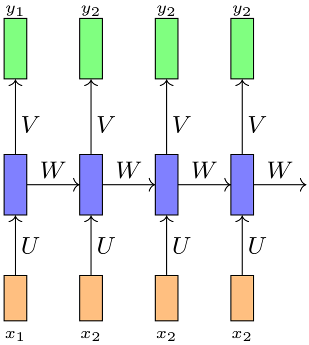
$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$



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$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
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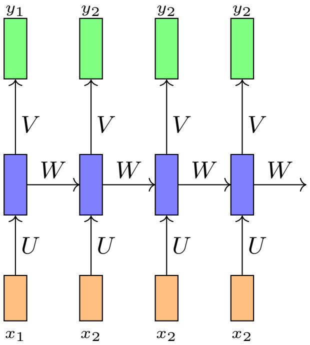

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$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
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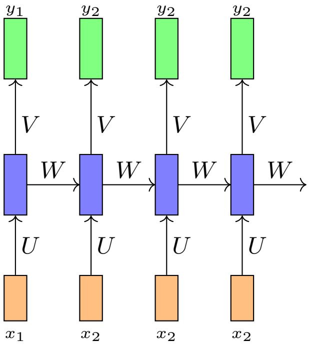
$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$



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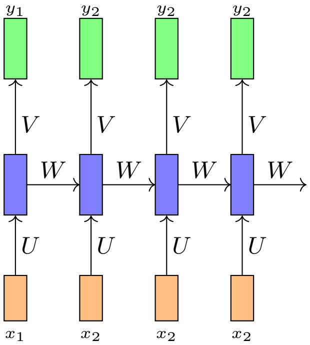

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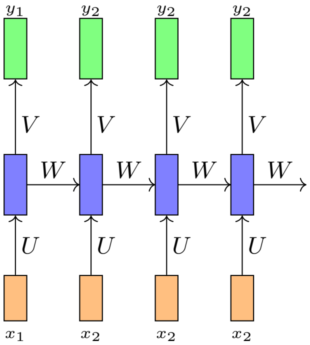

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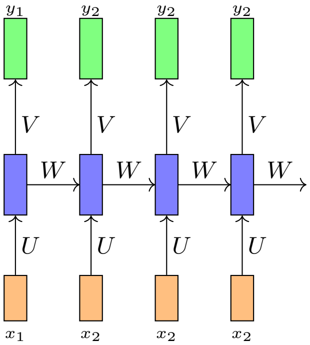

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$$U \in \mathbb{R}^{n \times d}$$

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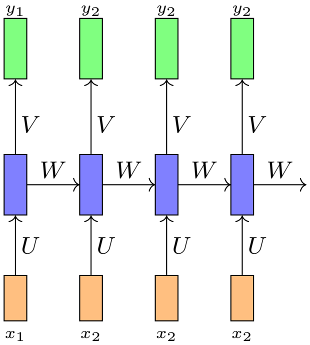

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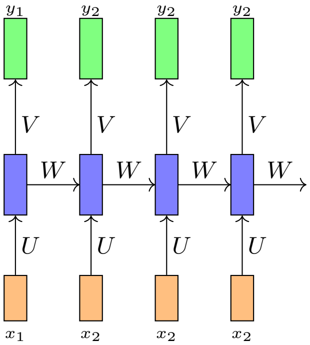
$$U \in \mathbb{R}^{n \times d}$$
$$V \in \mathbb{R}^{d \times k}$$
$$W \in$$



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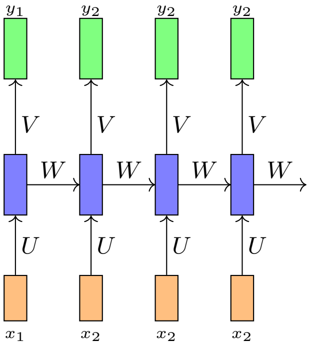
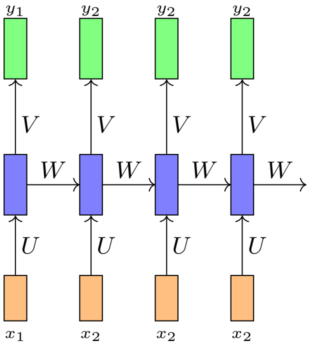
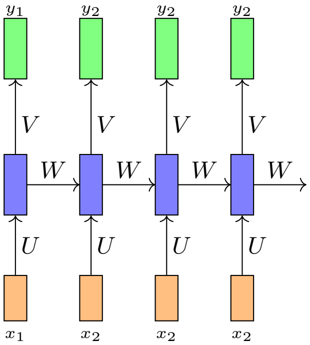
$$U \in \mathbb{R}^{n \times d}$$
$$V \in \mathbb{R}^{d \times k}$$
$$W \in \mathbb{R}^{d \times d}$$

- How do we train this network ?



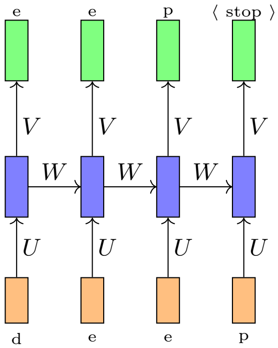
- How do we train this network ? (**Ans**: using backpropagation)

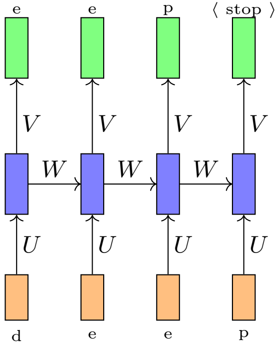




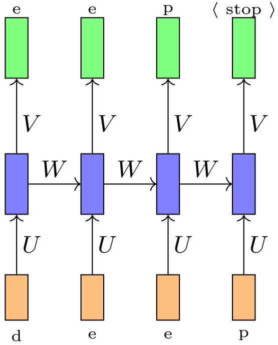
- How do we train this network ? (**Ans**: using backpropagation)
- Let us understand this with a concrete example

- Suppose we consider our task of auto-completion (predicting the next character)





- Suppose we consider our task of auto-completion (predicting the next character)
- For simplicity we assume that there are only 4 characters in our vocabulary (d,e,p, <stop>)



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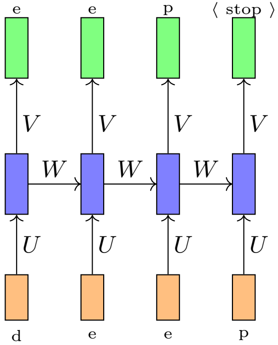

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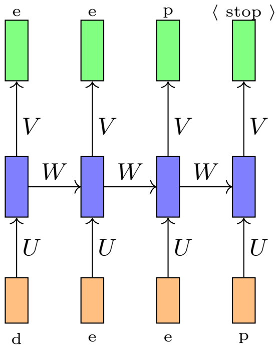

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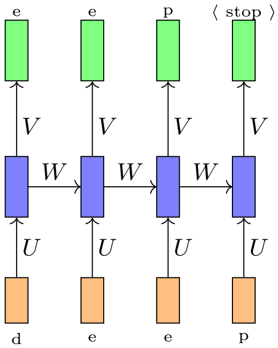

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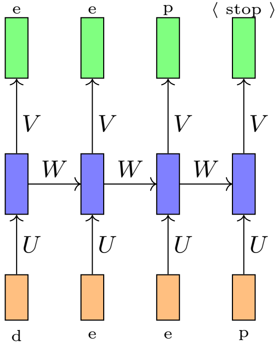

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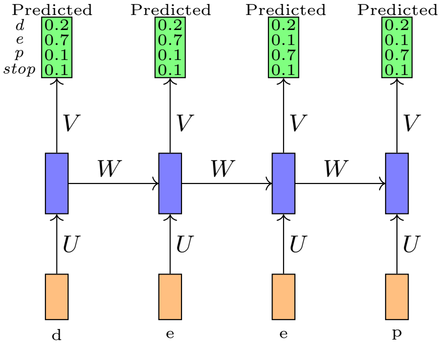

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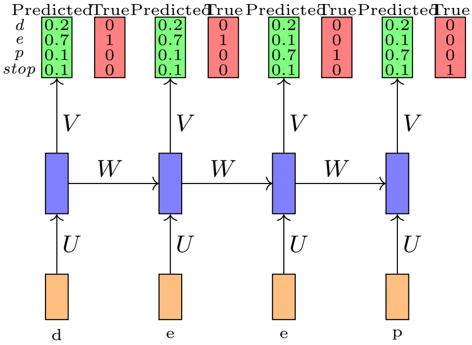

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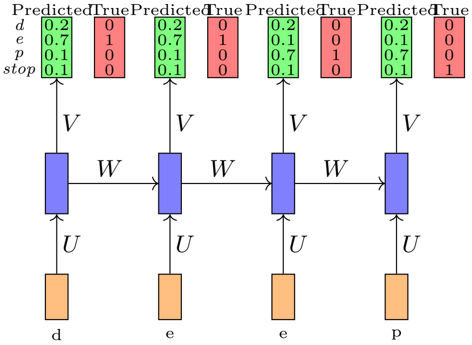

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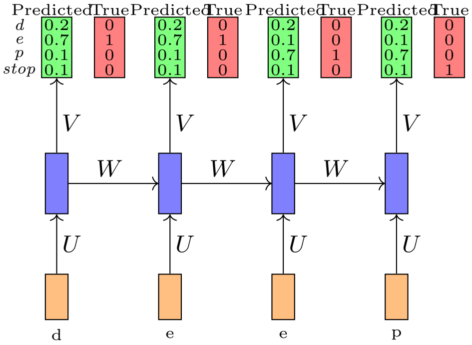

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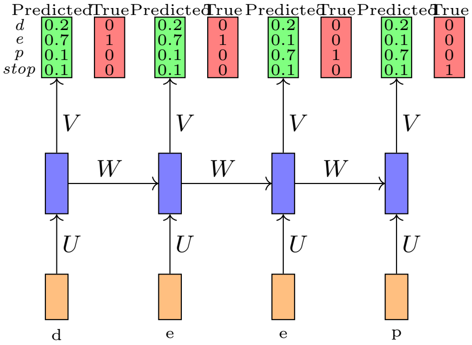

- Suppose we initialize U, V, W randomly and the network predicts the probabilities as shown
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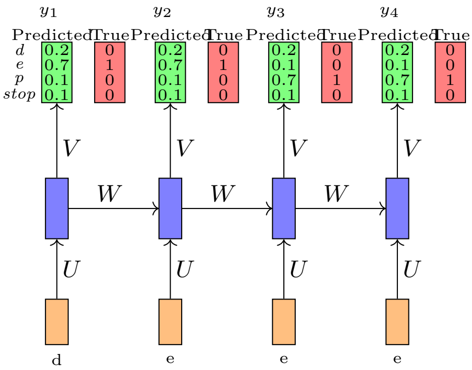
- We need to answer two questions
- What is the total loss made by the model ?
- How do we backpropagate this loss and update the parameters ($\theta = \{U, V, W, b, c\}$) of the network ?



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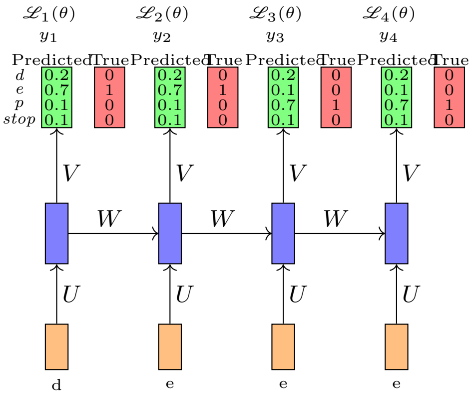

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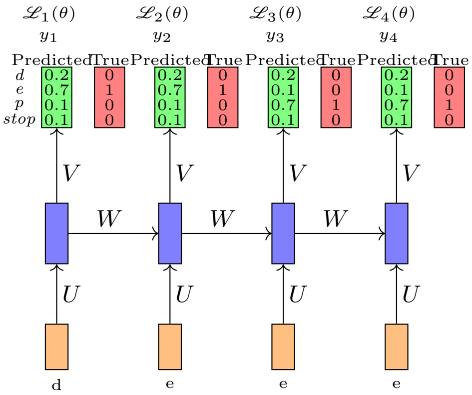

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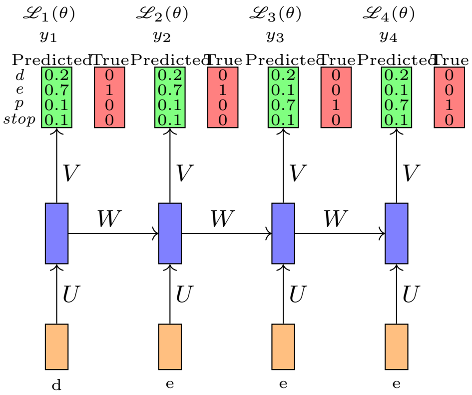

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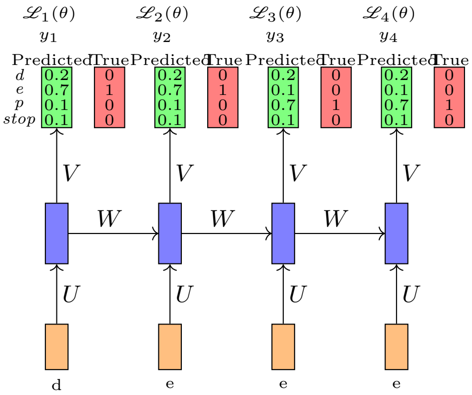

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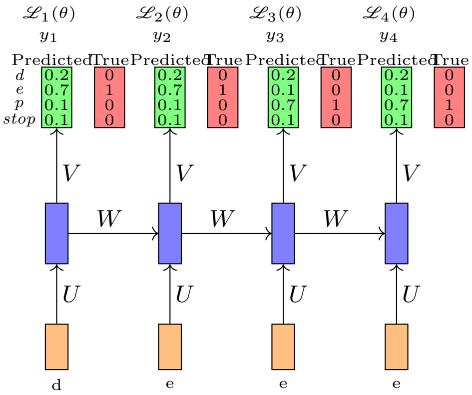

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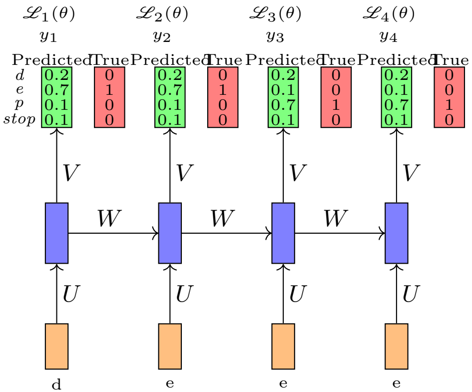

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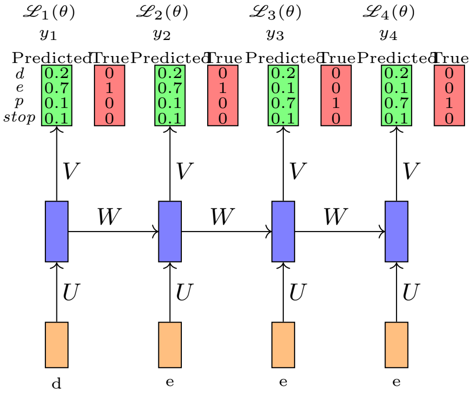

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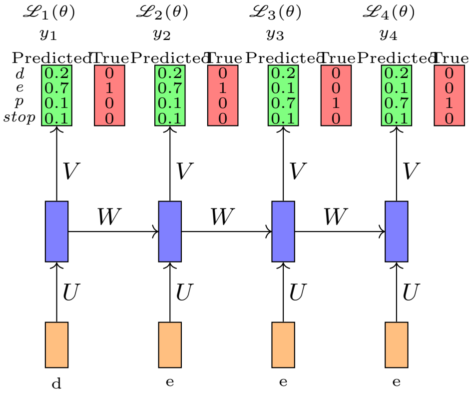

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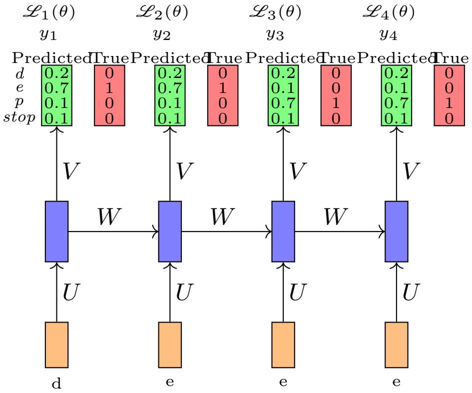

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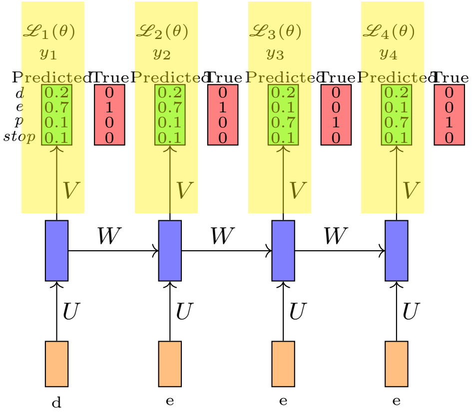

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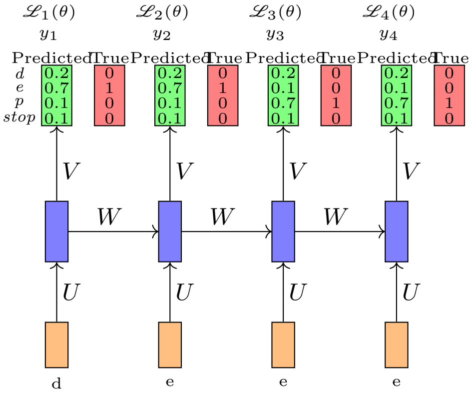
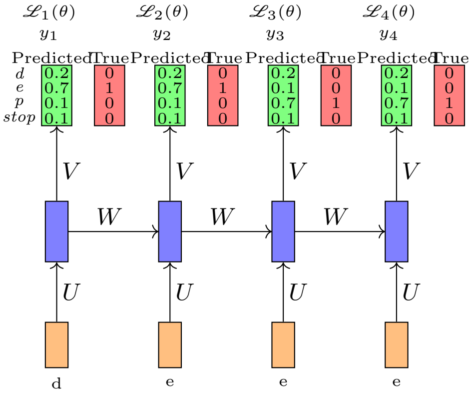
- Let us consider the derivative $\frac{\partial \mathscr{L}(\theta)}{\partial W}$





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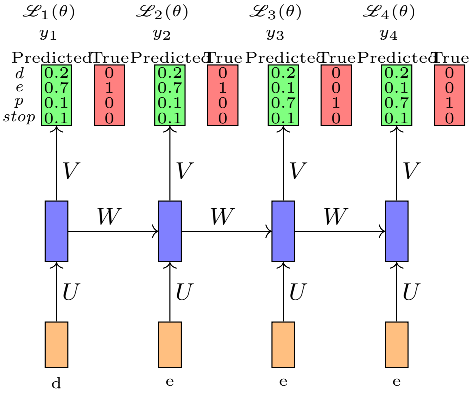

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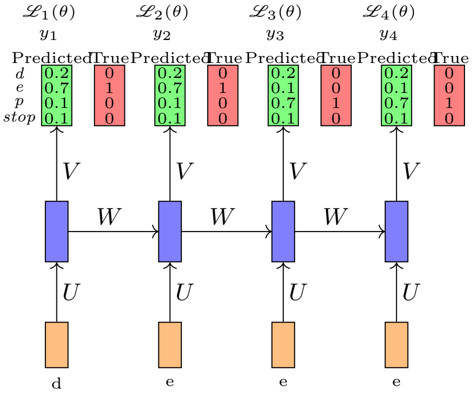

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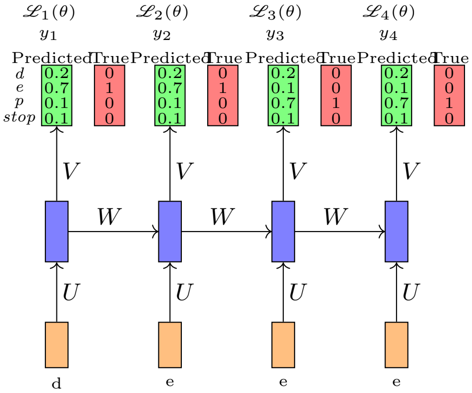

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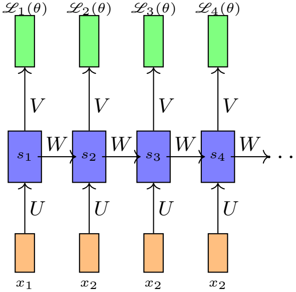
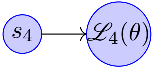
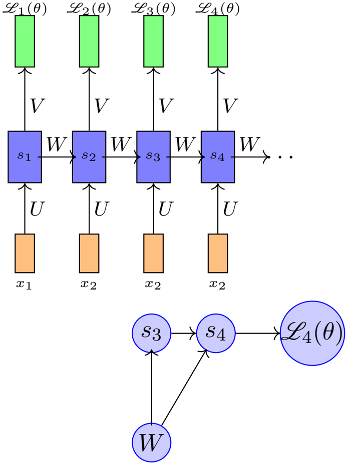
- Let us see this by considering $\mathscr{L}_4(\theta)$

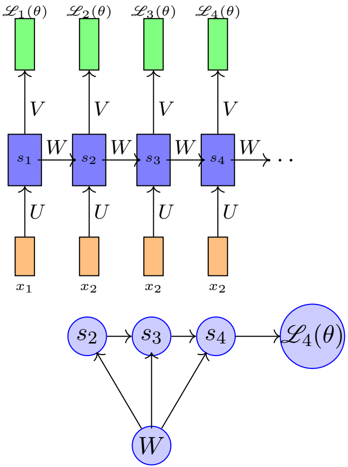
- $\mathscr{L}_4(\theta)$ depends on s_4





- $\mathscr{L}_4(\theta)$ depends on s_4
- s_4 in turn depends on s_3 and W





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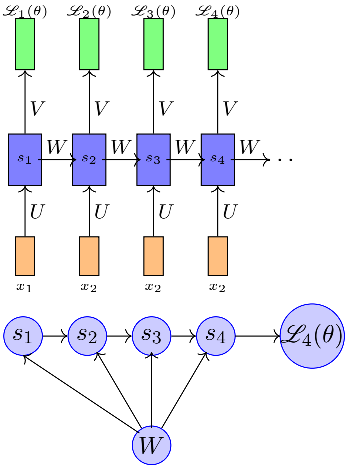

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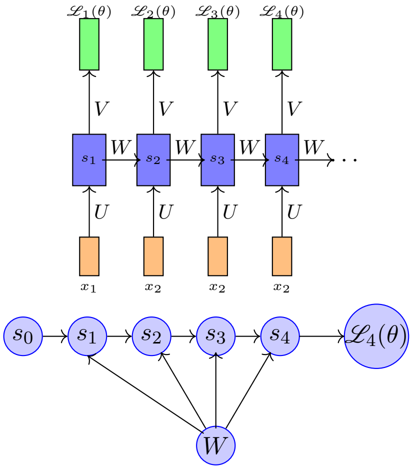

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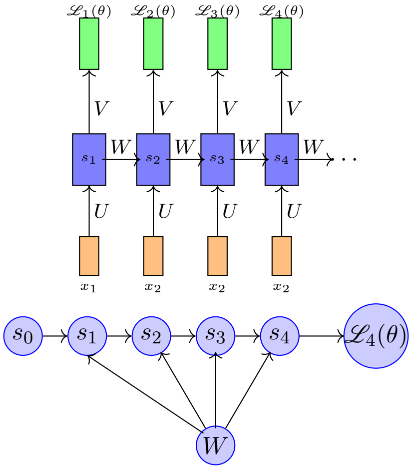
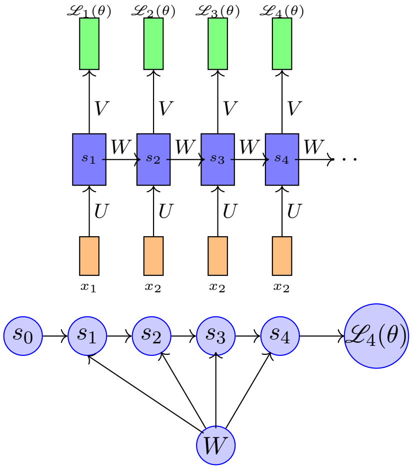
- s_3 in turn depends on s_2 and W
- s_2 in turn depends on s_1 and W
- s_1 in turn depends on s_0 and W

  where s_0 is a constant starting state.



- What we have here is an ordered network



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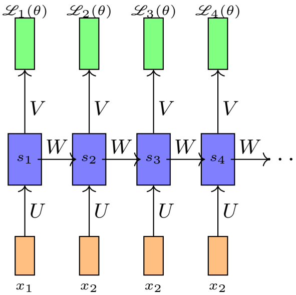
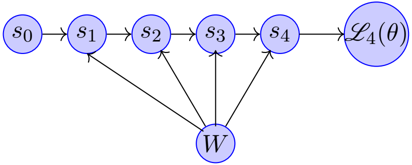
- In an ordered network each state variable is computed one at a time in a specified order (first s_1, then s_2 and so on)





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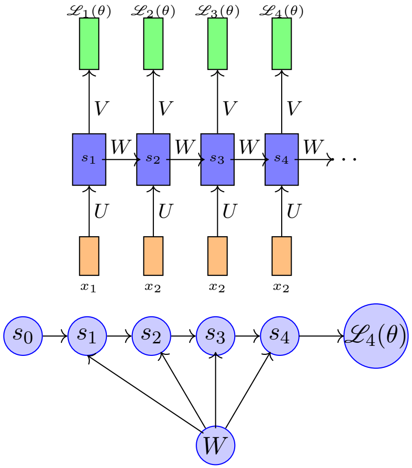

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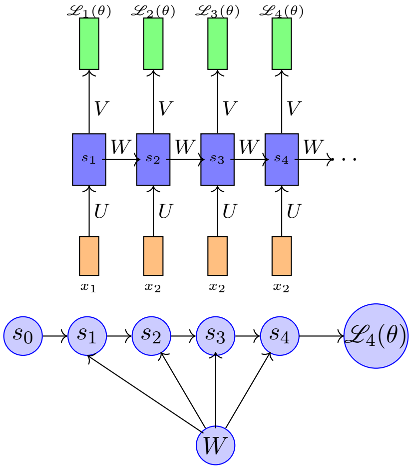

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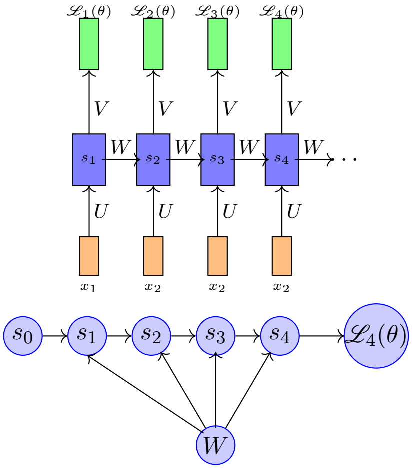
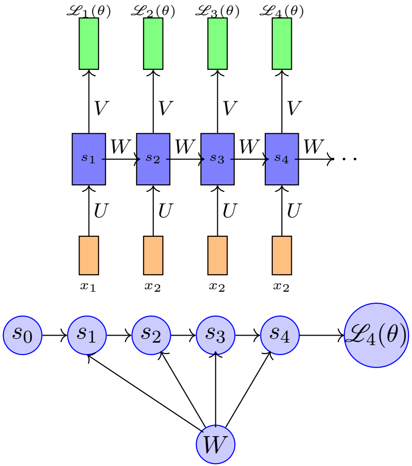
- But how do we compute $\frac{\partial s_4}{\partial W}$



- Recall that

$$s_4 = \sigma(Ws_3 + b)$$



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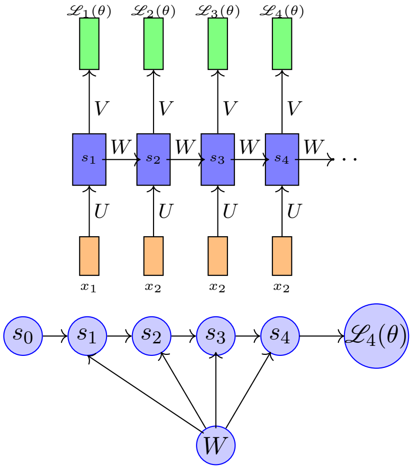

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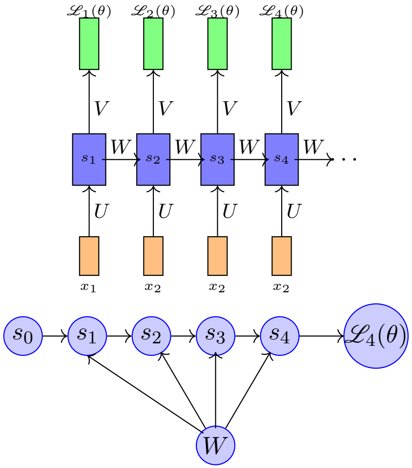

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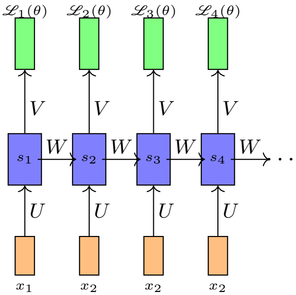

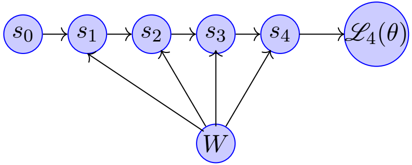

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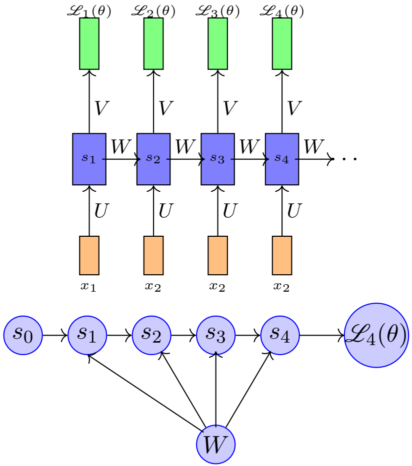

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$$\frac{\partial s_4}{\partial W} = \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}}$$

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$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right]$$

$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\left[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\Big[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\Big] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\Big[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\Big] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\Big[\frac{\partial^+ s_1}{\partial W}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\left[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\left[\frac{\partial^+ s_1}{\partial W}\right]
\end{aligned}
$$

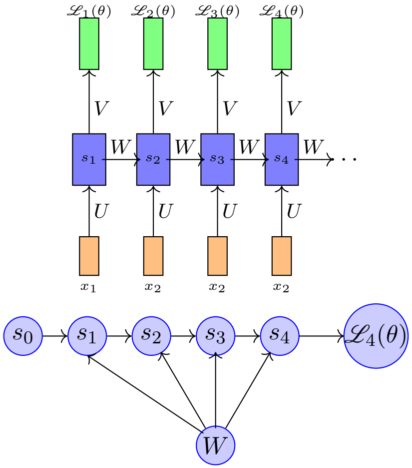
For simplicity we will short-circuit some of the paths

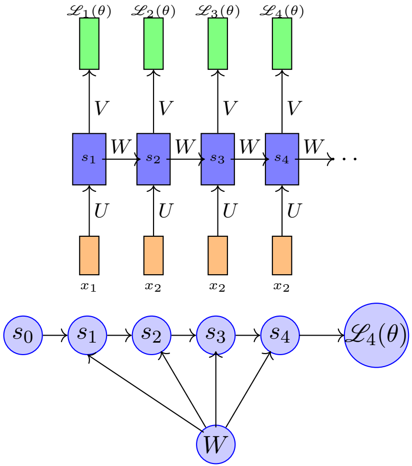
$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\Big[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\Big] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\Big[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\Big] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\Big[\frac{\partial^+ s_1}{\partial W}\Big]
\end{aligned}
$$

For simplicity we will short-circuit some of the paths

$$
\frac{\partial s_4}{\partial W} = \frac{\partial s_4}{\partial s_4}\frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_1}\frac{\partial^+ s_1}{\partial W} = \sum_{k=1}^{4}\frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}
$$

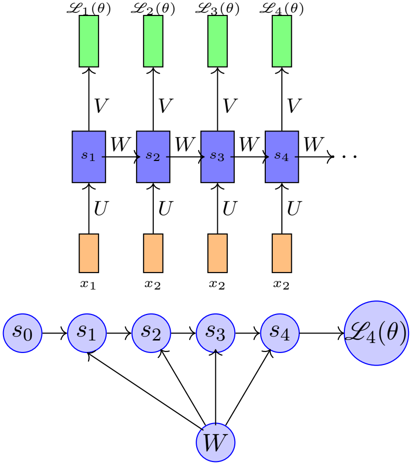
- Finally we have





- Finally we have

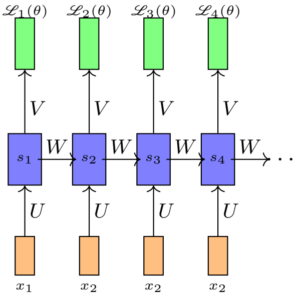
$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4} \frac{\partial s_4}{\partial W}$$

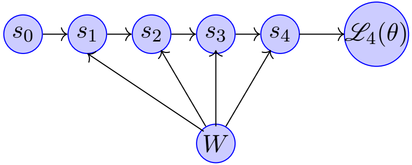


- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4}\frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4} \frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$



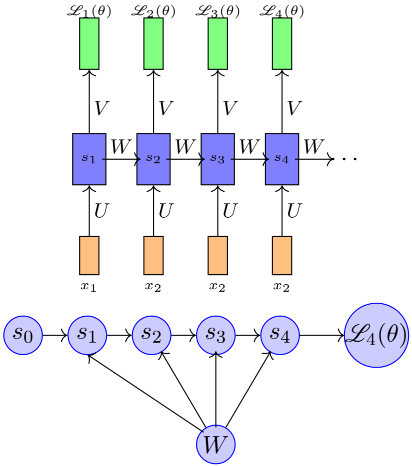


- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4}\frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4}\frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$

$$\therefore \frac{\partial \mathscr{L}_t(\theta)}{\partial W} = \frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}\sum_{k=1}^{t}\frac{\partial s_t}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$



- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4}\frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4} \frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$

$$\therefore \frac{\partial \mathscr{L}_t(\theta)}{\partial W} = \frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}\sum_{k=1}^{t} \frac{\partial s_t}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$

- This algorithm is called backpropagation through time (BPTT) as we backpropagate over all previous time steps

Module : The problem of Exploding and Vanishing Gradients

- We will now focus on $\frac{\partial s_t}{\partial s_k}$ and highlight an important problem in training RNN's using BPTT

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$$\frac{\partial s_t}{\partial s_k} = \frac{\partial s_t}{\partial s_{t-1}} \frac{\partial s_{t-1}}{\partial s_{t-2}} \cdots \frac{\partial s_{k+1}}{\partial s_k}$$

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$$= \prod_{j=k}^{t-1} \frac{\partial s_{j+1}}{\partial s_j}$$

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$$\frac{\partial s_t}{\partial s_k} = \frac{\partial s_t}{\partial s_{t-1}} \frac{\partial s_{t-1}}{\partial s_{t-2}} \dots \frac{\partial s_{k+1}}{\partial s_k}$$
$$= \prod_{j=k}^{t-1} \frac{\partial s_{j+1}}{\partial s_j}$$

- Let us look at one such term in the product (i.e., $\frac{\partial s_{j+1}}{\partial s_j}$)

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

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$$a_j = W s_j + b$$

$$s_j = \sigma(a_j)$$

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$$a_j = W s_j + b$$
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$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j}\frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
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$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & & \\ & & & \\ & & & \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
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$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
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$$a_j = W s_j + b$$
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$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ & & & \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j}\frac{\partial a_j}{\partial s_{j-1}}$$

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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ & & & \\ & & & \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

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$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \dots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma'(a_{jd}) \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = Ws_j + b$$
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$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \dots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma'(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma'(a_j))$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
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$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \ldots & \sigma'(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma'(a_j))$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$
$$= diag(\sigma'(a_j))W$$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

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$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma'(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma'(a_j))$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$
$$= diag(\sigma'(a_j))W$$

- We are interested in the magnitude of $\frac{\partial s_j}{\partial s_{j-1}}$ $\leftarrow$ if it is small (large) $\frac{\partial s_t}{\partial s_k}$ and hence $\frac{\partial \mathscr{L}_t}{\partial W}$ will vanish (explode)

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma'(a_j))W\right\|$$

$$\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| = \left\| diag(\sigma^{'}(a_j))W \right\|$$

$$\leq \left\| diag(\sigma^{'}(a_j) \right\| \|W\|$$

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\sigma^{'}(a_j) \leq \frac{1}{4} = \gamma\,[\text{if } \sigma \text{ is logistic }]$$

$$\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| = \left\| diag(\sigma^{'}(a_j))W \right\|$$
$$\leq \left\| diag(\sigma^{'}(a_j) \right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\sigma^{'}(a_j) \leq \frac{1}{4} = \gamma \,[\text{if } \sigma \text{ is logistic }]$$
$$\leq 1 = \gamma \,[\text{if } \sigma \text{ is tanh }]$$

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\sigma^{'}(a_j) \leq \frac{1}{4} = \gamma \,[\text{if } \sigma \text{ is logistic }]$$
$$\leq 1 = \gamma \,[\text{if } \sigma \text{ is tanh }]$$
$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| \leq \gamma \|W\|$$
$$\leq \gamma\lambda$$

$$\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| = \left\| diag(\sigma^{'}(a_j))W \right\|$$
$$\leq \left\| diag(\sigma^{'}(a_j) \right\| \|W\|$$

$$\left\| \frac{\partial s_t}{\partial s_k} \right\| = \left\| \prod_{j=k+1}^{t} \frac{\partial s_j}{\partial s_{j-1}} \right\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

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- If $\gamma\lambda > 1$ the gradient could explode

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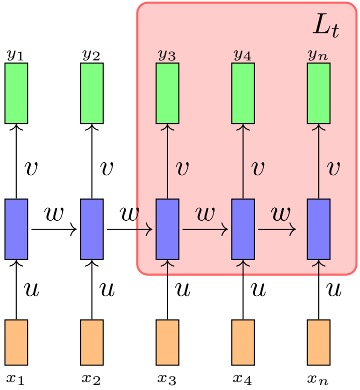
- If $\gamma\lambda < 1$ the gradient will vanish
- If $\gamma\lambda > 1$ the gradient could explode
- This is known as the problem of vanishing/ exploding gradients



- One simple way of avoiding this is to use truncated backpropogation where we restrict the product to $\tau(< t - k)$ terms

Module : Some Gory Details

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- We know how to compute $\frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}$ (derivative of $\mathscr{L}_t(\theta)$ (scalar) w.r.t. last hidden layer (vector)) using backpropagation

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- How do we compute $\frac{\partial^+ s_k}{\partial W}$? Let us see

- We just look at one element of this $\frac{\partial^+ s_k}{\partial W}$ tensor

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$$s_k = \sigma(a_k)$$

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$$\begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kp} \\ \vdots \\ a_{kd} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} & \dots & W_{1d} \\ \\ \vdots & \vdots & \vdots & \vdots \\ W_{p1} & W_{p2} & \dots & W_{pd} \\ \vdots & \vdots & \vdots & \vdots \\ \\ \end{bmatrix} \begin{bmatrix} s_{k-1,1} \\ s_{k-1,2} \\ \vdots \\ s_{k-1,p} \\ \vdots \\ s_{k-1,d} \end{bmatrix}$$

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