IMGS-633: HW-9: Irradiance, PSF, and MTF

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1. Irradiance

Formally the measured irradiance of a plane wave is given by the time-integrated Poynting vector:

$$I \ = < \vec{S}(\vec{r},t)\cdot\hat{k} > = < (\mathrm{Re}[\vec{E}]\times\mathrm{Re}[\vec{H}])\cdot\hat{k} > = < \frac{n}{Z_0}(\vec{A}\cdot\vec{A})\cos^2(\vec{k}\cdot\vec{r}-\omega t) > \tag{1}$$

where $\vec{A}$ is the amplitude of a plane wave electric field (where A has units of $[V/m]$), n is the refractive index of the material, $Z_0 = 377$ [Ohm] is the impedance of vacuum, irradiance has units of $[W/m^2]$, and the time average of a function $f(t)$ is given by

$$< f(t) > = \frac{1}{\tau}\int_0^\tau f(t)dt \ \text{ where } \ \tau \ \text{ is the detector integration time.}$$

(a) If $\vec{A} = A_x\hat{x} + A_y\hat{y}$, show that $I = (1/2)(n/Z_0)(A_x^2 + A_y^2)$, assuming τ tends toward infinity.

(b) If $\vec{E}(r,t) = (A_x\hat{x} + A_y\hat{y})e^{ikz-i\omega t}$, show that we can also write the irradiance as $I = (1/2)(n/Z_0)\vec{E}(r,t)\cdot\vec{E}^*(r,t)$ where $*$ represents complex conjugate.

(c) Two co-propagating plane waves interfere: $\vec{E}_a = (a_x\hat{x} + a_y\hat{y})e^{ikz-i\omega t+\phi_a}$ and $\vec{E}_b = (b_x\hat{x} + b_y\hat{y})e^{ikz-i\omega t+\phi_b}$, where the phase ϕ_a and ϕ_b are constants. Determine a simplified expression for the irradiance of the net field $\vec{E} = \vec{E}_a + \vec{E}_b$.

(d) The irradiance is often written in the short-hand form $I = \vec{E}(r,t)\cdot\vec{E}^*(r,t)$. What are the units of E in this cases if I has units of $[W/m^2]$?

2. Zernike Polynomials

A lens or aperture having an imperfect phase transfer function may be described in terms of Zernike polynomials. (This is the same Frits Zernike you learned about in your previous homework set. See for example: `https://spie.org/publications/fg10_p24-25_zernike-polynomials?SSO=1`.) These are beautiful modes of a circular diaphragm, typically expressed in circular coordinates. Note that unlike Seidel aberration polynomials, Zernike polynomial are independent object point positions and therefore do not describe all types of aberrations.

In general a transparent circular aperture function may be expressed $A(x_l, y_l) = \mathrm{circ}(r_l/a)\exp(i\Phi(r,\theta))$, where a is the radius of the circular aperture, and Φ describes the phase aberration. You are strongly encouraged to explore various Zernike modes. Here we will examine the "Oblique Trefoil" mode:

$\Phi(x_l, y_l) = \phi_0\ (r_l/a)^3\ \cos(3\theta_l)$ where $r_l = \sqrt{x_l^2 + y_l^2}$ and $\theta_l = \tan^{-1}(y_l/x_l)$

and where ϕ_0 is the magnitude of the phase aberration. The Marechal criterion roughly states: If the net wavefront distortion is less than $2\pi/14$ (i.e., $1/14^{th}$ of a wave) then the system produces "diffraction limited" images.

Let us numerically examine the "Oblique Trefoil" aberration function and its consequences.

Write a computer program using Python, Matlab, or similar to perform a two-dimensional ($N\times N$ pixels) "Fast Fourier Transform" (FFT) operation on the function

$E(x_l, y_l) = A(x_l, y_l)\exp(i\Phi(x_l, y_l))$ where $A(x_l, y_l) = \exp(-((x_l^2+y_l^2)/a^2)^{50})$ is the hyper-Gaussian aperture function.

Let $\phi_0 = 2\pi$ here, and $\Phi(x_l, y_l) = \phi_0\ (r_l/a)^3\ \cos(3\theta_l)$.

Note: $x_l = i_x - n_{cen}$ and $y_l = i_y - n_{cen}$, $a = \sqrt{N/\pi}$, $n_{cen} = 1 + N/2$, $i_x = 1, 2, 3, ..., N$ and $i_y = 1, 2, 3, ..., N$, $x = -N/2, -N/2+1, ... -1, 0, 1, ...N/2-1$, and $y = -N/2, -N/2+1, ... -1, 0, 1, ...N/2-1$.

If your computer memory allows, let $N = 4096$ so that $a = 36.1$ pixels.

Render cropped grayscale images of the following:
(a) Magnitude at the aperture function: $|E|$
(b) Phase at the aperture function: $\Phi_E = \text{atan2}(\text{IM}[E], \text{RE}[E])$.

Let h = FT[E] represent point response of the aperture function. Render cropped grayscale images of the following:
(c) Far-Field Magnitude: $|h|$
(d) Phase of h: $\Phi_h = \text{atan2}(\text{IM}[h], \text{RE}[h])$.

Let the Point Spread Function (PSF) be represented by the squared amplitude in (c): $\text{PSF} = |h|^2$.
(e) Render a cropped grayscale image of the PSF.

Let the Incoherent Optical Transfer Function $\tilde{H}$ be represented by the inverse Fourier transform of the PSF:
$\text{OTF} = \tilde{H} = \text{FT}^{-1}[PSF]$.

Physically the magnitude $|OTF|$ must not exceed unity, otherwise the system will exhibit gain. The maximum value of $|OTF|$ occurs at the zero frequency point, which corresponds to the numerical pixel value $(N/2+1, N/2+1)$. Therefore, you must normalize OTF by the value $|OTF(N/2+1, N/2+1)|$.

Render cropped grayscale images of the
(f) Magnitude of the OTF (called the Modulation Transfer Function): $MTF = |OTF|$.
(g) Phase of the OTF (called the Phase Transfer Function): $PTF = \text{atan2}(\text{IM}[OTF], \text{RE}[OTF])$.

3. Ideal PSF, MTF, OTF

Repeat parts (a)-(f) for the case of an ideal lens by setting $\phi_0 = 0$ above. Provide a brief qualitative description of each rendered grayscale image, with and without aberrations. Feel free to change ϕ_0 in problem (1) to a larger value to examine the aberration effects.

4. Imaging a Starburst Pattern With and Without Aberrations

Make an amplitude object that produces a starburst pattern containing radial arms:

$$E_{obj}(x_o, y_o) = \exp(-\,\gamma^{10}), \quad \text{where } \gamma = 4\cos(8\theta_o) \text{ and } \theta_o = \text{atan2}(y_o, x_o)$$

You may let this object function span the entire numerical grid. Place the origin $(x_o, y_o) = (0,0)$ at the grid point $(N/2+1, N/2+1)$.

Coherent Illumination
The electric field of the physical optics image $E_{\text{image}}(x_i, y_i)$ may be expressed as the convolution of the object field with the point response of the imaging system, h. Making use of the convolution theorem for Fourier transforms numerically determine the irradiance $|E_{\text{image}}(x_i, y_i)|^2$ for

(a) the unaberrated case in Problem 2, i.e., with $\phi_0 = 0$

(b) the aberrated case in Problem 2, i.e., with $\phi_0 = 2\pi$ (or higher).

(c) Describe the qualitative difference between these two images.

Incoherent Illumination
The irradiance of the physical optics image $I_{\text{image}}(x_i, y_i)$ may be expressed as the convolution of the object irradiance with the PSF of the imaging system, $|h|^2$. Making use of the convolution theorem for Fourier transforms numerically determine the irradiance $I_{\text{image}}(x_i, y_i)$ for

(d) the unaberrated case in Problem 2, i.e., with $\phi_0 = 0$

(e) the aberrated case in Problem 2, i.e., with $\phi_0 = 2\pi$ (or higher).

(f) Describe the qualitative difference between these two images.