11.2.9

AI25BTECH11003 - Bhavesh Gaikwad

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Question

AB is a line-segment. **P** and **Q** are points on opposite sides of AB such that each of them is equidistant from the points **A** and **B**. Show that the line PQ is the perpendicular bisector of AB.

Theoretical Solution

Since $\mathbf{P}$ has the same distance from $\mathbf{A}$ and $\mathbf{B}$,

$$\|\mathbf{P}-\mathbf{A}\| = \|\mathbf{P}-\mathbf{B}\| \tag{1}$$

We know

$$\|\mathbf{W}\|^2 = \mathbf{W}^\top\mathbf{W} \tag{2}$$

Squaring both sides in Equation 1,

$$(\mathbf{P}-\mathbf{A})^\top(\mathbf{P}-\mathbf{A}) = (\mathbf{P}-\mathbf{B})^\top(\mathbf{P}-\mathbf{B}) \tag{3}$$

After Simplifing, we get

$$\mathbf{P}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2) \tag{4}$$

Similarly for $\mathbf{Q}$,

$$\mathbf{Q}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2) \tag{5}$$

Theoretical Solution

From A.5.1(Book),
The equation of perpendicular bisector of AB can be represented as

$$\left(\mathbf{X} - \frac{\mathbf{A}+\mathbf{B}}{2}\right)^\top (\mathbf{B}-\mathbf{A}) = 0 \tag{6}$$

OR

$$\mathbf{X}^\top (\mathbf{B}-\mathbf{A}) = \frac{1}{2}\left(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2 \right) \tag{7}$$

A Point on line PQ can be represented as

$$\mathbf{X} = \mathbf{P} + \lambda(\mathbf{P}-\mathbf{Q}) \tag{8}$$

Theoretical Solution

Taking dot production with $\mathbf{B}-\mathbf{A}$ on both sides,

$$(\mathbf{B}-\mathbf{A})^\top\mathbf{X} = (\mathbf{B}-\mathbf{A})^\top\mathbf{P} + \lambda(\mathbf{B}-\mathbf{A})^\top(\mathbf{P}-\mathbf{Q}) \tag{9}$$

OR

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = \mathbf{P}^\top(\mathbf{B}-\mathbf{A}) + \lambda(\mathbf{P}-\mathbf{Q})^\top(\mathbf{B}-\mathbf{A}) \tag{10}$$

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = (1+\lambda)\mathbf{P}^\top(\mathbf{B}-\mathbf{A}) - \lambda\mathbf{Q}^\top(\mathbf{B}-\mathbf{A}) \tag{11}$$

From Equation 4 and 5,

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = (1+\lambda)\left[\frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2)\right] - \lambda\left[\frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2)\right] \tag{12}$$

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2) \tag{13}$$

Since, Equation 7 and 13 are same.

$\boxed{\text{Hence Proved , Line PQ is a perpendicular bisector to line segment AB.}}$

Example

Assuming $\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \& \quad \mathbf{P} - \mathbf{Q} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{14}$$

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{P} - \mathbf{Q}) = 0 \tag{15}$$

Thus, Line PQ is perpendicular to line segment AB.

Let $\mathbf{M}$ be the midpoint of line segment AB.

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \tag{16}$$

Example

Equation of line PQ,

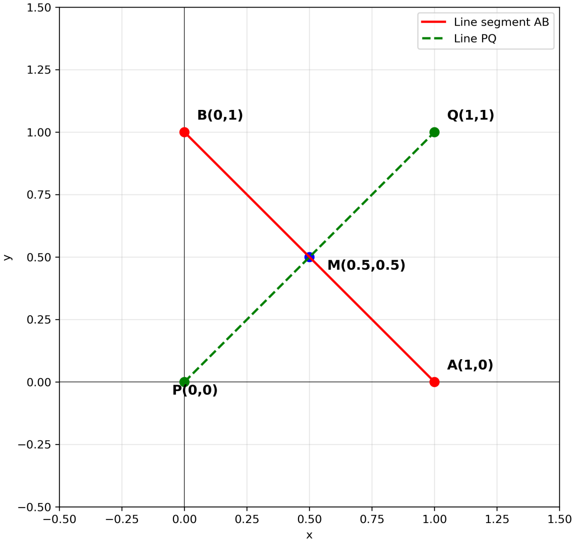
$$\mathbf{X} = \mathbf{P} + \lambda(\mathbf{P} - \mathbf{Q}) \tag{17}$$

$$\mathbf{X} = \lambda \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{18}$$

M satifies equation 18, thus Line PQ passes through midpoint of line segment AB. Thus PQ bisects AB.

$\therefore$ Line PQ is a perpendicular bisector to line segment AB.

Image



Line segment AB
Line PQ
B(0,1)
Q(1,1)
M(0.5,0.5)
A(1,0)
P(0,0)
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
y
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
x