Derivation of Mass of Star of a Binary Star System

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A Binary Star system is a star system which consists of two stars orbiting their Barycentre (Centre of Mass) bound together by gravitational attraction between them. Masses of the binary stars can be obtained from the measurements of their orbits as those will be related to each other. We will derive that relation.

Binary Star systems can be of many types. We will derive the relation for the case where the orbits are circular and orbit around their barycentre, as shown in Fig. 1 (a). And the forces experienced by one star in the system is shown in Fig. 1 (b).

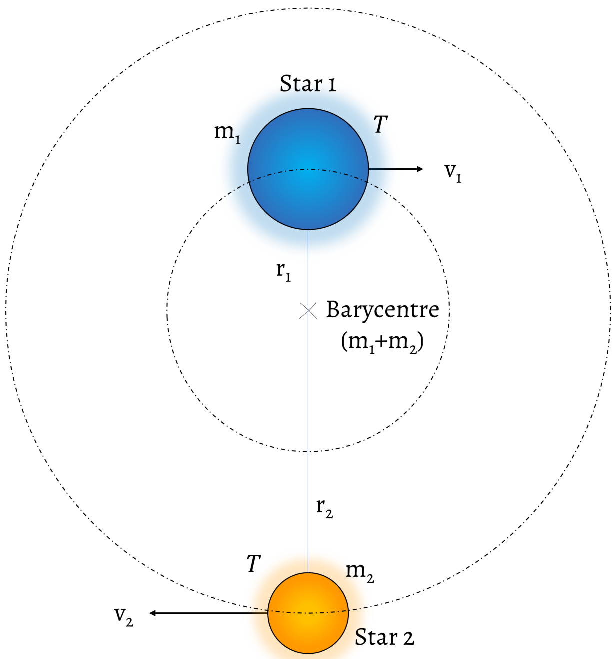

Figure 1 (a): A Binary Star System

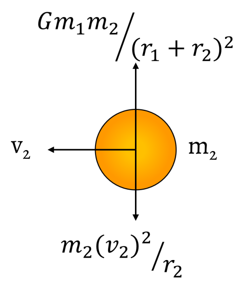

Figure 1 (b): F.B.D of Star 2

Let's take the system with two stars with mass $'m_1'$ and $'m_2'$ and lies at distance of $'r_1'$ and $'r_2'$ from the Barycentre. They are orbiting with $'v_1'$ and $'v_2'$ velocities around the barycentre with a time period of revolution 'T'. The time period is same for both the stars because the line joining the stars must pass through barycentre because the centre of mass lies on the line between the 2 masses.
Now, the Centre of Mass (COM) of any system is defined by,

$$M_{COM} \cdot R_{COM} = \sum_{i=1}^{n} m_i r_i$$

where, M_{COM} &R_{COM} is the Mass and Position of COM respectively.
Applying COM equation with respect to Star 1 for our system, we get,

$$\begin{aligned}(m_1 + m_2) \cdot r_1 &= m_1 \cdot (0) + m_2 \cdot (r_1 + r_2)\\ m_1 \cdot r_1 + m_1 \cdot r_2 &= m_2 \cdot r_1 + m_2 \cdot r_2\\ m_1 \cdot r_1 &= m_2 \cdot r_2\\ \frac{m_1}{m_2} &= \frac{r_2}{r_1} \qquad (1)\end{aligned}$$

Eq. 1 gives us a relation between mass and distance from the barycentre. It also shows that $\omega_1 = \omega_2$, hence the time period is same for both the stars. Now, let's see which forces are acting on Star 2. As we can see in Fig. 1 (b), the Gravitational Force is acting against the Centrifugal Force. Therefore, we can say,

$$\begin{aligned} F_G &= F_c \\ \frac{G \cdot m_1 \cdot m_2}{(r_1 + r_2)^2} &= \frac{m_2 \cdot (v_2)^2}{r_2} \\ \frac{G \cdot m_1}{(r_1 + r_2)^2} &= \frac{(v_2)^2}{r_2} \\ m_1 &= \frac{(r_1 + r_2)^2}{r_2} \cdot \frac{(v_2)^2}{G} \end{aligned} \tag{2}$$

Now, we can write the orbital velocity of a star as,

$$v = \frac{2\pi r}{T}$$

Therefore, the orbital velocity of Star 1 and Star 2 will be,

$$\begin{aligned} v_1 &= \frac{2\pi r_1}{T} \qquad \& \qquad v_2 = \frac{2\pi r_2}{T} \\ r_1 &= \frac{v_1 \cdot T}{2\pi} \qquad \& \qquad r_2 = \frac{v_2 \cdot T}{2\pi} \end{aligned} \tag{3}$$

Taking value of $r_1 \& r_2$ from eq. 3 and 4 and substituting in eq. 2, we get,

$$\begin{aligned} m_1 &= \left(\frac{v_1 \cdot T}{2\pi} + \frac{v_2 \cdot T}{2\pi} \right)^2 \cdot \frac{2\pi}{v_2 \cdot T} \cdot \frac{(v_2)^2}{G} \\ m_1 &= \frac{(v_1 + v_2)^2 \cdot T^2}{4\pi^2} \cdot \frac{2\pi}{v_2 \cdot T} \cdot \frac{(v_2)^2}{G} \\ m_1 &= \frac{v_2 \cdot T \cdot (v_1 + v_2)^2}{2\pi G} \end{aligned} \tag{4}$$

Similarly, the mass of Star 2 will be,

$$m_2 = \frac{v_1 \cdot T \cdot (v_1 + v_2)^2}{2\pi G} \tag{5}$$

Hence, we have an expression of Mass of Star in the terms of Orbital Velocity, Time Period of Revolution and Universal Constants. We will use this expression to calculate the mass of star in the assignment.
Also, using eq. 1 and 3, we can say that,

$$\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{v_2}{v_1} \tag{6}$$

This gives us the idea that how Mass of the star is inversely related with it's Radial Distance from COM and it's Orbital Velocity.