

12.81

AI25BTECH11003 - Bhavesh Gaikwad

Question: Let \mathbf{M} be a 3×3 real symmetric matrix with eigenvalues $-1, 1, 2$ and the corresponding unit eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively. Let \mathbf{x} and \mathbf{y} be two vectors in \mathbb{R}^3 such that

$$\mathbf{M}\mathbf{x} = \mathbf{u} + 2(\mathbf{v} + \mathbf{w}) \text{ and } \mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w})$$

Considering the usual inner product in \mathbb{R}^3 , the value of $|\mathbf{x} + \mathbf{y}|^2$, where $|\mathbf{x} + \mathbf{y}|$ is the length of the vector $\mathbf{x} + \mathbf{y}$, is

(ST 2022)

- a) 1.25 b) 0.25 c) 0.75 d) 1

Solution:

Let $\mathbf{P} = (\mathbf{u} \ \mathbf{v} \ \mathbf{w})$ be the 3×3 orthogonal matrix of unit eigenvectors. (0.1)

$$\text{Let } \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ be the diagonal matrix of eigenvalues.} \quad (0.2)$$

Since \mathbf{M} is symmetric with these eigenvectors and eigenvalues:
Eigen-Decomposition:

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (0.3)$$

Let

$$\mathbf{x} = \mathbf{P}\alpha, \quad \mathbf{y} = \mathbf{P}\beta \quad (0.4)$$

Given:

$$\mathbf{M}\mathbf{x} = \mathbf{u} + 2(\mathbf{v} + \mathbf{w}) \quad (0.5)$$

Express the right-hand side in the eigenbasis:

$$\mathbf{u} + 2(\mathbf{v} + \mathbf{w}) = \mathbf{P} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (0.6)$$

$$\mathbf{M}\mathbf{x} = \mathbf{P}\mathbf{D}\mathbf{P}^T\mathbf{P}\alpha = \mathbf{P}\mathbf{D}\alpha \quad (0.7)$$

$$\mathbf{PD}\alpha = \mathbf{P} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow \mathbf{D}\alpha = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (0.8)$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \alpha = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (0.9)$$

$$\alpha = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad (0.10)$$

$$\therefore \mathbf{x} = \mathbf{P} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad (0.11)$$

Given:

$$\mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w}) = \mathbf{P} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (0.12)$$

$$\mathbf{M}^2\mathbf{y} = (\mathbf{PDP}^\top)(\mathbf{PDP}^\top)\mathbf{P}\beta = \mathbf{PD}^2\beta \quad (0.13)$$

$$\mathbf{PD}^2\beta = \mathbf{P} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \Rightarrow \mathbf{D}^2\beta = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (0.14)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \beta = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (0.15)$$

$$\beta = \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix} \quad (0.16)$$

$$\therefore \mathbf{y} = \mathbf{P} \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix} \quad (0.17)$$

$$\mathbf{x} + \mathbf{y} = \mathbf{P}\alpha + \mathbf{P}\beta = \mathbf{P}(\alpha + \beta) \quad (0.18)$$

Since \mathbf{P} is an orthogonal matrix and \mathbf{u} , \mathbf{v} , \mathbf{w} are unit eigenvectors,

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{P}(\alpha + \beta)\|^2 = (\alpha + \beta)^\top (\alpha + \beta) \quad (0.19)$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = \alpha^\top \alpha + \alpha^\top \beta + \beta^\top \alpha + \beta^\top \beta \quad (0.20)$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\alpha\|^2 + 2\alpha^\top \beta + \|\beta\|^2 \quad (0.21)$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = 6 - 7 + \frac{9}{4} \quad (0.22)$$

$\|\mathbf{x} + \mathbf{y}\|^2 = 1.25$

Thus, Option-A is correct.

(0.23)