9.8.5

AI25BTECH11003 - Bhavesh Gaikwad

Question: Let $\mathbf{S}$ be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is

Solution:
Given:
Circle: $x^2 + y^2 - 2x - 4y = 0$
Parabola: $y^2 = 8x$

Parameters of the Circle:

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, f_1 = 0 \tag{0.1}$$

Parameters of the Parabola:

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_2 = 0, \mathbf{S} = \begin{pmatrix} 2e \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{0.2}$$

Points of Intersection of Circle and Parabola can be given as:

$$\mathbf{X}^\top(\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{X} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top\mathbf{X} + (f_1 + \mu f_2) = 0 \tag{0.3}$$

$$\mathbf{X}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1+\mu \end{pmatrix}\mathbf{X} - 2\begin{pmatrix} 1+4\mu & 2 \end{pmatrix}\mathbf{X} = 0 \tag{0.4}$$

To degenerate a conic into a line, we can find values of μ by $||\mathbf{M}_1 + \mu\mathbf{M}_2|| = 0$ (0.5)

where $\mathbf{M}_i = \begin{pmatrix} \mathbf{V}_i & \mathbf{u}_i \\ \mathbf{u}_i^\top & f_i \end{pmatrix}$

From Equation 0.5, we get

$$(4\mu + 1)^2(\mu + 1) = -4 \tag{0.6}$$

$$\implies \mu = \frac{-5}{4} \text{ (as the only real solution.)} \tag{0.7}$$

Substituting the value of μ in Equation 0.4,

$$\mathbf{X}^\top \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 8 & -4 \end{pmatrix}\mathbf{X} = 0 \tag{0.8}$$

$$(2x + y + 16)(2x - y) = 0 \tag{0.9}$$

$$\Rightarrow \; 2x + y + 16 = 0 \text{ OR } 2x - y = 0 \tag{0.10}$$

From Line $2x + y + 16 = 0$, we get no points of intersection with both the conics. Thus, Rejected this Case.

From Line $2x - y = 0$

$$\mathbf{X} = k\begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{0.11}$$

The Intersection of the given conic with the given line can be written as:

$$\mathbf{x}_j = \mathbf{h} + k_j\mathbf{m} \tag{0.12}$$

$$where,\ \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \&\ \mathbf{m} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{0.13}$$

$$k_j = \left(\frac{1}{\mathbf{m}^\top\mathbf{V}_i\mathbf{m}}\right)\left(-\mathbf{m}^\top(\mathbf{V}_i\mathbf{h} + \mathbf{u}_i) \ \pm \ \sqrt{[\mathbf{m}^\top(\mathbf{V}_i\mathbf{h} + \mathbf{u}_i)]^2 - g(h)(\mathbf{m}^\top\mathbf{V}_i\mathbf{m})}\right) \tag{0.14}$$

After Solving the Equation 0.12 with circle and parabola, We get common points of intersection as:

$$\mathbf{X}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \& \qquad \mathbf{X}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{0.15}$$

Therefore, Let $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

The Area of Triangle PQS is:

$$Area(\triangle PQS) = \frac{1}{2}\,\|\mathbf{SP} \times \mathbf{QP}\| = 4 \tag{0.16}$$

$$\boxed{\text{The Area of } \triangle PQS \text{ is 4 sq.units.}}$$

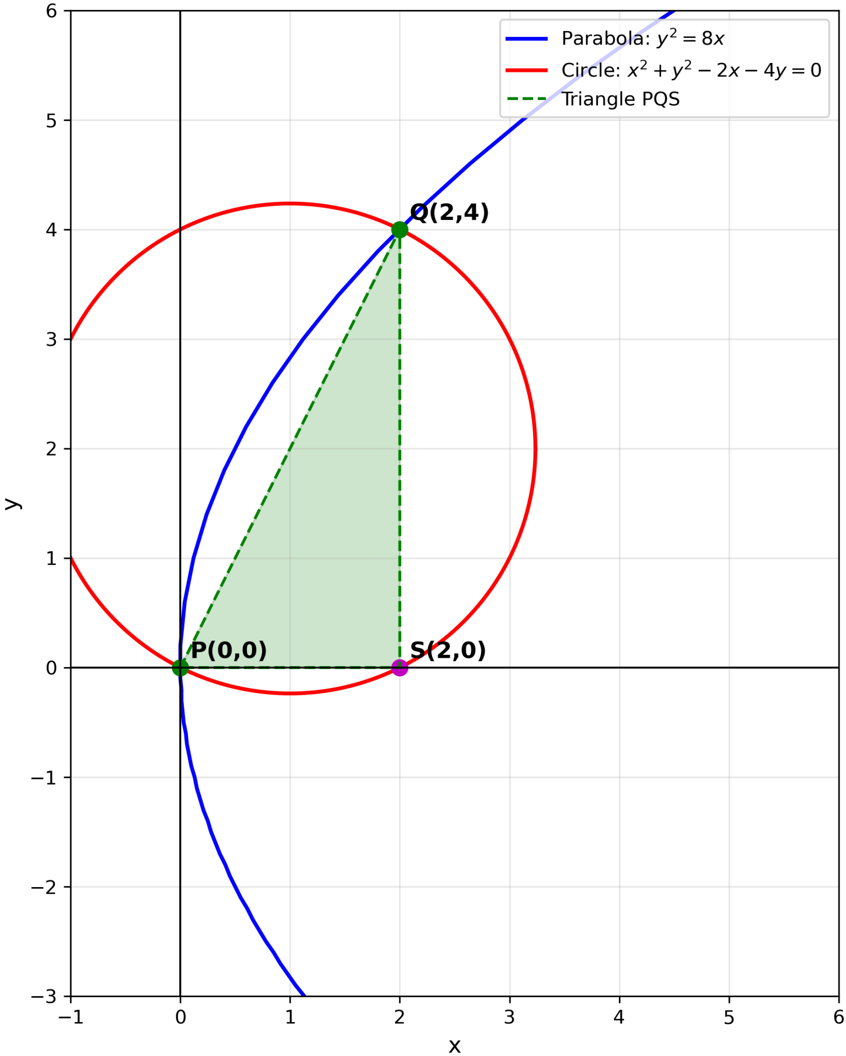

Fig. 0.1: Intersection of Two Conics and Triangle PQS