Probability and Statistics: Lecture-26

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
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» Online Quiz

1. Please login to gradescope
2. Attempt Quiz-6
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist for online class

1. Turn off your microphone, when you are listening
2. Turn on microphone only when you have question
3. Attend tutorials to practice problems or to discuss solutions or doubts
4. Chat is not always reliable, I may not look at chat

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1. Continuous Distributions
   * Gamma Distribution
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   * Joint Cumulative Distribution function
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   * Example of Joint PMF and Joint CDF
   * Computing Probability of a Rectangular Patch
   * Conditional PMF and Conditional CDF
   * Independent Random Variables

» Gamma Distribution...

Definition of Gamma Distribution

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1. That is, $\text{Gamma}(1, \lambda) = \text{Exponential}(\lambda)$
2. Sum of n independent Exponential(λ) RVs is $\text{Gamma}(n, \lambda)$ RV (proof: try!)

» Properties of Gamma ~~Function~~...

Distr.

Properties of Gamma Function

Let $X \sim \text{Gamma}(n, \lambda), \alpha > 0, \lambda > 0.$

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Prove the following:

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Prove the following:

1. $\int_0^\infty f_X(x) = 1$

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Let $X \sim \text{Gamma}(n, \lambda), \alpha > 0, \lambda > 0.$

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove the following:

1. $\int_0^\infty f_X(x) = 1$
2. $E[X] = \dfrac{\alpha}{\lambda}$
3. $\text{Var}(X) = \dfrac{\alpha}{\lambda^2}$

» Answer to previous problem...

① $\int_0^\infty f_X(x)\,dx = 1$

We have $\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \underbrace{\int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx}_{\to\ (*)}$

Recall prop. 2 of Gamma fn :

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \quad \text{for } \lambda > 0$$

Continuing from (*),

$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\cancel{\Gamma(\alpha)}}{\cancel{\lambda^\alpha}} = 1.$$

② $E[X] = \alpha/\lambda \to$ ②

$$E[X] = \int_0^\infty x\, f_X(x)\,dx = \int_0^\infty \frac{x \cdot \lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x \cdot x^{\alpha-1} e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \underbrace{\int_0^\infty x^\alpha e^{-\lambda x}\,dx}$$

Using prop 2

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \underset{\text{Prop 3}}{=} \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\alpha\,\cancel{\Gamma(\alpha)}}{\lambda^{\alpha+1}}$$

$$= \frac{\alpha}{\lambda} \ //$$

③ $E[X^2] = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}}$$

$$= \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2} \ //$$

» Solved Problem 1

Problem 1

Let $U \sim \text{Uniform}(0, 1)$ and $X = -ln(1 - U)$. Show that $X \sim \text{Exponential}(1)$.

Solution:

» Solved Problem 2

Problem 2

Let $X \sim N(2, 4)$ and $Y = 3 - 2X$.

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* Find $P(-2 < Y < 1)$

» Solved Problem 2

St. Normal $N(0,1)$

Problem 2

Let $X \sim N(2, 4)$ and $Y = 3 - 2X$.

* Find $P(X > 1)$
* Find $P(-2 < Y < 1)$
* Find $P(X > 2 \mid Y < 1)$

» Answer to previous problem...

Soln: (a) For $X \sim N(2,4)$, $\mu_x = 2, \sigma_x = 2$

$P(X>1) = 1 - P(X \le 1) = 1 - F_X(1)$

$= 1 - \phi\left(\frac{1-2}{2}\right) = 1 - \phi(-0.5) = \phi(0.5)$

(b) $P(-2 < Y < 1)$. Since $Y = 3 - 2X$

[From prev. result: if $Y = aX + b$, then $Y \sim N(\mu_y, \sigma_y^2)$
$\mu_y = a\mu_x + b$, $\sigma_y^2 = a^2 \sigma_x^2$]

$\Rightarrow Y \sim N(-1, 16)$

$P(-2 < Y < 1) = F_Y(Y \le 1) - F_Y(Y \le -2)$

$= \phi\left(\frac{1-(-1)}{4}\right) - \phi\left(\frac{(-2)-(-1)}{4}\right)$

Recall $P(X \le a) = \phi\left(\frac{a - \mu_x}{\sigma_x}\right)$

$P(Y \le a) = \phi\left(\frac{a - \mu_y}{\sigma_y}\right)$

$= \phi(0.5) - \phi\left(\frac{-1}{4}\right)$

$= \phi(0.5) - \phi(-0.25)$

(c) $P(X>2 \mid Y<1)$

$= P(X>2 \mid \underbrace{3-2X}_{<1}) = P(X>2 \mid X>1)$

$= \frac{P(X>2, X>1)}{P(X>1)} = \frac{P(X>2)}{P(X>1)}$

$= \frac{1 - P(X \le 2)}{1 - P(X \le 1)} = \frac{1 - \phi\left(\frac{2-2}{2}\right)}{1 - \phi\left(\frac{1-2}{2}\right)}$

Problem 3

Let $X \sim N(0, \sigma^2)$. Find $E[\,|X|\,]$.

Solution: Normal but not standard normal.

$X = \sigma Z$, where Z is std. Normal, i.e., $Z \sim N(0,1)$.

$\Rightarrow E[|X|] = \sigma E[|Z|]$

$$E[|Z|] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|\, e^{-t^2/2}\, dt \underset{\text{even}}{=} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} |t|\, e^{-t^2/2}\, dt = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} t\, e^{-t^2/2}\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2}\, d\left(\frac{t^2}{2}\right) = \sqrt{\frac{2}{\pi}} \left[-e^{-t^2/2}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

$$E[|X|] = \sigma E[|Z|] = \sigma \sqrt{\frac{2}{\pi}}.$$

» Solved Problem 4

Problem 4

Show that

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Already done!

» Mixed Random Variable...

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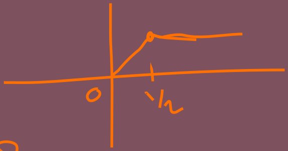
Example of mixed random variable

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Let

$f_Y(y)$?

$$Y = g(X) = \begin{cases} X & 0 \le X \le \frac{1}{2} \\ \frac{1}{2}, & X > \frac{1}{2} \end{cases}$$

Find the CDF of Y.

» Answer to previous problem...

$R_X = [0,1]$. For $x \in [0,1]$

$\Rightarrow 0 \le g(x) \le 1/2$

$\Rightarrow R_Y = [0, \frac{1}{2}]$

$\Rightarrow F_Y(y) = 0$ for $y < 0$

$F_Y(y) = 1$ for $y > \frac{1}{2}$

$$P(Y = \tfrac{1}{2}) = P(X > \tfrac{1}{2}) = \int_{1/2}^{1} 2x\,dx = \frac{3}{4}$$

$\int_{1/2}^{\infty}$

Also, for $0 < y < \frac{1}{2}$

$$F_Y(y) = P(Y \le y) = P(X \le y) = \int_0^y 2x\,dx = y^2$$

$\Rightarrow$ Summarize:

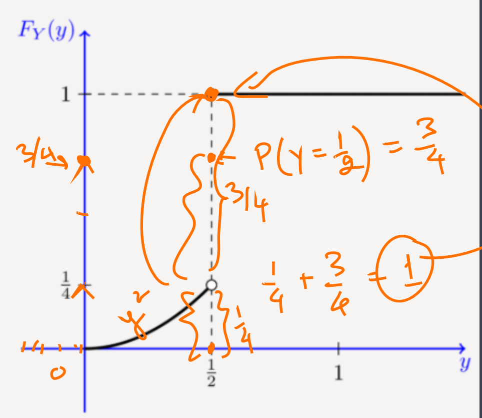
CDF of Y:

$$F_Y(y) = \begin{cases} 1 & y \ge \frac{1}{2} \\ y^2 & 0 \le y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$\Updownarrow$

$\boxed{y < 0}$

» Plot of the Mixed Random Variable Example



$$F_Y(\tfrac{1}{2}) = F_Y(y < \tfrac{1}{2}) + F_Y(y = \tfrac{1}{2})$$

$$= \frac{1}{4} + \frac{3}{4} = 1$$

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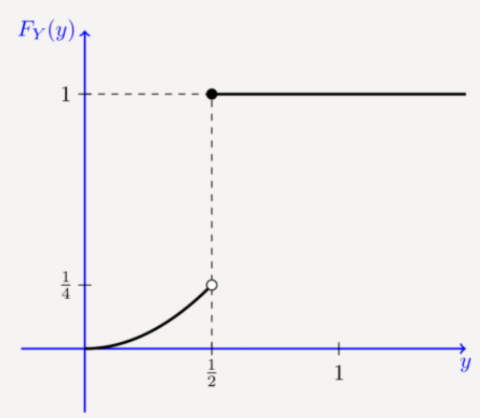

* the CDF is not continuous, so Y is not *a* continuous random variable

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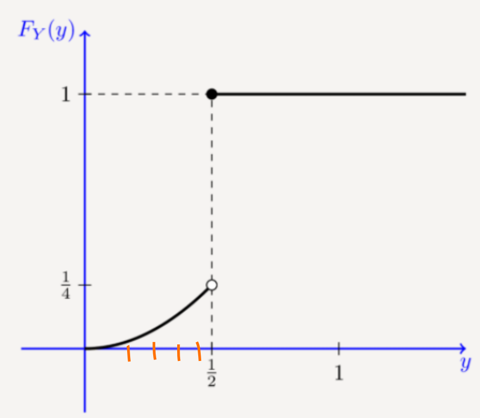

* the CDF is not continuous, so Y is not a continuous random variable
* the CDF is not in the staircase form, so it is not a discrete random variable either

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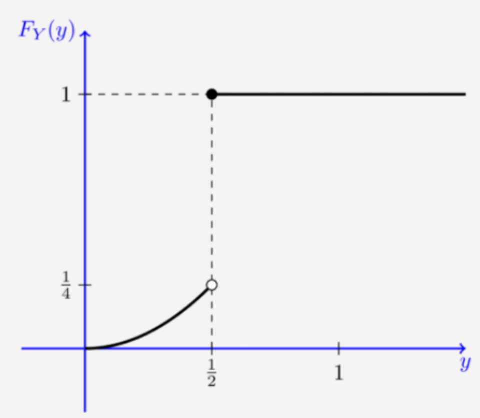

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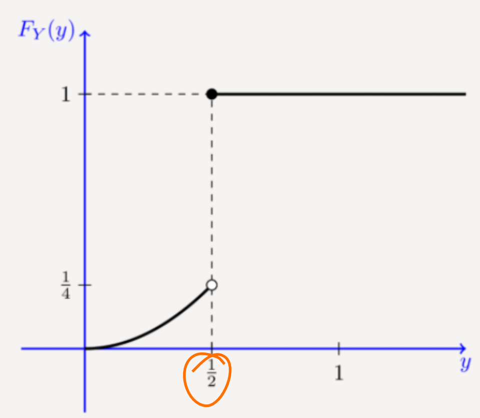

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- there is jump at $y = 1/2$

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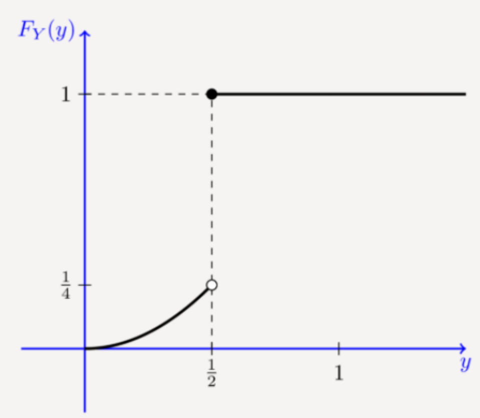

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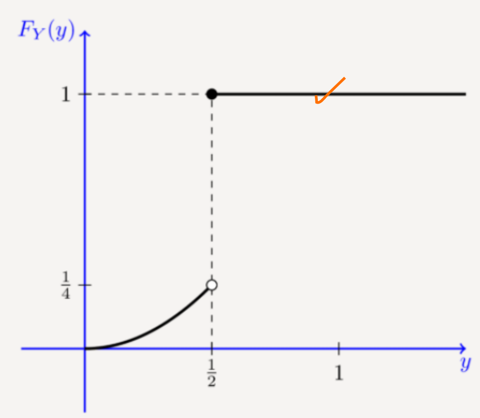

- the CDF is not continuous, so Y is not a continuous random variable
- the CDF is not in the staircase form, so it is not a discrete random variable either
- It is indeed a mixed random variable
- there is jump at $y = 1/2$
- amount of jump is 1 - 1/4 = 3/4
- CDF is continuous at other points