Probability and Statistics: Lecture-30

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad) on October 21, 2020



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$$E[X \mid Y] = \begin{cases} E[X \mid Y = y_1] & \text{with probability } P(Y = y_1) \\ E[X \mid Y = y_1] & \text{with probability } P(Y = y_2) \\ \vdots & \vdots \end{cases}$$

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- * For an example, let X = aY + b. Then $E[X \mid Y = y] = E[aY + b \mid Y = y] = ay + b$ $g(y) = ay + b, \quad E[X \mid Y] = aY + b$
- * Since $E[X \mid Y]$ is a RV, we can find its PMF, CDF, Variance, etc

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Let
$$Z = E[X \mid Y]$$
.

1. Find the Marginal PMFs of X and Y.

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- 4. Find E[Z], and check that E[Z] = E[X]

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- 5. Find Var(Z)

» Answer to previous problem... 2 3/5, y=0 } Boroulli X1 y independent? And No

We need
$$P_2(z)$$
, that what is the probability of $Z=Z$

(i.e. $P(Z=2/3)$ of $P(Z=0)$

We need $P_{2}(2)$, that what is the probable of Z = Z15 the probable of Z = ZThe P(Z = 2/3) + P(Z = 0)? P(Z = 0) = P(Z = 1)

* Answer to previous problem...

$$P_{2}(7) = \begin{cases} 3 \mid c \text{ if } 2 = d/3 \\ 2 \mid c \text{ if } 2 = 0 \end{cases}$$

$$0, \quad \text{otherwise}$$

» Fact...

Fact

Let X, Y be two RVs and g, h be two functions of X and Y respectively. Show that

$$E[g(X)h(Y) \mid X] = g(X)E[h(Y) \mid X]$$

Solution

$$|\nabla h| + |\nabla h$$



» Iterated Expectations...

Law of Iterated Expectations

Let X. Y be two RVs, then we have

 $E[X] = E[E[X \mid Y]]$





Proof

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.



Solved Example 2

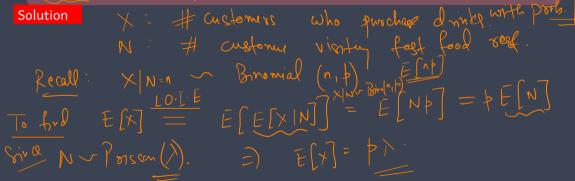
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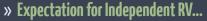
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Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?





» Expectation for Independent RV...



Expectation for Independent RVs

Let X, Y be two independent RVs. Then we have the following

$$J E[X \mid Y] = E[X]$$

$$2. E[g(X) \mid Y] = E[g(X)]$$

3.
$$E[XY] = E[X]E[Y]$$

$$\mathcal{L}[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

$$D = [x|1=1] = \sum_{x \in R_{x}} R_{x|1}(x|3) \\ = [g(x)|1] = \sum_{x \in R_{x}}$$

True for 4 y ERT

» Answer to previous problem...

= R R (2) Sy Ry (y)

- » Answer to previous problem...
 - 4) Easy. Try

Definition of Conditional Variance

Let X, Y be two RVs.

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$$Var(X | Y = y) = E[X^2 | Y = y] - \mu_{X|Y}(y)^2$$

Proof

