PROJECT #2 Graduate Optics

~~Due Tuesday, April 15~~

Here you will be learning how to perform numerical fast Fourier transforms to learn and visualize important concepts.

Note: if you use Matlab, the fft algorithm requires that you perform the operation "fftshift" BEFORE and AFTER each Fourier transform operation. Failure to do so will produce initially non-obvious errors, but these will haunt you and lead to hair loss. (ok .. joking about the hair loss). So don't make the mistake of countless fools ... always fftshift before and after calling fft.

Gary's MATLAB advice:
Create an fft function called myfft2 that handles the fftshift. This way you never have to worry about forgetting to shift the arrays. Save the following functions to the MATLAB default path. This is necessary if your function is centered about the $N/2+1$ index of your array, which is pretty much always the case in optical beam propagation simulations.

```
function out = myfft2( in )
out = fftshift(fft2(ifftshift(in)));
end

function out = myifft2( in )
out = ifftshift(ifft2(fftshift(in)));
end
```

Also note: at times you will want to view your numerical results as a linear grayscale image, other times you may want to "gamma" it or examine it in log scale (nonlinear scale). Be sure you always write on the image what kind of scaling you've used. Also state whether "white" corresponds to a high value or low value. (You may decide to invert the color map if it's useful to view the image that way.)

Gary's MATLAB advice:
Gamma the colormap. Do not gamma a saved plot afterwards. This will allow you to render the axis labels, ticks, colorbar, and annotations normally. A value of gamma between 0 and 1 brightens the darker features in your image.

Example: displaying a 2D floating point array

```
beta = 0.5;
imagesc(xp,yp,abs(field));
axis square;
set(gca,'YDir','normal')
colormap(gray(256).^beta);
colorbar;
xlabel('x (mm)');
ylabel('y (mm)');
title('Abs of field');
```

This is a general style you should stick to for displaying optical fields during beam propagation calculations. Be sure to include meaningful axis labels, units, and annotations if necessary.

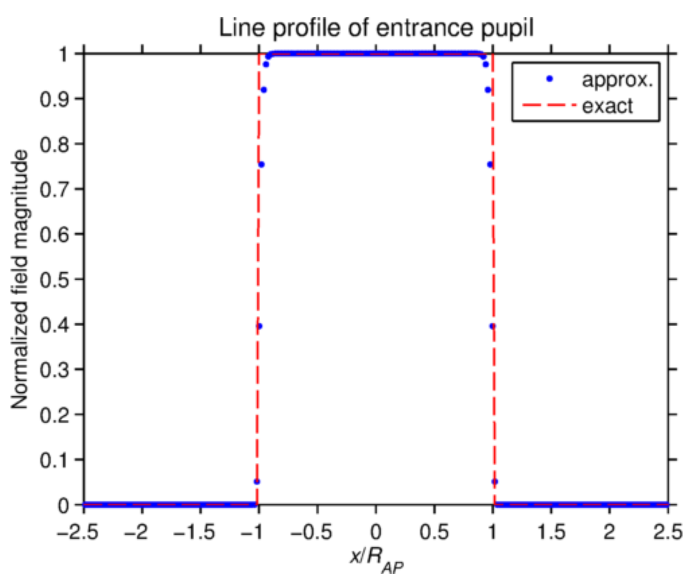
Generally, you will be computing an $N \times N$ array of data, where N must be (for fft's) equal to a power of two (512, 1024, 2048, ...). Note that there is a special characteristic size for this array, namely, the value $\sqrt{N/\pi}$. This magic value is of great use, as you will see.

If you are asked to create a circ function, please do NOT simply call circ as a MATLAB function. Instead, write a small section of code, defining $\mathrm{circ}(r/a)$ as $\exp\left[-(r/a)^{2c}\right]$, where c is a large number like 20 or more. Prove to yourself (plot the values) that this produces a circ function with a smooth (rather than abrupt) transition when eta is sufficiently large. (A trick in Fourier manipulation is to avoid introducing very high frequency noise that is introduced by abrupt step functions.) Use the value of a that corresponds to $\sqrt{N/\pi}$ pixels.

Gary's MATLAB advice:
A good way to start a beam propagation simulation script is to define arrays with coordinate values at each sample.

```
clear; close all;
N = 4096;              % Maximize this value for best results.
a = sqrt(N/pi) ;       % Entrance aperture radius [pixels]
lamoverd = pi*a/2;     % lambda/D at image plane [pixels]
[X,Y] = meshgrid(-N/2:N/2-1);% Create 2 arrays with x,y coordinates.
[THETA,RHO] = cart2pol(X,Y);% Create 2 arrays with r,theta coordinates.
EP = exp(-(RHO/a).^60);   % Define Hypergaussian entrance pupil
```

The value of N should be powers of 2 for maximum processing speed. However, MATLAB accepts arrays of any size. A line profile of the hypergaussian entrance pupil is shown below. Notice that there are a few samples along the edges to prevent aliasing.



1. Two mutually incoherent plane waves are transmitted through a single circular aperture:

$$E_1(x,y) = \exp(+i2\pi x/L)\,\mathrm{circ}(r/a),$$
$$E_2(x,y) = \exp(-i2\pi x/L)\,\mathrm{circ}(r/a).$$

You will determine the correct value of L to achieve the following condition. (That is, try different values of L until the following occurs.) Since the two fields are mutually incoherent, you will separately compute each fft, take the modulus squared to find the intensity, then add the intensities of the two distributions. Determine the value of L that produces "just barely resolvable" objects.

Setting the relation between the wave tilt parameter, L, the wavelength of light, λ, and the angular tilt of the wavefront, θ_L : $L = \lambda/\theta_L$ show that your numerical result agrees the with Rayleigh resolution criterion.

Solution:

Quick analytical answer: Owing to the location of the first zero of the Airy disk function, the value of L must correspond to an angular shift $\theta_L = 0.5(1.22\,\lambda/D)$. Thus, $L = \lambda/\theta_L = 1.64D$.

Full derivation: Using our knowledge of discrete Fourier transforms, we may find the value of L analytically in term of samples. Consider a one dimensional DFT. If we assume that we will only sample over a finite interval, the sample spacing in the spatial domain Δx and the spectral domain Δk_x are related by $\Delta x \Delta k_x N = 2\pi$ [1]. In two dimensions, the Airy disk function is of the form $J_1(k_r R_{AP})/k_r R_{AP}$, where k_r is the radial component of the wave vector in the image (x', y') plane. If we consider only a 1D profile along the x' axis, the argument of the Bessel function may be written as a discretely sampled sequence $k_r R_{AP} = (n_{kx}\Delta k_x)(a\Delta x) = 2n_{kx}/a$, where n_{kx} is the spatial frequency sample index and $a = \sqrt{N/\pi}$. Again, we find that the first zero of the Airy disk occurs when $2n_{kx,0}/a = 1.22\pi$. Solving for the index, we have $n_{kx,0} = 1.22\pi a/2$. A discretely sampled plane wave has the form

$$\tilde{\imath} \qquad = \exp(+i2\pi n_x n_{kx}/N),$$

where n_x is the spatial sample index. Its DFT is a discrete "delta function" (non-zero at one sample) located at n_{kx}. We want $n_{kx} = n_{kx,0}/2$, so that the images of the plane waves are separated by $n_{kx,0}$ samples. Thus, the discrete plane waves at the aperture may be written

$$\tilde{\imath} \qquad \exp\left(+i2\pi n_x \frac{n_{kx,0}}{2}\frac{1}{N}\right)\mathrm{circ}(r/a),$$
$$\tilde{\imath} \qquad \exp\left(-i2\pi n_x \frac{n_{kx,0}}{2}\frac{1}{N}\right)\mathrm{circ}(r/a).$$

By inspection, $L = 2N/n_{kx,0} = 4a/1.22 = 3.28a$.

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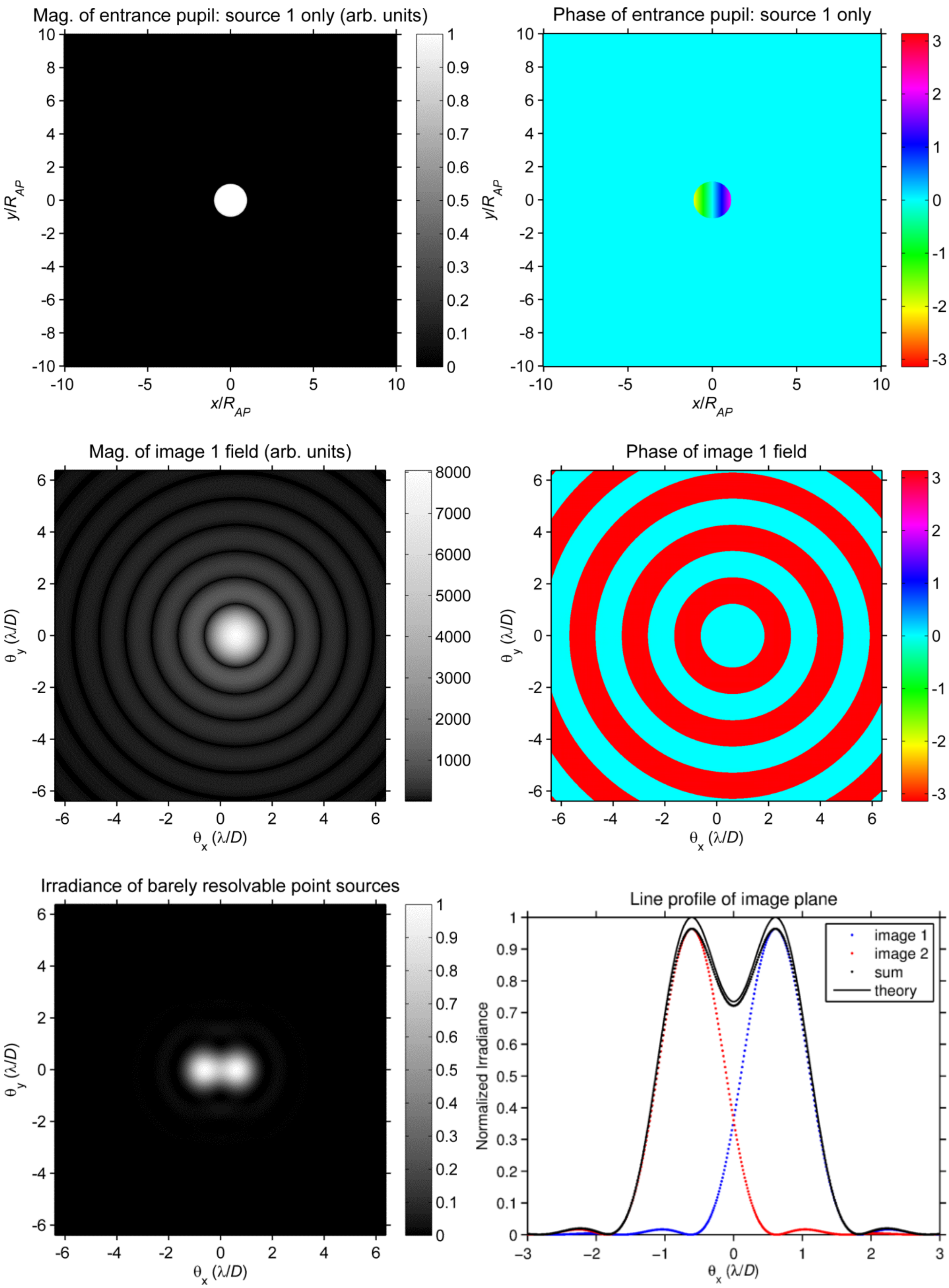
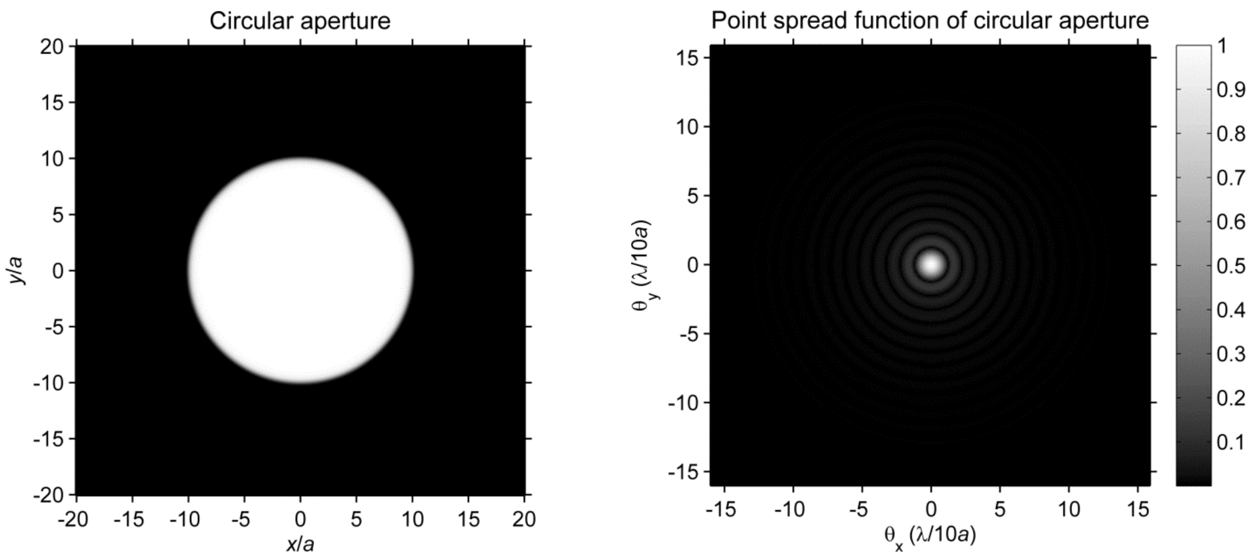
[1] See e.g. Roger Easton's "Fourier Methods in Imaging." In particular, the derivation leading up to equation 15.14 (p. 517).



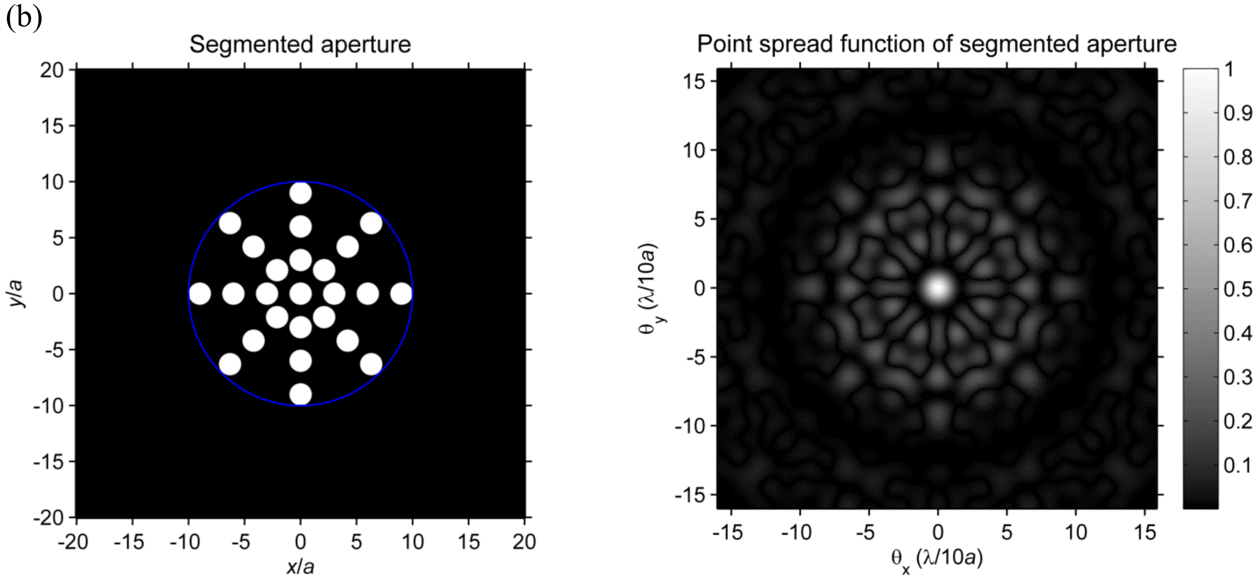
Mag. of entrance pupil: source 1 only (arb. units)
Phase of entrance pupil: source 1 only
x/R_{AP}
y/R_{AP}
Mag. of image 1 field (arb. units)
Phase of image 1 field
θ_x (λ/D)
θ_y (λ/D)
Irradiance of barely resolvable point sources
Line profile of image plane
Normalized Irradiance
image 1
image 2
sum
theory

2. A single plane wave is incident upon a large circular aperture of radius R (corresponding to $10\sqrt{N/\pi}$ pixels, where N is as large as possible for your computer (4096 or larger is possible). Note: be sure to terminate all other programs working on your computer because you will need all your memory to complete this task. (a) Compute and display the far-field intensity profile for this aperture. (b) Within this aperture region, set the transmission equal to zero, except for several smaller "sub-apertures" have unity transmission. Make each of these sub-apertures have a radius $a = \sqrt{N/\pi}$. Compute and display the far-field intensity profile for this system of apertures. (c) Same as (b), except use only two sub-apertures. Compare and contrast the images. Are the smallest feature sizes in (a) and (b) about the same? What dictates the characteristic size of the distributions? What accounts for the period and orientation of the interference lines you see in (c)?

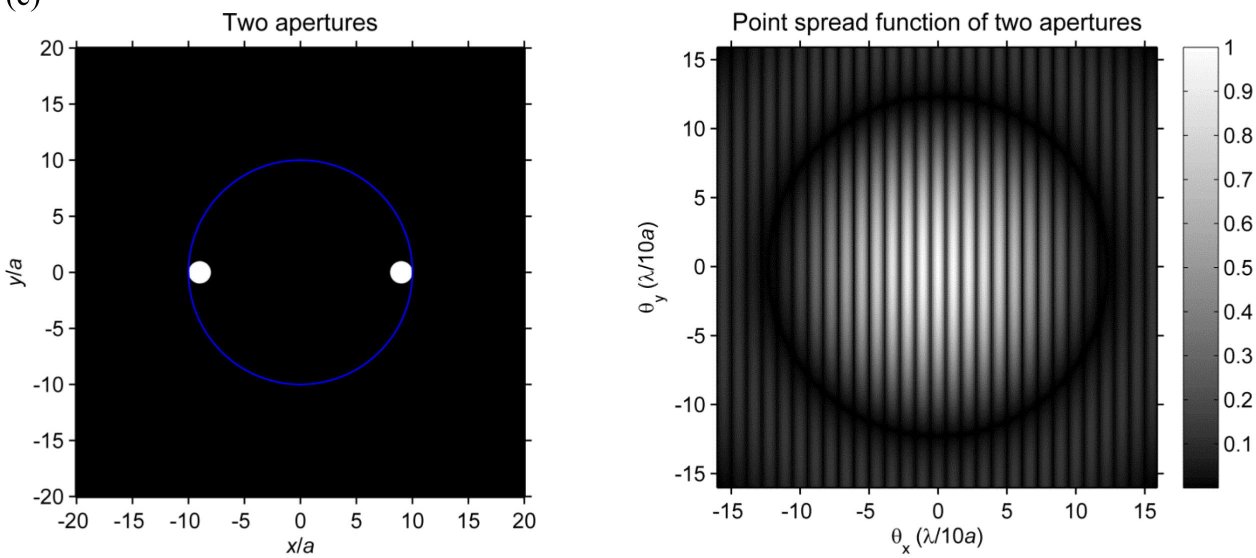
Solution:
(a) The peak normalized field magnitudes are shown below on nonlinear gray scales.



(b)



(c)



Discussion of results

The images in the right column represent the point spread function (PSF) of the corresponding aperture functions. The coherent PSF is the field formed when an infinitely distant point source is imaged. The width of the central spot of the PSF dictates image quality and resolution. The full aperture has the most compact PSF with a central spot diameter of $2.44\lambda/D$. The segmented aperture results in a distorted PSF with complicated interference patterns that may significantly degrade image quality. However, the characteristic size of central spot is roughly the same as the full aperture if the largest dimension (or “baseline”) is the same size. This can also be seen in the case of two apertures. The characteristic size of smallest feature is determined by the baseline. On the other hand, the PSF is much wider along the orthogonal direction, which is dictated by the size of the individual circular apertures. In other words, two apertures separated along the x axis have smaller PSF features and higher resolution along the x axis. There is no gain in resolution over a single aperture in the y direction.

3. The following are some interesting Zernike polynomials:

$$Z_4(r,\theta) = \sqrt{3}\left[2(r/a)^2 - 1\right], \qquad \text{(defocus)}$$

$$Z_5(r,\theta) = \sqrt{6}(r/a)^2 \sin(2\theta), \qquad \text{(astigmatism)}$$

$$Z_6(r,\theta) = \sqrt{6}(r/a)^2 \cos(2\theta), \qquad \text{(astigmatism)}$$

$$Z_7(r,\theta) = \sqrt{8}\left[3(r/a)^2 - 2(r/a)\right]\sin(\theta), \ \text{(coma)}$$

$$Z_8(r,\theta) = \sqrt{8}\left[3(r/a)^2 - 2(r/a)\right]\cos(\theta), \ \text{(coma)}$$

Imagine you have a circular aperture with phase aberration: $\mathrm{circ}(r/a)\exp(ia_n Z_n)$, where a_n is a coefficient. Assume an *x*-tilted input plane wave: $\exp(i2\pi\, x/a)$ or a *y*-tilted input plane wave: $\exp(i2\pi\, y/a)$.

Your task is to distinguish the PSF of a perfect aperture $(a_n = 0)$, with that of the aberrated aperture for each of the above Zernike polynomials. For each Zernike, produce three images, corresponding to $a_n = 2\pi$.

For example the first set of six images will be generated for the Z_4 Zernike polynomial and the following wave tilt and coefficients:

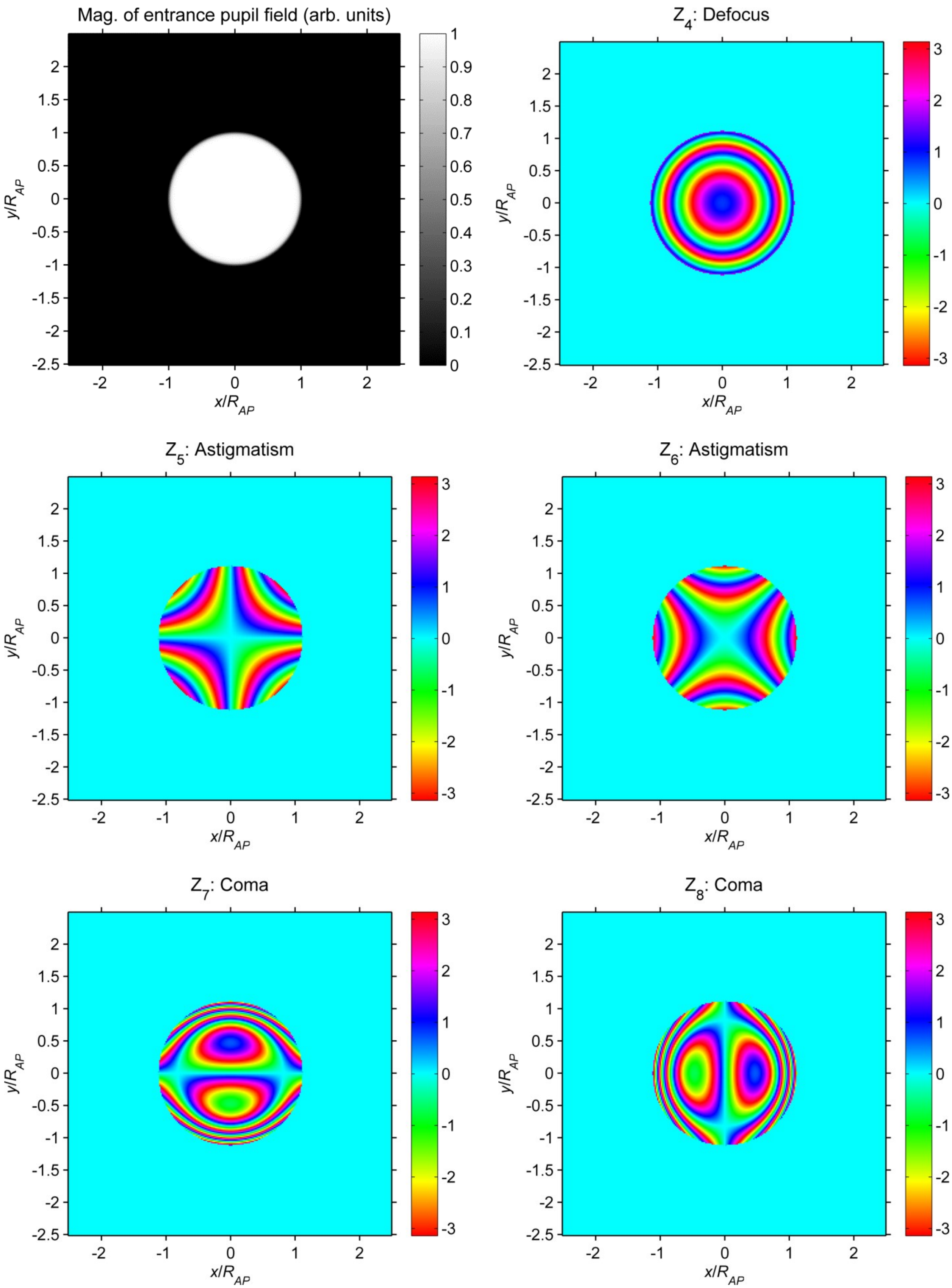
x tilt: $a_4 = 0,\ a_4 = \pi,\ a_4 = 2\pi$

y tilt: $a_4 = 0,\ a_4 = \pi,\ a_4 = 2\pi$

Be sure to crop your images to show only the "good stuff". Also, invert your image so that bright values appear dark so as to achieve better looking images (and to save black ink). Clearly label each image, and provide a paragraph under each set, describing what you observe. Feel free to "play" and try your own conditions.

Solution:

The magnitude of the aperture field and the phase due to Zernike aberrations are shown below:



Mag. of entrance pupil field (arb. units)
Z4: Defocus
Z5: Astigmatism
Z6: Astigmatism
Z7: Coma
Z8: Coma
x/R_AP
y/R_AP

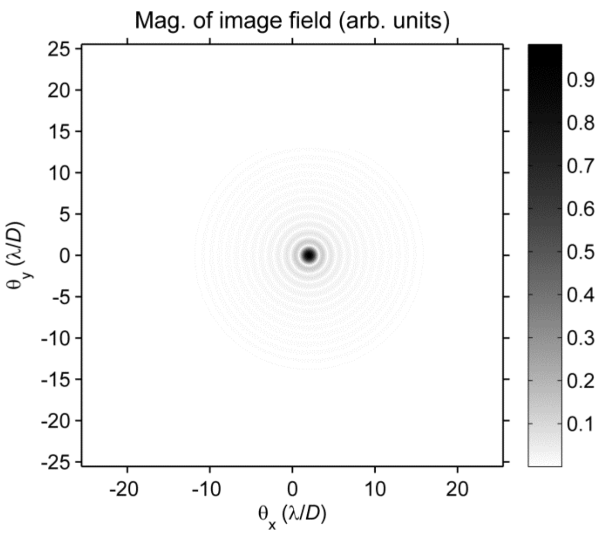
For each Zernike polynomial listed above, we will generated the image of a titled plane wave at the aberrated aperture for three different values for a_n. For clarity, the table below lists all combinations of interest. I exclude the combinations in red because they are redundant.

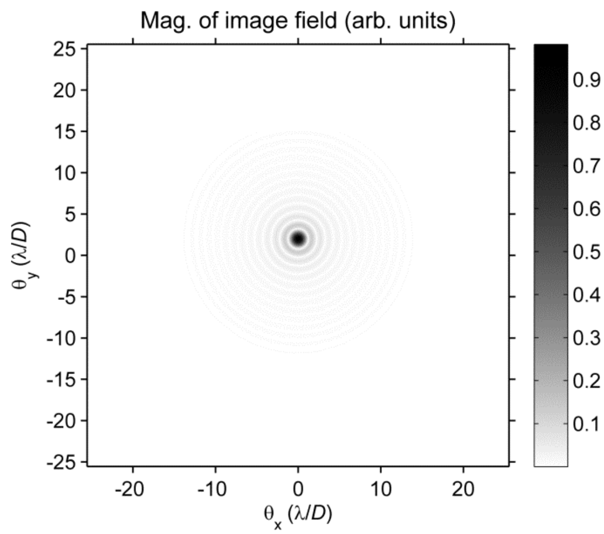
n	tilt	a_n	n	tilt	a_n	n	tilt	a_n	n	tilt	a_n	n	tilt	a_n
4	x	0	5	x	0	6	x	0	7	x	0	8	x	0
4	x	π	5	x	π	6	x	π	7	x	π	8	x	π
4	x	2π	5	x	2π	6	x	2π	7	x	2π	8	x	2π
4	y	0	5	y	0	6	y	0	7	y	0	8	y	0
4	y	π	5	y	π	6	y	π	7	y	π	8	y	π
4	y	2π	5	y	2π	6	y	2π	7	y	2π	8	y	2π

Z_4: Defocus

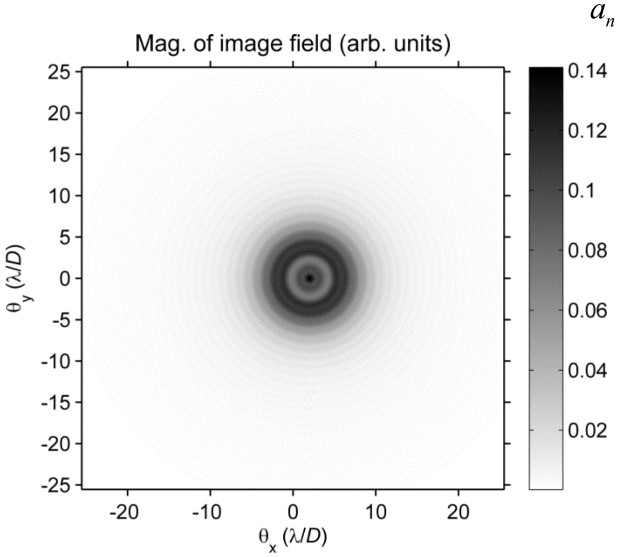
<u>x tilted plane wave</u> <u>y tilted plane wave</u>

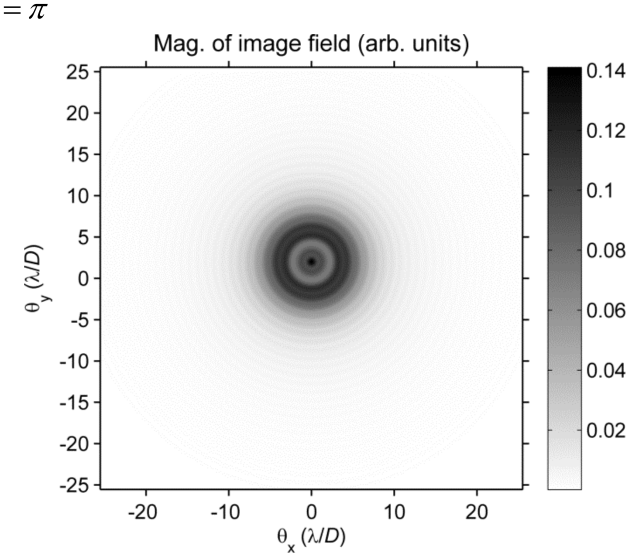
$a_n = 0$



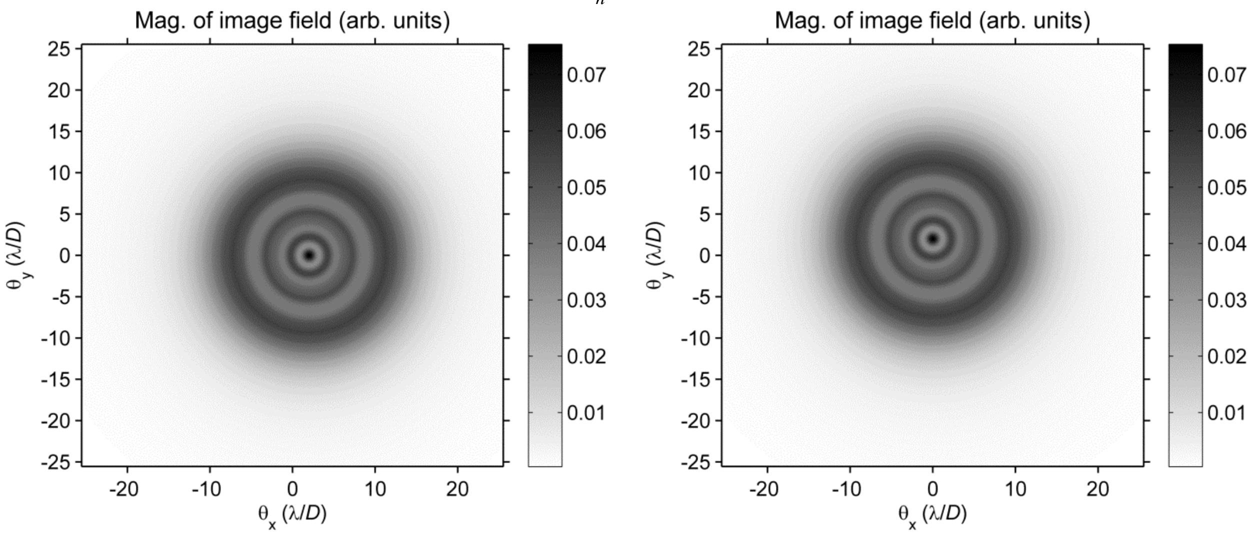


$a_n = \pi$



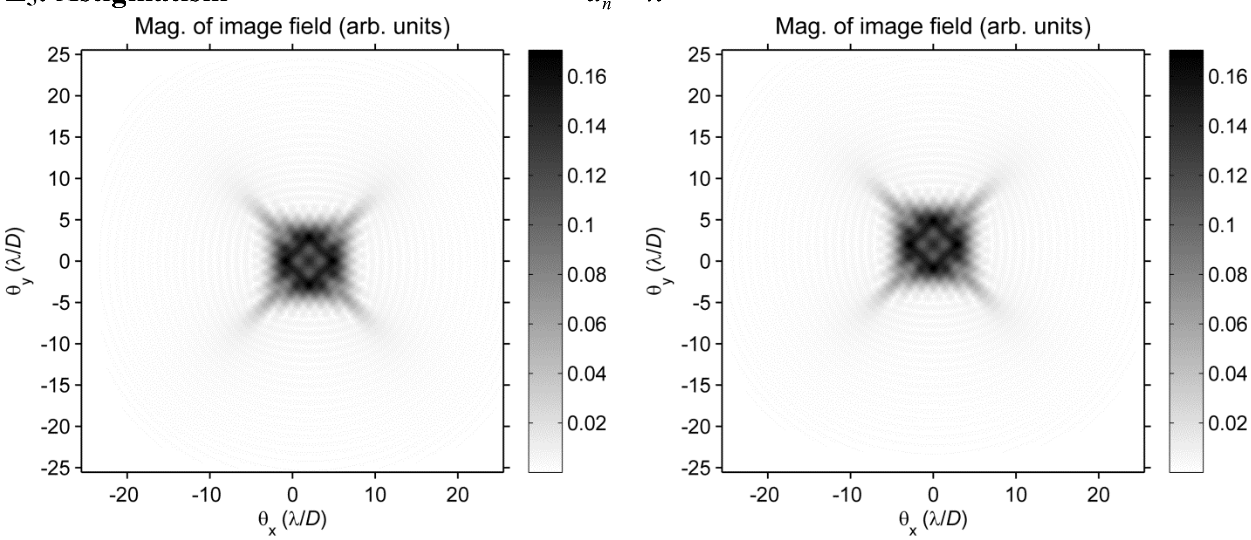


$$a_n = 2\pi$$

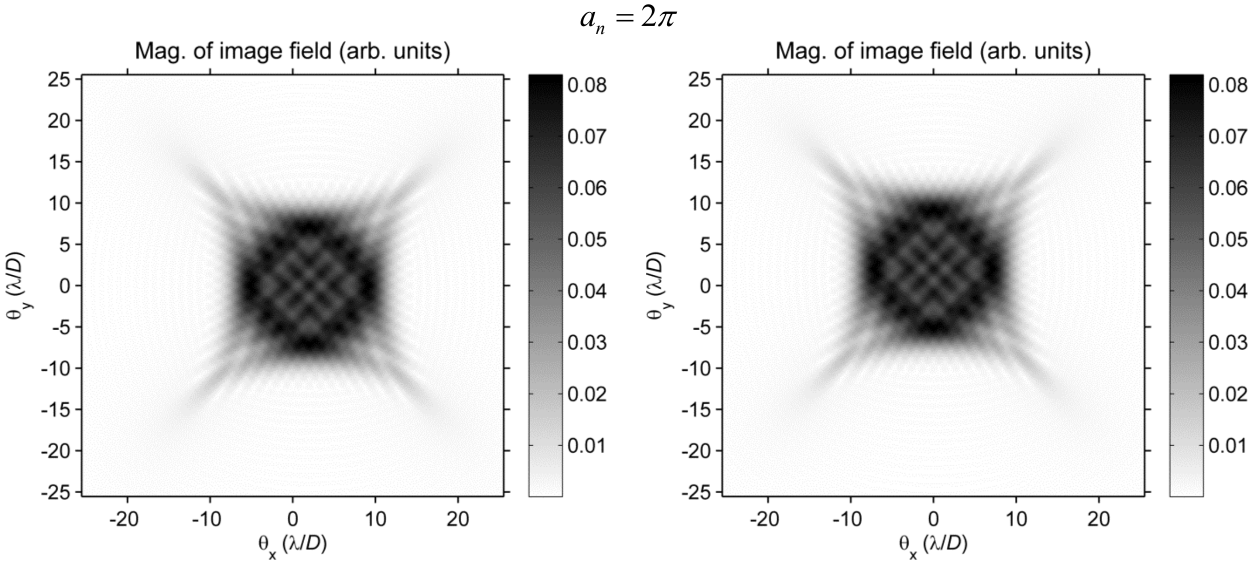


Z_5: Astigmatism

$$a_n = \pi$$

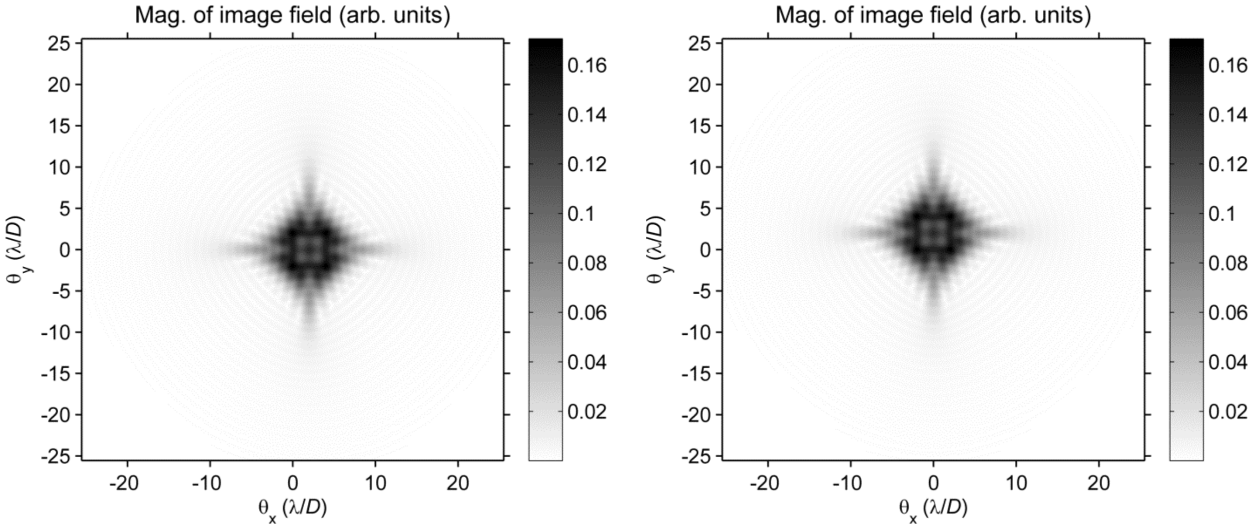


$$a_n = 2\pi$$

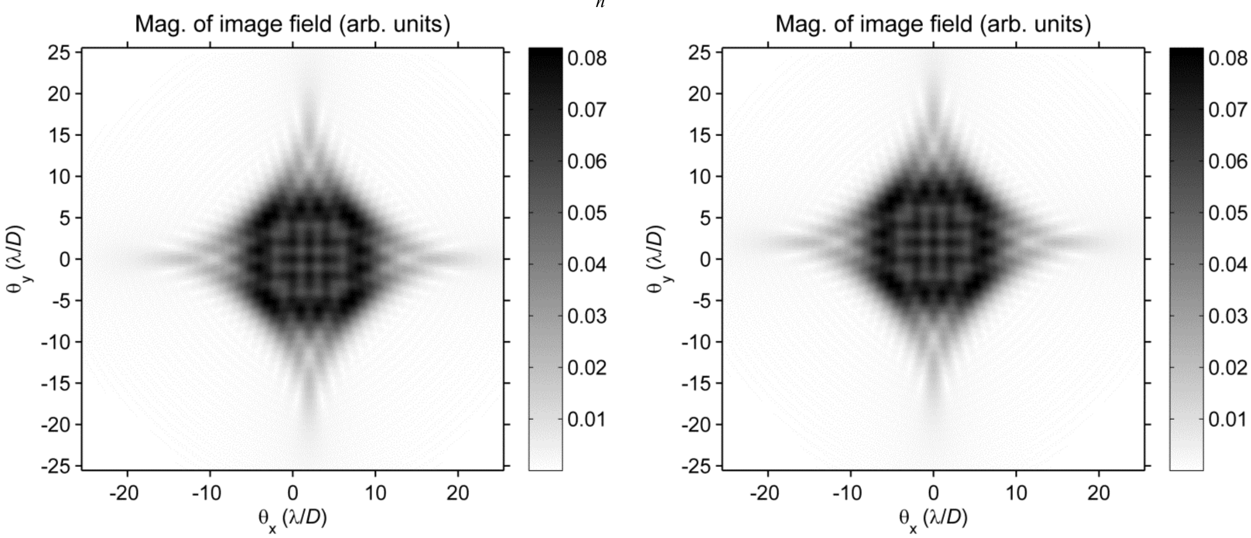


Z_6: Astigmatism

$a_n = \pi$

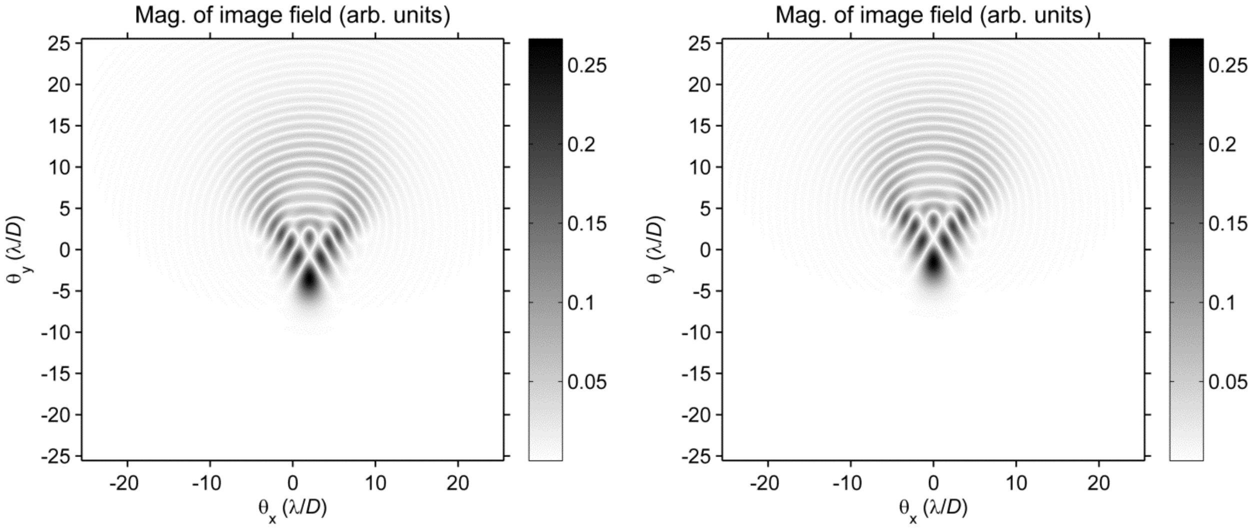


$a_n = 2\pi$

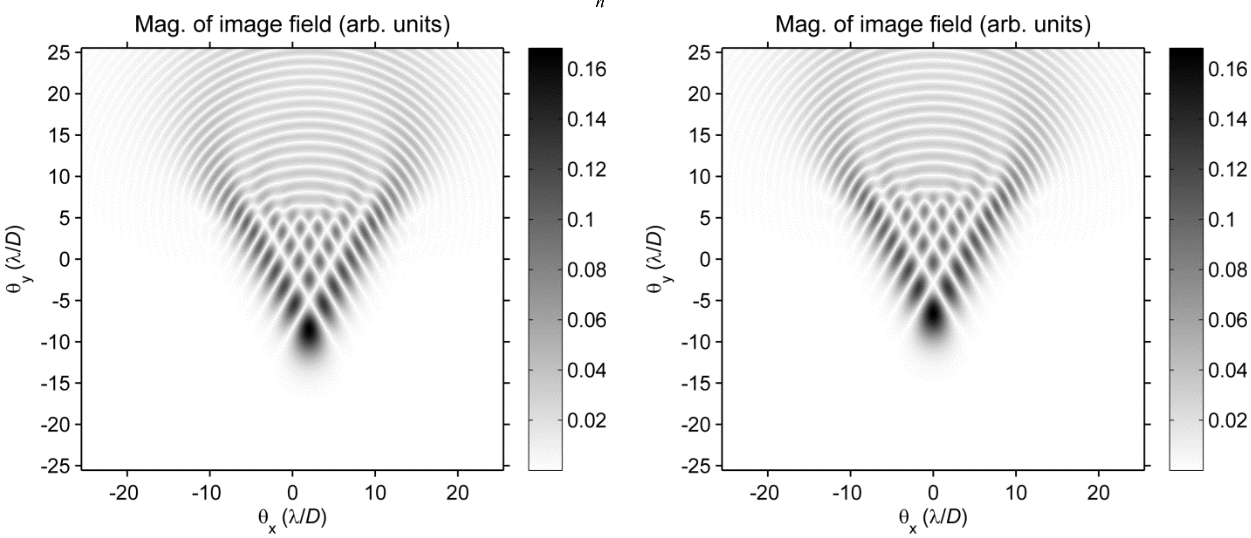


Z_7: Coma

$$a_n = \pi$$

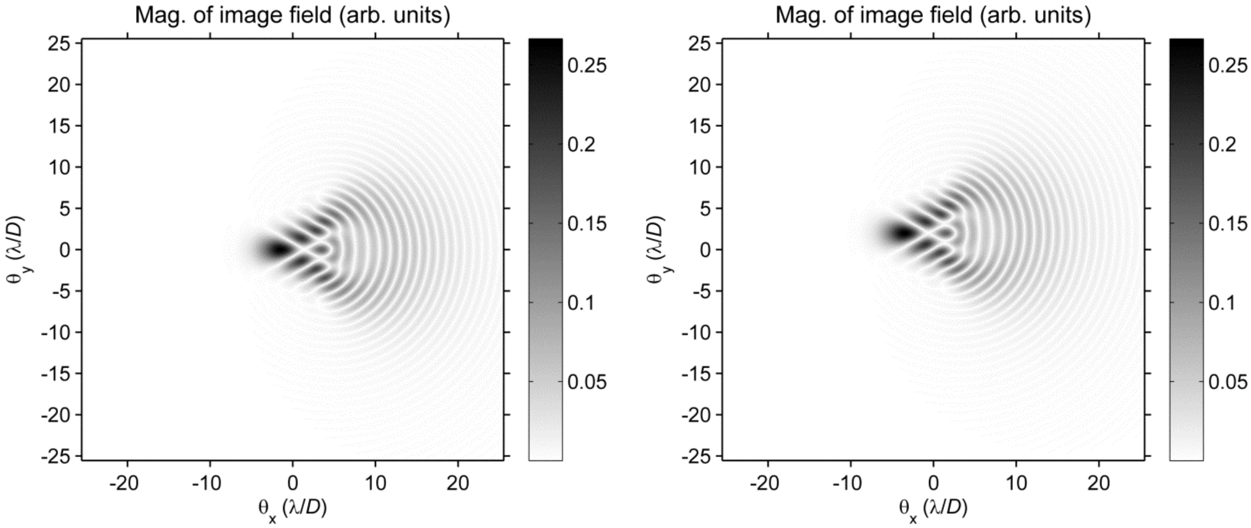


$$a_n = 2\pi$$

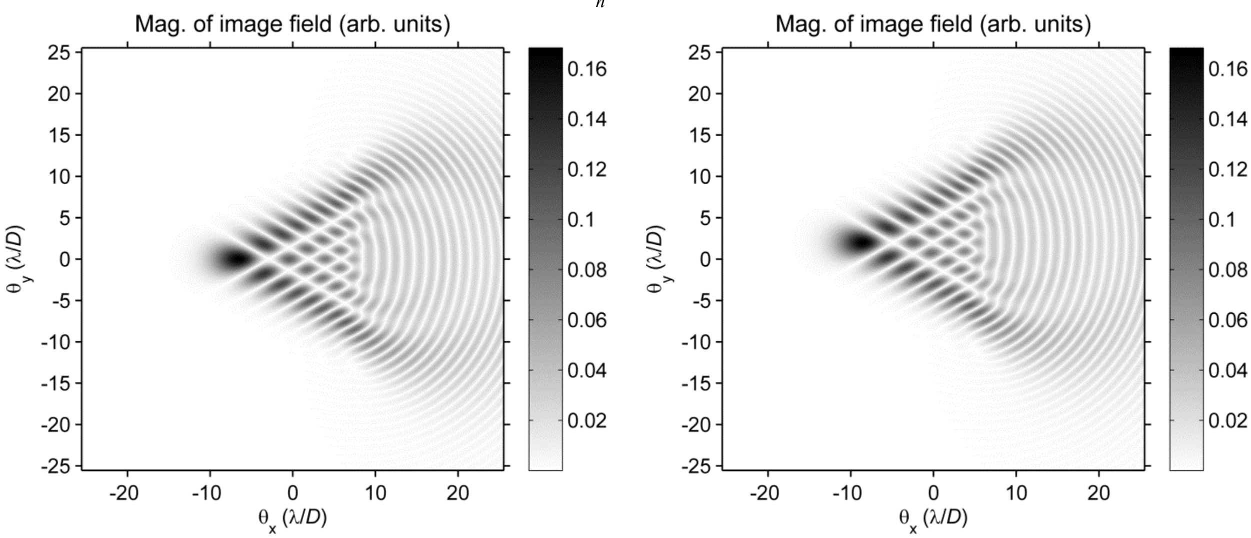


Z_8: Coma

$$a_n = \pi$$



$$a_n = 2\pi$$



General comments: Notice that the point spread function becomes larger in the presence of aberration. As we stated in the previous problem, a more compact point spread functions results in higher resolution. Thus, aberration degrades resolution and results in a distorted image.