Beam Propagation via FFT's

Grover Swartzlander (Apr. 2006)

Given an electric field distribution in the plane z=0, we may compute the spatial frequency spectrum, where $k_x = k\sin\theta_x$ and $k_y = k\sin\theta_y$, and (θ_x, θ_y) is the angular direction of a ray:

$$G(k_x,k_y) = \frac{1}{2\pi}\iint E(x,y,0)e^{-ik_x x}e^{-ik_y y}dxdy = FT\left[E(x,y,0)\right] \tag{1a}$$

$$E(x,y,0) = \frac{1}{2\pi}\iint G(k_x,k_y)e^{ik_x x}e^{ik_y y}dk_x dk_y = FT^{-1}\left[G(k_x,k_y)\right] \tag{1b}$$

We may describe propagation in two ways. The first way is shown here. The alternative approach is shown in the section on non-paraxial waves below. These two approaches are equivalent and may be applied for either the paraxial or non-paraxial propagation.

Here we describe the propagating paraxial beam as a convolution integral

$$E(x',y',z) = \frac{1}{2\pi}\iint E(x,y,0)h(x,y;x',y';z)dxdy \tag{2}$$

where

$$h(x,y;x',y';z) = \frac{\exp(ikz)}{i\lambda z}\exp\left\{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]\right\} \tag{3}$$

Eq. (2) may be derived by assuming Huygen wavelets at the source points (x,y,0) and summing their contributions at the field point(x',y',z). Note: the one-dimensional Fourier transform of the function $\exp(-px^2)$ is equal to $(2p)^{-1/2}\exp(-k_x^2/4p)$. Setting $p = -ik/2z$ and changing variables, we may express the Fourier transform of $h(x,y;x',y';z)$ as

$$FT\left[h\right] = H(k_x,k_y) = \frac{\exp(ikz)}{2\pi}\exp\left[i\frac{z}{2k}(k_x^2+k_y^2)\right] \tag{4}$$

Note: we are concerned with the envelop of the electric field and may therefore neglect the factor $\exp(ikz)$. Making use of the convolution theorem for Fourier transforms,

$$FT\left[a\otimes b\right] = FT\left[a\right]FT\left[b\right] \tag{5a}$$

$$a\otimes b = FT^{-1}\left\{FT\left[a\right]FT\left[b\right]\right\} \tag{5b}$$

where the symbol $\otimes$ represents convolution. Thus

$$E(x',y',z) = FT^{-1}\left\{G(k_x,k_y)\exp\left[iz(k_x^2+k_y^2)/2k\right]\right\} \tag{6}$$

Numerical considerations

If x has a range [-L,L] and there are N_x grid points, then $\Delta x = 2L/N_x$ is the spatial increment. The corresponding range in k_x-space is $[-k_{max},k_{max}]$ where $k_{max} = \pi/\Delta x$. Since there are also N_x grid points in k_x-space we obtain $\Delta k_x = 2k_{max}/N_x = \pi/L$.

Let the phase factor on the right hand side of Eq. (4) be written in a dimensionless format:

$$\phi_x = zk_x^2/2k = (zk/2)(k_x^2/k^2) \tag{7}$$

A similar expression may be written for ϕ_y. The j^{th} grid point in k_x-space will have a value

$$\phi_x(j) = (zk/2)(j\Delta k_x/k)^2 = (zk/2)(\pi/Lk)^2 j^2 = \alpha j^2 \tag{8}$$

where we define the dimensionless factor $\alpha = (zk/2)(\pi/Lk)^2$.

Limits to Propagation Distance

To properly sample the phases, we require

$$\phi_x(j+1) - \phi_x(j) = \alpha((j+1)^2 - j^2) = \alpha(1+2j) < \pi \tag{9}$$

Since $j:[-N_x/2,N_x/2]$ we insert $j=N_x/2$ into this condition to obtain

$$\alpha < \pi/N_x \tag{10}$$

We have assumed $N_x>>1$. From Eq. (8) we see that this imposes a restriction on the propagation distance:

$$z < z_{\max} = \frac{2kL^2}{\pi N_x} \tag{11}$$

Let's introduce a scale size: $w_0 = \sqrt{N_x/\pi}\Delta x$, where Δx corresponds to the physical size of a single pixel, i.e., $\Delta x = 2L/N_x$. Note that $L = N_x\Delta x/2 = w_0(\pi N_x)^{1/2}/2$. We can re-write our maximum propagation distance as $z_{\max} = \frac{2kL^2}{\pi N_x} = \frac{kw_0^2}{2}$. This is equal to the Rayleigh diffraction length corresponding to a feature of transverse size w_0.

In practice you may be able to propagate beyond z_{max} with very little error. Equation (10) was determined for the largest value of j in k_x-space, i.e., $N_x/2$. For many types of simulations there is very little power in $G(k_x,k_y)$at these large spatial frequencies, and thus the phase error from under sampling is moot. The practical maximum distance of propagation depends on the distribution of $G(k_x,k_y)$. For example, if $G(k_x,k_y)$ vanishes beyond $\pm\sqrt{N_x/\pi}$ pixels from the radial center, then Eq. (9) provides a new inequality: $\alpha' < \sqrt{\pi/N_x}/2 = \alpha\sqrt{N_x/\pi}/2$. This is a factor of $\sqrt{N_x/\pi}/2$ larger than α, and thus you may attempt to propagate the beam by as many as $\sqrt{N_x/\pi}/2$ Rayleigh ranges. First be sure to check the distribution of $G(k_x,k_y)$ before propagating this far.

Also be aware of aliasing in x-space for long propagation distances. The FFT algorithm acts as if your numerical grid is infinitely periodic. The "neighboring" beams interfere with each other when the beam sizes exceed your computation grid size.

Finally, if the input field distribution has a finite size, it's a good idea to define its scale to $\sqrt{N_x/\pi}$ pixels. For example, if the initial beam is Gaussian, let its waist, w_0 correspond to $\sqrt{N_x/\pi}$ pixels. In this case the Fourier transform of the Gaussian will be exactly the same number of pixels.

Non-Paraxial Case

Let us represent the propagating beam in a slightly different way than Eq. (2) and (3). Here we claim the field is given by a superposition of plane waves:

$$E(x,y,z) = \frac{1}{2\pi}\iint G(k_x,k_y)\exp(i\vec{k}\cdot\vec{r})dk_x dk_y \qquad (12)$$

where $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$, $k = |\vec{k}| = \sqrt{k_z^2 + k_\perp^2}$, and $k_\perp^2 = k_x^2 + k_y^2$. We have a monochromatic beam, $k = const$; hence k_z depends on the variables k_x and k_y. Thus we express Eq. (12) as