Probability & statistics - Assignment 2.

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Section - 1

Q.1 = Sol^n $\Rightarrow$

Given :

$$mgf_X(t) = \frac{1}{10}e^{-20t} + \frac{1}{5}e^{-3t} + \frac{3}{10}e^{4t} + \frac{2}{5}e^{5t} \quad \text{--- (1)}$$

Now, as we know : $mgf_X(t) = \sum_X e^{tX} P(X)$
(for discrete RV)

or $mgf_X(t) = E(e^{tX})$

therefore, converting given mgf_X to general form gives :-

$$mgf_X(t) = e^{t(-20)} P(X=-20) + e^{t(-3)} P(X=-3) + e^{t(4)} P(X=4) + e^{t(5)} P(X=5) + \sum e^{t(K)} P(X=k) \ ; \text{ where } K \in \mathbb{R} - \{-20, -3, 4, 5\}$$

On comparing with eq. (1) gives :-

$P(X=-20) = 1/10$
$P(X=-3) = 1/5$
$P(X=4) = 3/10$
$P(X=5) = 2/5$

On summing these 4 P's we get 1.
Hence, $\forall\, k \in \mathbb{R} - \{-20, -3, 4, 5\}$
$P(X=k) = 0$.

therefore, for $P(|X| \le 2)$ = ~~$P(X=-20) + P(X=-3)$~~ = ~~$1/10 + 1/5$~~ = ~~$3/10$ Ans~~ 0.

$= 0$ Ans

Q.2 = sol^n $\Rightarrow$

There is only 4 ordered pair for (Real parity, Guessed parity) and that is $\{(\text{Even}, \text{Even}), (\text{Even}, \text{Odd}), (\text{Odd}, \text{Even}), (\text{Odd}, \text{Odd})\}$.

Let, 'Z' be RV s.t. $Z = 1$ for correct guess otherwise, $Z = 0$.

So, $P(Z=1) = \frac{2}{4} = \frac{1}{2}$ & $P(Z=0) = \frac{2}{4} = \frac{1}{2}$.

Now, Assuming Another RV 'Y' which is money got after correct guess i.e. $Z=1$.

So, $Y \sim \text{Uniform}(0, 1000)$.

Therefore, Random Variable for profit i.e. X can be written as

$$X = Y.Z + (-200)(1-Z)$$

$$X = (Y+200)Z - 200.$$

$$X = Y'Z - 200 \quad \text{—— (1)} \qquad ; [\, Y' = Y + 200 \;\&\; Y' \sim U(200, 1200)\,]$$

So, Range of $X = \{-200\} \cup [0, 1000] = R_X$

Now, $P(X < -200) = 0$ (directly from R_X)

& $P(-200 < X < 0) = 0$ (" " ")

& $P(X > 1000) = 0$ (" " ")

And $P(X = -200) = P(Y'Z = 0)$ (from eq-(1))

$= P(Z=0)$ (because $Y' \geq 200$)

$= \frac{1}{2}$.

& for $0 \leq q \leq 100$;

$P(X = q) = P(Y'Z = 200 + q)$

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As, $Z=1$ is must for above probability

$$P(X=q) = P(Y'Z = 200+q) = P(Z=1).P(Y'=200+q)$$ (because Y' & Z are independent)

$[0 \le q \le 1000.]$

$$= \frac{1}{2} \cdot P(Y=q) \quad [\text{from } ☆]$$

$$= \frac{1}{2} \cdot \frac{1}{1000} \quad (\text{because } Y \sim U(0,1000))$$

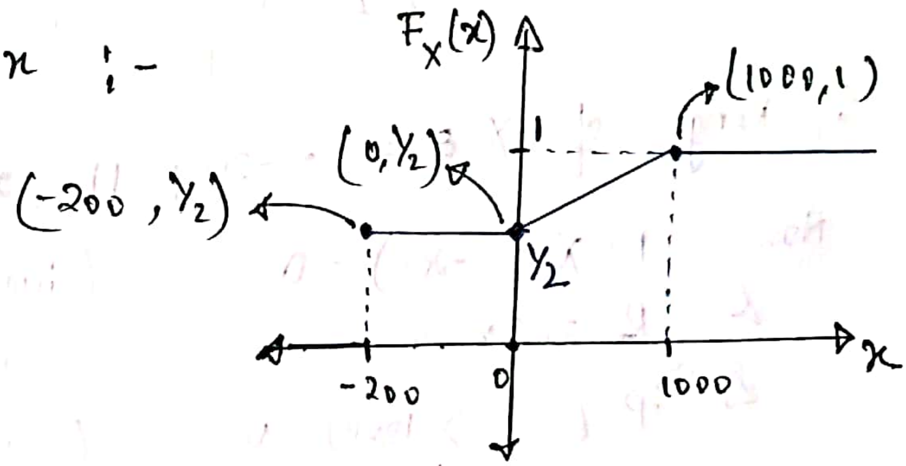
$$= 2000^{-1} = \frac{1}{2000}$$

Now, Let $F_X(x) = P(X \le x)$ be CDF.

So,

$$F_X(x) = \begin{cases} 0 & ; \quad x < -200 \\ 1/2 & ; \quad x \ge -200 \text{ and } x < 0 \\ 1/2 + x/2000 & ; \quad 0 \le x \le 1000 \\ 1 & ; \quad x > 1000 \end{cases}$$

Graph $F_X(x)$ Vs x :-



Now, Probability for not winning at least 500

$$= F_X(x=500) = \frac{1}{2} + \frac{x}{2000} = \frac{1}{2} + \frac{500}{2000}$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \quad \text{Ans}$$

$Q.3 = sol^n \Rightarrow$ $p \sim \text{Uniform}(0,3)$

$$Q = \log\left(\frac{p}{3-p}\right) = g(p)$$

$$E[Q] = ?$$

We can say, $P(p = q) = \frac{1}{3}$; $0 \leq q \leq 3$.

otherwise 0.

$$E[g(p)] = \int_{-\infty}^{\infty} g(p) \cdot P(p) \cdot dp$$

$$= \int_0^3 \log\left(\frac{p}{3-p}\right) \cdot \frac{1}{3} \cdot dp$$

(Assuming base 10 ~~natural~~ log)

$$= \frac{1}{3}\left[\int_0^3 \log(p)\,dp - \int_0^3 \log(3-p)\,dp\right]$$

Let for 2nd term assume: $3-p = a$

$\Rightarrow -dp = da$

$$\text{So, } 3 \cdot E[g(p)] = \left[\int_0^3 \log(p)\,dp\right] - \left[\int_3^0 \log(a)\,(-da)\right]$$

$$3 \cdot E[g(p)] = \int_0^3 \log(p)\,dp - \int_0^3 \log(a)\,da = 0.$$

So, $E[g(p)] = E[Q] = 0$ Ans

Q.4 = Sol^n ⇒

Let T be the random variable defining Ashish's waiting time.

Range of T = R_T = $[0, 5]$ minutes

because ~~to~~ waiting time cannot be negative (obviously) and next bus arrive at exactly ~~at~~ 5 minute.

So, it is sure that he will not wait for more that 5 min.

Let $P(t)$ be probability for waiting t min.

$$P(T=t) = \begin{cases} 2/3 & ; t=0 \quad \text{i.e. taxi present} \\ \frac{1}{3}\cdot\frac{1}{10} & ; 0<t<5 \quad \text{i.e. taxi come before 5 min. at } T=t. \\ \frac{1}{3}\cdot\underbrace{\frac{5}{10}}_{\text{this term indicates taxi did not come before 5 min}}\cdot 1 & ; t=5 \\ 0 & ; t<0 \text{ or } t>5. \end{cases}$$

Now, say $F_T(t)$ is CDF then

$$F_T(t) = P(T \le t) = \begin{cases} 0 & ; \ t < 0 \\ 2/3 & ; \ t = 0 \\ 2/3 + t/30 & ; \ 0 < t < 5 \\ \lim_{t \to 5} (2/3 + t/30) + \frac{1}{3} \cdot \frac{5}{10} + 1 & \\ = 2/3 + 1/6 + 1/6 & \\ = 1 & ; \ t = 5 \\ 1 & ; \ t > 5 \end{cases}$$

Finding, $E(T) = 0 \times P(T=0) + \int_0^5 t \cdot \frac{1}{30} \cdot dt$

$$+ 5 \cdot \frac{1}{3} \cdot \frac{5}{10}$$

$$= 0 + \frac{25}{60} + \frac{25}{30}$$

$$= 75/60 = 5/4 \text{ minute Ans}$$

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Q.5 = soln ⇒

$$P_G(g) = \begin{cases} 1/3 & ; g = -2 \\ 1/2 & ; g = 1 \\ 1/6 & ; g = 3 \\ 0 & ; \text{otherwise} \end{cases}$$

So, $P(G_i = -2)$ at i^{th} time $= 1/3$

& $P(G_i \neq -2)$ at i^{th} time $= 2/3$

because net gains at different time is independent.

(a) Let N be total rounds played.

~~[Let F_k be event that total k rounds played]~~

So, $P(N=1) = P(G_1 = -2) \times P(G_2 = -2)$

$= 1/3 \times 1/3 = 1/9$.

Now, $N > 1$

scenario will be :-

$G_1, G_2, G_3, G_4, \dots, G_{2m-1}, G_{2m}, \dots, G_{2N-1}, G_{2N}$.

Here, $G_{2N-1} = G_{2N} = 1$ is must.

Now, lets say k positions out of (N-1) odd positions have value -2, basically first (N-1) odd positions. & $0 \leq K \leq N-1$.

therefore, probability for above case will be = s.t. total round occur is N (>1)

$$= \frac{1}{9} \cdot {}^{(N-1)}C_K \left(\frac{1}{3}\right)^K \left(\frac{2}{3}\right)^K \cdot \left(\frac{2}{3}\right)^{N-1-k} (1)^{N-1-k}$$

for last round.

This part refers to those k rounds in which Vinay loses & then Mahesh does not loses.

This part refers to remaing N-1-k rounds in which Vinay does not loses.

$$= \frac{1}{9} \cdot \left(\frac{2}{3}\right)^{N-1} \cdot \left[{}^{(N-1)}C_K \left(\frac{1}{3}\right)^K \right] = P_K \ (\text{say})$$

Hence, required probability for total round to be N. $= \sum_{k=0}^{N-1} P_K$
(for case $N > 1$)

$$= \sum_{k=0}^{N-1} \frac{1}{9} \cdot \left(\frac{2}{3}\right)^{N-1} \cdot {}^{(N-1)}C_k \left(\frac{1}{3}\right)^k$$

$$= \frac{1}{9} \cdot \left(\frac{2}{3}\right)^{N-1} \sum_{k=0}^{N-1} {}^{(N-1)}C_k \left(\frac{1}{3}\right)^k$$

$$= \frac{1}{9} \cdot \left(\frac{2}{3}\right)^{N-1} \left[1 + \frac{1}{3}\right]^{N-1}$$

$$= \frac{1}{9} \cdot \left(\frac{2}{3}\right)^{N-1} \left(\frac{4}{3}\right)^{N-1} = \frac{2^{3N-3}}{3^{2N}}$$

$$= \frac{1}{8} \cdot \left(\frac{8}{9}\right)^N$$

Therefore, we can conclude that PMF of total number of N round played until Vinay loses then Mahesh looses $= \frac{1}{8}\left(\frac{8}{9}\right)^N$. Ans

Q.5 = Sol^n

(b) So, basically we have to find PMF for RV 'Z' which is number of trials until Mahesh has his 3rd loss.

So, sequence will be:

$$G_1, G_2, \ldots, G_{k_1-1}, \underbrace{G_{k_1}}_{\text{Mahesh's 1st loss}}, \ldots, G_{k_2-1}, \underbrace{G_{k_2}}_{\text{Mahesh's 2nd loss}}, \ldots, G_{Z-1}, \underbrace{G_Z}_{\text{Mahesh's 3rd loss.}}$$

* Clearly Z is even.

For odd times we don't need to worry as $\exists$ no restriction on them.

But we can have -2 at even times only for 2 places other than $G_Z = -2$.

Selecting those 2 places out of total even places i.e $\frac{Z}{2}$. $= {}^{(Z/2)-1}C_2$.

Probability that only above ~~given~~ 2 places will have (-2) & other even places not will be

$$= \underbrace{(1)^{z/2}}_{\text{For odd places can have any value i.e. } -2 \text{ or } 1 \text{ or } 3.} \cdot \underbrace{{}^{(z/2)-1}C_2 \left(\tfrac{1}{3}\right)^2}_{\text{For selected even places to have value } -2.} \cdot \underbrace{\left(\tfrac{2}{3}\right)^{(z/2)-3}}_{\text{For even places other than 3 required place.}} \cdot \underbrace{\left(\tfrac{1}{3}\right)}_{\text{For } G_z = -2.}$$

$$= \left(\frac{1}{3}\right)^3 {}^{(z/2)-1}C_2 \left(\frac{2}{3}\right)^{z/2} \left(\frac{2}{3}\right)^{-3}$$

$$= \left(\frac{1}{2}\right)^3 {}^{(z/2)-1}C_2 \left(\frac{2}{3}\right)^{z/2}$$

If $Q = z/2$

then PMF for Z(number of trials), defined as the time which Mahesh has his 3rd loss

$$= \frac{2^{k-3}}{3^k} \cdot {}^{(k-1)}C_2$$

$$; \; k = \frac{z}{2}.$$

Ans

Q.5 = C = Sol^n ⇒ let us define 4 events,

A_1 : Both win 1st round.

A_2 : Only Mahesh win 1st round

A_3 : Only Vinay win 1st round

A_4 : both loose 1st round

$$\Rightarrow P(A_1) = \frac{2}{3}\frac{2}{3} = \frac{4}{9}$$

$$P(A_2) = \frac{1}{3}\frac{2}{3} = \frac{2}{9}$$

$$P(A_3) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

$$P(A_4) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$

* Let E be the expected value of the RV N, which represents the no. of rounds, until each of them has won at least once.

$$E = E[N]$$

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Q.5 - C = Sol ⇒ (remaining)

* Expected value of N when A_1 occurs is 1.
As both of them won in the 1^{st} round.

* Expected value of N $E[N]$ when A_2 or A_3 occurs is $1+\frac{1}{p}$ where $p=\frac{2}{3}$

Proof: In the 1^{st} round, exactly one of them won, that means, from the second round, the distribution of the 1^{st} rounds loser winning will be geometric RV with $p=\frac{2}{3}$.

$$E(\text{geometric}) = \frac{1}{p} \Rightarrow E = \underbrace{1}_{\text{for } 1^{st} \text{ round}} + \underbrace{\frac{1}{p}}_{\text{for next rounds}} \quad ; \text{ for } A_2, A_3$$

* Expected val. of N $E[N]$ when A_4 occurs is $1+E$ as both lost in 1^{st} round & in the next rounds, similar events occur.

$$\Rightarrow E[N] = E[N/A].P(A_1) + E[N|A_2].P(A_2) + E[N|A_3]P(A_3) + E[N|A_4]P(A_4)$$

$$\Rightarrow E = 1\left(\frac{2}{3}\cdot\frac{2}{3}\right) + \left(1+\frac{3}{2}\right)\left(\frac{1}{3}\times\frac{2}{3}\right) + \left(1+\frac{3}{2}\right)\left(\frac{1}{3}\times\frac{2}{3}\right) + (1+E)\left(\frac{1}{3}+\frac{1}{3}\right)$$

$$\Rightarrow E = 15/8$$ Ans

Q.6 = Soln $\Rightarrow$

A book of 500 pages contains 500 misprints on average.

So, average no. of misprints on the page of same book $= 1$.

As, occurrence of errors is a poisson process we can take parameter λ involved to be 1.

So, $$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1}(1)^k}{k!} = \frac{1}{e(k!)}$$

; X is no. of misprints on a page

So, $$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - \frac{1}{e} - \frac{1}{e} - \frac{1}{2e}$$

$$= 1 - \left(\frac{5}{2}\right)\frac{1}{e}$$

$$= 1 - \frac{5}{2e}$$

$$\simeq 0.080301 \quad \text{Ans}$$

This is the chance that given page contain at least 3 errors.

Q.7 = Sol^n $\Rightarrow$ For Random Variable X, mgf $= M(t)$.

(a) $Y = kX$

For RV Y, $\text{mgf}_Y = M_Y(t) = M_{kX}(t) = E[e^{t(kX)}]$

For discrete RV.

$$\text{mgf}_Y = \sum_x e^{tkx} p(x)$$

$$= M(tk).$$

; $P(x)$ is PMF of X.

For continuous RV

$$\text{mgf}_Y = \int_x e^{tkx} f(x)\,dx$$

$$= M(tk)$$

; $f(x)$ is PDF of X.

Final Answer $= M(k \cdot t)$

(b) $Y = k+X$

For RV Y, $\text{mgf}_Y = M_Y(t) = M_{k+X}(t) = E[e^{t(k+X)}]$

For discrete RV

$$\text{mgf}_Y = \sum_x e^{t(k+x)} p(x)$$

$$= e^{tk} \sum_x e^{tx} p(x)$$

$$= e^{tk} M(t).$$

For Continuous RV

$$\text{mgf}_Y = \int_x e^{t(k+x)} f(x)\,dx$$

$$= e^{tk} \int_x e^{tx} f(x)\,dx$$

$$= e^{tk} M(t).$$

Final Answer $= M(t) \cdot e^{tk}$

(c) $Y = X_0 + X_1 + \dots + X_N$

$$\text{mgf}_Y = M_Y(t) = E(e^{tY}) = E[e^{t(X_0 + X_1 + \dots + X_N)}]$$

For discrete RV ! (Also for continuous RV)

$$\text{mgf}_Y = E[e^{tX_0} \times e^{tX_1} \times \dots \times e^{tX_N}]$$

$$= \prod_{i=0}^{N} E[e^{tX_i}] = \prod_{i=0}^{N} M(t) = [M(t)]^{N+1}$$ Ans

Q.7 = (d) $PDF(y) = PDF(x+k)$

Using Uniqueness theorem we can say that as PDF of 2 RV are same which are here (1) Y (2) X+k then their mgf will also be same.

Therefore, our final answer is equivalent to mgf of X+k only, which is $e^{tk} \cdot M(k \cdot t)$. (Found in Q.7(b)).

Q.7 =
(e) ⇒ $PDF(y) = PDF(2x)$

Following Uniqueness theorem, we have to find mgf of '2X' Random Variable.

We have found this result already in Q.7 part (a), i.e. $M(k \cdot t)$

Substituting, $k=2$ we get $M(2t)$. Ans

Q.8 = Sol^n ⇒

Given, X is RV which denotes the number of people that get their own hat. Let us write X as sum of N independent variables $(X_1, X_2 --- X_n)$

$X = \sum_{i=1}^{N} X_i$, where $X_i = \begin{cases} 1 ; & \text{if } i^{th} \text{ person select his/her own hat} \\ 0 ; & \text{otherwise} \end{cases}$

Now,

$P(X_i = 1) = P(i^{th} \text{ man selecting his own hat}) = \frac{1}{N}$

As all the hat are equally likely to be chosen.

⇒ $E[X_i] = 0 \times P(X_i = 0) + 1 \times P(X_i = 1)$

$= 0 + \frac{1}{N}$

$= 1/N$

So, $E[X] = E[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{N} E[X_i]$

As $E[X_1] = E[X_2] = E[X_3] = --- = E[X_N]$

⇒ $E[X] = N \times E[X_1]$

⇒ $E[X] = N \times \frac{1}{N} = 1.$

Hence, the expected value of the number of people who get back their own hat is 1.

Q. 9 = sol^n ⇒

$$f_X(x) = \begin{cases} x/2 & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Method - 1

Now, $Y = 6X - 3$ → strictly increasing

$R_X = [0, 2]$ → $R_Y = [-3, 9]$

Here;

$g(x) = 6x - 3 = y$.

For the given Range $g(x) = y$ is continuous & differentiable & strictly increasing,

So, using method of transformation:

$$f_Y(y) = \begin{cases} \dfrac{f_X(x)}{g'(x)} & ; \quad g(x) = y \\ 0 & ; \quad \text{if } g(x) = y \text{ doesnot have sol}^n. \end{cases}$$

So, $$\frac{f_X(x)}{g'(x)} = \frac{x/2}{6} = \frac{\left(\frac{y+3}{6}\right)/2}{6} = \frac{y+3}{72}.$$

So, $$f_Y(y) = \begin{cases} (y+3)/72 & ; \quad -3 \le y \le 9 \\ 0 & ; \quad \text{otherwise}. \end{cases}$$

Ans

Method : 2

We can calculated $F_X(x)$ from given $f_X(x)$.

$$F_X(x) = \begin{cases} 0 \; ; & x \le 0 \\ x^2/4 \; ; & 0 \le x \le 2 \\ 1 \; ; & x > 2 \end{cases}$$

Now, $F_Y(y) = P(Y \le y) = P(6X-3 \le y)$

$$F_Y(y) = P\left(X \le \frac{y+3}{6}\right).$$

$$F_Y(y) = F_X\left(\frac{y+3}{6}\right)$$

$$F_Y(y) = \begin{cases} 0 \; ; & y < -3 \\ [(y+3)/6]^2/4 \; ; & -3 \le y \le 9 \\ 1 \; ; & y > 9 \end{cases}$$

So, as $f_Y(y) = F_Y'(y)$

$$= \begin{cases} \frac{1}{2} \cdot \frac{y+3}{6} \cdot \frac{1}{6} = \frac{y+3}{72} \; ; & -3 \le y \le 9 \\ 0 \; ; & \text{otherwise} \end{cases}$$

Q.10 = Sol^n $\Rightarrow$

N denotes RV of no. of cars before U-turn

so we can say N = no. of τ_i time intervals between arrival of 2 cars that are less than τ.

Since, these τ_i's are independent, each one will be smaller than τ with probability P.

Also let k be random variable corresponding to no. of cars in τ time then

$$K \sim Poisson(\lambda\tau)$$

or $$P_K(x) = \frac{e^{-\lambda\tau}(\lambda\tau)^x}{x!} \quad \text{—— (1)}$$

$$P = P(\tau_i < \tau) = P(T < \tau) = 1 - P(T \geq \tau) = 1 - P_K(0)$$

$$P = 1 - (e^{-\lambda\tau}) \qquad \text{(from (1))}$$

Now, RV 'N' will simply be $N \sim Geometric(1-p)$

$$P_N(x) = (1-p)[1-(1-p)]^x \quad \text{for } x \in \{0,1,2 \ldots\}$$

It is because x cars passing means x independent time-intervals with probability p & last time interval has no car with probability $P_K(0) = 1 - P$.

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So, $P_N(x) = \begin{cases} e^{-\lambda\tau}(1-e^{-\lambda\tau})^x & ; \text{ for } x \in \{0,1,2 \dots\} \\ 0 & ; \text{ otherwise} \end{cases}$

then $E[N] = \sum_{x=0}^{\infty} x\, p^x (1-p) = (1-p) \sum_{x=0}^{\infty} x p^x$

$= (1-p)s$.

let $s = \sum_{x=0}^{\infty} x p^x = 1.p + 2.p^2 + 3.p^3 + \dots \infty$

$ps = 1.p^2 + 2.p^3 + 3.p^4 + \dots \infty$

$\Rightarrow (1-p)s = p + p^2 + p^3 + \dots \infty$

since $|p|<1$

$\Rightarrow (1-p)s = \frac{p}{1-p}$.

So, $E[N] = \frac{p}{1-p}$

$= \frac{1-e^{-\lambda\tau}}{1-(1-e^{-\lambda\tau})}$

$= e^{\lambda\tau} - 1$ Ans.

Section - 2

Q. 11 = Sol^n ⇒

(a) RY 'X' is the duration of time before 2 balloons of same color are shot i.e. total balloons shot when a color get repeated.

So, Range of X $= \{2, 3, \ldots, N+1\}$.

Any each stand are equiprobable.

Let total k shot happens, $2 \le k \le N+1$ then color for first k-1 shot will be different i.e. doesnot match pairwise.

So, number of ways of choosing k-1 stand $= {}^NC_{k-1}$

Probability for " " " $= {}^NC_{k-1}\left(\frac{1}{N}\right)^{k-1}\left(1-\frac{1}{N}\right)^{N-k+1}$

Final Probability for exactly k-shot before 2 matching color $= \left({}^NC_{k-1}\left(\frac{1}{N}\right)^{k-1}\left(1-\frac{1}{N}\right)^{N-k+1}\right)\left(\lfloor k-1\right)\left(Q\right)$

for arrangement of (k-1) colors

for Probability of k^{th} shot to one of the given (k-1) color

Here, $Q = \left(\frac{k-1}{N}\right)$

$$P(X=K) = {}^{N}C_{k-1} \cdot \left(\frac{1}{N}\right)^{k-1} \cdot \left(1-\frac{1}{N}\right)^{N-k+1} \cdot \left(\lfloor\underline{k-1}\right) \cdot \left(\frac{k-1}{N}\right)$$

$$= \frac{{}^{N}C_{k-1} \cdot \left(\lfloor\underline{k-1}\right) \left(N-1\right)^{N-k+1}}{N^2} (K-1)$$

$; 1 \leq k \leq N+1$ Ans

$0 \quad ; \text{ otherwise}$

Q.11 = b = sol^n ⇒

RV 'Y_i' is the duration of time before a new color of balloons is shot after i colors have already been shot

i.e. number of shots when $(i+1)^{th}$ color gets hit for the 1^{st} time (including $(i+1)^{th}$ color).

Let say k shots occur & the balloon at k^{th} has a color which occur 1^{st} time at k^{th} shot & 1 to $k-1$ shots comprises total 'i' different colors.

P_k = Probability for this case $= \left(\frac{N-i}{N}\right) \cdot \left({}^{N}C_{i}\right) \cdot \theta$

where ;

$$\theta = \left(\frac{i}{N}\right)^{k-1} - {}^{i}C_{i-1}\left(\frac{i-1}{N}\right)^{k-1} + {}^{i}C_{i-2}\left(\frac{i-2}{N}\right)^{k-1} - \cdots$$

$$+(-1)^{i+1}\, {}^{i}C_{1}\left(\frac{1}{N}\right)^{k-1}$$

so, $$P(Y_i) = \sum_{k=i+1}^{\infty} P_k = \frac{N-i}{N} \cdot {}^{N}C_{i} \sum_{k=i+1}^{\infty} \theta$$

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$$P(Y_i) = \left(\frac{N-i}{N}\right) {}^{N}C_i \sum_{k=i+1}^{\infty} \sum_{y=0}^{i-1} {}^{i}C_{i-y} \left(\frac{i-y}{N}\right)^{k-1} (-1)^y$$

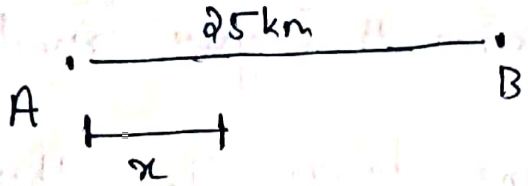
* k & y independent variables

$$P(Y_i) = \left(\frac{N-i}{N}\right)\left({}^{N}C_i\right) \sum_{y=0}^{i-1} {}^{i}C_{i-y} (-1)^y \left(\frac{i-y}{N-i+y}\right)$$

$$P(Y_i) = \left(\frac{N-i}{N}\right)\left({}^{N}C_i\right) \sum_{y=0}^{i-1} {}^{i}C_y (-1)^y \left(\frac{i-y}{N-i+y}\right)$$

Ans

Q.12 = Sol^n ⇒



Let T_A be duration in hours after which you leave college (Pt. A)

Let T_B be duration in hours after which your friend leaves (Pt. B)

T_A & T_B are random, $0 \leq T_A, T_B \leq 1$.

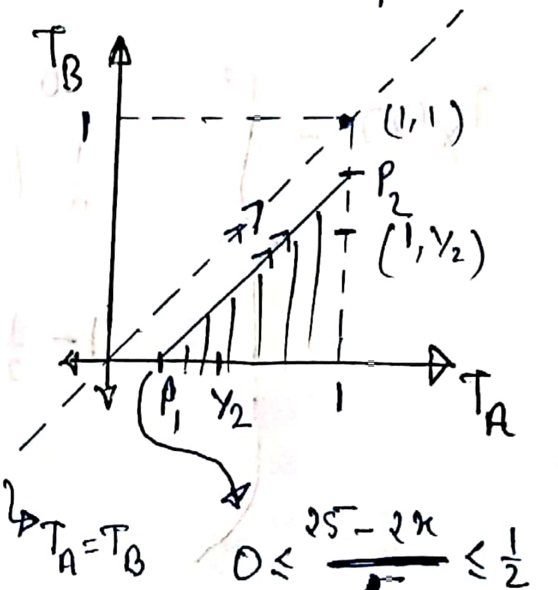
<u>Case - I</u> $T_A \geq T_B$ i.e. you leave later than or with your friend.

Here, $0 \leq x \leq 12.5$

$50 * (T_A - T_B) = 25 - 2x$ (Extra dist. covered by friend)

$\Rightarrow T_A - T_B = \dfrac{25-2x}{50}$

$\Rightarrow T_B = T_A - \dfrac{25-2x}{50}$



Area of graph $= \dfrac{1}{2}\left[\dfrac{25+2x}{50}\right]^2$

i.e. $F_X(x) = \dfrac{1}{2}\left[\dfrac{25+2x}{50}\right]^2$

$0 \leq \dfrac{25-2x}{50} \leq \dfrac{1}{2}$

$P_2\left(1, \dfrac{25+2x}{50}\right)$

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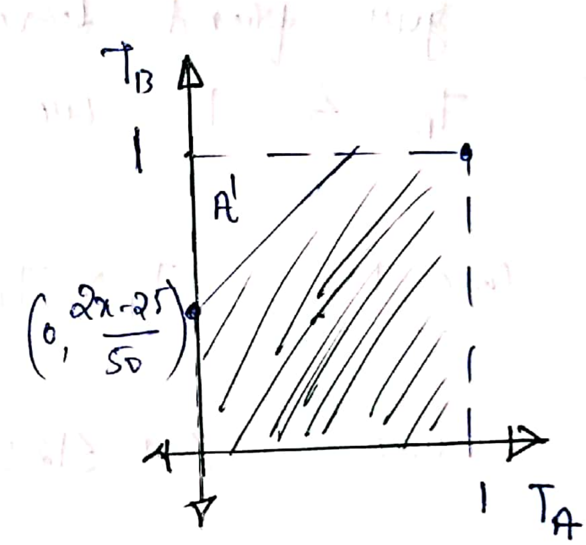
Case - 2 $T_B > T_A$ (you strictly leave sooner than your friend)

Here, if T_A leaves 30 mins before T_B, then $x = 25$ kms. ~~&~~ ~~but~~ for this case $12.5 < x \le 25$.

$\Rightarrow T_B - T_A = \frac{2x - 25}{50}$ [follow the process in Case - I]

Plotting the graph;

$F_X(x) =$ Area of shaded region $= 1 - A'$



$\Rightarrow F_X(x) = 1 - \frac{1}{2}\left[1 - \left(\frac{2x-25}{50}\right)\right]^2$

$\Rightarrow F_X(x) = 1 - \frac{1}{2}\left[\frac{25 - 2x}{50}\right]^2$

Hence,

$$F_X(x) = \begin{cases} 0 \; ; & x < 0 \\ \frac{1}{2}\left(\frac{25+2x}{50}\right)^2 \; ; & 0 \le x \le 12.5 \\ 1 - \frac{1}{2}\left(\frac{25-2x}{50}\right)^2 \; ; & 12.5 < x \le 25 \\ 1 \; ; & x > 25 \end{cases}$$

Q.13 = Sol ⇒

Let X & Y be RV denoting lifetime of bulbs A & B in years respectively.

Then $E[X] = 0.25$, $E[Y] = 0.5$

$$P(X=x) = \begin{cases} \dfrac{e^{-0.25}(0.25)^x}{x!} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$P(Y=y) = \begin{cases} \dfrac{e^{-0.5}(0.5)^y}{y!} & ; y \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

(1) 3 bulbs replacement means :

$$\begin{matrix} A & \rightarrow & B & \rightarrow & A & \rightarrow & B \\ A_1 & & B_1 & & A_2 & & B_2 \end{matrix}$$

Expected total illumination time

$$E[A_1 + B_1 + A_2 + B_2] = E[A_1] + E[B_1] + E[A_2] + E[B_2]$$

$$= 2\,[E[x] + E[y]]$$

$$= 2\,(0.25 + 0.5)$$

$$= 1.5 \text{ yrs} \quad \text{Ans}$$

Q. 12 = Sol^n →

(b) For n replacements, where replacement by Bulb A has probability p & by Bulb B has probability $1-p$.

1^{st} bulb is bulb A with expected span of life $= \frac{1}{4}$ yrs.

After that expected lifetime for i Bulbs of type A & $n-i$ bulbs of type B.

$$= \sum_{i=0}^{n} {}^{n}C_{i}\, p^{i} (1-p)^{n-i} \underbrace{(0.25 i + 0.5(n-i))}_{\text{Total bulbs time}}$$

Total expected lifetime

$$= 0.25 + \sum_{i=0}^{n} {}^{n}C_{i}\, p^{i} (1-p)^{n-i} \times \frac{i}{4}$$

$$+ \sum_{i=0}^{n} {}^{n}C_{i}\, p^{i} (1-p)^{n-i} \times \frac{n-i}{2}$$

$$= 0.25 + \frac{1}{4} np + \frac{1}{2} n (1-p)$$

($\because$ Binomial expectation is np)

$$= 0.25 + 0.5n - 0.25np$$

Hence,

$$E[X] = \frac{1 - np + 2n}{4}$$ Ans

Q. 14 = Soln ⇒

(a) Consider a graph with k components. Let's consider one of the component having |e| edges & |v| vertices.

Let us suppose that there exists a 'y' edge cycle containing monochromatic edge.
∵ it is a y-edge cycle, it has y-vertices.

Among choosing colors, we observe that each & every vertex in cycle should have same color & also, we do not consider coloring the y^{th} edge as coloring $y-1$ vertices in graph automatically generates an monochromatic.
So To disprove for a graph

$$P(\text{all monochromatic}) = \prod_{i=1}^{|e|} P(X_i \text{ is monochromatic})$$

X_i is an monochromatic edge.

$$\left(\frac{1}{3}\right)^{|v|-1} = \left(\frac{1}{3}\right)^{|e|}$$

We ~~all~~ see this is independent only when $|v| = |e|+1$ i.e. it is a tree.

∴ If ∃ a monochromatic cycle, it is the set X_i is not independent.

Hence proved.

Q. 14 = (b) ⇒

$$Y = |E \setminus E(A)|$$

$$E[Y] = E[|E| - |E_A|]$$

$$= |E| - E[|E_A|]$$

$$= |E| - \sum_{i=1}^{|E|} E[x_i]$$

where $x_i = \begin{cases} 1, & \text{if edge } x_i \text{ is monochromatic} \\ 0, & \text{otherwise.} \end{cases}$

$$E[Y] = |E| - |E| E[x_1] = |E| - |E| P(x_i)$$

Probability of monochromatic $= \dfrac{{}^3C_1}{3 \times 3} = \dfrac{1}{3}$

$$E[X] = \frac{2}{3}|E| \quad \text{Ans}$$

Q. 14 = (c) ⇒ If the graph contains an assignment having less that $\frac{2}{3}|E|$ monochromatic edges then there should be assigned such that monochromatic edges are greater than $\frac{2}{3}|E|$.

This can be seen by result of part (b)

Since, average is $\frac{2}{3}|E|$, if a number $< \frac{2}{3}|E|$ then it should have contain a value that is $> \frac{2}{3}|E|$

Also, If there is no assignment with monochromatic edges less than $\frac{2}{3}|E|$. Then each assignment should have monochromatic edge $= \frac{2}{3}|E|$.

Hence, ∃ assignment s.t. monochromatic edges $\geq \frac{2}{3}|E|$.

Q.15 = Sol^n ⇒ Let us denote location of ambulance at any moment of time as RV 'X' & accident location as 'Y'. X & Y are independent.

Also, $R_X = R_Y = [0, L]$.

& $f_X(x) = \begin{cases} \frac{1}{L} & ; 0 \le x \le L \\ 0 & ; \text{otherwise} \end{cases}$

& $f_Y(y) = \begin{cases} \frac{1}{L} & ; 0 \le x \le L \\ 0 & ; \text{otherwise} \end{cases}$

The time for ambulance to reach location of accident $= |X-Y|/v$.

Let RV, 'Z' $= |X-Y|$. & $R_Z = [0, L]$

Let us compute CDF of Z :-

$$F_Z(z) = P(Z \le z) = P(|X-Y| \le z)$$
$$= P(-z \le X-Y \le z)$$
$$= \int_0^L f_Y(y) \cdot P[-z+y \le X \le z+y \mid Y=y)\, dy$$

$$\Rightarrow F_Z(z) = \int_0^L \left[F_X(y+z) - F_X(y-z)\right] \cdot f_Y(y)\, dy$$

Now, $F_X(y+z) = \begin{cases} 0 & ; y < -z \\ \frac{y+z}{L} & ; -z \le y \le L-z \\ 1 & ; y \ge L-z \end{cases}$

$F_X(y-z) = \begin{cases} 0 & ; y < z \\ \frac{y-z}{L} & ; z \le y \le L+z \\ 1 & ; y \ge L+z \end{cases}$

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So,

$$F_Z(z) = \underbrace{\int_0^L F_X(y+z)\, f(y)\, dy}_{I_1} - \underbrace{\int_0^L F_X(y-z)\, f(y)\, dy}_{I_2}$$

Calculating, I_1 :-

$$I_1 = \int_0^{L-z} F_X(y+z) \cdot f(y)\, dy + \int_{L-z}^{L} F_X(y+z) \cdot f(y)\, dy$$

$$I_1 = \int_0^{L-z} \frac{(y+z)}{L^2} \cdot dy + \int_{L-z}^{L} 1 \cdot \frac{1}{L}\, dy$$

$$I_1 = \frac{1}{L^2}\left[\frac{y^2}{2} + zy\right]_0^{L-z} + \frac{1}{L}\left[y\right]_{L-z}^{L}$$

$$I_1 = \frac{(L-z)^2}{2L^2} + \frac{z(L-z)}{L^2} + \frac{z}{L}$$

Calculating, I_2 ;-

$$I_2 = \cancel{\int_0^z F_X(y-z) \cdot f(y)\, dy}^{\,0} + \int_z^L F_X(y-z) \cdot f(y)\, dy$$

$$I_2 = \int_z^L \frac{y-z}{L} \cdot \frac{1}{L}\, dy = \frac{1}{L^2}\left[\frac{y^2}{2} - zy\right]_z^L$$

$$I_2 = \frac{L^2 - z^2}{2L^2} - \frac{z}{L^2}(L-z)$$

Now, $$I_1 - I_2 = \frac{(L-z)^2 - (L^2 - z^2)}{2L^2} + \frac{z}{L} + \frac{2z(L-z)}{L^2}$$

$$F_{\cancel{x}Z}(z) = I_1 - I_2 = \frac{2z^2 - 2Lz}{\cancel{4}2L^2} + \frac{z}{L} + \frac{2Lz - 2z^2}{L^2}$$

$$= \frac{z^2 - Lz}{\cancel{2}L^2} + \frac{z}{L} + \frac{2Lz - (z^2 \times 2)}{L^2}$$

$$= \frac{Lz - z^2}{\cancel{2}L^2} + \frac{z}{L}$$

$$F_Z(z) = \frac{2Lz - z^2}{L^2} \longrightarrow \text{This is CDF of } Z$$

for pdf of Z, $f_Z(z) = F_Z'(z)$

$$f_Z(z) = \frac{2}{L} - \frac{2z}{L^2}.$$

Thus; ~~CDF~~

$$F_Z(z) = \begin{cases} 0 & ; z < 0 \\ (2Lz - z^2)/L^2 & ; 0 \le z \le L \\ 1 & ; z \geq L \end{cases} \qquad f_Z(z) = \begin{cases} \frac{2}{L} - \frac{2z}{L^2} & ; 0 \le z \le L \\ 0 & ; \text{otherwise} \end{cases}$$

Let us define RV, $T = \frac{Z}{V}$; V is speed of ambulance.
to be time taken by ambulance.

$$\text{So,} \quad f_T(t) = P(T = t) = P(Z = Vt) = f_Z(Vt)$$

$$\text{PDF} = f_T(t) = \begin{cases} \frac{2}{L} - \frac{2Vt}{L^2} & ; 0 \le t \le \frac{L}{V} \\ 0 & ; \text{otherwise} \end{cases}$$

& $F_T(t) = P(T \le t) = P(Z \le Vt) = F_Z(Vt)$

$$\text{CDF} = F_T(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t \geq L/V \\ (2LVt - (Vt)^2)/L^2 & ; 0 \le t \le \frac{L}{V} \end{cases}$$

$$\text{so,} \quad f_T(t) = F_T'(t) = \begin{cases} 2V(L - Vt)/L^2 & ; 0 \le t \le \frac{L}{V} \\ 0 & ; \text{Otherwise} \end{cases}$$

Q. 16 = Solⁿ ⇒

(a) The probability that in next two trials, there will be z tails is independent of prior trials. The probability z tails in 1st trial is $\left(\frac{1}{2}\right)^z$ & similarly for second trial, $\left(\frac{1}{2}\right)^z$ & for third trial also.

Therefore, the probability that the next two trials will have z tails is $\left(\frac{1}{2}\right)^z \cdot \left(\frac{1}{2}\right)^z$

i.e. $\left(\frac{1}{2}\right)^{2z}$ Ans

$= \left(\frac{1}{4}\right)^z$

(b) Probability of success for 1 trial $= \left(\frac{1}{2}\right)^z + \left(\frac{1}{2}\right)^z = \left(\frac{1}{2}\right)^{z-1}$

(i) PMF for k, number of trial up to, but not including the second success = ?

Suppose he did k trials before second success & exact one of which is success.

So, ways to select that 1 successfull trial $= {}^kC_1$

So, PMF $= \underbrace{{}^kC_1 \cdot \left(\frac{1}{2}\right)^{z-1}}_{\text{Prob. for 1 success}} \cdot \underbrace{\left(1-\left(\frac{1}{2}\right)^{z-1}\right)^{k-1}}_{\text{Prob. for (k-1) failure}} \cdot \underbrace{(1/2)^{z-1}}_{\text{for 2nd success}}$

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So, $PMF(k) = k \cdot p^2 \cdot (1-p)^{k-1}$

where $p = \left(\frac{1}{2}\right)^{Z-1}$. Ans

Q.16: b = (ii)

Already we know, probability of a successfull trial $= \left(\frac{1}{2}\right)^{Z-1}$.

$p = \left(\frac{1}{2}\right)^{3-1}$

$p = \frac{1}{4}$.

Say we did, before 1st success, N trails & X_i be the number of trails in X_i trail, which is failure.

So, Total number of trails $= M = \sum_{i=1}^{N} X_i$

Let us first compute $E[X_i]$, $Var(X_i)$, $E[N]$ & $Var(N)$;

Finding $E[N]$;-

i.e. expected no. of unsuccessfull trial before 1st success.

$$E[N] = \sum_{x=0}^{\infty} x \underbrace{(1-p)^x p}_{\hookrightarrow PMF(N=x)}$$

So, $E[N] = p \sum_{x=0}^{\infty} x(1-p)^x = ps$

Let $s = \sum_{x=0}^{\infty} x(1-p)^x = 1.(1-p)^1 + 2(1-p)^2 + \cdots \infty$

$(1-p)s = 1.(1-p)^2 + 3(1-p)^3 + \cdots \infty$

$\Rightarrow s - (1-p)s = (1-p) + (1-p)^2 + \cdots \infty$

$$\text{So,} \quad ps = \frac{1-p}{1-(1-p)} = \frac{1-p}{p} \quad \text{---} \ (*)$$

$$\text{So,} \quad E[N] = \frac{1-p}{p} = \frac{1-1/4}{1/4} = 3 \quad \text{---} \ (1)$$

$$\text{Now,} \quad Var(N) = E[N^2] - (E[N])^2$$

$$\text{finding} \quad E[N^2] = \sum_{x=0}^{\infty} x^2 (1-p)^x p$$

$$E[N^2] = p \sum_{x=0}^{\infty} x^2 (1-p)^x = p s'$$

$$\text{Let} \quad s' = \sum_{x=0}^{\infty} x^2 (1-p)^x = 0 + 1^2 \cdot (1-p)^1 + 2^2 \cdot (1-p)^2 + \cdots \infty$$

$$(1-p)s' = 1^2 \cdot (1-p)^2 + 2^2 \cdot (1-p)^3 + \cdots \infty$$

$$\Rightarrow s - (1-p)s' = 1^2 \cdot (1-p) + 3 \cdot (1-p)^2 + 5 \cdot (1-p)^3 + \cdots \infty$$

$$\Rightarrow ps' = \sum_{d=1}^{\infty} (2d-1)(1-p)^d$$

$$\Rightarrow ps' = 2\sum_{d=1}^{\infty} d(1-p)^d - \sum_{d=1}^{\infty} (1-p)^d$$

$$\Rightarrow ps' = 2s - \frac{1-p}{1-(1-p)}$$

$$\Rightarrow ps' = \frac{2(1-p)}{p^2} - \frac{1-p}{p} \quad (\text{from } *)$$

$$\Rightarrow E[N^2] = \frac{2(1-p)}{p^2} - \left(\frac{1-p}{p}\right) = 24 - 3 = 21.$$

$$\text{So,} \quad Var(N) = 21 - 9 = 12. \quad \text{---} \ (11)$$

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Now, computing $E[X_i]$;

All X_i's $(i \in \{1,2 \ldots, N\})$ are independent.

So, $\left.\begin{array}{l} E[x_i] = E[x_j] \\ \& \ Var(x_i) = Var(x_j) \end{array}\right\} \forall \ i,j \in \{1,2 \ldots, N\}$ —— (I)

Now;

$$E[X_i] = \sum_{k=0}^{Z} k \, {}^{Z}C_k \left(\frac{1}{2}\right)^{Z} = \sum_{k=1}^{Z} k \, {}^{Z}C_k \left(\frac{1}{2}\right)^{Z}$$

As $Z=3$; $E[X_i] = 1 \cdot {}^3C_1 \cdot \left(\frac{1}{2}\right)^3 + 2 \cdot {}^3C_2 \cdot \left(\frac{1}{2}\right)^3 + 3 \cdot {}^3C_3 \cdot \left(\frac{1}{2}\right)^3$

$\Rightarrow E[X_i] = \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$

$\Rightarrow E[X_i] = 12/8 = 3/2$. —— (III)

Calculating, $Var(X_i)$;

$Var(X_i) = E[X_i^2] - (E[X_i])^2$

finding; $E[X_i^2] = \sum_{k=0}^{Z} k^2 \, {}^{Z}C_k \left(\frac{1}{2}\right)^{Z} = \sum_{k=1}^{Z=3} k^2 \, {}^{Z}C_k \left(\frac{1}{2}\right)^{Z}$

$$E[X_i^2] = 1 \cdot {}^3C_1 \cdot \left(\frac{1}{2}\right)^3 + 4 \cdot {}^3C_2 \cdot \left(\frac{1}{2}\right)^3 + 9 \cdot {}^3C_3 \left(\frac{1}{2}\right)^3$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3. \quad \text{—— (\#\#)}$$

$Var(X_i) = 3 - (3/2)^2 = 3 - 9/4 = 3/4$.

$\Rightarrow Var(X_i) = 3/4$ —— (IV)

Let us now calculate expectation & variance of M :-

$$E[M] = E[\sum_{i=1}^{N'} X_i] = \sum_{i=1}^{N'} E[X_i]$$

$$E[M] = N' \cdot E[X_1] \quad \{\text{from } (\#)\}$$

; N' is expected value of N

So,

$$E[M] = E[N] \cdot E[X_1]$$

$$= 3 \cdot \frac{3}{2} \quad (\text{from } (i) \text{ \& } (iii))$$

$$E[M] = 9/2$$

Ans

Now, finding Var(M) :-

$$Var(M) = E[M^2] - (E[M])^2$$

finding $E[M^2] = E[(X_1+X_2+\cdots)^2]$

$$= E[(X_1^2+X_2^2+\cdots)+2(X_1X_2+X_2X_3\cdots)]$$

$$= E[X_1^2]+E[X_2^2]+\cdots+E[X_{N'}^2]+$$

Here,

N' is $E[N]$.

$$2E[X_1]E[X_2]+2E[X_2][X_3]+\cdots+2E[X_{N'}]E[X_1]$$

$$= E[N] \cdot E[X_1^2] + 2E[X_1]E[X_2] \cdot E[N]$$

from (# #)

$$= 3 \cdot 3 + \cancel{2} \cdot \frac{3}{\cancel{2}} \cdot \frac{3}{2} \cdot 3$$

$$E[X^2] = 9 + \frac{27}{2} = 45/2$$

So, $Var(M) = \frac{45}{2} - \left(\frac{9}{2}\right)^2 = \frac{90}{4} - \frac{81}{4} = 9/4$ Ans

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Q.16 = C = Sol^n ⇒

when there was not any trial performed we had M coins.

Number of trial until $(M-1)^{th}$ coins got removed (including $(M-1)^{th}$ coin removal trial also) $= X$.

We can write $X = Y_1 + Y_2 + Y_3 + \cdots + Y_{M-1}$;

where Y_1 is number of trial until 1^{st} success (including it)

Y_2 " " " " " 2^{nd} " " " & after 1^{st} success.

& so on.

finding, $E[Y_1] = \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$; $p = \left(\frac{1}{2}\right)^{M-1}$

$E[Y_2] = \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p$; $p = \left(\frac{1}{2}\right)^{M-2}$

Generalizing; $E[Y_j] = \sum_{i=1}^{\infty} i \, (1-p)^{i-1} \cdot p$; $p = \left(\frac{1}{2}\right)^{M-j}$

$\Rightarrow E[Y_j] = \frac{1}{p}$ $\forall \; j \in \{1, 2, \cdots, M-1\}$, $p = \left(\frac{1}{2}\right)^{M-j}$

Now, $E[X] = E\left[\sum_{j=1}^{M-1} Y_j\right] = \sum_{j=1}^{M-1} E[Y_j]$

$E[X] = \sum_{j=1}^{M-1} \frac{1}{p}$; $p = \left(\frac{1}{2}\right)^{M-j}$

$E[X] = \sum_{j=1}^{M-1} (2)^{M-j} = \left(-1 + (2)^{M-1}\right)2$

$= 2[2^{M-1} - 1]$ Ans

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Q·17 = Sol^n ⇒

(a)

There are n balls. & ~~2258~~ 2250 different colors.

The Probability of assigning a specified colour x to a ball $= \frac{1}{2250}$.

Now, consider probability of assigning a color to color to exactly k balls

$$P_k = {}^{n}C_k \left(\frac{1}{2250}\right)^k \left(\frac{224}{225}\right)^{n-k}$$

Therefore, for expected value we have to sum up this P_k for all colors.

Thus,

Expected number of colors which are assigned to exactly k balls $= n \times P_k$

$$= n \cdot \binom{n}{k} \cdot \left(\frac{1}{225}\right)^k \left(\frac{224}{225}\right)^{n-k}$$

Ans

Q.17 = (b) sol^n ⇒

Probability that an given color is assigned to more than one ball =

1 − Probability no such color − Probability of assigning that color to exactly 1 ball

$$P = 1 - \left(\frac{224}{225}\right)^n - {}^nC_1\left(\frac{1}{225}\right)\left(\frac{224}{225}\right)^{n-1}$$

So, Expected Number = $n \times P$

$$= n - n\left(\frac{224}{225}\right)^n - \frac{n^2 (224)^{n-1}}{(225)^n}$$

$$= \frac{n}{(225)^n}\left[(225)^n - (224)^n - n(224)^{n-1}\right]$$

Ans

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Qs. 17 = c = $sol^n \Rightarrow$

Part b expectation is

$$E = np = \frac{n}{(225)^n}\left[(225)^n - (224)^n - n(224)^{n-1}\right]$$

$$E = n\left[1 - \left(\frac{224}{225}\right)^n - \left(\frac{n}{224}\right)\left(\frac{224}{225}\right)^n\right]$$

so, for $E > 1$

n should be at least 50.

Actualy, plotting above function on graph for $n \in \mathbb{R}$ we found $(49.15, 1)$ satisfies the curve

but n can be integer only so,

$n \geq 50$. Ans

Q. 18 = Solⁿ ⇒

Let say I write code for k times & ~~then~~ it get submitted at k^{th} attempt.

So, total no. of tries = k.

Probability for total K tries = $(1-p)^{k-1} \cdot p$

$$\text{Mean} = E[k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p \cdot k$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

Let $$S = \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$S = 1 + 2(1-p) + 3(1-p)^2 + \cdots \infty$$

$$\Rightarrow (1-p)S = (1-p) + 2(1-p)^2 + \cdots \infty$$

$$\Rightarrow S - (1-p)S = 1 + (1-p) + (1-p)^2 + \cdots \infty$$

$$\Rightarrow pS = \frac{1}{p} \Rightarrow S = p^{-2} \quad \text{—— (1)}$$

So, $E[k] = \text{mean} = pS = 1/p$.

Now, Variance $= E[k^2] - (E[k])^2$

Calculating, $$E[k^2] = \sum_{k=1}^{\infty} (1-p)^{k-1} p\, k^2$$

$$E[k^2] = p \sum_{k=1}^{\infty} (1-p)^{k-1} k^2 = pS'$$

Let $$S' = \sum_{k=1}^{\infty} (1-p)^{k-1} k^2$$

$$S' = 1^2 + 2^2(1-p) + 3^2(1-p)^2 + 4^2(1-p)^3 + \cdots \infty$$
$$(1-p)S' = 1^2(1-p) + 2^2(1-p)^2 + 3^2(1-p)^3 + \cdots \infty$$

$$\Rightarrow pS' = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + \cdots \infty$$

$$\Rightarrow pS' = \sum_{k=1}^{\infty} (2k-1)(1-p)^{k-1}$$

$$\Rightarrow pS' = 2\sum_{k=1}^{\infty} k(1-p)^{k-1} - \sum_{k=1}^{\infty}(1-p)^{k-1}$$

$$\Rightarrow pS' = 2S - \frac{1}{p}$$
$$\Rightarrow pS' = 2/p^2 - 1/p \quad (\text{from } ①)$$

As $E[k^2] = pS'$
$$\Rightarrow E[k^2] = 2/p^2 - 1/p.$$

So, Variance $= E[k^2] - (E[k])^2$
$$= \left(\frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{1}{p}\right)^2$$
$$= \frac{1}{p^2} - \frac{1}{p}$$

Variance $= \dfrac{1-p}{p^2}$ & mean $= \dfrac{1}{p}$ Ans

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Q. 19 = Sol^n ⇒ The arrival of n students can be represented as n independent poisson process where arrival of i^{th} process is with rate λp_i. So arrival of all students is merging of all poisson process.

$P(X_i = k) = P_k$,

Let N_{t_i} be the number of i^{th} type students which arrived at time t. This is only possible iff, the students of ith type arrives between $[t-i, t]$

So, N_{t_i} is a poisson process with mean $\lambda i p_i$ and $N_t = \sum_{i=1}^{n} N_{t_i}$ (Total expectation)

$$E[N_t] = E\left[\sum_{i=1}^{n} N_{t_i}\right] = \sum_{i=1}^{n} E[N_{t_i}] \quad \left(\begin{array}{c}\text{linearity}\\ \text{of Expe.}\end{array}\right)$$

$$E[N_t] = \sum_{i=1}^{n} \lambda i p_i$$

$$\lambda_{new} = E[N_t] = \lambda \sum_{i=1}^{n} i p_i$$

$$PMF(N_t) = \begin{cases} \dfrac{e^{-\lambda_{new}} (\lambda_{new})^k}{k!} & ; k = 0, 1, 2 \ldots \\ 0 ; & \text{otherwise} \end{cases}$$

Ans

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Q.20 = Solⁿ ⇒

$$f_X(x) = \begin{cases} e^{-x} & ; \quad 0 \le x \le \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$Y = X^{1/2}$ then $f_Y(y) = ?$

$Y = g(X) = X^{1/2}$ (strictly increasing & differentiable) for $x \geq 0$

Lets compute $F_Y(y)$ first:

$$F_Y(y) = P_Y(Y \le y) = P(X^{1/2} \le y) = P(X \le y^2)$$

$$= P(\cancel{-y}\ 0 \le X \le y^2) \quad (\text{for this case})$$

$$= \int_0^{y^2} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^{y^2}$$

$$F_Y(y) = 1 - e^{-y^2}$$

And as $f_Y(y) = F_Y'(y) = \frac{d}{dy}(1 - e^{-y^2}) \quad \forall\, 0 \le y$.

$$f_Y(y) = \begin{cases} 2ye^{-y^2} & ; \quad y \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Ans