11.2.9

AI25BTECH11003 - Bhavesh Gaikwad

Question: AB is a line-segment. **P** and **Q** are points on opposite sides of AB such that each of them is equidistant from the points **A** and **B**. Show that the line PQ is the perpendicular bisector of AB.

Solution:
Since **P** has the same distance from **A** and **B**,

$$\|\mathbf{P}-\mathbf{A}\| = \|\mathbf{P}-\mathbf{B}\| \tag{0.1}$$

We know

$$\|\mathbf{W}\|^2 = \mathbf{W}^\top\mathbf{W} \tag{0.2}$$

Squaring both sides in Equation 0.1,

$$(\mathbf{P}-\mathbf{A})^\top(\mathbf{P}-\mathbf{A}) = (\mathbf{P}-\mathbf{B})^\top(\mathbf{P}-\mathbf{B}) \tag{0.3}$$

After Simplifing, we get

$$\mathbf{P}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\,\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2\,) \tag{0.4}$$

Similarly for **Q**,

$$\mathbf{Q}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\,\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2\,) \tag{0.5}$$

From A.5.1(Book),
The equation of perpendicular bisector of AB can be represented as

$$\left(\mathbf{X} - \frac{\mathbf{A}+\mathbf{B}}{2}\right)^\top(\mathbf{B}-\mathbf{A}) = 0 \tag{0.6}$$

OR

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\,\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2\,) \tag{0.7}$$

A Point on line PQ can be represented as

$$\mathbf{X} = \mathbf{P} + \lambda(\mathbf{P}-\mathbf{Q}) \tag{0.8}$$

Taking dot production with $\mathbf{B}-\mathbf{A}$ on both sides,

$$(\,\mathbf{B}-\mathbf{A}\,)^\top\mathbf{X} = (\,\mathbf{B}-\mathbf{A}\,)^\top\mathbf{P} + \lambda(\,\mathbf{B}-\mathbf{A}\,)^\top(\,\mathbf{P}-\mathbf{Q}\,) \tag{0.9}$$

OR

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = \mathbf{P}^\top(\mathbf{B}-\mathbf{A}) + \lambda(\mathbf{P}-\mathbf{Q})^\top(\mathbf{B}-\mathbf{A}) \tag{0.10}$$

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = (1+\lambda)\mathbf{P}^\top(\mathbf{B}-\mathbf{A}) - \lambda\mathbf{Q}^\top(\mathbf{B}-\mathbf{A}) \tag{0.11}$$

From Equation 0.4 and 0.5,

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = (1+\lambda)\left[\frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2)\right] - \lambda\left[\frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2)\right] \tag{0.12}$$

$$\mathbf{X}^\top(\mathbf{B}-\mathbf{A}) = \frac{1}{2}(\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2) \tag{0.13}$$

Since, Equation 0.7 and 0.13 are same.

$$\boxed{\text{Hence Proved , Line PQ is a perpendicular bisector to line segment AB.}}$$

Example:
Assuming $\mathbf{A} = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}0\\1\end{pmatrix}$, $\mathbf{P} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}1\\1\end{pmatrix}$

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}-1\\1\end{pmatrix} \quad \& \quad \mathbf{P}-\mathbf{Q} = \begin{pmatrix}-1\\-1\end{pmatrix} \tag{0.14}$$

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{P}-\mathbf{Q}) = 0 \tag{0.15}$$

Thus, Line PQ is perpendicular to line segment AB.

Let $\mathbf{M}$ be the midpoint of line segment AB.

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix}1/2\\1/2\end{pmatrix} \tag{0.16}$$

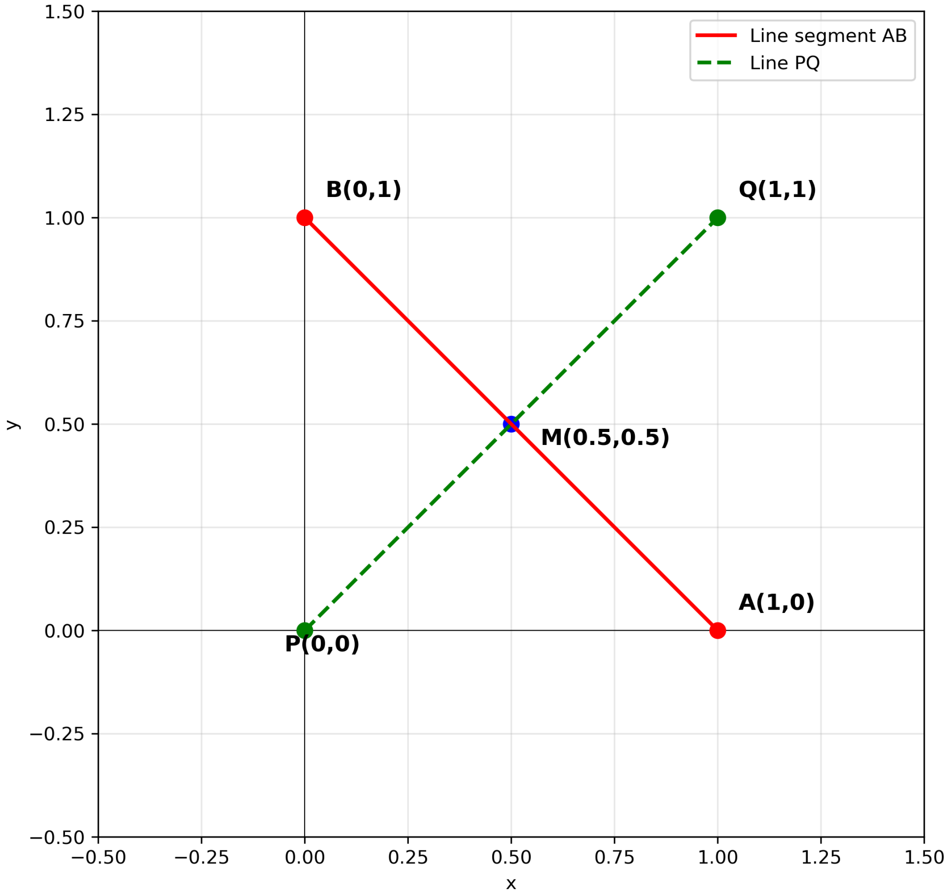
Equation of line PQ,

$$\mathbf{X} = \mathbf{P} + \lambda(\mathbf{P}-\mathbf{Q}) \tag{0.17}$$

$$\mathbf{X} = \lambda\begin{pmatrix}-1\\-1\end{pmatrix} \tag{0.18}$$

$\mathbf{M}$ satifies equation 0.18, thus Line PQ passes through midpoint of line segment AB. Thus PQ bisects AB.

$\therefore$ Line PQ is a perpendicular bisector to line segment AB.



Line segment AB
Line PQ
B(0,1)
Q(1,1)
M(0.5,0.5)
A(1,0)
P(0,0)
x
y
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50

Fig. 0.1