# Computer based Optimization Technique
# Lecture-II & III

LPP

Graphical Method

Types of Solution

Conversion of Normal LPP to standard LPP

Simplex method

# Brief History

- It is both Arts and Science
- It has started in World War-II (since 1930s) and then soon become very popular in mathematics, Industry, economics, management, and all most in any fields.
- Primary application areas include forecasting, production, scheduling, inventory control and project management
- It is also known as “Operation Research”, “Decision Science”, and Optimization Technique

**Optimization technique is the scientific study of operations for the purpose of making better decisions**

# Objective

- Decision making and improve its quality.
- Identify optimum solution.
- Integrating the systems.
- Improve the objectivity of analysis.
- Minimize the cost and maximize the profit.
- Improve the productivity.
- Success in competition and market leadership

# Different Methods

1.Simulation methods

- It gives ability to conduct sensitive study to -

    (a) search for improvements and

    (b) test the improvement ideas that are being made.

*2.Optimization methods*.

- Here goal is to enable the decision makers to identify and locate the very best choice, where innumerable feasible choices are available and comparing them is difficult.

- ***3.Data-analysis methods***
    - The goal is to aid the decision-maker in detecting actual patterns and inter-connections in the data set and
    - Use of this analysis for making solutions.
    - This method is very useful in Public Health.

# Process

1. Identification of program problem.
   - Most critical step in the process.
   - Unless problem is clearly defined it is impossible to develop good solutions.
2. Identification of possible reasons and solutions .
   - Once the problem has been identified , it is the job of the program implementer and researcher to determine the reasons for the problem and generate possible solutions

3.Testing of potential solution

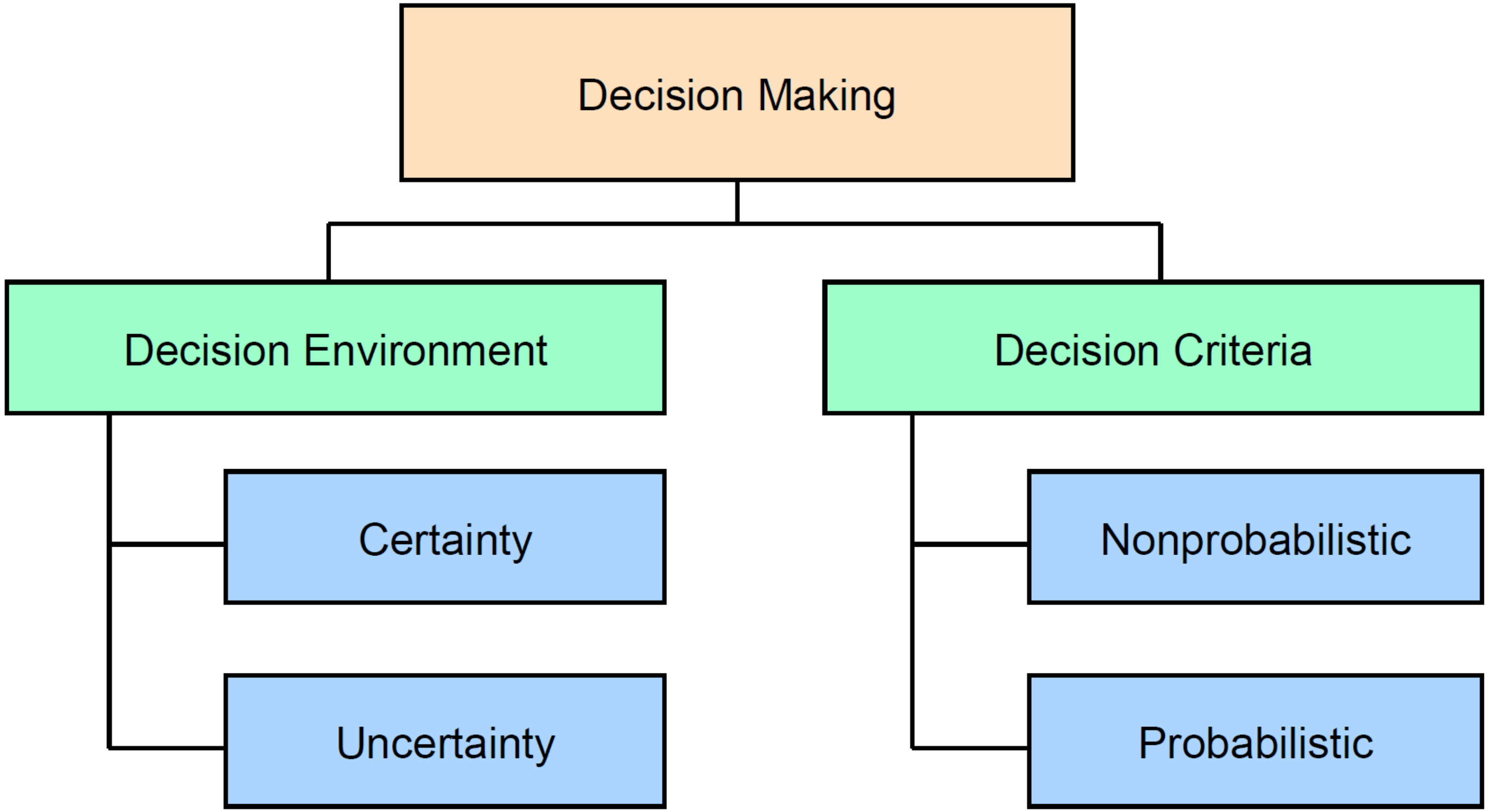
4.Result utilization

- It is necessary to decide how its results are meant to be used.

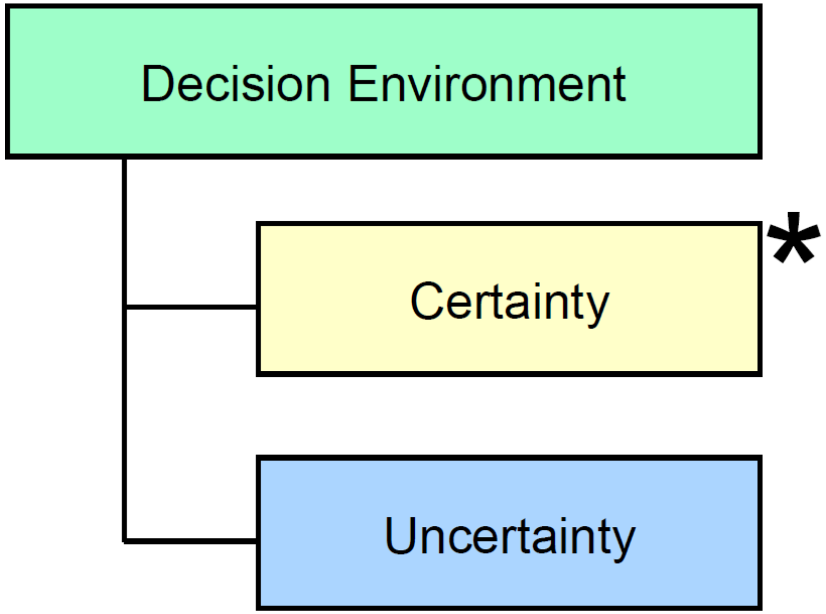
5.Results dissemination

- Results dissemination are done in the form of *seminars* or by *meeting* with decision makers.

# Decision Making Overview



# Decision Environment




Certainty: The results of decision alternatives are known

Decision Making Under Certainty When it is known for certain which is of the possible future conditions will happen, just choose the alternative that has the best payoff under the state of nature

Example:

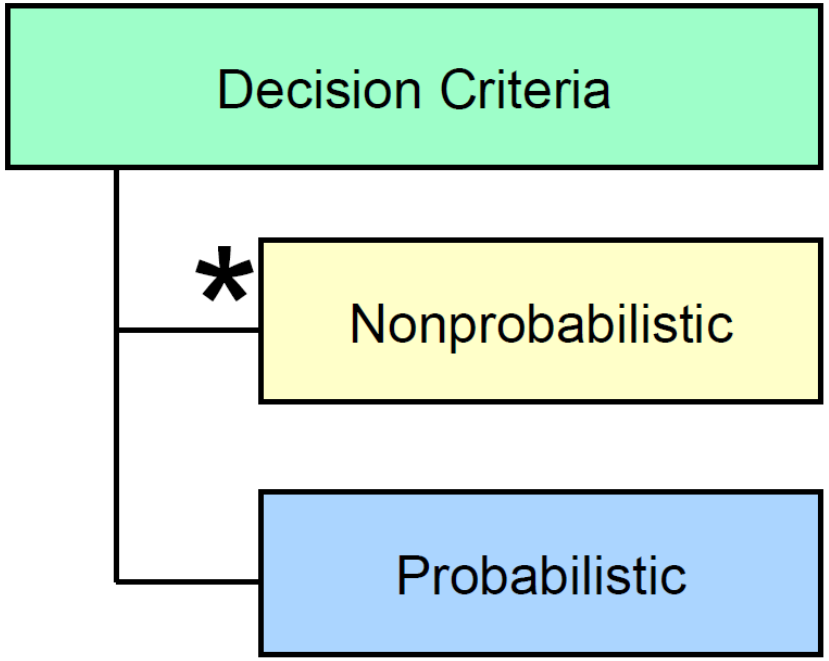
Must print 10,000 color brochures

Offset press A: 2,000 fixed cost
+ 0.24 per page

Offset press B: 3,000 fixed cost
+ 0.12 per page

# Decision Criteria

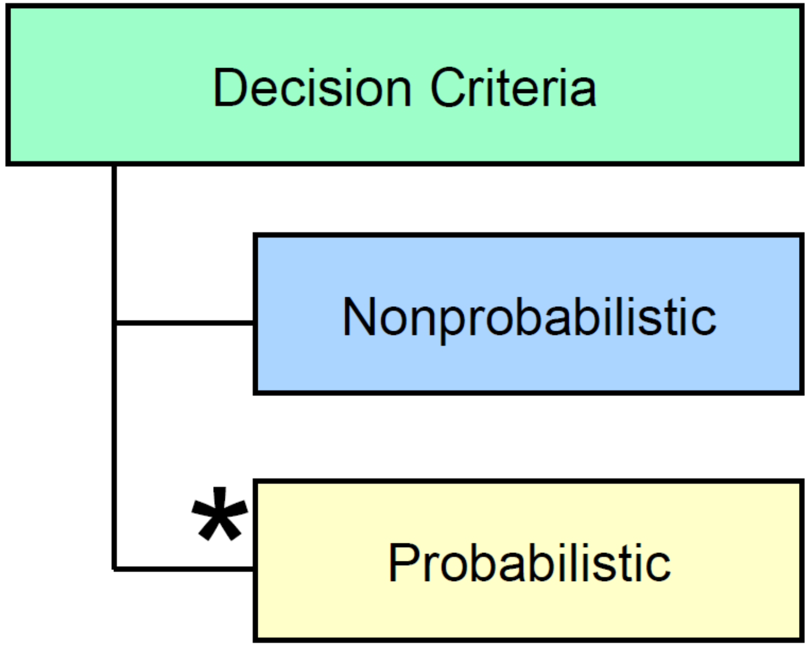
Nonprobabilistic Decision Criteria: Decision rules that can be applied if the probabilities of uncertain events are not known.




- maximax criterion
- maximin criterion
- minimax regret criterion

# Decision Criteria

Probabilistic Decision Criteria: Consider the probabilities of uncertain events and select an alternative to maximize the expected payoff of minimize the expected loss




- maximize expected value
- minimize expected opportunity loss

# Pay-off

- The **payoffs of a decision analysis problem are *the benefits or rewards that result from*** *selecting a particular decision alternative*

| | Strong Economy | Stable Economy | Weak Economy |
|---|---|---|---|
| Large Store | 200 | 50 | -100 |
| Average Store | 100 | 100 | -50 |
| Small Store | 50 | 30 | 25 |

- *Maximax, Minimax and Minimax regret (opportunity loss)*

# Linear Programming Problem (LPP)

- Objective function
- Constraints
  - Unequal
    - Greater than equal
    - Less than equal
  - Equal
- Solution Approach
  - Graphical
  - Simplex Method

# Linear Problem

- Objective Function
  - Optimization Criteria
    - Maximization
    - Minimization
- Constraints
  - Inequal
    - Greater than equal to
    - Less thank equal to
  - Equal

# Two-Dimensional LP

- There are only two decision variable

**2–dim Linear Programming Problem**

Min/ max $c_1x_1 + c_2x_2$

Subject to

$$a_{11}x_1 + a_{12}x_2 \le b_1$$

$$a_{21}x_1 + a_{22}x_2 \le b_2$$

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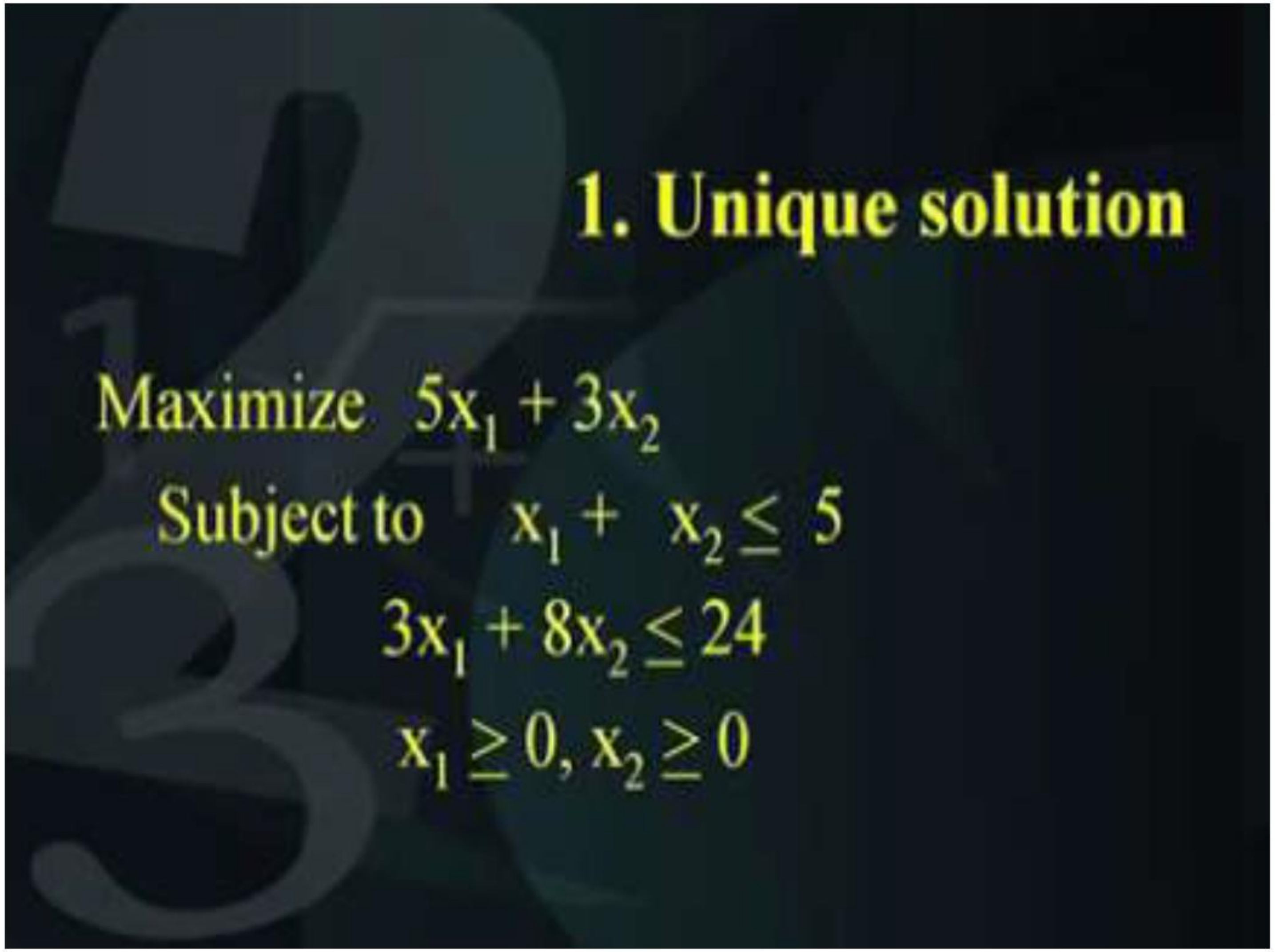
$$a_{m1}x_1 + a_{m2}x_2 \le b_m$$

$$x_1 , x_2, b_1, b_2 \ge 0$$

# Four possible Solution cases

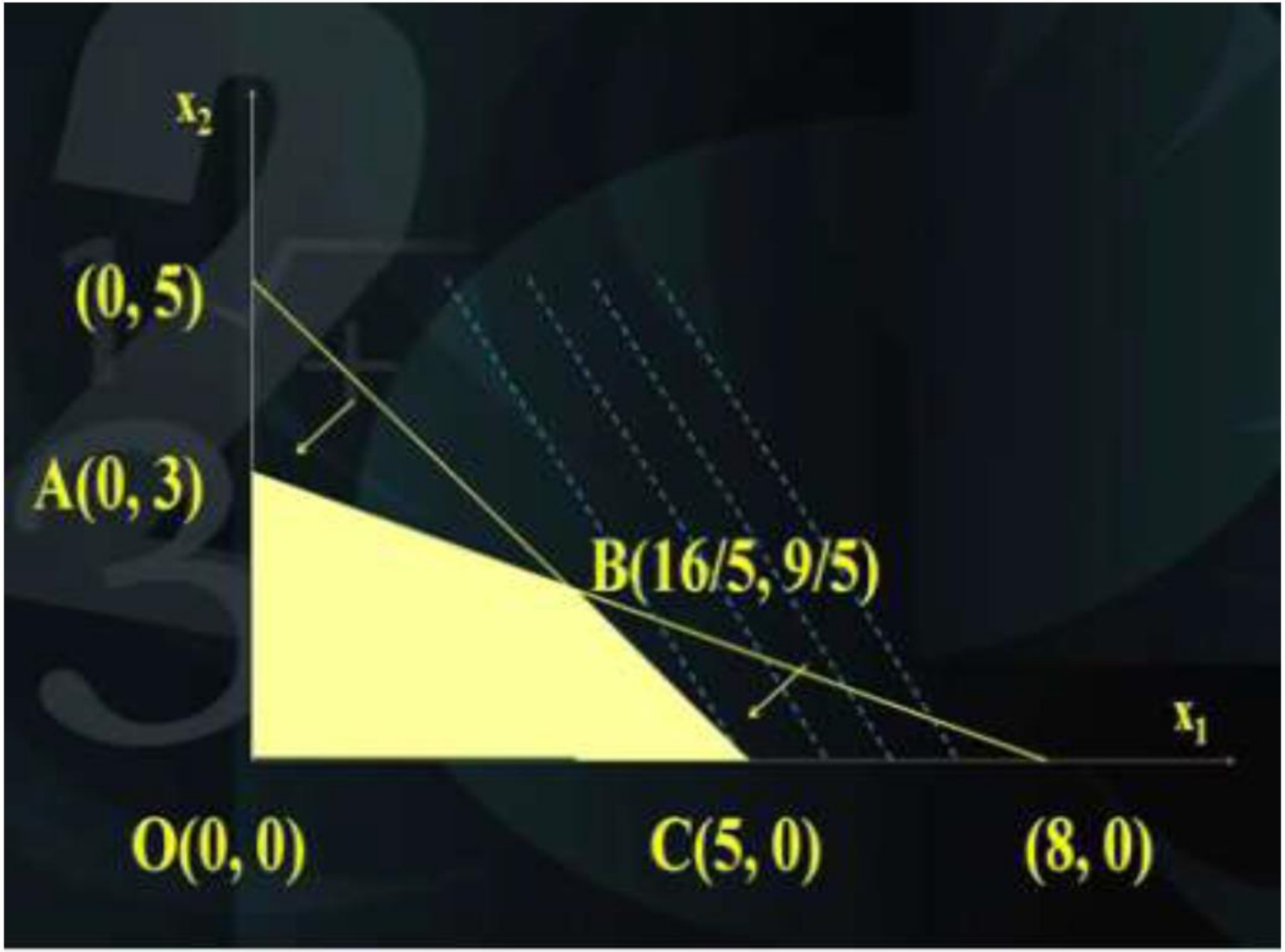
- Unique Solution
  - There is only one solution to the problem
- Multiple Solutions
  - There are multiple solution to the problem which gives the same answer
- Unbounded Solution
  - The solution is not bounded i.e almost infinite number of solutions
- Infeasible Solution
  - No solution exists

# Unique Solution



# Solution steps

- Plot the x-axis on the horizontal axis, that is, the x1 variable will be represented
- Plot the y-axis on the vertical axis, that is, the x2 variable will be represented O(0,0) is origin
- Plot the first constraint that is x1 + x2 < 5,
  - This is the straight line passing through the two points (0, 5) and (5, 0).
  - A line uniquely passes through two points and this can be obtained by putting x1 = 0 and getting the corresponding value of x2 similarly, putting x2 = 0 and getting the corresponding value of x1.
- Similarly plot for the second constraint
- Find out the feasible region
- Calculate the objective function value (At corner points)
- Find the Maximum


$x_2$
(0, 5)
A(0, 3)
B(16/5, 9/5)
$x_1$
O(0, 0)
C(5, 0)
(8, 0)

# The Best Solutions

| Vertices of feasible region | Objective function value $5x_1 + 3x_2$ |
|---|---|
| O(0, 0) | 0 |
| A(0, 3) | 9 |
| B(16/5, 9/5) | 21.4 |
| C(5, 0) | 25 Max |

# Multiple Solutions

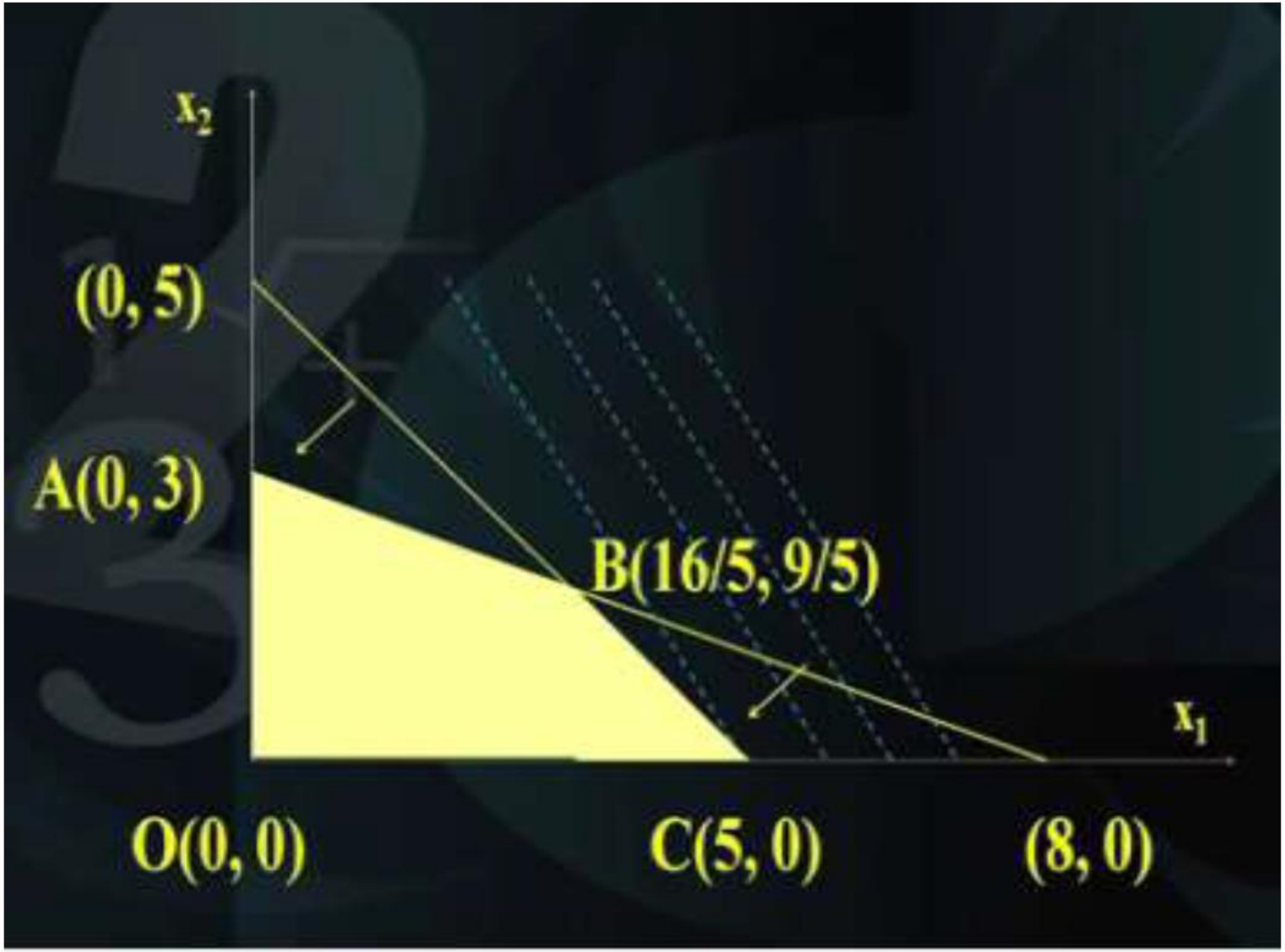
## 2. Multiple solutions

Maximize $3x_1 + 8x_2$

Subject to $x_1 + x_2 \leq 5$

$3x_1 + 8x_2 \leq 24$

$x_1 \geq 0, x_2 \geq 0$

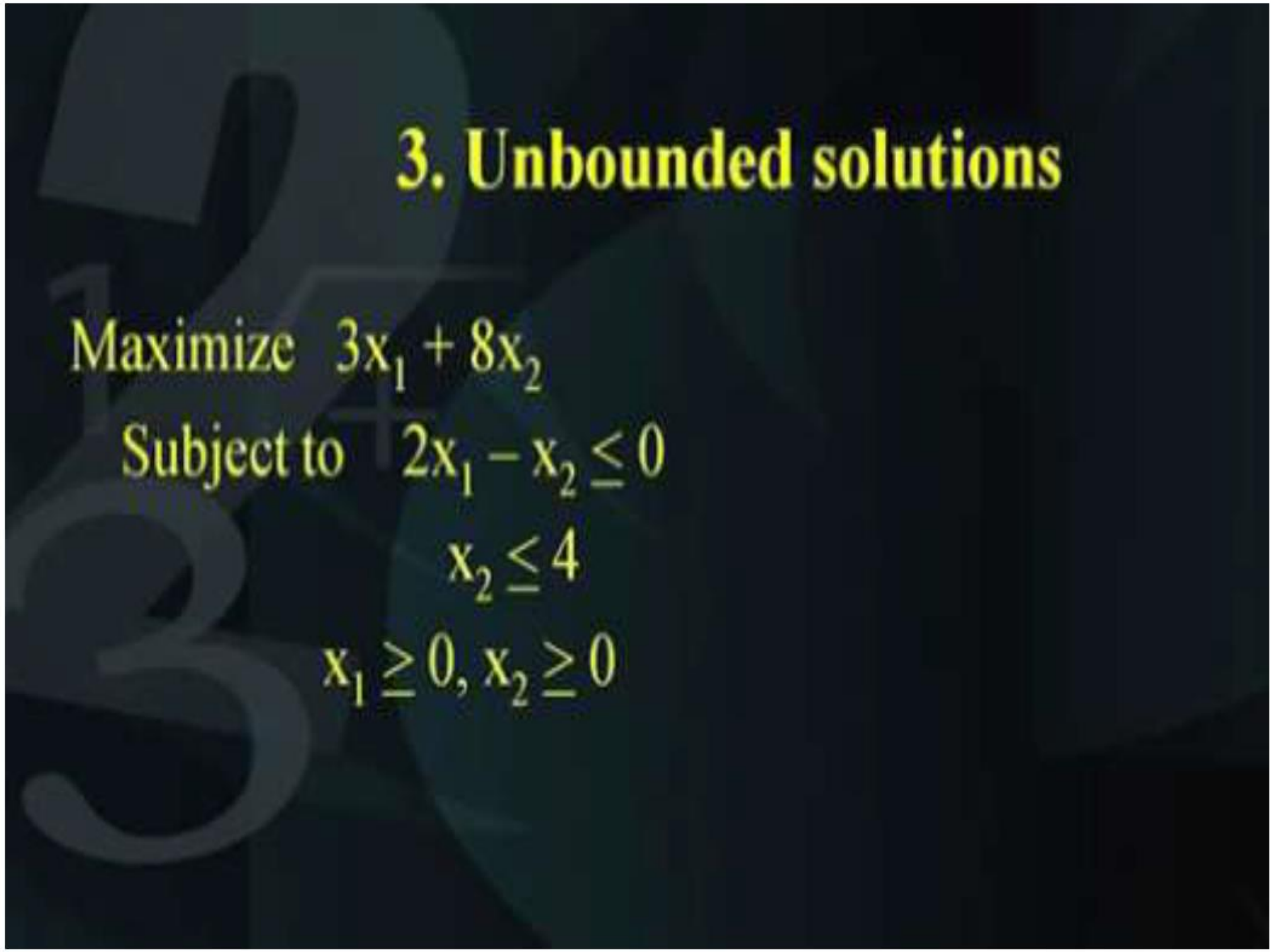


$x_2$
(0, 5)
A(0, 3)
B(16/5, 9/5)
$x_1$
O(0, 0)
C(5, 0)
(8, 0)

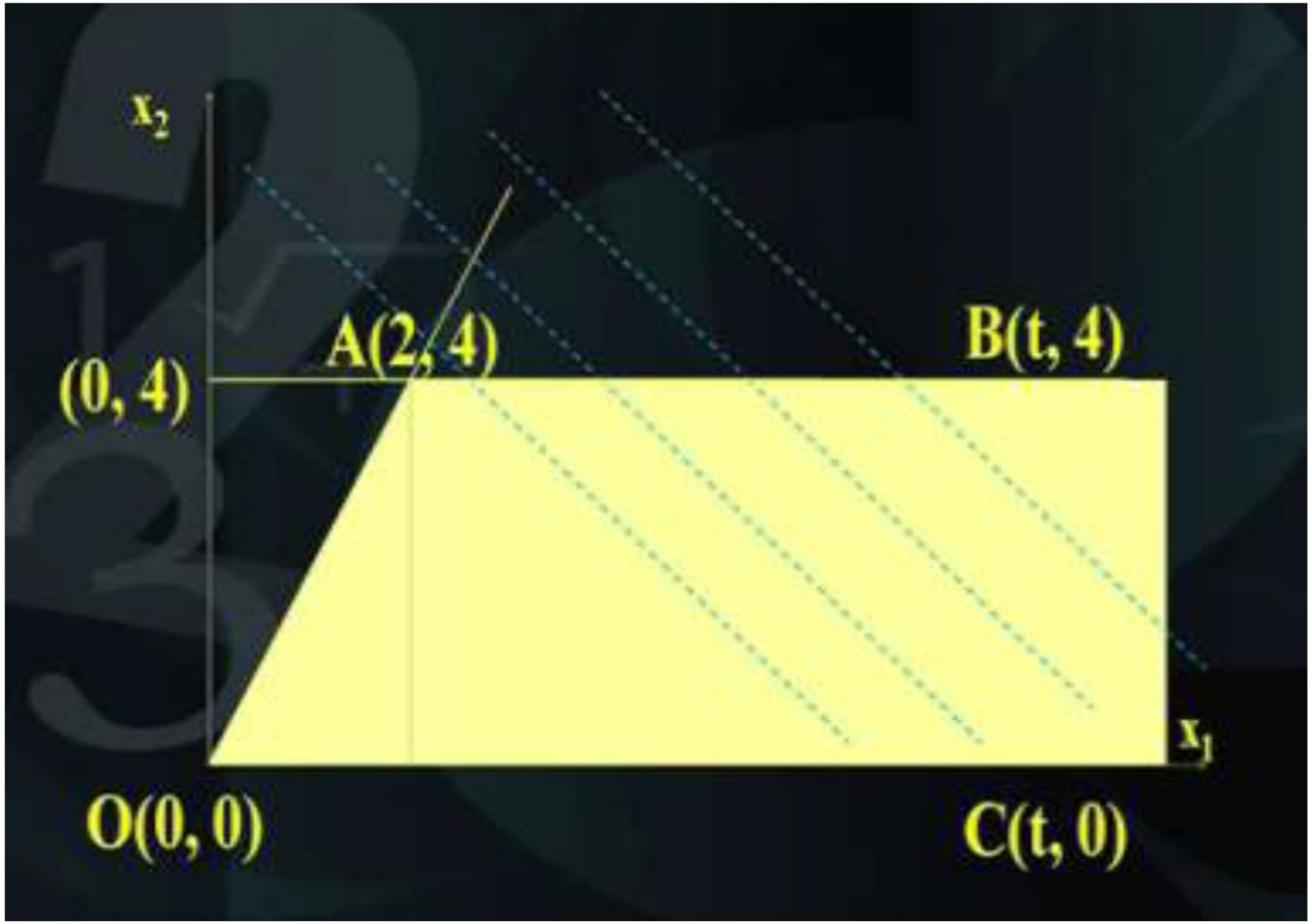
# Multiple best solutions

| Vertices of feasible region | Objective function value $3x_1 + 8x_2$ |
|---|---|
| O(0, 0) | 0 |
| A(0, 3) | 24 max |
| B(16/5, 9 /5) | 24 Max |
| C(5, 0) | 15 |

# Unbounded Solution



# The feasible solution Region

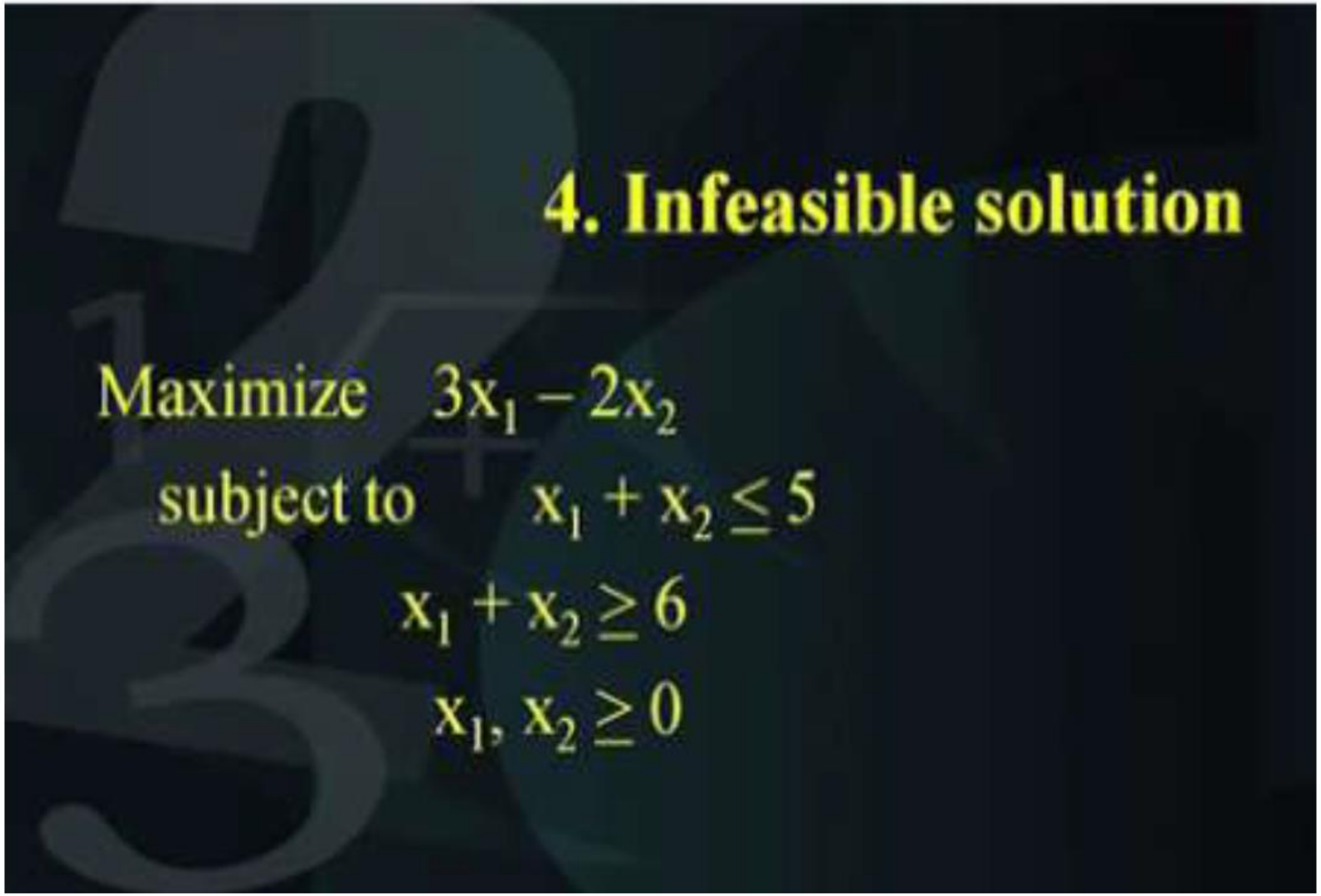


# Unbounded number of Solutions

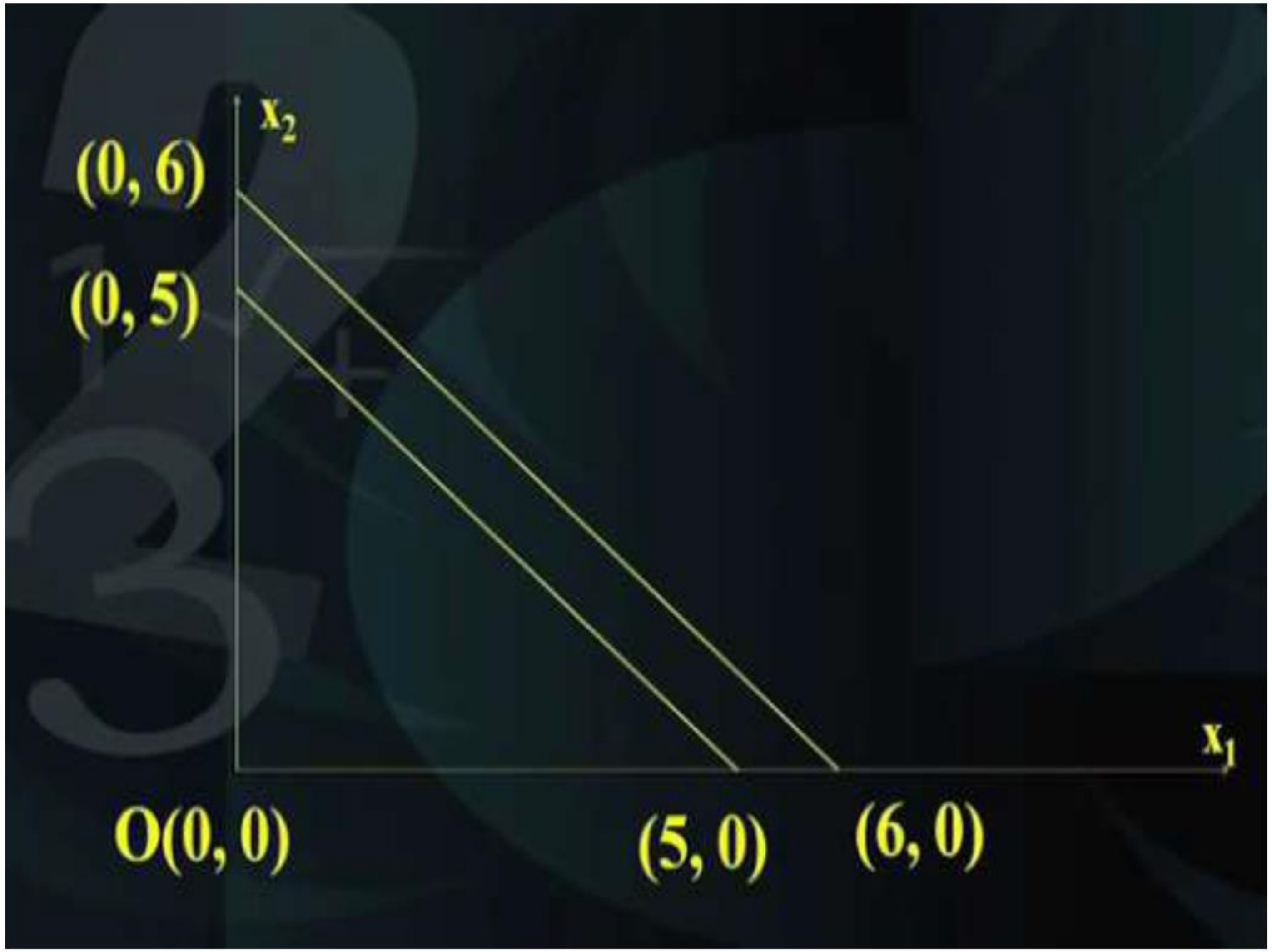
| Vertices of feasible region | Objective function value $3x_1 + 8x_2$ |
|---|---|
| O(0, 0) | 0 |
| A(2, 4) | 38 |
| B(t, 4) | 6 + 8t |
| C(t, 0) | 8t |

As t increases 6 + 8t also increases

# Infeasible Solution



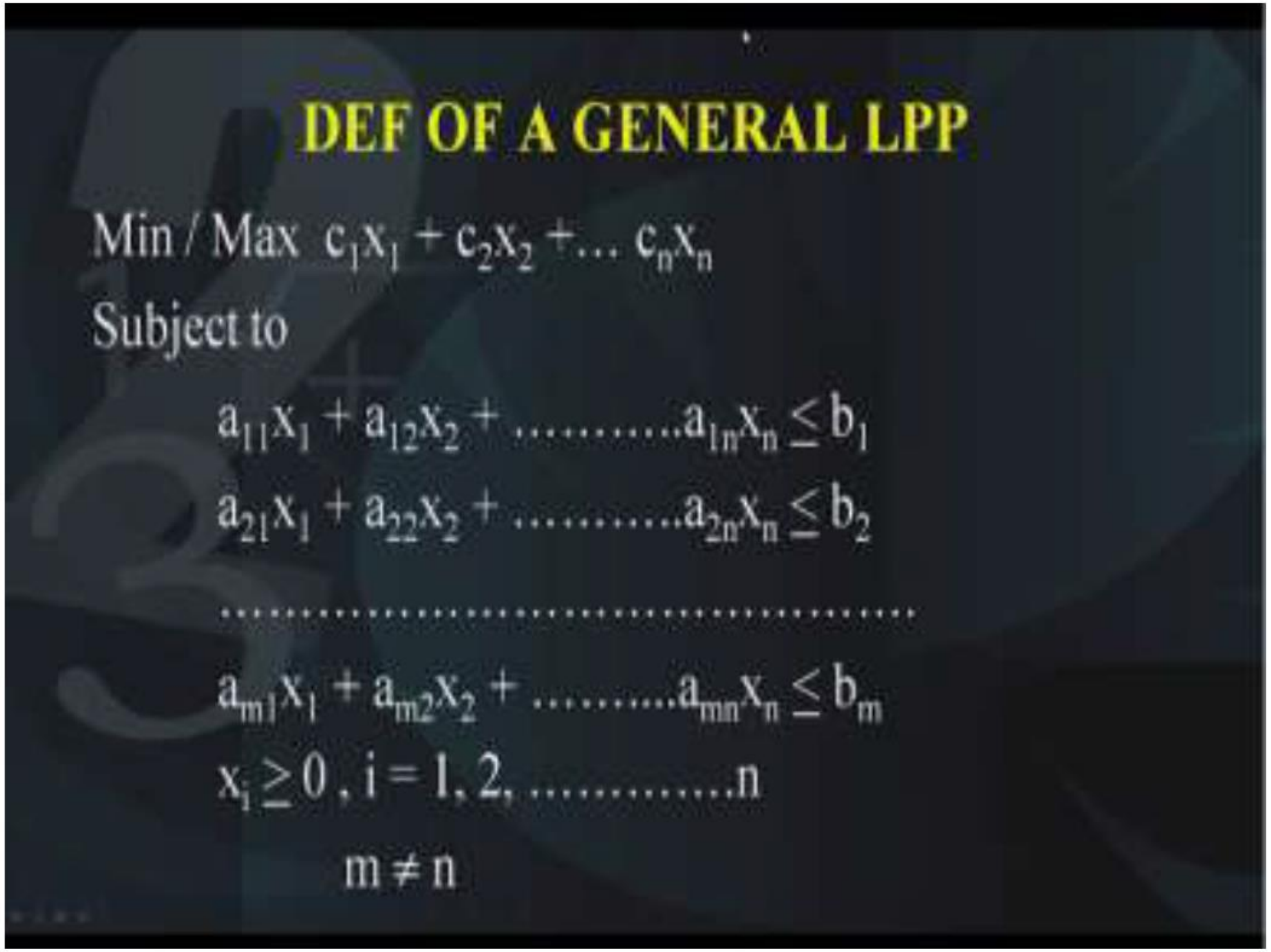
# No intersection of the constraint plots and feasible region is not bounded



# Simplex Method

- Definition of general LPP
- Conversion of general LPP to standard LPP
  - Basic variable
  - Slack and surplus variable
    - To convert the inquality constraints to equality constraints
- Find out the Incoming and outgoing variable
  - Identify the Pivot Column, Pivot Row and Pivot Element

# Definition of a general LPP



# Definition of standard LPP

**LPP IN STANDARD FORM**

Max $c_1x_1 + c_2x_2 + \ldots\ c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots a_{mn}x_n = b_m$$

$$x_i \geq 0,\ i = 1, 2, \ldots\ldots\ldots n$$

$$b_i \geq 0,\ j = 1, 2, \ldots\ldots\ldots m$$

$$m \neq n$$

# Use of Slack and Surplus Variable

- Slack variable is used to convert <= to equal
  - X1 <= 5   can be written as X1 + s1 = 5, where s1>=0
- Surplus Variable is used to convert >= to equal
  - X1>=5 can be written as X- s1 = 5, where s1>=0
- For unrestricted constraint (X1 can be positive or negative)
  - X1 + s1 – s2 = 0 (where s1, s2 >=0)
- Conversion of RHS to +ve
  - Multiply the row by -1 to make RHS +ve is it is –ve

# Example

**EXAMPLE**

Maximize $4x_1 + 2x_2$

Subject to

$7x_1 - 3x_2 \leq 15 \quad\longrightarrow\quad 7x_1 - 3x_2 + x_3 = 15$

$3x_1 + 4x_2 \geq 20 \quad\longrightarrow\quad 3x_1 + 4x_2 - x_4 = 20$

$5x_1 - 4x_2 = -60 \quad\longrightarrow\quad -5x_1 + 4x_2 = 60$

All $x_i \geq 0$

# Elementary Row Operations

- Multiply Row by a scalar
- Add/Subtract Row to/from another row
- Interchange two rows

# Terminology

- Basic variable: A variable appears with coefficient 1 in equation ‘I’ and with 0 in all other
- Non Basic Variable: All variable which are not basic
- Pivot Operation: By using elementary row operation a non-basic variable can be converted to basic variable
- Initial basic feasible solution obtained by setting all non-basic variables to zero and solving for basic variables where the basic variables are non –ve
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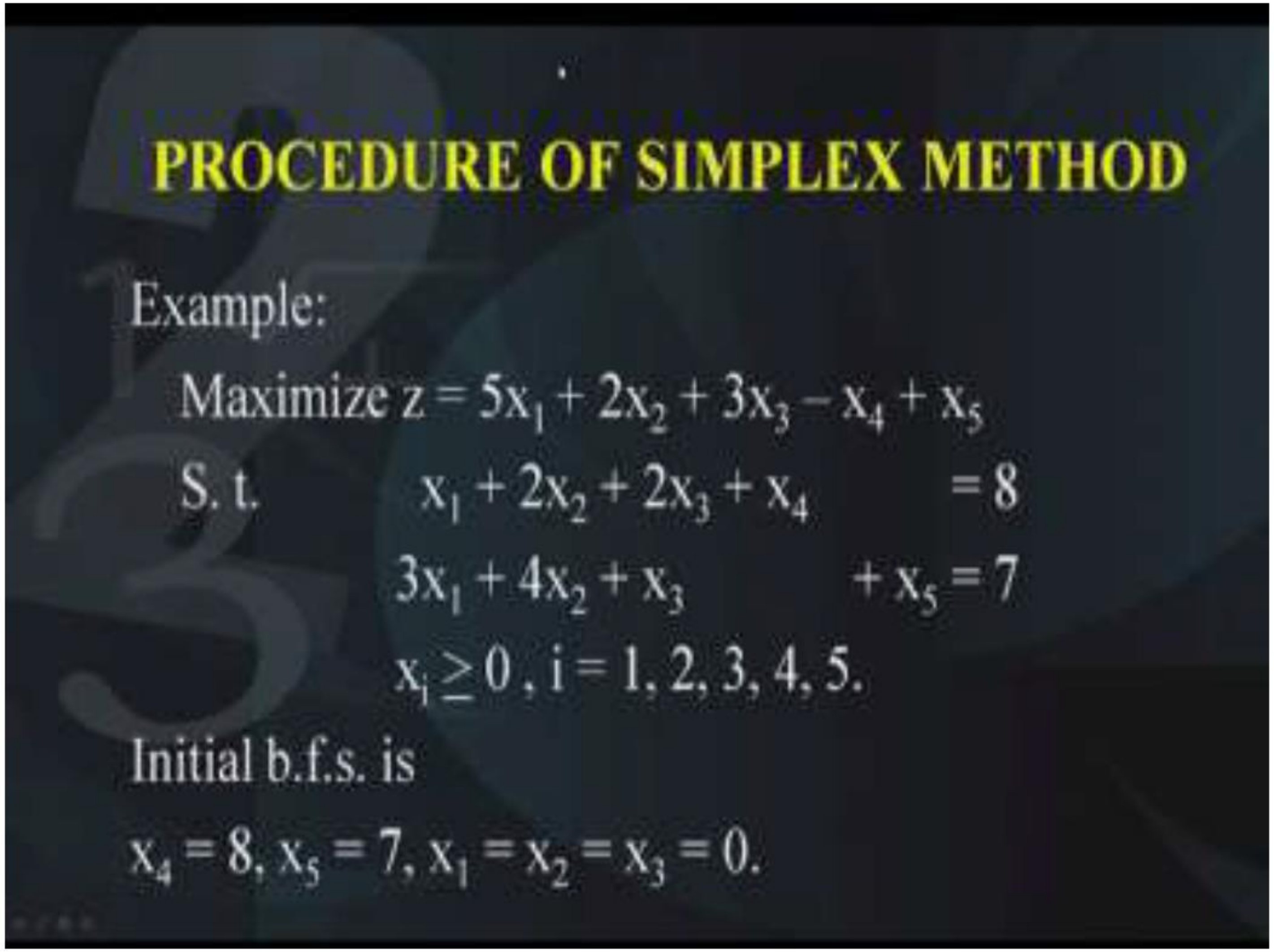
# Simplex Method Explanation



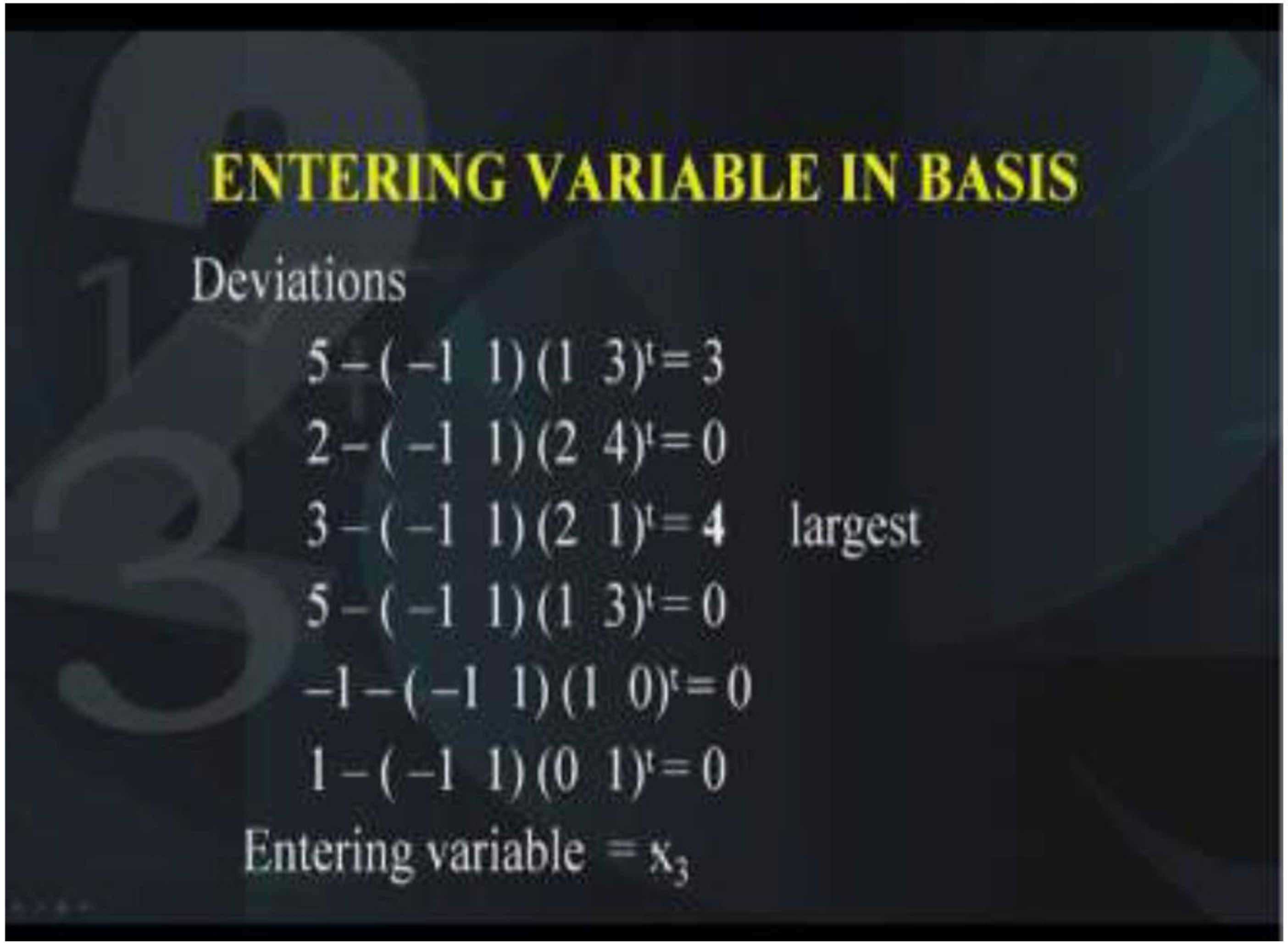
# Procedure for Simplex method

- Construct the initial simplex table

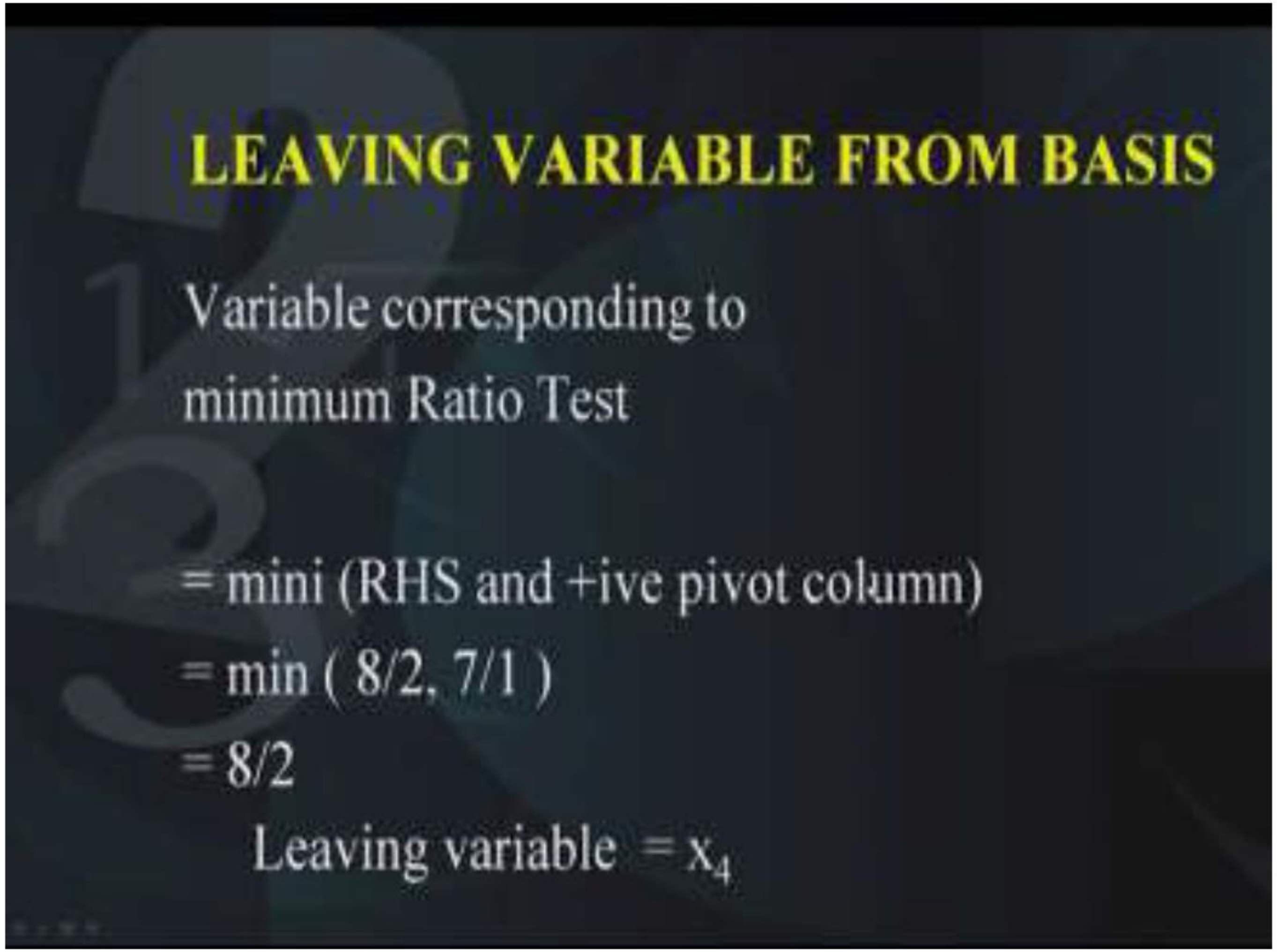
**INITIAL TABLE**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| . | | 5 | 2 | 3 | –1 | 1 | |
| $C_0$ | Basis | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
| –1 | $x_4$ | 1 | 2 | 2 | 1 | 0 | 8 |
| 1 | $x_5$ | 3 | 4 | 1 | 0 | 1 | 7 |
| | Dev. | 3 | 0 | 4 | 0 | 0 | Z=–1 |

# Identifying Incoming variable



# Outgoing Veriable using Minimum Ratio Test



# Second Iteration

SECOND TABLE

| | | 5 | 2 | 3 | –1 | 1 | |
|---|---|---|---|---|---|---|---|
| $C_0$ | Basis | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
| 3 | $x_3$ | 1/2 | 1 | 1 | 1/2 | 0 | 4 |
| 1 | $x_5$ | 5/2 | 3 | 0 | –1/2 | 1 | 3 |
| | Dev. | 1 | –4 | 0 | –2 | 0 | Z=15 |

# Third Iteration

THIRD TABLE

| | | 5 | 2 | 3 | –1 | 1 | |
|---|---|---|---|---|---|---|---|
| $C_0$ | Basis | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
| 3 | $x_3$ | 0 | 2/5 | 1 | 3/5 | –1/5 | 17/5 |
| 5 | $x_1$ | 1 | 6/5 | 0 | –1/5 | 2/5 | 6/5 |
| | Dev. | 0 | –26/5 | 0 | –9/5 | –2/5 | 81/5 |

# Computational Method

1. Convert all right side constant of constraints are positive
2. Convert unequal constraints to equations by introducing slack/surplus variable
3. Present the constraints equations in matrix form
4. Construct the starting simplex table (as discussed)
5. Test basic feasible solution for optimality

$$\Delta_j = c_j - x_j = c_j - C_B X_j$$

Optimality test

(i) If all $\Delta_j$ <= 0 the solution is optimal (alternate solution exists if any non-basic is also zero

(ii) At least one $\Delta_j$ is positive, The solution is non-optimal, proceed to next iteration

(iii) If corresponding to most positive $\Delta_j$, all elements of the column is negative or zero, the solution is unbounded

6. Find out the incoming and outgoing vector

# Contd…

7. Make the key element equal to 1 (unity), perform required matrix operations

8. Repeat steps 5 thru 7 until optimality condition is satisfied

Quick solution

1. Mark the incoming vector (max of $\Delta_j$), let it be column 'k'

2. Key element is found at the intersection of incoming and outgoing vector (Min $X_B$/ $X_k$)

3. ***Perform Row Operations***

# Contd..

4. $\Delta_{j}$ corresponding to unit matrix col. Is always zero

5. $\Delta_{j}$ Will be equal to $C_j$ in the intial solution with only slac or surplus variable in the initial basic solution matrix

6. While transforming the table, the value of ‘z’ and corresponding is also calculated $\Delta_{j}$

# Two phase simplex method

- Artificial Variable
  - To deal with >= and = constraints
  - Basic matrix can be obtained as an identity matrix in the simplex table
  - With slack/surplus variable, basic matrix can not be obtained
  - Use artificial variable (Fictitious/ no meaning)

Phase-I

- Remove the artificial variables
  - Start with artificial variable in the basic solution
  - Once the artificial variable is outgoing (drop the entire column)
  - Obtain the standard simplex table

- Phase-II
  - Proceed as discussed earlier

# Big-M (Penalty) method

- $X_1$, $X_2$....., $S_1$, $S_2$, ....., $A_1$, $A_2$.......
- Assign –M (M is very large +ve integer) as cost associated with Artificial Variable
- In the $\Delta_j$ row, choose the minimum negative M value (ignore other constants) as outgoing row

**Difficult for computation**

# Comparison between Two-phase and Big-M method

| Two-Phase | Big-M |
|---|---|
| Suitable for Computation | Not Suitable for Computation |
| No. of Iteration is large if there are more Artificial Variables | No. of iterations do not depend on the number of artificial variable |
| Preferable if less number of artificial variable | Preferable if less number of artificial variable |
| No choice of M is required | Choice of M is crucial |
| It is a two stage process | Single stage process |

# Other Mathematical Solutions

- Solution of Simultaneous linear equations using simplex method
  - Formulate the objective function with associated cost as zero
  - Simultaneous linear equations are written as equality constraint
  - Use two phase simplex method for solving the equations
  - The solution will give the solution of variables
- Finding matrix inversion problem using simplex method
  - Similar to solving the simultaneous linear equations
  - The values corresponding to artificial variable columns is the matrix inversion