Derivation of the Matched Filter Estimator

We aim to estimate the amplitude A from the observed data $\mathbf{y}$, where the model for the data is given by:

$$\mathbf{y} = A\mathbf{m} + \mathbf{n}$$

where:

- $\mathbf{y}$ is the observed data (a vector),
- A is the amplitude we are trying to estimate,
- $\mathbf{m}$ is the known template (or shape of the signal),
- $\mathbf{n}$ is the noise, which is Gaussian with a known covariance matrix $\mathbf{C}$.

We derive the matched filter estimator using the *maximum likelihood estimation* (MLE) approach.

The noise $\mathbf{n}$ is Gaussian with zero mean and covariance matrix $\mathbf{C}$, so the observed data $\mathbf{y}$ follows a multivariate Gaussian distribution:

$$\mathbf{y} \sim \mathcal{N}(A\mathbf{m}, \mathbf{C})$$

The probability density function (PDF) for a multivariate Gaussian distribution is:

$$p(\mathbf{y}|A) = \frac{1}{\sqrt{(2\pi)^N \det \mathbf{C}}} \exp\left(-\frac{1}{2}(\mathbf{y} - A\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{y} - A\mathbf{m})\right)$$

The log-likelihood function is given by:

$$\log \mathcal{L}(A) = -\frac{1}{2}(\mathbf{y} - A\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{y} - A\mathbf{m}) + \text{constant}$$

We can ignore the constant term since it doesn't depend on A.

Expanding the quadratic term:

$$\log \mathcal{L}(A) = -\frac{1}{2}\left(\mathbf{y}^T \mathbf{C}^{-1}\mathbf{y} - 2A\mathbf{y}^T \mathbf{C}^{-1}\mathbf{m} + A^2 \mathbf{m}^T \mathbf{C}^{-1}\mathbf{m}\right)$$

To find the MLE of A, we take the derivative of $\log \mathcal{L}(A)$ with respect to A and set it to zero:

$$\frac{d}{dA} \log \mathcal{L}(A) = \mathbf{y}^T \mathbf{C}^{-1}\mathbf{m} - A\mathbf{m}^T \mathbf{C}^{-1}\mathbf{m}$$

Setting this equal to zero and solving for A, we get:

$$A = \frac{\mathbf{y}^T \mathbf{C}^{-1} \mathbf{m}}{\mathbf{m}^T \mathbf{C}^{-1} \mathbf{m}}$$

Thus, the maximum likelihood estimator for A is:

$$\hat{A} = \frac{\mathbf{y}^T \mathbf{C}^{-1} \mathbf{m}}{\mathbf{m}^T \mathbf{C}^{-1} \mathbf{m}}$$

Or equivalently:

$$\hat{A} = \frac{(\mathbf{C}^{-1} \mathbf{m})^T \mathbf{y}}{\mathbf{m}^T \mathbf{C}^{-1} \mathbf{m}}$$

This is the matched filter formula. It provides the optimal estimate of the amplitude A given the observed data $\mathbf{y}$, the known signal template $\mathbf{m}$, and the noise covariance matrix $\mathbf{C}$.