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Optics for Imaging - Homework 9

Irradiance, PSF and MTF

(I have included both solutions with & without "i" with ϕ_a & ϕ_b terms) for 1C.

a. solution:

we have: $I = \langle \frac{n}{Z_0} (A_x^2 + A_y^2) \cos^2(\vec{k}\cdot\vec{r} - \omega t) \rangle$

$$= \frac{1}{\tau} \cdot \frac{n}{Z_0} \cdot (A_x^2 + A_y^2) \cdot \int_0^{\tau} \cos^2(\vec{k}\cdot\vec{r} - \omega t) dt \quad , \text{ using } \langle f(t) \rangle = \frac{1}{\tau} \int_0^{\tau} f(t) dt$$

$$= \frac{n}{Z_0} (A_x^2 + A_y^2) \frac{1}{2} \int_0^{\tau} 2\cos^2(\vec{k}\cdot\vec{r} - \omega t) dt$$

$$= \frac{n}{Z_0} (A_x^2 + A_y^2) \cdot \frac{1}{2\tau} \left[\lim_{\tau\to\infty} \int_0^{\tau} [1 + \cos 2(\vec{k}\cdot\vec{r} - \omega t)] dt \right]$$

$$= \frac{n}{Z_0} (A_x^2 + A_y^2) \cdot \frac{1}{2} \left[\lim_{\tau\to\infty} \left[\underbrace{\frac{1}{\tau} \int_0^{\tau} dt}_{\to 1} + \underbrace{\frac{1}{\tau} \int_0^{\tau} \cos 2(\vec{k}\cdot\vec{r} - \omega t) dt}_{\to 0} \right] \right]$$

$$= \frac{n}{Z_0} (A_x^2 + A_y^2) \cdot \frac{1}{2} \cdot 1$$

$$\therefore I = \left(\frac{1}{2}\right) \frac{n}{Z_0} (A_x^2 + A_y^2)$$

b. solution:

from above irradiance is: $I = \frac{1}{2} \frac{n}{Z_0} (A_x^2 + A_y^2)$ —— (i)

given $\vec{E}(r,t) = (A_x \hat{x} + A_y \hat{y}) e^{ikz - i\omega t}$

$$= (A_x \hat{x} + A_y \hat{y}) e^{i(kz - \omega t)}$$

$$\Rightarrow \vec{E}^*(r,t) = (A_x \hat{x} + A_y \hat{y}) e^{-i(kz - \omega t)}$$

Now, $\vec{E}(r,t) \cdot \vec{E}^*(r,t) = (A_x \hat{x} + A_y \hat{y}) \cancel{e^{i(kz - \omega t)}} \cdot (A_x \hat{x} + A_y \hat{y}) \cancel{e^{-i(kz - \omega t)}}$

i.e. $\vec{E}(r,t) \vec{E}^*(r,t) = (A_x^2 + A_y^2)$

so, if we replace $(A_x^2 + A_y^2)$ by $\vec{E}(r,t) \vec{E}^*(r,t)$ in (i) we get,

$$I = \frac{1}{2}\frac{n}{Z_0}\vec{E}(r,t)\,\vec{E}^*(r,t)$$

which is the required Expression.

c. solution: (i) {I didn't knew there should be "i" with "ϕ_a" & "ϕ_b" terms}

Here,

$$\vec{E_a} = (a_x\hat{x} + a_y\hat{y})e^{i(kz-\omega t)+\phi_a}$$

$$\vec{E_b} = (b_x\hat{x} + b_y\hat{y})\,e^{i(kz-\omega t)+\phi_b}$$

$$\vec{E} = \vec{E_a} + \vec{E_b} = (a_x\hat{x} + a_y\hat{y})e^{i(kz-\omega t)}e^{\phi_a} + (b_x\hat{x} + b_y\hat{y})e^{i(kz-\omega t)}e^{\phi_b}$$

$$= \{(a_x\hat{x} + a_y\hat{y})e^{\phi_a} + (b_x\hat{x} + b_y\hat{y})e^{\phi_b}\}\,e^{i(kz-\omega t)}$$

$$\therefore \vec{E} = \{(a_xe^{\phi_a} + b_xe^{\phi_b})\hat{x} + (a_ye^{\phi_a} + b_ye^{\phi_b})\hat{y}\}\,e^{i(kz-\omega t)}$$

Now,
$$\vec{E}\cdot\vec{E}^* = (a_xe^{\phi_a} + b_xe^{\phi_b})^2 + (a_ye^{\phi_a} + b_ye^{\phi_b})^2$$

So,
$$I = \frac{1}{2}\frac{n}{Z_0}(\vec{E}\cdot\vec{E}^*)$$

$$\Rightarrow I = \frac{1}{2}\frac{n}{Z_0}\{a_x^2e^{2\phi_a} + b_x^2e^{2\phi_b} + 2a_xe^{\phi_a}b_xe^{\phi_b} + a_y^2e^{2\phi_a} + b_y^2e^{2\phi_b}$$
$$+ 2a_ye^{\phi_a}b_ye^{\phi_b}\}$$

Rearranging the terms:

$$\therefore I = \frac{1}{2}\frac{n}{Z_0}\{(a_x^2 + a_y^2)e^{2\phi_a} + (b_x^2 + b_y^2)e^{2\phi_b} + 2(a_xb_x + a_yb_y)e^{\phi_a+\phi_b}\}$$

This is the required expression of I.

d. The unit of "E" is same as units of "A" given in above equation which is [V/m]. "E" is just same as "A" with a phase term. So, units of both "A" & "E" are same.

2.c: Solution :

Here,

$$\vec{E_a} = (a_x \hat{x} + a_y \hat{y}) e^{i(kz - \omega t + \phi_a)}$$

$$\vec{E_b} = (b_x \hat{x} + b_y \hat{y}) e^{i(kz - \omega t + \phi_b)}$$

So,

$$\vec{E} = \vec{E_a} + \vec{E_b}$$

$$= (a_x \hat{x} + a_y \hat{y}) e^{ikz - i\omega t + i\phi_a} + (b_x \hat{x} + b_y \hat{y}) e^{ikz - i\omega t + i\phi_b}$$

lly,

$$\vec{E}^* = (a_x \hat{x} + a_y \hat{y}) e^{-i(kz - \omega t + \phi_a)} + (b_x \hat{x} + b_y \hat{y}) e^{-i(kz - \omega t + \phi_b)}$$

Now,

$$\vec{E}.\vec{E}^* = a_x^2 + a_y^2 + a_x b_x e^{i(\phi_a - \phi_b)} + a_y b_y e^{i(\phi_a - \phi_b)} + b_x^2 + b_y^2$$

$$+ a_x b_x e^{i(\phi_b - \phi_a)} + a_y b_y e^{i(\phi_b - \phi_a)}$$

$$\Rightarrow \vec{E}.\vec{E}^* = a_x^2 + b_x^2 + a_y^2 + b_y^2 + 2a_x b_x \left\{ \frac{e^{i(\phi_a - \phi_b)} + e^{-i(\phi_a - \phi_b)}}{2} \right\}$$

$$+ 2a_y b_y \left\{ \frac{e^{i(\phi_a - \phi_b)} + e^{i(\phi_a - \phi_b)}}{2} \right\}$$

$$\Rightarrow \vec{E}.\vec{E}^* = a_x^2 + b_x^2 + a_y^2 + b_y^2 + 2a_x b_x \cos(\phi_a - \phi_b) + 2a_y b_y \cos(\phi_a - \phi_b)$$

Now,

$$I = \frac{1}{2} \frac{n}{Z_0} (\vec{E}.\vec{E}^*)$$

$$\Rightarrow I = \frac{1}{2} \frac{n}{Z_0} \left\{ (a_x^2 + b_x^2) + (a_y^2 + b_y^2) + 2(a_x b_x + a_y b_y) \cos(\phi_a - \phi_b) \right\}$$