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Question

Values of a, b, c which render the matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{3}} & 0 & b \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & c \end{pmatrix}$$

orthonormal are, respectively,

(AE 2013)

a) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$
b) $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$
c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
d) $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

Given: $\mathbf{Q}$ is an orthogonal matrix.

$$\mathbf{Q} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & \sqrt{6}a \\ \sqrt{2} & 0 & \sqrt{6}b \\ \sqrt{2} & -\sqrt{3} & \sqrt{6}c \end{pmatrix}$$

Condition for orthogonality,

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I} \tag{1}$$

Substituting $\mathbf{Q}$ in Equation 0.1,

$$\mathbf{Q}^\top \mathbf{Q} = \frac{1}{6} \begin{pmatrix} 6 & 0 & 2\sqrt{3}(a+b+c) \\ 0 & 6 & 3\sqrt{2}(a-c) \\ 2\sqrt{3}(a+b+c) & 3\sqrt{2}(a-c) & 6(a^2+b^2+c^2) \end{pmatrix} \tag{2}$$

$$\frac{1}{6} \begin{pmatrix} 6 & 0 & 2\sqrt{3}(a+b+c) \\ 0 & 6 & 3\sqrt{2}(a-c) \\ 2\sqrt{3}(a+b+c) & 3\sqrt{2}(a-c) & 6(a^2+b^2+c^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Theoretical Solution

On Comparing we get,

Let $\mathbf{P} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\mathbf{P}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \tag{4}$$

$$\mathbf{P}^\top \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \tag{5}$$

$$\mathbf{P}^\top \mathbf{P} = \|\mathbf{P}\|^2 = 1 \tag{6}$$

Theoretical Solution

From Equation 4 and 5,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

Row Transformation-1: $R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

Row Transformation-2: $R_1 \to R_1 + R_2$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

From this,

$$a = c \; \& \; b = -2a \tag{10}$$

Theoretical Solution

$$\therefore \ \mathbf{P} = a \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{11}$$

From Equations 6 and 11

$$a^2 \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1 \tag{12}$$

$$a^2 \left[1 + 4 + 1 \right] = 6a^2 = 1 \tag{13}$$

$$\boxed{a = \pm \frac{1}{\sqrt{6}}} \tag{14}$$

Theoretical Solution

Substituting value of a in Equation-11

$$\therefore \ \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \qquad OR \qquad \mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \tag{15}$$

$\boxed{\text{Thus, Option B and D are correct.}}$