5.2.70

AI25BTECH11003 - Bhavesh Gaikwad

Question: For which values of p does the pair of equations given below have a unique solution.

$$4x + py + 8 = 0$$
$$2x + 2y + 2 = 0$$

Solution:
Given:

$$4x + py + 8 = 0 \tag{0.1}$$
$$2x + 2y + 2 = 0 \tag{0.2}$$

Standard Form: $\mathbf{Ax} = \mathbf{b}$ where:

Coefficient Matrix: $\mathbf{A} = \begin{pmatrix} 4 & p \\ 2 & 2 \end{pmatrix}$

Constant Vector: $\mathbf{b} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}$

Augmented Matrix: $[\mathbf{A}|\mathbf{b}] = \begin{pmatrix} 4 & p & -8 \\ 2 & 2 & -2 \end{pmatrix}$

For Unique Solution:
Unique Solution: rank($\mathbf{A}$) = rank($[\mathbf{A}|\mathbf{b}]$) = n (number of variables)
For our system: $n = 2$ variables

Finding rank($\mathbf{A}$) - Rank of Coefficient Matrix

Initial Matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 4 & p \\ 2 & 2 \end{pmatrix} \tag{0.3}$$

Row Operations on $\mathbf{A}$:

$$R_2 \to R_2 - \frac{1}{2}R_1 \tag{0.4}$$

Row Echelon Form of $\mathbf{A}$:

$$\begin{pmatrix} 4 & p \\ 0 & 2 - \frac{p}{2} \end{pmatrix} \tag{0.5}$$

Rank Analysis:
Case 1: If $2 - \frac{p}{2} \neq 0$ (i.e., $p \neq 4$)
Both rows are non-zero and linearly independent
$\Rightarrow$ rank($\mathbf{A}$) = 2

Case 2: If $2 - \frac{p}{2} = 0$ (i.e., $p = 4$)
Second row is zero, only first row is non-zero
$\Rightarrow$ rank($\mathbf{A}$) = 1

Finding rank([$\mathbf{A}$|$\mathbf{b}$]) - Rank of Augmented Matrix
Initial Augmented Matrix:

$$\begin{pmatrix} 4 & p & -8 \\ 2 & 2 & -2 \end{pmatrix} \tag{0.6}$$

Row Operation on [$\mathbf{A}$|$\mathbf{b}$]:

$$R_2 \rightarrow R_2 - \frac{1}{2}R_1 \tag{0.7}$$

Row Echelon Form of Augmented Matrix:

$$\begin{pmatrix} 4 & p & -8 \\ 0 & 2 - \frac{p}{2} & 2 \end{pmatrix} \tag{0.8}$$

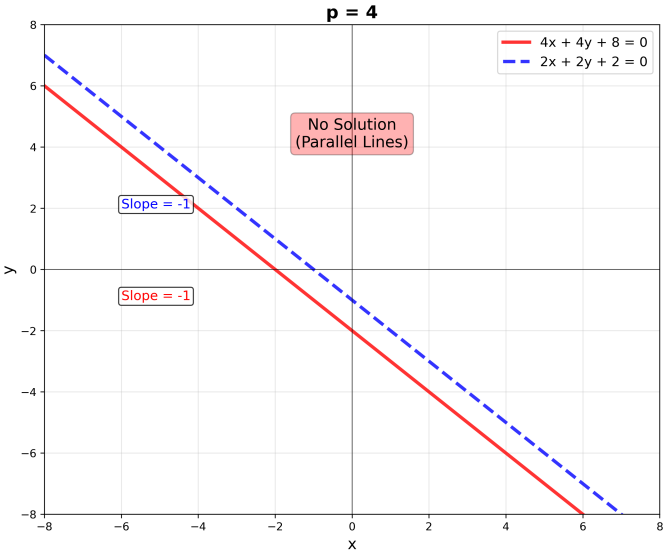
Rank Analysis:
Case 1: If $p \neq 4$:
$\Rightarrow$ rank([$\mathbf{A}$|$\mathbf{b}$]) = 2

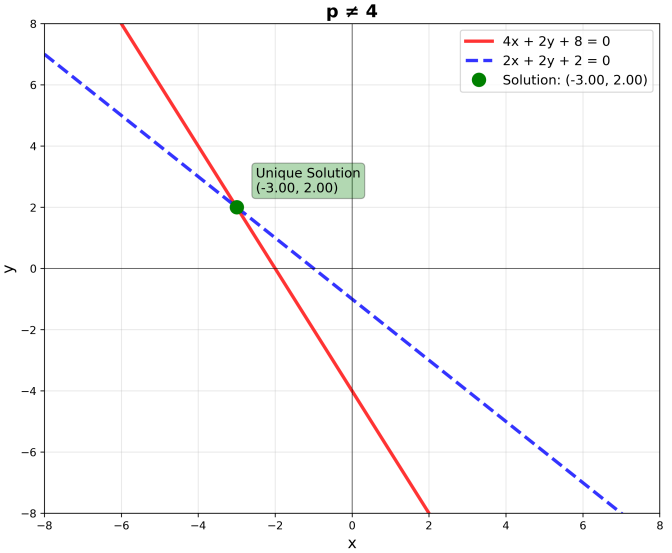
Case 2: If $p = 4$:
$\Rightarrow$ rank([$\mathbf{A}$|$\mathbf{b}$]) = 2

Comparing Ranks and Providing Solution Type

Value of p	rank($\mathbf{A}$)	rank([$\mathbf{A}$|$\mathbf{b}$])	Comparison	Solution Type
$p \neq 4$	2	2	rank($\mathbf{A}$) = rank([$\mathbf{A}$|$\mathbf{b}$]) = 2	UNIQUE
$p = 4$	1	2	rank($\mathbf{A}$) < rank([$\mathbf{A}$|$\mathbf{b}$])	NO SOLUTION

$$\therefore \text{ For } p \in \mathbb{R} - \{4\}, \text{ the pair of equations has an Unique Solution.} \tag{0.9}$$


p = 4
4x + 4y + 8 = 0
2x + 2y + 2 = 0
No Solution
(Parallel Lines)
Slope = -1
Slope = -1
x
y


p ≠ 4
4x + 2y + 8 = 0
2x + 2y + 2 = 0
Solution: (-3.00, 2.00)
Unique Solution
(-3.00, 2.00)
x
y