Assignment → 1 (Probability & statistics)

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Section : 1

Q.1 : sol^n ⇒ X is the number of hits.
Let p be the probability of hitting the target with each shot.
then p = 0.2

So, Range of X = {0, 1, 2, 3, ---, 10}

(1) PMF of X :

$$P(X=i) = \begin{cases} 0 & ; \; i \notin \{0,1,2,\ldots,10\} \\ {}^{10}C_i \, (p)^i (1-p)^{10-i} & ; \; i \in \{0,1,2,\ldots,10\} \end{cases}$$

(2) Expectation of X :

$$E(X) = \sum_{x_i = 0}^{10} x_i \, P(x_i) \quad ; \; x_i \in \{0,1,2,\ldots,10\}$$

$$= \sum_{x_i=0}^{10} (x_i) \left({}^{10}C_{x_i} \, (p)^{x_i} (1-p)^{10-x_i} \right)$$

& because $x_i \, {}^{10}C_{x_i} = 10 \times {}^{9}C_{(x_i - 1)}$, we can write;

$$E(X) = \sum_{x_i=0}^{10} 10 \cdot {}^{9}C_{x_i - 1} \cdot (p)^{x_i - 1} \cdot (p) \cdot (1-p)^{10 - x_i}$$

$$E(X) = 10p \sum_{x_i=1}^{10} {}^{9}C_{x_i - 1} \cdot (p)^{x_i - 1} \cdot (1-p)^{9-(x_i - 1)}$$

$$E(X) = 10p \, \left[(p) + (1-p)\right]^{9} \quad ; \text{Using Binomial expansion}$$

$$E(X) = 10 \times p$$

$$E(X) = 10 \times 0.2 = 2. \quad \text{Ans.}$$

Variance of X :

$$Var(X) = E(X^2) - (\mu)^2 \quad ; \quad \mu = E(X).$$

finding $E(X^2)$; $E(X^2) = \sum_{X_i=0}^{10} (X_i)^2 P(x_i)$

$$E(X^2) = \sum_{x_i=0}^{10} (x_i)^2 \left({}^{10}C_{x_i}\right) (p)^{x_i} (1-p)^{10-x_i}$$

$$= 10 \cdot p \sum_{x_i=1}^{10} x_i \, {}^{9}C_{(x_i - 1)} \, (p)^{x_i - 1} \, (1-p)^{10 - x_i}$$

$$= 10 \cdot p \sum_{j=0}^{9} (j+1) \, {}^{9}C_j \, (p)^{j} \, (1-p)^{9-j}$$

$= 10 \cdot p \; E(J+1)$; where J is Binomial of form $(9, p)$.

$= 10 \cdot p \left(E(J) + 1 \right)$ & $E(J) = 9p$

$= 10 \cdot p \left(9p + 1 \right)$ ——— (A)

& $E(X) = \mu = 2 = 10p$ (found already)

So, $Var(X) = 10 \cdot p \cdot (9 \cdot p + 1) - (10p)^2$

$\Rightarrow$ ~~$10 \cdot p \times 9 \cdot p$~~ $= 10 \cdot p \, (9p + 1 - 10p)$

~~$= 90p^2$~~ $= 10 \cdot p \cdot (1 - p)$

~~$= 3.6$~~ $= 1.6$ Ans

(3) In General from last part what we found for Binomial Random Variable $X \equiv (n, p)$

$E(X) = np$

& $Var(X) = np(1-p)$

We can observe & say Y is Binomial s.t. $\forall \; y_i \in Range(Y)$

$\Rightarrow y_i = 2x_i - 3$

$\forall \; x_i \in Range(X)$.

Therefore; $Y = 2X - 3$.

So, $E(Y) = E(2X-3)$

$= 2E(X) - 3$

$= 2 \times 2 - 3$ (from last part)

$= 1$

& $Var(Y) = E(Y^2) - (E(Y))^2$

$= E((2X-3)^2) - (E(Y))^2$

$= E(4X^2 + 9 - 12X) - 1$

$= 4E(X^2) + 9 - 12E(X) - 1$

(from Ⓐ)

$= 4[10 \cdot p(9 \cdot p + 1)] + 8 - 12[2]$

$= 4[2(2 \cdot 8)] + 8 - 24$

$= 22.4 + 8 - 24$

$= 6.4$ Ans

(4) In this case random variable $Z = X^2$

So, $E(Z) = E(X^2)$

$= 10p(9p+1)$ (from Ⓐ)

$= 5.6$ Ans

Q. 2 = sol ⇒

let x be no. of white ball withdrawn from 1^{st} bag & y be black ball from the same.

So, A.T.Q :- $x+y=2$
& $x = 0$ or 1 or 2
& $y = 0$ or 1 or 2

Only 3 possibility :-

① $\underbrace{x=0,\ y=2}_{\text{event } K_1}$ ② $\underbrace{x=y=1}_{\text{event } K_2}$ ③ $\underbrace{x=2,\ y=0}_{\text{event } K_3}$.

So, $P(K_1) = \frac{3}{8}\cdot\frac{2}{7} = 6/56 = {}^3C_2/{}^8C_2 = 3/28$

$P(K_2) = \frac{5}{8}\cdot\frac{3}{7} + \frac{3}{8}\cdot\frac{5}{7} = \frac{{}^5C_1 \cdot {}^3C_1}{{}^8C_2} = 15/28$

$P(K_3) = \frac{5}{8}\cdot\frac{4}{7} = \frac{{}^5C_2}{{}^8C_2} = 5/14$

Let E be event that 2 balls withdrawn from 2^{nd} bag be white and black.

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i.e. 1 black & 1 white.

As, $P(E|k_i) = \dfrac{P(E \cap k_i)}{P(k_i)}$

So, $\sum_{i=1}^{3} P(E \cap k_i) = \sum_{i=1}^{3} P(E|k_i)\, P(k_i)$

Now, As, $E = E \cap (k_1 \cap k_2 \cap k_3)$
because k_i's are mutual disjoint & exhaustive.

$\Rightarrow E = (E \cap k_1) \cup (E \cap k_2) \cup (E \cap k_3)$

we can say $(E \cap k_i)$'s are also mutual disjoint & exhaustive.

So, $P(E) = \sum_{i=1}^{3} P(E \cap k_i)$

$$P(E) = \sum_{i=1}^{3} P(E|k_i)\, P(k_i)$$

so, $P(E) = \left(\frac{{}^{3}C_1\, {}^{7}C_1}{{}^{10}C_2} \cdot \frac{3}{28}\right) + \left(\frac{{}^{4}C_1 \cdot {}^{6}C_1}{{}^{10}C_2} \cdot \frac{15}{28}\right) +$

$$\left(\frac{{}^{5}C_1\, {}^{5}C_1}{{}^{10}C_2} \cdot \frac{5}{14}\right)$$

$$P(E) = \frac{1}{45 \times 28}\left[63 + 360 + 250\right]$$

$$= \frac{673}{45 \times 28} = \frac{673}{1260} \quad \text{Ans}$$

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Q.3 = Sol ⇒ $x^2 + 2kx + m = 0$ has roots given by :

$$x = \frac{-2k \pm \sqrt{4k^2 - 4m}}{2} = -k \pm \sqrt{k^2 - m}.$$

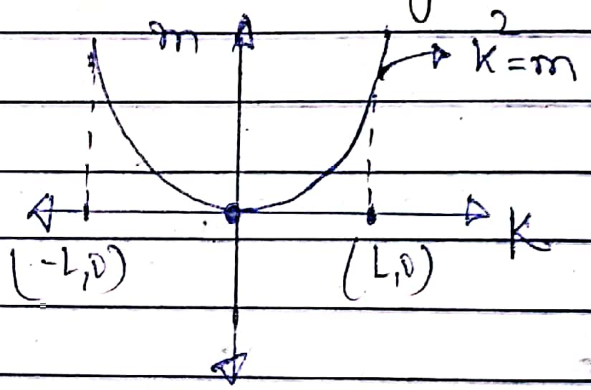
For at least 1 real root ; $D = k^2 - m \geq 0$

i.e. $k^2 - m \geq 0$.

In case $m < 0 \longrightarrow k^2 - m \geq 0$ clearly.

Now for $m \geq 0$;

Say $k \in [-L, L]$



Then, area under curve

$$= \int_{-L}^{L} k^2 \, dk$$

$$= \frac{2}{3} L^3$$

From Graph & also in general we can say $D = k^2 - m \geq 0$ for area below the curve $k^2 = m$.

If we consider only area formed by rectangle $(\pm L, L^2)$, $(\pm L, 0)$ then prob. that $D \geq 0$ is

$$\frac{\frac{2}{3}L^3}{(2L)(L^2)} = \frac{1}{3}$$

So, if we put limit $L \to \infty$ even then this will give probability of $1/3$.

Thus, final value of probability of at least one possible real root of

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3}$$

$$= 4/6 = 2/3 \text{ Ans}$$

Q. 4 = Sol^n $\Rightarrow$

say, Atsi's sister name is A.

So, sample space Gender of Atsi & A resp. will be

$$\{ (G,G), (G,B), (B,G), (B,B) \}$$

(i) $P(A = G \mid Atsi = G) = ?$

So, $\Rightarrow \dfrac{P((A=G) \cap (Atsi=G))}{P(Atsi=G)}$

$$\Rightarrow \frac{1/4}{1/2} = 1/2 \text{ Ans}$$

(2) $P(A = G \mid Atsi = B) = ?$

$$\Rightarrow \frac{P((A=G) \cap (Atsi = B))}{P((Atsi = B))}$$

$$\Rightarrow \frac{1/4}{1/2} = 1/2 \text{ Ans}$$

Q, 5 = Soln $\Rightarrow$

FLASH $\longrightarrow$ FAST

$P(F \text{ correct}) = 0.8 = P_1$
$P(L \text{ missed}) = 0.1 = P_2$
$P(A \text{ correct}) = 0.8 = P_3$
$P(S \text{ correct}) = 0.8 = P_4$

Now, there are two cases :

(1) T inserted as extra & then H missed

$P(T \text{ inserted extra}) = \frac{1}{26} \times 0.1 = A$

$P(H \text{ missed}) = 0.1 = B$

(2) H missed then T inserted as extra.

$P(H \text{ missed}) = 0.1 = B'$
$P(T \text{ inserted as extra}) = \frac{1}{26} \times 0.1 = A'$

Final Answer = $P_1 P_2 P_3 P_4 (AB + A'B')$

$= (0.8)^3 \times 0.1 \times \left(\frac{1}{26} \times (0.1)^2 + \frac{1}{26} \times (0.1)^2 \right)$

$= \frac{512}{13} \times 10^{-6}$ Ans

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Q.6 = Sol^n ⇒

P(atleast two people have same birthday) =

1 − P(no common birthdays)

$$= 1 - \left(\frac{{}^{365}P_{30}}{(365)^{30}} \right).$$ Ans

Q.7 = Sol^n ⇒

Total paths going through x,y = Q

$$Q = {}^{(x+y)}C_{x} \; {}^{[(n+m)-(x+y)]}C_{(n-x)}$$

Because Arya will have to travel through total (x+y) edges out of which x edges should be horizontal.

Total random path will be = ${}^{n+m}C_n$

for similar reason, as she cannot move beyon (n,m).

Thus, $P(\text{she passes } (x,y)) = \frac{\cancel{(x+y)}\, Q}{(n+m)_{C_n}}$. Ans

Q.7 ⇒

$$\text{Final Answer} = \frac{{}^{(x+y)}C_x \times {}^{(n+m-x-y)}C_{(n-x)}}{{}^{(n+m)}C_n}$$

Ans

Q.8 = Sol^n ⇒ Given :—

$P(E \cap F) = 1/6$ —— (i)

$P(E' \cap F') = 1/3$ —— (ii)

$[P(E) - P(F)][1 - P(F)] > 0$ —— (iii)

⇒ $P(E) \neq P(F)$ & $P(F) \neq 1$.

As E & F are independent then

$P(E|F) = P(E)$ & $P(F|E) = P(F)$. —— (A)

Also, as $0 \leq P(F) \leq 1$ $(P(F) \neq 1)$ from (iii).

⟶ $0 \leq 1 - P(F) \leq 1$

—— (iv)

So, from (iii) & (iv)

⇒ $P(E) - P(F) > 0$.

⇒ $P(E) > P(F)$. —— (*)

from set theory;

$(E \cap F)' = (E' \cup F')$

⇒ $P((E \cap F)') = P(E' \cup F')$

⇒ $1 - P(E \cap F) = P(E') + P(F') - P(E' \cap F')$

⇒ $5/6 = 1 - P(E) + 1 - P(F) - 1/3$

⇒ $P(E) + P(F) = 2 - 1/3 - 5/6 = 5/6$

⇒ $P(E) = 5/6 - P(F)$ —— (v)

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from (A)

$$P(E\cap F) = P(E|F)\,P(F) = P(E)\,P(F)$$

$$\Rightarrow P(E)\,P(F) = 1/6 \quad \text{--- (VI)}$$

So, from (V) & (VI);

Putting P(E) from (VI) in (V)

$$\frac{1}{6P(F)} = \frac{5}{6} - P(F)$$

$$\Rightarrow 6(P(F))^2 - 5P(F) + 1 = 0$$

$$\Rightarrow P(F) = \frac{1}{2} \text{ or } \frac{1}{3} \quad \text{--- (VII)}$$

As from (*) $P(E) > P(F)$

& from (VII) $P(E) = 1/3$ for $P(F) = 1/2$

& $P(E) = 1/2$ for $P(F) = 1/3$

Both can be satisfied only for

$$\boxed{P(E) = 1/2 \ \& \ P(F) = 1/3.}$$ Ans

Q.9 ⇒ Sol^n ⇒

Since, all the containers are identical and all the balls are identical.

So, different possible ways such that all container have at most 2 balls =

$$\begin{cases} (n/2)+1 \; ; & n \text{ is even} \\ n+1/2 \; ; & n \text{ is odd} \end{cases}$$

Ans

Q.10 = sol^n ⇒ Total number of possible outcome

$= {}^3C_1 \times {}^5C_1 \times {}^7C_1$

$= 105$.

For calculating favourable outcomes assume 3-digit no. such that

ones place can take value from 1 to 7

tens " " " " " 1 to 5

Hundred " " " " " 1 to 3.

i.e.

_ _ _

↓ hundreds: 1-3 ; ↓ tens: 1-5 ; ones → 1-7.

Final Answer $= 22/105$.

Now, finding three no. set s.t. that form AP from number 1 to 7. & validating them for above constraint.

1. k, k, k ; k = 1, 2, ..., 7 ⟶ Only k = 1, 2, 3 = 3

2. {1, 2, 3}, {2, 3, 4}, {3, 4, 5}, {5, 6, 7}

$\underline{|3} + \underline{|3} - 2 + 2 + 0 = 12$ → Permutations

3. {1, 3, 5}, {2, 4, 6}, {3, 5, 7}

$\underline{|3} - 2 + 1 + 1 = 6$ → Permutation

4. {1, 4, 7} → 1 permutation.

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Section : 2

B.11 = Sol^n ⇒

Given : ① C_i's are disjoint

② A & B are conditionally Independent

$$\Rightarrow P(A \cap B | C_i) = P(A|C_i)\, P(B|C_i)$$

$$\forall\, i \in 1, 2, \ldots, M.$$

③ $$P(B|C_i) = P(B)$$

$$\forall\, i \in 1, 2, \ldots, M.$$

To Prove : $$P(A \cap B) = P(A)\, P(B)$$

~~B/D~~

$\cancel{P(A|B)} =$

Proof :-

$$\cancel{\bigcup_{i=1}^{i=M} C_i = U \text{ (universal set)}}$$

$$\cancel{\text{i.e. } P\left(\bigcup_{i=1}^{M} C_i\right) = 1.}$$

so,

$$P(A \cap B | C_i) = P(A|C_i) \cdot P(B)$$

⇒ multiplying $P(C_i)$ on both side we get :

$\Rightarrow \quad P(A \cap B | C_i)\, P(C_i) = P(A|C_i) \cdot P(B) \cdot P(C_i)$

$\Rightarrow \sum_{i=1}^{M} P(A \cap B \cap C_i) = \sum_{i=1}^{M} P(A \cap C_i) \cdot P(B)$ —— (A)

Since, C_i's are disjoint partition of sample space S. Their union will be whole S & intersection will be ϕ.

Thus; $\sum_{i=1}^{M} P(A \cap C_i) = P(A \cap (C_1 \cup C_2 \cdots \cup C_M))$

$= P(A \cap S)$

$= P(A)$.

Similarly, $\sum_{i=1}^{M} P((A \cap B) \cap C_i) = P(A \cap B)$.

Thus, implementing this in eq. (A)

$\Rightarrow$

$P(A \cap B) = \sum_{i=1}^{M} P(A \cap C_i) \cdot P(B)$

$= P(B) \sum_{i=1}^{M} P(A \cap C_i) = P(B) \cdot P(A)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

Hence proved.

Q.12 = solⁿ ⇒ We have 4 possibility,

say E = Kaushik said truth

F = Neeraj said truth.

thus $S = \{ (E,F), (E,F'), (E',F), (E',F') \}$

A B C D

Total Probability that Neeraj wins given Kaushik's honesty $= P(A) + P(D)$

$= \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2}$

$= 5/10 = 1/2.$

Ans

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Q.13 = Soln ⇒

A = he copies the answer

B = he guesses the answer

C = answer is correct.

D = he knew the answer.

$P(B) = 1/3$, $P(A) = 1/6$, $P(C|A) = 1/8$

$P(D|C) = ?$

As, A, B & D are exhaustive & disjoint then,

$$P(A) + P(B) + P(D) = 1$$

$$\frac{1}{6} + \frac{1}{3} + P(D) = 1$$

$$P(D) = \frac{3}{6} = \frac{1}{2}$$

D ⟶ confirms correctness i.e C

B ⟶ correct or not-correct

$$\begin{array}{ccc} \downarrow & & \downarrow \\ C & & C' \end{array}$$

A ⟶ correct or not-correct

Total probability of correctness i.e

$$P(C) = P(C|A)\,P(A) + P(C|D)\,P(D) + P(C|B)\,P(B)$$

$$\therefore P(C) = \left(\frac{1}{8}\right)\left(\frac{1}{6}\right) + (1)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$$

$$P(C) = \frac{1}{48} + \frac{1}{2} + \frac{1}{12} = \frac{1}{48} + \frac{24}{48} + \frac{4}{48}$$

$$P(C) = \frac{29}{48}$$

Now, $$P(D|C) = \frac{P(C|D)\,P(D)}{P(C)}$$

$$= \frac{(1)\left(\frac{1}{2}\right)}{\left(\frac{29}{48}\right)} = \frac{24}{29}$$ Ans

Q. 14 = Total possible outcome $= 6^3 = 216$.

Total favourable outcome = at least 1 pawn on 1 dice.

Say this event by E

So, $P(E) = 1 - P(\text{no pawn on any dice})$

$$= 1 - \left(\frac{5^3}{6^3}\right)$$

$$= 1 - \frac{125}{216} = \frac{91}{216} \text{ Ans}$$

Q. 15 = Sol^n $\Rightarrow$ P(Head Appearing) $= p$

Say, at x^{th} toss head appeared $= (1-p)^{x-1}p$

Then total probability $= \sum_{x=1}^{\infty} p(1-p)^{x-1}$

$$= (p + (1-p)p + (1-p)^2 p + \cdots \infty)$$

$$\cancel{P(E)} = \frac{p}{1-(1-p)} = 1. = P(T) \text{ (say)}$$

Now, calculating for only even values of x =

$$((1-p)p + (1-p)^3 p + (1-p)^5 p + \cdots \infty)$$

$$= \sum_{k=0}^{\infty} (1-p)^{2k+1} p$$

$$= \frac{(1-p)p}{1-(1-p)^2} = P(E) \text{ (say)}.$$

$$\text{Answer} = \frac{P(E)}{P(T)} = \frac{\frac{(1-p)p}{1-(1-p)^2}}{1}$$

$$= \frac{(1-p)p}{(2-p)p}$$

$$= (1-p)/(2-p) \quad \text{Ans}$$

Q.16 = sol^n ⇒

Say the rod gets ~~broken~~ broke at x-distance from ~~purple~~ Green point & if we assume a small length at this point of length 'dx' then probability of breaking in this scenario will be $\frac{dx}{L}$; L is length of rod.

(1)

Now we have to find

$E(F(x))$ where $F(x) = \min(x, L-x)$.

$$E(F(x)) = \int_0^{\frac{L}{2}} \frac{x\,dx}{L} + \int_{\frac{L}{2}}^{L} \frac{(L-x)}{L}\,dx$$

$$= \left(\frac{1}{2L}\right)\frac{L^2}{4} + \frac{1}{L}\left(L\left(\frac{L}{2}\right) - \frac{1}{2}\frac{3}{4}L^2\right)$$

$$= \frac{L}{8} + L\left(\frac{1}{2} - \frac{3}{8}\right)$$

$$= L/8 + L/8$$

$$= L/4 \text{. Ans}$$

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Sol

(2) For average ratio of smaller to larger length :-

$$F(x) = \begin{cases} \text{for range } 0 \text{ to } L/2 \rightarrow \dfrac{x}{L-x} \\ \text{for range } L/2 \text{ to } L \rightarrow \dfrac{L-x}{x}. \end{cases}$$

So, $$E(F(x)) = \int_0^{L/2} \left(\frac{x}{L-x}\right) \frac{dx}{L} + \int_{L/2}^{L} \left(\frac{L-x}{x}\right) \frac{dx}{L}$$

$$= \int_L^{L/2} \left(\frac{L-y}{y}\right)\left(\frac{-dy}{L}\right) + \int_{L/2}^{L} \frac{L-x}{x} \frac{dx}{L}$$

[assume $y = L-x$]

$$= \frac{2}{L} \int_{L/2}^{L} \left(\frac{L-x}{x}\right) dx$$

$$= \frac{2}{L} \left[\int_{L/2}^{L} \frac{L}{x} dx - \int_{L/2}^{L} dx \right]$$

$$= (2/L) \left[L\,[\ln x]_{L/2}^{L} - [x]_{L/2}^{L} \right]$$

$$= \frac{2}{L}\left[L \ln 2 - \frac{L}{2} \right]$$

$$= 2 \ln 2 - 1 \quad \text{Ans}$$

(3) Assuming 3 part of the rod. We can say there is a rod of variable length 'x' & we have to find average length of one part of this rod of variable length x.

This will be given by $E_x(F(x))$ where

$$F(x) = \int_0^x \frac{y\,dy}{x} = \frac{1}{2x} \cdot [y^2]_0^x = \frac{x}{2}$$

So,

$$E(F(x)) = \int_0^L F(x) \left(\frac{dx}{L}\right)\left(\frac{x}{L}\right)$$

$$= \frac{1}{2L^2} \int_0^L x^2 dx$$

$$= \frac{L}{6} \quad \text{An}$$

Q.17 = sol ⇒ Probability that GG drawer produce Gold = 1 = $P(D_1)$
" " SS " " " = 0 = $P(D_2)$
" " GS " " " = $\frac{1}{2}$ = $P(D_3)$

We have to find $P(GG \mid \text{see } 1^{st} \text{ Gold}) = ?$

Using Baye's rule;

$$P(GG \mid \text{see } 1^{st} G) = \frac{P(\text{see } 1^{st} \text{ Gold} \mid GG) \times P(GG \text{ drawer})}{\sum_{i=1}^{3} P(\text{see } 1^{st} \text{ Gold} \mid D_i) P(D_i)}$$

$$= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}} = \frac{1}{1 + \frac{1}{2}}$$

$$= 2/3 \text{ Ans}$$

Q.18 = Sol^n → Probability of hitting = p

Now, say Anuja comes after X attempts.

For, Anuja to come out with confidence x^{th} attempt should be successful hit with no prior $\geq \beta$ failure.

i.e $X - \alpha < \beta$.

This probability can be expressed as following :-

$$\Rightarrow \underbrace{(p)}_{\text{for last hit i.e } x^{th}} \underbrace{\sum_{X=\alpha}^{X=\alpha+\beta-1} {}^{(X-1)}C_{(\alpha-1)} (p)^{\alpha-1} (1-p)^{X-\alpha}}_{\text{for the } (X-1) \text{ previous attempts}}$$

$$\Rightarrow (p)^{\alpha} \sum_{x=\alpha}^{X=\alpha+\beta-1} {}^{(X-1)}C_{(\alpha-1)} (1-p)^{X-\alpha}$$

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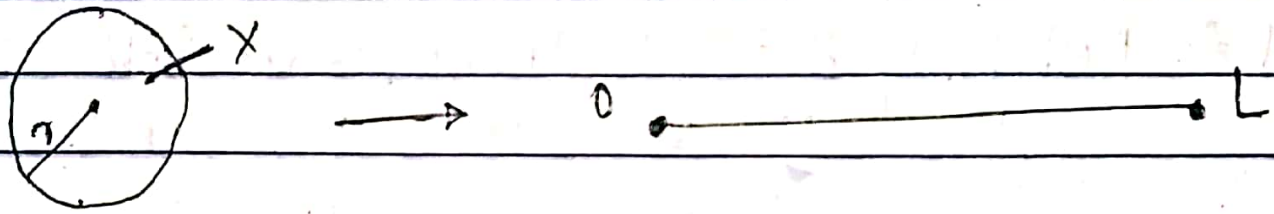
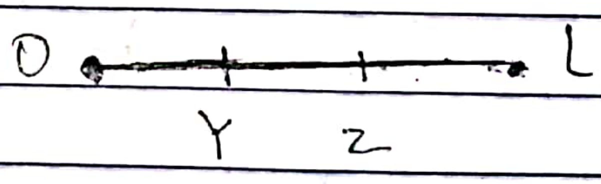
Q.19 = Sol^n ⇒ Suppose we break the circular wire at point X.



A wire of length $L = 2\pi r$

Now we took two random point Y, Z on this wire.

Case - 1 :- $Y < Z$



for non-degenerate Δ :-

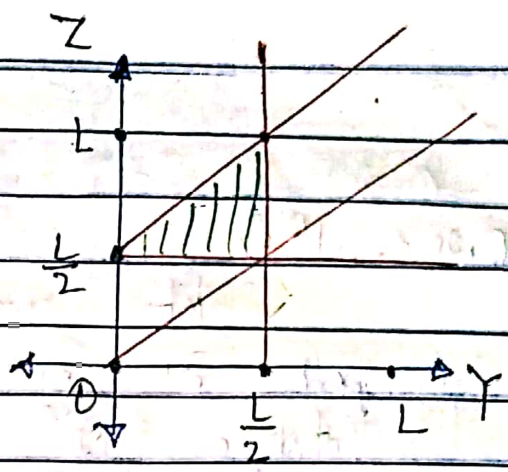
① $Y < (Z-Y) + (L-Z) \Rightarrow 2Y < L$

② $(-Y+Z) < Y + (L-Z) \Rightarrow 2Z < 2Y + L$

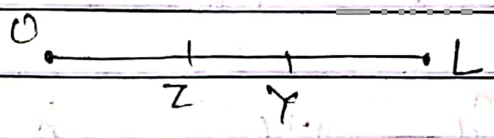
③ $(L-Z) < Y + (Z-Y) \Rightarrow L < 2Z$

Required area = shaded region
$= \frac{1}{2} \times \frac{L}{2} \times \frac{L}{2}$

$= \frac{L^2}{8}$



Case - 2 : $Y > Z$



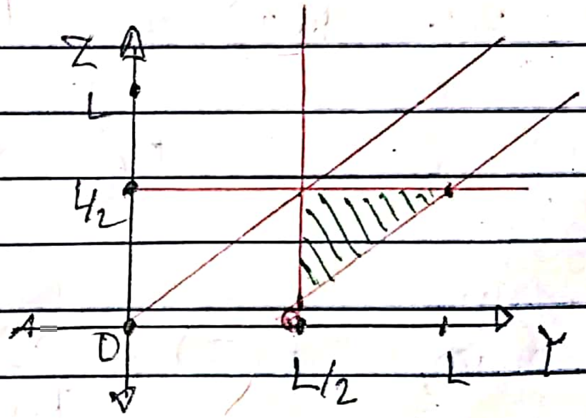
for non-degenerate Δ :-

① $Z < (Y-Z) + (L-Y) \Rightarrow 2Z < L$

② $(Y-Z) < Z + (L-Y) \Rightarrow 2Y < 2Z + L$

③ $(L-Y) < Z + (Y-Z) \Rightarrow L < 2Y$

Required Area = shaded region
$= \frac{1}{2} \times \frac{L}{2} \times \frac{L}{2}$
$= L^2/8$



So, Total possible (Y,Z) pairs lie in the area of shaded region which will give non-degenerated region. This area $= \frac{L^2}{8} + \frac{L^2}{8} = L^2/4$.

Total possible area which (Y,Z) points cover $= L^2$

Answer $= \frac{L^2}{4} / L^2 = \frac{1}{4} = 0.25$ Ans.

Q.20 = Sol^n ⇒

Let say before choosing the path 3 i.e. last road he choose 1st road p times & 2nd road q times.

In general time to reach the city $= (2p + 4q + 3) = T$

$; \; p, q \in \mathbb{N} + \{0\}$

& $P(T) = {}^{p+q}C_p \left(\frac{1}{3}\right)^{p+q+1}$

or ${}^{p+q}C_q \left(\frac{1}{3}\right)^{p+q+1}$

So, $E(T) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (2p + 4q + 3)\, P(T)$

We divide $E(T)$ in the three terms.

Say;

$E(T) = A + B + C.$

Solving : C i.e. $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 3\, P(T)$

⇒

$$C = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} {}^{(p+q)}C_q \left(\frac{1}{3}\right)^{p+q}$$

$$C = \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \sum_{q=0}^{\infty} {}^{(p+1)+q-1}C_q \left(\frac{1}{3}\right)^q$$

Using
$$(1-k)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, (k)^r \quad \text{—— (A)}$$

$$C = \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(1-\frac{1}{3}\right)^{-(p+1)}$$

$$C = \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{3}{2}\right)^{p+1} = \frac{3}{2} \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p$$

$$C = 3. \quad \text{—— (1)}$$

Solving : B i.e. $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 4q \, P(T) q$

$$B = \sum_{q=0}^{\infty} 4q \sum_{p=0}^{\infty} {}^{(p+q)}C_p \left(\frac{1}{3}\right)^{p+q+1}$$

$$B = \frac{4}{3} \sum_{q=0}^{\infty} q \left(\frac{1}{3}\right)^q \sum_{p=0}^{\infty} {}^{(q+1)+p-1}C_p \left(\frac{1}{3}\right)^p$$

Using (A)

$$B = \frac{4}{3} \sum_{q=0}^{\infty} q \left(\frac{1}{3}\right)^q \left(1-\frac{1}{3}\right)^{-(q+1)}$$

$$B = \frac{4}{3} \sum_{q=0}^{\infty} q \left(\frac{1}{3}\right)^{q} \left(\frac{3}{2}\right)^{q+1}$$

$$B = 2 \sum_{q=0}^{\infty} q \left(\frac{1}{2}\right)^{q}$$

Say, $S = \sum_{q=0}^{\infty} q (1/2)^{q} = 0 + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} \dots \infty$ —— #

$$\left(\frac{1}{2}\right) S = 0 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} \dots \infty$$

So,

$$S - \left(\frac{1}{2}\right) S = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + 1 \cdot \frac{1}{16} + \dots \infty$$

$$\frac{S}{2} = \sum_{a=1}^{\infty} \left(\frac{1}{2}\right)^{a} = 1$$

$$S = 2. \quad \text{——} \quad (*)$$

Hence $B = 2S = 2 \times 2 = 4.$ —— (11)

Solving : A i.e. $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} 2p \, P(T)$

$$A = \sum_{p=0}^{\infty} 2p \sum_{q=0}^{\infty} {}^{(p+q)}C_q \left(\frac{1}{3}\right)^{p+q+1}$$

$$A = \sum_{p=0}^{\infty} 2p \left(\frac{1}{3}\right)^{p+1} \sum_{q=0}^{\infty} {}^{(p+1)+q-1}C_q \left(\frac{1}{3}\right)^{q}$$

$$A = \frac{2}{3} \sum_{p=0}^{\infty} p\left(\frac{1}{3}\right)^{p} \left(1-\frac{1}{3}\right)^{-(p+1)}$$

$$A = \frac{2}{3} \sum_{p=0}^{\infty} p\left(\frac{1}{3}\right)^{p} \left(\frac{3}{2}\right)^{p+1}$$

$$A = \sum_{p=0}^{\infty} p\left(\frac{1}{2}\right)^{p} = S \quad (\text{from } \#)$$

$$= 2 \quad (\text{from } *)$$

$$A = 2 \quad \text{——— (iii)}$$

Thus, from eq - (i), (ii), (iii)

$$E(T) = A + B + C$$

$$= 2 + 4 + 3$$

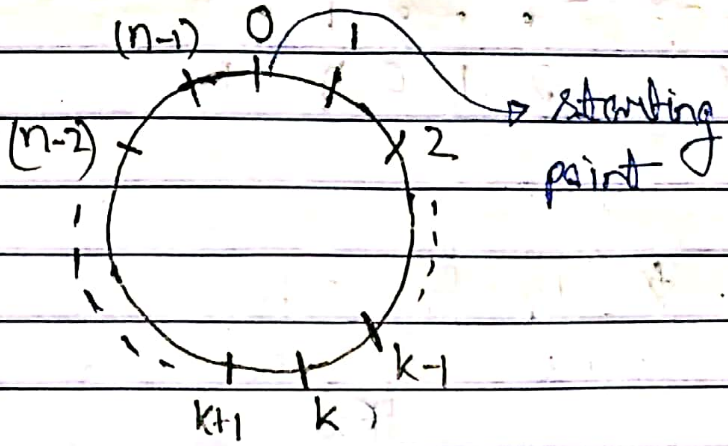
$$= 9 \text{ hours} \quad \text{Ans}$$

Section → 3

29

Q. = Solⁿ ⇒

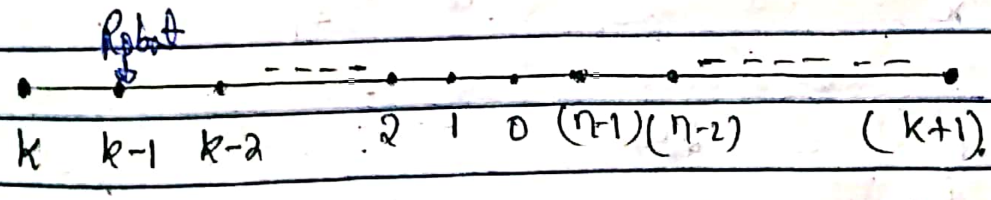
Circle is as follows :-



We have to find the probability that we visit point 'k' after all the other points are visited. ✪

Suppose, we reached point 'k-1' & Probability of reaching there is 1. We will prove this later.

Now, we have robot at 'k-1' point. Say, we break the circle as follows into line :-



Ultimately, we have find Probability of robot reaching (k+1) before point 'k'.

Let ~~us~~ we rename points as follows :-

k   k-1   k-2   - - - -   2   1   0   n   n-1   - - - - - - - -   k+1

$\downarrow$ $\downarrow$ $\downarrow$

i_0 i_1 i_2 - - - - - - - - - - - - i_{n-1}

& $i_r - i_{r-1} = 1.$

Let us assume P_{i_x} is the probability (that robot is at i_x point) that robot reaches 'k+1' before k.

We observe ;

$$P_{i_0} = 0 \quad , \quad P_{i_{n-1}} = 1 \qquad \text{——} \quad (1)$$

Now, $i_0 < i_x < i_{n-1}$

So, According to Total probability theorem ;

$$P_{i_x} = \frac{1}{2} P_{i_{x-1}} + \frac{1}{2} P_{i_{x+1}}$$

$$\Rightarrow P_{i_{x+1}} = 2P_{i_x} - P_{i_{x-1}}$$

Assume $y^x = P_{i_x}$;

$\Rightarrow \quad y^{n+1} = 2y^{n} - y^{n-1}$

$\Rightarrow \quad y^{2} = 2y - 1$

$\Rightarrow \quad y^{2} - 2y + 1 = 0$

$\Rightarrow \quad y = 1$ (equal roots)

thus;

$$\cancel{P_{i_x} = a \cdot (1)^{n} + b \cdot n (1)^{n} = a + b}$$

$$P_{i_x} = a \cdot (1)^{i_x} + b \cdot i_x \cdot (1)^{i_x}$$

$$P_{i_x} = a + b \cdot i_x$$

from eq. (1) : $P_{i_0} = 0 \Rightarrow a + b\,i_0 = 0$

$P_{i_{n-1}} = 1 \Rightarrow a + b\,i_{n-1} = 1$

$$b = \frac{1}{i_{n-1} - i_0} \cancel{= \frac{1}{n-1}}$$

$$a = -i_0 b = \frac{-i_0}{i_{n-1} - i_0}$$

So, $P_{i_1} = a + b \cdot i_1 = b(i_1 - i_0) = \dfrac{i_1 - i_0}{i_{n-1} - i_0} = \dfrac{1}{n-1}$

Ans

Hence, the required probability $= \frac{1}{n-1}$.

Now, proving (A) i.e. Probability of reaching $k-1$ or any point is 1.

1st find probability of moving from 0 ~~...~~ & reaching 1.
i.e $P(0 \rightarrow 1) = P_1$ (say)

$$P_1 = \frac{1}{2} + \frac{1}{2} \cdot P(\text{moving from } (n-1) \text{ \& reaching } 1).$$

$$2P_1 - 1 = P(\text{moving from } n-1 \text{ \& reaching } 0) \times P(\text{moving from } 0 \text{ \& reach } 1)$$

$\therefore$ Due to symmetry we can say;

$$2P_1 - 1 = P_1^2 \Rightarrow P_1^2 - 2P_1 + 1 = 0$$
$$\Rightarrow P_1 = 1.$$

& Due to independence $P(0 \rightarrow k) = \left[P(0 \rightarrow 1)\right]^k$
$$= [1]^k$$

Probability of reaching k & starting from 0. } $P(0 \rightarrow k) = 1.$

Hence proved

Q.28 = Soln ⇒

Initialy 1 Red ball & 2 blue ball.

(1) let P_i = P(Exactly 1 blue ball & which is in i^{th} trial)

So, $P_1 = \left(\frac{2}{3}\right)\cdot\left(\frac{1}{4}\right)\cdot\left(\frac{2}{5}\right) = 4/60$

$P_2 = \left(\frac{1}{3}\right)\cdot\left(\frac{2}{4}\right)\cdot\left(\frac{2}{5}\right) = 4/60$

$P_3 = \left(\frac{1}{3}\right)\cdot\left(\frac{2}{4}\right)\cdot\left(\frac{2}{5}\right) = 4/60$

Answer = $P_1 + P_2 + P_3 = 3 \times 4/60$

$= 1/5$ Ans

(2) E = All balls are same color

A = " " " Red "

B = " " " Blue "

$P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A)}{P(E)} = ?$

Now, $P(A) = \left(\frac{1}{3}\right)\left(\frac{2}{4}\right)\left(\frac{3}{5}\right) = 1/10$

Now, $P(B) = \left(\frac{2}{3}\right)\cdot\left(\frac{3}{4}\right)\cdot\left(\frac{4}{5}\right) = 2/5 = 4/10$

As $P(E) = P(A) + P(B) = 1/10 + 4/10 = 5/10$

So, $P(A|E) = \frac{1/10}{5/10} = 1/5$ Ans

(3) P(at least 1 blue ball) =

1 − P(no blue ball drawn)

= 1 − P(all red balls drawn)

= 1 − P(A) (from last part

= 1 − 1/10 P(A) = 1/10)

= 9/10 Ans

(4) P(at least 1 red ball) =

1 − P(all blue ball)

= 1 − P(B) (from (2) part

= 1 − 4/10 P(B) = 4/10)

= 6/10 = 3/5 Ans

Q.26 ⇒ Soln :—

First time he hits target
Second time he misses it.
Now, 98 trials remaining.
out of which 49 should be hits.

Now, say i^{th} hits occur at k_{i-1}^{th} trial.
$(2 \leq i \leq 50)$.
It's probability $= \dfrac{(i-1)}{k_{i-1} - 1}$.

Say $i = 2^{th}$ hit occur at $k_{i-1} = k_1 = 4^{th}$ trial
then it's probabity $= \dfrac{1}{3}$, which is true.

Similarly, for i^{th} miss occur at L_{i-1}^{th} trial $(2 \leq i \leq 50)$
It's probability $= \dfrac{(i-1)}{L_{i-1} - 1}$

Say $i = 2^{th}$ miss occur at $L_{i-1} = L_1 = 4^{th}$ trial
then it's probability $= \dfrac{1}{3}$. which is true.

If $i = 2^{th}$ miss & $L_{i-1} = L_1 = 3^{rd}$ trial.
then it's probability $= \dfrac{1}{2}$. which is tru..

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We can observe that for any combination of positions of 48 hits out of 98 trials

$$\prod_{i=2}^{50} (K_{i-1} - 1) (L_{i-1} - 1) = (98)!$$

And total no. of these combination $= {}^{98}C_{49} = T$

Now, Product of Prob. of all hits

$$= \prod_{i=2}^{50} \frac{(i-1)}{K_{i-1} - 1}$$

$$A = \frac{(49)!}{\prod_{i=2}^{50} K_{i-1} - 1}$$

Similarly, product of Prob. of all miss

$$B = \frac{(49)!}{\prod_{i=2}^{50} L_{i-1} - 1}$$

$$\text{Ans} = A \times B \times T = \frac{[(49)!]^2}{(99)!} \times {}^{98}C_{49} \quad \text{Ans}$$

$$= 1/99$$

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Q. 23 = sol^n ⇒

Probability of person's birthday on day $i = p_i$

& $\sum_{i=1}^{n} p_i = 1$.

Now probability that no two persons share their birthday in the room of k people =
$k! \times$ (k^{th} symmetric Polynomial)

$$P = e_k(x_1, x_2, x_3 \ldots x_n) \times (k!)$$

where $x_i = p_i$

$$P = (k!) \sum_{1 \le i_1 < i_2 \ldots < i_k \le n} (p_{i_1} p_{i_2} \ldots p_{i_k})$$

Now, we have to show that P will be maximum when $p_1 = p_2 = \ldots = p_n$.

i.e. all p_i's should be equal.

Method - 1 :-

According to AM-GM inequality;

$$\prod_{a=1}^{k} p_{i_a} \le \left[\frac{\sum_{a=1}^{k} p_{i_a}}{k} \right]^k$$

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And value of L.H.S is max i.e.

$\prod_{a=1}^{k} p_{i_a}$ will attain maximum only when

$p_{i_{a_1}} = p_{i_{a_2}} \ \forall \ a_1, a_2 = 1, 2, 3 \ldots, n$

this max value is value on the RHS i.e.

$$\left[\frac{\sum_{a=1}^{k} p_{i_a}}{k} \right]^{k}.$$

Now, As, $P = (k!) \sum \left(\prod_{a=1}^{k} p_{i_a} \right)$

One term P will be maximum for the same condition i.e.

$$p_{i_1} = p_{i_2} = p_{i_3} = \ldots = p_{i_k}.$$

~~On observing~~ For all terms we can conclude above equality.

Hence, implying $p_1 = p_2 = p_3 = \ldots = p_n$.

Another Method :—

We can write ; (by observation)

$$P = A p_i p_j + B(p_i + p_j) + C \ ; \ i<j$$

here A, B & C do not depend on either p_i or p_j.

Assume, $q_i = q_j = (p_i + p_j)/2$.

By AM-GM : $q_i q_j > p_i p_j$

& still $q_i + q_j = p_i + p_j$.

Now, say $P(>0)$ is max when not all p_i are equal i.e. $p_i \neq p_j$.

We can then also assume that P is nonzero & therefore that some p_i's are non-zero.

Then $A \neq 0$ (even though p_i or p_j might be zero). Now replace p_i & p_j by q_i & q_j. The sum of p_i's is still 1. & P has strictly increased.

This contradiction shows that
$p_i = p_j$ for all i & j.

Q.25 = Sol^n ⇒

Total number of ways to arrange
X girls & Y boys $= (X+Y)!$

If $X > Y$ then the required probability will be zero.

If $X = Y$ & if we represent boys with '+1' & girls with '-1', the formed sequence should be such that sum of prefix at any instant should greater than or equal to zero i.e. non-negative.

front position
If $a_1, a_2, a_3 ---, a_n$ is sequence where $a_i = +1$ or -1 ; $i = 1, 2, 3 --, n$
& $n = 2X$.
then $a_1 + a_2 + -- + a_k \geq 0$; $1 \leq k \leq n$.

Desired ways = Total ways to ~~arrange~~ select X out of 2X places for '+1's − Undesired ways.

Total ways to select x out of $2x$ places for '+1' s $=$

$$^{2x}C_x$$

For ~~Undesrse~~ undesired ways j
Assume number at j^{th} place is where 1^{st} time, $a_1+a_2+\dots+a_j<0$.

Obviously, $a_1+a_2+\dots+a_{j-1}=0$ & $a_j=-1$.

We can map these undesired results to no. of ways we can select $(n+1)$ '+1' s & $(n-1)$ '-1' s from $2n$ objects which has only +1 & -1 in domain.
This can be explain as following:

If multiply '-1' in first j terms then our above undesired sequence will contain $(x+1)$ '+1' s & $(x-1)$ '-1' s. This is reversible because if again multiply -1 to 1^{st} j number we get back the ± 1.
So, Undesired ways $= {}^{2x}C_{(x+1)}$

Thus, Desired ways $= {}^{2x}C_x - {}^{2x}C_{x+1}$
(Assuming Identical boys & Identical girls)

$$= \frac{(2x)!}{(x!)(x!)} - \frac{(2x)!}{(x+1)!\,(x-1)!}$$

$$= \frac{(2x)!}{x!\,(x-1)!}\left[\frac{1}{x} - \frac{1}{x+1}\right]$$

$$= \frac{(2x)!}{(x+1)(x!)(x!)}$$

$$= \left(\frac{1}{x+1}\right) {}^{2x}C_{x} \longleftarrow \bigstar$$

Therefore, if $x = r$ then Probability that number of boys ahead of each girls will atleast 1 more than no. of girls ahead her

$$= \frac{\left(\frac{1}{x+1}\right) {}^{2x}C_{x} \overbrace{(\lfloor x)(\lfloor x)}^{\text{for permutation}}}{(x+x)!}$$

$$= \left[(x+1)\cancel{(x+1)^2}\right]^{-1} \quad \text{Ans}$$

where $x \rightarrow$ no. of girls.

If $X < Y$, then following the similar logic as in $X = Y$ case;

Desired ways = Total no. of ways to select Y places out of $(X+Y)$ for '+1' − Undesired ways

Total no. of ways to select Y out of $X+Y$ places $= {}^{(X+Y)}C_X = {}^{(X+Y)}C_Y$

& for undesired ways:

$$\boxed{a_1 + a_2 + \cdots + a_{j-1}} + a_j + \boxed{a_{j+1} + \cdots + a_{X+Y}} = Y - X.$$

Say t '+1's & t '−1's; a_j: −1; $Y-t$ '+1's, $X-t-1$ '−1's

If $b_i = \begin{cases} -a_i & ; \ i \leq j \\ a_i & ; \ i > j \end{cases}$

$$\boxed{b_1 + b_2 + \cdots + b_{j-1}} + b_j + \boxed{b_{j+1} + \cdots + b_{X+Y}} = Y+1 - (X-1) = Y - X + 2.$$

t '+1's & t '−1's; b_j: +1; $Y-t$ '+1's, $X-t-1$ '−1's

We get $Y+1 \rightarrow$ '+1's & $X-1 \rightarrow$ '−1's

Thus, comparing from last logic
for case $x = Y$;

Undesired way $= {}^{[(Y+1)+(X-1)]}C_{(Y+1)}$

$$= {}^{(Y+X)}C_{(Y+1)}$$

Assuming: Identical boys & Identical Girls:—

Desired way $= {}^{Y+X}C_{X} - {}^{Y+X}C_{Y+1}$

$$= \frac{(Y+X)!}{(Y)!\,(X)!} - \frac{(Y+n)!}{(Y+1)!\,(X-1)!}$$

$$= \frac{(Y+n)!}{(Y)!\,(X-1)!}\left[\frac{1}{X} - \frac{1}{Y+1}\right]$$

$$= \frac{(Y+1-X)}{Y+1}\left[\frac{(Y+X)!}{(X!)\,(Y)!}\right]$$

$$= {}^{(X+Y)}C_{Y}\left[1 - \frac{X}{Y+1}\right]$$

Therefore, if $X < Y$, the Probability that no. of boys attend of each girls will atleast 1 more that no. of girls attend her

$$= \frac{{}^{X+Y}C_{Y}\left[1 - \frac{X}{Y+1}\right]}{(X+Y)!} \quad (x!)(Y!)$$

$$P = \frac{(Y+1-x)}{(Y+1)}$$; $Y \rightarrow$ no. of boys
$x \rightarrow$ no. of girls

$\bigcirc\!\!\!*$

So for case $X=2$ & $Y=3$

$$P = \frac{(3+1-2)}{(3+1)} = \frac{2}{4} = \frac{1}{2}$$

Ans

& for $X = Y$

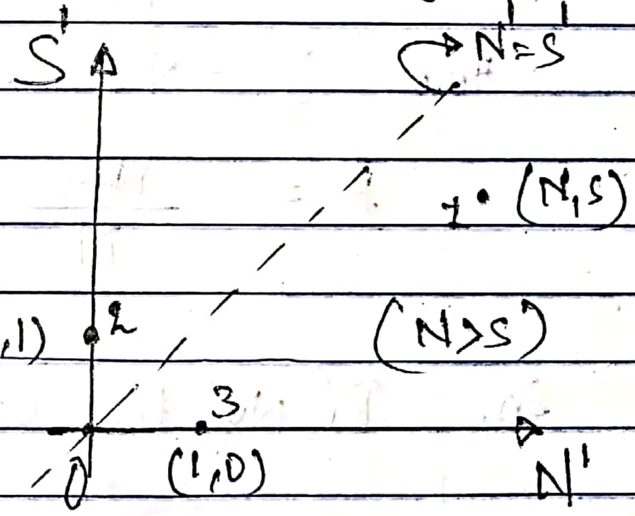
$$P = \frac{X+1-X}{X+1} = \frac{1}{X+1}$$ Ans.

Q.22 = Sol^n = (1) $N > S$

We have to find probability that there exist a point while counting, that they tied.

This require tie probability = Q = 1 − P(No tie possible)

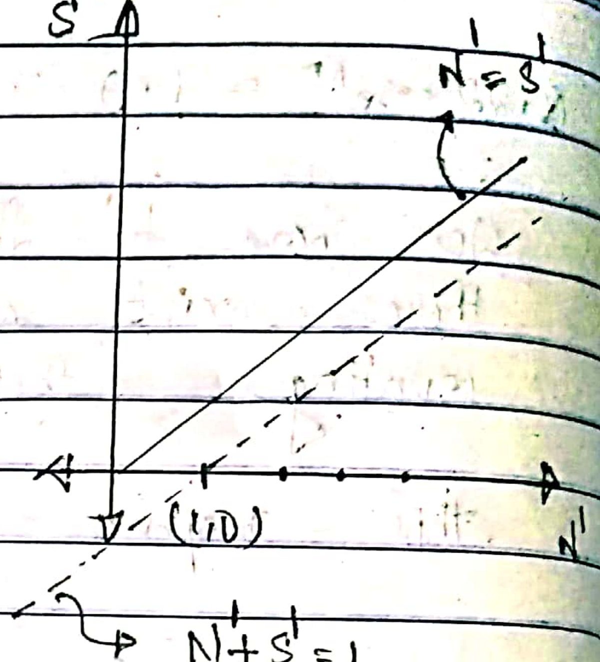
For P(No tie possible);
We can assume that if 1^{st} vote is one of the 'S' kind then surely tie will happen.



Because we have to finally reach point 1. And if we took 1^{st} step from 0 to point 2 then we have to cross $N' = S'$ line which ensures tie at a instant.

Now, if we took 1^{st} step from 0 to point '3' then we have to find the probability

that we never cross line $(N'-1)+S'=0$ but can touch it.
And only possible move is 1 up or 1 unit right.



this is the similar case of Q.25 for $Y > X$ (eq. ⊛ says)

$$P = \frac{Y+1-X}{1+Y}$$

Here, $Y = N-1$ & $X = S$

$$P = \frac{N-S}{N}$$

Probability for 1st vote as one of N type.

So, $$P(\text{No tie Occur}) = \left(\frac{N}{N+S}\right) P$$

$$= \frac{N-S}{N+S}$$

So, $$Q = 1 - \frac{N-S}{N+S} = \frac{2S}{N+S}$$ Ans

Q. 22 = sol ⟹ (2)

We have to find the probability that N (lets mention it as '−1') never gets ahead of S at any point in counting (lets mention S as '+1').

Say, there is sequence:

$a_1, a_2, a_3, ---, a_{N+S}$; where $a_i = \pm 1$, $i = 1, 2 ---, N+S$

On comparasion to Question we have to find probability of sequence such that sum of prefix of sequence at any instant should be non-negative i.e.

$$a_1 + a_2 + --- + a_k \geq 0 \; ; \text{ where } 1 \leq k \leq N+S$$

We have found no. of such sequence for N=S in Q. 25 in eq- ☆

which is $\frac{1}{N+1} \cdot {}^{2N}C_N = K$ — ①

$$\text{Required probability} = \frac{K}{\frac{(N+S)!}{(N)!\,(S)!}}$$

$$= \frac{K}{\frac{(2N)!}{(N!)(N!)}}$$

$$= \frac{K\,(N!)^2}{(2N)!} = \frac{K}{{}^{2N}C_N}$$

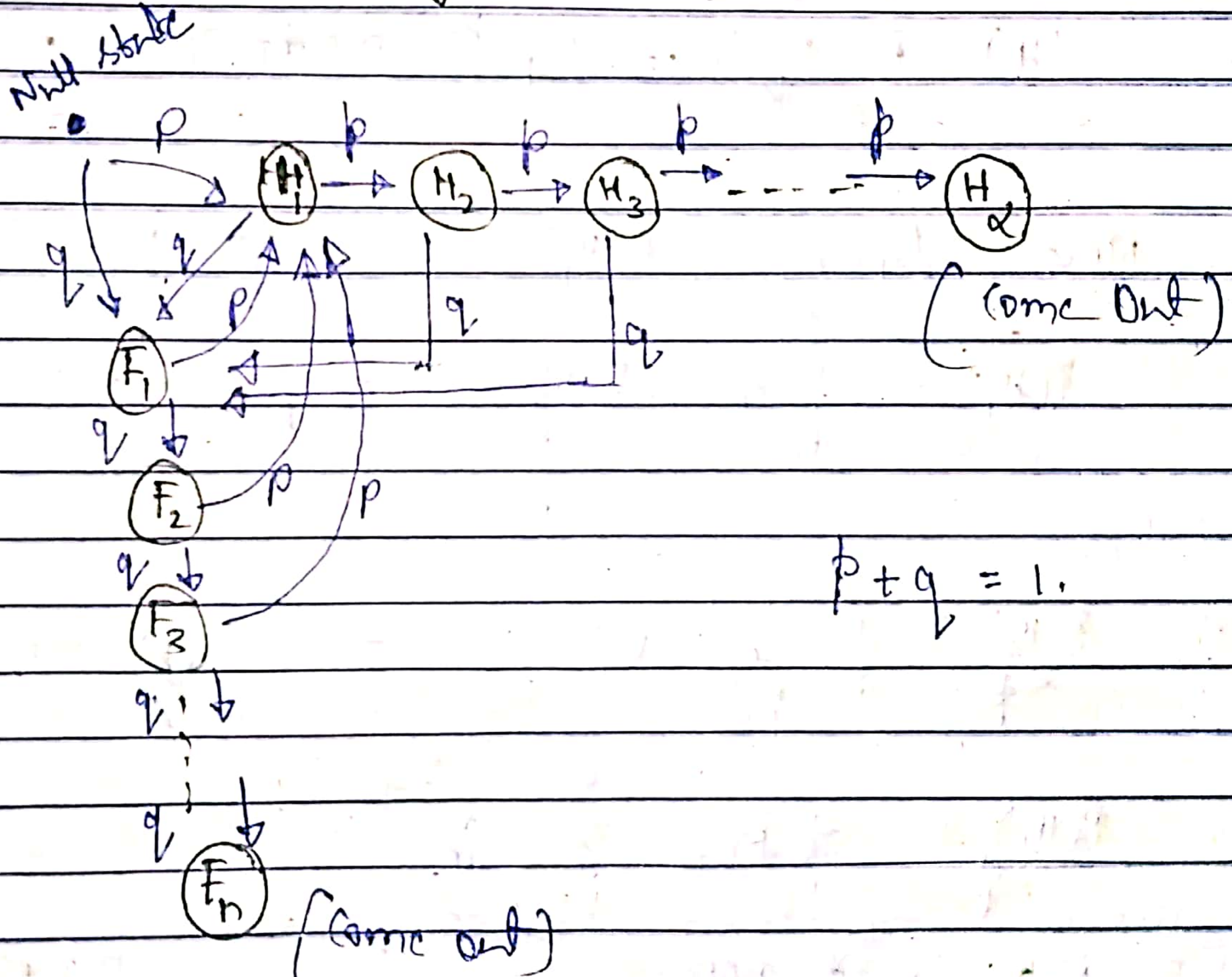
Putting K from ①

$$= \frac{\frac{1}{N+1} \cdot {}^{2N}C_N}{{}^{2N}C_N}$$

$$= \frac{1}{N+1} \quad \text{Ans}$$

Q.24 = Solⁿ ⇒

So, we have to find the probability that we reach state H_α for the following state diagram:-



$p + q = 1.$

Here H_i state is when continuous i hits occur & F_i state is when continuous i failure occurs.

From the above state diagram we clearly state that :-

(i) $P(H_k) = (p)^{k-1} P(H_1)$

(ii) $P(F_k) = (q)^{k-1} P(F_1)$

Now, finding $P(H_1)$ & $P(F_1)$:-

$$P(H_1) = p + P(F_1)p + P(F_2)p + \dots + P(F_{\beta-1})p$$

$$P(H_1) - p = p \sum_{i=1}^{\beta-1} P(F_i) = p \sum_{i=1}^{\beta-1} (q)^{i-1} P(F_1)$$

$$P(H_1) - p = p\,P(F_1) \sum_{i=1}^{\beta-1} (q)^{i-1}$$

$$P(H_1) - p = p\,P(F_1) \left[1 + q + q^2 + \dots + q^{\beta-2} \right]$$

$$P(H_1) - p = p\,P(F_1) \left[\frac{1 - q^{\beta-1}}{(1-q)} \right] ; \; p = 1-q$$

$$P(H_1) = p + P(F_1) \left[1 - q^{\beta-1} \right] \quad (1)$$

finding $P(F_1)$;

$$P(F_1) = q + P(H_1)q + P(H_2)q + \cdots + P(H_{\alpha-1})q$$

$$P(F_1) = q + q\sum_{i=1}^{\alpha-1} P(H_i)$$

$$P(F_1) - q = q\sum_{i=1}^{\alpha-1} (p)^{i-1} P(H_1)$$

$$P(F_1) = q + q\,P(H_1)\left[1+p+p^2+\cdots+p^{\alpha-2}\right]$$

$$P(F_1) = q + q\,P(H_1)\left[\frac{1-p^{\alpha-1}}{1-p}\right]$$

$$P(F_1) = q + P(H_1)\left[1-p^{\alpha-1}\right] \quad (11)$$

from (I) & (II)

$$P(H_1) = p + [q + P(H_1)[1 - p^{\alpha-1}]][1 - q^{\beta-1}]$$

$$\Rightarrow P(H_1) = p + q - q^{\beta} + P(H_1)[1 - p^{\alpha-1}][1 - q^{\beta-1}]$$

$$\Rightarrow P(H_1) = \frac{p + q - q^{\beta}}{1 - [1 - p^{\alpha-1}][1 - q^{\beta-1}]}$$

$$\Rightarrow P(H_1) = \frac{1 - q^{\beta}}{1 - [1 - p^{\alpha-1}][1 - q^{\beta-1}]}$$

Hence, probability that we reach state H_α :-

$$P(H_\alpha) = p^{\alpha-1} P(H_1)$$

$$P(H_\alpha) = (p^{\alpha-1})(1 - q^{\beta})\left[1 - [1 - p^{\alpha-1}][1 - q^{\beta-1}]\right]^{-1}$$

Ans