Probability and Statistics – Assignment 2

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Instructions:

- There is a total of 2 sections and each section consists of 10 questions.
- All questions are compulsory.
- Upload your submissions to Gradescope. Do not forget to assign page number(s) to each question in Gradescope when you submit.
- Use of calculator is permitted.
- Each question in Section 1 carries 4 marks and each question in Section 2 carries 6 marks.

Section 1

Question 1

The Moment Generating Function (MGF) for a random variable X, $\mathrm{mgf}_{\mathrm{X}}(t)$ is given. Find the probability:

$$P(|X| \leq 2)$$

Hint: use the uniqueness theorem for MGFs. Simply speaking, if any two MGFs are same, then their distributions are also exactly same [Wikipedia]

$$\mathrm{mgf}_{\mathrm{X}}(t) = \frac{1}{10}e^{-20t} + \frac{1}{5}e^{-3t} + \frac{3}{10}e^{4t} + \frac{2}{5}e^{5t}$$

Question 2

We are playing a game. I randomly say a whole number and if you guess its parity (odd or even) correctly, we spin a wheel which signifies the money you are going to win. (distributed uniformly randomly between 0 and 1000 Rupees). And if your guess is wrong, you must give me 200 rupees. Your profit is determined by the random variable X. Find and graph the distribution of X (CDF) and hence determine the probability of you not winning at least 500 rupees.

Question 3

Let P be a random variable having a uniform distribution with minimum 0 and maximum 3 i.e. $P \sim \text{Uniform}(0,3)$. Let $Q = \log\left(\frac{P}{3-P}\right)$. Find $E[Q]$. You are expected to solve this problem without using Method of Transformation.

Question 4

There is a taxi-stand cum bus-stop near Ashish's home. Ashish goes there at a given time and, if a taxi is waiting which happens with probability $\frac{2}{3}$, he boards it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected value of Ashish's waiting time.

Question 5

Vinay and Mahesh alternate playing at the casino table. (Vinay starts and plays at odd times $i = 1, 3, \dots$; Mahesh plays at even times $i = 2, 4, \dots$) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$p_G(g) = \begin{cases} \frac{1}{3}, & g = -2 \\ \frac{1}{2}, & g = 1 \\ \frac{1}{3}, & g = 3 \\ 0, & \text{otherwise} \end{cases}$$

Assume that the net gains at different times are independent. We refer to an outcome of −2 as a "loss."

a. They keep gambling until the first time where a loss by Mahesh immediately follows a loss by Vinay. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Vinay and then one by Mahesh.)
b. Write down the PMF for Z (number of trials), defined as the time at which Mahesh has his third loss.
c. Let N be the number of rounds until each one of them has won at least once. Find $E[N]$.

Question 6

A book of 500 pages contains 500 misprints on an average. Estimate the chances that a given page contains at least three misprints. Consider occurrence of errors as a Poisson process.

Question 7

Given a random variable X with Moment Generating function $M(t)$, compute the Moment generating functions for the following random variables (in terms of M). In each subpart, k is a scalar.

a. $k\, X$
b. $X + k$
c. $\sum_{i=0}^{N} X_i$ where all X_i are sampled independently from X
d. A random variable Y with $PDF(y) = PDF\ (x + k)$, both PDF defined over R.
e. A random variable Y with $PDF(y) = PDF\ (2x)$

Question 8

Suppose that n people throw their hats in a box and then each pick one hat at random. (Each hat can be picked by only one person, and each assignment of hats to persons is equally likely.) What is the expected value of X, the number of people that get back their own hat?

Question 9

Let X have the probability density function (PDF) given by

$$f_x(x) = \begin{cases} \tfrac{1}{2}\, x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = \Phi(X) = 6X - 3.$

(**Hint:** Use the method of Transformation)

Question 10

On a busy day you are heading to the office and want to reach early. You are on the wrong side of the road and the next intersection is far away, and you have decided to make a U-turn. But you do not want to get caught by the police cars which arrive according to a Poisson distribution with rate λ. You will make a U-turn only if the road has been clear for τ time units. Let N be the number of cars you see before the U-turn.

Find $E[N]$.

Section 2

Question 11

In a field there are N stands each with a bunch of balloons of a unique color. There is a random shooter shooting at those balloons.

a. Let X be the random variable representing the duration of time before 2 balloons of the same color are shot. Find the PMF of X.
b. Let Y_i be the random variable representing duration of time before a new color of balloon is shot after i colors have already been shot. Compute the PMF of Y_i.

Question 12

You are going from College (Point A) to some distant eastern part of Hyderabad (Point B) which is 25 kms away. While your friend starts from (Point B) towards

(Point A) with the goal of meeting you. Both of you travel at 50 km/h towards each other. Both your starting time is truly random and uniformly distributed from 1 pm to 2 pm and both your starting time is independent of each other. Let the random variable X denote the distance between college and the point where both of you meet. Find F_X i.e. $P(X \leq x)$.

Question 13

The lifetime of a bulb is modeled as a Poisson variable. You have two bulbs types A and B with expected lifetime 0.25 years and 0.5 years, respectively. When a bulb's life ends, it stops working. You start with new bulb of type A at the start of the year. When it stops working, you replace it with a bulb of type B. When it breaks, you replace with a type A bulb, then a type B bulb, and so on.

1. Find the expected total illumination time (in years), given you do exactly 3 bulb replacements.
2. Your replacements are now probabilistic. If your current bulb breaks, you replace it with a bulb of type A with probability p, and with type B with probability $(1 - p)$. Find the expected total illumination time (in years), given you do exactly n bulb replacements, and start with bulb of type A.

Answer for part 2 exists in closed form in terms of n and p.

Question 14

Given an undirected graph $G = (V, E)$, and a 3-color assignment colors R, G, B to the vertices of the graph a.

$$a: V \rightarrow \{R, G, B\}$$

Given an assignment a, the set of monochromatic edges

$$E(a) := \{(u, v) \in E: a(u) = a(v)\}$$

is the set of edges that has same colors for endpoints. Let a be randomly chosen, i.e. for every $v \in V$, it is chosen to be R, G, B uniformly and independent of the other vertices.

a. For any edge $e \in E$, let X_e be the random variable which is 1 when e is monochromatic and 0 otherwise. Even though the set of random variables $\{X_e\}_{e \in E}$ are pairwise independent, show that they are not independent of all other X_e (you are not required to prove the pairwise independence part).
b. Let Y be the random variable corresponding to the number of non-monochromatic edges. That is $Y := |E \setminus E(a)|$. Find $E[Y]$.
c. Show that there cannot be a graph for which all 3-color assignments make less than $\frac{2|E|}{3}$ edges nonmonochromatic. That is for any graph G, there exists an assignment a: $V \rightarrow \{R, G, B\}$ such that the number of non-monochromatic edges is at least $\frac{2|E|}{3}$.

Question 15

An ambulance travels back and forth, at a constant specific speed v, along a road of length L. We may model the location of the ambulance at any moment in time to be uniformly distributed over the interval $(0, L)$. Also at any moment in time, an accident (not involving the ambulance itself) occurs at a point uniformly distributed on the road; that is, the accidents distance from one of the fixed ends of the road is also uniformly distributed over the interval $(0, L)$. Assume the location of the accident and the location of the ambulance are independent. Supposing the ambulance is capable of immediate U-turns, compute the CDF and PDF of the ambulances travel time T to the location of the accident.

Question 16

Shashank performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of Z fair coins.

a. Given that Shashank has just had a trial with Z tails, what is the probability that next two trials will also have this result?
b. Whenever all the Z coins land on the same side in any given trial, Shashank calls the trial a success.
    i. Find the PMF for K, the number of trials up to, but not including the second success.

ii. Find the expectation and variance of M, the number of tails that occur before the first success where $Z = 3$ in this case.
(**Hint**: Write $M = X_1 + X_2 + \cdots + X_N$ Where X_i is number of tails in that occur on unsuccessful trail i and N is number of unsuccessful trails and find $E(M)$ in terms of $E(X)$ and $E(N)$ and for $\text{var}(M)$ first write in terms of $M \mid N$ (i.e. M given N) and later write $\text{var}(M)$ in terms of $E(N)$, $E(X)$, $var(N)$, $var(X)$ using the above derived variance equation in terms of $M \mid N$.)
You are expected to write all the steps involved in deriving $E(M)$, $\text{var}(M)$ as mentioned in the hint.

c. Sandeep conducts an experiment like Shashank's, except that he uses M coins for the first trial, and then he obeys the following rule: Whenever all the coins land on the same side in a trial, Sandeep permanently removes one coin from the experiment and continues with the trials. He follows this rule until the $(M - 1)^{th}$ time he removes a coin, at which point the experiment ceases. Find $E[X]$, where X is the number of trials in Sandeep's experiment.

Question 17

One of 225 different colors is assigned to a ball where each color is equally likely. For a group of n balls,

a. Find the expected number of colors which are assigned to exactly k balls.
b. Find the expected number of colors which have been assigned to more than one ball.
c. In part (b) how large should n be to make this expectation exceed 1? (Use calculator or code to do this part)

Question 18

You write a code over and over, and each time there is probability p that it gets accepted, independent of previous attempts. What is the mean and variance of X, the number of tries until the code gets accepted?

Question 19

Students arrive for the much dreaded JEE Advanced according to a Poisson process with rate lambda. For sanitization process they must stand in a queue, each student can take different time or same time compared with some other student for sanitization. Let us denote the time taken by i^{th} student as X_i.

X_i are independent identically distributed random variables. We assume that X_i takes integer values in range $1,2, \dots, n$, with probabilities $p_1, p_2, \dots, p_n$. Find the PMF for N_t, the number of students in sanitization queue at time t.

Question 20

Let X have the probability density function given by

$$f_x(x) = \begin{cases} e^{-x}, & 0 \leq x \leq \infty \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of $Y = X^{½}$.