6.4.3

AI25BTECH11003 - Bhavesh Gaikwad

Question: Find the shortest distance between the lines:

$$\mathbf{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$$
$$\mathbf{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

Solution:

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points on the given lines respectively.

$\mathbf{x}_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + k_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 7 \\ 0 \\ -6 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Let $\mathbf{A} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 0 \\ -6 \end{pmatrix}$

Let $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix}$

$$(\mathbf{M}\ \mathbf{B} - \mathbf{A}) = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 2 & 5 \\ 6 & 2 & -7 \end{pmatrix} \tag{0.1}$$

Row Transformation-1: $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 3 & 1 & 5 \\ 2 & 2 & 5 \\ 0 & 0 & -17 \end{pmatrix} \tag{0.2}$$

Row Transformation-2: $R_2 \to R_2 - \frac{2}{3}R_1$

$$\begin{pmatrix} 3 & 1 & 5 \\ 0 & 4/3 & 5/3 \\ 0 & 0 & -17 \end{pmatrix} \tag{0.3}$$

Therefore, The Rank is 3 $\Rightarrow$ The Lines are Skew Lines.

$$\text{Let } \mathbf{K} = \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} \tag{0.4}$$

$$(\mathbf{M}^\top \mathbf{M})\mathbf{K} = \mathbf{M}^\top(\mathbf{B} - \mathbf{A}) \tag{0.5}$$

$$\begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \mathbf{K} = \begin{pmatrix} -17 \\ 1 \end{pmatrix} \tag{0.6}$$

The Augmented Matrix from Equation 0.6,

$$\left(\begin{array}{cc|c} 49 & 19 & -17 \\ 19 & 9 & 1 \end{array}\right) \tag{0.7}$$

After Row Reductions,

$$\left(\begin{array}{cc|c} 1 & 0 & -43/20 \\ 0 & 1 & 93/20 \end{array}\right) \tag{0.8}$$

$$\therefore \mathbf{K} = \begin{pmatrix} -43/20 \\ 93/20 \end{pmatrix} \tag{0.9}$$

$$\therefore k_1 = -43/20 \, and \, k_2 = -93/20 \tag{0.10}$$

From Equation 0.10,

$$\mathbf{x}_1 = \begin{pmatrix} -89/20 \\ -93/10 \\ 119/10 \end{pmatrix} and \, \mathbf{x}_2 = \begin{pmatrix} -47/20 \\ -93/10 \\ -153/10 \end{pmatrix} \tag{0.11}$$

The Minimum Distance between the given skew lines is $\|\mathbf{x}_2 - \mathbf{x}_2\|$

$$\|\mathbf{x}_2 - \mathbf{x}_2\| = \sqrt{(\mathbf{x}_2 - \mathbf{x}_2)^\top (\mathbf{x}_2 - \mathbf{x}_2)} = \frac{17}{\sqrt{5}} \tag{0.12}$$

$$\boxed{\text{The Minimum Distance between the given Lines} = \frac{17}{\sqrt{5}} \, units} \tag{0.13}$$

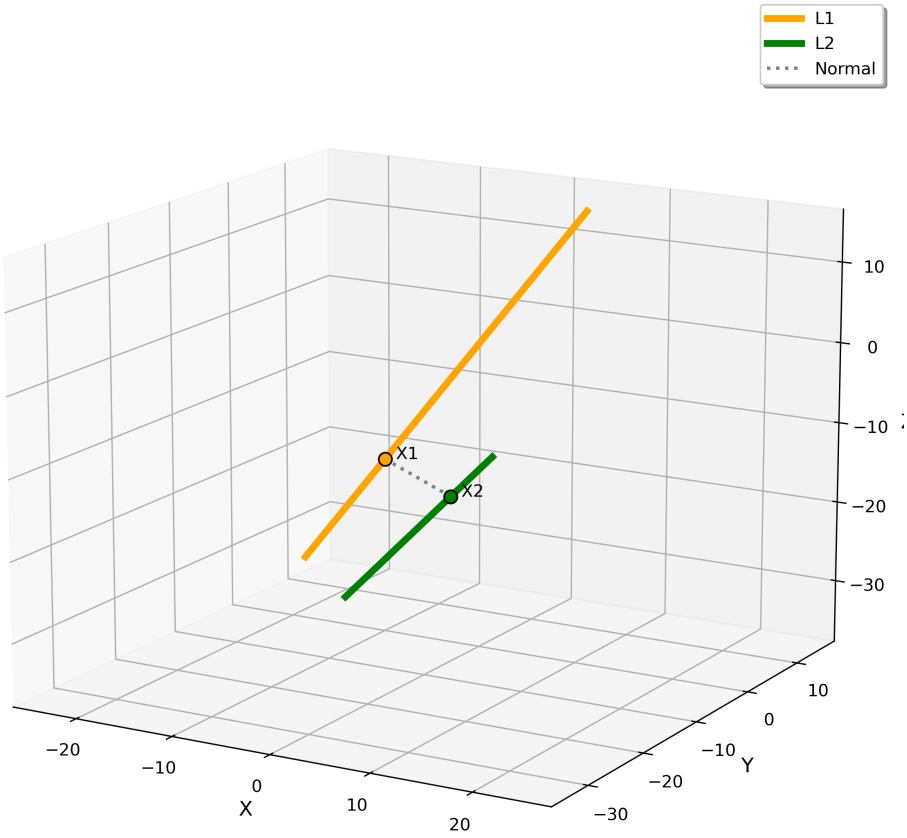

Fig. 0.1: Skew Lines