Barbara Hammer, Fabian Hinder
Universität Bielefeld

Applied Optimization (WS 2023/2024)
Exercise Sheet No. 3

Upload Date: 2023-11-09.
Submission Deadline: In groups of three until **2023-11-23, 7:55** to the Moodle.
Return date: 2023-11-30 in the tutorials.

In case you have questions do not hesitate to ask your tutor or to contact the tutor team at apopt@techfak.uni-bielefeld.de.

Remark Convexity is closed under the following operations: Sums, multiplications with positive scalars, concatenation with linear maps.

Remark Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times continuously differentiable function and $I \subset \mathbb{R}$ be an open interval. Then f is convex on I, i.e. $f(\lambda a + (1-\lambda)b) \leqslant \lambda f(a) + (1-\lambda)f(b)$ for all $a, b \in I,\ \lambda \in [0,1]$, if and only if $f''(x) \geqslant 0$ for all $x \in I$.

Remark: For each task, do not only provide the final solution but also a full derivation ("step-by-step") for your solution.

Remark: When handing in your solution, do not only provide the source code, but also all the other information we ask for (e.g. the output of the optimization, the runtime, etc.). We will *not* execute your code!

1 Shadow of a Box

(*3 points*)

Consider the constraint optimization problem:

$$\begin{aligned}
&\min_{\mathbf{x}\in\mathbb{R}^4} \begin{pmatrix} 2 & 9 & -3 & 1 \end{pmatrix} \cdot \mathbf{x} \\
&\text{s.t.} \ -5 \leqslant x_1 \leqslant 2 \\
&\quad\ 0 \leqslant x_2 \leqslant 7 \\
&\quad -18 \leqslant x_3 \leqslant 1 \\
&\quad -1 \leqslant x_4 \leqslant 1
\end{aligned}$$

(a) *(1 pts.)* Phrase the problem as a problem in standard from.

(b) *(1 pts.)* Find a simple and fast algorithm that computes the minimal/maximal value of a linear function with box constraints.

(c) *(1 pts.)* Solve the problem above.

2 Constraints that Make it Easy

(*3 points*)

Consider the function

$$f(x) = \frac{x^4}{12} - \frac{x\cos(8x)}{64} + \frac{\sin(8x)}{256}$$

Find the global minimum of f by:

(a) *(1 pts.)* Calculate the first and the second derivative of f.

(b) *(1 pts.)* Plot f, f', f'' for $x \in [-10, 10]$. Describe f; is it convex? Can you spot a global minimum? Is f convex near that minimum?

(c) *(1 pts.)* Find a constraint that turns optimization problem $\min_{x\in\mathbb{R}} f(x)$ into a convex problem with the same global minimum. Write down the new problem in standard form.

3 Lagrange Dual

(*6 points*)

Consider the following optimization problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^2} \quad & \frac{1}{\sqrt{2}} \|\mathbf{v}-\mathbf{x}\| \\ \text{s.t.} \quad & \|\mathbf{x}\| = 1, \end{aligned}$$

where $\mathbf{v} \in \mathbb{R}^2,\ \mathbf{v} \neq 0$ is some vector and $\|\cdot\|$ denotes the euclidean norm.

(a) *(1 pts.)* Explain the geometrical interpretation of the problem – what do you expect regarding a solution? Simplify the problem so that it has a linear objective, i.e. $\max_{\mathbf{x}\in\mathbb{R}^2} \mathbf{w}^\top\mathbf{x}$, s.t.... for some $\mathbf{w} \in \mathbb{R}^2$, and write it in standard form. Does this problem has a global optimum? If yes, is there a global optimum if we remove the constraint, and why?
Hint: Certain operations like adding or multiplying with a constant, taking roots or squaring, etc. do not change the solution of an optimization problem (Why?). You can use those to simplify the problem. Also recall that $\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 - 2\mathbf{a}^\top\mathbf{b} + \|\mathbf{b}\|^2$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$.

(b) *(2 pts.)* Calculate the Lagrange-dual of the *simplified* problem. Find the analytic solution to the inner minimization problem. Write down the Lagrange dual as a problem in standard form and solve the problem. Does the solution match the geometrical expectation?

(c) *(1 pts.)* Consider the map $\phi : \mathbb{R} \to \mathbb{R}^2, t \mapsto (\sin(t), \cos(t))^\top$. Why is the initial problem equivalent to

$$\max_{t\in\mathbb{R}} \frac{1}{\sqrt{2}} \|\mathbf{v} - \phi(t)\|?$$

Why can we drop the constraints? How to turn a solution to the new problem into a solution to the original one.

In the following we will refer to this as the *parametrization* approach.

(d) *(2 pts.)* Implement and visualize the problem and both approaches (Lagrange-dual and parametrization) using gradient descent for $\mathbf{v} = (5, 7)^T$: Plot the function as a heat map and mark the feasible set. Implement the Lagrange-dual and parametrization using gradient descent and plot the respective paths on the heat map as well as the obtained values for each step in a second plot. What do you observe?

4 Modeling

(*8 points*)

Imagine the following: You own a house. Since it is very dark in the attic, you decide to put a rectangular window in the gable wall, which has the shape of an isosceles triangle with width w, height h. Because of the eaves, a shadow falls on the wall, which reduces the light intensity on average approximately linearly from 100% to 0% towards the top of the gable with a factor d (see Figure 1. d is a property of your house, you can consider it as a parameter of the optimization problem). The price you have to pay for the window is given by its area (up to a constant p). Assuming you want to pay at most a fixed value C for the window, what are the best possible coordinates for the corners of the window if you want to get as much light inside as possible?

(a) *(2.5 pts.)* Formalize the problem described above. Explain your choices! What do you expect as a reasonable solution? And why?

(b) *(1 pts.)* Turn your formalization into an optimization problem in standard from.

(c) *(2 pts.)* Implement your approach: Write a Python-script that, given the parameters w, h, d, p, and C, randomly generates a starting point and optimizes your function using one of the algorithms from the lecture. Plot your results, i.e., gable wall with window.

(d) *(2.5 pts.)* Discuss your results. Do you obtain a good solutions? Can you do better by hand? Is the found solution what you expected (compare to part (a))? Also discuss all assumptions made.

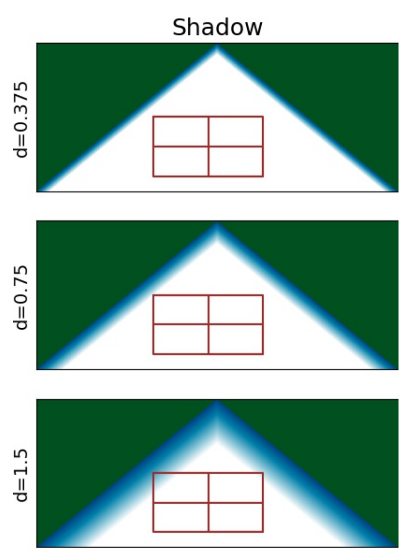

Figure 1: Shaddow on gable wall for different choices of d.