Probability and Statistics: Lecture-32

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on October 26, 2020

» Joint Probability Density Function...

Joint Continuous Density Functions

Two RVs X, Y are jointly continuous if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X, Y) \in A) = \int \int_A f_{XY}(x, y)\, dxdy$$

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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$$

valid PDF

» Solved Example...

Solved Example

Let X, Y be two jointly continuous RVs with joint *PDF*

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Let X, Y be two jointly continuous RVs with joint *PDF*

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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1. Find the constant c

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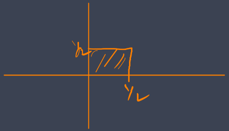

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1. Find the constant c
2. Find $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$

» Answer to previous problem...

Sol^n: (a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$\Rightarrow \int_0^1\int_0^1 (x + cy^2)\,dx\,dy = 1$

$\Rightarrow \int_0^1 \left[\frac{x^2}{2} + cy^2x\right]_0^1 dy = \int_0^1 \frac{1}{2} + cy^2\,dy = 1$

$\Rightarrow \left[\frac{1}{2}y + \frac{1}{3}cy^3\right]_0^1 = \frac{1}{2} + \frac{1}{3}c = 1$

$\Rightarrow c = 3/2$ //

(b) $P(0 \le X \le \frac{1}{2}, 0 \le Y \le 1/2)$

$= \int_0^{1/2}\int_0^{1/2} \left(x + \frac{3}{2}y^2\right) dx\,dy$

$= - - - .$

» Answer to previous problem...

» Marginal Continuous PDF...

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Let X, Y be two jointly continuous RVs with joint *PDF*

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the PDFs $f_X(x)$ and $f_Y(y)$

Marginal

» Answer to previous problem...

For $0 \le x \le 1$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy = \left[xy + \frac{1}{2}y^3\right]_0^1$$

$$= x + \frac{1}{2} \qquad 0 \le x \le 1$$

$$\Rightarrow f_X(x) = \begin{cases} x + \frac{1}{2} & \\ 0 & \text{otherwise} \end{cases}$$

Similarly for $0 \le y \le 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,\underline{dx}$$

$$= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx = \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^1$$

$$= \frac{3}{2}y^2 + \frac{1}{2} \quad //$$

$$f_Y(y) = \begin{cases} 3/2\, y^2 + \frac{1}{2}, & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Let X, Y be two jointly continuous RVs with joint PDFs

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$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find R_{XY} and plot it

» Solved Example...

$R_{XY} = \{(x, y) \mid 0 \le y \le x \le 1\}$

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4. Find $P\left(Y \leq \frac{X}{2}\right)$
5. Find $P\left(Y \leq \frac{X}{2} \mid Y \leq \frac{X}{2}\right)$

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(x, y)

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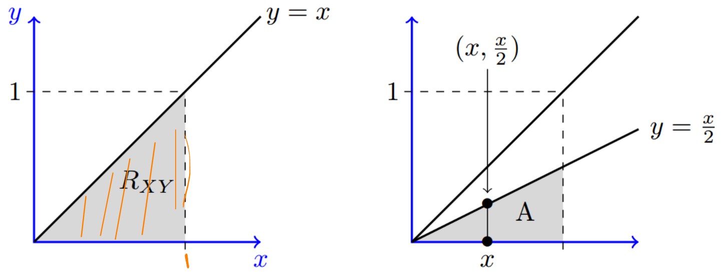

Figure showing R_{XY} and integration region $P(Y \leq \frac{X}{2})$

» Answer to previous problem...

5 $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$\displaystyle\Rightarrow \int_{0}^{1}\int_{0}^{x} c x^2 y \,dx\,dy = 1$

$\displaystyle\Rightarrow \int_{0}^{1}\left[c x^2 \frac{y^2}{2}\right]_0^x dx = 1$

$\displaystyle\Rightarrow \int_{0}^{1} \frac{c}{2} x^4\,dx = 1$

$\displaystyle\Rightarrow \left[\frac{c}{2}\frac{x^5}{5}\right]_0^1 = 1$

$\displaystyle\Rightarrow \frac{c}{10} = 1$

$\Rightarrow \boxed{c = 10}$

» Answer to previous problem...

(c) To find the marginal,

$R_X = R_Y = [0, 1]$

For $0 \le x \le 1$, we

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^x 10x^2 y\, dy = 5x^4$$

$f_{XY} = cx^2y$

For $0 \le y \le 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \ldots$$

(d) To find $P(Y \le X/2)$

$$P(Y \le X/2) = \int_{-\infty}^{\infty} \int_0^{x/2} f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^{x/2} 10x^2 y\, dy\, dx = \int_0^1 \ldots$$

» Joint Cumulative Distribution...

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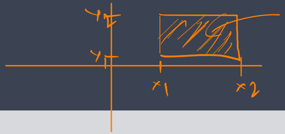

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3. $F_{XY}(\infty, \infty) = 1$
4. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
5. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$

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5. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$
6. If X, Y are independent, then $F_{XY} = F_X(x)F_Y(y)$

Solved Example

Let X, Y be two random variables with Uniform(0,1) distribution. Find $F_{XY}(x,y)$.

Since $X, Y \sim$ Uniform $(0,1)$, we have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Since X and Y are indep.

$$F_{XY}(x,y) = \underline{F_X(x)}\ \underline{F_Y(y)}$$

$$= \begin{cases} 0 & \text{for } x < 0 \ \& \ y < 0 \\ xy & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \\ y & x > 1,\ 0 \le y \le 1 \\ x & y > 1,\ 0 \le x \le 1 \\ 1 & x > 1,\ y > 1 \end{cases}$$

» Answer to previous problem...

» Figure for Solved Example...

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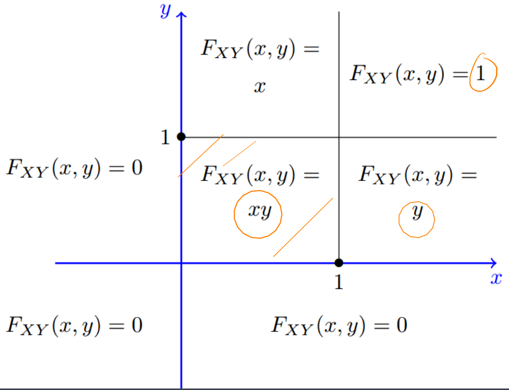

3D plots

Plot of joint CDF

» Relationship Between CDF and PDF...

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$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Relationship between CDF and PDF

Recall that for single RV we have

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

Relationship between CDF and PDF

Recall that for single RV we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

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Recall that for single RV we have

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Similarly, for two RVs we have

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$$f_X(x) = \frac{dF_X(x)}{dx}$$

Similarly, for two RVs we have

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v)\, du\, dv$$

$$f_{XY} = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) = \frac{\partial^2}{\partial y \partial x}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are cont. then mixed partial deriv. are equal

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$$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the joint CDF of X and Y

» Answer to previous problem...

(c) To find the const c

Recall $c = \frac{3}{2}$ (found before)

$$R_{XY} = \{(x,y) \mid 0 \le x \le 1, \quad 0 \le y \le 1\}$$

$$\begin{cases} F_{XY}(x,y) = 0 & x < 0, \; y < 0 \\ F_{XY}(x,y) = 1 & x \geq 1, \; y \geq 1 \end{cases}$$

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)\, du\, dv$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} \left(u + \frac{3}{2}v^2\right) dx\, dy$$

$$= \int_{0}^{y}\int_{0}^{x} \left(u + \frac{3}{2}v^2\right) du\, dv$$

$$= \int_{0}^{y} \left[\frac{1}{2}u^2 + \frac{3}{2}v^2 u\right]_0^x dv$$

$$= \int_{0}^{y} \left(\frac{1}{2}x^2 + \frac{3}{2}x v^2\right) dv$$

$$= \frac{1}{2}x^2 y + \frac{1}{2}x y^3$$

» Answer to previous problem...

For $0 \le x \le 1,\ y \geq 1$

For $0 \le y \le 1,\ x \geq 1$

Exercise

» Definition of Conditional PDF and Conditional CDF

$$A = \{\underbrace{a < x < b}\}$$

Let X be a continuous RV and A be an event that $a < X < b$ (where possibly $b = \infty$ or $a = -\infty$), then

$$f_{X|A}(x) = \begin{cases} 1 & x > b \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \le x < b \\ 0 & x < a \end{cases}$$

$$\frac{d}{dx} \frac{F_X(x) - F_X(a)}{\left(F_X(b) - F_X(a)\right)}$$

$$\left(F_X(b) - F_X(a)\right) = P(A)$$

$$= \frac{1}{P(A)} \frac{d}{dx}\left(F_X(x) - F_X(a)\right)$$

constant

$$= \frac{f_X(x)}{P(A)}$$

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(A)} & a \le x < b \\ 0 & \text{otherwise} \end{cases}$$

» Answer to previous problem...

Let $A = \{a \le X \le b\}$

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= P(X \le x \mid a \le X \le b)$$

$$= \frac{P(X \le x,\ a \le X \le b)}{P(A)}$$

If $x < a$, then $F_{X|A}(x) = 0$

because $P(X \le x,\ a \le X \le b) = 0$

If $\underbrace{a \le x \le b}$

$$F_{X|A}(x) = \frac{P(a \le X \le x)}{P(A)}$$

$$= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$$

If $x > b$, then $F_{X|A}(x) = 1$.

$$F_{X|A}(x) = \begin{cases} 1 & x > b \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

» Answer to previous problem...

» Conditional Expectation and Variance...

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Definition of Conditional Expectation and Variance

For a random variable X and event A, we have

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Definition of Conditional Expectation and Variance

For a random variable X and event A, we have

Condi. Expect

conditional PDF

$$E[X \mid A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$$

$$E[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$$

$$\mathrm{Var}(X \mid A) = E[X^2 \mid A] - (E[X \mid A])^2$$

» Solved Example: Conditional PDF and CDF...

» Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$.

» Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

» Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim$ Exponential(1). Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$

» Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim \text{Exponential}(1)$. Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$
2. Find $E[X \mid X > 1]$

» Solved Example: Conditional PDF and CDF...

Solved Example

Let $X \sim$ Exponential(1). Answer the following.

1. Find the conditional PDF and CDF of X given $X > 1$
2. Find $E[X \mid X > 1]$
3. Find $\mathrm{Var}(X \mid X > 1)$

» Answer to previous problem...

(a) Let A be the event that $X > 1$

$$P(A) = \int_1^\infty e^{-x}\,dx = \frac{1}{e}$$

" $P(X>1)$ "

$$f_{X|X>1}(x) = \frac{P(X=x,\ X>1)}{P(X>1)} = \frac{e^{-x}}{1/e} = e^{-x+1}$$

$$\boxed{\int e^{-x} = \frac{e^{-x}}{-1}}$$

For $x > 1$

$$F_{X|A} = \frac{F_X(x) - F_X(1)}{P(A)}$$

We have

~~$F_X(x) =$~~

$$F_{X|A}(x) = \int_1^x f_{X|A}(u)\,du$$

$$= \int_1^x e^{-u+1}\,du = \left[\frac{e^{-u+1}}{-1}\right]_1^x$$

$$= \left[\frac{e^{-x+1}}{-1} + e^{-1+1}\right] = 1 - e^{-x+1}$$

» Answer to previous problem...

(b) $E[X \mid \underbrace{X>1}]$

$$= \int_1^\infty x \, f_{X|X>1}(x)\,dx$$

$$= \int_1^\infty x\, e^{-x+1}\,dx = e \int_1^\infty \underbrace{x\, e^{-x}}\,dx$$

$$= \boxed{2}$$

(c) $E[X^2 \mid X>1]$

$$= \int_1^\infty x^2 f_{X|X>1}(x)\,dx$$

$$= \int_1^\infty x^2 e^{-x+1}\,dx \quad \ldots$$

$$\mathrm{Var}[X \mid X>1] = E[X^2 \mid X>1] - \big(\underbrace{E[X \mid X>1]}\big)^2$$