12.497

AI25BTECH11003 - Bhavesh Gaikwad

Question: If the rank of a (5×6) matrix $\mathbf{Q}$ is 4, then which one of the following statements is correct?

(EE 2008)

a) $\mathbf{Q}$ will have four linearly independent rows and four linearly independent columns.
b) $\mathbf{Q}$ will have four linearly independent rows and five linearly independent columns.
c) $\mathbf{QQ}^\top$ will be invertible.
d) $\mathbf{Q}^\top\mathbf{Q}$ will be invertible

Solution:

Primary Analysis:
Since rank($\mathbf{Q}$)=4 $\Rightarrow \therefore$ $\mathbf{Q}$ will have four linearly independent rows and four linearly independent columns.

Option-A:
Correct Option by Primary Analysis itself.

Option-B:
Incorrect Option by Primary Analysis itself.

Option-C:
$\mathbf{QQ}^\top$ is a 5×5 matrix.
Since, rank($\mathbf{QQ}^\top$) = rank($\mathbf{Q}$).
$\therefore$ rank($\mathbf{QQ}^\top$) = 4.
Since rank($\mathbf{QQ}^\top$)=4<5, Thus the 5×5 matrix $\mathbf{QQ}^\top$ is singular ($|\mathbf{QQ}^\top| = 0$), hence not invertible. Incorrect Option.

Option-D:
$\mathbf{Q}^\top\mathbf{Q}$ is a 6×6 matrix.
Since, rank($\mathbf{Q}^\top\mathbf{Q}$) = rank($\mathbf{Q}$).
$\therefore$ rank($\mathbf{Q}^\top\mathbf{Q}$) = 4.
Since rank($\mathbf{Q}^\top\mathbf{Q}$)=4<6, Thus the 6×6 matrix $\mathbf{Q}^\top\mathbf{Q}$ is singular ($|\mathbf{Q}^\top\mathbf{Q}| = 0$), hence not invertible. Incorrect Option.

Thus, Only Option-A is correct.

Proof by Example

Consider the 5×6 matrix $\mathbf{Q}$ of rank 4:

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{0.1}$$

Option (a): Four independent rows and columns

Clearly rows of $\mathbf{Q}$ are linearly independent. Thus row rank of $\mathbf{Q}$ = 4
Clearly columns of $\mathbf{Q}$ are linearly independent. Thus column rank of $\mathbf{Q}$ =4
Thus (a) holds.

Option (b): Four independent rows and Five independent columns

Column rank cannot exceed 4.
Hence (b) is false.

Option (c): Invertibility of $\mathbf{Q}\mathbf{Q}^\top$

$$\mathbf{Q}\mathbf{Q}^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 3 & 5 & 7 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 23 & 33 & 43 & 0 \\ 23 & 42 & 59 & 77 & 0 \\ 33 & 59 & 86 & 111 & 0 \\ 43 & 77 & 111 & 146 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{0.2}$$

Since the 5th row (and column) is zero, $|\mathbf{Q}\mathbf{Q}^\top| = 0$. Not invertible.
(c) is false.

Option (d): Invertibility of $\mathbf{Q}^\top\mathbf{Q}$

$$\mathbf{Q}^\top\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 3 & 5 & 7 & 9 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 & 9 \\ 2 & 4 & 6 & 8 & 120 & 154 \\ 3 & 5 & 7 & 9 & 154 & 197 \end{pmatrix} \tag{0.3}$$

Since, $|\mathbf{Q}^\top\mathbf{Q}| = 0$. Not invertible.
(d) is false.

Therefore, only option (a) is valid.