Algebraic Curves and the Bézout's Theorem

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Algebraic Curves and the Bézout's Theorem

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Supervisor's Certificate

This is to certify that the work presented in this project entitled *“Alegraic Curves and the Bézout's Theorem* " by *Bhaswati Saha*, 7473U206009, is a record of original review work carried out by her under my supervision and guidance in partial fulfilment of the requirements of the B. Sc. (Hons) in Mathematics and Computing. Neither this project nor any part of it has been submitted for any degree or diploma to any institute or university in India or abroad.

Dr. Archana S Morye

Dedicated

To my parents and my teachers

Declaration

I, Bhaswati Saha, 7473U206009 hereby declare that this project entitled "Algebraic Curves and the Bézout's Theorem" represents my original/review work carried out as a B. Sc. (Hons) of IMA Bhubaneswar and, to the best of my knowledge, it is not a complete copy of previously published or written by another person, nor any material presented for award of any other degree or diploma of IMA Bhubaneswar or any other institution. Any contribution made to this research by others, with whom I have worked at IMA Bhubaneswar or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the section Bibliography.

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Abstract

Algebraic geometry is the branch of mathematics that combines the principles of abstract algebra and geometry to study geometric objects defined by polynomial equations. The sets of zeros of these polynomial equations are called algebraic varieties and one dimensional algebraic varieties are called algebraic curves. Our objective is to study algebraic curves.

The main aim of this project is to provide a comprehensive examination of one of the earliest and most interesting results of algebraic geometry, the Bézout's Theorem . In its most general form, Bézout's theorem gives us a relation between the number of points of intersections of two polynomial curves and the degrees of the generating polynomials.

Keywords; Algebraic Curves, Bézout's Theorem, Intersection Multiplicity

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Chapter 1

Introduction

Algebraic geometry is a study of geometric objects such as curves, surfaces, using algebraic tools. For example if we want to study the unit sphere,

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

we need to study roots of $x^2+y^2+z^2-1$ in $\mathbb{R}^3$. More abstractly the unit sphere corresponds to an ideal generated by the polynomial $x^2+y^2+z^2-1$ in the polynomial ring $\mathbb{R}[x,y,z]$. The tangent space of unit sphere at a point (x_0,y_0,z_0) is given by a solutions to the linear equation $ux_0+vy_0+wz_0=0$, that is,

$$T_{(x_0,y_0,z_0)}S^2 = \{(u,v,w) \in \mathbb{R}^3 | ux_0 + vy_0 + wz_0 = 0\}.$$

Algebraic geometry is thus described as a study of geometric objects which are the zero sets of polynomial equations.

Algebraic curves are the zero sets of polynomials of types $p(x,y)$. We can see these zero sets either in the real plane $\mathbb{R}^2$, in the complex plane $\mathbb{C}^2$ or in the projective plane $\mathbb{P}^2$. The well known examples of algebraic curves are conic. In real plan upto an affine coordinate changes conic are circles or ellipses, parabolas and hyperbolas. But in the complex plane circle can be transformed to parabola by affine change. Further in the projective plane all conics are the same (upto an affine transformation) to the sphere. In the second chapter we study conics in the real and complex planes, and in the third chapter we studied the projective space and conics in the projective plane.

If a curve C is a zero set of $p(x,y)$, then the degree of $p(x,y)$ is a numerical invariant of the curve. Instead for line it is 1, for circle, ellipses, parabola it is 2. We know that when we take two lines which are not parallel intersect at one point, which is equal to 1×1. If we take two ellipses in the general position, they intersect in 4 points, which is 2×2. So it is natural to ask if a curve C_1 is of degree n, and a curve C_2 is of degree m, can the number of intersection points be nm. Bézout's theorem answers this question affirmatively, provided we see intersection in the projective plane and considering multiplicity of the intersection point.

In Chapter 4, we also studied the genus of the curve. Though we are not using the genus in Bézout's theorem, the genus is the invariant which is very important for the study of curves.

In Chapter 5, we have studied the Fundamental Theorem of Algebra, points of intersection of two curves counted with their multiplicity and the statement of the Bézout's theorem. We see some examples that show us how to find the Intersection Multiplicity, the use of Intersection Multiplicity and the Bézout's theorem.

Chapter 2

Conics

In this chapter we will be studying *Conics*. In general, we see zero sets of polynomial in the real plane but here we also study conics in the complex plane. We will show that upto affine transformation conics over the reals are only three(upto classes) while the Conics in Complex Plane are two (upto classes).

2.1 Conics over the Reals

We look at the different classes of conics over the reals.
Let us start with the polynomial, $P(x,y) = y - x^2$ and look at its zero set, i.e.,

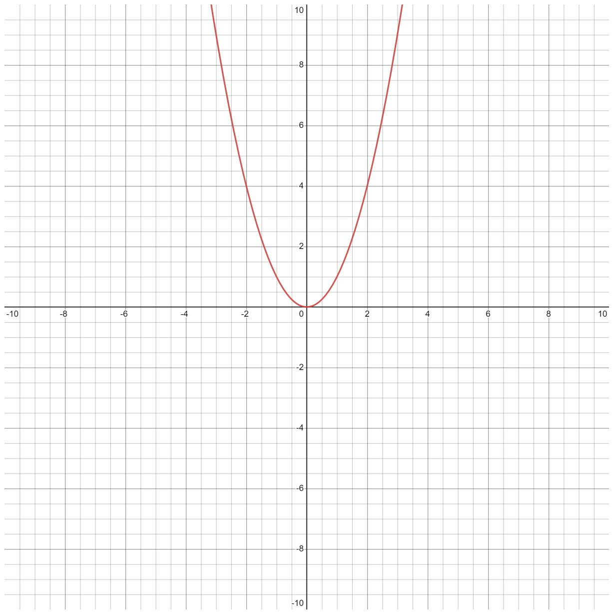
$$C = \{(x,y) \in \mathbb{R}^2 : P(x,y) = 0\}$$

Let us denote this set by *V(P)*.
Let us consider $(x,y) \in C$, we can easily observe the following,

- For any $(x,y) \in C$, we also have $(-x,y) \in C$. Thus, the curve C is symmetric about the y-axis.
- If $(x,y) \in C$, we get $y - x^2 = 0 \implies y = x^2$, hence $y \geq 0$.
- For every $y \geq 0$, $\exists x \in \mathbb{R}$, such that, $y = x^2 \implies y - x^2 = 0 \implies (x,y) \in C$.
- If $y \to \infty \implies x^2 \to \infty \implies x \to \infty, -x \to -\infty$.

Now, we sketch the curve,

$$C = \{(x,y) \in \mathbb{R}^2 : P(x,y) = 0\}$$



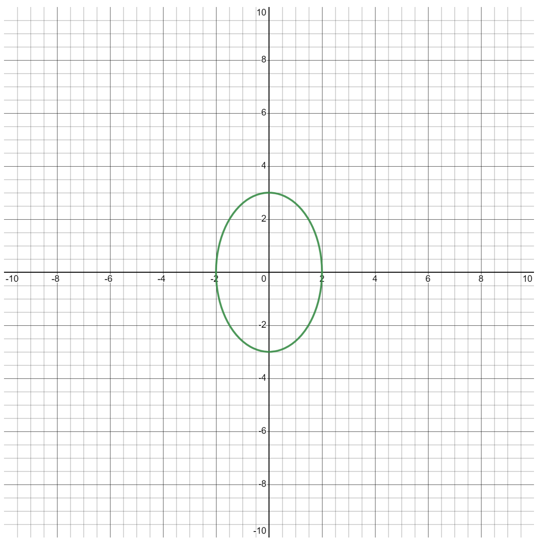
Conics that have these symmetry and boundedness properties and look like this curve C are called *parabolas*.

We can perform a similar analysis for the plane curve,

$$C = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\}$$

- If $(x,y) \in C$, the three points $(-x,y),(x,-y)$ and $(-x,-y)$ are also on C. Thus, C is symmetric about both the x- and y-axes.
- For every $(x,y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$. Hence, the curve C is bounded in both the positive and negative x- and y- directions.

Similarly sketching the graph,



Conics that have the above symmetry and boundedness properties and look like this curve C are called *ellipses*.

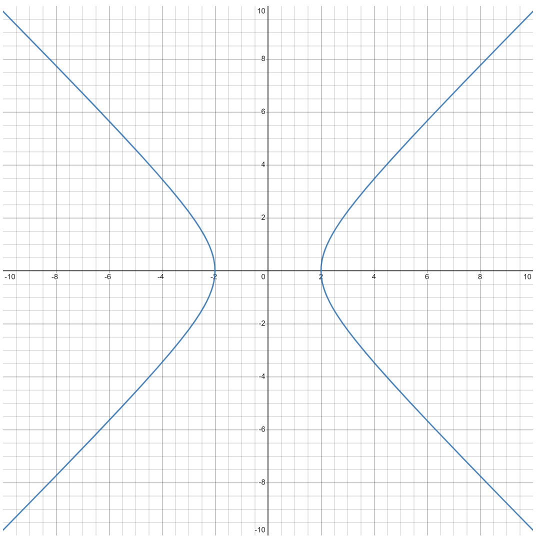
Now, we consider the curve,

$$C = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 - 4 = 0\}$$

.

- If $(x,y) \in C$, the three points $(-x,y),(x,-y)$ and $(-x,-y)$ are also on C. Thus, C is symmetric about both the x- and y-axes.
- For every $(x,y) \in C$, we have $|x| \geq 2$.

Similarly sketching the graph,



Conics that have the above symmetry, connectedness and boundedness properties and look like this curve C are called *hyperbolas*.

Now, let us consider the following second-degree polynomial in $\mathbb{R}^2$,

$$P(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

Expressing the above polynomial in one variable, let,

$$P(x,y) = Ax^2 + Bx + C$$

where,

$$A = a,$$

$$B = by + d,$$

$$C = cy^2 + ey + d$$

We know that roots of an equation, $Ax^2 + Bx + C = 0$ is given by,

$$(\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A})$$

To determine the nature of roots we need to look at the sign of the discriminant,

$$\Delta_x = B^2 - 4AC$$

Treating $P(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ as a polynomial in x, we get,

$$\Delta_x = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah)$$

. We observe the following,

$\Delta_x(y_0) < 0$	$\nexists$ point in *V(P)* such that (x, y_0).
$\Delta_x(y_0) = 0$	$\exists$ exactly one point in *V(P)* such that (x, y_0).
$\Delta_x(y_0) > 0$	$\exists$ exactly two points in *V(P)* (x, y_0).

2.2 Real Affine Change of Coordinates

In this section, we see how two conics can be called to be the same. A *real affine change of coordinates* in the real plane, $\mathbb{R}^2$, is given by,

$$u = ax + by + e$$

$$v = cx + dy + f$$

where, $a, b, c, d, e, f \in \mathbb{R}$, and $ad - bc \neq 0$ In matrix language we have,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$where, a, b, c, d, e, f \in \mathbb{R},$$

and

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

If $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then by some calculations we can see that the inverse change of coordinates is given by,

$$x = (\frac{1}{(ad-bc)})(du - bv) - (\frac{1}{(ad-bc)})(de - bf)$$

$$y = (\frac{1}{(ad-bc)})(-cu + av) - (\frac{1}{(ad-bc)})(-ce + af)$$

This is why we require that $ad - bc \neq 0$.
Suppose, we consider the ellipse $V(x^2 + y^2 - 1)$ in the xy-plane. Under the real affine change of coordinates

$$u = x + y$$

and

$$v = 2x - y$$

The above ellipse becomes $V(5u^2 - 2uv + 2v^2 - 9)$ in the uv-plane. To change the coordinates from the xy-plane to the uv-plane, we can use the inverse change of coordinates,

$$x = \frac{1}{3}u + \frac{1}{3}v$$

$$y = \frac{2}{3}u - \frac{1}{3}v$$

Let us now look at the following 3 examples,

Example 1: For a given ellipse in the xy-plane, defined by $V((x-1)^2 + y^2 - 1)$, let us find an a real affine change of coordinates that maps the above ellipse to the ellipse in the uv-plane, given by $V(16u^2 + 9(v+2)^2 - 1)$.
Considering, the ellipse $V((x-1)^2 + y^2 - 1)$, let us look at the following inverse change of coordinates,

$$x = 3v + 7$$

and

$$y = 4u$$

Substituting the above change of coordinates in the equation of the ellipse given for the xy-plane,

$$(x-1)^2 + y^2 - 1 = (3v+6)^2 + (4u)^2 - 1$$
$$= 9v^2 + 36v + 36 + 16u^2 - 1 = 9(v+2)^2 + 16u^2 - 1$$

Hence, we get the equation of the given ellipse in the uv-plane. Therefore, the real affine change of coordinates will be as follows,

$$u = \frac{y}{4}$$

and

$$v = \frac{(x-7)}{3}$$

Example 2: For a given hyperbola in the xy-plane, defined by $V(8xy-1)$, let us find an a real affine change of coordinates that maps the above hyperbola to the ellipse in the uv-plane, given by $V(2u^2-2v^2-1)$. Considering, the hyperbola $V(8xy-1)$, let us look at the following inverse change of coordinates,

$$x = \frac{(u+v)}{2}$$

and

$$y = \frac{(u-v)}{2}$$

Substituting the above change of coordinates in the equation of the hyperbola given for the xy-plane,

$$8xy - 1 = 8\frac{(u+v)(u-v)}{4} - 1$$
$$= 2(u+v)(u-v) - 1 = 2(u^2-v^2) - 1 = 2u^2 - 2v^2 - 1$$

Hence, we get the equation of the given hyperbola in the uv-plane. Therefore, the real affine change of coordinates will be as follows,

$$u = (x+y)$$

and

$$v = (x - y)$$

Example 3: For a given parabola in the xy-plane, defined by $V(x^2 - y)$, let us find an a real affine change of coordinates that maps the above parabola to the ellipse in the uv-plane, given by $V(u^2 + 2uv + v^2 - u + v - 2)$.
Considering, the parabola $V(x^2 - y)$, let us look at the following inverse change of coordinates,

$$x = (u + v)$$

and

$$y = (u - v) + 2$$

Substituting the above change of coordinates in the equation of the parabola given for the xy-plane,

$$\begin{aligned} x^2 - y &= (u + v)^2 - (u - v) - 2 \\ &= u^2 + 2uv + v^2 - u + v - 2 \end{aligned}$$

Hence, we get the equation of the given parabola in the uv-plane. Therefore, the real affine change of coordinates will be as follows,

$$u = \frac{(x + y)}{2} - 1$$

and

$$v = \frac{(x - y)}{2} - 1$$

The preceding three examples show us that we can transform ellipses into ellipses, hyperbolas into hyperbolas and parabolas into parabolas by way of real affine change of coordinates.

Two conics are said to be equivalent under a real affine change of coordinates if the defining polynomial for one of the conics can be transformed via a real affine change of coordinates into the defining polynomial of the other conic.

Under a real affine change of coordinates, all ellipses in $\mathbb{R}^2$ are equivalent, all hyperbolas in $\mathbb{R}^2$ are equivalent, and all parabolas in $\mathbb{R}^2$ are equivalent. Further,these three classes of conics are distinct; no conic of one class can be transformed via a real affine change of coordinates to a conic of a different class.

2.3 Conics over the Complex Numbers

Let us look at the polynomial, $P(x,y) = x^2 + y^2 + 1$ and its zero set,

$$V(P) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + 1 = 0\}$$

Whenever $x, y \in \mathbb{R}$, we have,

$$x^2 \geq 0, y^2 \geq 0$$

and hence, $x^2 + y^2 \geq 0$.
Thus, $V(P) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + 1 = 0\}$ must be empty. Now, let $x, y \in \mathbb{C}$. We want to look at the solutions for $x^2 + y^2 + 1 = 0$. Thinking of this as a one-variable polynomial in the y-coordinate, treating the x as a constant. Then we can use the quadratic equation to find the roots:

$$\frac{\pm\sqrt{4(x^2+1)}}{2}$$

If $x = \pm i$, then

$$y = \frac{\pm\sqrt{4(i^2+1)}}{2} = 0,$$

a unique solution for y. If $x \neq \pm 1$, then,

$$\sqrt{4(x2+1)} \neq 0,$$

giving us two different solutions for y.
Thus, if we only allow a solution to be a real number, some zero sets of second degree polynomials will be empty. This does not happen over the complex numbers.
Example: Let $C = V(\frac{x^2}{4} + \frac{y^2}{9} - 1 \subset \mathbb{C})$. Show that C is unbounded in both x and y. (Over the complex numbers $\mathbb{C}$, being unbounded in x, say, means, given any number M, there will be point $(x,y) \in \mathbb{C}^2$ such that $|x| > M$.)
For any $(x,y) \in C$, we must have that $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$, and hence, by solving for x,

$$x = \pm\sqrt{16 - (\frac{4y^2}{9})}$$

If we were working over the real numbers, then we could only allow y's such that,

$$16 - (\frac{4y^2}{9}) \geq 0,$$

but since we are working now over the complex numbers, where square roots are always defined, there is a solution $x \in \mathbb{C}$ for any $y \in \mathbb{C}$, no matter how large is $|y|$. The same argument works for showing that we can let $|x|$ be arbitrarily large.

Hyperbolas in $\mathbb{R}^2$ come in two pieces. In $\mathbb{C}^2$, it can be shown that hyperbolas are connected, meaning there is a continuous path from any point to any other point. The following shows this for a specific hyperbola.

Example: Let $C = V(x^2 - y^2 - 0) \subset \mathbb{C}^2$. Show that there is a continuous path on the curve C from the point $(1,0)$ to the point $(1,0)$, despite the fact that no such continuous path exists in $\mathbb{R}^2$.
We explicitly find the path. For any point $(x,y) \in C$, we have that

$$y = \pm\sqrt{x^2 - 1}.$$

For any real number $1 < x < 1$, we have that $x^2 - 1 < 0$ and hence that y is purely imaginary. Define the map

$$\gamma : [-1,1] \to C$$

by setting

$$\gamma(t) = (t, i\sqrt{1 - x^2})$$

Since γ is a continuous function, we are done.

These two examples demonstrate that in $\mathbb{C}^2$ ellipses are unbounded (just like hyperbolas and parabolas) and suggest the true fact that hyperbolas are connected (just like ellipses and parabolas).

2.4 Complex affine change of coordinates

A complex affine change of coordinates in the complex plane $\mathbb{C}^2$ is given by

$$u = ax + by + e$$

$$v = cx + dy + f,$$

where $a, b, c, d, e, f \in \mathbb{C}$ and $ad - bc \neq 0$. In matrix language we have,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$where, a, b, c, d, e, f \in \mathbb{C},$$

and

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

If $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then by some calculations we can see that the inverse change of coordinates is given by,

$$x = (\frac{1}{(ad - bc)})(du - bv) - (\frac{1}{(ad - bc)})(de - bf)$$

$$y = (\frac{1}{(ad - bc)})(-cu + av) - (\frac{1}{(ad - bc)})(-ce + af)$$

This is why we require that $ad - bc \neq 0$.

We can easily see that any ellipse is equivalent to the circle $x^2 + y^2 = 1$ under a real affine change of coordinates and that any hyperbola is equivalent to the hyperbola $x^2 y^2 = 1$ under a real affine change of coordinates. All real affine changes of coordinates are also complex affine changes of coordinates. Finally we have explicitly found a complex affine change of coordinates from the circle $x^2 + y^2 = 1$ to the hyperbola $x^2 y^2 = 1$. Thus given any ellipse, first map it to the circle, then map the circle to $x^2 y^2 = 1$ and finally map this hyperbola to any other hyperbola. Since we know that compositions of complex affine changes of coordinates are still complex affine changes of coordinates, we are done.

Now we shall see how parabolas, though, are still different through the following example:

Example : Let us show that $\{(x,y) \in \mathbb{C}^2 : x^2+y^2-1=0\}$ is not equivalent under a complex affine change of coordinates to the parabola $\{(u,v) \in \mathbb{C}^2 : u^2v=0\}$.
We assume that there is a complex affine change of coordinates

$$x = au+bv+e$$
$$y = cu+dv+f,$$

for some constants $a,b,c,d,e,f \in \mathbb{C}$ with $ad-bc \neq 0$ that takes the points on the parabola $\{(u,v) \in \mathbb{C}^2 : u^2v=0\}$. to the points on the circle $\{(x,y) \in \mathbb{C}^2 : x^2+y^2-1=0\}$, and then derive a contradiction. We have,

$$\begin{aligned} 1 &= x^2+y^2 \\ &= (au+bv+e)^2+(cu+dv+f)^2 \end{aligned}$$

We now use that $u^2 = v$ to put all of the above in terms of the variable u alone, to get

$$1 = (b^2+d^2)u^4 + (2ab+2cd)u^3 + (a^2+c^2+2be+2df)u^2$$
$$+(2ae+2cf)u+e^2+f^2$$

This looks like a polynomial in one variable of degree of at most four, meaning that there will at most four solutions u, which is absurd, as there are an infinite number of points on both curves. The only way that this could happen if all on the above coefficients, except for e^2+f^2, are zero. In particular, we would need:

$$b^2+d^2=0$$
$$ab+cd=0.$$

We will show that if these are true, then $ad-bc=0$, giving us our contradiction. Now $b^2+d^2=0$ means that either $d=ib$ or $d=-ib$. This means that if $b=0$ then $d=0$, which in turn means that $ad-bc=0$, which can-

not happen. Thus we can assume $b \neq 0$.
Assuming that $d = ib$. Then we have

$$\begin{aligned} 0 &= ab + cd \\ &= ab + ibc \\ &= b(a + ic), \end{aligned}$$

which means that we must have $a + ic = 0$, or, in other words, $c = ia$. Then we have $adbc = iabiab = 0$, which is forbidden. Then we must have $d = -ib$, which means that

$$\begin{aligned} 0 &= ab + cd \\ &= ab - ibc \\ &= b(a - ic). \end{aligned}$$

In this case,

$$c = -ia,$$

giving us,

$$ad - bc = -iab + iab = 0,$$

which is still forbidden. Thus there is no complex affine change of coordinates taking $u^2 = v$ to $x^2 + y^2 - 1 = 0$.

We now want to look more directly at $\mathbb{C}^2$ in order to understand more clearly why the class of ellipses and the class of hyperbolas are different as real objects but the same as complex objects. We start by looking more closely at $\mathbb{C}$. Algebraic geometers regularly use the variable x for a complex number. Complex analysts more often use the variable z, which allows a complex number to be expressed in terms of its real and imaginary parts.

$$z = x + iy,$$

where x is the real part of z and y is the imaginary part. Similarly, an algebraic geometer will usually use (x, y) to denote points in the complex plane $\mathbb{C}^2$ while a complex analyst will instead use (z, w) to denote points in the complex plane $\mathbb{C}^2$. Here the complex analyst will write $w = u + iv$.

There is a natural bijection from $\mathbb{C}^2$ to $\mathbb{R}^4$ given by,

$$(z,w) = (x+iy, u+iv) \to (x,y,u,v).$$

In the same way, there is a natural bijection from

$$\mathbb{C}^2 \cap \{(x,y,u,v) \in \mathbb{R}^4 : y = 0, v = 0\}$$

to the real plane $\mathbb{R}^2$, given by

$$(x+0i, u+0i) \to (x,0,u,0) \to (x,u).$$

Likewise, there is a similar natural bijection from $\mathbb{C}^2 = \{(z,w) \in \mathbb{C}^2\} \cap \{(x,y,u,v) \in \mathbb{R}^4; y = 0, u = 0\}$ to $\mathbb{R}^2$, given this time by

$$(x+0i, 0+vi) \to (x,0,0,v) \to (x,v).$$

One way to think about conics in 2 is to consider two dimensional slices of $\mathbb{C}^2$. Let $C = \{(z,w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}$.

Example : Let us construct a bijection from

$$C \cap \{(x+iy, u+iv) : x, u \in \mathbb{R}, y = 0, v = 0\}$$

to the real circle of unit radius in $\mathbb{R}^2$. (Thus a real circle in the plane $\mathbb{R}^2$ can be thought of as real slice of the complex curve C.)
We want to find a one-to-one onto map from $C \cap \{(x+iy, u+iv) : x, u \in \mathbb{R}, y = 0, v = 0\}$ to the circle

$$\{(x,u) \in \mathbb{R}^2 : x^2 + u^2 = 1\}.$$

Now we have,

$$\begin{aligned} 1 &= z^2 + w^2 \\ &= (x+0i)^2 + (u+0i)^2 \\ &= x^2 + u^2, \end{aligned}$$

Thus the desired map is the straightforward

$$(x+i0, u+i0) \to (x,u).$$

Taking a different real slice of C will yield not a circle but a hyperbola.

Example : Let us construct a bijection from

$$C \cap \{(x+iy, u+iv) \in \mathbb{R}^4 : x, v \in \mathbb{R}, y = 0, u = 0\}$$

to the hyperbola $(x^2 - v^2 = 1)$ in $\mathbb{R}^2$.

We have

$$\begin{aligned} 1 &= z^2 + w^2 \\ &= (x + 0i)^2 + (o + iv)^2 \\ &= x^2 u^2, \end{aligned}$$

Thus the desired map is the straightforward

$$(x + i0, 0 + iv) \to (x, u).$$

Thus the single complex curve C contains both real circles and real hyperbolas.

Chapter 3

Projective Space

The goal of this chapter is to introduce the complex projective plane $\mathbb{P}^2$, which is the natural ambient space (with its higher dimensional analog $\mathbb{P}^n$) for much of algebraic geometry. In $\mathbb{P}^2$, we will see that all ellipses, hyperbolas, and parabolas are equivalent.We shall also define the complex projective line $\mathbb{P}^1$ and show that it can be viewed topologically as a sphere. In the next section we will use this to show that ellipses, hyperbolas, and parabolas are also topologically spheres. Then we will extend our study of conics from ellipses, hyperbolas, and parabolas to the "degenerate" conics: crossing lines and double lines. Next we develop the idea of singularity. We'll show that all ellipses, hyperbolas, and parabolas are smooth, while crossing lines and double lines are singular.

3.1 The Complex Projective Plane $\mathbb{P}^2$

We will give the definition for the complex projective plane $\mathbb{P}^2$ together with examples to demonstrate its basic properties. It may not be immediately clear what this definition has to do with the "ordinary" complex plane $\mathbb{C}^2$. We will then see how $\mathbb{C}^2$ naturally lives in $\mathbb{P}^2$ and how the "extra" points in $\mathbb{P}^2$ that are not in $\mathbb{C}^2$ are viewed as points at infinity. In the next section we will look at the projective analogue of change of coordinates and see how we can view all ellipses, hyperbolas and parabolas as equivalent.

Defining a relation $\sim$ on points in $\mathbb{C}^3 - \{(0,0,0)\}$ as follows: $(x,y,z) \sim (u,v,w)$ if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x,y,z) = (\lambda u, \lambda v, \lambda w)$.

For any point $(x,y,z) \in \mathbb{C}^3 - \{(0,0,0)\}$ we have $(x,y,z) = (\lambda x, \lambda y, \lambda z)$

with $\lambda = 1$, thus $\sim$ is reflexive.
To see that $\sim$ is symmetric, suppose $(x,y,z) \sim (u,v,w)$ so that $(x,y,z) = (\lambda u, \lambda v, \lambda w)$ for some $\lambda \neq 0$. Then $(u,v,w) = (\frac{1}{\lambda}x, \frac{1}{\lambda}y, \frac{1}{\lambda}z)$, thus $(u,v,w) \sim (x,y,z)$.
Next assume $(x,y,z) \sim (u,v,w)$ and $(u,v,w) \sim (r,s,t)$. Then there are $\lambda, \mu \in \mathbb{C} - \{0\}$ such that $(x,y,z) = (\lambda u, \lambda v, \lambda w)$ and $(u,v,w) = (\mu r, \mu s, \mu t)$. Substituting we obtain $(x,y,z) = (\lambda\mu r, \lambda\mu s, \lambda\mu t)$ where $\lambda\mu \in \mathbb{C} - \{0\}$. This shows that $(x,y,z) \sim (r,s,t)$ and therefore $\sim$ is transitive. Thus $\sim$ is an equivalence relation.

Suppose that $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$. Let us show that

$$(x_1,y_1,z_1) \sim (\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1) \sim (\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1) \sim (x2, y2, z2).$$

If $z_1 \neq 0$, then we can set $\lambda = z_1$ and we have

$$(x_1,y_1,z_1) = (z_1\frac{x_1}{z_1}, z_1\frac{y_1}{z_1}, z_1 1)$$

thus

$$(x_1,y_1,z_1) \sim (\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1).$$

By the same argument,

$$(x_2,y_2,z_2) \sim (\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1)$$

Since $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$,by transitivity,

$$(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1) \sim (\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1)$$

Let $(x:y:z)$ denote the equivalence class of (x,y,z), i.e. $(x:y:z)$ is the following set.

$$(x:y:z) = \{(u,v,w) \in \mathbb{C}^3 - \{(0,0,0)\} : (x,y,z) \sim (u,v,w)\}$$

The complex projective plane, $\mathbb{P}^2(\mathbb{C})$,is the set of equivalence classes of the points in $\mathbb{C}^3 - \{(0,0,0)\}$. That is,

$$\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3(0,0,0))/\sim.$$

The set of points $\{(x:y:z) \in \mathbb{P}^2(\mathbb{C}) : z = 0\}$ is called the line at infinity.

We will write $\mathbb{P}^2$ to mean $\mathbb{P}^2(\mathbb{C})$ when the context is clear.

Example : The elements of $\mathbb{P}^2$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^3$.

An element of $\mathbb{P}^2$ is an equivalence class $(a:b:c)$, the set whose elements have the form $(x,y,z) \in \mathbb{C}^3 - \{(0,0,0)\}$ with $x = \lambda a, y = \lambda b, z = \lambda z$ for some complex number $\lambda \neq 0$. These elements correspond to the points, other than $(0,0,0)$, on the line through (a,b,c) and the origin.

Example : If $c \neq 0$, show, in $\mathbb{C}^3$, that the line $x = \lambda a, y = \lambda b, z = \lambda c$ intersects the plane $\{(x,y,z) : z = 1\}$ in exactly one point. We will show that this point of intersection is $(\frac{a}{c}, \frac{b}{c}, 1)$.

We assume $c \neq 0$ and (x,y,z) is a point on both the line $x = \lambda a, y = \lambda b, z = \lambda c$ and the plane $z = 1$. Then $z = \lambda c = 1$ and solving for the parameter λ we obtain $\lambda = \frac{1}{c}$. Substituting this parameter value back into our equations for the line we have $(x,y,z) = (\frac{a}{c}, \frac{b}{c}, 1)$.

In the next several examples we will use

$$\mathbb{P}^2 = \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\} \cup \{(x:y:z) \in \mathbb{P}^2 : z = 0\}$$

to show that $\mathbb{P}^2$ can be viewed as the union of $\mathbb{C}^2$ with the *line at infinity*.

Example : Let us show that the map $\phi : \mathbb{C}^2 \to \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\}$ defined by $\phi(x,y) = (x:y:1)$ is a bijection.

We want to show that ϕ is one-to-one and onto.
To show this map is one-to-one, suppose $\phi((a,b)) = \phi((c,d))$. Then $(a:b:1) = (c:d:1)$. For these two equivalence classes to be equal, there must be a non-zero λ with $(a,b,1) = (\lambda c, \lambda d, \lambda)$. Therefore $\lambda = 1$, so we have $a = c, b = d$ and $(a,b) = (c,d)$. Thus ϕ is one-to-one.
To show that ϕ is onto, let $(a:b:c) \in \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\}$. Then $c \neq 0$, so we may set $\lambda = \frac{1}{c}$ and write $(\frac{a}{c} : \frac{b}{c} : 1) = (a:b:c)$. We then have $(\frac{a}{c}, \frac{b}{c}) \in \mathbb{C}^2$band $\phi((\frac{a}{c}, \frac{b}{c})) = (a:b:c)$. Thus ϕ is also onto.

Since, ϕ is a bijection so we know that an inverse exists. Starting with a point $(a:b:c) \in \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\}$ we can write $(a:b:c) = (\frac{a}{c} : \frac{b}{c} : 1)$ as in the proof that ϕ is onto. Then $\phi^{-1}((a:b:c)) = (\frac{a}{c}, \frac{b}{c})$.

Considering the line $L = \{(x,y) \in \mathbb{C}^2 : ax+by+c = 0\}$ in $\mathbb{C}^2$. Assume $a,b \neq 0$. Let us see why, as $|x| \to \infty, |y| \to \infty$. (Here, $|x|$ is the modulus of x.)

Let (x,y) be a point on the line L, so we may write $y = \frac{-ax-c}{b}$. Then $|y| = \frac{1}{|b|}|ax+c|$. As $|x| \to \infty, |ax+c| \to \infty$, thus $|y| \to \infty$.
Considering again the line L. We know that a and b cannot both be 0, so we will assume without loss of generality that $b \neq 0$.

1. Let us show that the image of L in $\mathbb{P}^2$ under ϕ is the set
$$\{(bx : axc : b) : x \in \mathbb{C}\}.$$
2. Let us show that this set equals the following union.
$$\{(bx : axc : b) : x \in \mathbb{C}\} = \{(0 : c : b)\} \cup \{(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x})\}$$
3. Let us show that as $|x| \to \infty$, the second set in the above union becomes
$$\{(1 : -\frac{a}{b} : 0)\}$$
.

Thus, the points $(1 : -\frac{a}{b} : 0)$ are directions toward infinity and the set $\{(x:y:z) \in \mathbb{P}^2 : z = 0\}$ is the *line at infinity*.

1. As in the previous problem we can write $(x, \frac{-ax-c}{b})$ for an arbitrary point on $\mathbb{L}$. Then,
$$\phi((x, \frac{-ax-c}{b})) = (x : \frac{-ax-c}{b} : 1) = (bx : -ax-c : b)$$
since $b \neq 0$. Therefore
$$\phi(L) = \{(bx : -ax-c : b) : x \in \mathbb{C}\}$$
.

2. Let $(bx : -ax-c : b) \in \phi(L)$ and first suppose $x = 0$. Then substituting gives $(bx : -ax-c : b) = (0 : -c : b)$.
   Otherwise $x \neq 0$; since $b \neq 0$ we have $(bx : -ax-c : b) = (1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x})$. Thus
   $$\{(bx : axc : b) : x \in \mathbb{C}\} = \{(0 : c : b)\} \cup \{(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x})\}$$
3. As $|x| \to \infty, |\frac{1}{x}| \to 0$, thus
   $$(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}) \to (1 : -\frac{a}{b} : 0)$$

In order to consider zero sets of polynomials in $\mathbb{P}^2$, a little care is needed. We shall define the following,

A polynomial is *homogeneous* if every monomial term has the same total degree, that is, if the sum of the exponents in every monomial is the same. The degree of the homogeneous polynomial is the degree of one of its monomials. An equation is homogeneous if every nonzero monomial has the same total degree.

If the degree n homogeneous equation $P(x,y,z) = 0$ holds for the point (x,y,z) in $\mathbb{C}^3$, then we shall see that it holds for every point of $\mathbb{C}^3$ that belongs to the equivalence class $(x : y : z)$ in $\mathbb{P}^2$.

Let us suppose P is a homogeneous polynomial of degree n such that $P(x,y,z) = 0$ holds for the point (x,y,z). Let $(a,b,c) \in (x : y : z)$ with $a = \lambda x, b = \lambda y, c = \lambda z$ for some $\lambda \in \mathbb{C} - \{0\}$. Since P is homogeneous, we can write,
$$P(x,y,z) = \Sigma \alpha_{ijk} x^i y^j z^k$$
where the sum is taken over all triples i, j, k where $0 \leq i, j, k \leq n$ and $i + j + k = n$. Substituting (a,b,c) into $P(x,y,z)$, we have
$$P(x,y,z) = \Sigma \alpha_{ijk} \lambda x^i \lambda y^j \lambda z^k$$
$$= \Sigma \alpha_{ijk} \lambda^{i+j+k} x^i y^j z^k = \lambda^n \Sigma \alpha_{ijk} x^i y^j z^k$$
Thus,
$$P(a,b,c) = \lambda^n P(x,y,z) = 0.$$
Let us see the following problem
Example : Considering the non-homogeneous equation $P(x,y,z) = x^2 + 2y + 2z = 0$. Let us show that $(2,-1,-1)$ satisfies this equation, but not all

other points of the equivalence class $(2:-1:-1)$ satisfy the equation.

$P(2,-1,-1) = 4-2-2 = 0$. Take, for example, $(-2,1,1) \in (2:-1:-1)$. We have $P(-2,1,1) = 4+2+2 = 8$.
More generally, a point in the equivalence class $(2:-1:-1)$ will have the form $(2\lambda,-\lambda,-\lambda)$ for some non-zero complex number λ. Substituting into $P(x,y,z)$ we have $P(2\lambda,-\lambda,-\lambda) = 4\lambda^2 - 2\lambda - 2\lambda = 4\lambda(\lambda 1) \neq 0$ when $\lambda \neq 0,1$.

Thus the zero set of a non-homogeneous polynomials is not well- defined in $\mathbb{P}^2$. We can easily demonstrate that the only polynomials that are well-defined on $\mathbb{P}^2$ (and any projective space $\mathbb{P}^n$) are homogeneous polynomials.

In order to study the behavior at infinity of a curve in $\mathbb{C}^2$, we would like to extend the curve to $\mathbb{P}^2$. In order for the zero set of a polynomial over $\mathbb{P}^2$ to be well-defined we must, for any given a polynomial on $\mathbb{C}^2$, replace the original (possibly non-homogeneous) polynomial with a homogeneous one.
We start with an example. With a slight abuse of notation, the polynomial $P(x,y) = y-x-2$ maps to $P(x,y,z) = \frac{y}{z} - \frac{x}{z} - 2$. Since $P(x,y,z) = 0$ and $zP(x,y,z) = 0$ have the same zero set if $z \neq 0$ we clear the denominator and consider the polynomial $P(x,y,z) = y-x-2z$. The zero set of $P(x,y,z) = y-x-2z$ in $\mathbb{P}^2$ corresponds to the zero set of $P(x,y) = y-x-2 = 0$ in $\mathbb{C}^2$ precisely when $z = 1$.

Similarly, the polynomial x^2+y^2-1 maps to $(\frac{x}{z})^2 + (\frac{y}{z})^2 - 1$. Again, clear the denominators to obtain the homogeneous polynomial $x^2+y^2-z^2$, whose zero set,$V(x^2+y^2-z^2) \subset \mathbb{P}^2$ corresponds to the zero set, $V(x^2+y^2-1) \subset \mathbb{C}^2$ when $z = 1$.

Let $P(x,y)$ be a degree n polynomial defined over $\mathbb{C}^2$. The corresponding homogeneous polynomial defined over $\mathbb{P}^2$ is

$$P(x,y,z) = z^n P(\frac{x}{z}, \frac{y}{z}).$$

In $\mathbb{P}^2$, any two distinct lines will intersect in a point. Notice, this implies that parallel lines in $\mathbb{C}^2$, when embedded in $\mathbb{P}^2$, intersect at the line at infinity.
We know in the affine plane, two distinct lines are either parallel or inter-

sect in a point. Thus we need to show that parallel affine lines will meet in the projective plane. Let $ax+by+c=0$ and $dx+ey+f=0$ be two affine lines, which homogenize to $ax+by+cz=0$ and $cx+dy+ez=0$ in the projective plane. Since the affine lines are parallel, either $b=e=0$ or $\frac{a}{b}=\frac{c}{d}$. The projective lines will intersect at $(0:1:0)$ in the first case, and at $(-b:a:0)$ in the second.

Once we have homogenized an equation, the original variables x and y are no more important than the variable z. Suppose we regard x and z as the original variables in our homogenized equation. Then the image of the xz-plane in $\mathbb{P}^2$ would be $\{(x:y:z)\in\mathbb{P}^2 : y=1\}$.

3.2 Projective Change of Coordinates

A *projective change of coordinates* is given by

$$u = a_{11}x + a_{12}y + a_{13}z$$

$$v = a_{21}x + a_{22}y + a_{23}z$$

$$w = a_{31}x + a_{32}y + a_{33}z$$

where the $a_{ij}\in\mathbb{C}$ and

$$det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0$$

In matrix language,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $A=(a_{ij}), a_{ij}\in\mathbb{C}$, and $\det A\neq 0$.

Example : Let $C_1 = V(x^2+y^2-1)$ be an ellipse in $\mathbb{C}^2$ and let $C_2 = V(u^2-v)$ be a parabola in $\mathbb{C}^2$. Homogenize the defining polynomials for C_1 and C_2 and show that the projective change of coordinates

$$u = ix$$

$$v = y+z$$

$$w = y-z$$

transforms the ellipse in $\mathbb{P}^2$ into the parabola in $\mathbb{P}^2$.

The homogenized polynomials are $f_1 = x^2 + y^2 - z^2$ and $f_2 = u^2 - vw$ respectively. If we solve the above system for x, y and z we have

$$x = \frac{u}{i} = -ui$$
$$y = \frac{v+w}{2}$$
$$z = \frac{v-w}{2}$$

If we substitute these variables into f_1, we have

$$\begin{aligned} x^2 + y^2 - z^2 &= (-ui)^2 + (\frac{v+w}{2})^2 - (\frac{v-w}{2})^2 \\ &= -u^2 + \frac{1}{4}(v^2 + 2vw + w^2 - (v^2 - 2vw + w^2)) \\ &= -u^2 + vw \\ &= -(u^2 - vw) \end{aligned}$$

We may homogenize the affine change of coordinates to obtain a projective change of coordinates. The previous example shows that an ellipse is equivalent to a parabola under a projective change of coordinates. Therefore all ellipses, hyperbolas and parabolas are projectively equivalent.

3.3 The Complex Projective Line $\mathbb{P}^1$

Defining an equivalence relation $\sim$ on points in $\mathbb{C}^2 - \{(0,0)\}$ as follows: $(x,y) \sim (u,v)$ if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x,y) = (\lambda u, \lambda v)$. Let $(x : y)$ denote the equivalence class of (x, y). The *complex projective line* $\mathbb{P}^1$ is the set of equivalence classes of points in $\mathbb{C}^2 - \{(0,0)\}$. That is,

$$\mathbb{P}^1 = (\mathbb{C}^2 - \{(0,0)\})/\sim.$$

The point $(1 : 0)$ is called the *point at infinity*.

The elements of $\mathbb{P}^1$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^2$.
Let $(a,b) \in \mathbb{C}^2 - \{(0,0)\}$. Then the complex line through this point and

the origin $(0,0)$ can be described as all points (x,y) satisfying

$$x = \lambda a, y = \lambda b$$

for any complex number λ. Here λ can be thought of as an independent parameter. The point $(a:b) \in \mathbb{P}^1$ corresponds to the points $(\lambda a, \lambda b) \in \mathbb{C}^2$, which are indeed precisely the actual points on the line through the point (a,b) and the origin $(0,0)$, as desired.

Example : Considering the map $\phi : \mathbb{C} \to \mathbb{P}^1$ given by $\phi(x) = (x:1)$. Show that as $|x| \to \infty$, we have $\phi(x) \to (1:0)$.

We have,

$$\begin{aligned}
\lim_{|x|\to\infty} \phi(x) &= \lim_{|x|\to\infty} (x:1) \\
&= \lim_{|x|\to\infty} (1:\frac{1}{x}) \\
&= (1:0)
\end{aligned}$$

Hence we can think of $\mathbb{P}^1$ as the union of $\mathbb{C}$ and a single point at infinity. Now we want to see how we can regard $\mathbb{P}^1$ as a sphere, which means we want to find a *homeomorphism* between $\mathbb{P}^1$ and a sphere. A homeomorphism is a continuous map with a continuous inverse. Two spaces are topologically equivalent, or homeomorphic, if we can find a homeomorphism from one to the other. We know that the points of $\mathbb{C}$ are in one-to-one correspondence with the points of the real plane $\mathbb{R}^2$, so we will first work in $\mathbb{R}^2 \subset \mathbb{R}^3$. Let S^2 denote the unit sphere in $\mathbb{R}^3$ centered at the origin. This sphere is given by the equation

$$x^2 + y^2 + z^2 = 1.$$

Example : Let p denote the point $(0,0,1) \in S^2$, and let l denote the line through p and the point $(x,y,0)$ in the xy-plane, whose parametrization is given by

$$\gamma(t) = (1-t)(0,0,1) + t(x,y,0),$$

i.e.,

$$l = \{(tx, ty, 1-t) | t \in \mathbb{R}\}.$$

1. l clearly intersects S^2 at the point p. Show that there is exactly one other point of intersection q.

2. Find the coordinates of q.
3. Define the map $\psi : \mathbb{R}^2 \to S^2 - \{p\}$ to be the map that takes the point (x,y) to the point q. Show that ψ is a continuous bijection.
4. Show that as $|(x,y)| \to \infty$, we have $(x,y) \to p$.

We want to find the values of the real parameter t such that $\gamma(t) \in S^2$. Now the coordinates of the points on the line are given by

$$((tx,ty,(1-t)) \in l.$$

Thus we must find the real numbers t such that

$$(tx)^2+(ty)^2+(1-t)^2 = 1,$$

where x and y are fixed real numbers. Thus we must solve the quadratic

$$(x^2+y^2+1)t^2-2t = 0$$

and thus find the roots of

$$t(((x^2+y^2+1)t-2) = 0.$$

Certainly $t = 0$ is a root, which is the point $p = (0,0,1)$. The other root is a solution of

$$(x^2+y^2+1)t-2 = 0$$

and is hence

$$t = \frac{2}{x^2+y^2+1}.$$

Thus the other point of intersection is,

$$\begin{aligned}
\gamma(t) &= \gamma(\frac{2}{x^2+y^2+1}) \\
&= ((\frac{2}{x^2+y^2+1})x, (\frac{2}{x^2+y^2+1})y, 1-(\frac{2}{x^2+y^2+1})) \\
&= (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})
\end{aligned}$$

We thus define $\psi : \mathbb{R}^2 \to S^2 - \{p\}$ by setting

$$\psi(x,y) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$$

Since each of the components of ψ is continuous, the map ψ is continuous.

For part (4), we have

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2x}{x^2+y^2+1} = 0$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2y}{x^2+y^2+1} = 0$$

since both of the numerators grow linearly in x and y while the denominators grow quadratically in x and y. Also, we have

$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2-1}{x^2+y^2+1}$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{1-\frac{1}{x^2+y^2}}{1+\frac{1}{x^2+y^2}} = 1.$$

Thus as $|(x,y)| \to \infty$, we have $(x,y) \to p = (0,0,1)$.

This map from $S^2 - (0,0,1) \to \mathbb{R}^2$ is called the *stereographic projection* from the sphere to the plane. Note that the south pole $(0,0,1)$ on S^2 maps to the origin $(0,0)$ in $\mathbb{R}^2$. Also, there is an analogous map from $S^2 - (0,0,-1) \to \mathbb{R}^2$, where here it is the north pole $(0,0,1)$ on S^2 that maps to the origin $(0,0)$ in $\mathbb{R}^2$.

Let us see that $\mathbb{P}^1$ is homeomorphic to S^2.

We know that $\mathbb{P}^1 = \{(z:1) \in \mathbb{P}^1 : z \in \mathbb{C}\} \cup \{(1:0)\}$.Define

$$\alpha : \mathbb{P}^1 \to S^2,$$

using the notation of the previous exercise, by setting

$$\alpha(1:0) = p$$

where $p = (0,0,1)$ and

$$\begin{aligned}\alpha(z:1) &= \alpha(x+iy:1)\\ &= \psi(x,y)\\ &= \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)\end{aligned}$$

From the previous examples, we know that this is our desired homeomorphism.

The above argument does establish a homeomorphism, but it relies on co-

ordinates and an embedding of the sphere in $\mathbb{R}^3$. We now give an alternative method for showing that $\mathbb{P}^1$ is a sphere that does not rely as heavily on coordinates.
If we take a point $(x:y) \in \mathbb{P}^1$, then we can choose a representative for this point of the form $(\frac{x}{y}:1)$, provided $y \neq 0$, and a representative of the form $(1:\frac{y}{x})$, provided $x \neq 0$.
There are two points, namely $(1:0)$ and $(0:1)$ in $\mathbb{P}^1$ which do not have two representatives of the form $(x:1) = (1:\frac{1}{x})$.

Example : Our constructions needs two copies of $\mathbb{C}$. Let U denote the first copy of $\mathbb{C}$, whose elements are denoted by x. Let V be the second copy of $\mathbb{C}$, whose elements we'll denote y. Further let $U^* = U - \{0\}$ and $V^* = V - \{0\}$.
Mapping $U \to \mathbb{P}^1$ via $x \to (x:1)$ and map $V \to \mathbb{P}^1$ via $y \to (1:y)$. Show that there is a the natural one-to-one map $U^* \to V^*$.

The desired map $\mu : U^* \to V^*$ is,

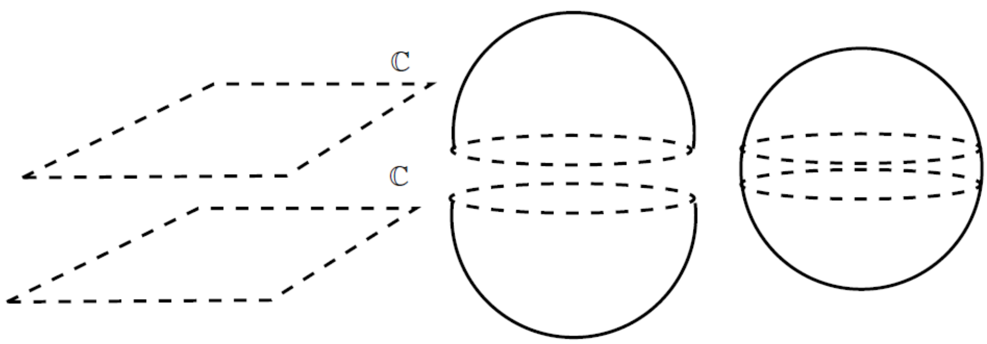
$$\mu(x) = \frac{1}{x}.$$

Note that μ is the composition of:

$$x \to (x:1) = (1:\frac{1}{x}) \to \frac{1}{x}.$$

A sphere can be split into a neighborhood of its northern hemisphere and a neighborhood of its southern hemisphere. Let us show that a sphere can be obtained by correctly gluing together two copies of $\mathbb{C}$.

We can identify S^2 - north pole with $\mathbb{R}^2$ which in turn can be identified with $\mathbb{C}$. The origin of $\mathbb{C}$ will map to the south pole. Similarly we also can identify S^2 - south pole with $\mathbb{R}^2$, which of course can be identified with another copy of $\mathbb{C}$. Here the origin of $\mathbb{C}$ will map to the north pole of S^2.



From the last example, we have a bijective map $U \to \mathbb{P}^1$ via $x \to (x:1)$ and a bijective map $V \to \mathbb{P}^1$ via $y \to (1:y)$. But we also have a bijective map from U to S^2 north pole and a bijective map from V to S^2 south pole. Putting these maps together we can say that $\mathbb{P}^1$ is topologically equivalent to a sphere. Note that the south pole will correspond to $(0:1)$ and the north pole with $(1:0)$.

3.4 Ellipses, Hyperbolas, and Parabolas as Spheres

Considering a conic $C = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\} \subset \mathbb{C}^2$ where $P(x,y)$ is a second degree polynomial. Our goal is to parametrize C with polynomial or rational maps. This means we want to find a map $\phi : \mathbb{C} \to C \subset \mathbb{C}^2$, given by $\phi(\lambda) = (x(\lambda), y(\lambda))$ such that $x(\lambda)$ and $y(\lambda)$ are polynomials or rational functions. In the case of a parabola, for example when $P(x,y) = x^2 - y$, it is easy to find a bijection from $\mathbb{C}$to the conic C.

For example : $\lambda \to (\lambda, \lambda^2)$, one-to-one: if $(\lambda, \lambda^2) = (\mu, \mu^2)$ then $\lambda = \mu$, onto: if $(x,y) \in C$ then $y = x^2$ so $x \to (x, x^2)$.

Example : Considering the ellipse $C = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$ and let p denote the point $(0,1) \in C$.

The slope of the line is $\frac{-1}{\lambda}$ so the equation of the line is $y = \frac{-1}{\lambda}x + 1$.
A parameterization for the line segment is then $(t, \frac{-1}{\lambda}t + 1)$ with t running from 0 to λ.
Substituting $y = \frac{-1}{\lambda}x + 1$ into $x^2 + y^2 - 1 = 0$ and solve for x to find that one solution is $x = 0$ which corresponds to the point p and the other solution is $x = \frac{2\lambda}{\lambda^2+1}$.
Solving for the y value to find the coordinates of $q : (\frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2-1}{\lambda^2+1})$
If $\lambda = \pm i$ when we substitute $y = \frac{-1}{\lambda}x + 1$ into $x^2 + y^2 - 1 = 0$ we'll find $2\lambda x = 0$ and so the only solution is $x = 0$ which corresponds to the point p.

Defining the map $\psi^* : \mathbb{C} \to C \subset \mathbb{C}^2$ by

$$\psi^*(\lambda) = (\frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1}).$$

We shall show that the above map can be extended to the map $\psi : \mathbb{P}^1 \to$

$\{(x:y:z) \in \mathbb{P}^2 : x^2+y^2-z^2=0\}$ given by

$$\psi(\lambda:\mu) = (2\lambda\mu : \lambda^2-\mu^2 : \lambda^2+\mu^2)$$

Restricting ψ to the affine chart $\mu = 1$ and scale the point in $\mathbb{P}^2$ that results by $\frac{1}{\lambda^2+1}$. We obtain,

$$(\lambda:1) \to (\frac{2\lambda}{\lambda^2+1} : \frac{\lambda^2-1}{\lambda^2+1} : 1)$$

which agrees with the map ψ^* on the affine copy of $\mathbb{C}$ that corresponds to $\mu = 1$.
We can easily observe that the map ψ is one-one and onto.

3.5 Degenerate Conics - Crossing lines and double lines

Considering the second degree polynomial

$$\begin{aligned} f(x,y,z) &= (-x+y+z)(2x+y+3z) \\ &= -2x^2+y^2+3z^2+xy-xz+4yz. \end{aligned}$$

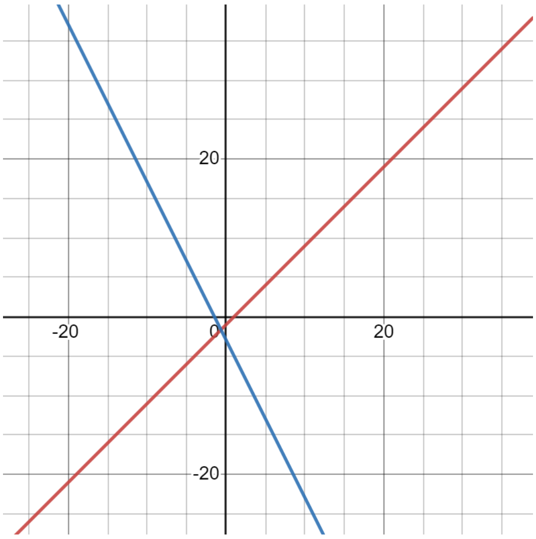
Dehomogenizing $f(x,y,z)$ by setting $z = 1$. Graph the curve

$$C(\mathbb{R}) = \{(x:y:z) \in \mathbb{P}^2 : f(x,y,1) = 0\}$$

in the real plane $\mathbb{R}^2$.
We have

$$\begin{aligned} f(x,y,1) &= (-x+y+1)(2x+y+3) \\ &= -2x^2+y^2+3+xy-x+4y. \end{aligned}$$

From the factored from we see that the graph is two crossing lines: $y = x-1$ and $y = -2x-3$.



Consider the two lines given by

$$(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)=0,$$

and suppose

$$det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$$

We can show that the two lines intersect at a point where $z \neq 0$.
Consider the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$. Since $det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$ it is easy to see this matrix will have full rank and hence non-trivial solution. In particular z will be the free variable. More explicitly if we perform row reduction we find the reduced row echelon form of the matrix is $\begin{pmatrix} 1 & 0 & \frac{-c_2b_2+b_1c_2}{b_2a_1-b_1a_2} \\ 0 & 1 & \frac{c_2a_1-a_2c_1}{b_2a_1-b_1a_2} \end{pmatrix}$.
Notice the non-zero determinant appears in the denominator of the entries in the third column. It is a simple matter to find the intersection for any value of $z \neq 0$.

Again, dehomogenizing the equation by setting $z=1$. Give an argument that, as lines in the complex plane $\mathbb{C}$, they have distinct slopes.
The slope of the line $a_1x+b_1y+c_1=0$ is $\frac{-a_1}{b_1}$. The slope of the line $a_2x+b_2y+c_2=0$ is $\frac{-a_2}{b_2}$. If $\frac{-a_1}{b_1}=\frac{-a_2}{b_2}$ then $a_1b_2-a_2b_1=0$ but by assumption the determinant is non-zero.

There is one other possibility. Consider the zero set

$$C = \{(x : y : z) \in \mathbb{P}^2 : (ax + by + cz)^2 = 0\}.$$

As a zero set, the curve C is geometrically the line

$$ax + by + cz = 0$$

but due to the exponent 2, we call C a *double line*.

Let

$$f(x, y, z) = (a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z),$$

where at least one of a_1, b_1, or c_1 is nonzero and at least one of the a_2, b_2, or c_2 is nonzero. Let us show that the curve defined by $f(x, y, z) = 0$ is a double line if and only if

$$det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0, det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} = 0, det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = 0$$

If f is a double line then $a_1 = ka_2, b_1 = kb_2$, and $c_1 = kc_2$ for some non-zero value of k. Clearly all the indicated determinants are then zero. Conversely if all three determinants are zero then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$ for some nonzero value of k. Then we can write $f(x, y, z) = (a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z)$ as $f(x, y, z) = k(ax + by + cz)^2$.

We now want to show that any two crossing lines are equivalent under a projective change of coordinates to any other two crossing lines and any double line is equivalent under a projective change of coordinates to any other double line. This means that there are precisely three types of conics: the ellipses, hyperbolas, and parabolas; pairs of lines; and double lines.

Example : Considering the crossing lines

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = 0,$$

with

$$det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$$

Let us find a projective change of coordinates from xyz-space to uvw-space so that the crossing lines become $uv = 0$.

We want to find a matrix M such that

$$M\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

gives $a_1x+b_1y+c_1z = u$ and $a_2x+b_2y+c_2z = v$ with this change of variables. If $d = det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$
we find

$$M = \frac{1}{d}\begin{pmatrix} b_2 - c_1b_2 + b_1c_2 & -b_1 - c_1b_2 + b_1c_2 & -c_1b_2 + b_1c_2 \\ -a_2 - c_2a_1 + a_2c_1 & a_1 - c_2a_1 + a_2c_1 & -c_2a_1 + a_2c_1 \\ d & d & d \end{pmatrix}$$

Example : Let us show that there is a projective change of coordinates from xyz-space to uvw-space so that the double line $(ax+by+cz)^2 = 0$ becomes the double line

$$u^2 = 0.$$

The tricky part here is finding a transformation matrix whose determinant is non-zero. If two of a, b, c are zero then simply renaming the appropriate variable u. Assuming then that two of a, b, c are non-zero, without loss of generality we'll assume a and c are non-zero. Solving systems similar to the two previous examples we'll have two free variables this time. One possible transformation is

$$\begin{pmatrix} \frac{1}{a} - \frac{b}{a} - \frac{c}{a} & -\frac{b}{a} - \frac{c}{a} & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -\frac{a}{c} - \frac{b}{c} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The determinant of this matrix is $\frac{-a+b+c}{ac}$. If $a+b+c = 0$, then there are two cases. If $b = 0$ then $ax+by+cz = a(x-z)$ and we rename $x-z$, u. If $b \neq 0$, we find a matrix analogous to the one given above, but with the assumption that a and b are non-zero.

Hence, we have seen that ellipses, parabola, and hyperbola are equivalent under projective transformations. In this section we have seen that crossed lines are double lines are distinct.

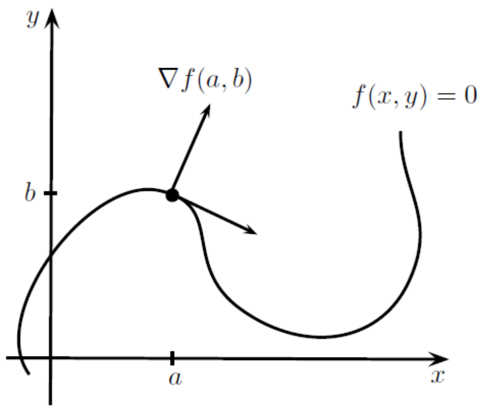
3.6 Tangents and Singular Points

Let $f(x,y)$ be a polynomial. Recall that if $f(a,b) = 0$, then the normal vector for the curve $f(x,y) = 0$ at the point (a,b) is given by the gradient vector

$$\nabla f(a,b) = (\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)$$

A tangent vector to the curve at the point (a,b) is perpendicular to $\nabla f(a,b)$ and hence must have a dot product of zero with $\nabla f(a,b)$. This observation shows that the tangent line is given by,

$$\{(x,y) \in \mathbb{C}^2 : (\frac{\partial f}{\partial x}(a,b))(x-a) + (\frac{\partial f}{\partial y}(a,b))(y-b) = 0\}$$



If both $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$, then every vector is orthogonal to ∇. Thus the direction of the tangent line is not unique, thus the tangent line cannot be well-defined.

Definition : A point $p = (a,b)$ on a curve $C = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$ is said to be *singular* if
$\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$
A point that is not singular is called *smooth*. If there is at least one singular point on C, then curve C is called a *singular curve*. If there are no singular points on C, the curve C is called a *smooth curve*.

Definition : A point $p = (a:b:c)$ on a curve $C = \{(x:y:z) \in \mathbb{P}^2 : f(x,y,z) = 0\}$, where $f(x,y,z)$ is a homogeneous polynomial, is said to be *singular* if

$$\frac{\partial f}{\partial x}(a,b,c) = 0, \frac{\partial f}{\partial y}(a,b,c) = 0, \frac{\partial f}{\partial z}(a,b.c) = 0$$

We have similar definitions, as before, for smooth point, smooth curve, and singular curve.

Example : Given,

$$C = \{(x:y:z) \in \mathbb{P}^2 : (x+y-z)(x-y-z) = 0\}$$

. We have $\frac{\partial f}{\partial x} = (x-y-z)+(x+y-z) = 2x-2z$, $\frac{\partial f}{\partial y} = (x-y-z)-(x+y-z) = -2y$ and $\frac{\partial f}{\partial z} = -(x-y-z)-(x+y-z) = 2z-2x$. This system has solution $y=0$ and $x=z$. We can scale this so that the singular point is $(1:0:1)$. Hence, the above pair of crossing lines have only one singular point.

Example : Given,

$$C = \{(x:y:z) \in \mathbb{P}^2 : (2x+3y-4z)^2 = 0\}$$

We have $\frac{\partial f}{\partial x} = 4(2x+3y-4z)$, $\frac{\partial f}{\partial y} = 6(2x+3y-4z)$ and $\frac{\partial f}{\partial z} = -8(2x+3y-4z)$. Every point on the curve satisfies the equation $2x+3y-4z=0$, so every point on the double line is singular.

For homogeneous polynomials, there is a clean relation between f, $\frac{\partial f}{\partial x}$,$\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ which we shall see.

Example : Considering, $f(x,y,z) = x^2+3xy+5xz+y^2-7yz+8z^2$, we see that

$$2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

Let $f(x,y,z) = x^2+3xy+5xz+y^2-7yz+8z^2$. Then

$$\frac{\partial f}{\partial x} = 2x+3y+5z$$

$$\frac{\partial f}{\partial y} = 3x+2y-7z$$

$$\frac{\partial f}{\partial z} = 5x-7y+16z$$

This means

$$x\frac{\partial f}{\partial x} = 2x^2+3xy+5xz$$

$$y\frac{\partial f}{\partial y} = 3xy + 2y^2 - 7yz$$
$$z\frac{\partial f}{\partial z} = 5xz - 7yz + 16z^2$$

And we have

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} &= 2x^2 + 6xy + 10xz + 2y^2 - 14yz + 16z^2 \\ &= 2f(x,y,z) \end{aligned}$$

Now, let us see what happens if $f(x,y,z)$ is a homogeneous polynomial of degree n.

Considering, $f(x,y,z)$ is a homogeneous polynomial of degree n. From the rules of differentiation, we only need to verify this for monic monomials. Consider this with a monomial of the form $x^j y^k z^l$, where $j + k + l = n$. Computing the partial derivatives yields

$$\frac{\partial f}{\partial x} = jx^{j-1}y^k z^l$$
$$\frac{\partial f}{\partial y} = kx^j y^{k-1} z^l$$
$$\frac{\partial f}{\partial z} = lx^j y^k z^{l-1}$$

This means

$$x\frac{\partial f}{\partial x} = jx^j y^k z^l$$
$$y\frac{\partial f}{\partial y} = kx^j y^k z^l$$
$$z\frac{\partial f}{\partial z} = lx^j y^k z^l$$

And we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = (j+k+l)x^j y^k z^l = nx^j y^k z^l = nf(x,y,z)$$

Example : Suppose that $f_1(a,b) = 0$, and $f_2(a,b) = 0$ for a point $(a,b) \in \mathbb{C}^2$

Let $f(x,y) = f_1 f_2$ and let $(a,b) \in \mathbb{C}^2$ with $f_1(a,b) = 0 = f_2(a,b)$. Now

$\frac{\partial f}{\partial x} = f_2 \frac{\partial f_1}{\partial x} + f_1 \frac{\partial f_2}{\partial x}$. . We have $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f_1}{\partial x}\Big|_{(a,b)} f_2(a,b) + f_1(a,b) \frac{\partial f_2}{\partial x}\Big|_{(a,b)} =$ 0.
Similarly, $\frac{\partial f}{\partial y} = 0$. Therefore, every point on the intersection of the two curves is a singular point.

All ellipses, hyperbolas and parabolas are smooth curves. All conics that are crossing lines have exactly one singular point, namely the point of intersection of the two lines. Every point on a double line is singular.

Example : Considering the curve

$$C = \{(u : v : w) \in \mathbb{P}^2 : u^2 - v^2 - w^2 = 0\}$$

. Suppose we have the projective change of coordinates given by

$$u = x + y$$

$$v = x - y$$

$$w = z.$$

We can see that C corresponds to the curve

$$C^* = \{(x : y : z) \in \mathbb{P}^2 : 4xy - z^2 = 0\}$$

. In other words, if $f(u,v,w) = u^2 - v^2 - w^2$, then $f^*(x,y,z) = 4xy - z^2$.

We have

$$\begin{aligned} u^2 - v^2 - w^2 &= (x+y)^2 + (x-y)^2 - z^2 \\ &= x^2 + 2xy + y^2 - (x^2 - 2xy + y2) - z^2 \\ &= 4xy - z^2 \end{aligned}$$

Therefore $f(u,v,w) = f^*(x,y,z)$.

We now want to show, under a projective change of coordinates, that singular points go to singular points and smooth points go to smooth points.

Example : Let

$$u = a_{11}x + a_{12}y + a_{13}z$$

$$v = a_{21}x + a_{22}y + a_{23}z$$

$$w = a_{31}x + a_{32}y + a_{33}z$$

be a projective change of coordinates. We shall see that $(u_0 : v_0 : w_0)$ is a singular point of the curve $C = \{(u : v : w) : f(u,v,w) = 0\}$ if and only if the corresponding point $(x_0 : y_0 : z_0)$ is a singular point of the corresponding curve $C^* = \{(x : y : z) : f^*(x,y,z) = 0\}$.

Since the inverse of a projective change of coordinates is also a projective change of coordinates, we can observe this for one direction and the converse will follow. Let $(u0 : v0 : w0) \in C$ be a singular point, so $\frac{\partial f}{\partial u} = 0, \frac{\partial f}{\partial v} = 0$ and $\frac{\partial f}{\partial w} = 0$. Consider $f^*(x,y,z)$. Now,

$$\begin{aligned} \frac{\partial f^*}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ &= a_{11}\frac{\partial f}{\partial u} + a_{21}\frac{\partial f}{\partial v} + a_{31}\frac{\partial f}{\partial w} \\ &= 0 \end{aligned}$$

Similarly, we can compute $\frac{\partial f^*}{\partial y} = 0$ and $\frac{\partial f^*}{\partial z} = 0$. Therefore a singular point is mapped to a singular point under a projective change of coordinates.

If we have an ellipse, hyperbola or parabola, then we have seen that they are projectively equivalent and there is no singular point on the curve. In the case of two lines crossing, there is one singular point,which is where the lines cross. In the case of a double line, every point on the line is singular.

Chapter 4

Higher Degree Curves

In this chapter we explore higher degree curves in $\mathbb{P}^2$.We define what it means for a curve to be irreducible and define the degree of a curve. We next show how curves in $\mathbb{P}^2$ can be thought of as real surfaces. Then, we introduce genus of a surface, which is an invariant that helps in the study of curves.

4.1 Higher Degree Polynomials and Curves

By now, we know that it is most natural to work in the complex projective plane, $\mathbb{P}^2$, which means in turn that we want our zero sets to be the zero sets of homogeneous polynomials. Suppose that $P(x,y,z) \in \mathbb{C}[x,y,z]$ is a homogeneous polynomial. We denote this polynomial's zero set by

$$V(P) = \{(a:b:c) \in \mathbb{P}^2 : P(a,b,c) = 0\}.$$

Example : Let $P(x,y,z) = (x+y+z)(x^2+y^2-z^2)$. We shall see that $V(P)$ is the union of the two curves $V(x+y+z)$ and $V(x^2+y^2-z^2)$.

Let $(a:b:c) \in V(P)$. Then we know that

$$0 = P(a,b,c) = (a+b+c)(a^2+b^2-c^2)$$

which can happen if and only if $a+b+c = 0 or a^2+b^2-c^2 = 0$, which in turn means that

$$(a:b:c) \in V(x+y+z) \cup V(x^2+y^2-z^2).$$

Thus, if we want to understand $V(P)$, we should start with looking at its two components: $V(x+y+z)$ and $V(x^2+y^2-z^2)$.

Example : Let $P(x,y,z) = (x+y+z)^2$. We may observe that $V(P) = V(x+y+z)$.

Let $(a:b:c) \in V(P)$. Then we know that

$$0 = P(a,b,c) = (a+b+c)^2$$

which can happen if and only if $a+b+c = 0$, which in turn means that

$$(a:b:c) \in V(x+y+z).$$

Both $(x+y+z)(x^2+y^2-z^2)$ and $(x+y+z)^2$ are *reducible*, meaning that both can be factored. We would prefer, for now, to restrict our attention to curves that are the zero sets of irreducible homogeneous polynomials.

Definition : If the defining polynomial P cannot be factored, we say the curve $V(P)$ is *irreducible*. When we are considering a factorization, we do not consider trivial factorizations, such as $P = 1 \cdot P$. For the rest of this chapter, all polynomials we use to define curves will be irreducible unless otherwise indicated.

Definition : The *degree* of the curve $V(P)$ is the degree of its defining polynomial, P.

The degree of a curve is the most basic number associated to a curve that is invariant under change of coordinates. The following is an example of this phenomenon.

Example : Let $P(x,y,z) = x^3+y^3-z^3$. Then $V(P)$ is a degree three curve. Considering the projective change of coordinates,

$$x = u - w$$

$$y = iv$$

$$z = u + v$$

Let us find the polynomial $P^*(u,v,w)$ whose zero set $V(P^*)$ maps to $V(P)$. We will find that $V(P^*)$ also has degree three.

We have

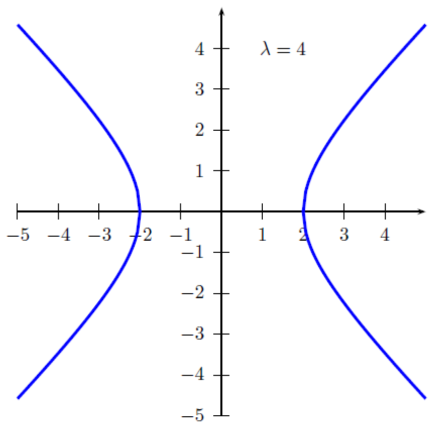
$$
\begin{aligned}
P(u,v,w) & \\
&= P(u-w, iv, u+v) \\
&= (u-w)^3 + (iv)^3 - (u+v)^3 \\
&= (u^3 - 3u^2w + 3uw^2 - w^3) - iv^3 - (u^3 + 3u^2v + 3uv^2 + v^3) \\
&= -3u^2v - 3u^2w - 3uv^2 + 3uw^2 - (1+i)v^3 - w^3,
\end{aligned}
$$

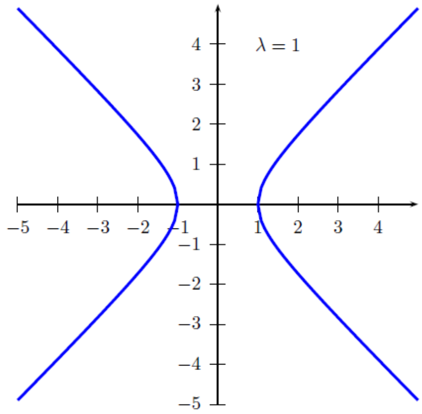
which has degree three.

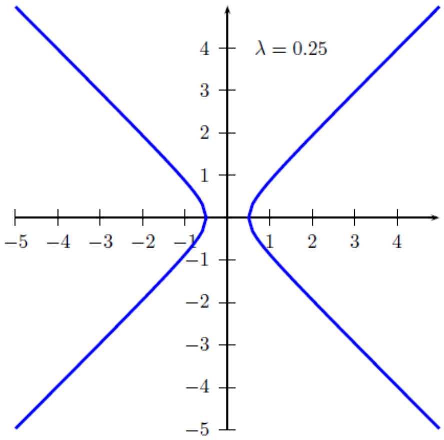
4.2 Higher Degree Curves as Surfaces

Suppose $f(x,y,z)$ is a homogeneous polynomial, so $V(f)$ is a curve in $\mathbb{P}^2$. We should recall that the degree of $V(f)$ is, by definition, the degree of the homogeneous polynomial f. We will see that this algebraic invariant of the curve is closely linked to the topology of the curve viewed as a surface over $\mathbb{R}$. Specifically, it is related to the "genus" of the curve, which counts the number of holes in the surface.

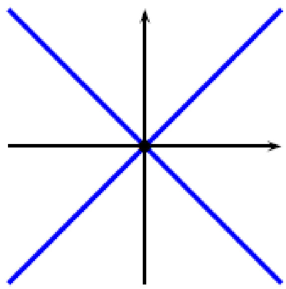
Example : Considering the conics defined by the homogeneous equation $x^2 - y^2 = \lambda z^2$, where λ is a parameter. We shall sketch affine patches of these in the chart $z = 1$ for $\lambda = 4, 1, 0.25$.



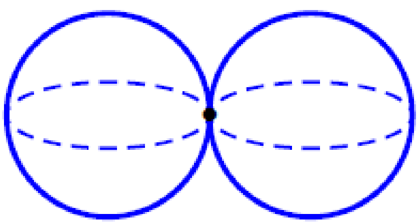




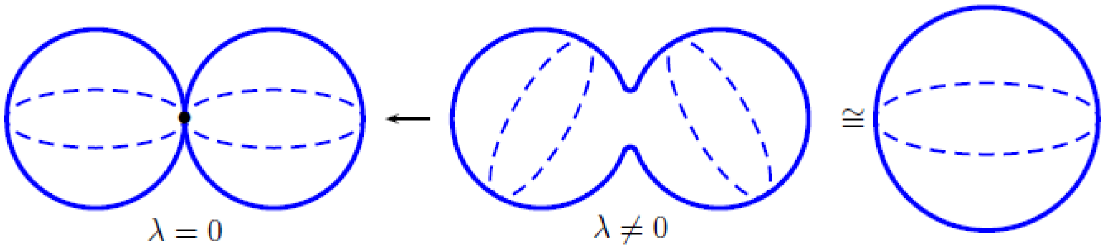
As $\lambda \to 0$, we get $x^2 - y^2 = 0$, or $(x-y)(x+y) = 0$. In $\mathbb{R}^2$, this looks like



but this picture isn't accurate over $\mathbb{C}$ in $\mathbb{P}^2$. Instead, topologically the picture looks like,

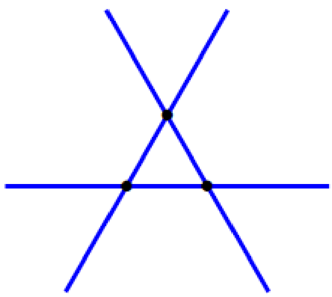


Thus, the topological version of the original equation, $x^2 - y^2 = \lambda z^2$, should be found by perturbing the spheres a little to account for $\lambda \neq 0$:

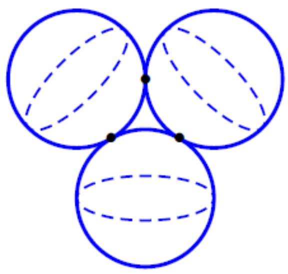


Therefore, by mildly perturbing the specialized, non-smooth conic, we find that topologically a smooth conic (those in this exercise for which $\lambda \neq 0$) is a sphere with no holes, which agrees with our work.

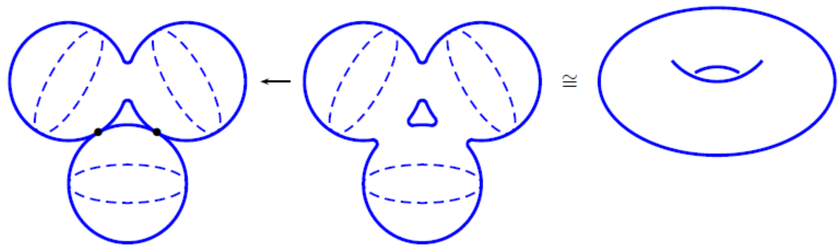
Following this same reasoning, we find another proof that a smooth cubic must be a torus when realized as a surface over $\mathbb{R}$. We begin with the highly degenerate cubic, $f(x,y,z) = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)$. In the real affine chart $z = 1$, the picture looks like



Again, our picture is not valid over $\mathbb{C}$ in $\mathbb{P}^2$. Instead, the correct topological picture is that of three spheres meeting at three points, as shown.

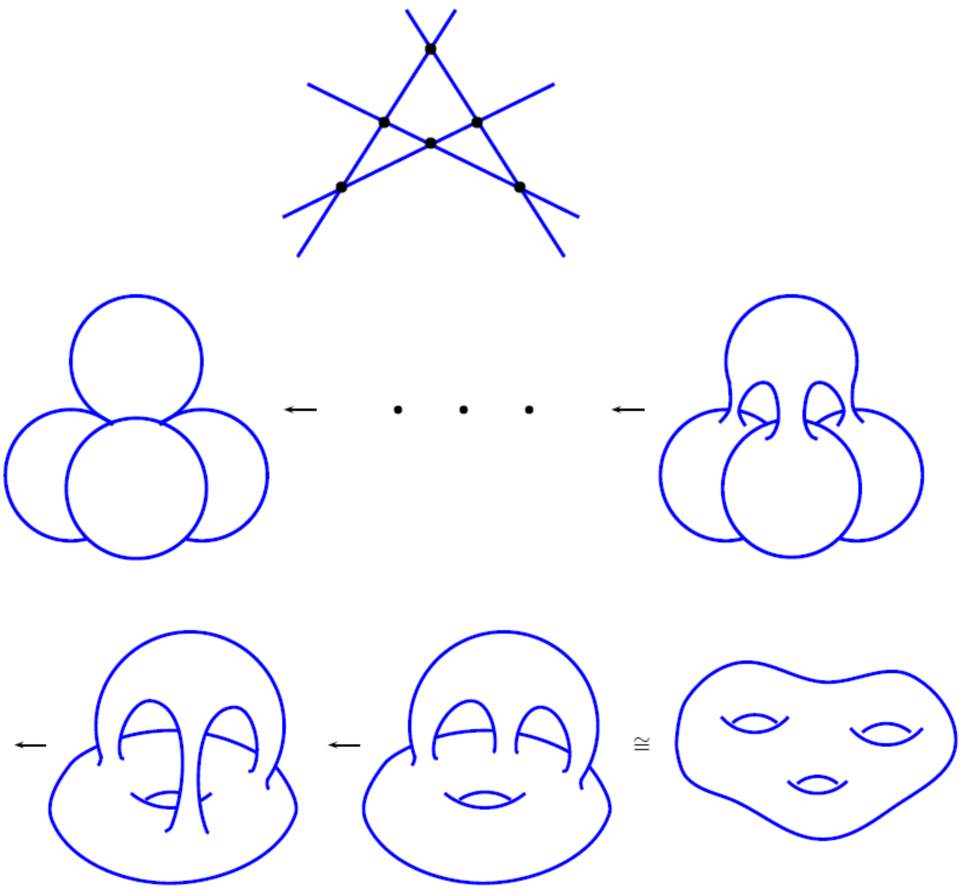


Perturbing the top two spheres slightly, we find they join into the topological equivalent of a single sphere, but that this new figure is joined to the third sphere at two points of contact. Perturbing each of these points of intersection independently of one another, we obtain a single surface with a hole through the middle as depicted in the sequence of figures below.



Thus a smooth cubic over $\mathbb{C}$ is topologically equivalent to a torus (a sphere with a hole through it) as a surface over $\mathbb{R}$.

Mimicing the arguments illustrated above to describe the real surface corresponding to a smooth quartic (fourth degree) curve over $\mathbb{C}$ in $\mathbb{P}^2$. Let us start with a highly degenerate quartic (the product of four pairwise non-parallel lines), draw the corresponding four spheres, and deform this surface by merging touching spheres two at a time. Let us see how many holes will the resulting figure possess! In $z = 1$:



It has 3 holes.

Definition : Let $V(P)$ be a smooth, irreducible curve in $\mathbb{P}^2(\mathbb{C})$. The number of holes in the corresponding real surface is called the *topological genus* of the curve $V(P)$.

Presently, this notion of genus only makes sense when we are working over the reals or an extension of them. However, by the discussion above, we see that there is a connection between the genus, g, and the degree, d, of a curve. That is, all smooth curves of degree d have the same genus, so we now wish to find a formula expressing the genus as a function of the degree.

Example : Let us find a quadratic function in d, the degree of a smooth curve, that agrees with the topological genus of curves of degrees $d = 2,3,4$ found earlier.

We will guess that the formula is

$$g = \frac{(d-1)(d-2)}{2}$$

For $d = 1$, we know that the genus is zero. We indeed have for $d = 1$

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} \\ &= \frac{(1-1)(1-2)}{2} \\ &= 0 \end{aligned}$$

For $d = 2$, we know that the genus is also zero, and we have for $d = 2$

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} \\ &= \frac{(2-1)(2-2)}{2} \\ &= 0 \end{aligned}$$

For $d = 3$, the genus is one, and we have for $d = 3$

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} \\ &= \frac{(3-1)(3-2)}{2} \end{aligned}$$

$$= 1$$

For $d = 4$, the genus is three, and we have for $d = 4$

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} \\ &= \frac{(4-1)(4-2)}{2} \\ &= 3 \end{aligned}$$

Definition : Let $V(P)$ be a curve of degree d. The number $\frac{(d-1)(d-2)}{2}$ is the *arithmetic genus* of the curve, which is an algebraic invariant of $V(P)$.

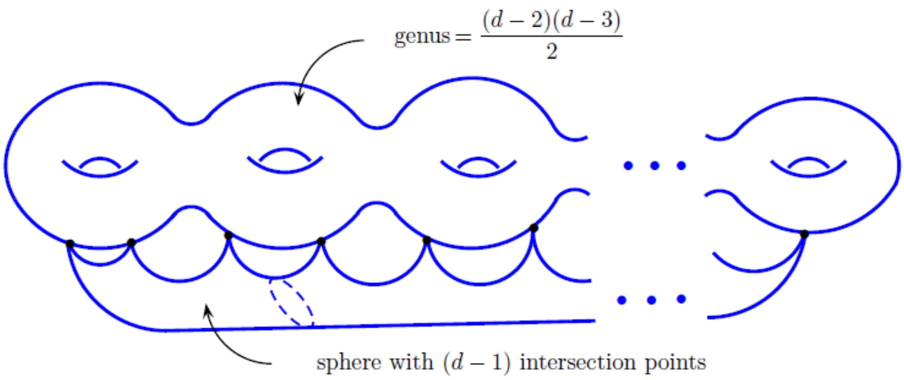
Let us see by induction on d, the degree, that the topological genus agrees with the arithmetic genus for smooth curves, or in other words that

$$g = \frac{(d-1)(d-2)}{2}$$

The base case is when $d = 1$, but we know that for $d = 1$ the genus is zero, and we have

$$\begin{aligned} g &= \frac{(d-1)(d-2)}{2} \\ &= \frac{(1-1)(1-2)}{2} \\ &= 0 \end{aligned}$$

Now supposing the topological genus agrees with the arithmetic genus for smooth curves of degree $d-1$ and consider a smooth curve of degree d. Notice that we can perturb the curve a little bit to obtain a smooth curve of degree $d-1$ which intersects a single line in $d-1$ points. By the induction hypothesis, the topological genus of this smooth curve of degree $d-1$ must agree with its arithmetic genus. Topologically, you now have a surface of genus $\frac{(d-2)(d-3)}{2}$ that intersects a single sphere in $d-1$ points.



Observing that the $d-1$ points of intersection of the surface and the sphere will add $d-2$ topological holes to the overall figure. Thus, a curve of degree d has a topological genus of

$$g = \frac{(d-2)(d-3)}{2} + (d-2)$$

Finally, we have that

$$\frac{(d-2)(d-3)}{2} + (d-2) = \frac{(d-1)(d-2)}{2}.$$

Chapter 5

Bézout's Theorem

In this chapter is to develop the needed definitions, such as, the Fundamental Theorem of Algebra and Intersection Multiplicity, that allow the statement and proof ofBézout's Theorem, which says that in $\mathbb{P}^2$ a curve of degree n will intersect a curve of degree m in exactly nm points, provided the points of intersection are "counted correctly."

5.1 Intuition behind Bézout's Theorem

Let us look at some examples.

Example : Let us show that the two lines $V(x-y+2)$ and $V(x-y+3)$ do not intersect in $\mathbb{C}^2$. Homogenize both polynomials and show that they now intersect at a point at infinity in $\mathbb{P}^2$.

Every point of $V(x-y+2) \subset \mathbb{C}^2$ is of the form $(x, x+2)$, so the point of intersection corresponds to the values of x such that $x-(x+2)+3=0$, but there is no x value, real or complex, that satisfies the equation $1=0$. After homogenizing, we see that every point of $V(x-y+2z) \subset \mathbb{P}^2$ is of the form $(x : x+2z : z)$, so the point of intersection corresponds to the values of x and z such that $x-(x+2z)+3z=0$. Thus, the point of intersection in $\mathbb{P}^2$ is $(1:1:0)$.

Example : Let us show that $V(y-\lambda)$ will intersect $V(x^2+y^2-1)$ in two points in $\mathbb{C}^2$, unless $\lambda = \pm 1$. Show that $V(y-1)$ and $V(y+1)$ are tangent lines to the circle $V(x^2+y^2-1)$ at their respective points of intersection. Explain why we say that $V(y-1)$ intersects the circle $V(x^2+y^2-1)$ in one point with multiplicity two.

Suppose $\lambda \neq 1$. The points of intersection correspond to the points whose x values satisfy $x^2+\lambda^2-1=0$, i.e. the two points $(\sqrt{1-\lambda^2},\lambda)$ and $(-\sqrt{1-\lambda^2},\lambda)$. Suppose $\lambda = 1$. The tangent to the circle $V(x^2+y^2-1)$ at the point $(0,1)$ is the line $V(y-1)$. Similarly, the tangent to $V(x^2+y^2-1)$ at $(0,-1)$ is $V(y+1)$. We know that the intersection multiplicity of the line $V(y-z)$ and the circle $V(x^2+y^2-z^2)$ in $\mathbb{P}^2$ is the multiplicity of the root $(0:1:1)$ of $x^2+y^2-z^2=0$, which is two.

Example : Let us show that there are no points in $\mathbb{C}^2$ in the intersection of $V(xy-1)$ with $V(y)$. Homogenizing both equations $xy=1$ and $y=0$. We will find that there is a point of intersection at infinity.

Every point of $V(y)$ has $y=0$, so $xy=0\neq 1$. Hence there is no point common to $V(xy-1)$ and $V(y)$. After homogenizing we have $V(xy-z^2)$ and $V(y)$ which have the point $(1:0:0)$ in common. The intersection multiplicity of $V(xy-z^2)$ and $V(y)$ is the multiplicity of the root $(1:0:0)$ of $z^2=0$, which is two.

5.2 Fundamental Theorem of Algebra

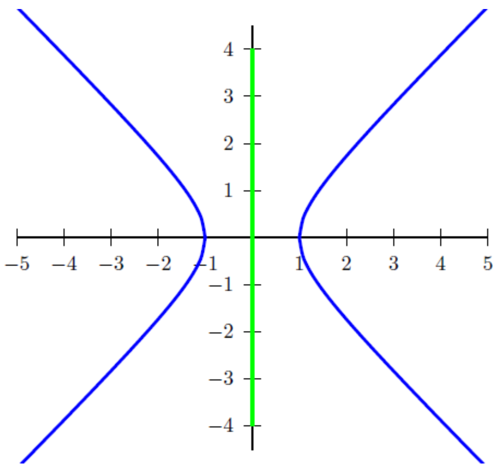
If $f(x)$ is a polynomial of degree d in $\mathbb{C}[x]$, then

$$f(x)=(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_r)^{m_r},$$

where each a_i is a complex root of multiplicity m_i and $\sum_{i=1}^{r} m_i = d$.

Example : Let $f(x,y)=x^2-y^2-1$ and $g(x,y)=x$. Let us sketch $V(f)$ and $V(g)$ in $\mathbb{R}^2$. Do they intersect? Let us find $V(f)\cap V(g)$ in $\mathbb{C}^2$.

They do not intersect in $\mathbb{R}^2$. They intersect at the points $(0,i)$ and $(0,-i)$ in $\mathbb{C}^2$.



Example : Let $g(x,y) = ax + by + c$, $b \neq 0$, in $\mathbb{C}[x,y]$. Let $f(x,y) = \sum_i a_i x^{r_i} y^{s_i}$ be any polynomial of degree d in $\mathbb{C}[x,y]$. Let us show that the number of points in $V(f) \cap V(g)$ is d, if the points are counted with an appropriate notion of multiplicity.

Since $b \neq 0$ we can write $y = \frac{-ax-c}{b}$ and now we want to find the number of roots of

$$f(x, \frac{-ax-c}{b}) = \sum_i a_i x^{r_i} (\frac{-ax-c}{b})^{s_i}$$

This is a single variable polynomial of degree $max_i(r_i + s_i) = d$, so by the Fundamental Theorem of Algebra, it has d roots, counted with multiplicity.

5.3 Intersection Multiplicity

Definition : Let f be a non-homogeneous polynomial (in any number of variables) and let p be a point in the set $V(f)$. The multiplicity of f at p, denoted by $m_p f$, is the degree of the lowest degree non-zero term of the Taylor series expansion of f at p.
Notice that if $p \notin V(f)$, then $f(p) \neq 0$, so the lowest degree non-zero term of the Taylor expansion of f at p is $f(p)$, which has degree zero. If $p \in V(f)$, then $f(p) = 0$, so $m_p f$ must be at least one.

Example : Let f be a non-homogeneous polynomial (in any number of variables) of degree n.

Let f be a polynomial in k variables $x_1, \cdots, x_k$. Suppose first that $m_p f > 1$. Then all of the first partial derivatives of f vanish at p, i.e.

$$\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_k}(p) = 0$$

But this is exactly what it means for p to be a singular point. Now suppose $m_p f = 1$. Then at least one of $\frac{\partial f}{\partial x_i}(p) \neq 0$. Hence p is a nonsingular point. Now $V(f)$ is nonsingular if and only if every point p is a nonsingular point, so $m_p f = 1$ for all $p \in V(f)$ if and only if $V(f)$ is nonsingular.

Let us suppose f is degree n polynomial in k variables $x_1, \cdots, x_k$. We will show that

$$\frac{\partial^m f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_k^{i_k}} = 0$$

for any $i_1 + i_2 + \cdots + i_k = m \geq n+1$, that is all the partial derivatives of order greater than n vanish identically. Hence, the first nonzero term of the Taylor series expansion must be of degree less than n+1. To show all of the higher order partial derivatives vanish, we observe that if f is of degree n, then for any i, $1 \leq i \leq k$, $\frac{\partial f}{\partial x_i}$ is a polynomial of degree at most $n-1$, and a straightforward induction argument shows that

$$\frac{\partial^m f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_k^{i_k}}$$

is a polynomial of degree at most $n-m$. We see then that if $m = n$, the result is a degree zero polynomial, i.e. a constant, perhaps zero. Differentiating once more gives the desired result.

Example : Let $f(x,y) = xy$. What is the multiplicity of f at the origin? Let $p = (0,1)$, and let us calculate $m_p f$.

First noting that the Taylor series expansion of f at the origin is $f(x,y) = xy$. Since the degree of the first nonvanishing term is two, the multiplicity of f at the origin is two. Now supposing $p = (0,1)$. The Taylor series expansion of f at $(0,1)$ is $f(x,y) = x + x(y-1)$. The lowest degree nonvanishing term is x, so $m_p f = 1$.

Example : Let $f(x,y) = x^2 + xy - 1$. Calculating the multiplicity of f at $p = (1,0)$.

We shall compute the Taylor series expansion of f at p.

$$\frac{\partial f}{\partial x}(p) = 2, \frac{\partial f}{\partial y}(p) = 1, \frac{\partial^2 f}{\partial x^2}(p) = 0,$$
$$\frac{\partial^2 f}{\partial y^2}(p) = 0, \frac{\partial^2 f}{\partial x \partial y}(p) = 1$$

We find that all higher derivatives are zero. The Taylor series expansion of f at p is $f(x,y) = 2(x-1) + y + (x-1)^2 + (x-1)y$, so $m_p f = 1$.

Theorem : Given polynomials f and g in $\mathbb{C}[x,y]$ and a point p in $\mathbb{C}^2$, there is a uniquely defined number $I(p, V(f) \cap V(g))$ such that the following axioms are satisfied.

1. $I(p, V(f) \cap V(g)) \cap \mathbb{Z}_{\geq 0}$, unless p lies on a common component of $V(f)$ and $V(g)$, in which case $I(p, V(f) \cap V(g)) = \infty$.
2. $I(p, V(f) \cap V(g)) = 0$ if and only if $p \notin V(f) \cap V(g)$.

3. Two distinct non-parallel lines meet with intersection number one at their common point of intersection.
4. $I(p,V(f)\cap V(g)) = I(p,V(g)\cap V(f))$.
5. $I(p,V(f)\cap V(g)) = \sum r_i s_j I(p,V(f_i)\cap V(g_j))$ when $f = \prod f_i^{r_i}$ and $g = \prod g_j^{s_j}$.
6. $I(p,V(f)\cap V(g)) = I(p,V(f)\cap V(g+af))$ for all $a \in \mathbb{C}[x,y]$.

Definition : The number $I(p,V(f)\cap V(g))$ is the *intersection multiplicity* of f and g at p.

Let $(a:b:c) \in V(f)$. The multiplicity of f at the point $(a:b:c)$ remains the same no matter how we dehomogenize. (This is quite a long problem to work out in full detail.)

Theorem : Given polynomials f and g in $\mathbb{C}[x,y]$ and a point p in $\mathbb{C}^2$, we have

$$I(p,V(f)\cap V(g)) \geq m_p(f)\cdot m_p(g),$$

with equality if and only if $V(f)$ and $V(g)$ have no common tangent at p.

5.4 Statement of Bézout's Theorem

Bezout's Theorem tells us how many points are in the intersection of two plane curves. We start with some examples.

Example : Let $f = x^2+y^2-1$ and $g = x^2-y^2-1$. Let us find all pointsof intersection of the curves $V(f)$ and $V(g)$. For each point of intersection p, send p to $(0,0)$ via a change of coordinates T.Let us find $I(p,f\cap g)$ by calculating $I((0,0),T(V(f))\cap T(V(g)))$. We will verify that

$$\sum_p I(p,V(f)\cap V(g)) = (deg(f))(deg(g)).$$

All points in $V(f)$ have $y^2 = 1-x^2$, so we have $g(x,y) = x^2-(1-x^2)-1 = 0$, which gives $x = \pm 1$. Then the two intersection points are $p_1 = (1,0)$ and $p_2 = (-1,0)$. Defining $T_1(x,y) = (x-1,y)$ and $T_2(x+1,y)$ so that $T_1(p_1) = (0,0)$ and $T_2(p_2) = (0,0)$. Under T_1 we have

$$\begin{aligned}
T_1(V(f)) &= V(f\cdot T_1^{-1}(x,y)) \\
&= V((x+1)^2+y^2-1) \\
&= V(2x+x^2+y^2)
\end{aligned}$$

$$\begin{aligned} T_1(V(g)) &= V(g \cdot T_1^{-1}(x,y)) \\ &= V((x+1)2 - y^2 - 1) \\ &= V(2x + x^2 - y^2) \end{aligned}$$

We know that these curves have a common tangent at (0, 0), so we need to use Axiom 7.

$$\begin{aligned} &I((0,0), T_1(V(f)) \cap T_1(V(g))) \\ &= I((0,0), V(2x+x^2+y^2) \cap V(2x+x^2-y^2)) \\ &= I((0,0), V(2x+x^2+y^2) \cap V((2x+x^2-y^2) \\ &\quad +(-1)(2x+x^2+y^2)) \\ &= I((0,0), V(2x+x^2+y^2) \cap V(-2y^2)) \end{aligned}$$

Since $V(2x+x^2+y^2)$ and $V(-2y^2))$ have no common tangent at $(0,0)$ we can apply Axiom 5 to obtain $I((0,0), V(2x+x^2+y^2) \cap V(-2y^2)) = 2$. A nearly identical calculation gives $I((0,0), T_2(V(f)) \cap T_2(V(g))) = 2$. Finally, we have $\sum_p I(p, V(f) \cap V(g)) = 2+2 = 4 = (2)(2) = (degf)(degg)$.

Example : Let $f = y - x^2$ and $g = x$. Let us verify that the origin is the only point of $V(f) \cap V(g)$ in $\mathbb{C}^2$ and that $I((0,0), V(f) \cap V(g)) = 1$.

Every point $p \in V(g)$ has $x = 0$, so the only point of $V(f) \cap V(g)$ has $y = 0$ also, i.e. $p = (0,0)$ is the only point of $V(f) \cap V(g)$ in $\mathbb{C}^2$. Since f and g are both smooth and f and g have no common tangent at p, we have $I((0,0), V(f) \cap V(g)) = (m_p f)(m_p g) = 1$.

Bézout's Theorem Let f and g be homogeneous polynomials in $\mathbb{C}[x,y,z]$ with no common factors, and let $V(f)$ and (g) be the corresponding curves in $\mathbb{P}^2(\mathbb{C})$. Then

$$\sum_{p \in V(f) \cap V(g)} I(p, V(f) \cap V(g)) = (deg(f))(deg(g)).$$

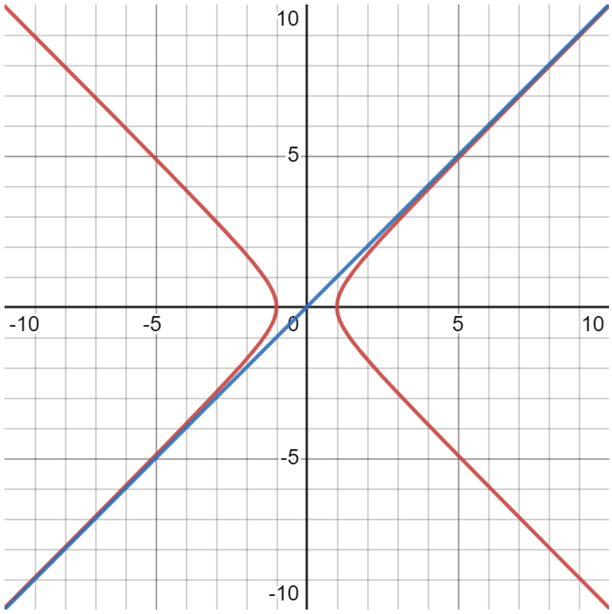
Example : Homogenizing the polynomials in the previous example and find the two points of $V(f) \cap V(g)$ in $\mathbb{P}^2(\mathbb{C})$.

After homogenizing we have $f(x,y,z) = yz - x^2$ and $g(x,y,z) = x$. Now if $p \in V(f) \cap V(g)$, then $x = 0$ and either $y = 0$ or $z = 0$, i.e. the two points of $V(f) \cap V(g)$ in $\mathbb{P}^2(\mathbb{C})$ are $p_1 = (0:0:1)$ and $p_2 = (0:1:0)$. We already found $I(p_1, V(f) \cap V(g)) = 1$ in the affine patch corresponding to $z = 1$. Now consider $f = z - x^2$ and $g = x$ in the affine patch corresponding to $y = 1$. The same analysis applies and we have $I(p_2, V(f) \cap V(g)) = 1$.

Example : Let $f = x^2 - y^2 - 1$ and $g = x - y$.Let us sketch $V(f)$ and $V(g)$ in $\mathbb{R}^2$. Homogenizing f and g and let us verify Bézout's Theorem in this case.

After homogenizing we have $f(x,y,z) = x^2 - y^2 - z^2$ and $g(x,y,z) = x - y$. Any point of $V(g)$ has $x = y$, so the only point of $V(f) \cap V(g)$ is $(1:1:0)$, a point at infinity. Now we dehomogenize in the $y = 1$ affine patch and consider $f = x^2 - z^2 - 1$ and $g = x - 1$, and we see $V(f) \cap V(g)$ consists of $p = (1,0)$. Since f and g are both smooth we know $m_p f = m_p g = 1$, but $V(f)$ and $V(g)$ have a common tangent, $x = 1$, at $(1,0)$.

$$
\begin{aligned}
&I(p, V(x^2 - z^2 - 1) \cap V(x-1)) \\
&\qquad = I(p, V(x-1) \cap V(x^2 - z^2 - 1)) \\
&\qquad = I(p, V(x-1) \cap V(x^2 - z^2 - 1 + (-x-1)(x-1))) \\
&\qquad = I(p, V(x-1) V(-z^2)) \\
&\qquad = 2
\end{aligned}
$$



The last inequality follows from $m_p(x-1) = 1$ and $m_p(-z^2) = 2$ and the fact that $V(x-1)$ and $V(-z^2)$ have no common tangent at p. We have thus verified Bézout's Theorem. A similar analysis yields the same result for $g = x+y$, but in this case, the point of intersection is $(1:1:0)$.

Note that in the real plane and the complex plane, this result is not true, i.e.,

$$V(x^2-y^2-1) \cap V(x-y) = \phi.$$

But in the Projective plane, we got their intersection point.

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