1 Introduction

Our solar system has eight planets and except for Mercury and Venus, all of them possess natural satellites called moons. The saturn system has around 146 moons[1] but the largest among them is Titan, the second largest moon in the solar system. Larger than Mercury, Titan is unique in a sense that it is the only moon known to have dense atmosphere and stable bodies of liquid on its surface, although it is believed that these are liquid hydrocarbon lakes.[1] This makes titan the most Earth-like celestial body in the solar system and therefore a primary focus of study for the astronomers.

The Stability of a celestial body refers to how likely it is to remain in its current orbit without colliding with other bodies or be flung out of its orbit. Gravitational forces are the primary drivers of stability in all celestial bodies, though other phenomenon like friction, drag can influence it as well. Studying the stability is important as it provides insights about the formation as well as the evolution of such systems. It also plays a crucial part in the assessing the long term habitability of a body, especially in case of Titan, which has been often poised as a potential future home for human civilization.

The force due to the spatial variations in the strength of the gravitational field, acting on a body is referred to as tidal force. Usually, its the larger bodies like planets & stars that cause significant changes in the smaller bodies like moons, asteroids etc due to tidal forces. This force creates an area of higher gravitational potential, called a tidal bulge, which generates a torque that tries to align the smaller body's rotation along its orbit. This process is called tidal locking. There is no longer any net change in the rotation rate over the course of a complete orbit. In particular, if the smaller body takes just as long as to rotate around its own axis as it does to revolve around the larger body, it is said to have synchronous rotation.

In this project, We will simulate the rotation of Titan around Saturn by utilizing the fourth order Runge-Kutta method in Fortran. We will study the variations in the rotation angle and the rotation rate over time to check whether Titan exhibits synchronous rotation. Additionally, we will explore the influence of orbital frequency and other physical parameters on this synchronous rotation.

[1] https://science.nasa.gov/saturn/moons/

2 Simulation, analysis and discussion

2.1 Synchronous Rotation

As discussed in 1, rotation of Titan around Saturn is synchronous and shows same face to a virtual observer located on Saturn. We will model the its rotation with following assumptions:

- Titan is a rigid triaxial body
- Its rotation is disturbed by the gravitational action of Saturn
- The rotation axis will be considered to be orthogonal to the orbital plane.

All the assumptions are valid, The first assumption means that Titan's shape is rougly ellipsoidal and that it has three distinct principal axes of rotation. It implies that the shape of the Titan does not change over time, which is reasonable, as the tidal forces exerted by Saturn are relatively low. The second assumption deals with the graviraional interaction of Saturn with Titan which cause variations in its rotation rate. These variations are known as librations and are caused due to the eccentricity of the Titan's orbit. [2]

The final assumption implies that Titan has no axial tilt, that means its rotation axis is perpendicular to the plane in which orbits around Saturn. It is again justified as the axial tilt of Titan is around 0.3 degrees only.[3]

This rotation is governed by a second order differential equation:

$$\ddot{p} + \frac{3}{4}n^2\frac{B-A}{C}H = 0 \tag{2.1}$$

with

$$H = \left(-e\cdot\sin(2p-\lambda)+2\left(1-\frac{5e^2}{2}\right)\cdot\sin(2p-2\lambda)+7e\cdot\sin(2p-3\lambda)+17e^2\cdot\sin(2p-4\lambda)\right) \tag{2.2}$$

where

- p is the rotation angle of Titan.$\ddot{p} = \frac{d^2p}{dt^2}$ is its second order time derivative.
- n is the orbital frequency of Titan[1]. It is a constant

[1] Even though here the orbital frequency has been represented by n but throughout the report as well as in scientific computations, it has been replaced by ω to avoid confusion with the number of steps used in computations, which is often universally reperesented by n

- e is the orbital frequency of Titan.
- $\lambda = nt$ is the mean longitude of Titan. It locates Titan on its elliptical orbit around Saturn.
- A, B and C are the moments of inertia of Titan with respect to axes x, y, and z respectively.

2.2 Numerical Integration

The force function 2.1 will be solved using a numerical integrator. In this case, we will go with the fourth order Runge-Kutta method. It is based on the principal of approximating the solution at each step by using the slope of the function at four points: the initial point, the midpoint and the endpoint of the interval, as well as a point near the midpoint. If we start with a ordinary differential equation of the form:

$$\frac{dy}{dx} = f(t, y) \tag{2.3}$$

$$y(t_0) = y_0 \tag{2.4}$$

y_0 is the initial condition. The discretization of the problem leads to the following set of equations which provide the solution to the differential equation.

$$\begin{aligned} y_{n+1} =& y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ t_{n+1} =& t_n + h \end{aligned} \tag{2.5}$$

where,

$$\begin{aligned} k_1 =& f(t_n, y_n), \\ k_2 =& f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right), \\ k_3 =& f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right), \\ k_4 =& f\left(t_n + h, y_n + h\right) \end{aligned}$$

The Runge-Kutta methods are easy to implement and are very stable for most of the problems but at the same time they require more function evaluations and computations per step than other methods, which makes than slow and computationally expensive. It can't adjust the step size automatically to acheive a desired accuracy.[4]

2.3 Computational description

Computationally, the problem comes down to solving of the function 2.1 with the help of subroutine that employs the fourth order Runge-Kutta method. The function 2.1 involves certain constants like eccentricity, rotational velocity etc and the values used in the study have been specified in the table below.

Parameter	Value
Eccentricity (e)	0.0289
Orbital frequency (n)	0.01643 rad hr^{-1}
Orbital rotation period	382.690 [2]
$\frac{B-C}{A}$(I)	1.35×10^{-4}

Table 2.1: Value of Dynamical Parameters

Even though values of all of the parameters have been taken from the paper **Titan's rotation, A 3-dimensional theory**[5], the units of the parameters used here are different than the ones stated in it as we have used different time scales for the simulation and adjusted the units accordingly.

Main internal data and variables:

Force Function: `force_func`
1 Input Parameters ***(t, p, dp)***

- **Type:** real(8)
- **Description:** Time, angular position, and angular velocity.

2 Local Variables

- **Type:** real(8)
- **Description:** Orbital parameters **(omega, e, l, I)** used in force calculation.

3 Output

- **Type:** real(8)
- **Description:** Computed force equation affecting the motion of Titan.

Runge Kutta Integrator: `rk4`
1.Input Parameters

- **1.1** `func`
  - **Type:** Function

  - **Description:** User-defined force function influencing the motion.
  - **Signature:** real(8) function func(t, p, dp)

- **1.2** $t, p,$ **dp**$, h, n,$ **file**
  - **Type:** real(8), integer, character(len=*)
  - **Description:** Variables related to time, rotation angle, rotation rate, and file handling.

2. Local Variables

- **Type:** real(8)
- **Description:** Coefficients (`k1p, k2p, k3p, k4p, k1dp, k2dp, k3dp, k4dp`) used in RK4 integration.

3. Output

- **Type:** File
- **Description:** Output file containing time, rotation angle, and rotation rate.

In the Runge Kutta numerical integration scheme in the program the interface block has been provide to declare the interface of the function within the scope of the integrator subroutine. This allows the compiler to check the consistency of the arguments and return the value of the function, when called later on. This is important in cases where function is defined outside the subroutine, as in our case. So, the integrator used is independent of the force function.

For the ease of the computation, we used the reduced units for various quantities, 1 hour was chosen as the unit of the time and accordingly the only quantity that had to be adjusted to the new set of units was the angular frequency, the modified value of which is given in the table 2.1.

The program computes the values of time step, rotation angle ($p - \omega \times t$) and rotation rate and stores it in a text file. The plotting of the parameters was done in python partly because of user friendly interface and partly due to more graphically formating options available.

The various scientific expliotations can be done by doing the following changes within the code:

- To check whether the system is synchronous at rotation rate equal to twice the orbital frequency, one simply needs to multiply 2 to the input parameter dp
- To increase the amplitude of the librations the e local variable inside the `force_func` needs to be adjusted for any value between 0 and 1.

3 Results and Scientific explotiations

As stated earlier the celestial dynamics are chaotic and are highly sensitive to the initial conditions, therefore care had to be taken in finding the appriopriate initial conditions. The realistic initial values for rotation angle and rotation rate were used with $p = \pi$ and $\dot{p} = 0.01643$ rad/h .

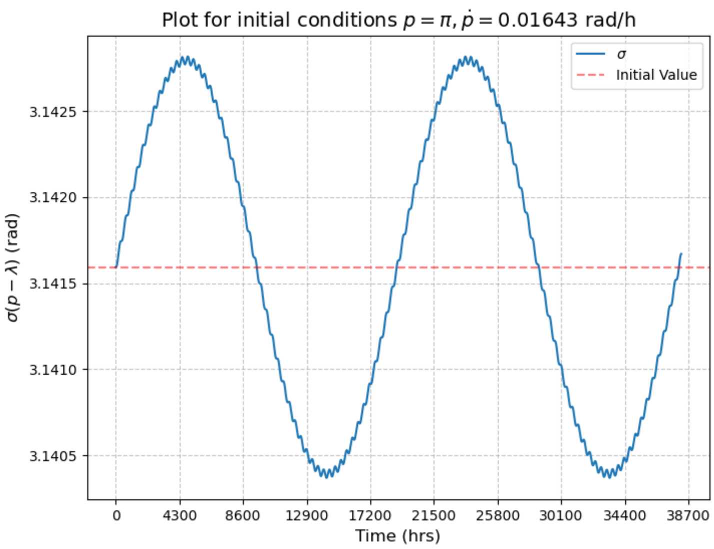

Figure 3.1: Variation of σ with time

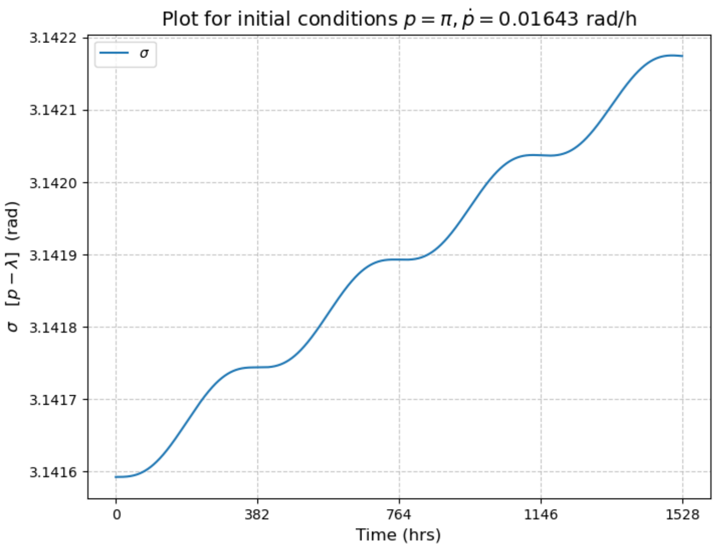

Figure 3.2: Close up view of a section of the σ variation with time

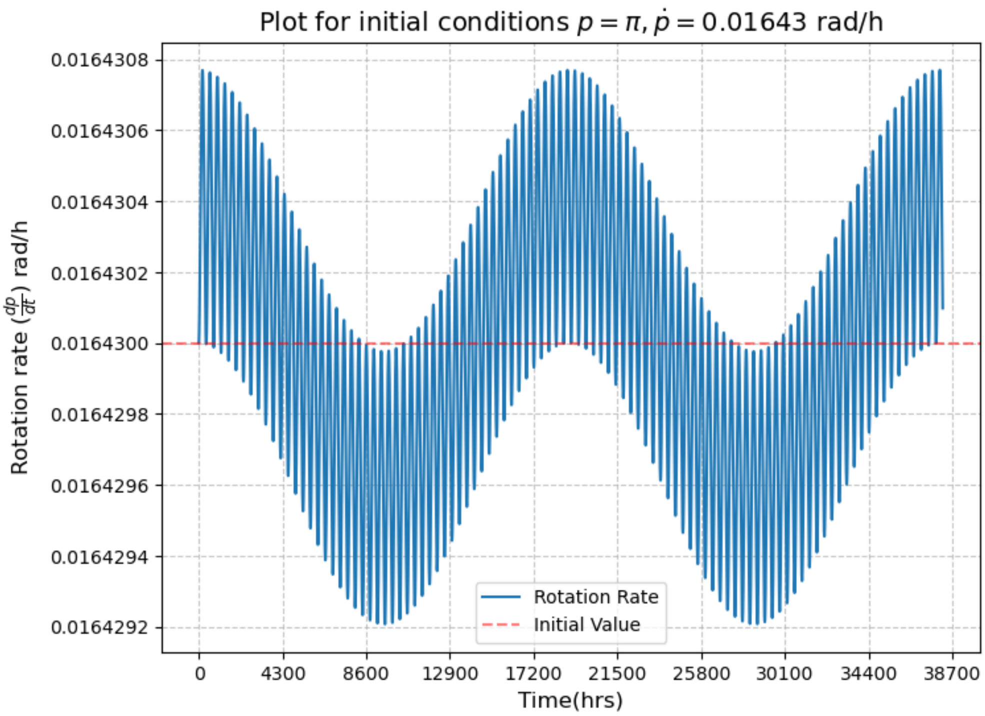

Figure 3.3: Rotation rate variation with time

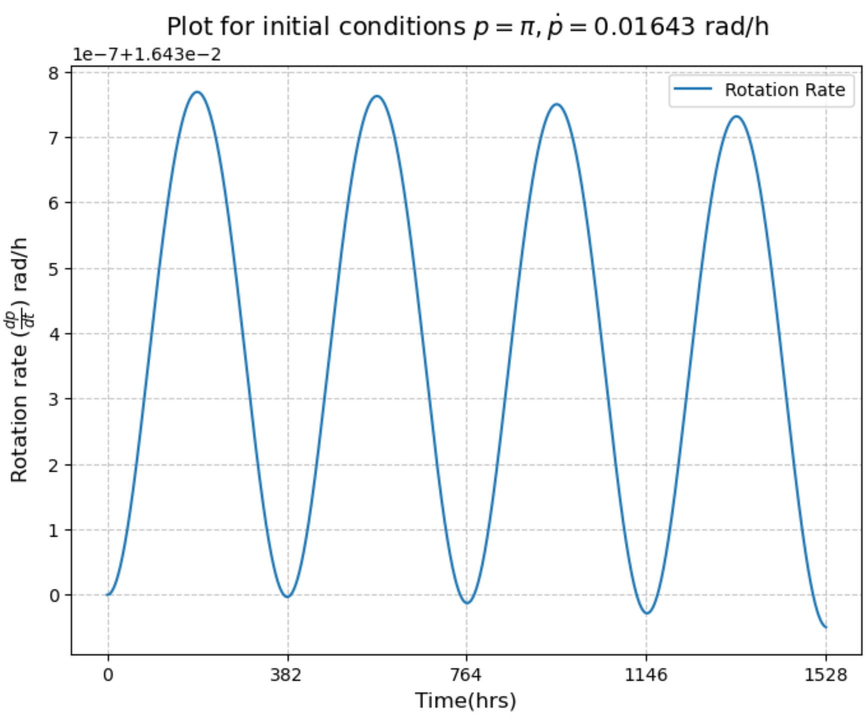

Figure 3.4: Zoomed in view of a section of the rotation rate variation with time

To see the variations in the rotation angle with time we have plotted σ which is $p - \lambda t$. The figures 3.1 and 3.2 give us the variation of the σ with the time while as the figures 3.3 and 3.4 give us the variation of the rotation rate (rotational angular velocity) with respect to time. In both of the cases we see an oscillatory behaviour of the variables whether it is the rotation angle[1] or the rotation rate around the initial conditions. The quantity σ varies between 3.141592 ± 0.001224 and the rotation rate varies between 0.01643 ± 0.0000008. Thus we see a nearly constant value for both the physical quantities but there is still a departure (however small it may be) from the uniform rotational motion and the reason for it has been discussed in the successive paragraphs.

If the rotation angle and rotation rate of a body has a period equal to the orbital period when plotted on the time axis, then the body is said to be in synchronous rotation.[2]. From the figures 3.2 and 3.4, the periodicity of the librations (i.e the periodic variations in the rotation of Titan) is equal to 382 hours which corresponds to 15.91 days, nearly equal to the orbital period of Titan around the Saturn. Thus, the simulations present the evidence for the synchronous rotation of Titan. The librations are caused due to the eccentricity of the Titan's orbit as well as its gravitational interactions that can cause tidal bulges rocking the body back and forth.

In the plots 3.1 and 3.3, another libration mode with a periodicity of 2.1731 years is observed. This is an example of the higher order resonance present in the system. Its period is roughly equal to 50 times that of the first order librations. These higher order librations are caused due to the non-linearity of the Titan-Saturn system caused by a non-zero eccentricity.

[1]although it does show the variations in rotational angle but the quantity plotted in graphs is σ.
[2]https://link.springer.com/referenceworkentry/10.1007/978-3-642-27833-$4_5488 - 1$

3.1 Variation of amplitude of the librations

The figures 3.5,3.6 and 3.7, 3.8 depict the variations in the amplitude of the librations with new eccentricities values (e^*) $2e$ and $5e$ respectively.

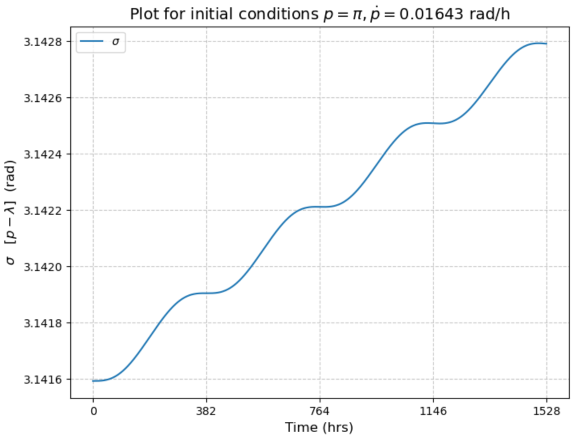

Figure 3.5: σ variation for eccentricity $e^* = 2e$

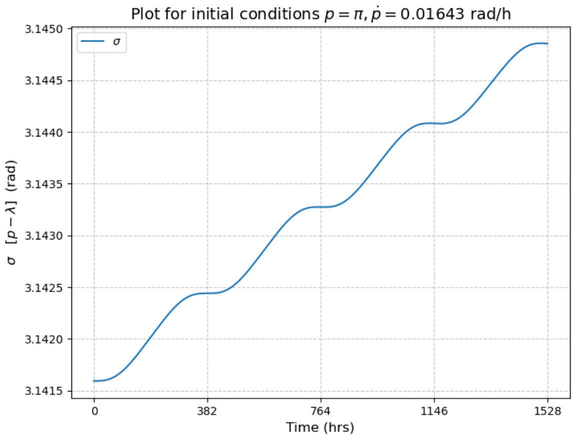

Figure 3.6: σ variation for eccentricity $e^* = 5e$

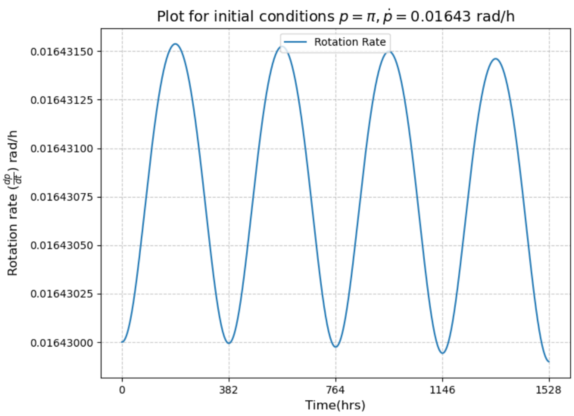

Figure 3.7: Rotation rate for eccentricity $e^* = 2e$

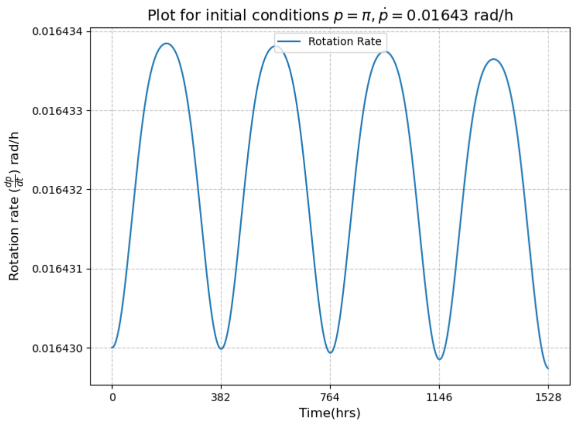

Figure 3.8: Rotation rate for eccentricity $e^* = 5e$

The amplitude of the librations increases with increase in the eccentricity. One of the reasons could be that a increasingly eccentric orbit leads to more variations in the gravitational field experienced by the Titan due to saturn, which will cause the body to wobble more and another one is that the tidal forces on Titan increase due to increased eccentricity. These forces cause it to experience tidal bulges that are offset from the line connecting the centers of the Titan and the Saturn. As Titan rotates, the bulge moves around it, causing it to rock back and forth more than usual, leading to more variations in the rotation rate.

3.2 Variation in the rotation rate

In this section, we will see whether the synchronous dynamics of Titan continues if the angular rotation velocity is twice that of the orbital one. The plots 3.9 and 3.10 show the evolution of physical quantities σ and angular rotational velocity with time for the new conditions.

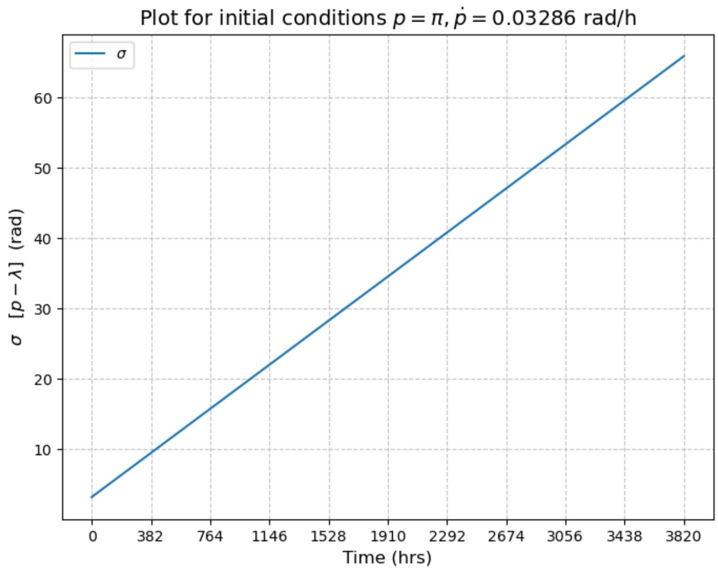

Figure 3.9: σ variations for $p = 2\omega$

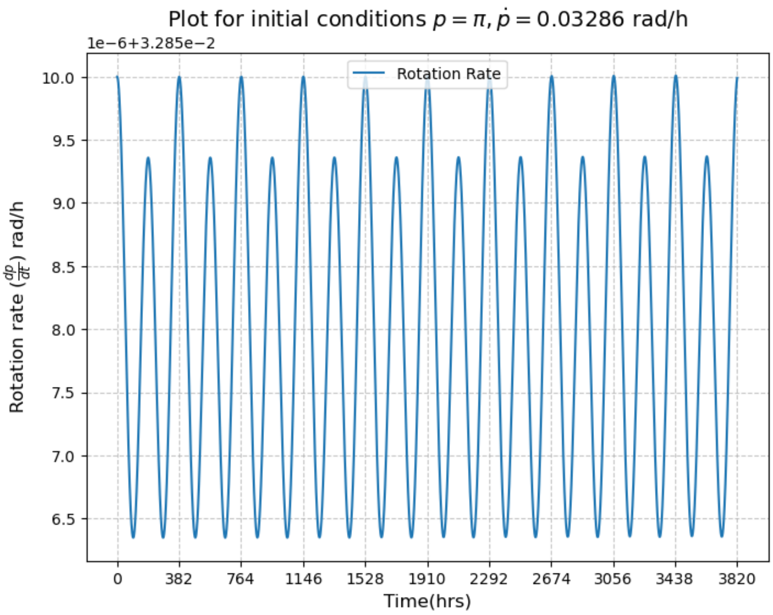

Figure 3.10: rotation rate variations for $p = 2\omega$

The rotate rate being twice that of the orbital frequency means that during one complete revolution of Titan around the Saturn, it completes two rotations around its own axis. Thus, there is no longer any 1:1 correspondence between the revolution and the rotation period, which means that the motion is no longer synchronous.

The rotation angle (more appropriately σ) shows a linear increase suggests that the body (Titan) is not keeping pace with its orbital motion and its rotation period is shorter than its orbital period. As a result, different parts of Titan are successively facing the Saturn, so an observer on Saturn will now be able to see the different faces of Titan. From the plot 3.10 we see that the librations now have a period of half of that of the orbital time period of Titan around Saturn. This means that Titan is in 1:2 spin orbit resonance.

3.3 Influence of Physical parameters

Now that we have analysed the results, an important question arises that which physical parameters affect this dynamics and how? In this section, we will try to shed some light on that. The dynamics is influenced by following factors.

- **Mass of the bodies:** If the mass of the central body (Saturn in our case) is large, it will exert a stronger gravitational force, meaning that the tidal forces on the orbiting body (Titan) will be larger which can aid in the achieving and maintaining synchronous rotation.
- **Eccentricity:** We studied the effect of eccentricity on the amplitude of librations, this suggests that as the eccentricity increases so does the deviation from a synchronous orbit. In highly eccentric orbits, the direction of tidal torque can change (due to precessional motion), potentially hindering the establishment of a stable synchronous motion.
- **Internal structure of Titan:** The tidal forces on Titan due to Saturn cause deformations in Titan's shape as well as lead to internal friction and heat generation within it, which is known as tidal heating. The rate at which this heat is dissipated affects how quickly Titan achieves synchronous rotation. In general, a deformable interior that dissipates energy efficiently may lead to a faster synchronization process. The same effect as discussed above can also be produced by the Titan's dense atmosphere.
- **Rotational Inertia:** The rotational inertia represents the resistance of a body to any changes in its angular rotation, determining how easily it can be slowed down or sped up by applied torques. Therefore, in a body with high rotational inertia (which comes from the distribution of the mass inside within the body itself), the tidal forces will have to overcome this first, which means a slower rate of synchronization.
- **Radius of the orbit:** A smaller orbit means a greater gravitational interaction of the orbiting body with the central body. This leads to stronger tidal forces, causing it to lose angular momentum faster, facilitating faster synchronization.

4 Conclusion

In this project, we simulated the synchronous rotation of Titan in Fortran by employing the fourth order Runge Kutta method. We found librations in both the rotational angle and the rotation rate with a period equal to the orbital period of Titan, which provided evidence for its synchronous rotation. Another set of resonant frequencies with a period of 2.1731 was also observed. The variations in the rotation rate became more pronounced with the increase in the eccentricity of the orbit, suggesting that the synchronous dynamics is influenced by the orbital eccentricities.

We explored how the rotation rate of Titan being twice that of its orbital frequency meant that it was no longer in a synchronous rotation but rather in a 1:2 spin orbit coupling. Furthermore,we studied the influence of physical parameters like mass of Saturn, eccentricity of Titan's orbit, internal and atmospheric compositions, the mass distribution within Titan and the separation between Titan and Saturn on the synchronous dynamics of Titan in particular and all other celestial bodies in general.

In conclusion, this study provides a sneak peek at the phenomenon of synchronization which is caused by the complex and often chaotic dynamics of celestial interactions. This study can be extended beyond Titan and applied to other heavenly bodies that are governed by synchronous rotations and of course the current study can be enhanced by incorporating 3 Dimensional model of Titan rather than a planar one, using a higher order integrator, taking in account the structural dynamics of Titan and its axial tilt.

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