Probability and Statistics: Lecture-33

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on October 28, 2020

» Conditional PDF, Conditional Probability, and Conditional CDF...

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Let X, Y be two continuous RVs. We define the following:

1. The conditional PDF of X given $Y = y$ is
$$f_{X\mid Y}(X \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$
2. The conditional probability that $X \in A$ given $Y = y$ is
$$P(X \in A \mid Y = y) = \int_A f_{X\mid Y}(x \mid y)\, dx$$
3. The conditional CDF of X given $Y = y$ is
$$F_{X\mid Y}(x \mid y) = P(X \leq x \mid Y = y) = \int_{-\infty}^{x} f_{X\mid Y}(x \mid y)\, dx$$

» Solved Example...

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Example (Solved Example)

Let X, Y be two jointly continuous RVs with joint PDF

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Let X, Y be two jointly continuous RVs with joint PDF

$$f_{XY} = \begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{5} + \dfrac{xy}{6} & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

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given

$$f_{X|Y}(a|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

For $0 \leq y \leq 2$, find the following

1. The conditional PDF of X given $Y = y$
2. $P\left(X < \frac{1}{2} \mid Y = y\right)$

» Answer to previous problem...

We find the marginal PDF of Y.

$$f_Y(y) = \int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}\right) dx$$

$$= \left[\frac{x^3}{3\cdot 4} + \frac{y^2 x}{4} + \frac{x^2 y}{2\cdot 6}\right]_0^1$$

$$= \frac{1}{12} + \frac{y^2}{4} + \frac{y}{12}$$

$$= \frac{3y^2 + y + 1}{12}$$

for $0 \le y \le 2$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{x^2}{4} + \dfrac{y^2}{5} + \dfrac{xy}{6}}{\dfrac{3y^2+y+1}{12}}$$

$$= \frac{\dfrac{30x^2 + 24y^2 + 20xy}{2\cdot 4\cdot 5\cdot \cancel{6}}}{\dfrac{3y^2+y+1}{\cancel{12}}}$$

$$= \frac{30x^2 + 24y^2 + 20xy}{10(3y^2+y+1)}$$

$$=$$

» Answer to previous problem...

(b) $P(X < \frac{1}{2} \mid Y = y)$

$$= \int_0^{1/2} \underbrace{f_{X|Y}(x|y)}_{\text{found}}\, dx$$

output is a fn of y

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$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \, f_{X \mid Y}(x \mid y) \, dx$$

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2. Conditional expectation of function of RV

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$$\mathrm{Var}(X \mid Y = y) = \underline{E[X^2 \mid Y = y]} - (E[X \mid Y = y])^2$$

» Solved Example on Conditional Expectation and Conditional Variance

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For $0 \leq y \leq 2$, find the following

1. Find $E[X \mid Y = 1]$ and $Var(X \mid Y = 1)$

» Answer to previous problem...

$$E[X \mid Y=1] = \int_{-\infty}^{\infty} \Big(x \underbrace{f_{X|Y}(x \mid y)}_{\uparrow\ \text{found}} \Big)\, dx \Big|_{y=1}$$

$$= \int_{0}^{1} \Big(x \underbrace{f_{X|Y}(x \mid y)}\Big)\, dx \Big|_{y=1}$$

$$= \int_{0}^{1} x \, \frac{\qquad}{\qquad}$$

$$\text{Var}(X \mid Y=1) = E[X^2 \mid Y=1] - \big(E[X \mid Y=1]\big)^2$$

$$E[X^2 \mid Y=1] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x \mid y)\, dx$$

$$= \int_{0}^{1} x^2 f_{X|Y}(x \mid y)\, dx$$

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$$F_{XY}(x, y) = F_X(x)\, F_Y(y), \quad \text{for all } x, y$$

If X and Y are independent, we have

$$E[XY] = E[X]\, E[Y]$$
$$E[g(X)h(Y)] = E[g(X)]\, E[h(Y)]$$

easily verified

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$$f_{XY}(x, y) = f_1(x)\, f_2(y),$$

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If X and Y are independent, we have

$$E[XY] = E[X]\, E[Y]$$

$$E[g(X)h(Y)] = E[g(X)]\, E[h(Y)]$$

* If we are given joint PDF of X and Y, $f_{XY}(x,y)$. If we can write

$$f_{XY}(x,y) = f_1(x)\, f_2(y),$$

then X and Y are independent.

» Solved Example...

$f_{XY}(x,y) = f_X(x)\, f_Y(y)$

Example (Solved Example)

Are the following RVs denoted by X and Y with the following PDF independent?

(a)

$2e^{-x} \cdot e^{-2y}$

$$f_{XY}(x,y) = \begin{cases} 2e^{-x-2y} & x,y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Yes, Independent

(b)

$$f_{XY}(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$(8x)(y)$

» Answer to previous problem...

Compute the Marginals $f_X(x)$, $f_Y(y)$

$$f_X(x) = \int_0^\infty 2e^{-x-2y}\,dy$$

$$= 2e^{-x}\int_0^\infty e^{-2y} = \cancel{2}e^{-x}\left[\frac{e^{-2y}}{-\cancel{2}}\right]_0^\infty$$

$$= -e^{-x}[-1] = e^{-x}$$

$$f_Y(y) = \int_0^\infty 2e^{-x-2y}\,dx$$

$$= 2e^{-2y}\int_0^\infty e^{-x}\,dx$$

$$= 2e^{-2y}\left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= -2e^{-2y}\left[e^{-x}\right]_0^\infty$$

$$= 2e^{-2y}$$

$$\Rightarrow f_X(x)\,f_Y(y) = f_{XY}(x,y)$$

» Answer to previous problem...

(b) $f_X(x) = \int_x^1 8xy \, dy$

$= 8x \left[\frac{y^2}{2}\right]_x^1 = 4x\left[y^2\right]_x^1$

$= 4x(1-x^2)$

$f_Y(y) = \int_0^y 8xy \, dx$

$= 8y \int_0^y x \, dx$

$= 8y \left[\frac{x^2}{2}\right]_0^y$

$= 4y^3$

Here $f_X(x) \cdot f_Y(y)$

$\neq f_{XY}(x,y)$

$\Rightarrow X, Y$ are not ind.

» Solved Example...

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Example (Solved Example)

Consider *a* point (X, Y) chosen uniformly *at* random from the following disc

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Example (Solved Example)

Consider a point (X, Y) chosen uniformly at random from the following disc

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

Area = π

» Solved Example...

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The joint PDF of X and Y is given by

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The joint PDF of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

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1. Find the constant c

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3. Find the conditional PDF of X given $Y = y$, where $-1 \leq y \leq 1$

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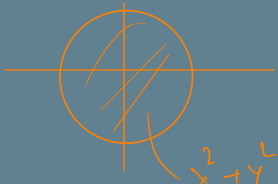

$x^2 + y^2 \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

The joint PDF of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

1. Find the constant c
2. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$
3. Find the conditional PDF of X given $Y = y$, where $-1 \leq y \leq 1$
4. Are X and Y independent?

» Answer to previous problem...

① We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \iint_D c\,dx\,dy = 1$$

$$\Rightarrow c\pi = 1$$

$$\Rightarrow c = 1/\pi \quad //$$

② $$f_X(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2\sqrt{1-x^2}}{\pi}$$

Similarly, by symmetry

$$f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi}$$

» Answer to previous problem...

③ $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

$$= \frac{1/\cancel{\pi}}{\frac{2\sqrt{1-y^2}}{\cancel{\pi}}} = \frac{1}{2\sqrt{1-y^2}}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{2\sqrt{1-y^2}}, & -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ 0 & \text{otherwise} \end{cases}$$

Note that given $Y=y$, X is uniformly distributed on $[-\sqrt{1-y^2}, \sqrt{1-y^2}]$

$X|Y=y \sim \text{Uniform}(-\sqrt{1-y^2}, \sqrt{1-y^2})$

④ No.

» Law of Total Probability...

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B_i's
Partition of sample space

$$P(A) = \sum_{i}^{k} \underbrace{P(A|B_i)}\ \underbrace{P(B_i)}$$

PDF

Law of total probability

1. Law of Total Probability

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$$P(A) = \int_{-\infty}^{\infty} \underbrace{P(A \mid X = x)} f_X(x)\, dx$$

» Law of Total Probability...

Law of total probability

1. Law of Total Probability

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x) f_X(x) \, dx$$

2. Law of Total Expectation

» Law of Total Probability...

Law of total probability

1. Law of Total Probability

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x) f_X(x)\, dx$$

$Y|X \sim \text{Expont}(\lambda)$

$E[Y|X] = \lambda$

2. Law of Total Expectation

$f_{X|Y}$

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X = x] f_X(x)\, dx = E[E[Y \mid X]]$$

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1. Law of Total Probability

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2. Law of Total Expectation

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X = x] f_X(x)\, dx = E[E[Y \mid X]]$$

3. Law of Total Variance

Law of total probability

1. Law of Total Probability

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x) f_X(x)\, dx$$

2. Law of Total Expectation

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X = x] f_X(x)\, dx = E[E[Y \mid X]]$$

3. Law of Total Variance

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y \mid X)] + \mathrm{Var}(E[Y \mid X])$$

» Solved Example...

» Solved Example...

$P(X|Y) = P(X)$ if X, Y ind

Example (Example)

Let X, Y be two independent Uniform$(0,1)$ RVs. Find $P(X^3 + Y > 1)$.

$$P(X^3+Y>1) = \int_{-\infty}^{\infty} P(X^3+Y>1 \mid X=x)\, f_X(x)\, dx$$ (Law of total Probability)

$$= \int_0^1 P(x^3+Y>1 \mid X=x)\, f_X(x)\, dx$$

because $X \sim$ Uniform$(0,1)$

because X, Y ind $=$

$$\int_0^1 P(Y > 1-x^3) \cdot 1 \cdot dx = \int_0^1 1 - P(Y \le 1-x^3)\, dx$$

$Y \sim$ Uniform$(0,1)$ $=$

$$\int_0^1 1 - (1-x^3)\, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

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Example (Solved example)

Let $X \sim \text{Uniform}(1, 2)$.

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1. Find $E[Y]$

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That is

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1. Find $E[Y]$
2. Find $\text{Var}(Y)$

» Answer to previous problem...

(a) $E[Y] = \int_{-\infty}^{\infty} E[Y|X=x]\, \underbrace{f_X(x)}\, dx$

(by Law of Total Expectation)

$= \int_{1}^{2} E[Y|X=x] \cdot 1 \cdot dx$

$= \int_{1}^{2} \frac{1}{x}\, dx = \ln 2$

Another way

$E[Y] = E[\underbrace{E[Y|X]}]$

$= E\left[\frac{1}{X}\right] = \int_{1}^{2} \frac{1}{x} \cdot f_X(x)\, dx$

$= \int_{1}^{2} \frac{1}{x}\, dx = \ln 2$

(b) $\mathrm{Var}(Y) = \underbrace{E[Y^2]} - (E[Y])^2$

We would prefer $\leftarrow$ Law of total var.

$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X])$

» Answer to previous problem...

we have

$$\mathrm{Var}(Y) = E\left[\frac{1}{\cancel{x}^2}\right] + \mathrm{Var}\left(\frac{1}{X}\right)$$

$$= E\left[\frac{1}{X^2}\right] + E\left[\frac{1}{X^2}\right] - \left(E\left[\frac{1}{X}\right]\right)^2$$

$$= E\left[\frac{2}{X^2}\right] - (\ln 2)^2$$

$$= \int_1^2 \frac{2}{x^2} \cdot f_X(x)\, dx$$

$$= \underline{\hspace{2cm}}$$

» Expectation of Function of Two RVs...

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$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y)\, dx\, dy$$

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$= E[g(x,y)]$

Find $E[XY^2]$.

$$E[xy^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy^2) f_{XY}(x,y)\,dx\,dy = \int_0^1\int_0^1 (xy^2)(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2y^2 + xy^3)\,dx\,dy = \int_0^1 \left(\frac{y^2}{3} + \frac{y^3}{3}\right) dy = \text{———}$$

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F_Z(z) &= P(Z \leq z) \\
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&= \int \int_D f_{XY}(x, y)\, dx\, dy,
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Example (Example)

Let X, Y be two independent Uniform$(0, 1)$ RVs, and $Z = XY$. Find the CDF and PDF of Z.