

Probability and Statistics: Lecture-29

Monsoon-2020

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on October 19, 2020

» Definition of Conditional Expectation...

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$$E[X | A] = \sum_{x_i \in R_X} x_i \underbrace{P_{X|A}}_{P_{X|Y}(x_i | y_j)}(x_i)$$
$$E[X | Y = \underbrace{y_j}_{y_j}] = \sum_{x_i \in R_X} x_i \underbrace{P_{X|Y}}_{P_{X|Y}(x_i | y_j)}(x_i | y_j)$$

» Example of Conditional Expectation

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1. Find $E[X \mid Y = 1]$

2. Find $E[X \mid -1 < Y < 2]$

✓ $1/13 \cdot (x+1)$

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1. Find $E[X \mid Y = 1]$
2. Find $E[X \mid -1 < Y < 2]$
3. Find $E[|X| \mid -1 < Y < 2]$

» Answer to previous problem...

Recall from last class that

Given $y=1$, x is uniformly distributed over the set $\{-1, 0, 1\}$

$$\underline{P_{x|y}} = \begin{cases} \frac{1}{3} & \text{for } x = -1 \\ \frac{1}{3} & \text{for } x = 0 \\ \frac{1}{3} & \text{for } x = 1 \end{cases} \quad \text{Uniform distr.}$$

Note $x = -1, 0, 1$ because $y = 1$
 $\nwarrow (|x| + |y| \leq 2)$

$$\Rightarrow E[x|y=1] = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$$

$$\textcircled{b} E[x | -1 < y < 2]$$

$$-1 < y < 2 \Rightarrow y \in \{0, 1\} \equiv A$$

$$P(\underbrace{-1 < y < 2}_A) = P_y(0) + P_y(1) = \frac{5}{13} + \frac{3}{13} = \frac{8}{13}$$

$$P_{x|A}(k) = \frac{P(x=k, A)}{P(A)} = \frac{P(x=k, A)}{8/13} = \frac{13}{8} P(x=k, A)$$

» Answer to previous problem...

So we can write

$$P_{X|A}(-2) = \frac{13}{8} P(X=-2, A) = \frac{13}{8} P(\cancel{2}, 0) \\ = \frac{13}{8} \cdot \frac{1}{13} = \frac{1}{8}$$

©

Try!

$$P_{X|A}(-1) = \frac{13}{8} P(X=-1, A) = \frac{13}{8} \left[P_{X,Y}(-1, 0) + P_{X,Y}(-1, 1) \right] \\ = \frac{13}{8} \cdot \frac{2}{13} = \frac{1}{4}$$

Similarly,

$$P_{X|A}(0) = \frac{1}{4}, \quad P_{X|A}(2) = \frac{1}{8}$$

$$E[X|A] = \sum_{x_i \in R_X} x_i P_{X|A}(x_i)$$

$$= -2 \cdot \frac{1}{8} + (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} = 0$$

» Law of Total Probability...

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- * Law of Total Probability

» Law of Total Probability...

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i)$$

Handwritten notes:
- B_i 's form a partition
- $P(B_i)$ is circled in the original image

Law of Total Probability

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$$P(X \in A) = \sum_{y_j \in R_Y} P(X \in A | Y = y_j) P_Y(y_j), \quad \text{for any set } A$$

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1. If B_1, B_2, \dots is a **partition** of sample space S

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Proof. $E[X|B_i] \stackrel{\text{def}}{=} \sum_j x_j P(x_j|B_i)$ $E[X] = \sum_i \underbrace{E[X|B_i] P(B_i)}$

Mult by $P(B_i)$ & sum over i

$$\begin{aligned} \sum_i E[X|B_i] P(B_i) &= \sum_i \sum_j x_j P(x_j|B_i) P(B_i) \\ &= \sum_j x_j \frac{\sum_i P(x_j|B_i) P(B_i)}{P(x_j)} = \underline{\underline{E[X]}} \end{aligned}$$

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1. If B_1, B_2, \dots is a **partition** of sample space S

$$E[X] = \sum_i E[X \mid B_i] P(B_i)$$


2. For a RV X and a **discrete** RV Y
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$$E[Y] = \sum_{y_j \in R_Y} E[X \mid \underbrace{Y = y_j}] P_Y(y_j)$$


» Solved Example

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Solved Example 1

Let $X \sim \text{Geometric}(p)$. Find $E[X]$. [Hint: condition on first coin toss.]

Geometric(p): toss the coin repeatedly until 1st head.

Here $P(H) = p$, $X = \text{total no. of coin tosses}$.

Two possible cases: H or T.

Using Law of total Expectation

$$\begin{aligned} E[X] &= E[X|H] P(H) + E[X|T] P(T) \\ &= p E[X|H] + (1-p) E[X|T] \\ &= p \cdot 1 + (1-p) (1 + E[X]) \end{aligned}$$

Solving for $E[X] = \frac{1}{p}$.

$$E[X|H] = 1$$

$$E[X|T] = 1 + E[X]$$

» Solved Example

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in day

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution
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Uniform, Bernoulli, Geometric, Binomial

Solved Example 2

Number of customers N visiting a fast food restaurant follows Poisson distribution $N \sim \text{Poisson}(\lambda)$. Each customer arriving in this restaurant purchases a drink with probability p , which is independent from other customers. What is the average number of customers who purchase drinks?

» Answer to previous problem... $[X:]$ no of customizing purchasig drinks

Solⁿ: Given $N=n$, then X is a sum of n ind. Bernoulli(p), which means it is

$X|N=n \sim \text{Binomial}(n, p)$

$$P_{X|N}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X|N=n] = np$$

$E[X] =$ expected no of customizing purchasig drink.

$$= \sum_{n=0}^{\infty} E[X|N=n] P_N(n)$$

$$= \sum_{n=0}^{\infty} np P_N(n)$$

$$= p \sum_{n=0}^{\infty} n P_N(n)$$

$$= p E[N] = \underline{\underline{p\lambda}}$$

» PMF and Expectation of Two Random Variables...

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PMF and Expectations of Two Random Variables

- * Let X, Y be two RVs and suppose $Z = g(X, Y), g : \mathbb{R}^2 \rightarrow \mathbb{R}$.

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$$P_Z(z) = P(\underbrace{g(X, Y)}_Z = z) = \sum_{(\underline{x_i}, \underline{y_j}) \in A_z} P_{XY}(x_i, y_j),$$

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- * The **expectation** is given as follows

$$E[\underbrace{g(X, Y)}] = \sum_{(x_i, y_j) \in R_{XY}} g(x_i, y_j) \underbrace{P_{XY}(x_i, y_j)}$$

» Linearity of Expectation for Two Random Variable...

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Linearity of Expectation for Two RV

Let X, Y be two discrete RVs. Then $E[X + Y] = E[X] + E[Y]$.

Exercise

» PMF of Difference of Two Geometric Distributions...

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PMF of Difference

Let $X, Y \sim \text{Geometric}(p)$ be two random variables. Let $Z = X - Y$. Find the PMF of Z .

» Answer to previous problem...

$$Z = X - Y$$

Ans. $R_X = R_Y = \{1, 2, 3, \dots\} = \mathbb{N}$

$\Rightarrow R_Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
 $= \mathbb{Z}$.

Since $X, Y \sim \text{Geometric}(p)$

$P_X(k) = P_Y(k) = \underbrace{p q^{k-1}}$, $k = 1, 2, 3, \dots$
 $q = 1 - p$.

$P_Z(k) = P(Z=k) = P(X-Y=k)$
 $= P(X=Y+k) \stackrel{\text{L.T.P.}}{=} \sum_{j=1}^{\infty} \underbrace{P(X=Y+k | Y=j)}_{\text{L.T.P.}} P(Y=j)$

$= \sum_{j=1}^{\infty} P(X=j+k | Y=j) P(Y=j)$
 $= \sum_{j=1}^{\infty} \underbrace{P(X=j+k) P(Y=j)}_{(X, Y \text{ ind})}$

$k \geq 0$
 $P_Z(k) = \sum_{j=1}^{\infty} \underbrace{p q^{j+k-1}}_{P(X=j+k)} \underbrace{p q^{j-1}}_{P(Y=j)}$

$k < 0$

» Answer to previous problem...

$$\text{For } k < 0$$
$$p_z(k) = \sum_{j=1}^{\infty} \underbrace{p_x(j+k)}_{=0} p_y(j)$$

$$= \sum_{j=-k+1}^{\infty} \underbrace{p_x(j+k-1)}_{=0} \cdot p_y(j-1)$$

==