Matched Filter Maximization of Signal-to-Noise Ratio (SNR)

We are given the following signal model:

$$\mathbf{y} = A\mathbf{m} + \mathbf{n}$$

Where:

- $\mathbf{y}$ is the observed signal (a vector).
- A is the unknown scalar amplitude of the signal.
- $\mathbf{m}$ is the known signal template (a vector).
- $\mathbf{n}$ is the noise, which is assumed to be a zero-mean Gaussian random vector with covariance matrix $\mathbf{C}$, i.e., $\mathbb{E}[\mathbf{n}] = 0$ and $\mathbb{E}[\mathbf{n}\mathbf{n}^T] = \mathbf{C}$.

The goal is to find the estimator $\hat{A}$ for A that maximizes the SNR.
The matched filter is designed to estimate A as:

$$\hat{A} = \frac{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{y}}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}$$

This filter applies a weighting based on the inverse of the noise covariance $\mathbf{C}$ to suppress the impact of noise. The term $\mathbf{C}^{-1}\mathbf{m}$ is referred to as the filter.

We start by computing the expectation of $\hat{A}$. Using the signal model $\mathbf{y} = A\mathbf{m} + \mathbf{n}$, we have:

$$\hat{A} = \frac{\mathbf{m}^T\mathbf{C}^{-1}(A\mathbf{m} + \mathbf{n})}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}$$

Now, take the expectation of $\hat{A}$ with respect to the noise $\mathbf{n}$:

$$\mathbb{E}[\hat{A}] = \frac{\mathbf{m}^T\mathbf{C}^{-1}\mathbb{E}[A\mathbf{m} + \mathbf{n}]}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}$$

Since $\mathbb{E}[\mathbf{n}] = 0$ and $A\mathbf{m}$ is deterministic, we get:

$$\mathbb{E}[\hat{A}] = \frac{A\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}} = A$$

Thus, $\hat{A}$ is an unbiased estimator of A, meaning $\mathbb{E}[\hat{A}] = A$.

Next, we compute the variance of $\hat{A}$. The variance is given by:

$$\text{Var}(\hat{A}) = \mathbb{E}\left[(\hat{A} - \mathbb{E}[\hat{A}])^2\right]$$

Since $\hat{A}$ is unbiased ($\mathbb{E}[\hat{A}] = A$), we can rewrite this as:

$$\text{Var}(\hat{A}) = \mathbb{E}\left[(\hat{A} - A)^2\right]$$

Substitute the expression for $\hat{A}$:

$$\text{Var}(\hat{A}) = \mathbb{E}\left[\left(\frac{\mathbf{m}^T\mathbf{C}^{-1}(A\mathbf{m} + \mathbf{n})}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}} - A\right)^2\right]$$

Since $\mathbf{m}^T\mathbf{C}^{-1}A\mathbf{m} = A\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}$, the terms involving A cancel out, leaving:

$$\text{Var}(\hat{A}) = \mathbb{E}\left[\left(\frac{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{n}}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}\right)^2\right]$$

Factor out the constant:

$$\text{Var}(\hat{A}) = \frac{1}{(\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m})^2}\mathbb{E}\left[\left(\mathbf{m}^T\mathbf{C}^{-1}\mathbf{n}\right)^2\right]$$

Now, we use the fact that $\mathbf{n}$ is a zero-mean Gaussian vector with covariance matrix $\mathbf{C}$. Therefore, $\mathbb{E}[\mathbf{n}\mathbf{n}^T] = \mathbf{C}$, and:

$$\mathbb{E}\left[\left(\mathbf{m}^T\mathbf{C}^{-1}\mathbf{n}\right)^2\right] = \mathbf{m}^T\mathbf{C}^{-1}\mathbf{C}\mathbf{C}^{-1}\mathbf{m}$$

Simplifying:

$$\mathbb{E}\left[\left(\mathbf{m}^T\mathbf{C}^{-1}\mathbf{n}\right)^2\right] = \mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}$$

Thus, the variance becomes:

$$\text{Var}(\hat{A}) = \frac{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}{(\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m})^2} = \frac{1}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}$$

The signal-to-noise ratio (SNR) for $\hat{A}$ is given by the ratio of the expected value of $\hat{A}$ (which represents the signal power) to the standard deviation of $\hat{A}$ (which represents the noise power):

$$\text{SNR} = \frac{\mathbb{E}[\hat{A}]}{\sqrt{\text{Var}(\hat{A})}}$$

Substitute $\mathbb{E}[\hat{A}] = A$ and $\text{Var}(\hat{A}) = \frac{1}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}$:

$$\text{SNR} = \frac{A^2}{\frac{1}{\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}}} = A^2\mathbf{m}^T\mathbf{C}^{-1}\mathbf{m}$$

The matched filter maximizes the SNR because it minimizes the variance of the estimator $\hat{A}$ (assuming Gaussian noise; in case the model does not adhere to this, the bias-variance tradeoff is back on the table). This results in an optimal filter for extracting the signal amplitude A from the noisy observation $\mathbf{y}$.

Thus, the matched filter provides the maximum possible signal-to-noise ratio, making it equivalent to a maximum likelihood estimator for A.