GRAPH THEORY

1. GRAPH THEORY-1

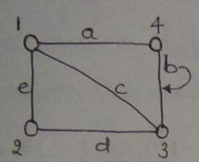
A Graph is represented as $G = (V, E)$

V = Vertices set = $\{v_1, v_2 --- v_n\}$

E = Edges set = $\{v_i \rightarrow v_j ---\}$

Now, $|V|$ = No. of vertices in the Graph = order of the Graph

$|E|$ = No. of edges = size of the Graph



$G = (V, E)$

$V = \{1, 2, 3, 4\}$

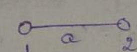
$E = \{a, b, c, d, e\}$

$a = \{1, 4\}$ $d = \{2, 3\}$

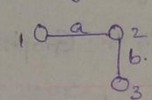
$b = \{4, 3\}$ $e = \{1, 2\}$

$c = \{1, 3\}$

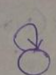
⇒ Adjacent Vertices :- Vertices having common edge are adjacent



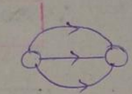
⇒ Advacent edges : If two vertices have an edge in common then the edges are called Adjacent edges.



⇒ Self-loop :



⇒ Multiedge :



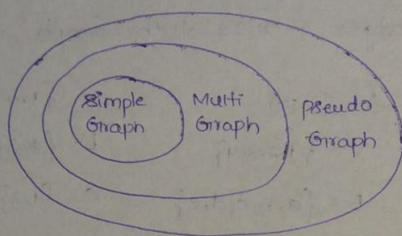
⇒ Simple Graph is the one which donot have selfloops and multiedges.

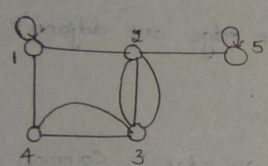
Graph	Self loops	Multiedges
General/pseudo Graph	✓	✓
MultiGraph	✗	✓
Simple Graph	✗	✗

2. GRAPH THEORY- 2

Degree of the vertex : No. of edges incident on it (a particular vertex) is called the degree of that vertex (counting loops twice)

⇒ Every simple Graph is a multigraph aswell as pseudo Graph. similarly a multiGraph is a pseudo Graph.





deg(1) = 4   deg(4) = (3)

deg(2) = 5   deg(5) = 3

deg(3) = 5

∴ sum of the degrees = 20.

No. of edges in the Graph = 10.

∴ $\boxed{\text{sum of the degrees} = 2(\text{No. of edges})}$ → Hand Shaking Lemma

∴ $\boxed{\sum_{v \in V} d(v) = 2|E|}$

2. The no. of vertices with odd degree in a Graph is always even.

$\sum \deg(v) = 2|E|$

$\sum_{odd} \deg(v) + \underbrace{\sum_{even} \deg(v)}_{even} = even$

⇒ $\sum_{odd} \deg(v) = even - even = even$

∴ $\boxed{\sum_{odd} \deg(v) = even}$

∴ $\boxed{\text{The no. of vertices with odd degree in a Graph is always even}}$

3. GRAPH TH

Degree Seque

(Descending

Ex: d = 3

d = 3

⇒ Now, Giv

a deg se

or not

solution

HAVEL- H

① ⇒ put the

② ⇒ Remove

③ ⇒ Substrac

⇒ Repeat

Stop this p

⇒ we get

⇒ If we ge

⇒ Not eno

Ex:

1) (3, 2, 1, 1,

2)

3) (6, 5, 4,

⇒ 6

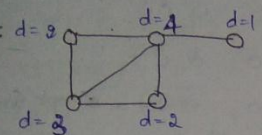
Now

sub

next

3. GRAPH THEORY-3

Degree Sequence: The Arrangement of degrees in Non-Ascending order (Descending order) is called the Degree sequence.

Ex:


Degree Sequence = (4,3,2,2,1)

⇒ Now, Given a graph finding the degree sequence is easy but given a deg sequence finding whether there exits atleast one simple Graph or not is difficult. we use "HAVEL-HAKIMI PROCEDURE" to find solution to such problem.

HAVEL- HAKIMI PROCEDURE

1. ⇒ put the degree sequence in Descending order
2. ⇒ Remove the highest degree (Let it be K)
3. ⇒ Substract one from the remaining K vertices

⇒ Repeat the steps ①②③

Stop this procedure when

⇒ we get all 0 entries ⇒ simple Graph exits

⇒ If we get Negative values ⇒ Simple Graph Not possible

⇒ Not enough edges ⇒ No Simple Graph.

Ex:

1) (3,2,1,1,0) ⇒ sum of degrees should be even ⇒ Here we get '7'

∴ No such Graph exists.

2) ($\cancel{3}$ 2 1 0 0)

(1 0 -1 0)

↳ Negative edge = No such Graph exists.

3) (6,5,4,3,3,1)

⇒ $\cancel{6}$ 5 4 3 3 1

↓

Now cut '6' and substract '1' from the next 6 nodes but there are only 5 ⇒ so No Simple Graph Exists.

4) which of the following degree sequence doesnot correspond a Simple Graph

i) $(7,6,5,4,4,3,2,1) \Rightarrow (\cancel{7}\ 6\ 5\ 4\ 4\ 3\ 2\ 1)$ ④

$(\cancel{5}\ 4\ 3\ 3\ 2\ 1\ 0)$

$(\cancel{3}\ 2\ 2\ 1\ 0\ 0)$

$(\cancel{1}\ 1\ 0\ 0\ 0)$

$(0\ 0\ 0\ 0)$ = simple Graph exists.

2) $(6,6,6,6,3,3,2,2) = (\cancel{6}\ 6\ 6\ 6\ 3\ 3\ 2.\ 2)$

$(\cancel{5}\ 5\ 5\ 2\ 2\ 1\ 2)$

$(\cancel{4}\ 4\ 1\ 1\ 0\ 2)$

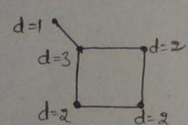
$(3\ 0\ 0\ -1\ 1)$

$\hookrightarrow$ (-ve) = No simple Graph

4. GRAPH THEORY-4

Min degree = "δ" = Min of all the degrees of the particular Graph

Max degree = "Δ" = Max of all the degress of the Graph is called Max deg.



$\boxed{\delta = 1 \quad \Delta = 3}$

$\Rightarrow$ If 'G' is a graph with V- vertices and e- Edges then,

$$\boxed{\delta \leq \frac{2e}{V} \leq \Delta}$$

Proof:

$\Sigma \deg(v) = 2|E|$

$\Rightarrow$ Replacing each degree with min. degree

$\underbrace{\delta + \delta + \delta + \delta \dots + \delta}_{\Rightarrow\ V \text{ times}} \leq 2e$

$\therefore \boxed{V \cdot \delta \leq 2e}$

$\Rightarrow$ Replacing with max degree we get

$= \Delta + \Delta + \Delta + \dots + \Delta \geq 2e$

$= \boxed{V \cdot \Delta \geq 2e} \Rightarrow \boxed{2e \leq V \cdot \Delta}$

$\therefore \quad V \cdot \delta \le 2e \le V \cdot \Delta$

$\Rightarrow \boxed{\delta \le \frac{2e}{V} \le \Delta}$

$\Rightarrow$ G is a Graph with 11 edges and min deg = 3 then what is the max no. of vertices ?

Now, $\delta \le \frac{2e}{V}$

$\Rightarrow 3 \le \frac{2 \times 11}{V} \Rightarrow V \le \frac{2 \times 11}{3}$

$\Rightarrow V \le 7.33$

$\Rightarrow \boxed{V = 7}$

$\Rightarrow$ G is a Graph with 12 vertices and max degree = 4, Then max no. of edges in the Graph G is ——?

$= 2e \le V \cdot \Delta$

$= 2e \le 12 * 4$

$= e \le 24 \Rightarrow \boxed{e = 24}$ (max no)

5. GRAPH THEORY 5

Special Graphs:

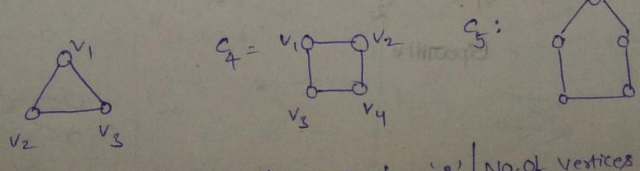
Null Graph: A graph with no edges and n-vertices is called Null Graph.

$\boxed{N_1 = \ 0} \quad \boxed{N_2 = \ 0 \quad 0} \quad \boxed{N_3 = \ 0 \ 0 \ 0} \quad \boxed{N_4 = \ 0 \ 0 \ 0 \ 0}$

Cycle Graph

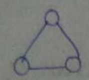
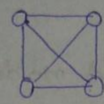
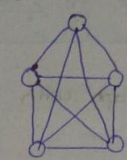
The cycle Graph is a simple Graph with n-vertices $\{v_1 v_2 --- v_n\}$ and edges $\{v_1 v_2\} \{v_2 v_3\}$ ---- $\{v_{n-1} v_n\} \{v_n, v_1\}$

$C_3 =$ $C_4 =$ $C_5:$



$\Rightarrow$ In cycle Graph 'C_n' deg of each vertex will be '2'. | No. of vertices = edges = n |

Complete Graph : complete Graph is a Simple Graph in which every pair of vertices are adjacent, Complete Graph denoted by 'K_n'. (6)

K_3 = 
K_4 = 
K_5 = 

	Vertices	Edges	deg(v)
K_n	n	n_{C_2}	$(n-1)$

$\Rightarrow$ Max no. of edges in simple Graph with 'n' edges $= n_{C_2} = \frac{n(n-1)}{2}$

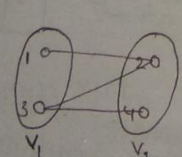
$\Rightarrow V = \{v_1\ v_2\ v_3 \text{----} v_n\}$ of n- vertices then how many simple Graphs are possible?

1	2	3	- - - - - - -	$(n(n-1)/2)$
v_1	v_2	v_3	- - - - - - - -	v_n
$\downarrow$ 2	$\downarrow$ 2	$\downarrow$ 2		$\downarrow$ 2

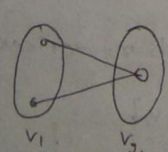
$\therefore$ $\boxed{\text{Total possible no. of simple graphs} = 2^{\left(\frac{n(n-1)}{2}\right)}}$

Bipartite Graphs :

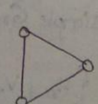
A Graph $G(V,E)$ is Bipartite Graph if the vertex set can be partitioned into two sets V_1 and V_2 such that every edge is in between vertex of V_1 and vertex of V_2.



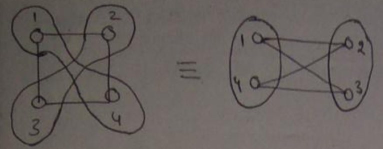
$\Rightarrow$ There should not be any edges between the vertices in a vertex set. (i.e. No edge between (1,3) and (2,4)



Bipartite



{ It is complete and cannot be Bipartite)



$\therefore$ The given graph is Bipartite Graph.

C_n is Bipartite if n = even | A complete Graph K_n can never be
C_n is not Bipartite if n = odd | Bi partite Graph.

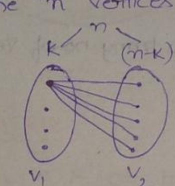
6. MAX NO. OF EDGES IN A COMPLETE BIPARTITE GRAPH

$\Rightarrow$ which of the following is the max no. of edges in Bipartite graph with n-vertices

(a) n^2 (b) $n^2/2$ ~~(c)~~ $n^2/4$ (d) $n/2$

$\Rightarrow$ ~~Bipartite Graph with 6 Vertices.~~

Now, let us assume the 'n' vertices are divide into 2 sets having K and $(n-K)$ vertices



$\Rightarrow$ To get the max. no. of edges then every vertex in V_1 should be matched with all the vertex in V_2

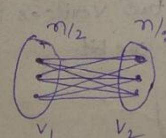
$\therefore$ The No. of edges = $K(n-K)$

$\therefore$ To get the max value of K, $\frac{d}{dK}(K(n-K)) = 0$

$= (n-K) + K(-1) = 0$

$\Rightarrow \boxed{K = n/2}$

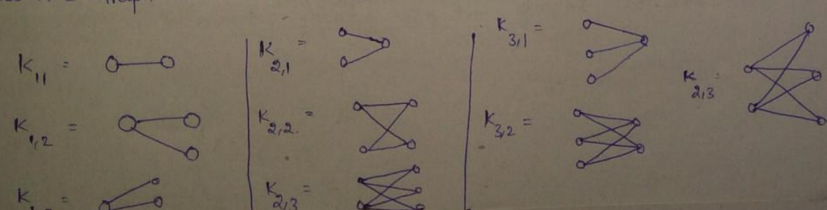
$\therefore$



$\Rightarrow$ No. of edges = $n/2 \times n/2 = n^2/4$.

7. COMPLETE BIPARTITE GRAPH AND REGULAR GRAPHS.

Complete Bipartite Graph: If we try to put every possible edge in the Bi-partite Graph then it is called Complete Bi-partite Graph ($K_{m,n}$)



	v	e	d(v)
K_{mn}	m+n	m*n	$\left\{\begin{matrix} m: v\in V_2 \\ n: v\in V_1 \end{matrix}\right\}$

$|V_1| = m$

$|V_2| = n.$

Regular Graph:

A Graph in which every vertex has same degree is called Regular graph

⇒ Every complete Graph (K_n) is always a Regular Graph.

Null Graph

Ex: $N_n = 0$ (0- Regular graph)

$C_n =$ (2- Regular Graph)

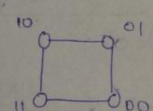
8. N- Cube

Now 1 cube means with 1 bits how many no. of vertices are possible $= 2 = 2^1$

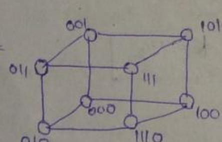
1- cube =

Now, 2cube means with 2-bits how many no. of vertices are possible $= 4 = 2^2$

2-cube =



similarly 3- cube =



An edge is present between two vertices if they differ by one bit.

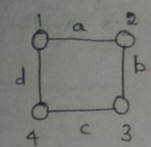
∴ $\boxed{\text{No. of Vertices in N-cube Graph} = (2)^N}$

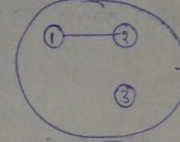
$\boxed{\text{Deg. of Each Vertex} = N}$

$\boxed{\text{No. of Edges} = N * 2^{N-1}}$

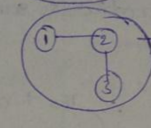
9. SUB-GRAPHS

A Graph $H(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$
$E' \subseteq E$

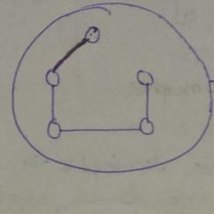




→ Sub Graph of given Graph



→ Induced sub graph (If i take all the possible edges between the choosen vertices that are given in the Actual Graph).



→ Spanning subGraph (If i take all the vertices of a Graph it is called spanning sub Graph. It may contain edges also.

10. ADJACENCY MATRIX

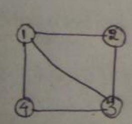
A Graph $G(V, E)$, $|V| = n$ it can be represented as $n \times n$ - matrix

$$A[G] = \begin{cases} 0 & \text{if } \{i,j\} \notin E \\ 1 & \text{if } \{i,j\} \in E \end{cases}$$

⇒ No provision for parllel edges.

⇒ Graph has no self loops iff the matrix diagnol entries are zero's.

→ For Graph having no self loops $\deg(v_i)$ = sum of entries in the i_{th} row.



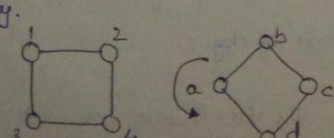
$$A[G] = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

→ deg of (1) = 3 (No. of 1's in that row)
→ d(2) = 2
→ d(3) = 3
→ d(4) = 2

11. ISOMORPHISIM INTRODUCTION

Isomorphisim: Isomorphic graphs are the graphs that are drawn differently.

Ex:



The time complexity for checking whether the Graphs are Isomorphic or not is $O(n!)$

⇒ Isomorphic means two graphs should have same no. of vertices and their adjacencies should be preserved.

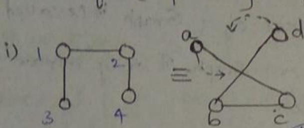
⇒ For the above examples. the adjacency matrix is same

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \equiv \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

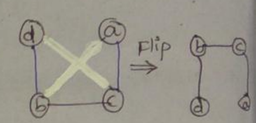
∴ The two graphs are Isomorphic

Which of the following Graphs are Isomorphic

i)

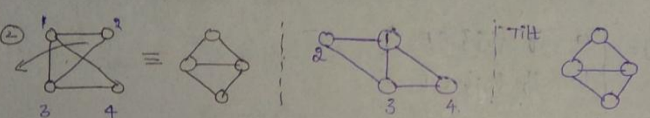


⇛ They are Isomorphic ⇒

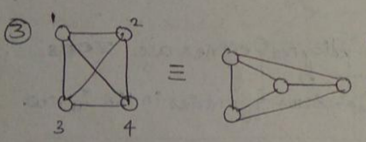


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \equiv \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ & & & \\ & & 0 & 0 \\ & & & \end{bmatrix} \end{array}$$

2)

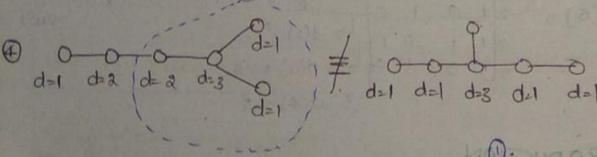


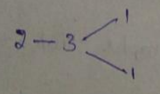
3)



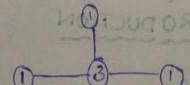
lift the node 4 and put in centre

4)





∴ The node Having deg=3 is adjacent to nodes of deg=2 and deg=1, deg=1

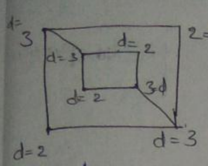


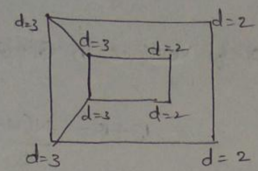
Here the node of deg=3 is not adjacent to the node of deg=2

∴ The two Graphs are not Isomorphic

12. ISOMORPHISIM EXAMPLES

Are these Graphs Isomorphic?

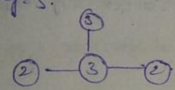




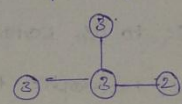
First check
1) No. of vertices and Edges $|V|(G_1) = |V|(G_2)$
2) $|E|(G_1) = |E|(G_2)$
3) check for Adjacency

↓

In this graph every node of deg = 3 is adjacent to nodes of deg = 2, deg = 2, deg = 3.

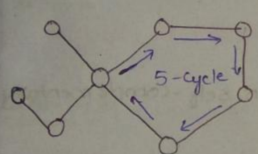


Every node of deg = 3 is adjacent to nodes of deg : 2, deg = 3, deg = 3.

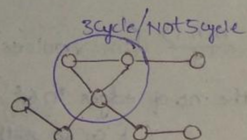


∴ There is a conflict ∴ These Graphs are not Isomorphic

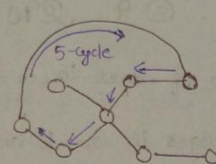
⇒ which of the graphs is Isomorphic to the Graph



(A) ×



(B)



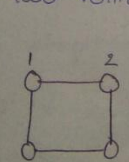
✓ And you can check degree sequence also.

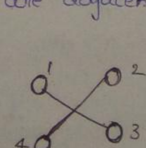
13. SELF COMPLEMENTING GRAPHS

Complement of a Graph

Let $G(V,E)$ be a simple Graph the the compliment of Graph $G = G^C = (V, E^C)$ such that two vertices are adjacent in G^C if they are not adjacent in G.

Ex





No. of edges in (G) + No. of edges in $G' = {}^{n}C_2$ (∵ G, G' forms Complete Graph)

⇒ If the no. of edges in $G_1 = 13$ and No. of edges in $G_1^c = 15$ then what is the no. of vertices in G_1? (12)

⇒ we know that the no. of edges in $G_1 + G_1^c = \frac{N(N-1)}{2}$ [N = No. of vertices]

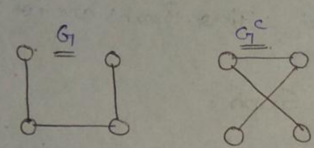
$= 13 + 15 = \frac{N(N-1)}{2}$

$= \boxed{N = 8}$

Self-complementing Graph

A graph which is isomorphic to its complement is called self-complementary graph. which means if G^c is isomorphic to 'G' then it is called Self Complement graph.

Ex:



$G_1^c \equiv G_1$ (Isomorphic Graphs)

⇒ which of the following cannot be the no. of vertices in self-complementary Graph. (A) 4 (B) 5 (C) 9 (D) 10

We know that the no. of edges in $(G_1 + G_1^c) = \frac{n(n-1)}{2}$ (n = vertices)

⇒ Let the no. of edges in $G_1 = K$ and so the no. of edges in $G_1^c = K$

$\therefore 2K = \frac{n(n-1)}{2}$ (if 'G' is self complementary)

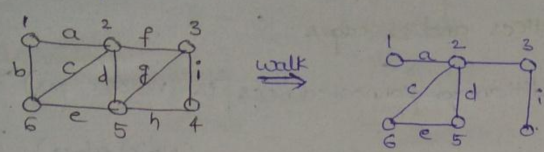
⇒ $K = \frac{n(n-1)}{4}$ | Here K should be integer not fractions.

$\therefore$ n should be multiple of 4 or $(n-1)$ should be ÷ble by 4.

14. CONNECTED COMPONENTS

Edge sequence: Sequence of edges starting and ending with vertex

Walk: Edge sequence in which no edge is Repeated (vertices can be repeated but not the edges).



(vertext '2' is visited twice (vertex may be repeated but not the edge).

Closed walk: A walk in which start and end at same vertex is called closed walk otherwise it is open walk. (vertex may be repeated).

path: A path is an open walk in which no vertex is repeated.

cycle: A cycle is a closed walk in which no other vertex is repeated.

Results:

1. A graph is Bipartite if every cycle in the graph is even cycle.

Connected: Two vertices are said to be connected if there exists atleast one path between them.

⇒ The maximal connected subgraph is called "component".

Results:

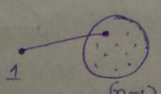
1) A simple Graph with n-vertices and k-components has atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

15. SUFFICIENT CONDITION FOR CONNECTEDNESS

The sufficient condition for connectedness is.

→ G is a simple Graph with n-vertices the (no. of edges) $> \frac{(n-1)(n-2)}{2}$

The 'G' is connected.



$(n-1)$ vertices

$(n-1)C_2$ edges $= \frac{(n-1)(n-2)}{2}$

⇒ which of the following Simple graphs are always connected.

a) 'G' with 5 vertices and 5 edges

b) 'G' with 6 vertices and 9 edges.

c) 'G' with 7 vertices and 13 edges.

d) ✓ 'G' with 8 vertices and 22 edges

⇒ The sufficient condition for connectedness is (no. of edges) $> {}^{(n-1)}C_2$

$$(\text{No of edges}) > \frac{(n-1)(n-2)}{2}$$

i) 5V, 5E ⇒ $5 > \frac{\cancel{4}^{2}(3)}{\cancel{2}}$ $= \boxed{5>6}$ false } Disconnected Graph. (Not connected)

ii) 6V, 9E ⇒ $9 > \frac{(5)(4)}{2}$ $= \boxed{9>10}$ = false } Disconnected

iii) 7V, 13E ⇒ $13 > \frac{(6)(5)}{2}$ $= \boxed{13>15}$ false }

iv) 8V, 22E ⇒ $22 > \frac{7\times6}{2}$ $\boxed{22>21}$ } Connected

16. COMPLEMENTATION AND CONNECTEDNESS

Result: Atleast one of the Graphs 'G' or G^c is connected.

G	G^c	
C	C	possible cases.
D	C	
C	D	
D	D	Not possible

C = Connected
D = Disconnected

⇒ which of the following is always true?

S_1: G is connected, then G^c is disconnected

S_2: G is disconnected, then G^c is connected.

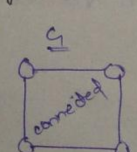
a) only S_1 is True (Not Necessarily true)

b) ✓ only S_2 is True

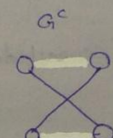
c) Both

d) None

S_1: G

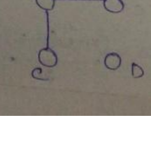


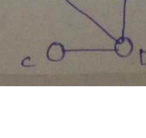
G^c



= disconnected

S_2





= connected

17. CUT VE

⇒ To make

may remo

Removal of a

Removal of

CUT EDGE (

the Graph

⇒ If there

⇒ If there

CUT VERTEX

A single ve

cut vertex

1)

2)

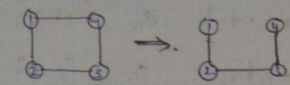
⇒ A Gra

Biconne

17. CUT VERTEX AND CUT EDGE

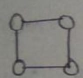
⇒ To make a Graph disconnected we may remove some edges or we may remove some vertices.

Removal of an edge :

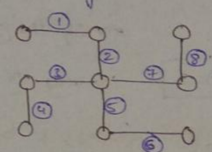


Removal of an vertex:

CUT EDGE (OR) BRIDGE : A single edge where removal of it makes the Graph dis-connected is called cut edge.



⇒ No cut edges are present



No. of cut edges = 8.

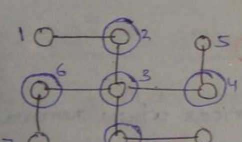
⇒ If there is a tree every edge is a cut edge

⇒ If there is no cycle then every edge is a cut edge

CUT VERTEX AND ARTICULATION POINT

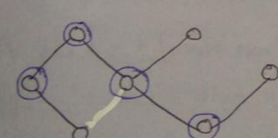
A single vertex whose removal disconnects the connected Graph is called cut vertex or "ARTICULATION POINT."

1)



cut vertices / Articulation points are = $\{2, 3, 4, 6, 8\}$

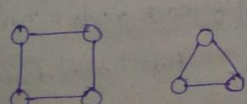
2)



No. of cut vertices are (a) 3 ~~(b)~~ 4 (c) 5 (d) 6

⇒ A Graph having no Articulation point / cut vertex is called a Biconnected Graph.

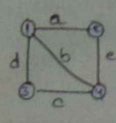
= All cycles are Bi-connected Graphs.



18. CUT SET AND EDGE CONNECTIVITY

Cut set: A set of edges where removal disconnects the Graph is called cutset. There can be more cutset for a given Graph.

Ex:



$c_1 = \{abcde\}$ $c_4 = \{abc\}$

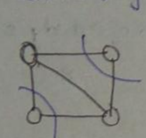
$c_2 = \{abd\}$ $c_5 = \{cd\}$

$c_3 = \{ae\}$

$\rightarrow$ Minimum cut set

Edge- connectivity (λ):

The min. no. of edges where removal disconnects the graph G is called Edge- connectivity (λ).



$\boxed{\lambda = 2}$

$\Rightarrow$ If the Graph contains cycle then edge connectivity λ will always be greater than one $\boxed{\lambda > 1}$.

$\Rightarrow$ The edge connectivity is always upper bounded by min degree $\therefore$ $\boxed{\lambda \le \delta}$

W.K.T $\left.\begin{array}{l} V\cdot\delta \le 2E \\ \delta \le \frac{2E}{V} \end{array}\right\} \Rightarrow \boxed{\lambda \le \frac{2E}{V}}$

19. VERTEX CONNECTIVITY AND EDGE CONNECTIVITY

Vertex Connectivity (K = Kappa) : The min no. of vertices where removal disconnects the graph (or) leaves trivial graph is called Vertex Connectivity.

0 $\rightarrow$ Trivial Graph (K_1).

$$\boxed{K \le \lambda \le \delta \le \frac{2E}{V}}$$

G	λ	K
C_n	2	2
W_n	3	3
K_n	$(n-1)$	$(n-1)$
$K_{m,n}$	min (m,n)	min (m,n)

wheel Graph W_n contains $(n+1)$ vertices

W_n has $2(n-1)$ edges

deg of each node = n_3 (which are in cycle)

Hub vertex degree = $n-1$

20. WHITNEY'S THEOREM

In any Graph $\boxed{K \le \lambda \le \delta}$

$\boxed{\text{Vertex Connectivity} \le \text{Edge connectivity} \le \text{Mindegree}}$

⇒ A Graph with 11 edges and min degree is '4'. what is the max value of vertex connectivity?

Acc. to whitney theorem $K \le \lambda \le \delta \le \frac{2E}{V}$

⇒ $K \le \frac{2E}{V}$ | $\delta \le \frac{2E}{V}$

⇒ $K \le \frac{2 \times 11}{V}$ | $\delta \le$

⇒ $K \le \delta$

⇒ $\boxed{K \le 4}$

∴ max value $\boxed{K = 4}$

⇒ A Graph 'G' with 11 edges and 7 vertices is given what is the max value of K (vertex connectivity)?

Sol $K \le \frac{2E}{V}$

$K \le \frac{2 \times 11}{7}$

$K \le 3.1$ ⇒ max value = 3 ∴ $\boxed{K = 3}$

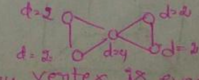
21. EULER'S GRAPH

⇒ If you can start from a vertex and if you can travel all edges exactly once and again comeback to the same vertex then such a walk is called Euler circuit and such a Graph is called Euler Graph.

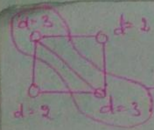
⇒ If you start from a node and come back to the same node by covering all the vertices then it is called Eulerian circuit / Euler path

⇒ If you start from a node and cover all the nodes and reach some other node which is final then it is called unicursal path and the Graph is called unicursal Graph

⇒ If a graph has to be Euler Graph it should contain Evendegree. (The vertices should have Even degree.)

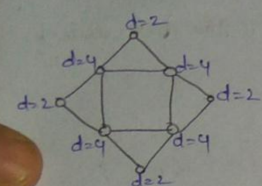


Result: A multigraph is Euler Graph iff degree of every vertex is even.



⇒ so Not Euler graph. so uni cursal Graph.

⇒ Is this Graph a Euler Graph



All degrees are Even ∴ The Graph is Euler Graph.

Result: A Multigraph is unicursal graph if there are exactly 2 vertices of odd degree.

22. HAMILTONIAN GRAPH (H- Graph)

→ In Euler Graph every edge should be covered and vertex can be repeated

→ In "Hamiltonian Graph" all the vertices should be covered and nothing should be Repeated (Need not cover all the edges)

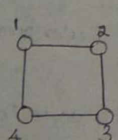
E-E (Euler Graph - should cover all the Edges).

⇒ You have to draw a cycle in the Graph such that it covers all the Vertices

⇒ A Graph containing Hamiltonian Cycle is called Hamiltonian-Graph.

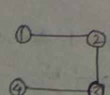
⇒ A path containing all the vertices of the Graph is called "Hamitonian Path".

Eg:

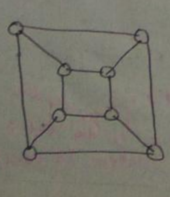


1-2-3-4-1 = Hamiltonian cycle

= Hamiltonian Graph (H-Graph)



= 1-2-3-4 = Hamiltonian path (Remove one edge from Hamiltonian-cycle then you get H-path).



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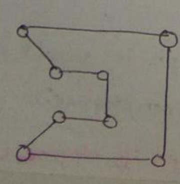


= Hamiltonian Cycle = Hamiltonian path

Results:

1) If 'G' is a Hamiltonian Graph with n-vertices

   1) Hamiltonian cycle, H-path contains n-vertices.
   2) Hamiltonian cycle contains n-edges.
   3) H-path contains $(n-1)$ edges.
   4) In a Hamiltonian cycle each vertex should have $(deg = 2)$.

The sufficient condition for Hamiltonian cycle is (Not Necessary)

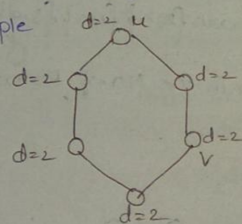
a) Diracs theorem: (simple Graph)

If min degree $\boxed{(\delta) \geq n/2}$ then G is Hamiltonian Graph

b) Ores Theorem: If $deg(u) + deg(v) \geq n$ (for $n \geq 3$) then for every pair of non adjacent vertices, then 'G' is H-Graph.

⇒ If a Graph satisfies Dirac, Ores theorem G is Hamiltonian but if a Graph doesnt satisfy above Theorems we cannot say 'G' is Non-H Graph.

For example



Here $\delta = 2$ $n = 6$

$\boxed{\delta \geq n/2}$ ⇒ $2 \geq 6/2$ ⇒ $(2 \geq 3)$ false

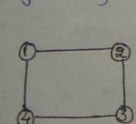
but the Graph is Hamiltonian.

$d(u) + d(v) = 4$ $(4 \geq n)$ false but 'G' is H-graph.

23. POWERS OF ADJACENCY MATRIX

Let $A[G]$ be the Adjacency matrix. then $[A(G)]^n$ Represents powers of Adjacency matrix.

Ex:



$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow A^2 = \begin{array}{c c} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \end{array}$$

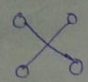
→ Represents the no. of paths of length 1 from 1 to 2

→ Represents the no of paths of length 2 from 1 to 3. $\left\{ \begin{array}{l} 1-2-3 \\ 1-4-3 \end{array} \right.$

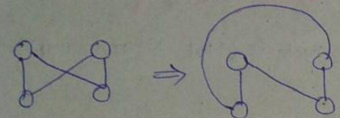
⇒ A^n Represents the no. of paths btn two vertices of length 'n'.

24. PLANAR GRAPHS

⇒ If we can draw a Graph on a paper without Crossovers then the Graph is planar. This concept is used in VLSI design.

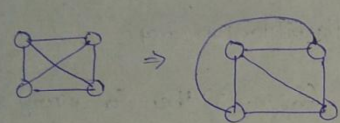
⇒ Cross over: 

Ex:



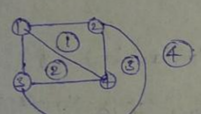
∴ The given graph is planar.

K_4 =



⇒ ∴ K_4 is planar.

⇒



Internal Region = {1,2,3}

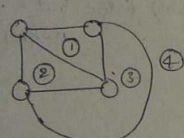
External Region = {4}.

The edge that divides two Regions is called Boundary edge.

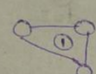
Here the Boundary edges are {2,4} {1,4} {3,4}

Degree of the Region: Degree of the Region is the No. of Boundary edges touching it.

Ex



Region	deg(R)
1	3
2	3
3	3
4	3.



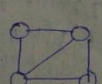
= 3 edges.

⇒ sum of degrees of the Regions = Twice the no. of Boundary edges.

25. EULER FORMULA FOR SIMPLE GRAPH

Let 'G' be a connected planar graph with v - vertices, e = edges, r = regions then

$$\boxed{v - e + r = 2}$$



v = 4
e = 5
r = 3

$v - e + r = 4 - 5 + 3$
$= 7 - 5 = 2$ //.

$\Rightarrow$ Min degree of a Region in simple Graph = 3.

$\Rightarrow$ Now, $\sum deg(r) = 2e$

$= r_1 + r_2 + r_3 + \dots + r_n = 2e$

$= 3 + 4 + 5 + \dots + x = 2e$

$= 3 + 3 + 3 + \dots + 3 \le 2e$

$= 3|r| \le 2e \Rightarrow \boxed{3r \le 2e}$

$\hookrightarrow \boxed{r \le 2e/3}$ (or) $\boxed{e \ge \frac{3r}{2}}$

Now, Euler formula

$\Rightarrow v - e + r = 2$

$\Rightarrow v - e + \frac{2e}{3} \ge 2$

$\Rightarrow \boxed{e \le 3v - 6}$

$3r \le 2e$

$\Rightarrow \boxed{r \le 2e/3}$

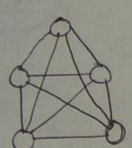
Now, $v - e + r = 2$

$\Rightarrow v - \frac{3r}{2} + r \le 2$

$\Rightarrow \boxed{2v - r \le 4}$

26. K_5 is NOT PLANAR

$K_5 =$



K_5 = Kuratowski's Graph = $K_{3,3}$ is also called Kuratowski's Graph.

$v = 5$

$e = 10$

By Euler formula $v - e + r = 5 - 10 + r = 2$

$\Rightarrow \boxed{r = 7}$

Now, $3r \le 2e$

$\Rightarrow 21 \le 2 \times 10$

$= \boxed{21 \le 20}$ false $\therefore$ K_5 is Not planar / It must not be connected

$\Rightarrow$ Since K_5 is connected, K_5 is Not-planar Graph.

27. $K_{3,3}$ IS NOT PLANAR PART 1 AND PART 2

W.K.T. $v - e + r = 2$, Now let us assume that the min. deg = k then.

W.K.T. sum of degree = $2e$ and $v - e + r = c + 1$ $\hookrightarrow$ No. of connected components.

$= k + k + k + \dots + k = 2e$

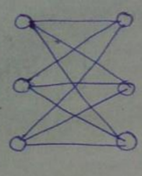
$= k|r| \le 2e \Rightarrow \boxed{kr \le 2e}$

$\Rightarrow \boxed{r \le 2e/k}$

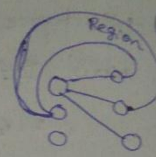
$\Rightarrow v - e + \frac{2e}{k} \ge 2 \Rightarrow \boxed{e \le \frac{k(v-2)}{k-2}}$ (or) $\boxed{v - \frac{kr}{2} + r \le 2}$

Now, let us check if K_{33} is planar or not

$K_{3,3}$:



Min. deg of any region = 4



Now, $v-e+r=2$

$\Rightarrow 6-9+r=2$

$\boxed{r=5}$ Now $e \leq \frac{4(4)}{2}$

$\boxed{9 \leq 8}$ - false

Note: There wont be any Region of deg 3 in Bipartite Graph.

$\therefore K_{3,3}$ is not planar

$\therefore$ The Graphs that contain $K_{3,3}$ and K_5 as subgraphs are not planar.

1) The Graph which is Non-planar with min. no. of edges is 'K_5' $\therefore$ The Graphs with 2,3,4 vertices are planar. (simple Graphs)

2) $K_{3,3}$ is Non-planar with min. no. of edges $\therefore$ Any simple Graph with edges <8. is planar.

So,

K_n is planar iff $n \leq 4$

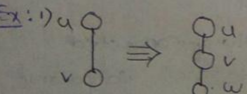
$K_{m,n}$ is planar iff $m \leq 2$ and $n \leq 2$

29. CHECKING FOR PLANARITY

Homomorphic operations

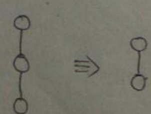
$\Rightarrow$ when ever you take a Graph and if we apply Homomorphic operations and we get another Graph then we can say that the two graphs are Homomorphic to each other.

Ex: 1)

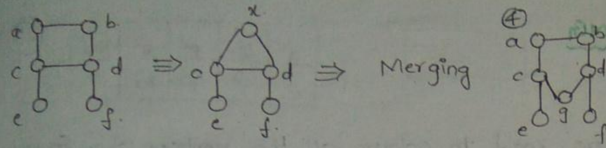


$\Rightarrow$ Insertion of vertex of $d=2$ in series.

2)

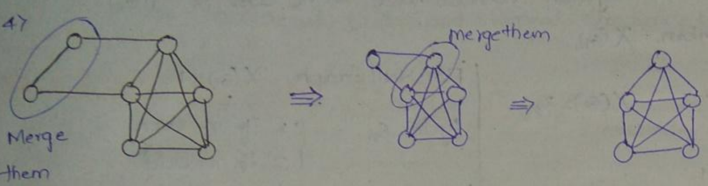


$\Rightarrow$ Removing a vertex

3) 

⇒ Merging

Insertion and pulling.

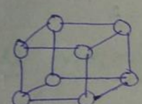
4) 

∴ The given Graph is Not planar.

<u>Kuratowskis</u> <u>Theorem</u>

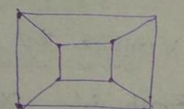
A connected graph is planar iff it is not homomorphic to K_5 or $K_{3,3}$.

<u>Ex</u>:



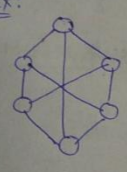
whether the Graph is Homomorphic or not?

⇒

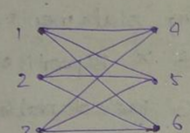


= planar (No cross overs)

<u>Ex</u>:



⇒



$= K_{3,3}$ ∴ NOT planar Graphs.

<u>Ex</u>:

1) $V = 20$, $deg(v_i) = 3$ for all $v_i \in V$ then sum of degrees $= 2e = 60$ then what are the no. of Regions.

<u>Sol</u> $V = 20$ sum of degree ⇒ $2e = 60$

$d(v_i) = 3$.   ⇒ $\boxed{e = 30}$

at min case each region will have min deg of '3'

∴ $3 + 3 + 3 + \cdots 3$ (20 times) ← each vertex degree

$= 60$

$= \boxed{2e}$

Now, $V - e + r = 2$

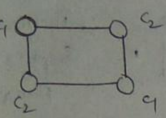
$= 20 - 30 + r = 2$

$= \boxed{r = 12}$

30. GRAPH COLOURING

Graph colouring:

⇒ The min no. of colours reqd to colour all the vertices of a Graph such that no two adjacent vertices have same color is called chromatic number. $\chi(G)$.



$\chi(G) = 2$

For Null Graph $\chi(G) = 1$

Cycle $C_n = \begin{cases} 2 & \text{if } n = \text{even} \\ 3 & \text{if } n = \text{odd} \end{cases}$

Wheel $W_n = \begin{cases} 3 & \text{if } n = \text{even} \\ 4 & \text{if } n = \text{odd} \end{cases}$

⇒ $K_n = \{n\}$

⇒ $K_{m,n} = 2$

Any Bipartite Graph, $\chi(G) = 2$

Properities

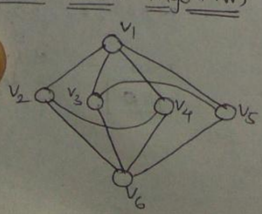
* 'G' is a graph with n-vertices then $\chi(G) \le n$
* 'K_n' is a sub-graph of Graph 'G' $\chi(G) \ge n$
* $\chi(G) \le 1 + \Delta$ → max degree
* $\chi(G) = \dfrac{|V|}{|V| - \delta}$

The following statements are Equivalent
1> 'G' is Bipartite
2> 'G' is 2-colourable
3> Every cycle in 'G' is Even cycle.

31. GRAPH COLOURING EXAMPLE-1

There is no Algo which can find the chromatic no. of a graph. (NP-Complete)
But we use some Greedy methods (Algo) like

Welsh-Powell-Algorithm



$d(v_1) = 4$
$d(v_6) = 4$

$d(v_2) = d(v_3) = d(v_4) = d(v_5) = 3.$

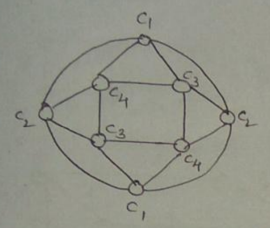
Now, sort the vertices in descending order of the degrees. = $v_1\, v_6\, v_2\, v_3\, v_4\, v_5$

V_1	V_6	V_2	V_4	V_3	V_5
c_1	c_1	c_2	c_2	c_3	c_3

$\therefore \boxed{\chi(G) = 3.}$

$\Rightarrow$ start from 'v_1'. Now apply c_1 color to v_1 and the vertices that are not adjacent to it, and Repeat the procedure.

32. GRAPH - COLOURING EXAMPLE - 2

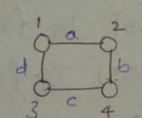


$\boxed{\chi(G) = 4}$

$\Rightarrow$ The Graph contains 'K_3' so the value of $\boxed{\chi(G) \geq 3.}$

33. MATCHING AND EDGE COVER

Matching: Set of Non-Adjacent edges.



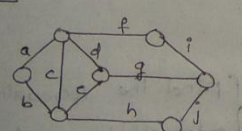
$M_1 = \{a\}$ $M_3 = \{b, d\}$

$M_2 = \{a, c\}$

Matching no: $[\alpha'(G)]$: Max no. of Non-Adjacent edges. For above Graph

$\boxed{\alpha'(G) = 2.}$

Ex:



$M_1 = \{a, i, h\}$

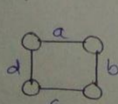
$M_2 = \{f, g, h\}$

$M_3 = \{i, e, a\}$

$\boxed{\alpha'(G) = 3}$

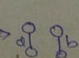
Edge cover:

What are all the edges that you should choose that covers all the vertices (Not isolated vertex)



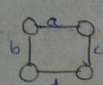
$\Rightarrow$ Edge cover $= \{abc\}$ $\rightarrow \{abcd\}$

$= \{bcd\}$ $\rightarrow \{ac\}$

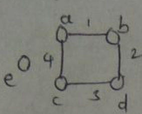
$= \{adc\}$ $\rightarrow \{db\} \Rightarrow$ 

Edge covering NO: Min no. of edges that are Required to cover all the vertices is called Edge covering NO. $[\beta'(G)]$

Ex



$\Rightarrow \beta'(G) = \{2\} - \{ad\}$, $\{bc\}$



$\Rightarrow \beta'(G) = \{2\} + 1$ (for isolated vertex)

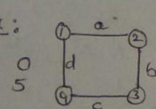
$\Rightarrow \boxed{\beta'(G) = 3}$

For any Graph $\boxed{\alpha'(G) + \beta'(G) = n}$

34. VERTEX COVER AND INDEPENDENT SET

Independent set: set of Non-Adjacent vertices is called Independent set

Ex:



Independent set $I_1 = \{1, 3\}$ $I_3 = \{1, 5\}$ $I_5 = \{3, 5\}$

$I_2 = \{2, 4\}$ $I_4 = \{2, 5\}$ $I_6 = \{4, 5\}$

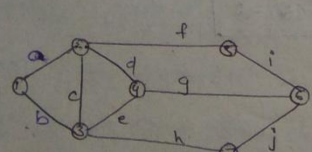
$I_7 = \{1, 3, 5\}$ $I_8 = \{2, 4, 5\}$

Independence no: $\alpha(G)$

Max No. of elements in the largest Independent set is called Max NO.

For the above graph $\alpha(G) = 3$ $[I_8, I_7]$

Ex:



$I_1 = \{1, 4, 5, 7\}$

$I_2 = \{2, 6\}$

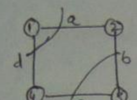
{Find the decreasing order of degree of vertices}

or

{Increasing order of vertices}

VERTEX COVER : set of vertices that cover all the edges is called vertex cover.

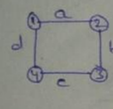
Ex:



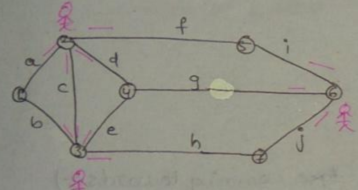
$V_1 = \{1,2,3,4\}$ $V_3 = \{2,4\}$

$V_2 = \{1,3\}$

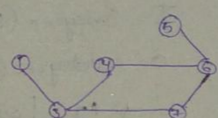
Vertex covering number : The min no. of vertices which can cover all the edges



$\beta(G) = 2$

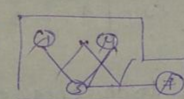


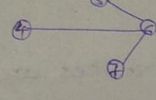
⇒ To get the vertex covering number remove the vertex with highest degree $= \{2,3\}$ so we can remove any one.



⇒ Now remove the vertex with next highest degree $= 3$ (In ~~main Graph not the Graph~~ the Remaining Graph)

⇒





⇒ Now Remove $= 6$ then all vertices get Removed.

∴ we have removed three vertices ∴ $\beta(G) = 3$

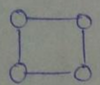
For any Graph "G" $\boxed{\alpha(G) + \beta(G) = n}$

Independence no + Vertex covering no = n.
Matching no + Edge covering no = n → $\alpha'(G) + \beta'(G) = n$

35. PERFECT MATCHING

A matching which can cover all the vertices of a Graph

Ex:



$M_1 = \{a, c\}$ can cover all vertices

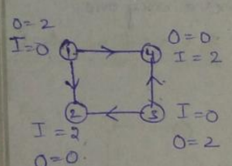
⇒ A graph 'G' has perfect matching only if 'G' has even no. of vertices.

*) No. of perfect matchings in $K_{2n} = \dfrac{(2n)!}{2^n (n!)}$

⇒ Find the no of perfect matchings in K_3 ⇒ $n = 3$ ⇒ $\dfrac{6!}{2^3\, 3!} = \underline{\underline{15}}$

⇒ No. of perfect matching in $K_{n,n} = n!$

36. DIRECTED GRAPH



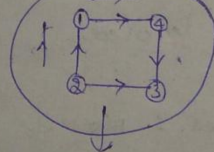
I = Indegree (No. of edges comming towards it)

O = outdegree (No. of edges outgoing from it)

V	In	out
1	0	2
2	2	0
3	0	2
4	2	0
sum	4	4

In any Graph sum of Indegrees = sum of outdegrees = No. of Edges

⇒ Two vertices are said to be connected if there exists a "directed path" between them, In the below Graph ② and ④ are connected.



→ This Graph is unilaterally connected because between every pair there exists a directed path.

possible pairs $= {}^4C_2 = 6$

(1,2)
(1,3)
(1,4)
(2,4)
(2,3)
(3,4)

= (1,2) (2,1) (3,1) (4,1)
(1,3) (2,3) (3,2) (4,2)
(1,4) (2,4) (4,2) (4,3)

(2,3)

~~(1,4)~~
(1,2)

$\boxed{\begin{matrix}(2,1) & (1,4) \\ (2,4) & (1,3) \\ (2,3) & (4,3)\end{matrix}}$