4.13.72

AI25BTECH11003 - Bhavesh Gaikwad

Question: A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors $\hat{\imath}$, $\hat{\imath}+\hat{\jmath}$ and the plane determined by the vectors $\hat{\imath}-\hat{\jmath}$, $\hat{\imath}+\hat{k}$. The angle between **a** and the vector $\hat{\imath}-2\hat{\jmath}+2\hat{k}$ is ______.

(1996)

Solution:

Let $\mathbf{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\text{Let the Equation of Plane-1 be: } \mathbf{n}^\top \mathbf{x} = 0 \tag{0.1}$$

Since **A** and **B** satisfy Equation 0.1,

$$\mathbf{n}^\top \mathbf{A} = 0 \text{ And } \mathbf{n}^\top \mathbf{B} = 0 \tag{0.2}$$

OR

$$\mathbf{A}^\top \mathbf{n} = 0 \text{ And } \mathbf{B}^\top \mathbf{n} = 0 \tag{0.3}$$

From Equation 0.3,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}^\top \mathbf{n} = 0 \tag{0.4}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{n} = 0 \tag{0.5}$$

$$\therefore \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{0.6}$$

$$\text{Therefore, The Equation of Plane-1 is } \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{0.7}$$

$$\text{Let the Equation of Plane-2 be: } \mathbf{m}^\top \mathbf{x} = 0 \tag{0.8}$$

Since **C** and **D** satisfy Equation 0.8,

$$\mathbf{m}^\top \mathbf{C} = 0 \text{ And } \mathbf{n}^\top \mathbf{D} = 0 \tag{0.9}$$

OR

$$\mathbf{C}^\top \mathbf{m} = 0 \text{ And } \mathbf{D}^\top \mathbf{m} = 0 \tag{0.10}$$

From Equation 0.10,

$$\begin{pmatrix} \mathbf{C} & \mathbf{D} \end{pmatrix}^\top \mathbf{m} = 0 \tag{0.11}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{0.12}$$

$$\therefore \ \mathbf{m} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{0.13}$$

$$\text{Therefore, The Equation of Plane-2 is } \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{0.14}$$

Let the parallel vector of line of intersection of Plane-1 and Plane-2 be $\mathbf{r}$.

As $\mathbf{r}$ satisfies the Equation of both the Planes,

$$\mathbf{n}^\top \mathbf{r} = 0 \ \& \ \mathbf{m}^\top \mathbf{r} = 0 \tag{0.15}$$

From Equation 0.15,

$$\begin{pmatrix} \mathbf{n} & \mathbf{m} \end{pmatrix}^\top \mathbf{r} = 0 \tag{0.16}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \mathbf{r} = 0 \tag{0.17}$$

$$\therefore \ \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{0.18}$$

$$\text{Therefore from Equation 0.18, } \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{0.19}$$

Let $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ [Already given in the Question]

We know,

$$\mathbf{a}^\top \mathbf{u} = \|\mathbf{a}\| \, \|\mathbf{u}\| \cos(\theta) \tag{0.20}$$

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top \mathbf{a}} = \sqrt{2}, \ \|\mathbf{u}\| = \sqrt{\mathbf{u}^\top \mathbf{u}} = 3 \tag{0.21}$$

From Equation 0.20 and 0.21,

$$\cos(\theta) = \frac{3}{3\sqrt{2}} \quad \Rightarrow \theta = 45^\circ \tag{0.22}$$

$$\boxed{\text{The angle between } \mathbf{a} \text{ and } 1\hat{i} - 2\hat{j} + 2\hat{k} \text{ is } 45^\circ.} \tag{0.23}$$

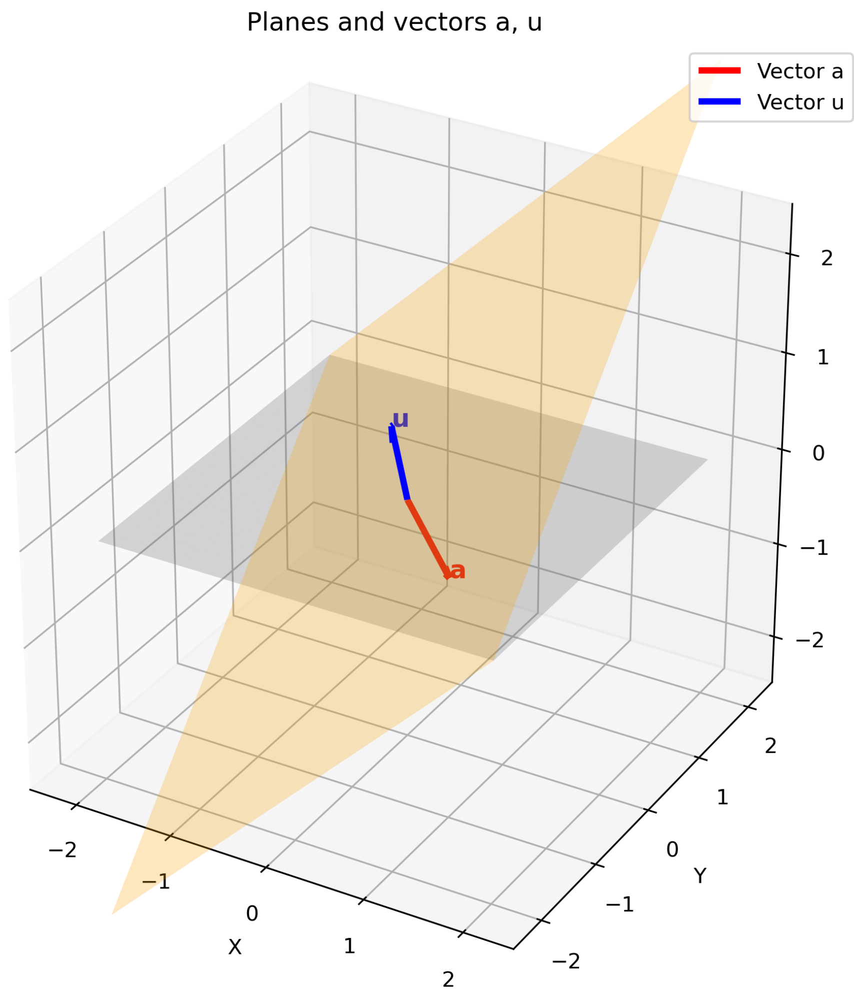

Fig. 0.1: Plane