12.81

AI25BTECH11003 - Bhavesh Gaikwad

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Question

Let $\mathbf{M}$ be a 3×3 real symmetric matrix with eigenvalues -1, 1, 2 and the corresponding unit eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^3$ such that

$$\mathbf{MX} = \mathbf{u} + 2(\mathbf{v} + \mathbf{w}) \text{ and } \mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w})$$

Considering the usual inner product in $\mathbb{R}^3$, the value of $|\mathbf{x} + \mathbf{y}|^2$, where $|\mathbf{x} + \mathbf{y}|$ is the length of the vector $\mathbf{x} + \mathbf{y}$, is

(ST 2022)

a) 1.25 b) 0.25 c) 0.75 d) 1

Theoretical Solution

$$\text{Let } \mathbf{P} = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} \text{ be the } 3 \times 3 \text{ orthogonal matrix of unit eigenvectors.} \quad (1)$$

$$\text{Let } \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ be the diagonal matrix of eigenvalues.} \quad (2)$$

Since $\mathbf{M}$ is symmetric with these eigenvectors and eigenvalues:
Eigen-Decomposition:

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \quad (3)$$

Let

$$\mathbf{x} = \mathbf{P}\alpha, \quad \mathbf{y} = \mathbf{P}\beta \quad (4)$$

Given:

$$\mathbf{M}\mathbf{x} = \mathbf{u} + 2\left(\mathbf{v} + \mathbf{w}\right) \quad (5)$$

Theoretical Solution

$$\mathbf{u} + 2\left(\mathbf{v} + \mathbf{w}\right) = \mathbf{P}\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{6}$$

$$\mathbf{M}\mathbf{x} = \mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{P}\alpha = \mathbf{P}\mathbf{D}\alpha \tag{7}$$

$$\mathbf{P}\mathbf{D}\alpha = \mathbf{P}\begin{pmatrix}1\\2\\2\end{pmatrix} \implies \mathbf{D}\alpha = \begin{pmatrix}1\\2\\2\end{pmatrix} \tag{8}$$

$$\implies \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}\alpha = \begin{pmatrix}1\\2\\2\end{pmatrix} \tag{9}$$

$$\alpha = \begin{pmatrix}-1\\2\\1\end{pmatrix} \tag{10}$$

Theoretical Solution

$$\therefore \ \ \mathbf{x} = \mathbf{P}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \tag{11}$$

Given:

$$\mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w}) = \mathbf{P}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{12}$$

$$\mathbf{M}^2\mathbf{y} = \left(\mathbf{P}\mathbf{D}\mathbf{P}^\top\right)\left(\mathbf{P}\mathbf{D}\mathbf{P}^\top\right)\mathbf{P}\beta = \mathbf{P}\mathbf{D}^2\beta \tag{13}$$

Theoretical Solution

$$\mathbf{P}\mathbf{D}^2\beta = \mathbf{P}\begin{pmatrix}1\\-1\\-2\end{pmatrix} \implies \mathbf{D}^2\beta = \begin{pmatrix}1\\-1\\-2\end{pmatrix} \tag{14}$$

$$\implies \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 4\end{pmatrix}\beta = \begin{pmatrix}1\\-1\\-2\end{pmatrix} \tag{15}$$

$$\beta = \begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix} \tag{16}$$

$$\therefore\ \mathbf{y} = \mathbf{P}\begin{pmatrix}1\\-1\\-\frac{1}{2}\end{pmatrix} \tag{17}$$

Theoretical Solution

$$\mathbf{x} + \mathbf{y} = \mathbf{P}\alpha + \mathbf{P}\beta = \mathbf{P}\left(\alpha + \beta\right) \tag{18}$$

Since $\mathbf{P}$ is an orthogonal matrix and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are unit eigenvectors,

$$\left\|\mathbf{x} + \mathbf{y}\right\|^2 = \left\|\mathbf{P}(\alpha + \beta)\right\|^2 = \left(\alpha + \beta\right)^\top \left(\alpha + \beta\right) \tag{19}$$

$$\left\|\mathbf{x} + \mathbf{y}\right\|^2 = \alpha^\top\alpha + \alpha^\top\beta + \beta^\top\alpha + \beta^\top\beta \tag{20}$$

$$\left\|\mathbf{x} + \mathbf{y}\right\|^2 = \left\|\alpha\right\|^2 + 2\alpha^\top\beta + \left\|\beta\right\|^2 \tag{21}$$

$$\left\|\mathbf{x} + \mathbf{y}\right\|^2 = 6 - 7 + \frac{9}{4} \tag{22}$$

$$\boxed{\left\|\mathbf{x} + \mathbf{y}\right\|^2 = 1.25}$$

$$\boxed{\text{Thus, Option-A is correct.}} \tag{23}$$