Probability and Statistics: Lecture-37

Monsoon-2020

by Dr. Pawan Kumar (IIIT, Hyderabad)
on November 9, 2020

» Multiple Random Variable: Joint PDF, Joint CDF...

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» Example

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1. Find the constant c
2. Find the marginal PDF of X

» Answer to previous problem...

① We have

$$\underline{\underline{1}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 c\,(x+2y+3z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 c\left[\left(\frac{x^2}{2}+2yx+3zx\right)\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 c\left[\frac{1}{2}+2y+3z\right] dy\,dz$$

$$= \int_0^1 c\left[\frac{1}{2}y+\frac{2y^2}{2}+3zy\right]_0^1 dz$$

$$= \int_0^1 c\left[\frac{3}{2}+3z\right] dz$$

$$= c\left[\frac{3}{2}z+\frac{3z^2}{2}\right]_0^1$$

$$= c\cdot 3$$

$$\Rightarrow c = \frac{1}{3}$$

② $$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{3}(x+2y+3z)\,dy = \underline{\qquad}$$

» Independence of Multiple Random Variables...

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* If random variables $X_1, X_2, \ldots, X_n$ are independent, then we have

$$E[X_1, X_2, \cdots, X_n] = E[X_1] E[X_2] \cdots E[X_n]$$

» Independent and Identically Distributed Random Variables (I.I.D.)

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n^{th} flip

- If we flip the same coin N times and record the outcome, then $X_1, \ldots, X_n$ are I.I.D.
- Verify that these I.I.D. variables will have same mean and variances

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$Cov(X, Y) = E[XY] - E[X]E[Y]$

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» Example...

Example (Who will receive the present?)

N people sit *around a* round table, where $N > 5$.

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Try

» Answer to previous problem...

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» PDF of the Sum of Multiple Random Variables...

PDF of the Sum of Multiple RVs

We recall that if $Y = X_1 + X_2$, and X_1 and X_2 being independent, we have

$\bar{X} = [X_1, X_2, \cdots X_n]$
$\bar{Y} = [Y_1, Y_2, \cdots, Y_n]$

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Convolution

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)\, dx$$

For multiple variable case, i.e., if $Y = X_1 + X_2 + \cdots + X_n$, we have

$$f_Y(y) = f_{X_1}(y) * f_{X_2}(y) * \cdots * f_{X_n}(y)$$

* However, it is computationally difficult!

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$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad E[X] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

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We call X a random vector and $E[X]$ is the expectation of random vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

* CDF is $F_X(x) = F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$
* If X is jointly continuous, the PDF is denoted as

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Shorthand notation

» Random Matrix and Expectation...

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* Also, if $X_1, X_2, \cdots, X_k$ are k n-dimensional RVs, then

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Variance of Multiple Random Variables

Let $X_1, X_2, \ldots, X_n$ be n random variables. Then we have the following

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j)$$

If $X_1, X_2, \ldots, X_n$ are independent, then we have

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$

» Correlation and Covariance Matrix...

Definition of Correlation and Covariance Matrix

For a random vector X, the correlation matrix R_X and covariance matrix C_X is

Outer product

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ x_nx_1 & x_nx_2 & \cdots & x_nx_n \end{bmatrix}$$

Definition of Correlation and Covariance Matrix

For a random vector X, the correlation matrix R_X and covariance matrix C_X is

$$R_X = E[XX^T] = E\begin{bmatrix} X_1^2 & X_1X_2 & \dots X_1X_n & X_1X_n \\ X_2X_1 & X_2^2 & \dots & X_2X_n \\ \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & \dots & X_n^2 \end{bmatrix} = \begin{bmatrix} E[X_1^2] & E[X_1X_2] & \dots & E[X_1X_n] \\ E[X_2X_1] & E[X_2^2] & \dots & E[X_2X_n] \\ \vdots & \vdots & \vdots & \vdots \\ E[X_nX_1] & E[X_nX_2] & \dots & E[X_n^2] \end{bmatrix}$$

$$C_X = E[(X - E[X])(X - E[X])^T] = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \dots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \dots & \mathrm{Cov}(X_2, X_n) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Cov}(X_n, X_1) & \mathrm{Cov}(X_n, X_2) & \dots & \mathrm{Var}(X_n) \end{bmatrix}$$

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$U = \begin{bmatrix} X \\ Y \end{bmatrix}$

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1. $C_X = R_X - E[X]E[X]^T$

Recall : Cov(X, Y) = Cov(Y, X)

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(i, j) = (j, i)

1. $C_X = R_X - E[X]E[X]^T$
2. If $Y = AX + b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, then $C_Y = AC_XA^T$

» Answer to previous problem...

» Example of Correlation and Covariance Matrices...

Example (Example of correlation and covariance matrices)

Let X and Y be jointly continuous random variables with joint PDF

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$$f_{X,Y} = \begin{cases} \frac{3}{2}x^2 + y & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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Let $U = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the random vector. Find the correlation and covariance matrices of U.

» Answer to previous problem...

Solution First find the marginal PDFs of X & Y.

$$f_X(x) = \int_0^1 \left(\frac{3}{2}x^2 + y\right) dy = \frac{3}{2}x^2 + \frac{1}{2} \qquad 0 < x < 1.$$

Similarly,

$$f_Y(y) = \int_0^1 \left(\frac{3}{2}x^2 + y\right) dx$$

$$= y + \frac{1}{2} \qquad 0 < y < 1.$$

We recall

$$R = \begin{bmatrix} E[X^2] & E[XY] \\ E[YX] & E[Y^2] \end{bmatrix}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^1 x^2 \cdot \left(\frac{3}{2}x^2 + \frac{1}{2}\right) dx$$

IIIy

$$E[Y^2] = \ldots$$

$$E[XY] = \int_0^1 \int_0^1 (xy)\left(\frac{3}{2}x^2 + y\right) dx\, dy$$

$$\mathrm{Cov}(X,Y) = E[\underline{XY}] - E[\underline{X}]\,E[\underline{Y}]$$

<u>Covariance Matrix</u>

$$C_u = E\left[(U - E[U])(U - E[U])^T\right]$$

$$= \begin{bmatrix} \mathrm{Var}(\underline{X}) & \mathrm{Cov}(\underline{X,Y}) \\ \mathrm{Cov}(\underline{X,Y}) & \mathrm{Var}(\underline{Y}) \end{bmatrix}$$

» Properties of Covariance Matrix...

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2. The covariance matrix C_X is positive semi-definite (PSD)

Recall definition of semi Positive definite (SPD). Assume A symmetric.

A matrix A is SPD if $x^T A x \geq 0 \quad \forall x \neq 0$

Properties of Covariance

$x^T E[z z^T] x \geq 0$

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3. The covariance matrix is positive definite if and only if all its eigenvalues are larger than zero

$$C_x = E\Big[\underbrace{(x - E[x])}_{z}\,(x - E[x])^T\Big] = E[\underbrace{z z^T}_{A}]$$

$x^T(z z^T)x = (z^T x)^T (z^T x)$

$\|z^T x\|_2 \geq 0$

① Symmetry: $A^T = (z z^T)^T = z z^T = A \Rightarrow A$ is symmetric

② S.P.D.

arb. $x^T A x > 0 \quad \forall x \neq 0$

If λ is eig. val. of $A \Rightarrow \quad Au = \lambda u, \quad u$ eig. vec.

u eig. vec. $\Rightarrow u \neq 0 \Rightarrow u^T A u > 0$ (A is P.D) $\Rightarrow u^T \lambda u > 0$

$\Rightarrow \lambda \|u\|_2 > 0$

$\Rightarrow \boxed{\lambda > 0}$

Properties of Covariance

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3. The covariance matrix is positive definite if and only if all its eigenvalues are larger than zero
4. The covariance matrix is positive definite if and only if $\det(C_X) > 0$

$\det(C_X) = \prod \lambda_i$

But $\lambda_i > 0 \Rightarrow \det(C_X) > 0$

» Answer to previous problem...

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Are the matrices C_U and C_V positive definite?

» Answer to previous problem...

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» Denition of Cross-Correlation and Cross-Covariance Matrix...

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$$C_x = E[(x - E[x])(x - E[x])^T]$$

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$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} H_1(Y_1, Y_2, \ldots, Y_n) \\ H_2(Y_1, Y_2, \ldots, Y_n) \\ \vdots \\ H_n(Y_1, Y_2, \ldots, Y_n) \end{bmatrix}$$

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$$f_Y(y) = f_X(H(y))\,|J|, \qquad \text{where } J = \det \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \cdots & \frac{\partial H_1}{\partial y_n} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \cdots & \frac{\partial H_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial H_n}{\partial y_1} & \frac{\partial H_n}{\partial y_2} & \cdots & \frac{\partial H_n}{\partial y_n} \end{bmatrix}$$

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$$X = A^{-1}(y-b) = H(y)$$

$$J = \det(A^{-1}) = \frac{1}{\det(A)}$$

Example (Example of Method of Transform for Function of Random Vector)

invertible

Let $Y = AX + b$, where X is a n dimensional random vector, A be a fixed (non-random) n by n matrix, and b be a fixed n-dimensional vector. Find the PDF of Y in terms of X.

$$f_Y(y) = \underline{f_X}(H(y))\,|J|$$

$$= f_X(A^{-1}(y-b))\,|J|. = \frac{1}{\det(A)} f_X(A^{-1}(y-b))$$

» Answer to previous problem...

Definition of Jointly Normal Random Vectors

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» Definition of Jointly Normal or Gaussian Random Vectors...

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$$f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \exp\left\{-\frac{1}{2}(x-m)^T C^{-1}(x-m)\right\}$$