Probability and Statistics: Lecture-36

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on November 6, 2020

» Solved Example

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$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

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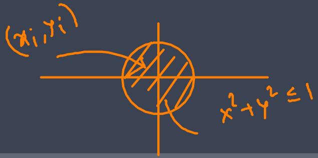

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Are X and Y uncorrelated? → $\mathrm{Cov}(X,Y) = 0$?

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\,E[Y]$$

» Answer to previous problem...

Earlier we found that X & Y are not independent &

$$X \mid Y \sim \text{Uniform}\left(-\sqrt{1-y^2}, \sqrt{1-y^2}\right)$$

We want

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[X] = E\left[\cancel{E[X|Y]}\right] \qquad [1 \cdot 0 \cdot 1 \cdot 1 \cdot \ldots]$$

$= 0$ Since $X|Y \sim \text{Uniform}(\ldots)$

$$E[XY] = E\big[E[XY|Y]\big] = E\big[Y\,E[X|Y]\big] = 0$$

$\Rightarrow \text{Cov}(X,Y) = \cancel{E[XY]} - \cancel{E[X]}E[Y] = 0$

$\Rightarrow \rho_{XY} = 0$

$\Rightarrow X, Y$ uncorrelated.

» Bivariate Normal Distribution...

Example (Sum of Two Normal Distribution May Not be Normal)

Let RVs $X \sim N(0,1)$ and $W \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ be two independent RVs.

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$$Y = h(X, W) = \begin{cases} X & \text{if } W = 0 \\ -X & \text{if } W = 1 \end{cases}$$

Find the PDF of Y

» Answer to previous problem...

Due to symmetry of $N(0,1)$ about zero, $-X$ is also a $N(0,1)$.

$$F_Y(y) = P(Y \le y)$$
$$= P(Y \le y \mid W=0)\,P(W=0) + P(Y \le y \mid W=1)\,P(W=1)$$
$$= \frac{1}{2} P(Y \le y \mid W=0) + \frac{1}{2} P(Y \le y \mid W=1)$$
$$= \frac{1}{2} P(X \le y) + \frac{1}{2} (-X \le y)$$
$$= \frac{1}{2}\Phi(y) + \frac{1}{2}\Phi(y)$$
$$= \Phi(y)$$

CDF of Std. Normal.

$$\Rightarrow Y \sim N(0,1)$$

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6. Can we provide a simple way to generate jointly normal random variables?

» Bivariate Normal Distribution and Properties...

$N(0,1)$ → $X \sim N(\mu_X, \sigma_X^2)$

$Z = \frac{X - \mu_X}{\sigma_X}$

$X = \sigma_X Z + \mu_X$

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6. Can we provide a simple way to generate jointly normal random variables?
7. We first introduce standard bivariate normal distribution

» Example of Bivariate Normal...

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2. What is the joint PDF of X and Y?

» Example of Bivariate Normal...

$g : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}$

Example (Bivariate Normal Variable)

Let Z_1 and Z_2 be two independent $N(0, 1)$ RVs. We define

$$\begin{cases} X = Z_1 \\ Y = \rho Z_1 + \sqrt{1 - \rho^2} Z_2, \end{cases}$$

where ρ is a real number in $(-1, 1)$.

1. Is X and Y bivariate normal?
2. What is the joint PDF of X and Y?
3. Find $\rho(X, Y)$

Note that $\underline{z_1}$ and $\underline{z_2}$ are normal and independent,

$$\underline{f_{z_1 z_2}(z_1, z_2)} = f_{z_1}(z_1)\, f_{z_2}(z_2)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z_2^2}{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)}$$

(a) Need to show that $ax + by$ is normal $\forall a, b \in \mathbb{R}$

$$ax + by$$

$$= a z_1 + b\left(\rho z_1 + (1-\rho^2) z_2\right)$$

$$= \underbrace{(a + b\rho)}\, z_1 + \underbrace{b(1-\rho^2)}\, z_2$$

which is a linear combination of z_1 and z_2 $\Rightarrow$ it is $\underline{\text{normal}}$.

» Answer to previous problem...

(b) Using the method of transform. to find the joint PDF of X & Y.

(i) Inverse:

$$z_1 = x = h_1(x,y)$$

$$z_2 = \frac{-\rho}{\sqrt{1-\rho^2}}x + \frac{y}{\sqrt{1-\rho^2}} = h_2(x,y)$$

Recall:

$$f_{XY}(z_1,z_2) = f_{Z_1Z_2}\left(h_1(x,y), h_2(x,y)\right)|J| \longrightarrow (*)$$

$$J = \det\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{1-\rho^2}}.$$

$$\Rightarrow f_{XY}(x,y) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[x^2 + \frac{1}{1-\rho^2}(-\rho x+y)^2\right]\right\}\cdot\frac{1}{\sqrt{1-\rho^2}}$$

» Standard Bivariate Normal Distribution...

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[x^2 - 2\rho xy + y^2\right] \right\}$$

If $\rho = 0 \Rightarrow f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$

This is a joint PDF of Std. Normal

Definition of Standard Bivariate Normal Distribution

Two RVs X and Y are said to have the standard bivariate normal distribution with correlation coefficient ρ if their joint PDF is given by

$Cov(X,Y) = E[XY] - E[X]E[Y]$

X, Y ind $\Rightarrow E[XY] = E[X]E[Y]$

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where $\rho \in (-1,1)$. If $\rho = 0$, then we call X and Y to have standard ~~normal~~ bivariate normal distribution with $\rho = 0$

» Definition of Bivariate Normal Distribution...

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$N(0,1) \rightarrow N(\mu_x, \sigma_x^2)$

Definition of Bivariate Normal Distribution

Two random variables X and Y are said to have a **bivariate normal distribution** with parameters $\mu_x, \sigma_X^2, \mu_Y, \sigma_Y^2$, and ρ,

» Definition of Bivariate Normal Distribution...

$X \sim N(\mu_X, \sigma_X^2)$ $\quad \boxed{X = \sigma_X Z_1 + \mu_X}$

$Y \sim N(\mu_Y, \sigma_Y^2)$ $\quad \boxed{Y = \sigma_Y Z_2 + \mu_Y}$

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Two random variables X and Y are said to have a bivariate normal distribution with parameters $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$, and ρ, if their joint PDF is given by

$$f_{XY} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\},$$

$Z_1 = \frac{X-\mu_X}{\sigma_X}$, $\quad Z_2 = \frac{Y-\mu_Y}{\sigma_Y}$

Z_1, Z_2 are $N(0,1)$.

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where $\mu_X, \mu_Y \in \mathbb{R}, \sigma_X, \sigma_Y > 0$ and $\rho \in (-1, 1)$ are all constant.

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we start with two independent standard normal RVs Z_1 and Z_2 and define

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y \end{cases}$$

and follow the above procedure: solve for Z_1, Z_2, and apply method of transformation