* How to change a matrix to its echelon form?

Echelon forms :-

A matrix is in row echelon form (ref) when it satisfies the following conditions:-

- The first non-zero element in each row, called the leading entry is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements are, if any, are below rows having non-zero elements.

A matrix is in reduced row echelon form (rref) when it satisfies the following conditions:-

- the matrix is in row echelon form (i.e. it satisfies the three conditions listed above).
- the leading entry in each row is the only non zero entry in the column.

A matrix in echelon form is called an echelon matrix. Matrix A and B are examples of echelon matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A B

Matrix A is in row echelon form, and matrix B is in reduced row echelon form.

How to transform a matrix into its echelon forms :

Any matrix can be transformed into its echelon forms using a series of elementary row operations :-

1. Pivot the matrix.
   a. Find the pivot, the first non zero entry in the first column of the matrix.
   b. Interchange rows, moving the pivot to the first row.
   c. Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.
   d. Add multiples of pivot rows to each of the lower rows so that every element ~~of pivot row~~ in the pivot column of the lower rows equals ~~one~~ 0.

2. To get the matrix in row echelon form, repeat the pivot.
   a. Repeat the procedure from step 1 above, ignoring previous pivot rows.
   b. Continue until there are no more pivots to be processed.

3. To get the matrix in reduced row echelon form, process non zero entries above each pivot.
   a. Identify the last row having pivot equal to 1, and let this be the pivot row.
   b. Add multiples of the pivot row to each of the upper rows, until every element above the pivot equals 0.
   c. Moving up the matrix, repeat this process for each row.

Transforming a matrix to its echelon forms : example.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \Rightarrow \underset{\text{Aref}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}.$$

$$\underset{\text{Aref}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \Rightarrow \underset{\text{Arref}}{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}.$$

Note :—

The row echelon matrix which results from a series of elementary row operations is not necessarily unique. A different set of row operations could result in a different row echelon matrix.

However, the reduced row echelon matrix is unique; each matrix has only one reduced row echelon matrix.

Matrix Rank

You can think of a $r \times c$ matrix as a set of r row vectors, each having c elements; or you can think of it as a set of c column vectors each having r elements.

The rank of a matrix is defined as :-

(a) maximum number of linearly independent column vectors in the matrix or

(b) maximum number of linearly independent row vectors in the matrix.

BOTH DEFINITIONS ARE EQUIVALENT.

For an $r \times c$ matrix,

- if $r < c$; maximum rank of matrix is r
- if $r > c$; maximum rank of matrix is c.

* The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be 1.

* The maximum number of linearly independent vectors in a matrix is equal to the number of non zero rows in its row echelon form matrix. Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of rows which are non zero.

eg: $\underset{A}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}} \Rightarrow \underset{A_{ref}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$ Because A_{ref} has 2 non zero rows, rank of matrix is $\underline{\underline{2}}$.

Row 1, Row 2 of matrix are linearly independent. However row 3 is linear combination of rows 1 & 2.

Row 3 = 3^*(Row 1) + 2^*(Row 2).

Full Rank matrices:

When all the vectors in a matrix are linearly independent, the matrix is said to be full rank. Consider matrices A & B below:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Row 2 of matrix A is scalar multiple of row 1; i.e. row 2 is equal to twice row 1. Therefore rows 1 and 2 are linearly dependent. Matrix A has only one linearly independent row, so its rank is 1, hence it is not full rank.

Now, matrix B has all of its rows linearly independent. So, rank of matrix ~~to~~ B is 3. Matrix B is full rank.

(1) $X = \begin{bmatrix} 1 & 2 & 44 \\ 3 & 4 & 80 \end{bmatrix}$ — Rank 2.

(2) $Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 9 \end{bmatrix}$ — Rank 2.

~~Cols~~ Since matrix has non zero elements, min rank is 1. Since it has fewer columns than rows, its maximum rank equal to the maximum number of linearly independent columns. Column 1 & 2 are " ", but column 3 = column 1 + column 2. So, rank of matrix is 2.

Computing rank using determinants

Definition

Let A be an $m \times n$ matrix. A minor of A of order k is a determinant of a $k \times k$ sub-matrix of A.

We obtain the minors of order k from A by first deleting $m - k$ rows and $n - k$ columns, and then computing the determinant. There are usually many minors of A of a given order.

Example

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Computing minors

Solution

We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \cdot (-4) + 2 \cdot 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4×4 sub-matrices of A. There are $3 \cdot 6 = 18$ minors of order 2 and $3 \cdot 4 = 12$ minors of order 1

Computing rank using minors

Proposition

Let A be an $m \times n$ matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof: If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank

Start with the minors of maximal order k. If there is one that is non-zero, then $\text{rk}(A) = k$. If all maximal minors are zero, then $\text{rk}(A) < k$, and we continue with the minors of order $k-1$ and so on, until we find a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then $\text{rk}(A) = 0$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\text{rk}(A) = 3$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

Rank: Examples using minors

Solution

The maximal minors have order 3*, so we compute the* 4 *minors of order* 3*. The first one is*

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7 \cdot (-1) + (-1) \cdot (-7) = 0$$

The other three are also zero. Since all minors of order 3 *are zero, the rank must be* $\mathrm{rk}(A) < 3$*. We continue to look at the minors of order two. The first one is*

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 5 - 18 = -13 \neq 0$$

It is not necessary to compute any more minors, and we conclude that $\mathrm{rk}(A) = 2$*. In fact, the first two columns of A are linearly independent.*

Application: Linear independence

Example

Show that the vectors are linearly independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Solution

The vectors are linearly independent if and only if $\mathrm{rk}(A) = 2$*, where* A *is the matrix with* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *as columns. Since we have*

$$\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$$

it follows that $\mathrm{rk}(A) = 2$.

Rank and linear systems

Theorem

Let $A_\mathbf{b} = (A|\mathbf{b})$ *be the augmented matrix of a linear system* $A\mathbf{x} = \mathbf{b}$ *in* n *unknowns. Then we have:*

1. *The linear system is consistent if and only if* $\mathrm{rk}\, A_\mathbf{b} = \mathrm{rk}\, A$.
2. *If the linear system is consistent, then it has a unique solution if and only if* $\mathrm{rk}(A) = n$*. Moreover, if* $\mathrm{rk}(A) < n$*, then the system has* $n - \mathrm{rk}(A)$ *free variables.*

Idea of proof: Think of the pivots in the reduced echelon form of the system.

Linear system: Example using rank

Example

Is the following linear system consistent? Does it have a unique solution?

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 4x_1 & & & + & 2x_3 & = & 2 \\ & & 6x_2 & - & 3x_3 & = & 4 \end{array}$$

Linear system: Example using rank

Solution

We form the matrices

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -1 \end{pmatrix}, \quad A_{\mathbf{b}} = \begin{pmatrix} 2 & 2 & -1 & 1 \\ 4 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \end{pmatrix}$$

We compute that $\det(A) = -24 \neq 0$*, so* $\operatorname{rk}(A) = 3$ *and* $\operatorname{rk} A_{\mathbf{b}} = 3$ *(since the determinant is a maximal minor of the augmented matrix). Hence the system is consistent. In fact,* $n - \operatorname{rk} A = 3 - 3 = 0$*, so the system has a unique solution.*

Linear system: Explicit solutions using minors

Interpretation of minors

We consider a consistent linear system $A\mathbf{x} = \mathbf{b}$ and let $k = \mathrm{rk}(A)$. Then there is a non-zero minor of order k. We can interpret this minor in the following way:

- The deleted rows are not essential, and we may disregard them. Hence we only regard the rows (equations) that are *in* the minor.
- The variables corresponding to deleted columns represent free (independent) variables. The variables corresponding to columns *in* the minor are basic (dependent) variable.
- We may write down the solution of the system by solving the equations *in* the minor for the basic (dependent) variables.

Linear system: Example solved using minors

Example

Solve the following (consistent) linear system using minors:

$$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 2x_3 & + & 3x_4 & = & 0 \\ & & x_2 & + & x_3 & & & = & 0 \\ x_1 & & & + & 3x_3 & + & 3x_4 & = & 0 \end{array}$$

We remark that this system is consistent, since it has the trivial solution $\mathbf{x} = \mathbf{0}$.

Linear system: Example solved using minors

Solution

We compute the rank of the coefficient matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 3 \end{pmatrix}$$

After some computations, we see that all maximal (order 3) minors are zero. However, the minor of order 2 obtained by deleting the last row and the last two columns from A is

$$\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

This means that $\operatorname{rk} A = 2$, and that the linear system has $4 - 2 = 2$ free variables.

Linear system: Example solved using minors

Solution

The free variables are x_3, x_4, and we may express x_1, x_2 in terms of the free variables using the first two equations:

$$\begin{aligned} x_1 - x_2 &= -2x_3 - 3x_4 \\ x_2 &= -x_3 \end{aligned}$$

This gives

$$\begin{aligned} x_1 &= -3x_3 - 3x_4 \\ x_2 &= -x_3 \\ x_3 &= \text{ free variable} \\ x_4 &= \text{ free variable} \end{aligned}$$

(1)

$$\begin{matrix}1&1&0\\0&1&1\\1&2&1\end{matrix} \qquad \cancel{\begin{matrix}0&1&1\\1&1&0\\1&2&1\end{matrix}} \qquad \begin{matrix}1&1&0\\0&1&1\\0&1&1\end{matrix} \qquad \underset{A_{rref}}{\begin{matrix}1&1&0\\0&1&1\\0&0&0\end{matrix}} \qquad \underset{A_{rref}}{\begin{matrix}1&0&-1\\0&1&1\\0&0&0\end{matrix}}$$

(2)

$$\begin{bmatrix}0&1\\1&2\\0&5\end{bmatrix} \Rightarrow \begin{bmatrix}1&2\\0&1\\0&5\end{bmatrix} \Rightarrow \begin{bmatrix}1&0\\0&1\\0&5\end{bmatrix} \Rightarrow \underset{A_{rref}}{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}.$$

(3) $\begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}$ – This is not in rref because the leading entry in row 2 is to the left of leading entry in row 1, it should be to the right.

(4) $\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}$ – This is not in rref because column 2 has more than one non zero entry.

(5) $\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$ – This is not in rref because row 2 with all zeros is followed with a row with a non zero element; all zero rows must follow non-zero rows.