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Question

If the rank of a (5×6) matrix $\mathbf{Q}$ is 4, then which one of the following statements is correct?

(EE 2008)

a) $\mathbf{Q}$ will have four linearly independent rows and four linearly independent columns.
b) $\mathbf{Q}$ will have four linearly independent rows and five linearly independent columns.
c) $\mathbf{QQ}^\top$ will be invertible.
d) $\mathbf{Q}^\top\mathbf{Q}$ will be invertible

Theoretical Solution

Primary Analysis:
Since rank($\mathbf{Q}$)=4 $\Rightarrow \therefore$ $\mathbf{Q}$ will have four linearly independent rows and four linearly independent columns.

Option-A:
Correct Option by Primary Analysis itself.

Option-B:
Incorrect Option by Primary Analysis itself.

Option-C:
$\mathbf{Q}\mathbf{Q}^\top$ is a 5$\times$5 matrix.
Since, rank$(\mathbf{Q}\mathbf{Q}^\top) =$ rank$(\mathbf{Q})$.
$\therefore$ rank$(\mathbf{Q}\mathbf{Q}^\top) = 4$.
Since rank$(\mathbf{Q}\mathbf{Q}^\top)$=4<5, Thus the 5$\times$5 matrix $\mathbf{Q}\mathbf{Q}^\top$ is singular $\left(|\mathbf{Q}\mathbf{Q}^\top| = 0\right)$, hence not invertible. Incorrect Option.

Theoretical Solution

Option-D:
$\mathbf{Q}^\top\mathbf{Q}$ is a 6×6 matrix.
Since, $\text{rank}(\mathbf{Q}^\top\mathbf{Q}) = \text{rank}(\mathbf{Q})$.
$\therefore$ $\text{rank}(\mathbf{Q}^\top\mathbf{Q}) = 4$.
Since $\text{rank}(\mathbf{Q}^\top\mathbf{Q})=4<6$, Thus the 6×6 matrix $\mathbf{Q}^\top\mathbf{Q}$ is singular $\left(|\mathbf{Q}^\top\mathbf{Q}| = 0\right)$, hence not invertible. Incorrect Option.

Thus, Only Option-A is correct.

Proof by Example

Consider the 5×6 matrix $\mathbf{Q}$ of rank 4:

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Option (a): Four independent rows and columns
Clearly 4 rows of $\mathbf{Q}$ are linearly independent. Thus row rank of $\mathbf{Q} = 4$
Clearly 4 columns of $\mathbf{Q}$ are linearly independent.
Thus column rank of $\mathbf{Q}$ $=4$
Thus (a) holds.

Option (b): Four independent rows and Five independent columns
Column rank cannot exceed 4.
Hence (b) is false.

Proof by Example

Option (c): Invertibility of $\mathbf{Q}\,\mathbf{Q}^\top$

$$\mathbf{Q}\,\mathbf{Q}^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 3 & 5 & 7 & 9 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{Q}\mathbf{Q}^\top = \begin{pmatrix} 14 & 23 & 33 & 43 & 0 \\ 23 & 42 & 59 & 77 & 0 \\ 33 & 59 & 86 & 111 & 0 \\ 43 & 77 & 111 & 146 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

Since $|\mathbf{Q}\,\mathbf{Q}^\top| = 0$. Not invertible.
(c) is false.

Proof by Example

Option (d): Invertibility of $\mathbf{Q}^\top\mathbf{Q}$

$$\mathbf{Q}^\top\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 3 & 5 & 7 & 9 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$$\mathbf{Q}^\top\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 2 & 4 & 6 & 8 & 120 & 154 \\ 3 & 5 & 7 & 9 & 154 & 197 \end{pmatrix} \tag{5}$$

Since, $|\mathbf{Q}^\top\mathbf{Q}| = 0$. Not invertible.
(d) is false.