8.2.37

AI25BTECH11003 - Bhavesh Gaikwad

Question: Find the equation of the conic, that satisfies the given conditions. vertex (-3, 0), directrix x + 5 = 0.

Solution:
Given:

- Vertex: $\mathbf{V_o} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$
- Directrix: $x + 5 = 0$, which gives us $\mathbf{n}^\top \mathbf{x} = c$ where $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = -5$

The general matrix equation of a conic is:

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

where the matrices are defined as:

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 (\mathbf{n}\mathbf{n}^\top) \tag{0.2}$$

$$\mathbf{u} = (ce^2)\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{0.4}$$

For a vertex at $(-3, 0)$ and using the vertex-focus-directrix geometry, the focus $\mathbf{F}$ is at:

$$\mathbf{F} = \begin{pmatrix} -3 + 2e \\ 0 \end{pmatrix} \tag{0.5}$$

Case 1: $e < 1$ (Ellipse)
Let $e = \frac{1}{2}$

Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = \frac{1}{2}$
- $\mathbf{F} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
- $\|\mathbf{n}\|^2 = 1$, $\|\mathbf{F}\|^2 = 4$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.6}$$

$$\mathbf{u} = \frac{-5}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 0 \end{pmatrix} \tag{0.7}$$

$$f = 4 - \frac{25}{4} = -\frac{9}{4} \tag{0.8}$$

Putting Values of $\mathbf{V}$, $\mathbf{u}$, f in Equation 0.1, we get

$$\mathbf{x}^\top \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 & 0 \end{pmatrix}\mathbf{x} - 9 = 0 \tag{0.9}$$

Case 2: $e = 1$ (Parabola)

Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = 1$
- $\mathbf{F} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- $\|\mathbf{F}\|^2 = 1$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.10}$$

$$\mathbf{u} = -5\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{0.11}$$

$$f = 1 - 25 = -24 \tag{0.12}$$

Putting Values of $\mathbf{V}$, $\mathbf{u}$, f in Equation 0.1, we get

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 & 0 \end{pmatrix} \mathbf{x} - 24 = 0 \tag{0.13}$$

Case 3: $e > 1$ (Hyperbola)
Let $e = \frac{3}{2}$

Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = \frac{3}{2}$
- $\mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- $\|\mathbf{F}\|^2 = 0$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{9}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -5/4 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.14}$$

$$\mathbf{u} = \frac{-45}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -45/4 \\ 0 \end{pmatrix} \tag{0.15}$$

$$f = 0 - \frac{225}{4} = -\frac{225}{4} \tag{0.16}$$

Putting Values of $\mathbf{V}$, $\mathbf{u}$, f in Equation 0.1, we get

$$\mathbf{x}^\top \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 90 & 0 \end{pmatrix}\mathbf{x} + 225 = 0 \tag{0.17}$$

Therefore, The Possible conics with the vertex at (-3,0) and Directrix as x+5=0 are

$$\mathbf{x}^\top \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 & 0 \end{pmatrix}\mathbf{x} = 9$$

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -8 & 0 \end{pmatrix}\mathbf{x} = 24$$

$$\mathbf{x}^\top \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 90 & 0 \end{pmatrix}\mathbf{x} + 225 = 0$$

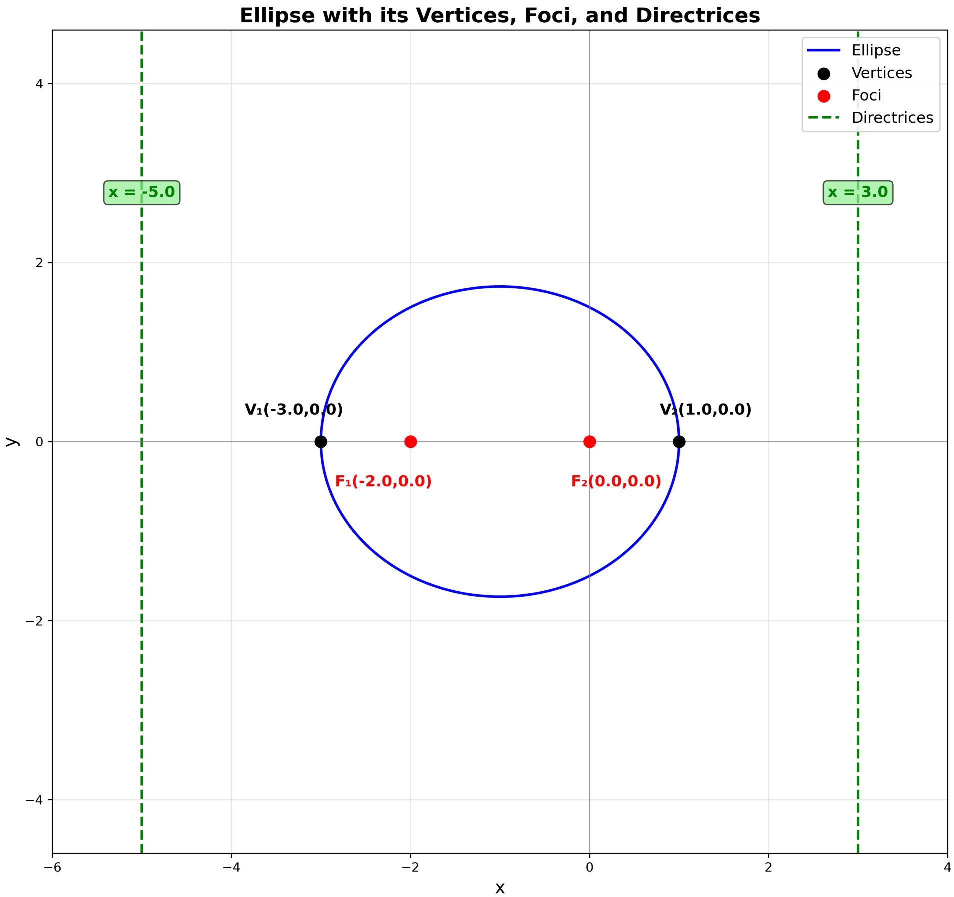

Fig. 0.1: Ellipse

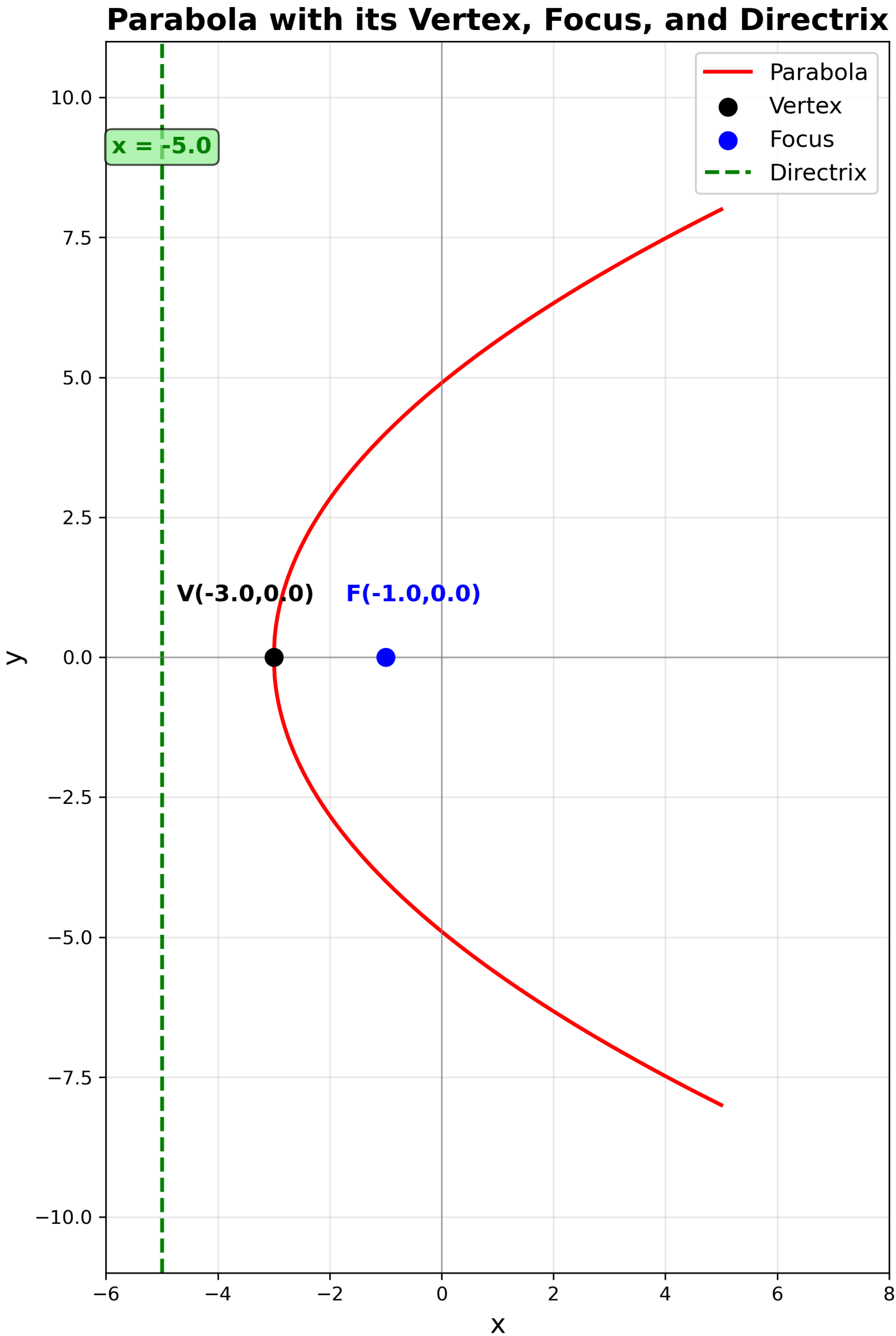

Fig. 0.2: Parabola

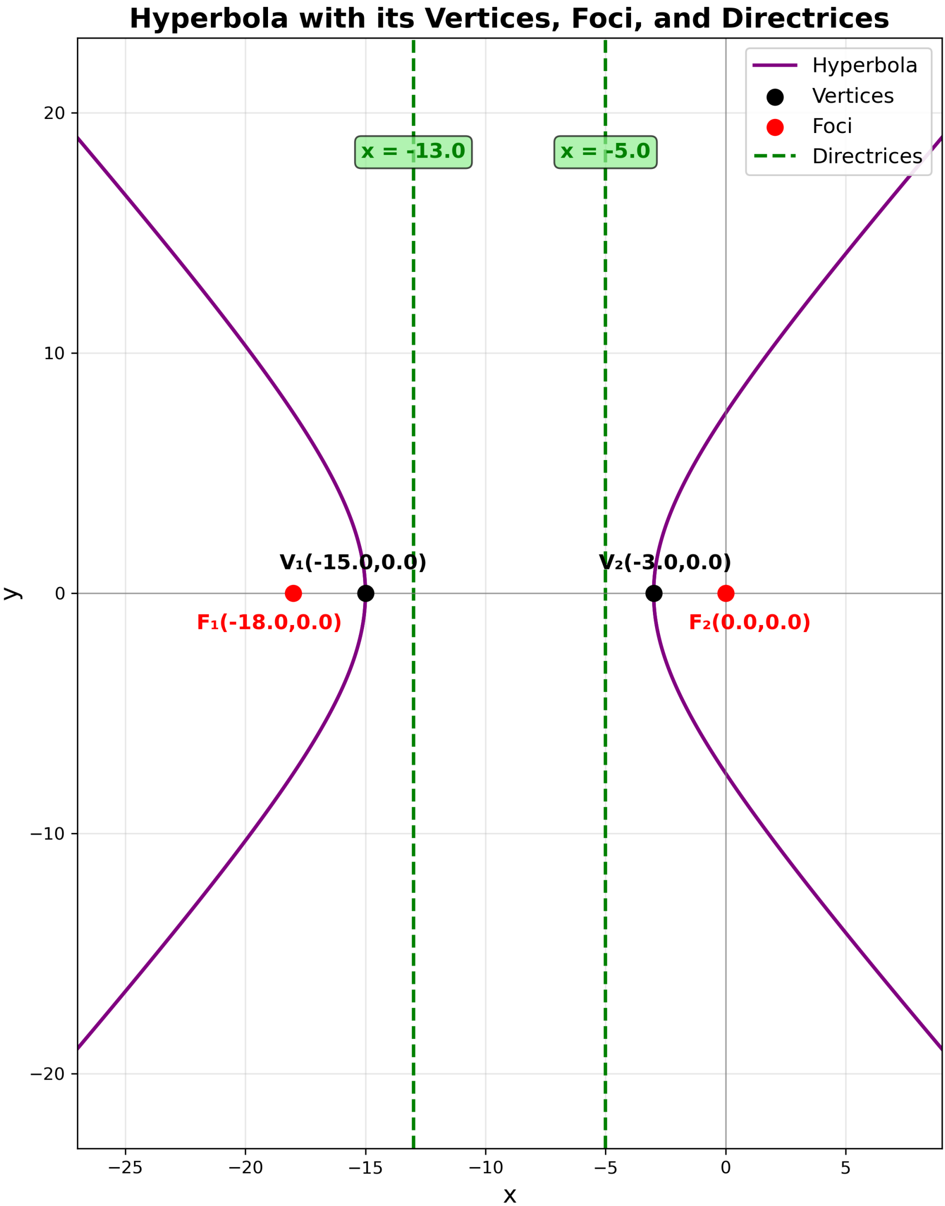

Fig. 0.3: Hyperbola