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Probability Assignment 5

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Problem 1 solⁿ:

$R_X = R_Y = \{0,1,2,---,10\}$ but as $X+Y=10$

so, $R_{XY} = \{(0,10),(1,9),---,(10,0)\}$

$\Rightarrow R_{XY} = \{(x,y) ; x+y=10, x,y \geq 0, x,y \in \mathbb{Z}\}$

thus, joint PMF $= P_{XY}(x,y)$

$$\Rightarrow P_{XY}(x,y) = \begin{cases} \dfrac{{}^{40}C_x \; {}^{60}C_y}{{}^{100}C_{10}} ; & (x,y) \in R_{XY} \\ 0 ; & \text{otherwise} \end{cases}$$

Problem 2 solⁿ:

$R_{XY} = \{(i,j) \in \mathbb{Z}^2 \mid (i,j \geq 0), |i-j| \leq 1\}$

& $P_{XY}(i,j) = \dfrac{1}{6 \cdot 2^{\min(i,j)}}, (i,j) \in R_{XY}$

(1) As $i,j \geq 0 \rightarrow i,j \in I^{st}$ quadrant

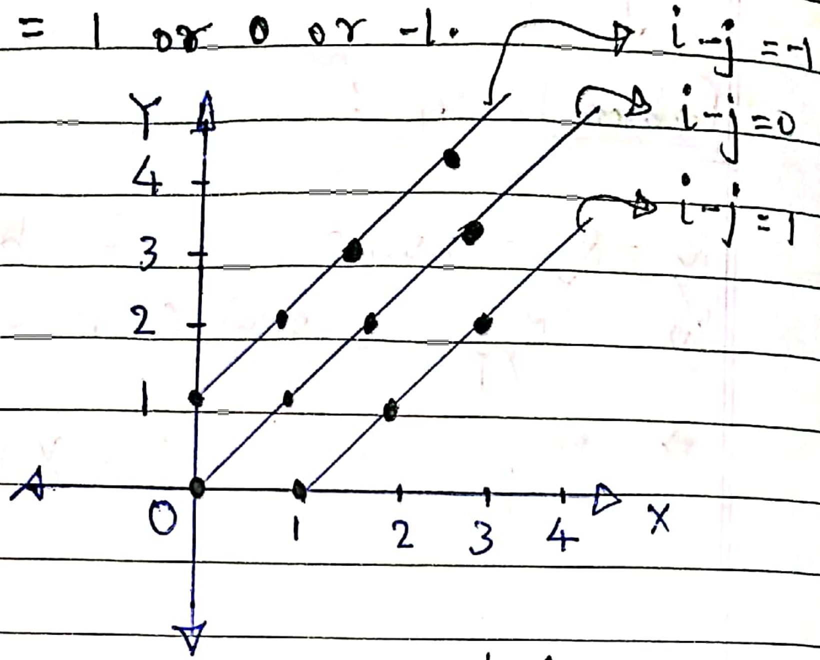
& As $|i-j| \leq 1 \rightarrow$

$-1 \leq (i-j) \leq 1$

But since i and j are integer points so,

$$i - j = 1 \text{ or } 0 \text{ or } -1.$$

[X-Y Plane]



* All points marked as '•' will come under R_{XY}.

(2) Symmetrically X & Y will have same PMF.

$P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P_X(1) = P_{XY}(1,0) + P_{XY}(1,1) + P_{XY}(1,2) = \frac{1}{3}$

$P_X(2) = P_{XY}(2,1) + P_{XY}(2,2) + P_{XY}(2,3) = \frac{1}{6}$

$P_X(3) = P_{XY}(3,2) + P_{XY}(3,3) + P_{XY}(3,4) = \frac{1}{12}$

In general we can say, marginal PMF

$$P_X(i) = P_Y(i) = \begin{cases} 1/3 & ; i = 0 \\ 1/3 \cdot 2^{k-1} & ; i = 1, 2, 3 \ldots \\ 0 & ; \text{otherwise} \end{cases}$$

(3) $P(X=Y \mid X<2) = \dfrac{P((X=Y) \text{ AND } (X<2))}{P(X<2)}$

As $P(X<2) = P(X=1) + P(X=0)$

$= \frac{1}{3} + \frac{1}{3}$ (from marginal PMF)

$= 2/3$

And set where $X=Y$ & $X<2$ is $(x,y) =$

$\{ (0,0), (1,1) \}$

So, $P((X=Y) \text{ AND } (X<2)) = P(0,0) + P(1,1)$

$= \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$

So, $P(X=Y \mid X<2) = \dfrac{1/4}{2/3} = 3/8$ Ans

(4) $P(1 \leq X^2+Y^2 \leq 5) = P(E)$

For event E, (x,y) can take values from set $\{ (0,1), (1,0), (1,1), (1,2), (2,1) \}$.

So,

$P(E) = P_{XY}(0,1) + P_{XY}(1,0) + P_{XY}(1,1) +$

$P_{XY}(1,2) + P_{XY}(2,1)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{7}{12}$ Ans

(5) $P(X=Y) = \sum_{k=0}^{\infty} P_{XY}(k,k)$

$= \sum_{k=0}^{\infty} \frac{1}{6 \cdot 2^k}$

$= \frac{1}{6} \sum_{k=0}^{\infty} \frac{1}{2^k}$

$= \frac{1}{6} \cdot 2 = \frac{1}{3}$ Ans

(6) $E[X|Y=2] = ?$

When $Y=2$, $X = 1$ or 2 or 3.

So,

$E[X|Y=2] = \sum_{x_i \in R_X} x_i P_{X|Y}(x_i|Y=2)$

; $R_X = \{0,1,2 \ldots\}$

We know

For $x_i \neq 1,2,3$, $P_{X|Y}(x_i|Y=2) = 0$.

So,

$E[X|Y=2] = 1 \cdot P_{X|Y}(1|Y=2) +$

$2 \cdot P_{X|Y}(2|Y=2) + 3 \cdot P_{X|Y}(3|Y=2)$

$E[X|Y=2] = \frac{P_{XY}(1,2)}{P_Y(2)} + \frac{2 \cdot P_{XY}(2,2)}{P_Y(2)} +$

$3 \cdot P_{XY}(3,2) / P_Y(2)$

As $P_Y(2) = 1/6$.

$$\Rightarrow E[X|Y=2] = \frac{1/12}{1/6} + 2 \cdot \frac{1/24}{1/6} + 3 \cdot \frac{1/24}{1/6}$$

$$\Rightarrow E[X|Y=2] = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{7}{4}$$ Ans

(7) $\quad Var(X|Y=2) = E[X^2|Y=2] - \left(E[X|Y=2]\right)^2$

As,

$$E[X^2|Y=2] = (1)^2 \cdot P_{X|Y}(1|Y=2) + (2)^2 \cdot P_{X|Y}(2|Y=2) + (3)^2 \cdot P_{X|Y}(3|Y=2)$$

$$\Rightarrow E[X^2|Y=2] = 1 \cdot \frac{1/12}{1/6} + (2)^2 \cdot \frac{1/24}{1/6} + (3)^2 \cdot \frac{1/24}{1/6}$$

$$= 1/2 + 1 + 9/4 = 15/4$$

So, $Var(X|Y=2) = \frac{15}{4} - \left(\frac{7}{4}\right)^2$

$$= \frac{15}{4} - \frac{49}{16} = \frac{11}{16}$$ Ans

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Problem 3 Solⁿ:

Given: X and Y are bivariate normal and uncorrelated.

i.e. $\rho(X,Y) = 0$.

To proof: X and Y are independent

Proof:

[Method : 1]

As we know for 2 jointly normal RV X & Y with parameters $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2$ & ρ, we can write

$$E[Y|X=x] = \mu_y + \rho\sigma_y \frac{x-\mu_x}{\sigma_x} \quad \text{— (i)}$$

$$\& \quad Var(Y|X=x) = (1-\rho^2)\sigma_y^2 \quad \text{— (ii)}$$

As $\rho(X,Y) = \rho = 0$. putting it in (i) & (ii) we get,

$$E[Y|X=x] = \mu_y \quad \& \quad Var(Y|X=x) = \sigma_y^2$$

Also, since, Y is normally distributed we can say $f_{Y|X}(y|x) = f_Y(y) \ \forall x,y \in R$

Thus, X & Y are independent.

[Method : 2]

$X \sim N(\mu_X, \sigma_X^2)$ & $Y \sim N(\mu_Y, \sigma_Y^2)$

Two RV X & Y are said to have a bivariate normal distribution with parameter $\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2$ & ρ, if their joint PDF is given by

$$f_{XY}(x,y) = \left[2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\right]^{-1} \cdot$$

$$\exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right]$$

; where $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ &

$\rho \in (-1,1)$ are all constant.

As X & Y are uncorrelated $\Rightarrow \rho = 0$.

$$\Rightarrow f_{XY}(x,y) = \left[2\pi\sigma_X\sigma_Y\right]^{-1} \cdot \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

As $f_Q(q) = \frac{1}{\sigma_q\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{q-\mu_q}{\sigma_q}\right)^2\right\}$; $Q = X, Y$

$q = x, y$

$$\Rightarrow \boxed{f_{XY}(x,y) = f_X(x)\, f_Y(y)};$$

X & Y are Independent.

Problem 4 Soln :

$X \rightarrow$ n-Dimensional Vector
$Y = AX+B$;
A is fixed $m \times n$ matrix
& B is m-Dimensional
fixed vector.

To proof : $C_Y = A C_X A^T$

Proof :
Covariance of matrix $X = C_X = E[(X-EX)(X-EX)^T]$
Covariance of matrix $Y = C_Y$
And
$$C_Y = E[(Y-E[Y])(Y-E[Y])^T]$$
As
$Y = AX+B \Rightarrow$ Using linearity of Expectation
$$E[Y] = A\,E[X] + B.$$

So, $C_Y = E[(AX+B-AE[X]-B)$
$(AX+B-AE[X]-B)^T]$
$\Rightarrow C_Y = E[A(X-E[X])(X-E[X])^T A^T]$
$= A\,E[(X-E[X])(X-E[X])^T]\,A^T$

$\Rightarrow C_Y = A\,C_X\,A^T$ Hence proved

Problem 5 Sol^n:

Given: $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$ $\Big\}$ $\rho(x,y) = \rho$

For normal Random vector X with mean m & covariance matrix C pdf is

$$f_X(x) = a e^{-(b/2)}$$

where $a = \dfrac{1}{(2\pi)^{n/2}\sqrt{\det C}}$; n = dimension of X.

& $b = (X-m)^T C^{-1} (X-m)$

$\lambda = \begin{bmatrix} X \\ Y \end{bmatrix}$ i.e. normal random variable

& $m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ & $C = \begin{bmatrix} Var(X) & cov(X,Y) \\ cov(X,Y) & Var(Y) \end{bmatrix}$

$$= \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

To proof:- $f_\lambda(\lambda) = f_{XY}(x,y) = \dfrac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$.

$$\exp\left\{\frac{-1}{2(1-\rho)^2}\left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 + \left(\frac{y-\mu_y}{\sigma_Y}\right)^2 - 2\rho(x-\mu_x)(y-\mu_y)/\sigma_x\sigma_y\right]\right\}$$

basically, $a = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}}$

& $b = \left\{ \frac{1}{(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right] \right\}$

for $f_\lambda(\lambda)$ where $\lambda = \begin{bmatrix} X \\ Y \end{bmatrix}$

Proof :—

As $a = \frac{1}{(2\pi)^{n/2} \sqrt{\det C}}$ & $\det C = \sigma_X^2 \sigma_Y^2 (1-\rho^2)$

& $n = 2$

So, $a = \frac{1}{2\pi \sqrt{\sigma_X^2 \sigma_Y^2 (1-\rho^2)}} = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}}$ — (1)

Now, for λ RV ;

$b = (\lambda - m)^T C^{-1} (\lambda - m)$

As $C = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$

$C^{-1} = \frac{1}{\sigma_X^2 \sigma_Y^2 (1-\rho^2)} \begin{bmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix}$

So,

$$b = (\lambda - m)^T C^{-1} (\lambda - m)$$

$$= \frac{1}{\sigma_X^2 \sigma_Y^2 (1-\rho^2)} \cdot \begin{bmatrix} X - \mu_X \\ Y - \mu_Y \end{bmatrix}^T \cdot \begin{bmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix} \cdot \begin{bmatrix} X - \mu_X \\ Y - \mu_Y \end{bmatrix}$$

$$b = \frac{1}{(1-\rho^2)} \left[\left(\frac{x - \mu_x}{\sigma_x} \right)^2 + \left(\frac{y - \mu_y}{\sigma_y} \right)^2 - 2\rho \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} \right] \quad \text{--- (ii)}$$

Thus $f_{XY}(x,y) = f_\lambda(\lambda) = a e^{-b/2}$

putting a & b from (i) & (ii).

So, $$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 + \left(\frac{y-\mu_y}{\sigma_Y} \right)^2 - 2\rho \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} \right] \right\}.$$

Hence proved

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Problem 6 : Sol^n :-

Random variable X , $E[X] = 10$, $X > 0$,

say $h(x) = \ln\sqrt{x} = \frac{1}{2}\ln(x)$; $x > 0$

then $h'(x) = \frac{d}{dx}h(x) = (2x)^{-1}$; $x > 0$

& $h''(x) = \frac{d^2}{dx^2}h(x) = \frac{-1}{2x^2}$; $x > 0$.

So, we can say 'h' is concave on $(0, \infty)$.

Thus,

$$E[\ln\sqrt{x}] = E[\tfrac{1}{2}\ln(X)] = Q.$$

using Jensen's inequality;

$$Q \leq \frac{1}{2}\ln(E[X]) = \frac{1}{2}\ln(10)$$

Thus,

$$E[\ln\sqrt{X}] \leq \frac{1}{2}\ln 10.$$

Ans

Problem : 8

$X \sim$ Binomial (n, p).

Acc. to chebyshev inequality ; we can write

$$P(|X - E[X]| \geq b) \leq \frac{Var(X)}{b^2} ; b > 0$$

As for $X \sim$ Binomial (n,p)

$\Rightarrow Var(X) = np(1-p)$.

So, $P(X \geq \alpha n) = P(X - np \geq \alpha n - pn)$

$$\Rightarrow P(X \geq \alpha n) \leq P(|X - np| \geq \alpha n - np)$$

$$\leq \frac{Var(X)}{(\alpha n - np)^2}$$

So, $P(X \geq \alpha n) \leq \frac{p(1-p)}{n(\alpha - p)^2}$; $p < \alpha < 1$

putting $p = 1/2$ & $\alpha = 3/4$

$\Rightarrow \alpha n = 3n/4$

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{(1/2)(1/2)}{(n)(3/4 - 1/2)^2} = \frac{4}{n}$$

Problem 9 : solⁿ

Let $X \sim \text{Binomial}(n, p)$.
Find upper bound for $P(X \geq \alpha n)$
using Markov, chebyshev & chernoff bounds.

Markov: If X is any non-negative RV then

$$P(X \geq a) \leq \frac{E[X]}{a}.$$

; $a > 0$.

Here $E[X] = np$.
So,

$$P(X \geq \alpha n) \leq \frac{np}{\alpha n} = \frac{p}{\alpha}$$

$$\Rightarrow \boxed{P(X \geq \alpha n) = p/\alpha}.$$

Chebyshev: $P(|X - E[X]| \geq b) \leq \frac{Var(X)}{b^2}$

; $b > 0$.

Here, $Var(X) = np(1-p)$.
Now,

$$P(X \geq \alpha n) = P(X - np \geq \alpha n - np)$$
$$\leq P(|X - np| \geq \alpha n - np)$$
$$\leq Var(X) / (\alpha n - np)^2$$

So, $$\boxed{P(X \geq \alpha n) \leq \frac{p(1-p)}{n(\alpha - p)^2}}.$$

Chernoff :

$$P(X \geq a) \leq e^{-sa} M_X(s) \; ; \; s > 0$$
$$P(X \leq a) \leq e^{-sa} M_X(s) \; ; \; s < 0$$

Finding $M_X(s) = E[e^{sX}] = (pe^s + 1 - p)^n$
because $X \sim$ Binomial(n, p).

Let Q.

So,
$$P(X \geq \alpha n) \leq e^{-s\alpha n}(1 - p + pe^{s})^n$$
; for any $s > 0$

Finding minimum of Q w.r.t s :—

$$\frac{d}{ds} Q = 0 \Rightarrow e^{-s\alpha n}(1-p+pe^{s})^{n-1}\left[-n\alpha(1-p+pe^{s}) + npe^{s}\right] = 0$$

So,
$1 - p + pe^s = 0$ or $\left[-n\alpha(1-p+pe^s) + npe^s\right] = 0$

$\Rightarrow e^s = 1 - \frac{1}{p} < 0$
not possible

$$-n\alpha + n\alpha p - n\alpha pe^s + npe^s = 0$$
$$npe^s(1-\alpha) = n\alpha(1-p)$$
$$e^s = \frac{\alpha(1-p)}{p(1-\alpha)}$$

So, $P(X > \alpha n) \leq (e^s)^{-\alpha n}(1 - p + pe^s)^n$

putting $e^s = \frac{\alpha(1-p)}{p(1-\alpha)}$

$\Rightarrow$

$$P(X \geq \alpha n) \leq \left(\frac{\alpha(1-p)}{p(1-\alpha)}\right)^{-\alpha n}\left(1-p+\frac{\alpha(1-p)}{(1-\alpha)}\right)^{n}$$

$\Rightarrow$

$$P(X \geq \alpha n) \leq \left(\frac{\alpha(1-p)}{p(1-\alpha)}\right)^{-\alpha n}\left(\frac{1-p}{1-\alpha}\right)^{n}.$$

$\Rightarrow$

$$\boxed{P(X \geq \alpha n) \leq \left(\frac{1-p}{1-\alpha}\right)^{n(1-\alpha)} \cdot \left(\frac{p}{\alpha}\right)^{n\alpha}}$$

Problem 10 :- Sol^n :-

First of all we number the people say $1, 2, 3 \dots N$ where $N \geq 5$.

Let Random Variable $X_i = 1$ where i^{th} person receives present where $1 \leq i \leq N$.

So,

$$X = X_1 + X_2 + \dots + X_N = \sum_{i=1}^{N} X_i$$

We can say $P(X_i = 1) = \underbrace{\frac{1}{2}}\cdot\underbrace{\frac{1}{2}} = \frac{1}{4}$

different outcome for { left person / right person

So, $E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1)$

$= P(H_{i-1} T_i H_{i+1}) + P(T_{i-1} H_i T_{i+1})$

$= 1/8 + 1/8 = 1/4.$

Thus, $E[X] = E\left[\sum_{i=1}^{N} X_i\right]$

$= \sum_{i=1}^{N} E[X_i]$ } using linearity of expectation

$= \sum_{i=1}^{N} 1/4$

$$\boxed{E[X] = N/4}$$

Now since;

$$VAR(X) = \sum_{i=1}^{N} VAR(X_i) + \sum_{i=1}^{N}\sum_{j\neq i} Cov(X_i, X_j)$$

And $VAR(X_i) = \frac{1}{4}\cdot\frac{3}{4} = 3/16$ because $X_i \sim Bernoulli(1/4)$.

And for i,j s.t. $2 < |i-j| < N-2$, X_i & X_j are independent i.e. $Cov(X_i, X_j) = 0$

So,

$$\sum_{i=1}^{N}\sum_{j\neq i} Cov(X_i, X_j) = 2\sum_{i=1}^{N}\left[Cov(X_i, X_{(i+1)\%N}) + Cov(X_i + X_{(i+2)\%N})\right]$$

$$= 2\sum_{i=1}^{N}\left[Cov(X_i, X_{((i+1)\%N)}) + Cov(X_i, X_{(i+2)\%N})\right]$$

; % is modulus.

& due to symmetry, $\forall\ i,j \in \{1,2,\ldots,N\}$

$$Cov(X_i, X_{(i+1)\%N}) = Cov(X_j, X_{(j+1)\%N})$$

$$Cov(X_i, X_{(i+2)\%N}) = Cov(X_j, X_{(j+2)\%N})$$

Thus,

$$\sum_{i=1}^{N} \sum_{j \neq i} Cov(x_i, x_j) =$$

$$2N\left[Cov(X_1, X_2) + Cov(X_1, X_3)\right].$$

Now, $Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$

$= P(X_1 = 1, X_2 = 1) - E[X_1]E[X_2]$

$= P(H_N, T_1, H_2, T_3) + P(T_N, H_1, T_2, H_3)$

$- E[X_1]E[X_2]$

$= \frac{1}{16} + \frac{1}{16} - \frac{1}{4} \cdot \frac{1}{4}$

$= 1/16.$

&

$Cov(X_1 X_3) = E[X_1 X_3] - E[X_1]E[X_3]$

$= P(X_1 = 1, X_3 = 1) - E[X_1]E[X_3]$

$= P(H_N, T_1, H_2, T_3, H_4) +$

$P(T_N, H_1, T_2, H_3, T_4) - E[X_1]E[X_3]$

$= \cancel{\frac{1}{32} + \frac{1}{32} - \frac{1}{4} \cdot \frac{1}{4}}$

$= 0.$

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Thus,

$$\sum_{i=1}^{N} \sum_{j \neq i} Cov(X_i, X_j) = 2N\left[Cov(X_1 + X_2) + 0\right]$$

$$= 2N\left[\frac{1}{16} + 0\right] = N/8.$$

So,

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + \frac{N}{8}.$$

$$= \sum_{i=1}^{N} \frac{3}{16} + \frac{N}{8}$$

$$= \frac{3N}{16} + \frac{N}{8}$$

$$\boxed{Var(X) = \frac{5N}{16}}$$

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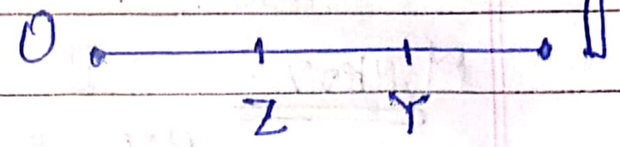
Problem : 11 Sol^n :-

Let we first cut at point Y then at Z.

So, for Δ :-



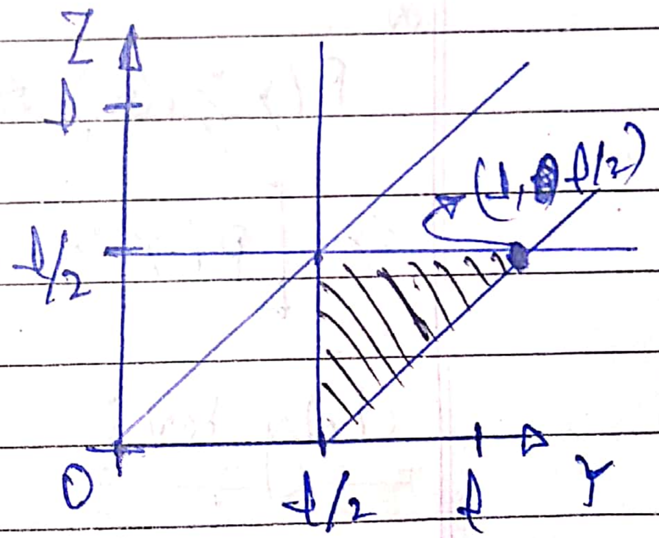
① $Z < (Y-Z)+(l-Y) \Rightarrow 2Z < l$

② $(Y-Z) < Z+(l-Y) \Rightarrow 2Y < 2Z+l$

③ $(l-Y) < Z+(Y-Z) \Rightarrow l < 2Y$

Required Area - shaded region $= \frac{1}{2} \times \frac{l}{2} \times \frac{l}{2}$

$= l^2/8$.



Total possible area which cover point (Y, Z)

$= l^2/2$

Probability $= \dfrac{l^2/8}{l^2/2} = 1/4 = \cancel{0.25}$ 0.25

Ans

Problem 12: Sol^n

$(l < d)$

Let x be the distance of centre of needle from closest line and θ be acute angle of needle w.r.t one of the lines.

Now we can say, $0 \leq x \leq d/2$
; d is distance between lines.
& x is uniformly distributed.

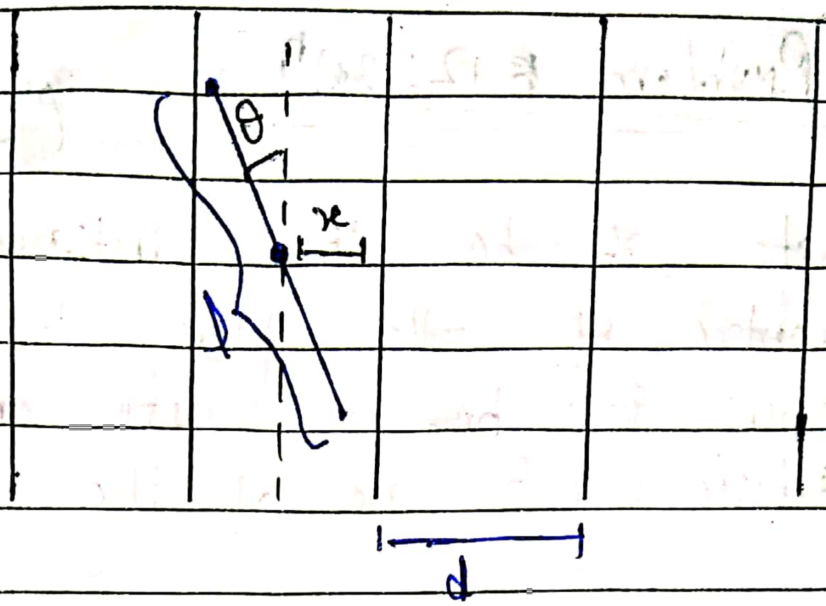
So,

$$P_X(x) = \begin{cases} 2/d \; ; & 0 \leq x \leq d/2 \\ 0 \; ; & \text{otherwise.} \end{cases}$$

Similarly,
θ is uniformly distributed from $0 \to \pi/2$.

Thus,

$$P_\theta(\theta) = \begin{cases} 2/\pi \; ; & 0 \leq \theta \leq \pi/2 \\ 0 \; ; & \text{otherwise} \end{cases}$$



Since, x & θ are independent then joint pdf of X & Θ will be

$$f_{X,\Theta}(x,\theta) = \begin{cases} 4/\pi d \; ; 0 \le x \le d/2 \; , 0 \le \theta \le \pi/2 \\ 0 \; ; \text{ otherwise} \end{cases}$$

We can observe from figure that needle will intersect when

$$x \le \frac{l}{2} \sin\theta .$$

As $l < d$;

So, Required probability $P = \int_{0}^{\pi/2} \int_{x=0}^{\left(\frac{l}{2}\sin\theta\right)} \frac{4}{\pi d} \, dx \, d\theta$

$$P = \int_0^{\pi/2} \frac{2l}{\pi d} \sin\theta \, d\theta = \frac{2l}{\pi d} \quad \text{Ans}$$

Problem 7

Generalised Union Bound or Bonferroni Inequality :-

Suppose (S, f, P) is probability space & $A_1, A_2 \dots A_n \in f$ are events.

We Define

$$S_{1,n} = \sum_{i=1}^{n} P(A_i)$$

$$S_{2,n} = \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$S_{k,n} = \sum_{1 \le i_1 < \dots < i_k \le n} P(A_{i_1} \cap \dots \cap A_{i_k}), \quad k = 3, \dots n$$

So, Bonferroni Inequality :-

1. For odd k in $\mathbb{N}$: $(k \le n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{j=1}^{k} (-1)^{j-1} S_{j,n}$$

2. For even k in $\mathbb{N}$: $(k \le n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \ge \sum_{j=1}^{k} (-1)^{j-1} S_{j,n}$$

* proof for $k = 1$ i.e. $k=1$ in case 1 has been done in later part.

We will treat $k=1$ as base case.

Let's do some S' conversion :-

$$S_{1,n} = \sum_{i=1}^{n} P(A_i) = S_{1,n+1} - P(A_{n+1})$$

$$S_{2,n} = \sum_{1 \le i < j \le n} P(A_i \cap A_j) = S_{2,n+1} - \sum_{1 \le i \le n} P(A_i \cap A_{n+1})$$

$$S_{k,n} = \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

$$S_{k,n} = S_{k,n+1} - \sum_{1 \le i_1 < i_2 \cdots < i_{k-1} \le n} P(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_{k-1}} \cap A_{n+1})$$

$$S_{k,n} = S_{k,n+1} - L.$$

Proving for case 1 :- (by Induction)
($k =$ odd)

Say for n it is true, we have to then prove true for $n+1$.

so, let

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{j=1}^{k} (-1)^{j-1} S_{j,n}$$

Using s' conversion :-

We can write :

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq S_{1,n} - S_{2,n} + S_{3,n} \cdots + (-1)^{k-1} S_{k,n}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq \left(S_{1,n+1} - P(A_{n+1})\right) - \left(S_{2,n+1} - Q\right) \cdots$$

$$\cdots + (-1)^{k-1}\left(S_{k,n+1} - L\right).$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq \left(\sum_{j=1}^{k} (-1)^{j-1} S_{j,n+1}\right) - P(A_{n+1})$$

$$+ Q + \cdots - (-1)^{k-1} L.$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - Q + \cdots + (-1)^{k-1} L$$

$$\leq \sum_{j=1}^{k} (-1)^{j-1} S_{j,n+1}$$

We can observe LHS is the ~~same~~ greater ~~than~~ the expression we obtain on expanding $P\left(\bigcup_{i=1}^{n+1} A_i\right)$ using Inclusion - Exclusion principle.

So,
$$\boxed{P\left(\bigcup_{i=1}^{n+1} A_i\right) \leq \sum_{j=1}^{k} (-1)^{j-1} S_{j,n+1}}$$

$k \leq n$ & $k = \text{odd}$.

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Proving for case - 2 i.e k = even.

Assuming base case $k=2$ which will prove later.

Let assume for $k \geq 2$ & k = even.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{k} (-1)^{j-1} S_{j,n}$$

Now, using same substitution as in case 1 we can directly say that

$$\boxed{P\left(\bigcup_{i=1}^{n+1} A_i\right) \geq \sum_{j=1}^{k} (-1)^{j-1} S_{j,n+1}}.$$

$k \leq n$ & k = even.

Proving all base case i.e. $k=2$ and $k=1$.

1 $k=2$.

~~We know~~,

We have to show;

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq S_{1,n} - S_{2,n}.$$

We can show this by induction too.

Let base case be $n=2$ because for $n=1$, $P(A_1) \geq S_{1,n} = P(A_1)$ already.

Also, $P(A_1 \cup A_2) = \underbrace{P(A_1) + P(A_2)}_{\geq S_{1,n}} - \underbrace{P(A_1 \cap A_2)}_{S_{2,n}}$

$\Rightarrow P(A_1 \cup A_2) \geq S_{1,n} - S_{2,n}$

Now, let it be true for $n \geq 2$.
we have to show it for $n+1$.

so, $P\left(\bigcup_{i=1}^{n} A_i\right) \geq S_{1,n} - S_{2,n}$

$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \left(S_{1,n+1} - P(A_{n+1})\right) -$

$\left(S_{2,n+1} - \sum_{i=1}^{n} P(A_i \cap A_{n+1})\right)$

$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - \sum_{i=1}^{n} P(A_i \cap A_{n+1})$

$\geq S_{1,n+1} - S_{2,n+1}$

We can see LHS $\leq P\left(\bigcup_{i=1}^{n+1} A_i\right)$

because $\sum_{i=1}^{n} P(A_i \cap A_{n+1}) = P(X \cup A_{n+1})$

; $X = A_1 \cap A_2 \cdots \cap A_n$.

$\Rightarrow \boxed{P\left(\bigcup_{i=1}^{n+1} A_i\right) \geq S_{1,n+1} - S_{2,n+1}}$

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Problem: 7 Solⁿ :- (Remaining)

2

K=1

The union bound states that in general for any events $A_1, A_2 \ldots A_n$ we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \text{— (1)}$$

Proof by Induction :-

Base case : $n=1, 2$.

for $n=1$; $P(A_1) = P(A_1)$

Hence, the correctness of Union Bound.

& for $n=2$;

$$P(A_1 \cup A_2) = [P(A_1) + P(A_2)] - \underbrace{P(A_1 \cap A_2)}_{\Rightarrow\ \geq 0}$$

$$\Rightarrow P(A_1 \cup A_2) \leq [P(A_1) + P(A_2)]$$

Hence, the correctness of Union Bound.

claim : If eq. (1) is true for any $n = k \geq 2$ then eq. (1) is true for $n = (k+1)$ also.

Proof : Assume eq. (1) is true for $n = k \geq 2$.

so,

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i) \quad \text{— (ii)}$$

Now on observing the quantity : $P\left(\bigcup_{i=1}^{k+1} A_i\right)$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i + A_{k+1}\right).$$

Now, we reduce it from $(k+1)$ terms to 2 terms by letting $X = \bigcup_{i=1}^{k} A_i$

So, $P\left(\bigcup_{i=1}^{k+1} A_i\right) = P(X + A_{k+1})$

$\Rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq P(X) + P(k+1)$ {already proved for $n=2$ i.e. base case

$\Rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(k+1)$

$\Rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq \sum_{i=1}^{k} P(A_i) + P(k+1)$ {from eq - (1)

Thus,

$$\boxed{P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq \sum_{i=1}^{k+1} P(A_i) \; ; \; k = \mathbb{Z}^+}$$

Hence proved.

Example on slide 4 :— (Lec. 38)

We can say that $B_n = \bigcup_{i=1}^{n} A_i$

From inclusion - exclusion :—

$$P(B_n) = P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

(*)

We can say from symmetry ;

$$\sum_{i=1}^{n} P(A_i) = nP(A_1) \quad \text{—— (I)}$$

&

$$\sum_{i<j} P(A_i \cap A_j) = {}^{n}C_{2} \, P(A_1 \cap A_2) \quad \text{—— (II)}$$

Now since the event A_1 occurs if node 1 is not connected to any of the other $(n-1)$ nodes. Since, the connections are independent, we can say

$$P(A_1) = (1-p)^{n-1} \quad \text{—— (III)}$$

Now for $P(A_1 \cap A_2)$, both nodes 1 & 2 are isolated, total edges will be
$2(n-2)+1 = 2n-3$.

So, $P(A_1 \cap A_2) = (1-p)^{2n-3}$ —— (IV)

From eq: ($\cancel{*}$), (I), (II), (III) & (IV)

$$\boxed{P(B_n) \geq n(1-p)^{n-1} - {}^{n}C_{2}(1-p)^{2n-3}}$$