Probability and Statistics: Lecture-40

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on November 16, 2020

» Motivation for Statistical Inference...

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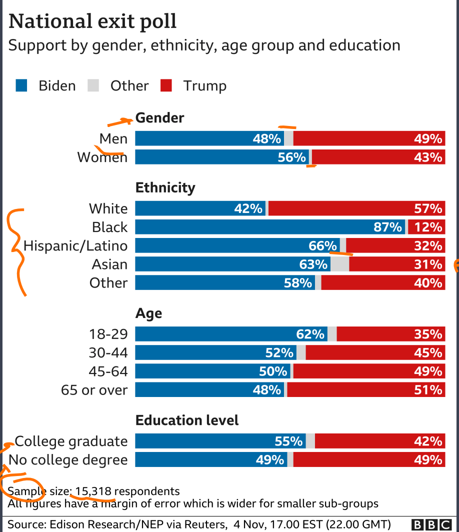

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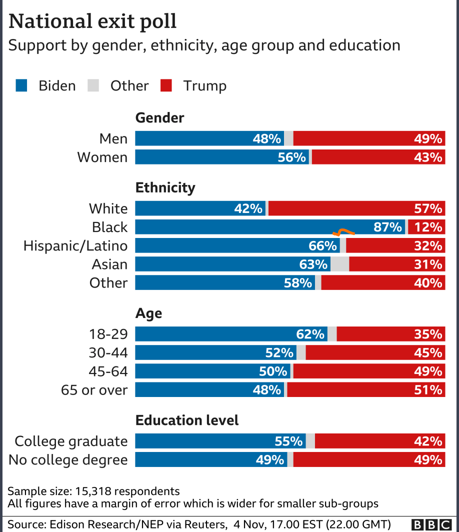
» Motivation for Statistical Inference...



* On the left, US exit poll results
* Poll on Trump Vs Biden
* Sample size of 15,318
* Error margin shown in grey
* Draw conclusions from the sample data
* Will inference fail? How much it can fail?
* How confident we are of this?

» Motivation for Statistical Inference...

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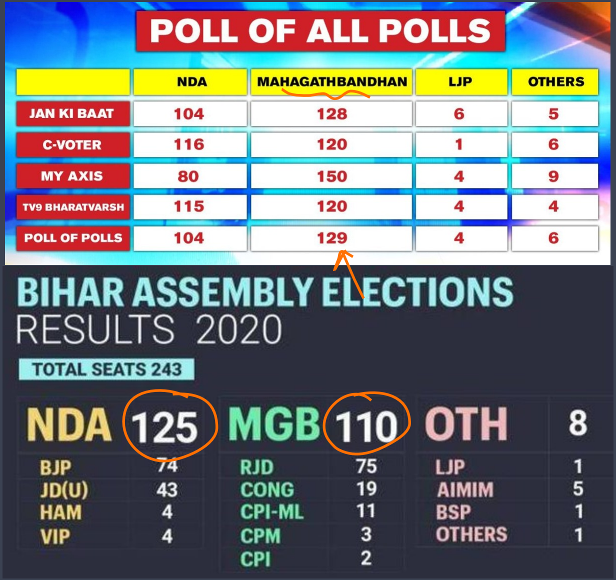
POLL OF ALL POLLS

	NDA	MAHAGATHBANDHAN	LJP	OTHERS
JAN KI BAAT	104	128	6	5
C-VOTER	116	120	1	6
MY AXIS	80	150	4	9
TV9 BHARATVARSH	115	120	4	4
POLL OF POLLS	104	129	4	6



* On the left, poll of polls showing clear majority for MAHAGATHBANDHAN
* After election, NDA has full majority
* How do we estimate such errors?

» Definition of Statistical Inference...

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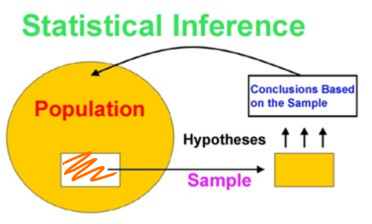

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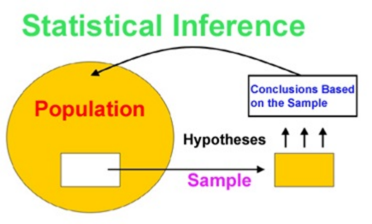

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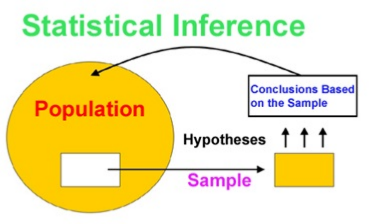

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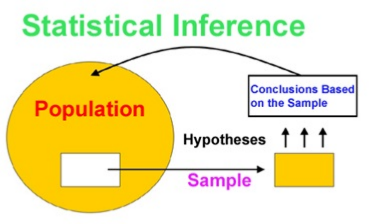

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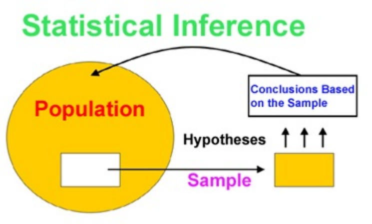

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Statistical Inference Problem

To determine an unknown quantity, get some data, and then estimate the required quantity using this data.

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  * Here $\hat{\Theta}$ is random variable, because it depends on random sample

» Bayesian Inference...

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* We use the prior knowledge that $\Theta \sim \text{Bernoulli}(p)$

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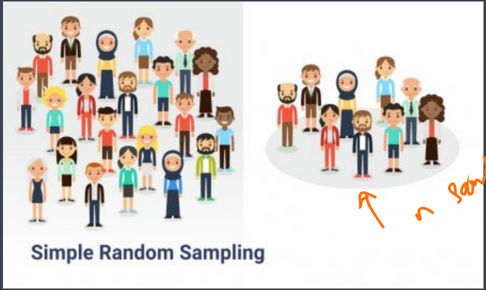
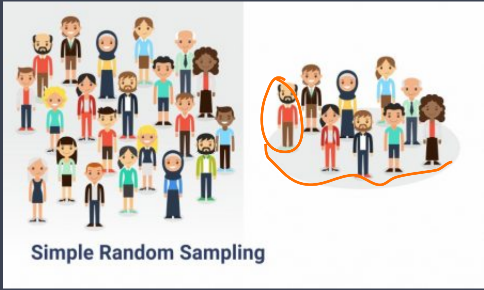


n sample

1. Choose a random sample of size n : $X_1, \ldots, X_n$ with replacement



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  * that is, working with independently and identically distributed makes analysis simpler

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» Point Estimator and Sample Mean...

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$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

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$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = \begin{cases} n!\, f_X(x_1, f_X(x_2) \cdots f_X(x_n)) & \text{for } x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & \text{otherwise} \end{cases}$$

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Find the PDFs of $X_{(1)}, X_{(2)}$, and $X_{(4)}$.

$$f_{X_{(2)}}(x) = \frac{4!}{(2-1)!\,(4-2)!} f_X(x) \left[F_X(x)\right]^{2-1} \left[1 - F_X(x)\right]^{4-2}$$

$=$

» Answer to previous problem...

» Point Estimator, Biased and Unbiased Estimators...

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$$B(\hat{\Theta}) = 0, \quad \text{for all possible values of } \theta$$

» Unbiased Estimator is not Necessarily a Good Estimator...

Fact

Show that unbiased estimator is not necessarily a good estimator.

Pf: $x_1, x_2, \cdots x_n$ Random sample, $\theta = E[x_i] = E[x]$

$\hat{\Theta} = \bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$

If we choose $\underbrace{\hat{\Theta}_1 = x_1}$, then $\hat{\Theta}_1$ is also an unbiased estimator of θ:

$B(\hat{\Theta}_1) = E[\hat{\Theta}_1] - \theta = \underbrace{E[x_1]} - \theta = \theta - \theta = 0$

Observe: $\hat{\Theta}_1$ is probably not as good as sample mean $\bar{x}$.

need other measures to ensure that estimator is "good" estimators.

. Better: $E[(\hat{\Theta} - \theta)^2]$

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Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean $E[X_i] = \theta$, and variance $\text{Var}(X_i) = \sigma^2$.

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Sol^n^ We have

$MSE[\hat{\theta}_1] = E[(\hat{\theta}_1 - \theta)^2]$

$= E[(x_1 - E[x_1])^2] = Var(x_1) = \sigma^2$

To find $MSE(\hat{\theta}_2) = E[(\hat{\theta}_2 - \theta)^2]$

$= E[(\bar{x} - \theta)^2] =$

$Var(\bar{x} - \theta) + (E[\bar{x} - \theta])^2$

$= Var(\bar{x}) + 0$

$\Rightarrow MSE[\hat{\theta}_2] = Var(\bar{x}) = \frac{\sigma^2}{n}$

As $n > 1$

$MSE[\hat{\theta}_1] > MSE[\hat{\theta}_2]$

» Answer to previous problem...

» Relationship of MSE, Variance, and Bias...

Property

If $\hat{\Theta}$ is a point estimator for θ,

$$\mathrm{MSE}(\hat{\Theta}) = \mathrm{Var}(\hat{\Theta}) + B(\hat{\Theta})^2$$

Pf.

$$\begin{aligned}\mathrm{MSE}(\hat{\Theta}) &= E[(\hat{\Theta}-\theta)^2] \\ &= \mathrm{Var}(\hat{\Theta}-\theta) + (E[\hat{\Theta}-\theta])^2 \\ &= \mathrm{Var}(\hat{\Theta}) + B(\hat{\Theta})^2\end{aligned}$$

» Consistent Estimator...

Definition of Consistent Estimator

Let $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_n, \ldots,$ be *a* sequence of point estimators of θ.

$\hat{\Theta}_2 = \frac{X_1 + X_2}{2}$

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Pf. $\lim_{n\to\infty} P(|\hat{\Theta}_n - \theta| \geq \epsilon) = P(|\hat{\Theta}_n - \theta|^2 \geq \epsilon^2) \leq \frac{E[\hat{\Theta}_n - \theta]^2}{\epsilon^2}$ [Markov] $\to 0$ as $n \to \infty$

$= \lim_{n\to\infty} \frac{\text{MSE}[\hat{\Theta}_n]}{\epsilon^2} = 0.$

» Answer to previous problem...

» Definition of Sample Variance and Sample Standard Deviation...

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Sample Variance and Sample Standard Deviation

Let $X_1, X_2, \ldots, X_n$ be a random ~~variable~~ sample with mean $E[X_i] = \mu < \infty$, and variance $0 < \text{Var}(X_i) < \sigma^2 < \infty$.

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Let $X_1, X_2, \ldots, X_n$ be a random variable with mean $E[X_i] = \mu < \infty$, and variance $0 < \mathrm{Var}(X_i) < \sigma^2 < \infty$. The sample variance of this random sample is defined as

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$$S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2 = \frac{1}{n-1}\left(\sum_{k=1}^{n} X_k^2 - n\bar{X}\right)$$

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Find the values of the sample mean, the sample variance, and the sample standard deviation for the observed sample.

Sample mean

$$\overline{T} = \frac{T_1 + \cdots + T_6}{6} = \frac{18 + 21 + \cdots + 20}{6}$$

Sample Variance

$$S^2 = \frac{1}{6-1} \sum_{k=1}^{6} (T_k - \overline{T})^2 = \cdots$$

$$S = \sqrt{S^2}.$$

» Answer to previous problem...

Example

I have a bag that contains 3 balls.

Example

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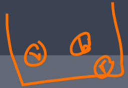

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2. Find the value of θ that maximizes the probability of the observed sample

» Answer to previous problem...

Sol: Since $X_i \sim \text{Bernoulli}\left(\frac{\theta}{3}\right)$.

$$P_{X_i}(x) = \begin{cases} \theta/3 & \text{for } x=1 \quad (\text{blue}) \\ 1-\theta/3 & \text{for } x=0 \quad (\text{red}) \end{cases}$$

Since X_i's are ind., the joint PMF of X_1, X_2, X_3 and X_4 is

$$P_{X_1X_2X_3X_4}(x_1, x_2, x_3, x_4)$$

$$= P_{X_1}(x_1)\, P_{X_2}(x_2)\, P_{X_3}(x_3)\, P_{X_4}(x_4)$$

This is called max. likelihood estimate of θ

$p = \frac{\theta}{3}$: prob. of picking blue.

$\Rightarrow$

$$P_{X_1X_2X_3X_4}(1,0,1,1) = \frac{\theta}{3}\left(1-\frac{\theta}{3}\right)\frac{\theta}{3}\cdot\frac{\theta}{3}$$

$$= \left(\frac{\theta}{3}\right)^3\left(1-\frac{\theta}{3}\right).$$

Joint PMF depends on θ.

Table

θ	$P_{X_1X_2X_3X_4}(1,0,1,1;\theta)$
0	0
1	0.02 ←
→ 2	0.09 ←
3	0

$\underline{\underline{\theta = 2}}$ max the probab.

» Answer to previous problem...

» Likelihood and log likelihood Function...

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$$\ln L(x_1, x_2, \ldots, x_n; \theta)$$

» Example

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Find the likelihood function for the following random sample

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1. $X_i \sim$ Binomial$(3, \theta)$ and we have observed $(x_1, x_2, x_3, x_4) = (1, 3, 2, 2)$

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2. $X_i \sim \text{Exponential}(\theta)$ and we have observed $(x_1, x_2, x_3, x_4) = (1.23, 3.32, 1.98, 2.12)$

» Answer to previous problem...

Solution: Recall that Random sample: X_1, X_2, X_3, X_4 are i.i.d.

$\Rightarrow$ the joint PMF & (PDF)

= product of marginal PMFs (& PDFs respect.)

① If $X_i \sim$ Binomial$(3, \theta)$,

then $P_{X_i}(x_i; \theta) = \binom{3}{x} \theta^x (1-\theta)^{3-x}$

$$L(x_1, x_2, x_3, x_4; \theta)$$

$$= P_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4; \theta)$$

$$= P_{X_1} P_{X_2} P_{X_3} P_{X_4}$$

$$= \binom{3}{x_1}\binom{3}{x_2}\binom{3}{x_3}\binom{3}{x_4} \theta^{x_1+x_2+x_3+x_4} (1-\theta)^{12-(x_1+x_2+x_3+x_4)}$$

Since $(x_1, x_2, x_3, x_4) = (1, 3, 2, 2)$

$$\Rightarrow L(1, 3, 2, 2; \theta) = \binom{3}{1}\binom{3}{3}\binom{3}{2}\binom{3}{2} \theta^8 (1-\theta)^4$$

$$= 27\, \theta^8 (1-\theta)^4.$$

» Answer to previous problem...

Definition of maximum likelihood estimator

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$$L(x_1, x_2, \ldots, x_n; \theta) : \quad f(\theta)$$

A **maximum likelihood estimator (MLE)** of the parameter θ, shown by $\hat{\theta}_{ML}$ is a random variable $\hat{\theta}_{ML} = \hat{\theta}_{ML}(X_1, X_2, \ldots, X_n)$ whose value when $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is given by $\hat{\theta}_{ML}$.

» Example of Maximum Likelihood Estimator...

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For the following examples, find the maximum likelihood estimator (MLE) of θ :

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1. $X_i \sim \text{Binomial}(m, \theta)$, and we have observed $X_1, X_2, \ldots, X_n$ $(1, 3, 2, 2)$
2. $X_i \sim \text{Exponential}(\theta)$ and we have observed $X_1, X_2, \ldots, X_n$ $(1.23, 3.32, \ldots)$

$$L(\underbrace{1,3,2,2}; \theta) = 27\, \theta^8 (1-\theta)^4$$

$$\frac{dL}{d\theta} = 27\left[8\theta^7(1-\theta)^4 - 4\theta^8(1-\theta)^3\right] = 0$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{2}{3}$$

» Answer to previous problem...

» Example of Maximum Likelihood Estimators...

Example (Example of maximum likelihood estimator)

Suppose that we have observed the random sample $X_1, X_2, X_3, ..., X_n$, where $X_i \sim N(\theta_1, \theta_2)$ so

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Find the maximum likelihood estimators for θ_1 and θ_2.

Better to use log likelihood.

① Find $L(\cdot)$ ② Take $\ln L(\ )$

③ maximize $\ln L(\ )$

» Answer to previous problem...

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(CLT)

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Exercise