Probability and Statistics: Lecture-25

Monsoon-2020

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» Online Quiz

1. Please login to gradescope
2. Attempt Quiz-6
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist for online class

1. Turn off your microphone, when you are listening
2. Turn on microphone only when you have question
3. Attend tutorials to practice problems or to discuss solutions or doubts
4. Chat is not always reliable, I may not look at chat

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* Widely used distribution

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* Related to exponential *and* normal

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Gamma Function: Extension of Factorial Function

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Generally, for any positive number $\alpha, \Gamma(\alpha)$ is defined as

$\mathbb{R}$ $\mathbb{R}_{++}$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \quad \text{for } \alpha > 0.$$

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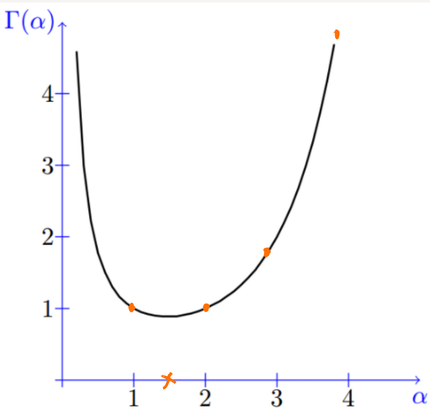

Gamma function for positive real values

» Properties of the Gamma Function...

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5. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

» Proof of Properties of Gamma Function...

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$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \leftarrow \text{Gamma}$$

2. $\underbrace{\int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx}_{L\cdot H\cdot S} = \underbrace{\frac{\Gamma(\alpha)}{\lambda^\alpha}}_{R\cdot H\cdot S}$, for $\lambda > 0$

In $\Gamma(\alpha)$, do change of variable : $\boxed{x = \lambda y}$ $\Rightarrow dx = \lambda\, dy$

Limits: $x = 0 \Rightarrow y = 0$ & $x = \infty \Rightarrow y = \infty$

$$\Gamma(\alpha) = \int_0^\infty (\lambda y)^{\alpha-1} e^{-\lambda y} \lambda\, dy = \lambda^\alpha \int_0^\infty y^{\alpha-1} e^{-\lambda y}\, dy \equiv \lambda^\alpha (L\cdot H\cdot S)$$

$$\Rightarrow \int y^{\alpha-1} e^{-\lambda y}\, dy = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

change back $y \to x$ to get the result.

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$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ ← Gamma fn

Recursive

3. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
4. $\Gamma(n) = (n-1)!$, for $n = 1, 2, 3, \ldots$

$$\Gamma(\alpha) = \lambda^\alpha \int_0^\infty \underbrace{y^{\alpha-1}}_{\text{2nd}} \underbrace{e^{-\lambda y}}_{\text{1st}} dy = \lambda^\alpha \left(\left[e^{-\lambda y} \frac{y^\alpha}{\alpha} \right]_0^\infty - \int_0^\infty -\lambda e^{-\lambda y} \cdot \frac{y^\alpha}{\alpha} dy \right)$$

$$= \lambda^\alpha \left(\frac{\lambda}{\alpha} \int_0^\infty y^\alpha e^{-\lambda y} dy \right) = \frac{\lambda^{\alpha+1}}{\alpha} \underbrace{\int_0^\infty y^\alpha e^{-\lambda y} dy}$$

Previously: Prop. 2

$$\int_0^\infty y^{\alpha-1} e^{-\lambda y} dy = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$= \frac{\cancel{\lambda^{\alpha+1}}}{\alpha} \frac{\Gamma(\alpha+1)}{\cancel{\lambda^{\alpha+1}}}$$

$$\Rightarrow \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

Prop. 4 is obvious

Sol: $\alpha = n$ apply 3 recursiv

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5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\left(\frac{1}{2}-1\right)! = \sqrt{\pi} \Rightarrow \boxed{-\frac{1}{2}! = \sqrt{\pi}}$

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx = \sqrt{\pi}$$

Integral Calculus

5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

* We show this in three steps:

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  1. First we show a fact from calculus that $d\,xdy = r\,dr\,d\theta$

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  1. First we show a fact from calculus that $d\,xdy = r\,dr\,d\theta$
  2. Second we show that the constant in normal distribution is $1/\sqrt{2\pi}$

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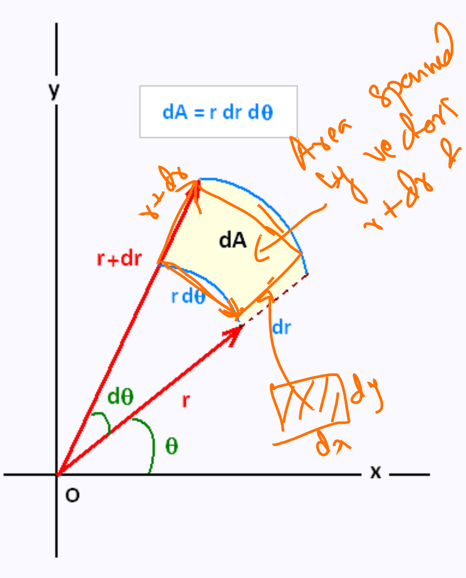
* We show this in three steps:
    1. First we show a fact from calculus that $dxdy = r\,dr\,d\theta$
    2. Second we show that the constant in normal distribution is $1/\sqrt{2\pi}$
    3. Finally, using above, we then show the final result stated above

» Step-1: Proof that $dxdy = r\,dr\,d\theta$



For polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

Cross product to find area in polar

cross prod.

$$dx\,dy = r\,dr\,d\theta$$ (after simplifi

» Step-2: Proof that Constant in the Normal Distribution is $1/\sqrt{2\pi}$

Recall $X \sim N(0,1)$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

Since f_X is P.D.F

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$$

$$\Rightarrow \underbrace{\boxed{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}}}_{I}$$

$\Rightarrow I^2 = 2\pi$

$\Rightarrow I = \underline{\underline{\sqrt{2\pi}}}$

Show that: $I^2 = 2\pi$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx\, dy$$

Switch to Polar coord: $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\, dy = r\, dr\, d\theta$

$$\Rightarrow I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\, r\, dr\, d\theta$$

$$= \int_0^{\infty} e^{-r^2/2}\, r\, dr \underbrace{\int_0^{2\pi} d\theta}_{} = 2\pi \int_0^{\infty} r e^{-r^2/2}\, dr$$

$$= 2\pi \left[-e^{-r^2/2}\right]_0^{\infty} = \underline{\underline{2\pi}}$$

$\Rightarrow$

» Step-3: Proof of $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx \equiv I$$

$$x = u^2 \Rightarrow dx = 2u\, du$$

$$I_1 = \int_0^\infty (u^2)^{-1/2} \cdot e^{-u^2} \cdot 2u\, du$$

$$= 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du$$

Change: $u \to y/\sqrt{2}$

$$du = \frac{dy}{\sqrt{2}}$$

$$\Rightarrow I_1 = \int_{-\infty}^\infty e^{-y^2/2} \cdot \frac{dy}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \underbrace{\int_{-\infty}^\infty e^{-y^2/2} dy}_{\sqrt{2\pi}}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}} = \sqrt{\pi}$$

Problem on Gamma Function

1. * Find $\Gamma(7/2)$
2. * Find the value of the following integral $\int x^{\alpha-1} e^{-\lambda x} \, dx$

$$I = \int_0^\infty x^6 e^{-5x} \, dx$$

Ans 1) $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$

2) Prop. 2 $\int_0^\infty x^{\alpha-1} e^{-\lambda x} \, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \lambda > 0$

Here: $\alpha = 7, \lambda = 5$

$\Rightarrow I = \frac{\Gamma(7)}{5^7} = \frac{6!}{5^7}$

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Exponential is a special case of Gamma distribution

For $\alpha = 1$, we obtain

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall :

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* Sum of n independent Exponential(λ) RVs is $\text{Gamma}(n, \lambda)$ RV (proof later)