Introduction of curves

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Analytic curves

- Mathematically the curves can be described using algebraic equations or in terms of a parameter (parametric).
- Analytic equations are used: lines, circles, ellipses, parabolas, hyperbolas or other general conics
- provide the compact form and more convenient for computations of secondary properties such surface area, volume
- But, not attractive for interactive computation

Synthetic curves

- Described by a set of data points (called as **control points**) and parametric polynomials that either interpolate or approximate those points
- They provide greater flexibility and control of a curve by changing the positions of control points
- Global as well as local control can be obtained

Curve Representation Methods

- Non-parametric representation:
  - Explicit representation of a general 3-D curve
    - $P = [\,x \;\; y \;\; z\,]^T = [\,x \;\; f(x) \;\; g(x)\,]^T$
  - Implicit representation of a curve as intersection of two surfaces defined
    - $F(x, y, z) = 0 \qquad\qquad G(x, y, z) = 0$
- Parametric representation:
  - Each point on a curve is expressed as a function of a parameter u.
  - The parameter acts as a local coordinate for points on the curve.

$$P(u) = [x \;\; y \;\; z]^T = [x(u) \;\; y(u) \;\; z(u)]^T \quad \text{where } 0 \le u \le 1$$

Curve Representation Methods

Non-parametric Representations

1. Precision
2. Compact Storage – we store only the equations
3. Ease of Interpolation
4. Any point on the curve can be precisely determined

Parametric Representations

1. Slope of the curve represented by tangent vectors
2. Infinite slope results when one of the components of tangent vector is zero
3. Parametric representation is independent of axis
4. The curve end points and length are fixed by the range

Limitations of Non-parametric Curves

1. If the slope of a curve at a point is vertical or near vertical, its value becomes infinity or very large, a difficult condition to deal with both computationally and programming-wise. Other ill-defined mathematical conditions may result.
2. Shapes of most engineering objects are intrinsically independent of any coordinate system. What determines the shape of an object is the relationship between its data points and not the relationship between these points and some arbitrary coordinate system.
3. If the curve is to be displayed as a series of points or straight-line segments, the computations involved could be extensive.

Parametrization

- In parametric form each coordinate of a point is represented as a function of a single parameter

$$P(u) = [x \;\; y \;\; z]^T = [x(u) \;\; y(u) \;\; z(u)]^T \quad \text{where } 0 \le u \le 1$$

- The derivative or tangent vector on the curve is given by

$$P'(u) = [x'(u) \;\; y'(u) \;\; z'(u)]^T$$

- The slope of the curve

$$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{y'(u)}{x'(u)}$$

Analytic curves: Parametric form

- Line: parametric equation of line in 3D is

$$P(u) = P_1 + u(P_2 - P_1) \text{ where } 0 \le u \le 1$$

- This is equivalent to $x(u) = x_1 + u(x_2 - x_1)$

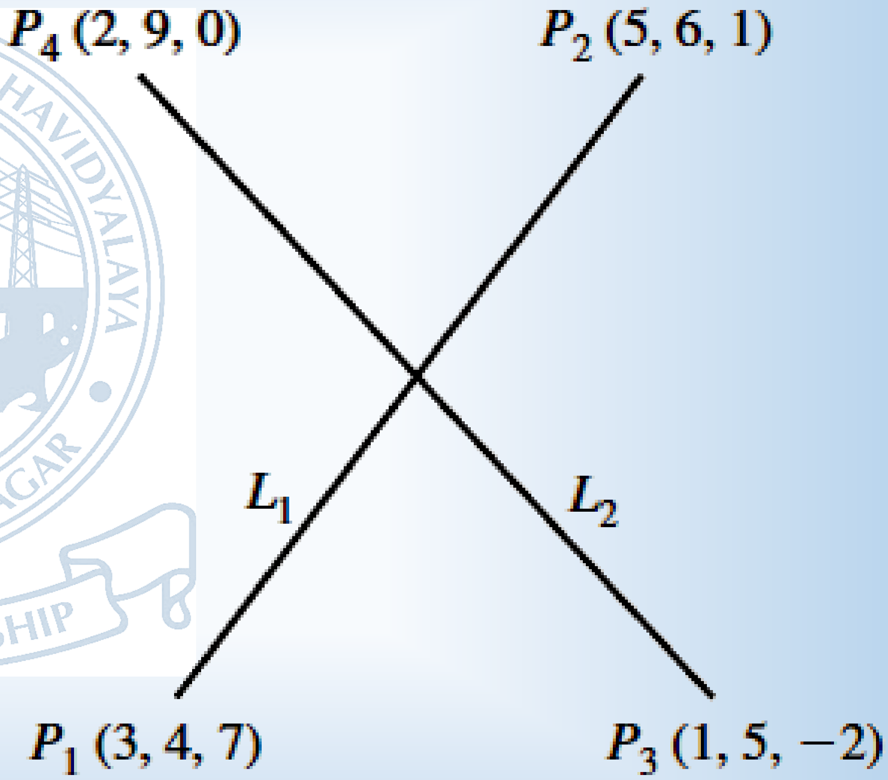
$$y(u) = y_1 + u(y_2 - y_1)$$

$$z(u) = z_1 + u(z_2 - z_1)$$

- Exercise: Determine the line segment between the position vectors $(1, 2)$ $(4, 3)$. Also determine the slope and tangent vector.

Exercise

- Given the two lines L_1 and L_2 and their end points shown on the right;
    - Find the equations of the lines. Show the parameterization directions.
    - Are the two lines parallel or perpendicular?
    - Find the coordinates of the intersection point.



Solution

For L_1,

$$P = P_1 + u(P_2 - P_1) = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} + u \begin{bmatrix} 2 \\ 2 \\ -6 \end{bmatrix}$$

For L_2,

$$P = P_3 + v(P_4 - P_3) = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

The parametrization directions for L_1 and L_2 go from P_1 to P_2 and P_3 to P_4, respectively.

If lines L_1 and L_2 are perpendicular, then

$$P'_{L_1} \cdot P'_{L_2} = 0$$

However, $P'_{L_1} \cdot P'_{L_2} = -2$, hence they are not perpendicular.

Solution

If L_1 and L_2 are parallel, then

$$\hat{n}_1 = \hat{n}_2$$

where, $\hat{n}_1$ and $\hat{n}_2$ are unit vector in the direction of line L_1 and L_2 respectively.

$$\hat{n}_1 = \frac{P_2 - P_1}{|P_2 - P_1|} = \frac{P_2 - P_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$$\therefore \hat{n}_1 = [0.3 \quad 0.3 \quad -0.6]^T$$

$$\hat{n}_2 = \frac{P_4 - P_3}{|P_4 - P_3|} = \frac{P_4 - P_3}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

$$\therefore \hat{n}_2 = [0.2 \quad 0.87 \quad 0.43]^T$$

hence they are not parallel.

Solution

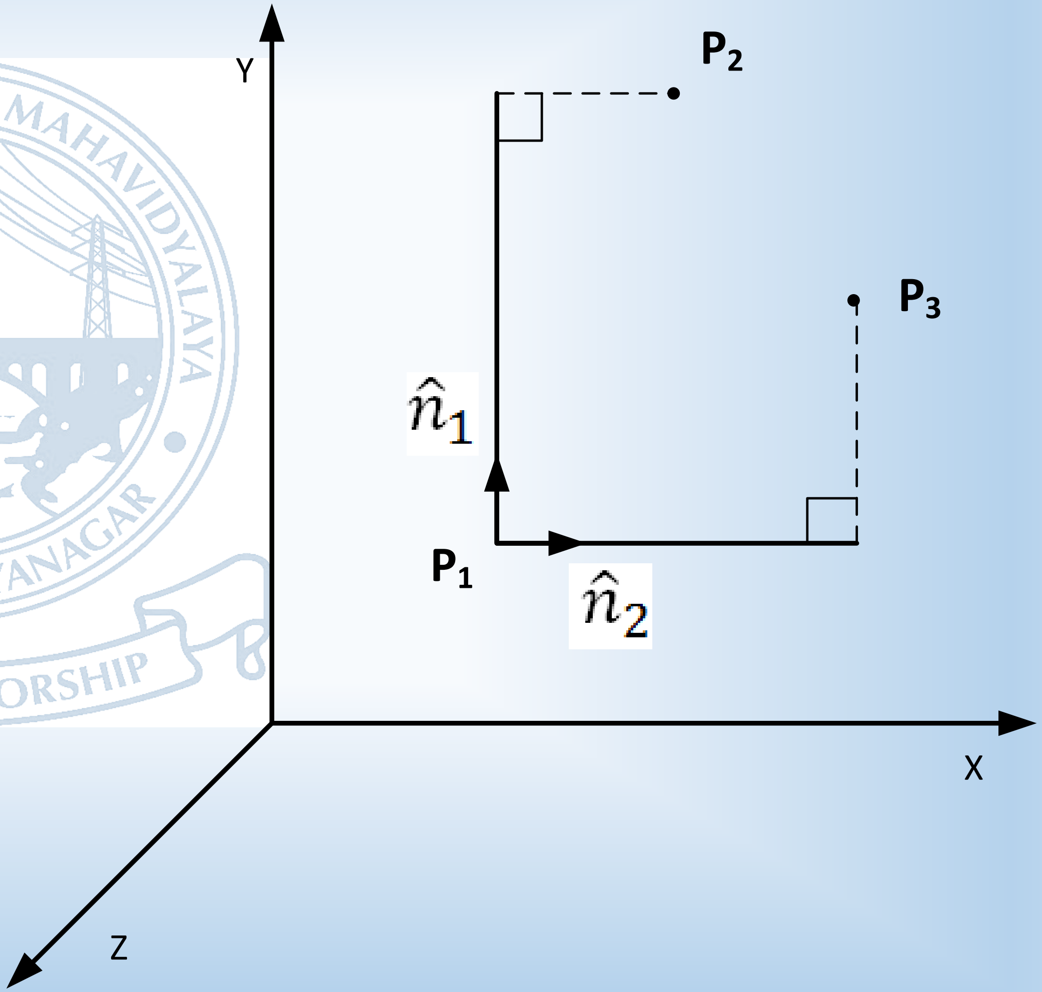
To find the intersection point of the two lines, we equate the two line equations. This produces three equations (x, y, z) in two unknowns u and v. let us use the x an y component to solve for u and v.

$$3 + 2u = 1 + v$$
$$4 + 2u = 5 + 4v$$
$$\therefore u = -1.5\ and\ v = -1$$

Substituting u in L_1 equation or v in L_2 equation gives intersection point as $(0, 1, 16)$.

Exercise

- Find the equations and end points of two lines, one horizontal and the other vertical. Each line begins at and passes through a given point and is clipped by another given point.



Solution

Assume that P_1, P_2 and P_3 are given. The vertical line passes through P_1 and ends at P_2, while the horizontal line passes through P_1 and ends at P_3. In general, two lines cannot pass through P_2 and P_3. Therefore, the ends are determined by projecting the points to the lines as shown in figure.

We have, line equation

For, vertical line $P = P_1 + u(P_2 - P_1) = P_1 + L\hat{n}_1 \qquad 0 \le L \le l_1$

where, $P_1 = [x_1 \quad y_1 \quad z_1], P_2 = [x_2 \quad y_2 \quad z_2],\ l_1 = y_2 - y_1$

$$\hat{n}_1 = \frac{(P_2 - P_1)}{l_1}, unit\ vector\ in\ direction\ of\ l_1$$

Solution

for each point from P_1 to P_2 on line L_1

$$x_2 = x_1, z_2 = z_1$$

And end point $y_2 = y_1 + L$

So, endpoints are $[x_1 \quad y_1 \quad z_1]$ and $[x_1 \quad y_1 + l_1 \quad z_1]$

For Horizontal line, $P = P_1 + v(P_3 - P_1) = P_1 + L\hat{n}_2 \qquad 0 \le L \le l_2$

where, $P_1 = [x_1 \quad y_1 \quad z_1], P_3 = [x_3 \quad y_3 \quad z_3], l_2 = x_2 - x_1$

$$\hat{n}_2 = \frac{(P_3 - P_1)}{l_2}, unit\ vector\ in\ direction\ of\ l_1$$

Solution

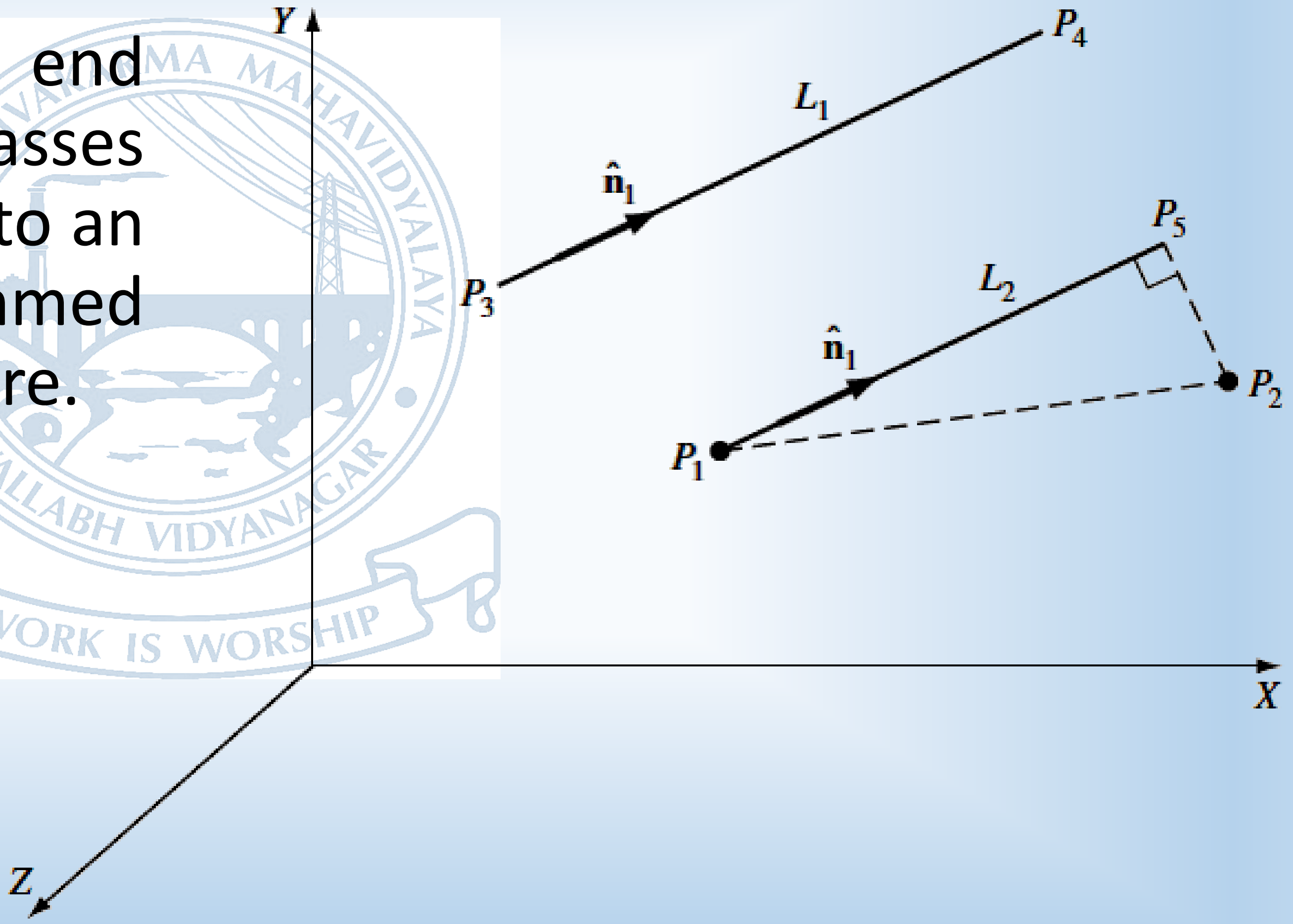
for each point from P_1 to P_3 on line L_2

$$x_3 = x_1 + L$$
$$y_3 = y_1$$
$$z_3 = z_1$$

So, endpoints are $[x_1 \quad y_1 \quad z_1]$ and $[x_1 + l_2 \quad y_1 \quad z_1]$

Exercise

- Find the equation and end points of a line that passes through a point P_1, parallel to an existing line L_1, and is trimmed by point P_2 as shown in Figure.



Solution

Assume that the existing line has the two end points P_3 and P_4, a length L_1, and a direction defined by the unit vector $\hat{n}_1$. The new line has the same direction $\hat{n}_1$, a length l_2, points P_1 and P_5. P_5 is projection of P_2 on the line.

The equation of new line is found by substituting the proper vector into equation

$$P = P_1 + L\hat{n}_1$$

The unit vector is given by

$$\hat{n}_1 = \frac{P_4 - P_3}{l_1}$$

Solution

l_2 can be found as

$$l_2 = \hat{n}_1 \cdot (P_2 - P_1) = \frac{(P_4 - P_3)}{l_1} \cdot (P_2 - P_1)$$

$$= \frac{(x_4 - x_3)(x_2 - x_1) + (y_4 - y_3)(y_2 - y_1) + (z_4 - z_3)(z_2 - z_1)}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

The equation of the new line becomes

$$P = P_1 + L \cdot \hat{n}_1 \qquad 0 \leq L \leq l_2$$

and two end points are $P_1(x_1, y_1, z_1)$ and $P_5\,(x_1 + l_2 n_{1x}, y_1 + l_2 n_{1y}, z_1 + l_2 n_{1z})$.

Exercise

Relate the following CAD commands to their mathematical formulations

a) The command that measures the angle between two lines

b) The command that measures the distance between a point and a line.

Solution

(a) If the end point of the two lines are P_1, P_2 and P_3, P_4, the angle measurement command uses the equation.

$$\cos\theta = \frac{(P_2 - P_1)\cdot(P_4 - P_3)}{|P_2 - P_1|\cdot|P_4 - P_3|}$$

$$= \frac{(x_2 - x_1)\cdot(x_4 - x_3) + (y_2 - y_1)\cdot(y_4 - y_3) + (z_2 - z_1)\cdot(z_4 - z_3)}{\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]\cdot[(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2]}}$$

(b) Let D is distance from point P3 to the line whose end points are P1 and P2, direction is $\hat{n}$ and length L. Then, D is given by

$$D = |(P_3 - P_1) \times \hat{n}| = \left|(P_3 - P_1) \times \frac{(P_2 - P_1)}{L}\right|$$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ \frac{x_2 - x_1}{L} & \frac{y_2 - y_1}{L} & \frac{z_2 - z_1}{L} \end{matrix} \right\|$$

Exercise

Find the unit tangent vector in the direction of a line.

(a) Parallel to and existing line.

(b) Perpendicular to an existing line.

Solution.

(a) For L_1 and L_2 to be parallel

$$\hat{n}_1 = \hat{n}_2$$

$$or\ [n_{2x} \quad n_{2y} \quad n_{2z}] = [n_{1x} \quad n_{1y} \quad n_{1z}]$$

This equation defines an infinite number lines in an infinite number of planes in space. Additional geometric conditions are required to define a specific line.

(b) For L_1 and L_2 to be perpendicular

$$\hat{n}_1 \cdot \hat{n}_2 = 0$$

$$or\ n_{1x} \cdot n_{2x} + n_{1y} \cdot n_{2y} + n_{1z} \cdot n_{2z} = 0 \qquad (1)$$

Additional equations are needed to solve for $\hat{n}_2$. They are

$$|\hat{n}_1 \times \hat{n}_2| = 1$$

$$or\ \left(n_{1y} \cdot n_{2z} - n_{1z} \cdot n_{2y}\right)^2 + \left(n_{1z} \cdot n_{2x} - n_{1x} \cdot n_{2z}\right)^2 + \left(n_{1x} \cdot n_{2y} - n_{1y} \cdot n_{2x}\right)^2 = 1 \qquad (2)$$

and

$$|\hat{n}_2 \times \hat{n}_2| = |\hat{n}_2|^2$$

$$or\ n_{2x}^{\ 2} + n_{2y}^{\ 2} + n_{2z}^{\ 2} = 1 \qquad (3)$$

However, only two equations are needed

Parametric equation of Circle

- Circle: parametric equation of a circle (in the XY plane)

$$x_i = R * \cos(u)$$

$$y_i = R * \sin(u) \qquad \text{. . . OR . . .} \qquad x_i = \frac{1-u^2}{1+u^2} \quad y_i = \frac{2u}{1+u^2}$$

$$z = z_c \qquad 0 \leq \mathrm{u} \leq 2\pi \qquad \text{where } 0 \leq \mathrm{u} \leq 1$$

- Radius (R) and center of circle are enough….
- Use of parametric equations in Computer Graphics

$$x_{i+1} = R * \cos(u + \Delta u)$$

$$y_{i+1} = R * \sin(u + \Delta u)$$

$$z = z_c \qquad 0 \leq \mathrm{u} \leq 2\pi$$

Parametric equation of Circle (contd..)

$$x_{i+1} = R\cos(u + \Delta u)$$

$$x_{i+1} = R\cos u \cos\Delta u - R\sin u \sin\Delta u$$

$$x_{i+1} = x_i \cos\Delta u - y_i \sin\Delta u$$

Similarly

$$y_{i+1} = R * \sin(u + \Delta u)$$

$$y_{i+1} = R\sin u \cos\Delta u + R\cos u \sin\Delta u$$

$$y_{i+1} = y_i \cos\Delta u + x_i \sin\Delta u \qquad \text{where } \Delta u = \frac{2\pi}{n}$$

A non-origin centered circle can be generated by translating the origin centered circle

Parametric equation of Ellipse

$$x_i = a*\cos(u) \quad \text{and} \quad y_i = b*\sin(u) \qquad 0 \le \mathrm{u} \le 2\pi$$

$$x_{i+1} = a*\cos(u+\Delta u) \quad \text{and} \quad y_{i+1} = b*\sin(u+\Delta u)$$

$$x_{i+1} = a(\cos u \cos \Delta u - \sin u \sin \Delta u)$$

$$y_{i+1} = b(\sin u \cos \Delta u + \cos u \sin \Delta u)$$

substituti ng

$$x_{i+1} = x_i \cos \Delta u - \frac{a}{b} y_i \sin \Delta u$$

$$y_{i+1} = y_i \cos \Delta u + \frac{b}{a} x_i \sin \Delta u \quad \text{where } \Delta u = \frac{2\pi}{n}$$

Parametric equation of Ellipse (contd..)

Note:

- If $a=b$, it reduces to circle
- Quantity Δu is constant, hence $\sin\Delta u$ and $\cos\Delta u$ need to be calculated only once
- A non-origin centered ellipse can be generated by translating the origin centered ellipse.

Parametric equation of Parabola

- it is an open curve we need to calculate the limits to display the curve

Parabola with vertex at (0,0)

Cartesian form : $x^2 = 4ay$

where a is focal length (constant)

Parametric form : $x_i = 2au$

$y_i = au^2 \qquad 0 \le \mathrm{u} \le \infty$

Parametric equation of Parabola (contd..)

$$x_{i+1} = 2a(u + \Delta \mathrm{u})$$

$$= 2au + 2a\Delta \mathrm{u}$$

$$x_{i+1} = x_i + 2a\Delta \mathrm{u}$$

$$y_{i+1} = a(u + \Delta \mathrm{u})^2$$

$$= \mathrm{au}^2 + 2\mathrm{au}\Delta \mathrm{u} + \mathrm{a}\Delta \mathrm{u}^2$$

$$= y_i + x_i\Delta \mathrm{u} + \mathrm{a}\Delta \mathrm{u}^2 \qquad \text{where } \Delta u = \frac{2\pi}{(n-1)}$$

Parametric equation of Hyperbola

- A hyperbola is defined as the locus of points whose distances from two fixed points (foci) have a constant difference equal to the transverse axis of the hyperbola.
- Hyperbola with vertex at $(0,0)$
- Cartesian form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Parametric form: $x = a \cosh u$

$$y = b \sinh u$$

- The above equations are very similar to that obtained for an ellipse.

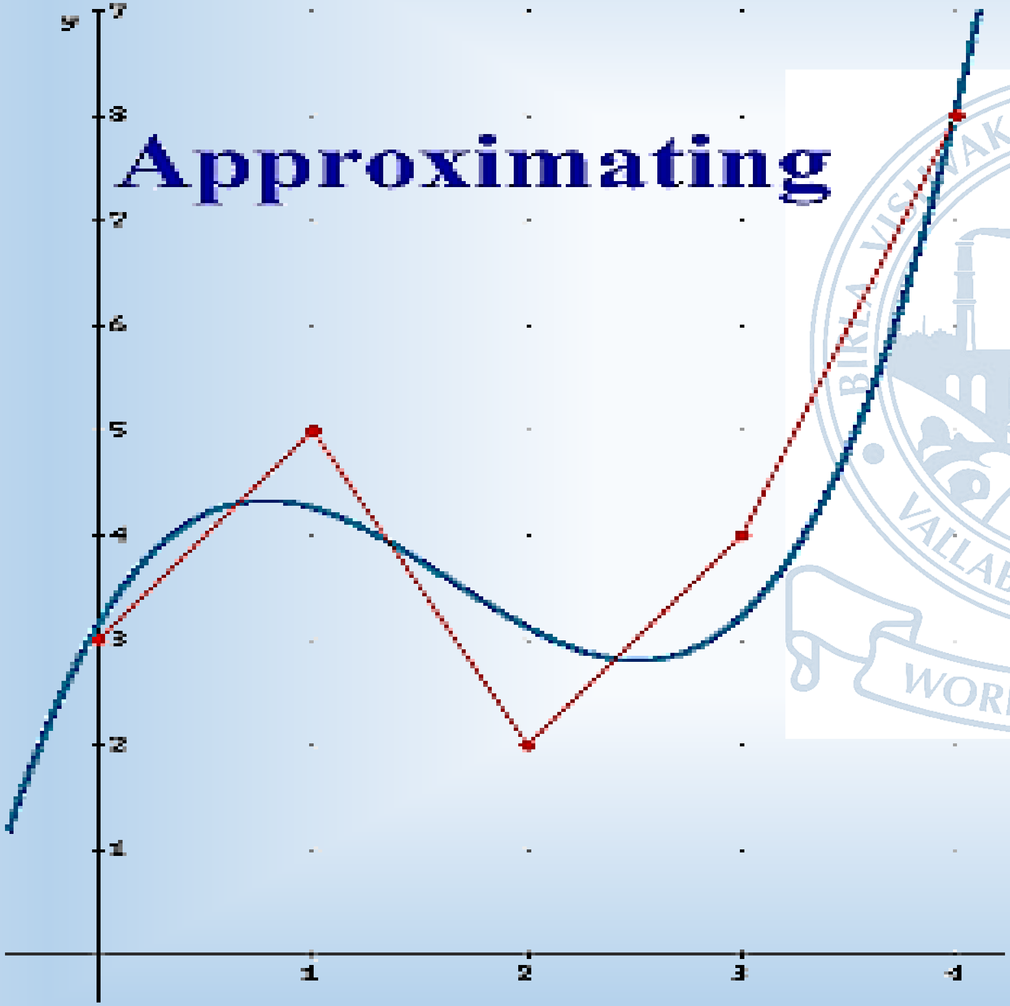
What are Synthetic curves ???

- Limitations of the analytic curves prompt us to study the synthetic curves - efficient at complex geometric design
- They represent a curve fitting problem to construct a smooth curve that passes through given data points,
- When we combine polynomial segments to represent a desired curve, it is called a synthetic curve
  - Automobile: Car bodies
  - Ship building: Ship hulls
  - Aeronautics: Airplane fuselage and wings, propeller blades
  - Architectural designs, Civil engineering designs
  - Shoe insoles and aesthetically designed bottles

Generation of Synthetic curves

- low degree polynomials are combined to construct a curve. Low degree polynomials reduce both the computational effort and numerical instabilities that arise with higher degree curves.
- However, as low degree polynomials cannot span a large number of points, small curve segments are blended together to construct any desired curve for the practical design applications
- A common technique is to use series of cubic spline segments with each segment spanning only two points.

Synthetic curves



- Interpolating curve:

  the curve passes through the data points

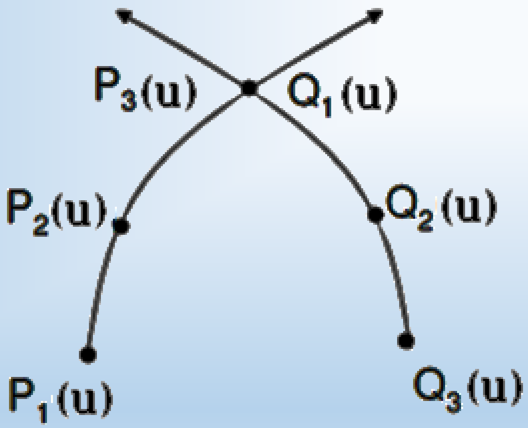
- Approximating curve: the curve does not actually pass through the data points and the data points are used to control the general shape of the curve

Control of shape of curve

- For efficient design, shape of the curve should be controllable most effectively in the easiest possible way
- Two types of control exist: Local control and Global control
- **Local control** is said to be present if change in one control point or tangent vector results in change of shape of curve local to that point
- **Global control** is said to be present if change in one control point or tangent vector results in change of overall shape of the curve segment

Order of continuity: C_0 continuity

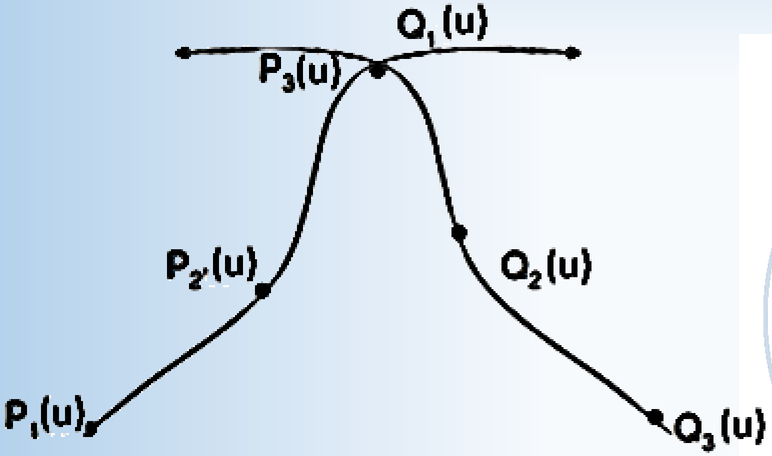
- Curves represented by segments of splines (piecewise polynomials) by connecting them end to end. Hence, the type or order of continuity becomes important for accepting them in design applications. The minimum continuity requirement is position continuity. This ensures the physical connectivity between different segments of the curve



Position Continuity (C_0 continuity)

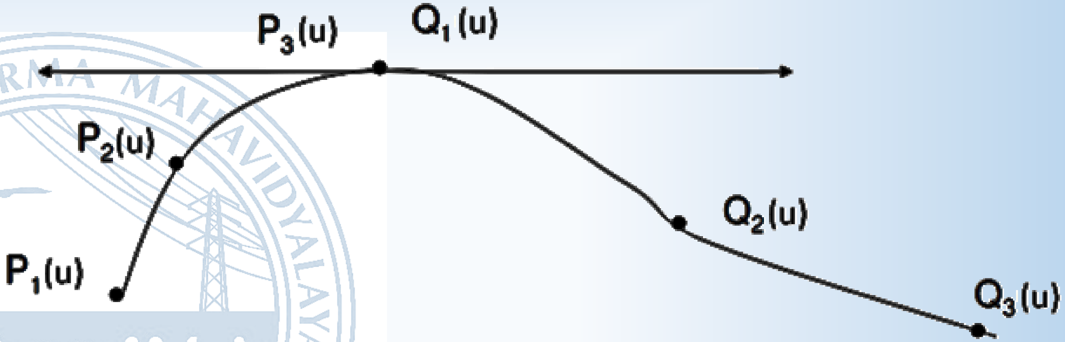
$$P_3(u)=Q_1(u)$$

Tangent [C_1] and Curvature [C_2] continuity



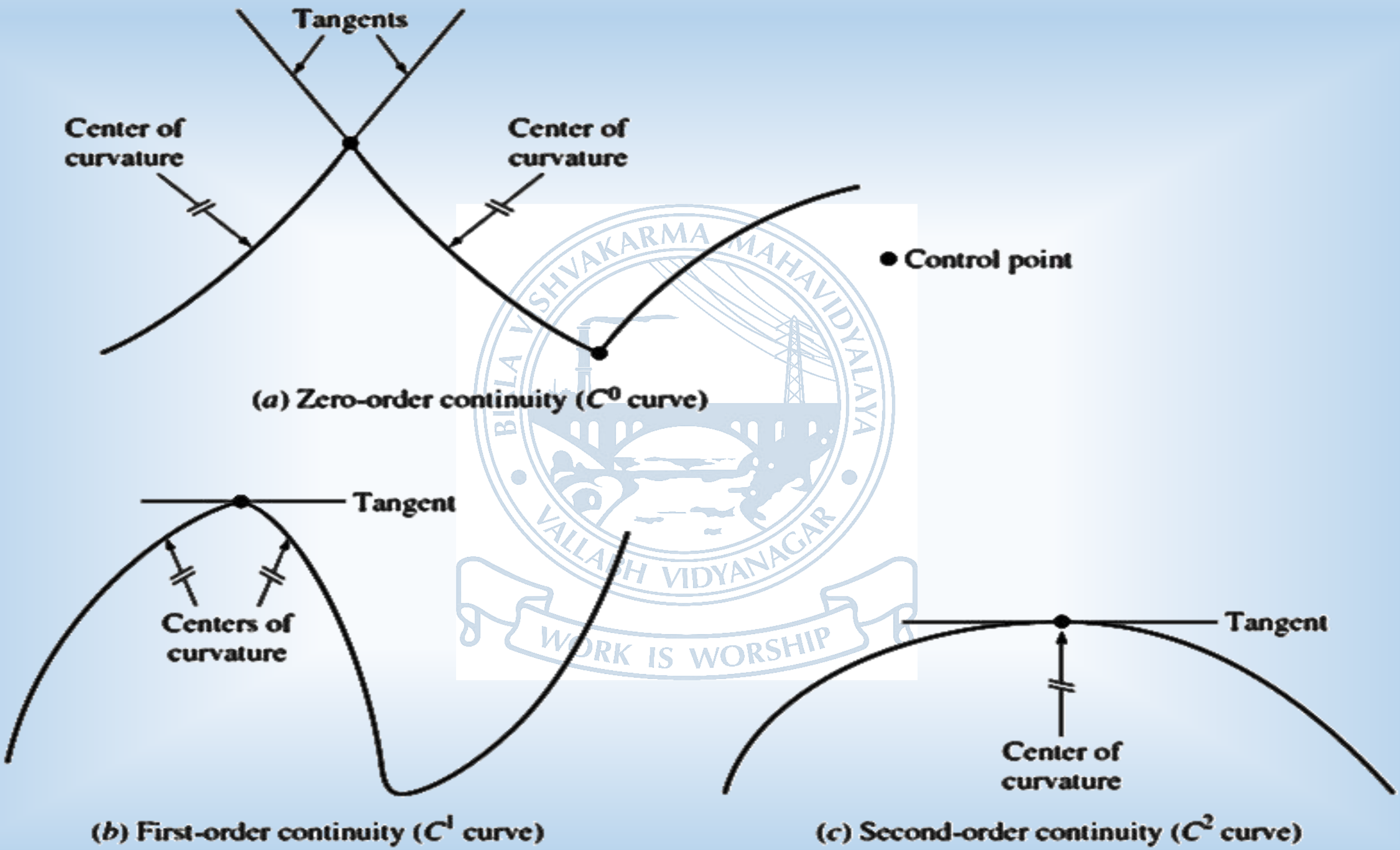
C_1 continuity

- $P_3(u)=Q_1(u)$
- $P_3'(u)=Q_1'(u)$



C_2 continuity

- $P_3(u)=Q_1(u)$
- $P_3'(u)=Q_1'(u)$
- $P_3''(u)=Q_1''(u)$


Tangents
Center of curvature
Center of curvature
Control point
(a) Zero-order continuity (C^0 curve)
Tangent
Centers of curvature
(b) First-order continuity (C^1 curve)
Tangent
Center of curvature
(c) Second-order continuity (C^2 curve)

Continuity between Curve Segments

- C_1 curve is minimum acceptable curve for engineering design
- A third order (cubic) polynomial is the minimum-order polynomial that can guarantee the all three C_0, C_1 and C_2 continuities
- A cubic polynomial is the lowest degree curve that allows representation of non-planar (twisted) 3-D curves in space
- Higher order polynomials (>3) not used in CAD:
  - They tend to oscillate about the control points
  - computationally expensive and inconvenient
  - uneconomical in terms of storage of curve and surface representations in the computer

Common synthetic curves

1. *Hermite Cubic Spline*
   - It passes through the control points and therefore it is an interpolating curve
2. *Bezier Curve*
   - It does not pass through the control points but only approximates the trend
3. *B-Spline Curve*
   - Generally an approximating curve, however, an interpolating B-Spline curve is also possible

Hermite Cubic Spline Curve Segment

- Used to interpolate the given data but not to design free-form curves.
- Word (Cubic) *Splines* derived from "French curves or splines"
- Hermite cubic spline is one type of general parametric cubic spline – a 3rd order curve determined by two data points and tangent vectors at the data points.
- It can be a 3-D planar curve or 3-D twisted curve

Hermite Cubic Spline Curve Segment

- In order to assure C_2 continuity at two extremities, the function must be of at least degree 3

$$P(u)=\sum_{i=0}^{3} C_i u^i = \mathrm{au}^3+\mathrm{bu}^2+\mathrm{cu}+\mathrm{d} \qquad 0 \le \mathrm{u} \le 1$$

- In scalar form, this equation is written as

$$P(u)=[x(u) \quad y(u) \quad z(u)]^T$$

where

$$x(u)=a_x u^3+b_x u^2+c_x u+d_x$$

$$y(u)=a_y u^3+b_y u^2+c_y u+d_y$$

$$z(u)=a_z u^3+b_z u^2+c_z u+d_z$$

Hermite Cubic Spline Curve Segment

- In expanded vector form $P(u) = au^3 + bu^2 + cu + d$
- In matrix form $P(u) = [u]^T.[C]$

  where $[u] = [u^3 \quad u^2 \quad u \quad 1]^T$

  $[C] = [a \quad \text{b} \quad \text{c} \quad \text{d}]^T$ is coefficien t vector

- The tangent vector to the curve at any point is

$$P'(u) = \frac{dP(u)}{du} = 3au^2 + 2bu + c$$

Derivation of coefficients

Hermite Specification

- cubic curve segment with the two end points (P_0 , P_1) and tangents at end points (P'_0, P'_1)

Derivation of coefficients

$$@u=0$$

$$P_0 = d$$

$$P_0' = c$$

$$@u=1$$

$$P_1 = a+b+c+d$$

$$P_1' = 3a+2b+c$$

Solving four simultaneous linear equations

$$d = P_0$$

$$c = P_0'$$

$$b = -3(P_0 - P_1) - 2P_0' - P_1'$$

$$a = 2(P_0 - P_1) + P_0' + P_1'$$

Cubic spline equation

- Substituting in the parametric equation and rearranging

$$P(u) = (2u^3 - 3u^2 + 1)P_0 + (-2u^3 + 3u^2)P_1 +$$
$$(u^3 - 2u^2 + u)P_0' + (u^3 - u^2)P_1'$$

$$P(u) = F_1(u)P_0 + F_2(u)P_1 + F_3(u)P_0' + F_4(u)P_1'$$

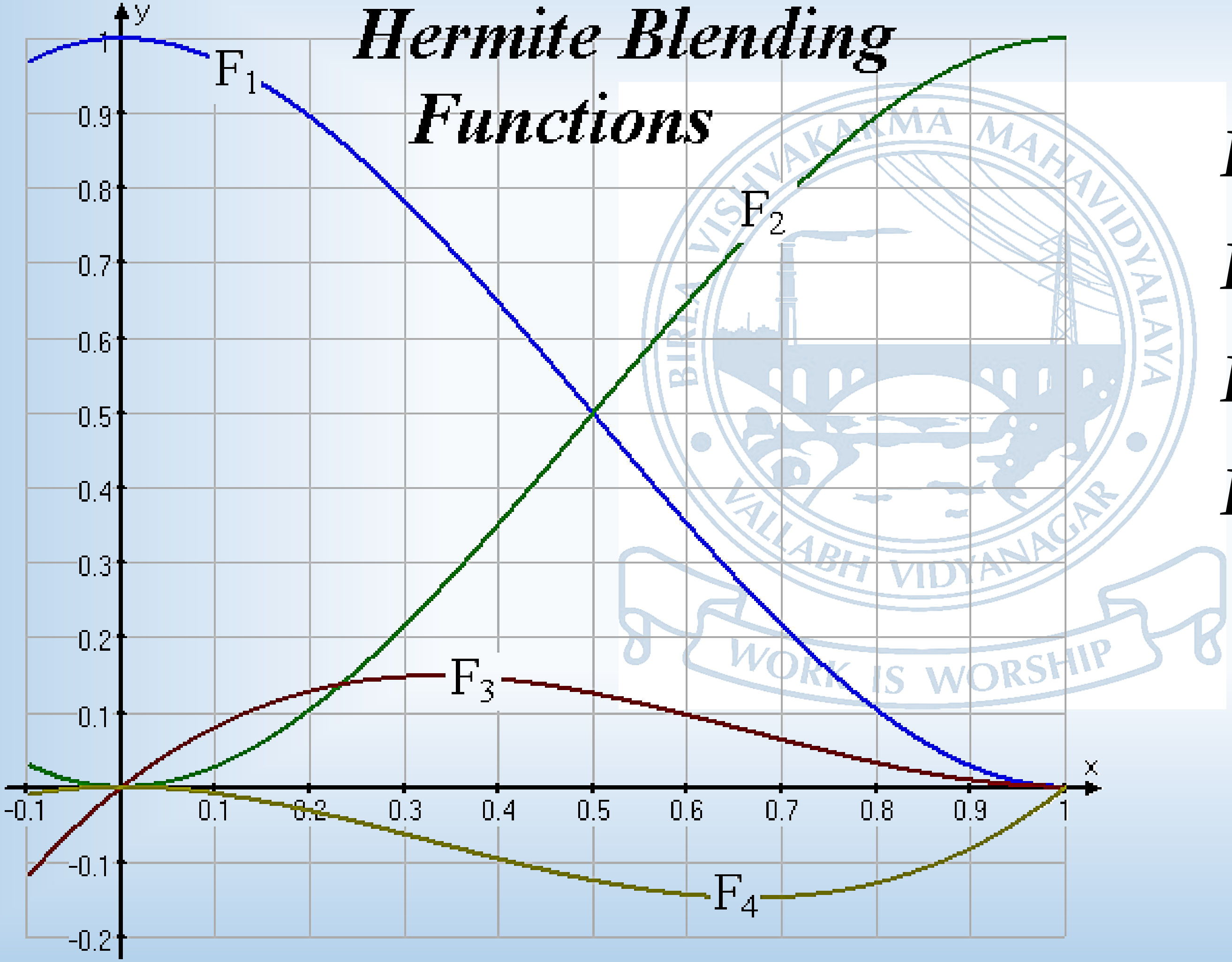
where

P_0, P_1, P_0' *and* P_1' are Geometric coefficients and

$F_1(u), F_2(u), F_3(u)$ and $F_4(u)$ are Blending functions

$$\text{Tangent Vector}: P'(u) = (6u^2 - 6u)P_0 + (-6u^2 + 6u)P_1$$
$$+ (3u^2 - 4u + 1)P_0' + (3u^2 - 2u)P_1'$$

Hermite Cubic spline: Blending functions



$$F_1(u) = 2u^3 - 3u^2 + 1$$

$$F_2(u) = -2u^3 + 3u^2$$

$$F_3(u) = u^3 - 2u^2 + u$$

$$F_4(u) = u^3 - u^2$$

Hermite Cubic spline – Matrix form

$$P(u) = [U]^T.[M_H].[P]$$

Where Hermite Matrix $[M_H] = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$[x \quad y] = [u^3 \;\; u^2 \;\; u \;\; 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_0' & y_0' \\ x_1' & y_1' \end{bmatrix}$$

Exercise-1

For points $P_0 = [1,2]$ and $P_1 = [3,1]$ with corresponding slopes 60° and 30°, write the formulation of Hermite cubic spline

$$x(0) = 1 \; x(1) = 3 \;\; x'(0) = \cos 60° \; x'(1) = \cos 30°$$
$$y(0) = 2 \;\; y(1) = 1 \;\; y'(0) = \sin 60° \; y'(1) = \sin 30°$$

$$x(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ \cos 60 \\ \cos 30 \end{bmatrix}$$

$$x(u) = -2.63u^3 + 4.13u^2 + 0.5u + 1$$

Exercise-1

$$y(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ \sin 60 \\ \sin 30 \end{bmatrix}$$

$$y(u) = 3.36u^3 - 5.23u^2 + 0.86u + 2$$

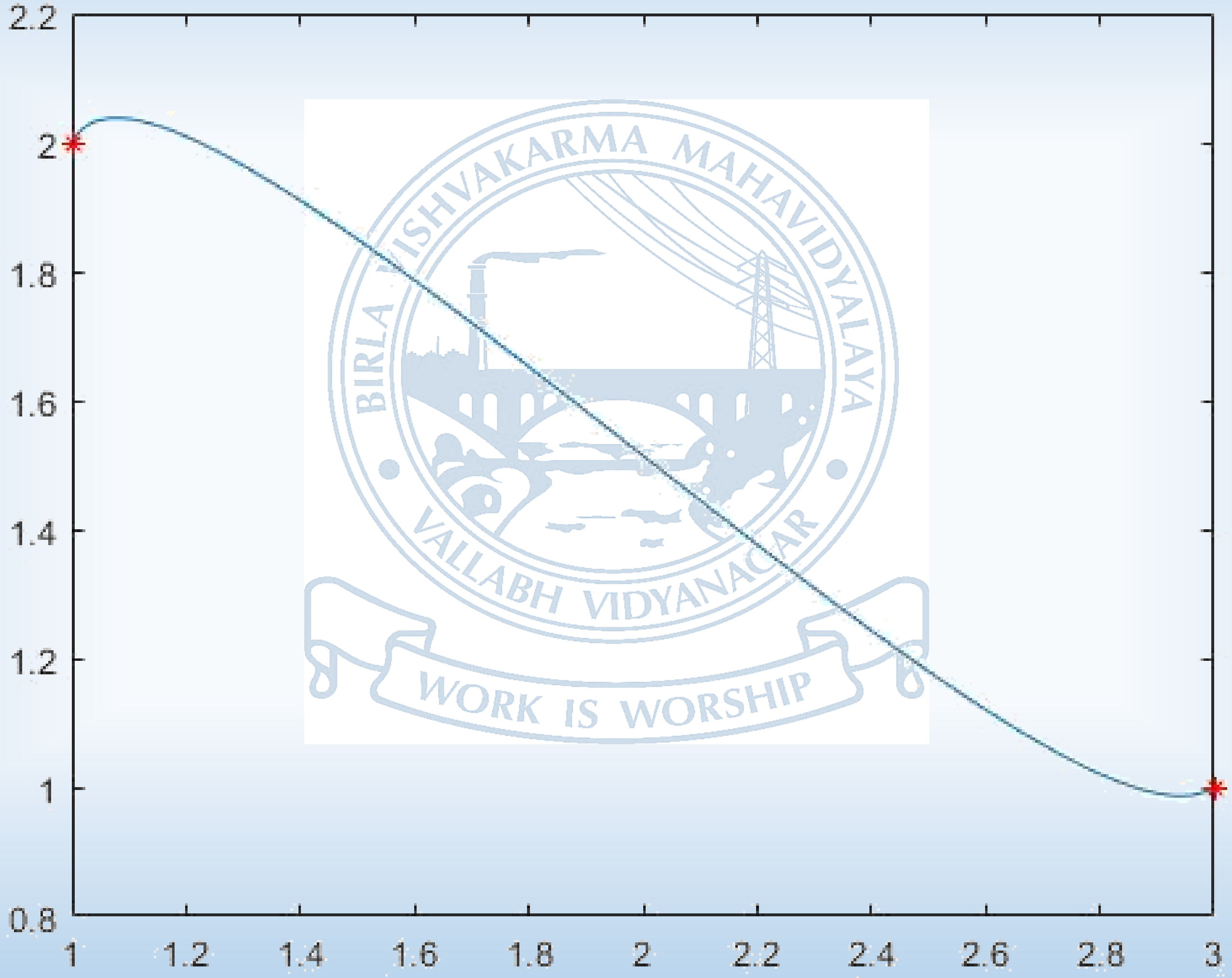
$$x'(u) = -7.89u^2 + 8.26u + 0.5$$

$$y'(u) = 10.08u^2 - 10.46u + 0.86$$

Exercise-1

u	$-2.63u^3$	$4.13u^2$	0.5u	X(u)	$3.36u^3$	$-5.23u^2$	0.86u	Y(u)
0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	2.00
0.20	-0.02	0.17	0.10	1.24	0.03	-0.21	0.17	1.99
0.40	-0.17	0.66	0.20	1.69	0.22	-0.84	0.34	1.72
0.60	-0.57	1.49	0.30	2.22	0.73	-1.88	0.52	1.36
0.80	-1.35	2.64	0.40	2.70	1.72	-3.35	0.69	1.06
1.00	-2.63	4.13	0.50	3.00	3.36	-5.23	0.86	0.99

Exercise-1



Composite Hermite Cubic spline

- Blending: construction of composite curves by joining either the same type of curves or different types of curves
- Cubic spline curve is created as a blend of cubic spline segments by connecting a set of points with the requirements of tangent vectors at the intermediate points.
- For curvature continuity between the first two segments

$$P''_{(u_1=1)} = P''_{(u_2=0)}$$

Composite Hermite Cubic spline

$$P(u) = (2u^3 - 3u^2 + 1)P_0 + (-2u^3 + 3u^2)P_1 + (u^3 - 2u^2 + u)P_0' + (u^3 - u^2)P_1'$$

$$P'(u) = (6u^2 - 6u)P_0 + (-6u^2 + 6u)P_1 + (3u^2 - 4u + 1)P_0' + (3u^2 - 2u)P_1'$$

$$P''(u) = (12u - 6)P_0 + (-12u + 6)P_1 + (6u - 4)P_0' + (6u - 2)P_1'$$

$$P''_{(u_1=1)} = 6P_0 - 6P_1 + 2P_0' + 4P_1'$$

$$P''_{(u_2=0)} = -6P_1 + 6P_2 - 4P_1' - 2P_2'$$

$$6P_0 - 6P_1 + 2P_0' + 4P_1' = -6P_1 + 6P_2 - 4P_1' - 2P_2'$$

$$2P_0' + 8P_1' + 2P_2' = 6P_2 - 6P_1 + 6P_1 - 6P_0$$

$$P_0' + 4P_1' + P_2' = 3(P_2 - P_1) + 3(P_1 - P_0)$$

Composite Hermite Cubic spline

For more than two segments, a matrix equation can result from repeating this procedure

$$\begin{bmatrix} 1 & 4 & 1 & 0 & .. & .. \\ 0 & 1 & 4 & 1 & .. & .. \\ 0 & 0 & 1 & 4 & 1 & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & 0 & 1 & 4 & 1 \\ .. & .. & .. & .. & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0' \\ P_1' \\ P_2' \\ .. \\ .. \\ P_n' \end{bmatrix} = \begin{bmatrix} 3(P_2 - P_1) + 3(P_1 - P_0) \\ 3(P_3 - P_2) + 3(P_2 - P_1) \\ .. \\ .. \\ .. \\ .. \end{bmatrix}$$

Exercise 2

- The position vectors of a normalized cubic spline are given as (0 0), (1 1), (2 -1) and (3 0). The tangent vectors at the ends are both (1 1). Calculate:

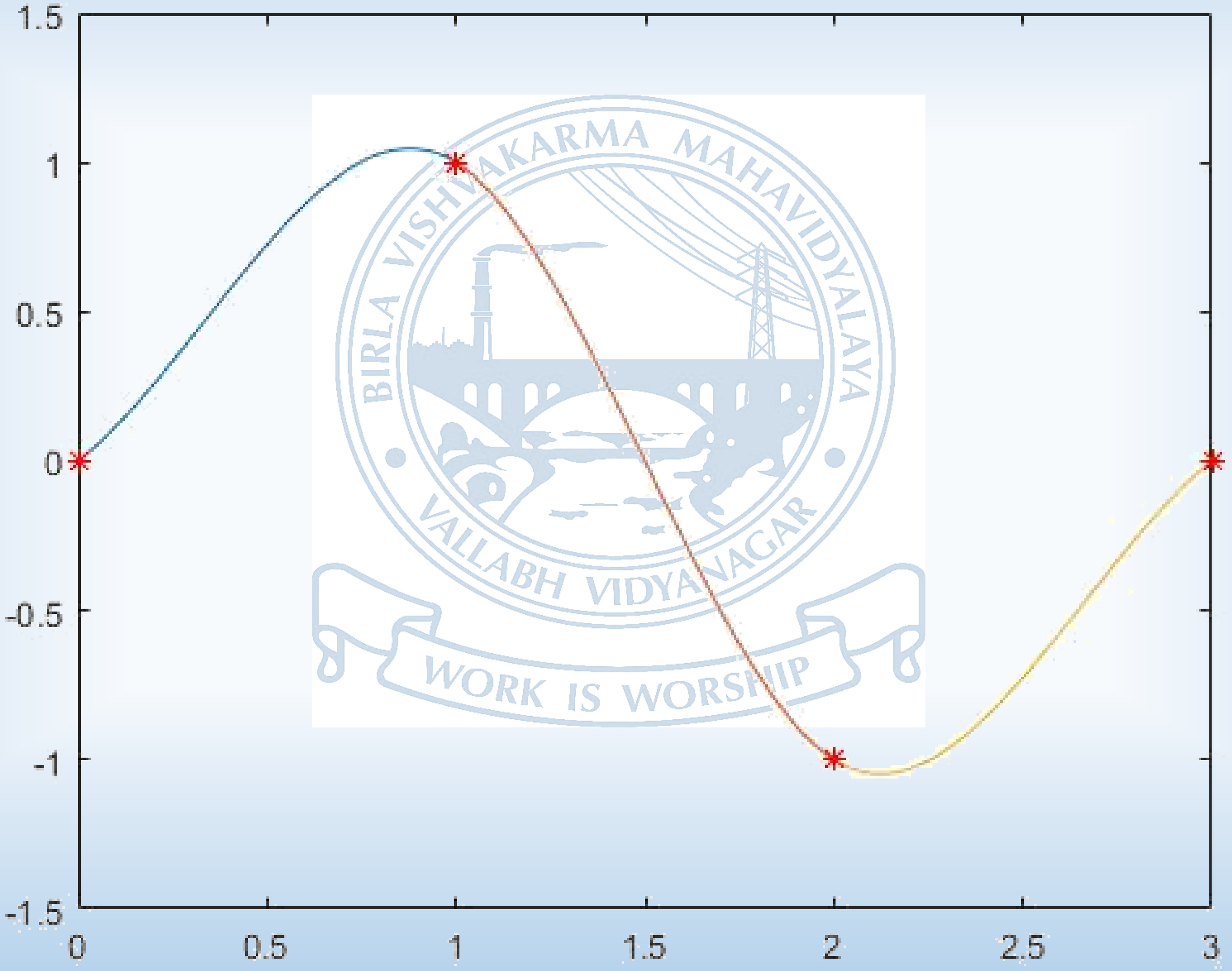
(i) 2 internal tangent vectors, and

(ii) points on the curve at u=¼, ½ and ¾ for first segment

Ans: (i) (1, -0.8) (1, -0.8)

(ii) at u=0.25, (0.25, 0.33); at u = 0.5, (0.5, 0.725); at 0.75, (0.75, 1.003).

Exercise 2



Limitations of Cubic splines

- If the number of data points are large, computation time required can be excessive.
- Parametric cubic curves cannot represent conic sections exactly.
- Poorly approximate asymptotic curves.
- If not controlled properly, exhibit spurious oscillations.