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Assignment 4
Probability and statistics
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Q.1

$$f(x) = 4xe^{-2x} = \frac{(2)^2 x^{(2-1)} e^{-2x}}{\Gamma(2)} ; x>0$$

Comparing with $X \sim \text{Gamma}(\alpha,\lambda)$; $\alpha,\lambda>0$

i.e. $$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} ; & x>0 \\ 0 ; & \text{otherwise} \end{cases}$$

we get, $\alpha=\lambda=2$ for our $f(x)=f_X(x)$

(a) Finding $E(X)$; $\quad$ * $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^\infty x f(x)\,dx$$

$$E(x) = \int_0^\infty 4x^2 e^{-2x}\,dx = 4\cdot\frac{\Gamma(3)}{(2)^3} = 1.$$ Ans

(because $\int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$)

& $\Gamma(3) = 2\Gamma(2) = 2\times(1!) = 2.$

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Similarly ;

Finding $E(X^2)$;

$$E[X^2] = \int_0^\infty x^2 f(x)dx = \int_0^\infty 4x^3 e^{-2x}$$

$$E[x^2] = \frac{4\,\Gamma(4)}{(2)^4} = \frac{4}{16} \times (3!)$$

$$E[x^2] = 3/2 \quad \text{Ans}$$

Similarly finding $E[X^3]$;

$$E[x^3] = \int_0^\infty x^3 f(x)dx = \int_0^\infty 4x^4 e^{-2x} dx$$

$$E[x^3] = \frac{4\,\Gamma(5)}{(2)^5} = \frac{4}{32} \times (4!)$$

$* \Gamma(n) = (n-1)!$

$$E[X^3] = 3 \quad \text{Ans}$$

(b) Deriving $E(X^n)$

$$E[X^n] = \int_0^\infty x^n f(x)dx = \int_0^\infty 4x^{(n+1)} e^{-2x} dx$$

Using Gamma function properties as in part a; $E[X^n] = \frac{4\,\Gamma(n+2)}{(2)^{n+2}}$

Thus,

$$E(x^n) = (2)^{-n} (n+1)!$$

Q.2

(a) $\Gamma(7/2) = (5/2)\,\Gamma(5/2)$

* by using $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$

$$\Gamma(7/2) = (5/2)(3/2)\,\Gamma(3/2)$$
$$= (5/2)(3/2)(1/2)\,\Gamma(1/2)$$

As $\Gamma(1/2) = \sqrt{\pi}$

So, $\Gamma(7/2) = (5/2)(3/2)(1/2)\sqrt{\pi}$
$$= \frac{15\sqrt{\pi}}{8}$$ Ans

(b) $I = \int_0^\infty x^6 e^{-5x}\,dx$

~~As we know for X ~ Gamma(α,λ), $x>0$~~

~~$f_X(x) = \{$~~

Definition of Gamma function :

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

& $$\frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx \quad ; \lambda>0$$

comparing we get $\alpha=7$ & $\lambda=5$

So, $I = \dfrac{\Gamma(\alpha)}{\lambda^{\alpha}}$; $\alpha = 7, \lambda = 5$

$I = \dfrac{\Gamma(7)}{5^7}$

$I = \dfrac{(6)!}{(5)^7}$ (using $\Gamma(\alpha) = (\alpha-1)!$)

$I = \dfrac{144}{15,625} \simeq 0.009216$ Ans

Q.3

$$f_{XY}(x,y) = \begin{cases} c(x^2+y^2) ; & \text{if } x \in \{1,2,4\} \\ & \& \ y \in \{1,3\} \\ 0 ; & \text{otherwise} \end{cases}$$

(a) For RV X, Y sample space S is $\{(1,1), (1,3), (2,1), (2,3), (4,1), (4,3)\}$

So, $\sum_{(x,y) \in S} f_{XY}(x,y) = 1$

$\Rightarrow c[(1^2+1^2) + (1^2+3^2) + (2^2+1^2) + (2^2+3^2) + (4^2+1^2) + (4^2+3^2)] = 1$

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$$\Rightarrow \frac{1}{c} = 2 + 10 + 5 + 13 + 17 + 25$$

$$\Rightarrow \frac{1}{c} = 72$$

$$\Rightarrow c = 1/72 \text{ Ans}$$

(b) For sample space S
$P(Y < X)$ means equivalent to finding $\Sigma P(x,y)$ for 3 ordered pair (x,y) i.e. $(2,1), (4,1), (4,3)$

So, $P(Y<X) = P(x=2, y=1) + P(x=4, y=1) + P(x=4, y=3)$

$$= f_{XY}(2,1) + f_{XY}(4,1) + f_{XY}(4,3)$$

$$= c[(4+1) + (16+1) + (16+9)]$$

$$= \frac{1}{72}[5 + 17 + 25]$$

$$P(Y<X) = 47/72 \text{ Ans}$$

(c) $Y > X$ is satisfied by only 2 (x,y) ordered pair in S & they are $(1,3), (2,3)$

$$P(Y>X) = f_{XY}(1,3) + f_{XY}(2,3)$$

$$= c\,[(1+9) + (4+9)]$$

$$= \frac{1}{72}(23)$$

$$P(Y>X) = 23/72 \quad \text{Ans}$$

(d) Only 1 (x,y) ordered pair in S satisfy $X=Y$ i.e. $(1,1)$

So, $P(X=Y) = f_{XY}(1,1) = c(1+1)$

$$P(X=Y) = 2c = 1/36 \quad \text{Ans}$$

(e) $P(Y=3) = f_{XY}(1,3) + f_{XY}(2,3) + f_{XY}(4,3)$

$$P(Y=3) = c\,[(1+9) + (4+9) + (16+9)]$$

$$= \frac{1}{72} \times 48$$

$$P(Y=3) = \frac{48}{72} = 4/6 = 2/3 \quad \text{Ans}$$

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(f) Finding marginal PMF $P_X(x)$.

Since, $P_X(x) = \sum_{y=-\infty}^{\infty} P_{XY}(x,y)$

So, $P_X(1) = f_{XY}(1,1) + f_{XY}(1,3) = \frac{12}{72}$

$P_X(2) = f_{XY}(2,1) + f_{XY}(2,3) = \frac{18}{72}$

$P_X(4) = f_{XY}(4,1) + f_{XY}(4,3) = \frac{42}{72}$

So, $P_X(x) = \begin{cases} 12/72 ; & x=1 \\ 18/72 ; & x=2 \\ 42/72 ; & x=4 \\ 0 ; & \text{otherwise} \end{cases}$

Similarly,

$P_Y(1) = f_{XY}(1,1) + f_{XY}(2,1) + f_{XY}(4,1) = 24/72$

$P_Y(3) = f_{XY}(1,3) + f_{XY}(2,3) + f_{XY}(4,3) = 48/72$

So, $P_Y(y) = \begin{cases} 24/72 ; & y=1 \\ 48/72 ; & y=3 \\ 0 ; & \text{otherwise} \end{cases}$

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(g) $E[X] = \sum_{x=-\infty}^{\infty} x \cdot P_X(x)$; for any RV X.

using Marginal PMF for X & Y as found in part (f) we can write

$E[X] = 1 \cdot \frac{12}{72} + 2 \cdot \frac{18}{72} + 4 \cdot \frac{42}{72} = 3$ Ans

& $E[Y] = 1 \cdot \frac{24}{72} + 3 \cdot \frac{48}{72} = \frac{7}{3}$ Ans

Let $Z = XY$ then range of Z will be $R_Z = \{1, 2, 3, 4, 6, 12\}$

$P_Z(1) = P_{XY}(1,1) = 2/72$
$P_Z(2) = P_{XY}(2,1) = 5/72$
$P_Z(3) = P_{XY}(1,3) = 10/72$
$P_Z(4) = P_{XY}(4,1) = 17/72$
$P_Z(6) = P_{XY}(2,3) = 13/72$
$P_Z(12) = P_{XY}(4,3) = 25/72$

so, $E[XY] = E[Z] = \sum_{z \in R_Z} z P_Z(z)$

$E[XY] = 1 \cdot \frac{2}{72} + 2 \cdot \frac{5}{72} + 3 \cdot \frac{10}{72} + 4 \cdot \frac{17}{72} + 6 \cdot \frac{13}{72} + 12 \cdot \frac{25}{72}$

$= 61/9$ Ans

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(h) $E[X^2] = \sum_{x=-\infty}^{\infty} (x)^2 \cdot P_X(x)$; for any RV 'x'.

Using Marginal PMF for X & Y as found in part (f) we can write

$E[X^2] = (1)^2 \cdot \frac{12}{72} + (2)^2 \cdot \frac{18}{72} + (4)^2 \cdot \frac{42}{72} = \frac{21}{2}$ —— (A)

$E[Y^2] = (1)^2 \cdot \frac{24}{72} + (3)^2 \frac{48}{72} = \frac{57}{9}$

So, $Var(X) = E[X^2] - (E[X])^2 = \frac{21}{2} - (3)^2$

$Var(X) = 3/2$ Ans (found in (g))

& $Var(y) = E[Y^2] - (E[Y])^2 = \frac{57}{9} - \left(\frac{7}{3}\right)^2$

$Var(y) = 8/9$ Ans ($E[y]$ from (g))

Now, $Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$ —— (1)

Finding $(E[X+Y])^2$;

Using linearity, $E[X+Y] = E[X] + E[Y]$

So, $(E[X+Y])^2 = \left(3 + \frac{7}{3}\right)^2$ (from part (g))

$= 256/9$

& Now, $E[(X+Y)^2] = E[X^2 + Y^2 + 2XY]$

Using linearity again,

$E[(X+Y)^2] = E[X^2] + E[Y^2] + 2E[XY]$

put $E[X^2]$ & $E[Y^2]$ from above eq ⓐ

& $E[XY]$ from part (g)

So, $E[(X+Y)^2] = \frac{21}{2} + \frac{57}{9} + 2 \cdot \frac{61}{9}$

$= \frac{547}{18}$

Therefore, $Var(X+Y) = \frac{547}{18} - \frac{256}{9}$

$Var(X+Y) = \frac{35}{18}$ Ans

(i) Event 'A' is $Y \leq X$.

Sample space for A is S_A then

$$S_A = \{ (2,1), (1,1), (4,1), (4,3) \}$$

~~$P_X(x=2|A)=$~~ Finding $P(A)$;

$$P(A) = f_{XY}(2,1) + f_{XY}(1,1) + f_{XY}(4,1) + f_{XY}(4,3)$$

$$P(A) = \frac{5}{72} + \frac{2}{72} + \frac{17}{72} + \frac{25}{72} = \frac{49}{72}.$$

$$P_{X|A}(x=2|A) = \frac{P((x=2)\cap A)}{P(A)} = \frac{5/72}{49/72} = 5/49$$

Similarly,

$$P_{X|A}(x=1|A) = \frac{2/72}{49/72} = 2/49$$

$$P_{X|A}(x=4|A) = \frac{17/72 + 25/72}{49/72} = 42/49$$

So, we can use ~~def~~ the of ~~def~~ conditional expectation that is

$$E[X|A] = \sum_{x_i = R_X} x_i \, P_{X|A}(x_i|A)$$

here, $R_X = \{2, 1, 4\}$

So, $E[X|A] = 2 \cdot \frac{5}{49} + 1 \cdot \frac{2}{49} + 4 \cdot \frac{42}{49}$

$E[X|A] = 180/49$ Ans

Now,

$\text{Var}(X|A) = E[X^2|A] - (E[X|A])^2$

since, $E[X^2|A] = (2)^2 \cdot \frac{5}{49} + (1)^2 \cdot \frac{2}{49} +$

$(4)^2 \cdot \frac{42}{49}$

$= 694/49$

So, $\text{Var}(X|A) = \frac{694}{49} - \left(\frac{180}{49}\right)^2$

$= \frac{1606}{2401}$

Ans

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Q.4

Given: $f_Q(q) = \begin{cases} 6q(1-q) & ; 0 \le q \le 1 \\ 0 & ; \text{otherwise} \end{cases}$

& $P(X=1 | Q=q) = q.$

$\Rightarrow P(X=0 | Q=q) = 1-q$

Let's first find $P(X=1)$.

According to law of total probability:

$$P(X=1) = \int_0^1 P(X=1 | Q=q)\, dq$$

$$P(X=1) = \int_0^1 q\, dq = 1/2$$

* because $R_Q \in [0,1]$.

Similarly, $P(X=0) = \int_0^1 1-q\, dq = 1/2.$

Now,

$$P_{Q|X}(q|x) = \frac{P(Q=q \text{ And } X=x)}{P(X=x)} \quad -(I)$$

&

$$P_{X|Q}(x|q) = \frac{P(X=x \text{ And } Q=q)}{P(Q=q)} \quad -(II)$$

(I) ÷ (II) we get:

$$\frac{P_{\Theta|X}(q|x)}{P_{X|\Theta}(x|q)} = \frac{P(\Theta=q)}{P(X=x)} \quad \text{--- (iii)}$$

for $x=1$; in eq. (iii)

$$P_{\Theta|X}(q|1) = \frac{P(\Theta=q)}{P(X=1)} P_{X|\Theta}(1|q)$$

$$= \frac{6q(1-q)}{1/2} \cdot q$$

$$= 12q^2(1-q) \quad ; 0 \le q \le 1$$

& for $x=0$; in eq. (iii)

$$P_{\Theta|X}(q|0) = \frac{P(\Theta=q)}{P(X=0)} P_{X|\Theta}(0|q)$$

$$= \frac{6q(1-q)}{1/2} \cdot (1-q)$$

$$= 12q(1-q)^2 \quad ; 0 \le q \le 1$$

Therefore, ~~$\forall q \in [0,1]$~~

$$P_{\Theta|X}(q,x) = \begin{cases} 12q^2(1-q) & ; x=1 \,\&\, 0 \le q \le 1 \\ 12q(1-q)^2 & ; x=0 \,\&\, 0 < q \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Q.5

Let the dart has its centre at origin. Then, for any point on the dart $(x,y) \rightarrow x^2+y^2 \le r^2$ i.e. circle (Say C)

Range of $X \in [-r, r]$
" " $Y \in [-r, r]$
} X & Y are RV representing x & y co-ordinates resp.

Since, all the points of impact are equally likely hence,

$$f_{XY}(x,y) = \begin{cases} 1/\pi r^2 & ; \ x^2+y^2 \le r^2 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Now finding conditional PDF $f_{X|Y}(x/y)$.

$$f_{X|Y}(X=x \mid Y=y) = \frac{f_{X,Y}(X=x \text{ and } Y=y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{1/\pi r^2}{f_Y(y)}$$

finding $f_Y(y)$; ~~that~~ marginal PDF.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$f_Y(y) = \int_{-t}^{t} \frac{1}{\pi r^2}\,dx \quad ; \quad t = \sqrt{r^2 - y^2}$$

& $|y| < r$

$$f_Y(y) = 2t / \pi r^2$$

Therefore,

$$f_{X|Y}(x,y) = \frac{1/\pi r^2}{f_Y(y)}$$

$$= \frac{1/\pi r^2}{2t/\pi r^2}$$

$$= 1/2t$$

$$f_{X|Y}(x|y) = \frac{1}{2\sqrt{r^2-y^2}}, \text{ if } x^2+y^2 \le r^2$$

For each value y, $f_{X|Y}(x|y)$ is uniform.

So, $$f_{X|Y}(x|y) = \begin{cases} \frac{1}{2\sqrt{r^2-y^2}} & ; x^2+y^2 \le r^2 \\ 0 & ; \text{Otherwise} \end{cases}$$

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Q.6

Total ways to distribute 4 balls in 4 bins $= 4^4$
because each ball have 4 options.

(i) Let X be number of balls in 3^{rd} bin.

Favourable cases $= {}^4C_x \cdot 3^{4-x}$

Choosing 'x' balls for 3^{rd} bin

distributing other (4-x) balls in bin 1,2,4.

Thus, $$P_X(x) = \begin{cases} \dfrac{{}^4C_x \cdot 3^{4-x}}{4^4} & ; x \in \{0,1,2,3,4\} \\ 0 & ; \text{otherwise.} \end{cases}$$

Thus,
$$P_X(x) = \begin{cases} 81/256 & ; x = 0 \\ 27/64 & ; x = 1 \\ 27/128 & ; x = 2 \\ 3/64 & ; x = 3 \\ 1/256 & ; x = 4 \\ 0 & ; \text{otherwise} \end{cases}$$

(ii) Let N be number of non-empty bins.
n=0 is not possible because every ball needs to be in a bin.

Favourable cases for (n=1) = selecting 1 bin to store all ball $= {}^4C_1 = 4$

Favourable cases for (n=2) = selecting 2 bin × (ways of assignment)

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$= {}^4C_2 [2^4 - 2]$ → substracted because of 2 case exist when all balls goes in 1 bin.

$= 84$

Favourable cases for $(n=3)$ = Select 3 bin × (way of Assignment)

Using Inclusion - Exclusion Principle ;

For this case, way of assignment =

$$3^4 - {}^3C_1 2^4 + {}^3C_2 (1)^4 = 36$$

So, Favourable case $(n=3)$ $= {}^4C_3 \times 36 = 144$

Favourable cases for $(n=4)$ = distributing balls s.t. no bin is empty

$$= 4^4 - {}^4C_1 3^4 + {}^4C_2 2^4 - {}^4C_3 1^4$$ Inclusion Exclusion P.

$= 24$

So, $P_N(n) = \begin{cases} 1/64 ; & n=1 \\ 21/64 ; & n=2 \\ 9/16 ; & n=3 \\ 3/32 ; & n=4 \\ 0 ; & \text{otherwise} \end{cases}$

(iii) A = 1^{st} ball sent to same label bin

So, $P(A) = \frac{4^3}{4^4} = \frac{1}{4}$ → 1 bin is favourable → 4 bin available

$P(A) = \frac{1}{4}$.

* Now $\underbrace{x=1}_{\text{3rd bin has 1 ball only.}}$ & $\underbrace{n=2}_{2 \to \text{2 empty bins}}$

Due to A, 1st bin has at least 1 ball.
Thus, # balls in bin 1 & 3 is 3, 1 resp.

So, Favourable count $= {}^3C_1 = 3$ = choosing which ball goes to bin 3 out of 2nd, 3rd & 4th ball.

* Now, $\underbrace{x=2}_{\to \text{3rd bin has 2 balls only}}$ & $n=2 \} \to$ 2 empty bins

Due to A, 1st bin has ball no. 1.
Favourable count = Giving 3rd bin 2 ball out of 2nd, 3rd & 4th ball $= {}^3C_2 = 3$.

* $x=3$ & $n=2$ (2 empty bins)
1st bin $\Rightarrow$ 1 ball (ball no. 1) due to A
3rd bin has exactly 3 ball.
Only 1 favourable case = 1st ball in bin 1 & all other in 3rd bin $= 1$.

* $x=1$ & $n=3$ (only 1 empty bin)
1st bin $\Rightarrow$ ball no. 1.
3rd bin $\Rightarrow$ 1 ball only, ways $= {}^3C_1$ (out of 2, 3, 4th)
Selecting other filled bin $= {}^2C_1$
Favourable count $= {}^3C_1 \, {}^2C_1 \, (2^2-1) = 18$

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* $x=2$ & $n=3$ (only 1 empty bin)

1^{st} bin $\rightarrow$ 1^{st} ball
3^{rd} bin $\Rightarrow$ 2 ball only, ways $= {}^3C_2$ out of 2^{nd}, 3^{rd} & 4^{th} ball.

Selecting other bin $= {}^2C_1$

Favourable count $= {}^3C_2 \, {}^2C_1 = 6$.

* $x=3$ & $n=3$ (only 1 empty bin)
1^{st} bin $\rightarrow$ 1^{st} ball
3^{rd} $\Rightarrow$ 3 balls only ways $= {}^3C_3$, 3 out of 2^{nd}, 3^{rd} & 4^{th} ball.

Selecting other bin is not possible because no such case can exist because all the ball gets exhausted.

Now, As $P_{X,N|A}(x,n) = \dfrac{P((x,n)\cap A)}{P(A)} = \dfrac{c(x,n)}{4^4 \cdot \frac{1}{4}}$

; $c(x,n)$ is favourable count.
we have following table giving $P_{X,N|A}(x,n)$ for corr. x & n values:—

$x \backslash y$	2	3
1	$3/4^3 = \frac{3}{64}$	$18/4^3 = \frac{9}{32}$
2	$3/4^3 = \frac{3}{64}$	$6/4^3 = \frac{3}{32}$
3	$1/4^3 = \frac{1}{64}$	0.

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Q.7

(a)

Let 'S' be RV denoting total no. of steps. And 'T' is time when he passes out.

Acc. to law of total expectation:

$$E[S] = \sum_{t=1}^{3} E[S|T=t]\, P_T(t)$$

Finding $E[S|T=1]$? At $T=1$:-
All possible steps $= \{0, 1, 2\}$
So,

$$E[S|T=1] = 0\cdot\left(1-\frac{1}{4}-\frac{1}{2}\right) + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4}$$
$$= 1$$

& $P_T(1) = 1/3$.

Finding $E[S|T=2]$? at $T=2$:-
All possible steps $= \{0, 1, 2, 3, 4\}$
So,

$$E[S|T=2] = 0\cdot\frac{1}{4}\cdot\frac{1}{4} + 1\cdot\left(\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4}\right) +$$

$$2\cdot\left(\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2}\right) +$$

$$3\cdot\left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}\right) + 4\cdot\left(\frac{1}{4}\cdot\frac{1}{4}\right)$$

$$= \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{2}\right) + \left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)$$

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$E[S|T=2] = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + 1 = 2$

& $P_T(2) = 1/3$.

Finding $E[S|T=3]$? RV 'S_i'

At $T=3$:-

All possible steps $= \{0,1,2,3,4,5,6\}$

$P_{S_i}(0) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = 1/64$ $\underbrace{(0,0,0)}_{1\times}$

$P_{S_i}(1) = 3 \times \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = 3/32$ $\underbrace{(0,1,0)}_{3\times}$

$P_{S_i}(2) = 3\times\left(\frac{1}{4}\frac{1}{4}\frac{1}{4}\right) +$ $(0,0,2)$,

$3\times\left(\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{15}{64}$ $\underbrace{(0,1,1)}_{3\times}$

$P_{S_i}(3) = 6\times\left(\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4}\right) +$ $(0,1,2)\}\ 6\times$

$\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) = 10/32$ $(1,1,1)$

$P_{S_i}(4) = 3\times\left(\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}\right) +$ $(0,2,2)\}\ 3\times$

$3\times\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4}\right) = 15/64$ $(1,1,2)\}\ 3\times$

$P_{S_i}(5) = 3\times\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = 3/32$ $(1,2,2)\}\ 3\times$

$P_{S_i}(6) = 1 \cdot \left(\frac{1}{4}\frac{1}{4}\frac{1}{4}\right) = 1/64$ $(2,2,2)$

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So, $E[S|T=3] = 0 \cdot P_{S_i}(0) + 1 \cdot P_{S_i}(1) +$

$2 \cdot P_{S_i}(2) + 3 \cdot P_{S_i}(3) + 4 \cdot P_{S_i}(4) +$

$5 \cdot P_{S_i}(5) + 6 \cdot P_{S_i}(6)$

$\Rightarrow E[S|T=3] = 0 + \frac{3}{32} + \frac{15}{32} + \frac{30}{32} +$

$\frac{30}{32} + \frac{15}{32} + \frac{3}{32}$

$= \frac{96}{32} = 3$

& $P_T(3) = 1/3$

Thus,

$E[S] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3}$

$E[S] = 6/3 = 2$ Ans

Q.7

(b) Let 'S' be RV denoting his position when he passes out & T be that time.

$$E[S^5] = \sum_{s_i \in R_S} (s_i)^5 \, P(S = s_i)$$

from law of total Probability;

$$P(S = s_i) = \sum_{t=1}^{3} P(S = s_i \mid T = t) \, P(T = t) \xrightarrow{} 1/3 \qquad (1)$$

Let A_1, A_2, A_3 represents set of position of drunker when passes out at $T = 1, 2, 3$ respectively. R_S will be union of A_1, A_2, A_3

i.e. $R_S = A_1 \cup A_2 \cup A_3$

We can observe, $A_1 = \{0, 1, -2\}$

& $A_2 = \{-4, -2, -1, 0, 1, 2\}$

& $A_3 = \{-6, -4, -3, -2, -1, 0, 1, 2, 3\}$

Thus, $R_S = \{-6, -4, -3, -2, -1, 0, 1, 2, 3\}$

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Now using eq. (1) ;

*

$$P(S=-6) = \sum_{i=1}^{3} P(S=-6 \mid T=i)\frac{1}{3} = P(S=-6 \mid T=3)\frac{1}{3}$$

$$\Rightarrow P(S=-6) = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{3 \cdot 64}$$

Similarly ;

*

$$P(S=-4) = \frac{1}{3} P(S=-4 \mid T=2) + \frac{1}{3} P(S=-4 \mid T=3)$$

$$= \frac{1}{3} \cdot \frac{1}{16} + \frac{1}{3} \cdot \frac{3}{64} = \frac{1}{3} \frac{7}{64}$$

*

$$P(S=-3) = \frac{1}{3} P(S=-3 \mid T=3) = \frac{1}{3} \cdot \frac{3}{32}$$

*

$$P(S=-2) = \frac{1}{3} P(S=-2 \mid T=1) + \frac{1}{3} P(S=-2 \mid T=2) + \frac{1}{3} P(S=-2 \mid T=3)$$

$$= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{3}{64}$$

$$= \frac{1}{3} \cdot \frac{16+8+3}{64} = \frac{1}{3} \cdot \frac{27}{64}$$

*

$$P(S=-1) = \frac{1}{3} P(S=-1 \mid T=2) + \frac{1}{3} P(S=-2 \mid T=3)$$

$$= \frac{1}{3} \frac{1}{4} + \frac{1}{3} \cdot \frac{6}{32} = \frac{1}{3} \cdot \frac{14}{32}$$

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*

$$P(S=0) = \frac{1}{3}P(S=0|T=1) + \frac{1}{3}P(S=0|T=2) + \frac{1}{3}P(S=0|T=3)$$

= let say K.

*

$$P(S=1) = \frac{1}{3}\sum_{i=1}^{3} P(S=1|T=i)$$

$$= \frac{1}{3}\left[\frac{1}{2}+\frac{1}{4}+\frac{3}{32}\right] = \frac{1}{3}\cdot\frac{16+8+3}{32}$$

$$= \frac{1}{3}\cdot\frac{27}{32}$$

*

$$P(S=2) = \frac{1}{3}\sum_{i=2}^{3} P(S=2|T=i)$$

$$= \frac{1}{3}\left[\frac{1}{4}+\frac{3}{16}\right] = \frac{1}{3}\cdot\frac{7}{16}$$

*

$$P(S=3) = \frac{1}{3}P(S=3|T=3) = \frac{1}{3}\cdot\frac{1}{8}$$

Now,

$$E[S^5] = (-6)^5 P(S=-6) + (-4)^5 P(S=-4) + (-3)^5 P(S=-3) + (-2)^5 P(S=-2) + (-1)^5 P(S=-1) + (0)^5 P(S=0) + (1)^5 P(S=1) + (2)^5 P(S=2) + (3)^5 P(S=3)$$

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$$= \left(\frac{-7776}{3.64}\right) + \left(\frac{-1024.7}{3.64}\right) + \left(\frac{-243.3}{3.32}\right) + \left(\frac{-32.27}{3.64}\right)$$

$$+ \left(\frac{-14}{3.32}\right) + 0 + \left(\frac{27}{3.32}\right) + \left(\frac{32.7}{3.16}\right) + \left(\frac{\cancel{81}\,243}{3.2}\right)$$

$$= \frac{1}{3.64}\left[-7776 - 7168 - 1458 - 864 - 28 + 0 + 54 + 896 + 1944\right]$$

$$= \frac{1}{3.64}\left(-14400\right)$$

$$= -75 \quad \text{Ans}$$

Q.9

Total throws = 4

Range of RV 'X' = $R_X = \{0,1,2,3,4\}$

Range of RV 'Y' = $R_Y = \{0,1,2,3,4\}$

and $X+Y \leq 4$.

Let joint PMF of X,Y = $f_{XY}(x,y)$

; $x \in R_X$, $y \in R_Y$, $x+y \leq 4$.

Let $y=0$ then $f_{XY}(x,0) = {}^4C_x \left(\frac{1}{6}\right)^x \left(\frac{4}{6}\right)^{4-x}$

; $x \in R_X$

Similarly for $y=1$, $f_{XY}(x,1) = {}^4C_1 \, {}^3C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right) \left(\frac{4}{6}\right)^{3-x}$

; $x \in R_X - \{4\}$

$y=2$, $f_{XY}(x,2) = {}^4C_2 \, {}^2C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^2 \left(\frac{4}{6}\right)^{2-x}$

; $x \in R_X - \{3,4\}$

$y=3$, $f_{XY}(x,3) =$

${}^4C_3 \, {}^1C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^3 \left(\frac{4}{6}\right)^{1-x}$

; $x \in R_X - \{2,3,4\}$

for $y=4$; $f_{XY}(0,4) = \left(\frac{1}{6}\right)^4$
(x will be 0 only)

Therefore, we can write

$$f_{XY}(x,y) = \begin{cases} {}^4C_x \left(\frac{1}{6}\right)^x \left(\frac{4}{6}\right)^{4-x} & ; x \in R_X, y=0 \\ {}^4C_1 \cdot {}^3C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right) \left(\frac{4}{6}\right)^{3-x} & ; x \in R_X - \{4\}, y=1 \\ {}^4C_2 \cdot {}^2C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^2 \left(\frac{4}{6}\right)^{2-x} & ; x \in R_X - \{3,4\}, y=2 \\ {}^4C_3 \cdot {}^1C_x \left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^3 \left(\frac{4}{6}\right)^{1-x} & ; y=3, x \in \{0,1\} \\ \left(\frac{1}{6}\right)^4 & ; y=4 \\ 0 & ; \text{otherwise.} \end{cases}$$

Q.9 remaining

Hence, Joint PMF is

X \ Y	0	1	2	3	4
0	$\frac{16}{81}$	$\frac{16}{81}$	$\frac{2}{27}$	$\frac{1}{81}$	$\frac{1}{1296}$
1	$\frac{16}{81}$	$\frac{4}{27}$	$\frac{1}{27}$	$\frac{1}{324}$	0
2	$\frac{2}{27}$	$\frac{1}{27}$	$\frac{1}{216}$	0	0
3	$\frac{1}{81}$	$\frac{1}{324}$	0	0	0
4	$\frac{1}{1296}$	0	0	0	0

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Q. 10

$E[N] = e$ & $Var(N) = v$.
$E[X_i] = k$ & $Var(X_i) = m$

since, $Y = \sum_i X_i$

We can write, (if from N customers) using law total probability for expectation;

$$E[Y] = \sum_{n_i \in R_N} E[Y|N=n_i] \, P_N(n_i)$$

$$E[Y] = E[E[Y|N]]$$

because $E[Y|N]$ will be function of N i.e. $g(N)$.

So,
$$E[Y] = E\left[E\left[\sum_{i=1}^{N} X_i \middle| N\right]\right]$$

$$= E\left[\sum_{i=1}^{N} E[X_i|N]\right]$$

; linearity of expectation

$$= E\left[\sum_{i=1}^{N} E[X_i]\right]$$

; X_i's & N are independent

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$\Rightarrow E[Y] = E[N\ E[X_i]]$

$\because E[X_i]$ are equal

$\Rightarrow E[Y] = E[X_i]\ E[N]$ $\because$ since $E[X]$ does not vary for give N & is not random

$\Rightarrow E[Y] = ke.$

Ans

Now, using law of total variance;

$Var(Y) = E(Var(Y|N)) + Var(E[Y|N])$

$= E[Var(Y|N)] + Var(N\ E[X])$

$= E(Var(Y|N)) + k^2 Var(N)$ (as above)

$= E(Var(Y|N)) + k^2 v$

As, $Var(Y|N) = \sum_{i=1}^{N} Var(X_i|N) = \sum_{i=1}^{N} Var(X_i)$

($\because X_i$'s are independent of N) $= N\ Var(X_i) = Nm$

So, $Var(Y) = E(Nm) + k^2 v = m E(N) + k^2 v$

$Var(Y) = me + k^2 v$ Ans

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Q.11

Let T be the random temperature (in °C) of the city, s.t. $T \sim N(\mu, \sigma^2)$ where $\mu = \sigma = 10$ °C.

So,

$$T \sim N(10, 100)$$

$P(T \leq 59°F) = P(T \leq 15°C) = ? = Q.$

We can write,

$$f_T(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

$$f_T(t) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-10}{10}\right)^2} \quad \forall t \in \mathbb{R}$$

So, $$Q = \int_{-\infty}^{15} f_T(t)\, dt$$

$$Q = \int_{-\infty}^{15} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-10}{10}\right)^2} dt$$

put $\frac{t-10}{10} = z \Rightarrow dt = 10\,dz$

So, $$Q = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \phi(0.5) \text{ Ans}$$

CDF of standard normal distribution.

$Q = 0.6915$ Ans

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Q.12

Let 'R' be RV denoting amount of rice.
Let 'C' be RV denoting curry served.
Let 'W' be RV denoting weight (total).

ATQ: $R \sim N(370g, (24)^2 g^2)$

$C \sim N(170g, (7)^2 g^2)$

& $W = R + C$.

We have to find $P(W \leq 575g) = ?$

As proved in Q.13;

$$W = R + C \sim N(\mu_R + \mu_C, \sigma_R^2 + \sigma_C^2)$$

where $\mu_R = 370g$, $\mu_C = 170g$
& $\sigma_R = 24g$ & $\sigma_C = 7g$

So, if $Z \sim N(0,1)$ i.e. standard Normal RV
then, $W = (\sigma_R^2 + \sigma_C^2)^{1/2} Z + (\mu_R + \mu_C)$

Now;

$$P(W \leq 575) = P\left(Z \leq \frac{575 - (\mu_R + \mu_C)}{(\sigma_R^2 + \sigma_C^2)^{1/2}}\right)$$

$$= \phi\left(\frac{575 - (\mu_R + \mu_C)}{(\sigma_R^2 + \sigma_C^2)^{1/2}}\right) = \phi\left(\frac{35}{25}\right) = \phi(1.4)$$

$= 0.919$ Ans

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Q.13

$X \sim N(\mu_X, \sigma_X^2) \equiv E[X] = \mu_X$
$\& \ Var(X) = \sigma_X^2$

$Y \sim N(\mu_Y, \sigma_Y^2) \equiv E[Y] = \mu_Y$
$\& \ Var(Y) = \sigma_Y^2$

Let $Z \sim N(0,1)$

RTP:

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Proof:

We have to proove that for RV $Q = X + Y$, mean $= \mu_X + \mu_Y$ and $Var(Q) = \sigma_X^2 + \sigma_Y^2$ i.e.

$Var(Q) = Var(X) + Var(Y)$.

Finding m.g.f for RV 'X' i.e. $M_X(t)$

$$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} dx$$

$\therefore X = \sigma_X Z + \mu_X$

$$M_X(t) = \int_{-\infty}^{\infty} e^{(\sigma_X Z + \mu_X)t} \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} dZ$$

$$M_X(t) = \frac{e^{\mu_X t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma_X Z t} \cdot e^{-Z^2/2} dZ$$

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Let $I = \int_{-\infty}^{\infty} e^{(\sigma_x Z)t} \cdot e^{-Z^2/2} \, dZ$

So,

$$I = \int_{-\infty}^{\infty} e^{\left[(\sigma_x t) Z - \frac{Z^2}{2}\right]} dZ$$

$$I = \int_{-\infty}^{\infty} e^{\left[\frac{(\sigma_x t)^2}{2} - \left(\frac{Z}{\sqrt{2}} - \frac{\sigma_x t}{\sqrt{2}}\right)^2\right]} dZ$$

Let $\lambda = (Z - \sigma_x t)/\sqrt{2}$

$\Rightarrow d\lambda = dZ/\sqrt{2}$

So,

$$I = \int_{-\infty}^{\infty} \sqrt{2}\, e^{\left[\frac{(\sigma_x t)^2}{2} - \lambda^2\right]} d\lambda$$

$$I = \exp\left[\frac{(\sigma_x t)^2}{2}\right] \int_{-\infty}^{\infty} \sqrt{2}\, e^{-\lambda^2} d\lambda$$

$$I = \sqrt{2}\, e^{\frac{(\sigma_x t)^2}{2}} \int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda \rightarrow I_1$$

$$(I_1)^2 = \left[\int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda\right]\left[\int_{-\infty}^{\infty} e^{-k^2} dk\right]$$

Let $x = r\sin\theta$, $k = r\cos\theta$

$$(I_1)^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2} r\,d\theta\,dr$$

$$= 2\pi\int_0^{\infty} r e^{-r^2} dr = -\pi\int_{r=0}^{r=+\infty} e^{-r^2} d(-r^2)$$

$$= -\pi\left[e^{-r^2}\right]_{r=0}^{r=\infty}$$

$$I_1^2 = \pi \Rightarrow I_1 = \sqrt{\pi}$$

So, $I = \sqrt{2\pi}\, e^{(\sigma_x t)^2/2}$

So, $M_X(t) = e^{\mu_x t} \cdot e^{(\sigma_x t)^2/2}$.

Similarly, $M_Y(t) = e^{\mu_Y t} \cdot e^{(\sigma_Y t)^2/2}$

Now, $M_Q(t) = E(e^{Qt}) = E[e^{(x+y)t}]$

$\Rightarrow$ $M_Q(t) = E[e^{xt} \cdot e^{yt}]$ $\Big\{$ by linearity

$= E[e^{xt}] \cdot E[e^{yt}]$ —— (I)

&

$\Rightarrow$ $M_Q(t) = e^{\mu_Q t} \cdot e^{(\sigma_Q t)^2/2}$ —— (II)

where μ_Q is mean & σ_Q^2 is variance of Q.

Solving, eq. (1)

$$M_Q(t) = e^{\mu_X t} \cdot e^{(\sigma_X t)^2/2} \cdot e^{\mu_Y t} \cdot e^{(\sigma_Y t)^2/2}$$

$$\Rightarrow M_Q(t) = \exp[(\mu_X + \mu_Y) t] \cdot \exp\left[(\sigma_X^2 + \sigma_Y^2)\frac{t^2}{2}\right] \longrightarrow (iii)$$

On comparing above eq. (iii) with eq (ii) we find;

$$\mu_Q = (\mu_X + \mu_Y)$$

and $\sigma_Q^2 = (\sigma_X)^2 + (\sigma_Y)^2$.

Also, from eq. (iii) & due to uniqueness theorem, we can say Q will be normal distribution Random variable with mean μ_Q & variance σ_Q^2.

So,

$$Q = X + Y \sim N(\mu_Q, \sigma_Q)$$

$$\equiv N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

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Q.14

$$f_{XY}(x,y) = \begin{cases} 4y(x-y)e^{-(x+y)} & ; \; 0 < x < \infty \\ & \quad 0 \le y \le x \\ 0 & ; \text{ otherwise} \end{cases}$$

For,

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\, dx$$

(due to constraint i.e. Domain consideration) $= \int_{y}^{\infty} x \, f_{X|Y}(x|y)\, dx$

Finding $f_{X|Y}(x|y)$:

$$f_{X|Y}(x|y) = f_{XY}(x,y) \,/\, f_Y(y).$$

We can say, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

(due to constraint) $= \int_{y}^{\infty} f_{XY}(x,y)\, dx$

So, $f_Y(y) = \int_{y}^{\infty} 4y(x-y)\, e^{-(x+y)}\, dx$

$$f_Y(y) = 4ye^{-y} \underbrace{\int_{y}^{\infty} (x-y)\, e^{-(x)}\, dx}_{\mathrm{I}}$$

$$I = \int_y^\infty x e^{-x}\,dx - y\int_y^\infty e^{-x}\,dx$$

$$I = \left[-xe^{-x}\right]_y^\infty + \int_y^\infty e^{-x}\,dx - y\left[-e^{-x}\right]_y^\infty$$

$$I = ye^{-y} + (1-y)\left(e^{-y}\right) = e^{-y}$$

So, $f_Y(y) = 4ye^{-2y}$. $\forall\ y \geq 0$

Thus,

$$f_{X|Y}(x,y) = \frac{4y(x-y)e^{-(x+y)}}{4y\,e^{-2y}}$$

$$f_{X|Y}(x,y) = \begin{cases} (x-y)\,e^{(y-x)}; & x \geq y \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

Thus,

$$E[X|Y=y] = \int_y^\infty x(x-y)e^{-(x-y)}\,dx$$

$$E[X|Y=y] = e^{y}\underbrace{\int_y^\infty (x^2-xy)e^{-x}\,dx}_{I_1}$$

$$I_1 = \int_y^\infty x^2 e^{-x}\,dx - y\int_y^\infty xe^{-x}\,dx$$

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$$I_1 = \left[-x^2 e^{-x}\right]_y^\infty + (2-y)\int_y^\infty x e^{-x}\,dx$$

$$I_1 = y^2 e^{-y} + (2-y)\left[\left[-xe^{-x}\right]_y^\infty + \int_y^\infty e^{-x}\,dx\right]$$

$$I_1 = y^2 e^{-y} + (2-y)\left[y e^{-y} + e^{-y}\right]$$

$$I_1 = y^2 e^{-y} + (2-y)(y+1) e^{-y}$$

$$I_1 = e^{-y}\left(y^2 + 2y - y^2 + 2 - y\right)$$

$$I_1 = e^{-y}(y+2)$$

So, $E[X|Y=y] = e^{y} I_1 = y+2$

Ans

For $Var(X|Y=y)$:-

Now Finding $E[X^2|Y=y]$?

$$E[X^2|Y=y] = \int_{-\infty}^{\infty} (x)^2 f_{X|Y}(x|y)\,dx$$

considering domain

$$= \int_y^\infty x^2 (x-y) e^{-(x-y)}\,dx$$

$$= e^{y} \underbrace{\int_y^\infty x^2 (x-y) e^{-x}\,dx}_{I_2}$$

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$$I_2 = \int_y^\infty x^3 e^{-x}\,dx - y\int_y^\infty x^2 e^{-x}\,dx$$

$$I_2 = \left[-x^3 e^{-x}\right]_y^\infty + (3-y)\int_y^\infty x^2 e^{-x}\,dx$$

$$I_2 = y^3 e^{-y} + (3-y)\Big[\left[-x^2 e^{-x}\right]_y^\infty + 2\int_y^\infty x e^{-x}\,dx\Big]$$

$$I_2 = y^3 e^{-y} + (3-y)\Big[y^2 e^{-y} + 2\left[-x e^{-x}\right]_y^\infty + 2\int_y^\infty e^{-x}\,dx\Big]$$

$$I_2 = y^3 e^{-y} + (3-y)\left[y^2 e^{-y} + 2y e^{-y} + 2e^{-y}\right]$$

$$I_2 = e^{-y}\left[y^3 + (3-y)(y^2+2y+2)\right]$$

$$I_2 = e^{-y}\left[y^2 + 4y + 6\right]$$

So, $E[X^2 | Y=y] = y^2+4y-6$

So, $Var(X|Y=y) = E[X^2|Y=y] - E[X|Y=y]^2$

$= (y^2+4y+6) - (y^2+4y+4) = 2$ Ans

Q.15

$X_1 \sim N(2, 3)$

& $X_2 \sim N(1, 4)$

As found in Q.13,
mgf of a 'Normal' RV 'x' is

$$M_X(t) = e^{\mu_X t} \cdot e^{(\sigma_X t)^2/2} \quad \text{—— (\#)}$$

Therefore,

$$M_{X_1}(t) = \exp[2t] \cdot \exp[3t^2/2]$$

&

$$M_{X_2}(t) = \exp[t] \cdot \exp[2t^2]$$

Thus,

$$\left.\begin{aligned} M_{X_1}(t) &= e^{[2t + \frac{3t^2}{2}]} \\ \& \quad M_{X_2}(t) &= e^{[t + 2t^2]} \end{aligned}\right\} \quad (\star)$$

(a) $Y = 2X_1 + 3X_2$

Since, $M_Y(t) = E[e^{tY}] = E[e^{t(2X_1 + 3X_2)}]$

$$M_Y(t) = E[e^{2tX_1} \cdot e^{3tX_2}]$$

using linearity ;

$$M_Y(t) = E[e^{2tX_1}] \cdot E[e^{3tX_2}]$$

Thus,

$$M_Y(t) = M_{X_1}(2t) \cdot M_{X_2}(3t)$$

$$M_Y(t) = e^{\left[2(2t)+3\frac{(2t)^2}{2}\right]} \cdot e^{\left[(3t)+2(3t)^2\right]}$$ using (*)

$$\Rightarrow M_Y(t) = e^{[7t]} \cdot e^{[24t^2]}$$

On comparing with eq. (#) we find that, (also using uniqueness theorem) $M_Y(t)$ refers to mgf of a Normal distribution whose mean $= 7$ and variance $= 48$.

So, $Y \sim \cancel{N(6,28)}\ N(7,48)$ Ans

(b) $M_Y(t) = E[e^{Yt}] = E[e^{(X_1-X_2)t}]$

$$M_Y(t) = E[e^{X_1 t} \cdot e^{-X_2 t}]$$

using linearity;

$$M_Y(t) = E[e^{X_1 t}] \cdot E[e^{-X_2 t}]$$

Thus,

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(-t)$$

Using ☆

$$M_Y(t) = e^{\left[2t + \frac{3t^2}{2}\right]} \cdot e^{\left[-t + \frac{4t^2}{2}\right]}$$

$$= e^{[t]} \cdot e^{\left[\frac{7t^2}{2}\right]}$$

On comparing with eq. # and also using uniqueness theorem, $M_Y(t)$ refers to mgf of a Normal distribution whose mean $= 1$ and variance $= 7$

So, $Y \sim N(1, 7)$ An

Q.15 remaining

Therefore for part (a):

$$f_Y(y) = \frac{1}{4\sqrt{6\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-7}{4\sqrt{3}}\right)^2\right).$$

and for part (b):

$$f_Y(y) = \frac{1}{\sqrt{14\pi}} \exp\left(-\frac{1}{2} \cdot \frac{(y-1)^2}{7}\right).$$

Ans