Probability and Statistics: Lecture-30

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on October 21, 2020

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- Since $E[X \mid Y]$ is a RV, we can find its PMF, CDF, Variance, etc

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	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
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Let $Z = E[X \mid Y]$.

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	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
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need $P_{X|Y}(x|y)$

$\sum_x P_{X|Y}(x|y)$

$P(0,0) = \frac{1}{5}$

$= P_X(0)\, P_Y(0)$?

$= \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$

$\Rightarrow P(0,0) \neq P_X(0)\, P_Y(0)$

$\Rightarrow X, Y$ not independent.

Let $Z = E[X \mid Y]$.

1. Find the Marginal PMFs of X and Y.
2. Find the conditional PMF of X given Y=0 and Y=1, i.e., find $P_{X|Y}(x|0)$ and $P_{X|Y}(x|1)$
3. Find the PMF of Z
4. Find $E[Z]$, and check that $E[Z] = E[X]$
5. Find $\mathrm{Var}(Z)$

» Answer to previous problem...

a) $P_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

$P_X(1) = \frac{2}{5} + 0 = \frac{2}{5}$

$P_Y(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, $P_Y(1) = \frac{2}{5} + 0 = \frac{2}{5}$

$$P_X(x) = \begin{cases} 3/5, & x = 0 \\ 2/5, & x = 1 \end{cases} \quad \text{Bernoulli}$$

$$P_Y(y) = \begin{cases} 3/5, & y = 0 \\ 2/5, & y = 1 \end{cases} \quad \text{Bernoulli}$$

Q Are X, Y independent? Ans No

Recall: $P_{XY}(x,y) = P_X(x) P_Y(y) \quad \forall x, y$

found in table

(b) We need to find $P_{X|Y}(x|y)$

$y = 0$

$$P_{X|Y}(0|0) = \frac{P_{XY}(0,0)}{P_Y(0)} = \frac{1/5}{3/5} = \frac{1}{3}$$

$$P_{X|Y}(1|0) = \frac{P_{XY}(1,0)}{P_Y(0)} = \frac{2/5}{3/5} = 2/3$$

Similarly, find

$$P_{X|Y}(0|1) = \frac{P_{XY}(0,1)}{P_Y(1)} = \frac{2/5}{2/5} = 1$$

$$P_{X|Y}(1|1) = 0$$

In words: Given $Y = 1$, we always have $X = 0$.

» Answer to previous problem...

(c) $Z = E[X|Y]$, Noting Y takes 0, 1 values.

$$= \begin{cases} E[X|Y=0] & \text{if } y=0 \\ E[X|Y=1] & \text{if } y=1 \end{cases}$$

$$E[X|Y=0] = \sum_{x_i} x_i P_{X|Y}(x|y=0)$$

$$= 0 \cdot P_{X|Y}(0|0) + 1 \cdot P_{X|Y}(1|0)$$

$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

$$E[X|Y=1] = \sum_{x_i} x_i P_{X|Y}(x|Y=1)$$

$$= 0 \cdot P_{X|Y}(0|1) + 1 \cdot P_{X|Y}(1|1)$$

$$= 0 \cdot 1 + 1 \cdot 0 = 0$$

Since, $P(Y=0) = \frac{3}{5}$, $P(Y=1) = \frac{2}{5}$

$$Z = \begin{cases} 2/3 & \text{if } y=0 \\ 0 & \text{if } y=1 \end{cases}$$ (*)

We need $P_Z(z)$, that what is the probab. of $Z=z$?

i.e. $P(Z=2/3)$ & $P(Z=0)$?

(*) ~~i.e.~~ $P(Z=2/3) = P(Y=0)$

$P(Z=0) = P(Y=1)$

» Answer to previous problem...

$$P_Z(z) = \begin{cases} 3/5 & \text{if } z = 2/3 \\ 2/5 & \text{if } z = 0 \\ 0, & \text{otherwise} \end{cases}$$

(c) $E[Z] = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{5} + 0 \cdot \frac{2}{5} = \frac{2}{5}$

$E[X] = 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{2}{5}$.

$\Rightarrow E[Z] = E[X]$

$\Rightarrow E[\underbrace{E[X|Y]}_{Z}] = E[X]$

true for $\forall X, Y$?

even for X, Y ind ?

$E[\underbrace{E[X|Y]}] = \underbrace{E[X]}$ (Called Law of Iterated Expectation)

$Var(Z) = \underbrace{E[Z^2]} - \underline{\underline{(E[Z])^2}}$

$E[Z^2] = \sum_i z_i^2 P_Z(z_i)$

$= \left(\frac{2}{3}\right)^2 \cdot \frac{3}{5} + \cancel{0^2 \cdot \frac{2}{5}} + \cancel{0}$

$= \frac{4}{\cancel{9}_3} \cdot \frac{\cancel{3}}{5} = \frac{4}{15}$.

$Var(Z) = \frac{4}{15} - \left(\frac{2}{5}\right)^2$

» Fact...

$E[c\,g(x)] = c\,E[g(x)]$

Fact

Let X, Y be two RVs and g, h be two functions of X and Y respectively. Show that

$$E[g(X)h(Y) \mid X] = g(X)E[h(Y) \mid X]$$

Solution

$$E[\underbrace{g(x)}_{\text{becomes a constant}} h(y) \mid \underbrace{x = x}_{\text{fixed instantiated}}] \overset{\text{Lin. of Exp.}}{=} g(x)\, E[h(y) \mid x = x].$$

Above True $\forall\, x = x$

$$\Rightarrow \quad E[g(x)h(y) \mid x] = g(x)\, E[h(y) \mid x].$$

» Iterated Expectations...

Partition: $\{Y = y_i\} = B_i$, $\bigcup_i B_i = S$

Law of Iterated Expectations

Let X, Y be two RVs, then we have

$$E[X] = E[\underbrace{E[X \mid Y]}]$$

Proof

Let $g(Y) = E[X|Y]$. Applying Law of Total Probability for Expectation.

$$E[X] = \sum_{y_j \in R_Y} E[X|Y = y_j] P_Y(y_j)$$

$$= \sum_{y_j \in R_Y} g(y_j) P_Y(y_j) \overset{\text{def}}{=} E[g(Y)] = E[\underbrace{E[X|Y]}_{g(Y)}]$$

» Solved Example on Application of Iterated Expectation...

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Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$.

$X \mid N = n \sim \text{Binomial}(n, p)$

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Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p, which is independent from other customers. What is the average number of customers who purchase drinks?

Solution

X : # customers who purchase drinks with prob. p.

N : # customer visiting fast food rest.

Recall: $X | N=n \sim \text{Binomial}(n, p)$ $E[np]$

To find $E[X] \overset{L.O.I.E}{=} E[E[X|N]] = E[Np] = pE[N]$ $X|N \sim \text{Bin}(n,p)$

Since $N \sim \text{Poisson}(\lambda)$. $\Rightarrow E[X] = p\lambda$.

» Expectation for Independent RV...

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Recall X, Y ind

$\Rightarrow P_{X|Y}(x|y) = P_X(x)$.

Expectation for Independent RVs

Let X, Y be two independent RVs. Then we have the following

1. $E[X \mid Y] = E[X]$
2. $E[g(X) \mid Y] = E[g(X)]$
3. $E[XY] = E[X]E[Y]$
4. $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

» Answer to previous problem...

① $E[X|Y=y] = \sum_{x \in R_X} x\, P_{X|Y}(x|y) \rightarrow (*)$

Since $P_{X|Y}(x|y) = P_X(x)$

$\Rightarrow (*)\ \sum_{x \in R_X} x\, P_X(x) = E[X].$

True for $\forall y \in R_Y$.

$\Rightarrow E[X|Y] = E[X]$, when X, Y ind.

② $g(X,Y) = XY$.

Similarly

$E[g(X)|Y] = \underline{E[g(X)]}$

③ $E[XY] = \sum_x \sum_y xy\, P_{XY}(x,y)$

$= \sum_x \sum_y xy\, P_X(x) P_Y(y)$

$= \sum_x x\, P_X(x) \sum_y y\, P_Y(y)$

$= \underline{E[X]\, E[Y]}$

» Answer to previous problem...

4 Easy: Try!

» Conditional Variance...

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Let X, Y be two RVs. By $\mathrm{Var}(X \mid Y = y)$ the conditional variance of X given $Y = y$. Let $\mu_{X \mid Y}(y) = E[X \mid Y = y]$. Then

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Let X, Y be two RVs. By $\mathrm{Var}(X \mid Y = y)$ the conditional variance of X given $Y = y$. Let $\mu_{X\mid Y}(y) = E[X \mid Y = y]$. Then

$$\mathrm{Var}(X \mid Y = y) = E[X^2 \mid Y = y] - \mu_{X\mid Y}(y)^2$$

Proof

Assignment.