8.2.37

AI25BTECH11003 - Bhavesh Gaikwad

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Question

Find the equation of the conic, that satisfies the given conditions. vertex (-3, 0), directrix $x + 5 = 0$.

Theoretical Solution

Given:

- Vertex: $\mathbf{V} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$
- Directrix: $x + 5 = 0$, which gives us $\mathbf{n}^\top \mathbf{x} = c$ where $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = -5$

The general matrix equation of a conic is:

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where the matrices are defined as:

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2(\mathbf{n}\mathbf{n}^\top) \tag{2}$$

$$\mathbf{u} = (ce^2)\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{4}$$

Theoretical Solution

For a vertex at $(-3, 0)$ and using the vertex-focus-directrix geometry, the focus $\mathbf{F}$ is at:

$$\mathbf{F} = \begin{pmatrix} -3 + 2e \\ 0 \end{pmatrix} \tag{5}$$

Case 1: $e < 1$ (Ellipse)

Let $e = \frac{1}{2}$

Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = \frac{1}{2}$
- $\mathbf{F} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
- $\|\mathbf{n}\|^2 = 1$, $\|\mathbf{F}\|^2 = 4$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\mathbf{u} = \frac{-5}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 0 \end{pmatrix} \tag{7}$$

$$f = 4 - \frac{25}{4} = -\frac{9}{4} \tag{8}$$

Putting Values of $\mathbf{V}$, $\mathbf{u}$, f in Equation 0.1, we get

$$\mathbf{x}^\top \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{9}$$

Case 2: $e = 1$ (Parabola)

Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = 1$
- $\mathbf{F} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- $\|\mathbf{F}\|^2 = 1$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

$$\mathbf{u} = -5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{11}$$

$$f = 1 - 25 = -24 \tag{12}$$

Theoretical Solution

Putting Values of **V**, **u**, f in Equation 1, we get

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 & 0 \end{pmatrix} \mathbf{x} = 24 \tag{13}$$

Case 3: $e > 1$ (Hyperbola)
Let $e = \frac{3}{2}$
Parameters:

- $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -5$, $e = \frac{3}{2}$
- $\mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- $\|\mathbf{F}\|^2 = 0$

Matrix Calculation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{9}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -5/4 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

$$\mathbf{u} = \frac{-45}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -45/4 \\ 0 \end{pmatrix} \tag{15}$$

Theoretical Solution

$$f = 0 - \frac{225}{4} = -\frac{225}{4} \tag{16}$$

Putting Values of **V**, **u**, f in Equation 1, we get

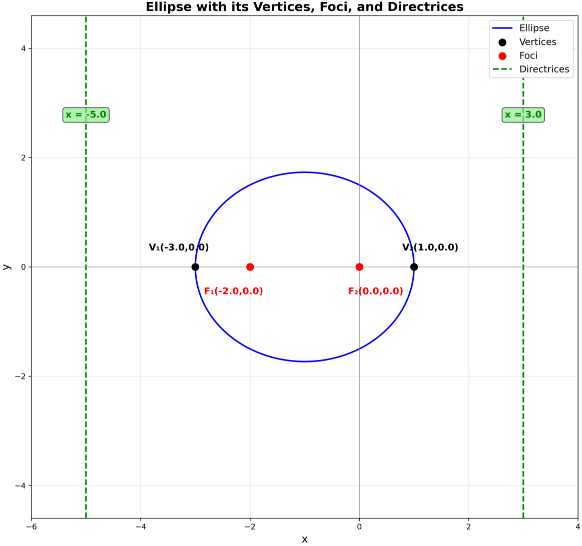
$$\mathbf{x}^\top \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 90 & 0 \end{pmatrix} \mathbf{x} + 225 = 0 \tag{17}$$

Theoretical Solution

Therefore, The Possible conics with the vertex at (-3,0) and Directrix as x+5=0 are

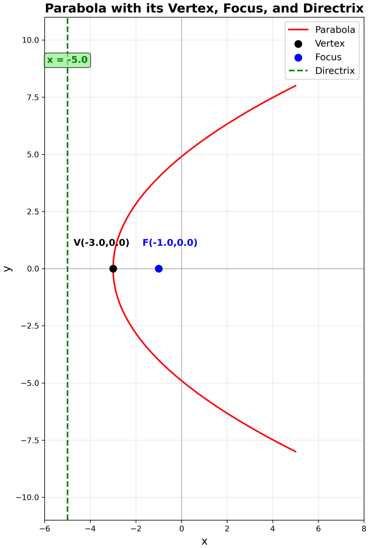
- $\mathbf{x}^\top \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6 & 0 \end{pmatrix} \mathbf{x} = 9$
- $\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 & 0 \end{pmatrix} \mathbf{x} = 24$
- $\mathbf{x}^\top \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 90 & 0 \end{pmatrix} \mathbf{x} + 225 = 0$

Ellipse



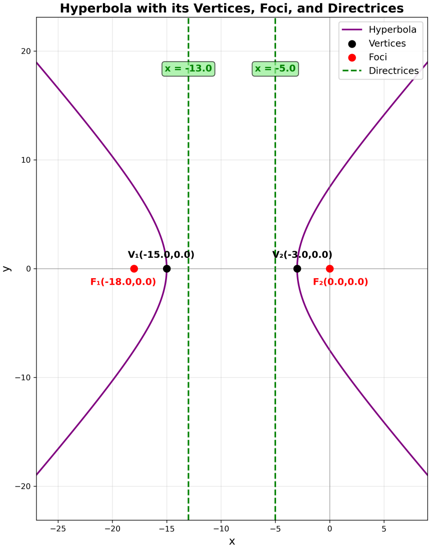
Ellipse with its Vertices, Foci, and Directrices
Ellipse
Vertices
Foci
Directrices
x = -5.0
x = 3.0
V₁(-3.0,0.0)
V₂(1.0,0.0)
F₁(-2.0,0.0)
F₂(0.0,0.0)
x
y

Parabola



Parabola with its Vertex, Focus, and Directrix
Parabola
Vertex
Focus
Directrix
x = -5.0
V(-3.0,0.0)
F(-1.0,0.0)
x
y
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−6
−4
−2
0
2
4
6
8

Hyperbola



Hyperbola with its Vertices, Foci, and Directrices
Hyperbola
Vertices
Foci
Directrices
x = -13.0
x = -5.0
V₁(-15.0,0.0)
V₂(-3.0,0.0)
F₁(-18.0,0.0)
F₂(0.0,0.0)
20
10
0
−10
−20
−25
−20
−15
−10
−5
0
5
x
y