AKTU

B. Tech II-Year

Gateway Classes

GATEWAY CLASSES

Semester -IV **ENGG.Mathematics-IV**

BAS-403 ENGG- Mathematics-IV

UNIT-2 : ONE SHOT

Applications of Partial Differential

Gateway Series for Engineering

- Topic Wise Entire Syllabus
- Long - Short Questions Covered
- AKTU PYQs Covered
- DPP
- Result Oriented Content

Download App

For Full Courses including Video Lectures

BAS-403 ENGG. MATHEMATICS-IV

Unit-2-ONE SHOT

Introduction to Applications of Partial Differential

Syallbus

Module II: Applications of Partial Differential Equations and Fourier Transform: Method of separation of variables,Solution of one dimensional heat equation, wave equation, Two dimensional heat equation (only Laplace Equation) and their application, Complex Fourier transform, Fourier sine transform, Fourier cosine transform, Inverse transform, convolution theorem, Application of Fourier Transformto solve partial differential equation

Maths-IV One Shot

Unit-2 in 2 Hrs

Applications of PDE and Fourier Transform

LIVE

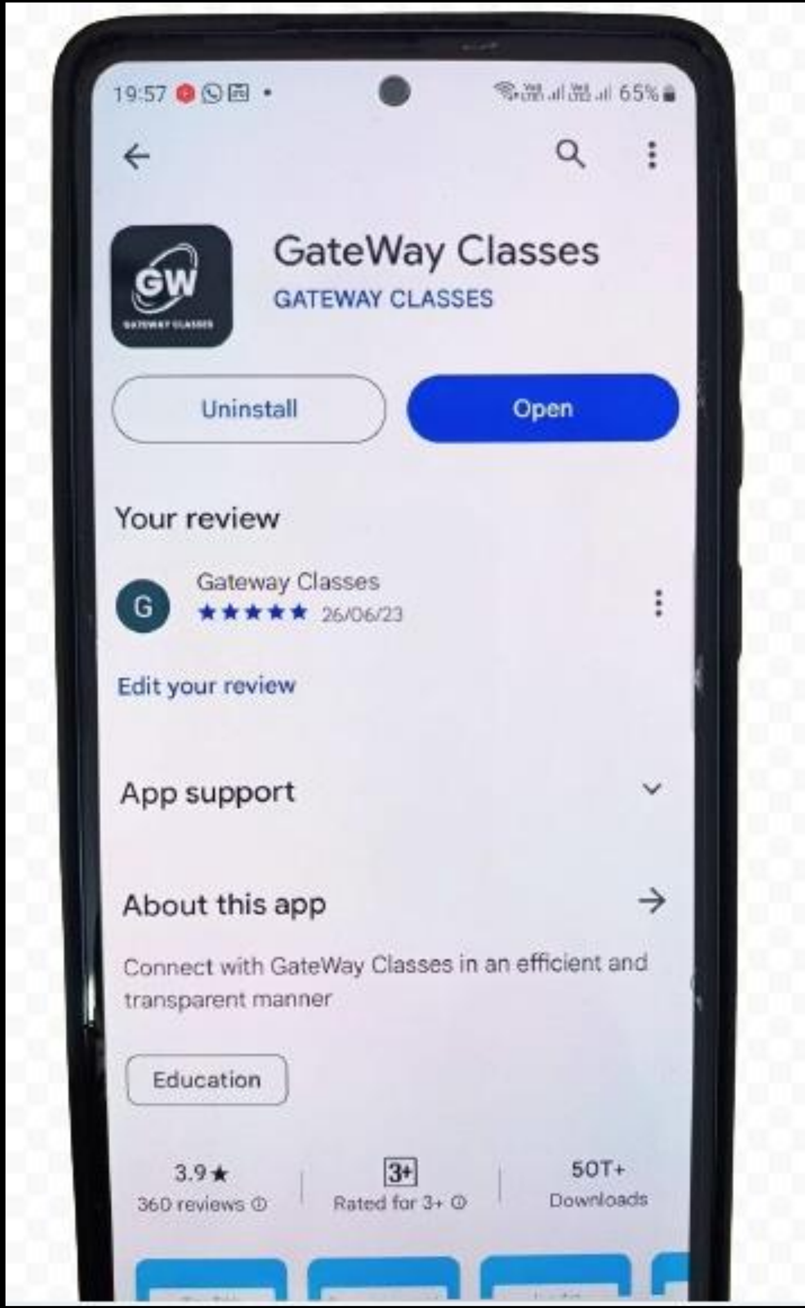
9 March 9 PM

Questions यही से आएंगे

B.Tech -II-Year (4^{th} Sem)
Courses will be available
in
March Last Week

B.Tech IV-Sem

- ➢CS, IT & CS Allied
- ➢EC & EC Allied
- ➢ME & ME Allied
- ➢EE & EE Allied



Engineering Mathematics-IV

Module -II: Applications of Partial Differential Equations and Fourier Transform

Chapter-1 :- Applications of Partial Differential Equations

- Method of separation of variables
- Solution of one dimensional heat equation, wave equation
- Two dimensional heat equation (only Laplace Equation) and their application

Chapter-2 :- Fourier Transform

- Complex Fourier transform, Fourier sine transform, Fourier cosine transform, Inverse transform, convolution theorem
- Application of Fourier Transformto solve partial differential equation.

UNIT : Applications of Partial Differential Equations and Fourier Transform

Today's Target

- Method of separation of variables(When boundary conditions are not given)
- PYQs
- DPP

Calculation of Complementary Function (C.F)

CASE-1 : When A.E has Real and distinct roots

Nature of roots	*C.F*
(i) One real root m_1	$CF = c_1 e^{m_1 x}$
(ii) Two real and distinct roots m_1, m_2	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
(iii) Three real and distinct roots m_1, m_2, m_3	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

CASE-2 : When A.E has Real and equal roots

Nature of roots	*C.F*
(i) Two real and equal roots m, m	$CF = (c_1 + c_2 x)e^{mx}$
(ii) Three real and equal roots m, m, m	$CF = (c_1 + c_2 x + c_3 x^2)e^{mx}$

GW

CASE-3 : When A.E has *complex* roots

Nature of roots	*C.F*
(i) One pair of complex roots $\alpha \pm i\beta$	$CF = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$
(ii) Two pair of complex and equal roots $\alpha \pm i\beta,\ \alpha \pm i\beta$	$CF = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

CASE-~~3~~4 : When A.E has *Irrational* roots

Nature of roots	*C.F*
(i) One pair of irrational roots $\alpha \pm \sqrt{\beta}$	$CF = c_1 e^{(\alpha+\beta)x} + c_2 e^{(\alpha-\beta)x}$
(ii) Two pair of irrational and equal roots $\alpha \pm \sqrt{\beta},\quad \alpha \pm \sqrt{\beta}$	$CF = (c_1 + c_2 x)e^{(\alpha+\beta)x} + (c_3 + c_4 x)e^{(\alpha-\beta)x}$

Method of separation of variables

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- *In this method, we assume the solution of the given PDE as the product of the two functions.*
- *Each function is the function of one variable only.*
- *So two ordinary differential equations are formed.*

Method of separation of variables

When boundary conditions are not given

L-1

When boundary conditions are given

L-2

When boundary conditions are not given

Q.1. Use the method of separation of variables to solve the equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$ (GBTU-2012)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{--- (1)}$$

Let $u = X(x)Y(y)$

$$u = XY \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = YX''$$

$$\frac{\partial u}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$YX'' - 2YX' + XY' = 0 \quad \text{--- (3)}$$

Divide by XY

$$\frac{\cancel{Y}X''}{X\cancel{Y}} - 2\frac{\cancel{Y}X'}{X\cancel{Y}} + \frac{\cancel{X}Y'}{\cancel{X}Y} = \frac{0}{XY}$$

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0$$

$$\frac{X''}{X} - 2\frac{X'}{X} = -\frac{Y'}{Y}$$

$$\frac{X''-2X'}{X} = -\frac{Y'}{Y} = -p^2 \text{ (say)} \quad \text{--- (4)}$$

$$\frac{X''-2X'}{X} = -p^2$$

$$X''-2X' = -p^2X$$

$$X''-2X'+p^2X = 0$$

$$D^2X-2DX+p^2X = 0$$

$$(D^2-2D+p^2)X = 0$$

Aux. Eq^n

$$D^2-2D+p^2 = 0$$

$$D^2 - 2D + p^2 = 0$$

$$D = \frac{2 \pm \sqrt{(-2)^2 - 4 \times p^2}}{2 \times 1}$$

$$D = \frac{2 \pm \sqrt{4 - 4p^2}}{2}$$

$$D = \frac{2 \pm 2\sqrt{1-p^2}}{2}$$

$$D = 1 \pm \sqrt{1-p^2}$$

$$CF = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}$$

$$PI = 0$$

$$X = CF + PI$$

$$\boxed{X = c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}}$$

From (4)

$$\frac{-Y'}{Y} = -p^2$$

$$+Y' = +p^2 Y$$

$$Y' = p^2 Y$$

$$Y' - p^2 Y = 0$$

$$DY - p^2 Y = 0$$

$$(D - p^2) Y = 0$$

Aux. Eq^n

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = c_3 e^{p^2 y}$$

$$PI = 0$$

$$Y = CF + PI$$

$$\boxed{Y = c_3 e^{p^2 y}}$$

Put X and Y in (2)

$$\boxed{u = \left[c_1 e^{(1+\sqrt{1-p^2})x} + c_2 e^{(1-\sqrt{1-p^2})x}\right] c_3 e^{p^2 y}}$$

Q.2. *Use the method of separation of variables to solve the equation* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u.$

(UPTU-2014)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{--- (1)}$$

Let $u = X(x)\,Y(y)$

$$u = XY \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = YX''$$

$$\frac{\partial u}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$YX'' = XY' + 2XY$$

Divides by XY

$$\frac{\cancel{Y}X''}{X\cancel{Y}} = \frac{\cancel{X}Y'}{\cancel{X}Y} + \frac{2\cancel{XY}}{\cancel{XY}}$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = -p^2 \ (\text{say}) \quad \text{--- (3)}$$

$$\frac{X''}{X} = -p^2$$

$$X'' = -p^2 X$$

$$X'' + p^2 X = 0$$

$$D^2 X + p^2 X = 0$$

$(D^2+p^2)X = 0$

Aux. Eq^n

$D^2+p^2 = 0$

$D = \pm ip$

$D = 0 \pm ip$

$CF = e^{0x}(c_1 \cos px + c_2 \sin px)$

$CF = c_1 \cos px + c_2 \sin px$

$PI = 0$

$X = CF + PI$

$$\boxed{X = c_1 \cos px + c_2 \sin px}$$

From 3

$\frac{Y'}{Y} + 2 = -p^2$

$\frac{Y' + 2Y}{Y} = -p^2$

$Y' + 2Y = -p^2 Y$

$$Y' + 2Y + p^2 Y = 0$$

$$DY + 2Y + p^2 Y = 0$$

$$(D + 2 + p^2)Y = 0$$

Aux. Eq^n

$$D + 2 + p^2 = 0$$

$$D = -(p^2 + 2)$$

$$CF = c_3 e^{-(p^2+2)y}$$

$$PI = 0$$

$$Y = CF + PI$$

$$\boxed{Y = c_3 e^{-(p^2+2)y}}$$

Put X and Y in (2)

$$\boxed{u = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-(p^2+2)y}}$$

Unit-2 : Applications of Partial Differential Equations and Fourier Transform

DPP-1

Topic : Method of separation of variables(When boundary conditions are not given)

Q .1. *Use the method of separation of variables to solve the equation* $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ *UPTU-2012*

Ans. $u(x,\ y) = c_1 c_2\ e^{k(x+y)}$

Q.2. *Use the method of separation of variables to solve the equation* $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$ *UPTU-2007*

Ans. $u(x,\ y) = c_1\ e^{-p^2 y}\ (c_2 \cos px + c_3 \sin px)$

Engineering Mathematics

UNIT : Application of Partial Differential Equation and Fourier Transform

Today's Target

- Method of Separation of Variables (When Boundary conditions are given)
- PYQs
- DPP

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Method of Separation of Variables

- In this method, we assume the solution of the given PDE as the product of two functions.
- Each function is the function of one variable only.
- So two ordinary differential equations are formed.

Method of Separation of Variables

When Boundary conditions are not given

L-1

When Boundary conditions are given

L-2

Q.1:- *Solve the partial differential equation :* $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0;\quad z(0,\ y) = 2e^{-y}$ *by the method of separation of variables.* **(AKTU-2017)**

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0 \quad \text{—— (1)}$$

Let $z(x,y) = XY$ —— (2)

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx} = YX'$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy} = XY'$$

Put these values in (1)

$$2YX' + 3XY' + 5XY = 0$$

Divide by XY

$$\frac{2\cancel{Y}X'}{X\cancel{Y}} + \frac{3\cancel{X}Y'}{\cancel{X}Y} + 5\frac{\cancel{XY}}{\cancel{XY}} = 0$$

$$2\frac{X'}{X} + 3\frac{Y'}{Y} + 5 = 0$$

$$2\frac{X'}{X} = -\left(3\frac{Y'}{Y}+5\right) = K \text{ (say)}$$

$$2\frac{X'}{X} = K$$

$$2X' = KX$$

$$2DX = KX$$

$$2DX - KX = 0$$

$$(2D-K)X = 0$$

Aux. Eqn

$$2D - K = 0$$

$$D = \frac{K}{2}$$

$$CF = c_1 e^{\frac{K}{2}x}$$

$$PI = 0$$

$$\boxed{X = c_1 e^{\frac{K}{2}x}}$$

$$-\left(\frac{3Y'}{Y}+5\right) = K$$

$$\frac{3Y'+5Y}{Y} = -K$$

$$3Y' + 5Y = -KY$$

$$3DY + 5Y + KY = 0$$

$$(3D+5+K)Y = 0$$

Aux. Eqn

$$3D+5+K = 0$$

$$D = -\frac{(K+5)}{3}$$

$$CF = c_2 e^{-\left(\frac{K+5}{3}\right)y}$$

$$PI = 0$$

$$Y = CF + PI$$

$$\boxed{Y = c_2 e^{-\left(\frac{k+5}{3}\right)y}}$$

Put X and Y in (2)

$$z(x,y) = c_1 e^{\frac{k}{2}x} \times c_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$\boxed{z(x,y) = c_1 c_2 e^{\frac{k}{2}x - \left(\frac{k+5}{3}\right)y}} \quad (3)$$

Apply Boundary condition

Put $x = 0$

$$z(0,y) = c_1 c_2 e^{-\left(\frac{k+5}{3}\right)y}$$

$$2e^{-y} = c_1 c_2 e^{-\left(\frac{k+5}{3}\right)y}$$

Where

$$c_1 c_2 = 2$$

$$+1 = +\left(\frac{k+5}{3}\right)$$

$$1 = \frac{k+5}{3}$$

$$k + 5 = 3$$

$$k = -2$$

Put $c_1 c_2$ and k in (3)

$$z(x,y) = 2e^{-x-y}$$

$$\boxed{z(x,y) = 2e^{-(x+y)}}$$

Q.2:- Solve by method of separation of variables $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u; \quad u(x,\ 0) = 10e^{-x} - 6e^{-4x}$ *(UPTU-2018)*

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u \quad \text{—— (1)}$$

Let $u = TX$ —— (2)

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} = XT'$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx} = TX'$$

Put these values in (1)

$$XT' = TX' - 2TX$$

Divide by TX

$$\frac{\cancel{X}T'}{T\cancel{X}} = \frac{\cancel{T}X'}{\cancel{T}X} - \cancel{\frac{2TX}{TX}}$$

$$\frac{T'}{T} = \frac{X'}{X} - 2 = p^2 \text{ (say)}$$

$$\frac{T'}{T} = p^2$$

$$T' = p^2 T$$

$$T' - p^2 T = 0$$

$$DT - p^2 T = 0$$

$$(D - p^2)T = 0$$

Aux. Eqn

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = c_1 e^{p^2 t}$$

$$PI = 0$$

$T = CF + PI$

$$\boxed{T = c_1 e^{p^2 t}}$$

$$\frac{X'}{X} - 2 = p^2$$

$$\frac{X' - 2X}{X} = p^2$$

$$X' - 2X = p^2 X$$

$$DX - 2X - p^2 X = 0$$

$$(D - 2 - p^2)X = 0$$

Aux. Eq^n

$$D - 2 - p^2 = 0$$

$$D = p^2 + 2$$

$$CF = c_2 e^{(p^2+2)x}$$

$$PI = 0$$

$$\boxed{X = c_2 e^{(p^2+2)x}}$$

Put T and X in (2)

$$u(t,x) = c_1 e^{p^2 t} \times c_2 e^{(p^2+2)x}$$

$$u(t,x) = c_1 c_2 e^{p^2 t + (p^2+2)x}$$

Most General solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{p^2 t + (p^2+2)x} \quad (3)$$

Apply Boundary conditions

Put $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{(p^2+2)x}$$

$$10e^{-x} - 6e^{-4x} = b_1 e^{(p^2+2)x} + b_2 e^{(p^2+2)x} + b_3 e^{(p^2+2)x} + \cdots$$

Where

$b_1 = 10$	$b_2 = -6$
$p^2+2 = -1$	$p^2+2 = -4$
$p^2 = -3$	$p^2 = -6$

From (3)

$$u(x,t) = b_1 e^{p^2 t + (p^2+2)x} + b_2 e^{p^2 t + (p^2+2)x}$$

$$u(x,t) = 10 e^{-3t-x} - 6 e^{-6t-4x}$$

$$\boxed{u(x,t) = 10 e^{-(x+3t)} - 6 e^{-2(2x+3t)}}$$

Q.3. Solve by method of separation of variables $\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}=0$; $z(x,\ 0)=0,\ \ z(x,\ \pi)=0,\ \ z(0,\ y)=4\sin 3y$

(UPTU-2014)

$$\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}=0 \quad \text{—— (1)}$$

Let $Z(x,y)=XY$ —— (2)

$$\frac{\partial z}{\partial x}=Y\frac{dX}{dx}=YX'$$

$$\frac{\partial z}{\partial y}=X\frac{dY}{dy}=XY'$$

$$\frac{\partial^2 z}{\partial y^2}=X\frac{d^2Y}{dy^2}=XY''$$

Put these values in (1)

$$YX'+XY''=0$$

Divide by XY

$$\frac{\cancel{Y}X'}{X\cancel{Y}}+\frac{\cancel{X}Y''}{\cancel{X}Y}=0$$

$$\frac{X'}{X}+\frac{Y''}{Y}=0$$

$$\frac{X'}{X}=-\frac{Y''}{Y}=p^2\ (\text{say})$$

$$\frac{X'}{X} = p^2$$

$$X' = p^2 X$$

$$X' - p^2 X = 0$$

$$DX - p^2 X = 0$$

$$(D - p^2)X = 0$$

Aux. Eq^n

$$D - p^2 = 0$$

$$D = p^2$$

$$CF = c_1 e^{p^2 x}$$

$$PI = 0$$

$$\boxed{X = c_1 e^{p^2 x}}$$

$$\frac{-Y''}{Y} = p^2$$

$$-Y'' = p^2 Y$$

$$Y'' + p^2 Y = 0$$

$$D^2 Y + p^2 Y = 0$$

$$(D^2 + p^2)Y = 0$$

Aux. Eq^n

$$D^2 + p^2 = 0$$

$$D^2 = -p^2$$

$$D = \sqrt{-p^2}$$

$$D = \pm i p$$

$$D = 0 \pm i p$$

$$CF = e^{\alpha y}(c_2 \cos \beta y + c_3 \sin \beta y)$$

$$CF = c_2 \cos py + c_3 \sin py$$

$$PI = 0$$

$$\boxed{Y = c_2 \cos py + c_3 \sin py}$$

Put X and Y in 2

$z(x,y) = c_1 e^{p^2 x} \times (c_2 \cos py + c_3 \sin py)$ —— (3)

Given Boundary conditions

$z(x,0) = 0$ —— (i)

$z(x,\pi) = 0$ —— (ii)

$z(0,y) = 4\sin 3y$ —— (iii)

Apply Boundary conditions (i)

Put $y = 0$

$z(x,0) = c_1 e^{p^2 x} c_2$

$0 = c_1 c_2 e^{p^2 x}$

$c_1 c_2 e^{p^2 x} = 0$

$c_1 \neq 0$

$e^{p^2 x} \neq 0$

$\Rightarrow c_2 = 0$

Put $c_2 = 0$ in (3)

$z(x,y) = c_1 e^{p^2x} \times c_3 \sin py$

$z(x,y) = c_1 c_3 e^{p^2x} \sin py$ — (4)

Apply Boundary condition (ii)

Put $y = \pi$

$z(x,\pi) = c_1 c_3 e^{p^2x} \sin p\pi$

$c_1 c_3 e^{p^2x} \sin p\pi = 0$

$c_1 \neq 0$

$c_3 \neq 0$

$e^{p^2x} \neq 0$

$\Rightarrow \sin p\pi = 0$

$p\cancel{\pi} = n\cancel{\pi}$

$p = n$

Put $p = n$ in (4)

$$\boxed{z(x,y) = c_1 c_3 e^{n^2x} \sin ny} \quad \text{— (5)}$$

Apply (iii) Boundary condition

Put $x = 0$

$$z(0,y) = c_1 c_3 e^0 \sin ny$$

$$4 \sin 3y = c_1 c_3 \sin ny$$

Where

$$c_1 c_3 = 4$$

$$n = 3$$

$$\boxed{z(x,y) = 4 e^{9x} \sin 3y}$$

DPP-2

➢ Topic : Method of Separation of Variables (When Boundary conditions are given)

Q .1 *Solve by the method of separation of variables* $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0;\ u(x,\ 0) = 4e^{-x}$ (AKTU-2017)

Ans. $u(x,\ y) = 4e^{-x+3y/2}$

Q .2 *Solve by the method of separation of variables* $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u;\ u(0,\ y) = 4e^{-y} - e^{-5y}$ (AKTU-2022)

Ans. $u(x,\ y) = 4e^{+x-y} - e^{2x-5y}$

Q .3 *Use the method of separation of variables to solve the equation* $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$ *given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$. (GBTU-2013)

Ans. $u(x,\ y) = (c_1 \cos px + c_2 \sin px)c_3 e^{-(p^2+2)y}$

Q .4 *Solve the P.D.E. by separation of variables method* $u_{xx} = u_y + 2\,u,\ u(0,\ y) = 0, \frac{\partial}{\partial x} u(0,\ y) = 1 + e^{-3y}$. (GBTU-2013)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (PART-1
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

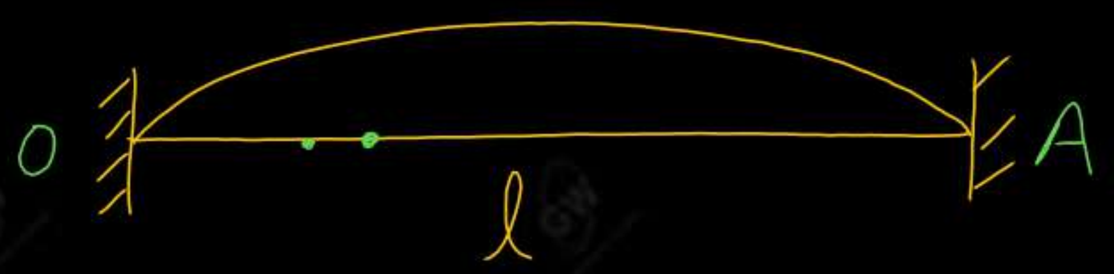
One Dimensional wave equation

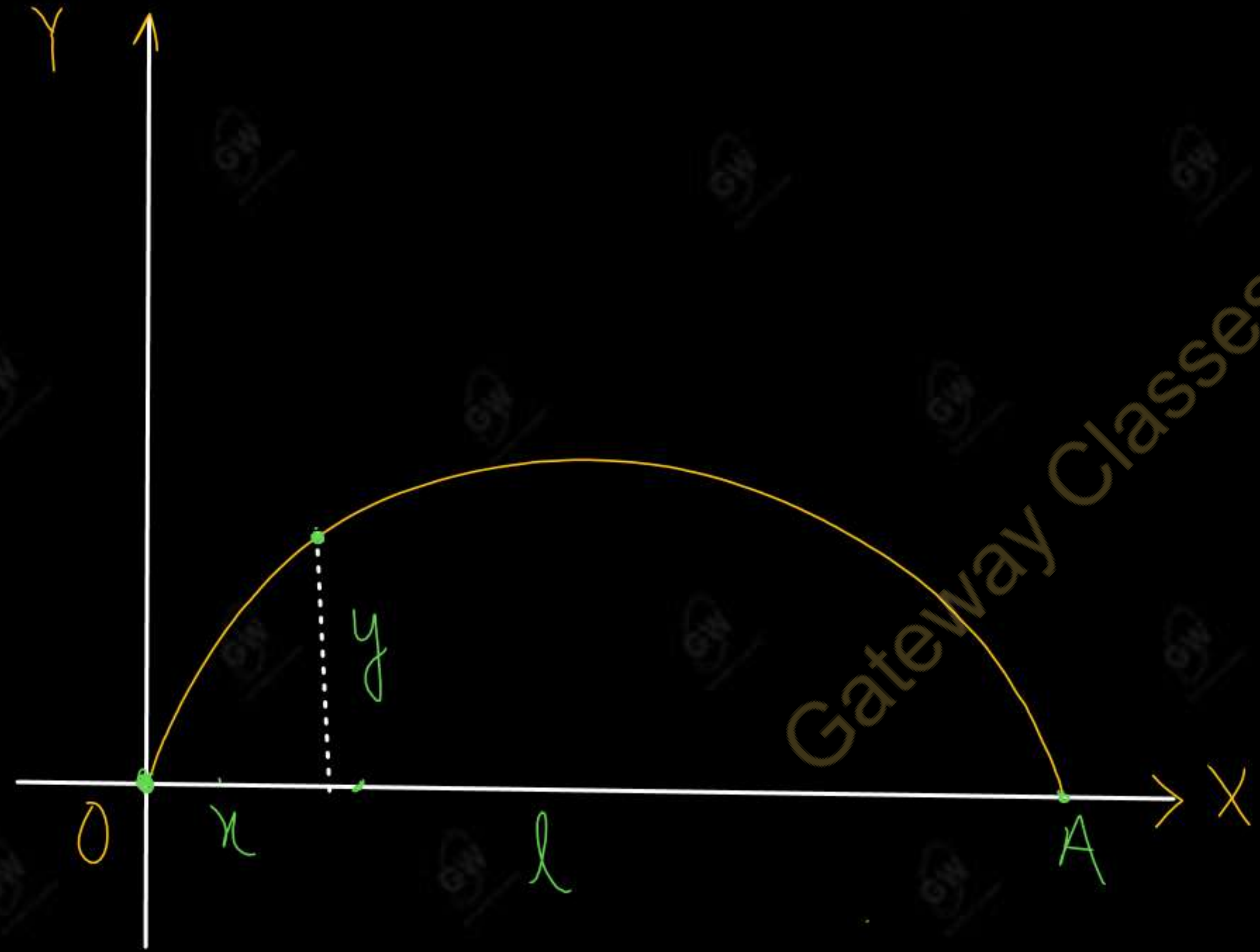
OR

Equation of vibrating string

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

Vibrations of a stretched string





Boundary conditions

(i) At $x = 0$, $y = 0$

(ii) At $x = l$, $y = 0$

Initial condition

At $t = 0$, $y = f(x)$

At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Q.1. Solve one dimensional wave equation by using Method of separation of variables.

one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} \; (1)$$

Let $y = X(x)T(t)$

$$y = XT \quad \text{——} \; (2)$$

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$XT'' = c^2 TX''$$

Divide by XT

$$\frac{\cancel{X}T''}{\cancel{X}T} = c^2\frac{\cancel{T}X''}{X\cancel{T}}$$

$$\frac{T''}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{---} \ (3)$$

$$\frac{1}{c^2}\frac{T''}{T} = K$$

$$T'' = c^2KT$$

$$T'' - c^2KT = 0$$

$$D^2T - c^2KT = 0$$

$$(D^2 - c^2K)T = 0 \quad \text{---} \ (4)$$

CASE-1: $K = 0$

From (3)

$$D^2X = 0$$

Aux Eq^n

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2 x$$

From (4)

$$D^2T = 0$$

Aux. Eqn

$$D^2 = 0$$

$$D = 0, 0$$

$$T = c_3 + c_4 t$$

Put X and T in (2)

$$\boxed{y = (c_1 + c_2 x)(c_3 + c_4 t)} \quad \text{—— (5)}$$

CASE-2: $K = p^2$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

From (4)

$$(D^2 - c^2p^2)T = 0$$

$$D^2 - c^2p^2 = 0$$

$$D = \pm cp$$

$$T = c_3 e^{cpt} + c_4 e^{-cpt}$$

Put X and T in (2)

$$\boxed{y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})} \quad \text{—— (6)}$$

CASE -3 : $K = -p^2$

$$(D^2+p^2)X = 0$$

$$D^2+p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$X = c_1 \cos px + c_2 \sin px$$

From (4)

$$(D^2+c^2p^2)T = 0$$

$$D^2+c^2p^2 = 0$$

$$D = \sqrt{-c^2p^2}$$

$$D = \pm icp$$

$$D = 0 \pm icp$$

$$T = c_3 \cos cpt + c_4 \sin cpt$$

Put X and T in (2)

$$\boxed{y = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos cpt + c_4 \sin cpt)}$$

(7)

Most suitable solution

$$\boxed{y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)} \quad \text{— (8)}$$

Apply Boundary condition

(i) At $x = 0$, $y = 0$ ✓

(ii) At $x = l$, $y = 0$

$$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$$

$$\boxed{c_1 = 0}$$

Updated solution

$$\boxed{y = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)} \quad \text{— (9)}$$

$$0 = c_2 \underline{\sin pl} (c_3 \cos cpt + c_4 \sin cpt)$$

✓

$$\sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

Initial conditions

(i) At $t=0$, $y=f(x)$

(ii) At $t=0$, $\frac{\partial y}{\partial t}=0$

Updated solution

$$\boxed{y = c_2 \sin\frac{n\pi}{l}x\left(c_3\cos\frac{cn\pi}{l}t + c_4\sin\frac{cn\pi}{l}t\right)} \quad \text{— (10)}$$

$$\frac{\partial y}{\partial t} = c_2\sin\frac{n\pi x}{l}\left(-c_3\sin\frac{cn\pi t}{l}\times\frac{cn\pi}{l} + c_4\cos\frac{cn\pi t}{l}\times\frac{cn\pi}{l}\right)$$

$$0 = c_2\sin\frac{n\pi x}{l}\left(0 + c_4\times\frac{cn\pi}{l}\right)$$

$$0 = c_2 c_4\frac{nc\pi}{l}\sin\frac{n\pi x}{l}$$

$$\Rightarrow c_4 = 0$$

Updated solution

$$y = c_2\sin\frac{n\pi x}{l}\, c_3\cos\frac{cn\pi t}{l}$$

$$y = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{nc\pi t}{l}$$

Most General solution

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{nc\pi t}{l}$$

At $t=0$, $y = f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

$$b_n = ?$$

Comparison

Fourier series

(Half range sine series)

DPP- 3

Topic : Solution of one Dimensional Wave Equation

Q.1. *Derive one dimensional wave equation.*

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 2)
- PYQ
- DPP

Q.1 : Find the deflection $y(x,t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x, \quad 0 \le x \le 1$$

(G.B.T.U. 2013)

The equation of string

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

length of string

$l = 1$ unit

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = 1$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ IC-1

At $t = 0$, $y = \sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x$

IC-2

Most Suitable solution

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (8)}$$

Apply BC-1

$$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$$

$$c_1 = 0$$

Updated solution

$$y = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (9)}$$

Apply BC-2

$$0 = c_2 \sin p (c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \sin p = 0$$

$$\sin p = \sin n\pi$$

$$p = n\pi$$

Updated solution

$$y = c_2 \sin n\pi x\left(c_3 \cos n\pi c t + c_4 \sin n\pi c t\right) \quad \text{---} (10)$$

$$\frac{\partial y}{\partial t} = c_2 \sin n\pi x\left(-c_3 \times n\pi c \sin n\pi c t + c_4 \times n\pi c \cos n\pi c t\right)$$

Apply I C-1

$$0 = c_2 \sin n\pi x\left(0 + n\pi c\, c_4\right)$$

$$0 = c_2 c_4 n\pi c \sin n\pi x$$

$$\Rightarrow c_4 = 0$$

Updated solution

$$y = c_2 \sin n\pi x \times c_3 \cos n\pi c t$$

$$y = c_2 c_4 \sin n\pi x \cos n\pi c t$$

Most General solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos n\pi c t \quad \text{— (11)}$$

Apply I C-2

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + b_4 \sin 4\pi x + b_5 \sin 5\pi x + \cdots$$

By comparison

$b_1 = 1$

$b_2 = 0$

$b_3 = \frac{1}{3}$

$b_4 = 0$

$b_5 = \frac{1}{5}$

From (11)

$$y(x,t) = b_1 \sin \pi x \cos \pi ct + b_2 \sin 2\pi x \cos 2\pi ct + b_3 \sin 3\pi x \cos 3\pi ct + b_4 \sin 4\pi \cos 4\pi ct + b_5 \sin 5\pi x \cos 5\pi ct$$

$$\boxed{y(x,t) = \sin \pi x \cos \pi ct + \frac{1}{3} \sin 3\pi x \cos 3\pi ct + \frac{1}{5} \sin 5\pi x \cos 5\pi ct}$$

Q.2. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$.
Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$$

(AKTU-2018, 2013; UKTU-2011, 2012,2022)

The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

length of string

$l = l$

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $\frac{\partial y}{\partial t} = 0$ IC-1

At $t = 0$, $y = A \sin \frac{\pi x}{l}$ IC-2

Most Suitable solution

$$y=(c_1\cos px+c_2\sin px)(c_3\cos cpt+c_4\sin cpt) \quad \text{--- (8)}$$

Apply BC-1

$$0=c_1(c_3\cos cpt+c_4\sin cpt)$$

$$\Rightarrow c_1=0$$

Updated solution

$$y=c_2\sin px\,(c_3\cos cpt+c_4\sin cpt) \quad \text{--- (9)}$$

Apply BC-2

$$0=c_2\sin pl\,(c_3\cos cpt+c_4\sin cpt)$$

$$\Rightarrow \sin pl=0$$

$$\sin pl=\sin n\pi$$

$$pl=n\pi$$

$$p=\frac{n\pi}{l}$$

Updated Solution

$$y = c_2 \sin\frac{n\pi}{l}x\left(c_3 \cos\frac{n\pi c t}{l} + c_4 \sin\frac{n\pi c t}{l}\right) \quad \text{—— (10)}$$

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \times \frac{n\pi c}{l} \sin\frac{n\pi c t}{l} + c_4 \times \frac{n\pi c}{l} \cos\frac{n\pi c t}{l}\right)$$

Apply IC-1

$$0 = c_2 \sin\frac{n\pi x}{l} \times c_4 \times \frac{n\pi c}{l}$$

$$\Rightarrow c_4 = 0$$

Updated solution

$$y = c_2 \sin\frac{n\pi x}{l} \times c_3 \cos\frac{n\pi ct}{l}$$

$$y = c_2 c_4 \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

Most General solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \quad \text{(11)}$$

Apply IC-2

$$A \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$A \sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots$$

on comparison

$$b_1 = A$$

From (11)

$$y(x,t) = b_1 \sin\frac{\pi x}{l} \cos\frac{\pi c t}{l} + \cdots$$

$$\boxed{y(x,t) = A \sin\frac{\pi x}{l} \cos\frac{\pi c t}{l}}$$

Hence proved

DPP- 4

Topic : Solution of one Dimensional Wave Equation (Part -2)

Q.1. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $sin^3\left(\frac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$ *(U.P.T.U. 2014)*

Q.2 A string *is* stretched and fastened to two points distance l apart. Find the displacement $y(x,t)$ at any point at a distance x from one end at time t given that:

$$siny(x,0) = A\,sin\left(\frac{2\pi x}{l}\right)$$

(M.T.U. 2013)

Q.3 A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0\, sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$.

(GBTU-2011,2020)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Solution of one Dimensional Wave Equation (Part – 3)
- PYQ
- DPP

Q.1 : A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t. (MTU-2011)

$t=0$

$t=0$

The equation of String

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} \quad (1)$$

Boundary conditions

At $x = 0$, $y = 0$ BC-1

At $x = l$, $y = 0$ BC-2

Initial conditions

At $t = 0$, $y = 0$ IC-1

At $t = 0$, $\frac{\partial y}{\partial t} = \lambda x(l - x)$ IC-2

Let $y = XT$ —— (2)

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} = XT''$$

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2} = TX''$$

Put these values in (1)

$$XT'' = c^2 TX''$$

$$\frac{\cancel{X}T''}{\cancel{X}T} = \frac{c^2 \cancel{T}X''}{X\cancel{T}}$$

$$\frac{T''}{T} = c^2\frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2}\times\frac{T''}{T} = K \text{ (Say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \quad \text{—— (3)}$$

$$\frac{1}{c^2}\times\frac{T''}{T} = K$$

$$T'' = c^2KT$$

$$T'' - c^2KT = 0$$

$$D^2T - c^2KT = 0$$

$$(D^2 - c^2K)T = 0 \quad \text{—— (4)}$$

CASE-1 When $K = 0$

$D^2 X = 0$

$D^2 = 0$

$D = 0, 0$

$CF = c_1 + c_2 x$

$PI = 0$

$X = c_1 + c_2 x$

$D^2 T = 0$

$D^2 = 0$

$D = 0, 0$

$CF = c_3 + c_4 t$

$PI = 0$

$T = c_3 + c_4 t$

Put X and T in (2)

$$\boxed{y = (c_1 + c_2 x)(c_3 + c_4 t)} \quad \text{— (5)}$$

CASE-2 : When $K = p^2$

$(D^2 - p^2) X = 0$

$D^2 - p^2 = 0$

$D = \pm p$

$C.F = c_1 e^{px} + c_2 e^{-px}$

$PI = 0$

$X = c_1 e^{px} + c_2 e^{-px}$

$(D^2 - c^2 p^2) T = 0$

$D^2 - c^2 p^2 = 0$

$D = \pm cp$

$CF = c_3 e^{cpt} + c_4 e^{-cpt}$

$PI = 0$

$T = c_3 e^{cpt} + c_4 e^{-cpt}$

Put X and T in (2)

$$\boxed{y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})} \quad \text{— (6)}$$

CASE-3: When $k = -p^2$

$(D^2+p^2)X = 0$

$D^2+p^2 = 0$

$D = \sqrt{-p^2}$

$D = \pm ip$

$D = 0 \pm ip$

$X = c_1\cos px + c_2\sin px$

$(D^2+c^2p^2)T = 0$

$D^2+c^2p^2 = 0$

$D = \sqrt{-c^2p^2}$

$D = \pm icp$

$D = 0 \pm icp$

$T = c_3\cos cpt + c_4\sin cpt$

Put X and T in (2)

$$\boxed{y = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt)} \quad (7)$$

Most suitable solution from solutions (5), (6) and (7)

$$y = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt) \quad (8)$$

Apply BC-1

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow c_1 = 0$$

Updated Solution

$$y = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \text{--- (9)}$$

Apply BC-2

$$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Updated solution

$$y = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \quad \text{--- (10)}$$

Apply IC-1

$$0 = c_2 \sin\frac{n\pi x}{l} \times c_3$$

$$\Rightarrow c_3 = 0$$

Updated Solution

$$y = c_2 \sin\frac{n\pi x}{l} \times c_4 \sin\frac{n\pi ct}{l} \quad (11)$$

Most General Solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l} \quad (12)$$

Where $b_n = c_2 c_4$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \cos\frac{n\pi ct}{l} \times \frac{n\pi c}{l}$$

Apply IC - 2

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \frac{n\pi c}{l}$$

$$\lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$$

$$\lambda x(l-x) = \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{l}$$

Half range sine series

$$f(x) = \sum b_n \sin\frac{n\pi x}{l}$$

Where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{\pi c}{l} n b_n = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\int_0^l \underset{\mathrm{I}}{(lx-x^2)}\,\underset{\mathrm{II}}{\sin\frac{n\pi x}{l}}\,dx$$

$$\boxed{\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots}$$

Formula

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[(lx-x^2)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{\left(\cos\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[0 - 0 - \frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} - \left(0 - 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$\frac{\pi c}{l} n b_n = \frac{2\lambda}{l}\left[-\frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} + \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$\frac{\pi c}{\cancel{l}} n b_n = \frac{2\lambda}{\cancel{l} \times \left(\frac{n\pi}{l}\right)^3}\left[2 - 2\cos n\pi\right]$$

$$b_n = \frac{2\lambda \times l^3}{\pi c n \times n^3 \pi^3} \times 2\left(1 - \cos n\pi\right)$$

$$b_n = \frac{4\lambda l^3}{c n^4 \pi^4}\left[1 - (-1)^n\right]$$

From (11)

$$\boxed{y = \sum_{n=1}^{\infty} \frac{4\lambda l^3}{c n^4 \pi^4}\left[1 - (-1)^n\right] \sin\frac{n\pi x}{l} \sin\frac{n\pi c t}{l}}$$

$$y(x,t) = \frac{4\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}\left[1-(-1)^n\right] \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

If n is even

$$y(x,t) = 0$$

If n is odd

$$\boxed{y(x,t) = \frac{8\lambda l^3}{c\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}}$$

DPP- 5

Topic : Solution of one Dimensional Wave Equation

Q.1. Show how the wave equation $c^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}$ *Can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between $x=0$ and $x=l$ are given by $y=f(x)$ and $\frac{\partial y}{\partial t}=g(x)$, determine the constants in the series solution. .* *(UPTU 2014)*

Q.2 A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $F(x)=\mu x(l-x), \mu$ is a constant and then released. Find the displacement $y(x,t)$ of any point x of the string at any time $t>0$. *(M.T.U. 2015)*

ENGINEERING MATHEMATICS

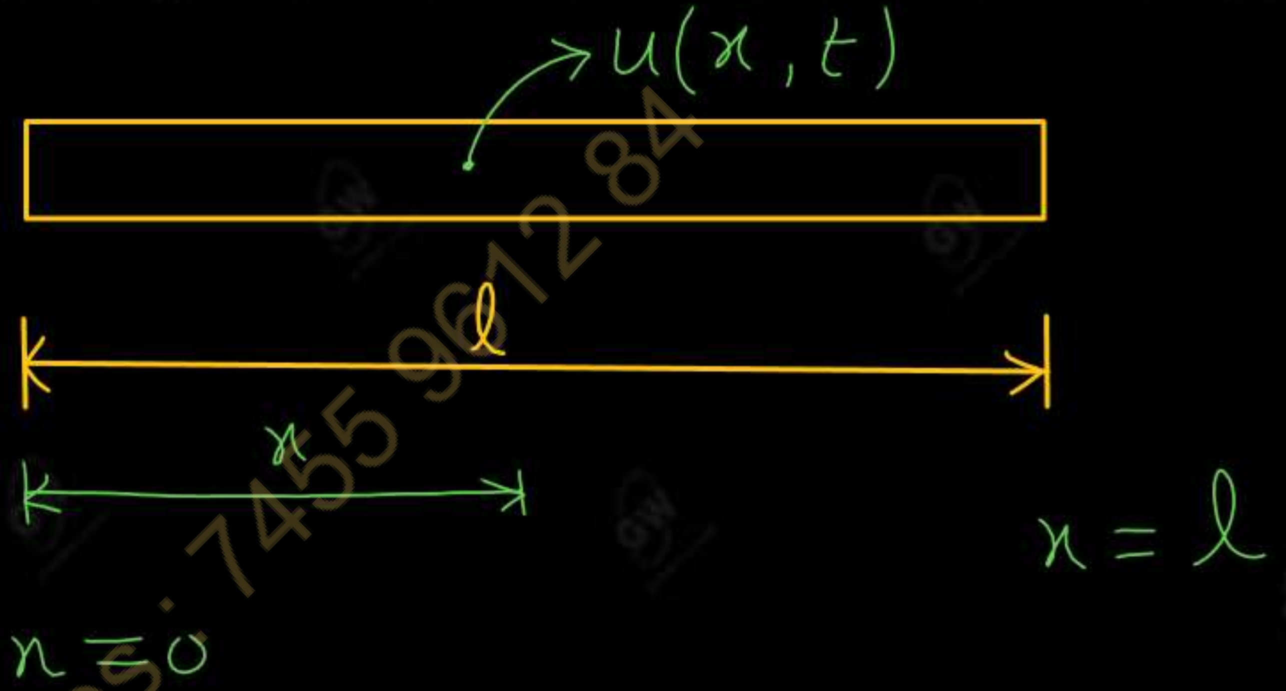
UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 1)
- PYQ
- DPP

One Dimensional Heat Equation

Consider the flow of heat in a homogeneous rod of length l and variation of temperature with position and time.



Assumptions

(i) Rod has uniform cross-section

(ii) Rod has constant density

(iii) Lateral surface of rod is insulated

(iv) Heat flow only in one direction

(v) The rod is sufficiently thin so that the temperature is same at all points of any cross-section area of rod

Let $u(x,t)$ be the temperature distribution of the cross-section at the point x and at any time t

Mathematically

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

This partial differential equation is known as one-dimensional Heat equation

Q. 1:- *Find the solution of One – dimensional heat equation by the method of separation of variables*

One - Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $u = XT$ —— (2)

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = TX''$$

Put these values in (1)

$$XT' = c^2 TX''$$

Divide by XT

$$\frac{\cancel{X}T'}{\cancel{X}T} = \frac{c^2 \cancel{T} X''}{X\cancel{T}}$$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \times \frac{T'}{T} = K \text{ (say)}$$

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

$$D^2X - KX = 0$$

$$(D^2 - K)X = 0 \longrightarrow (3)$$

$$\frac{1}{c^2} \times \frac{T'}{T} = K$$

$$T' = c^2 KT$$

$$DT - c^2KT = 0$$

$$(D - c^2K)T = 0 \longrightarrow (4)$$

CASE-1: When $K = 0$

$$D^2X = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2 x$$

$$DT = 0$$

$$D = 0$$

$$T = c_3 e^{0t}$$

$$T = c_3$$

Put X and T in (2)

$$u(x,t) = (c_1 + c_2 x) c_3$$

$$u(x,t) = c_1 c_3 + c_2 c_3 x$$

$$\boxed{u(x,t) = a + bx} \quad \text{--- (5)}$$

Time-independent solution

$\Rightarrow$ steady-state solution

CASE-2: When $K = p^2$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$(D - p^2 c^2)T = 0$$

$$D - p^2 c^2 = 0$$

$$D = p^2 c^2$$

$$T = c_3 e^{p^2 c^2 t}$$

Put these values in (2)

$$\boxed{u(x,t) = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{p^2 c^2 t}}$$

(6)

CASE-3: When $K = -p^2$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D = \sqrt{-p^2}$$

$$D = \pm ip$$

$$D = 0 + iP$$

$$X = c_1 \cos Px + c_2 \sin Px$$

$$(D + c^2P^2)T = 0$$

$$D + c^2P^2 = 0$$

$$D = -c^2P^2$$

$$T = c_3 e^{-c^2P^2t}$$

Put these values in (2)

$$\boxed{u(x,t) = (c_1 \cos Px + c_2 \sin px)\, c_3 e^{-c^2P^2t}} \quad \text{— (7)}$$

Most Suitable Solution

$$u(x,t) = (c_1 \cos Px + c_2 \sin px)\, c_3 e^{-c^2P^2t} \quad \text{— (8)}$$

Steady state solution

$$u(x,t) = a + bx$$

comple solution

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t} + a + bx}$$

$$u(x,t) = u_t(x,t) + u_s(x)$$

$u_t(x,t) \downarrow$ Transient solution

$u_s(x) \downarrow$ Steady state solution

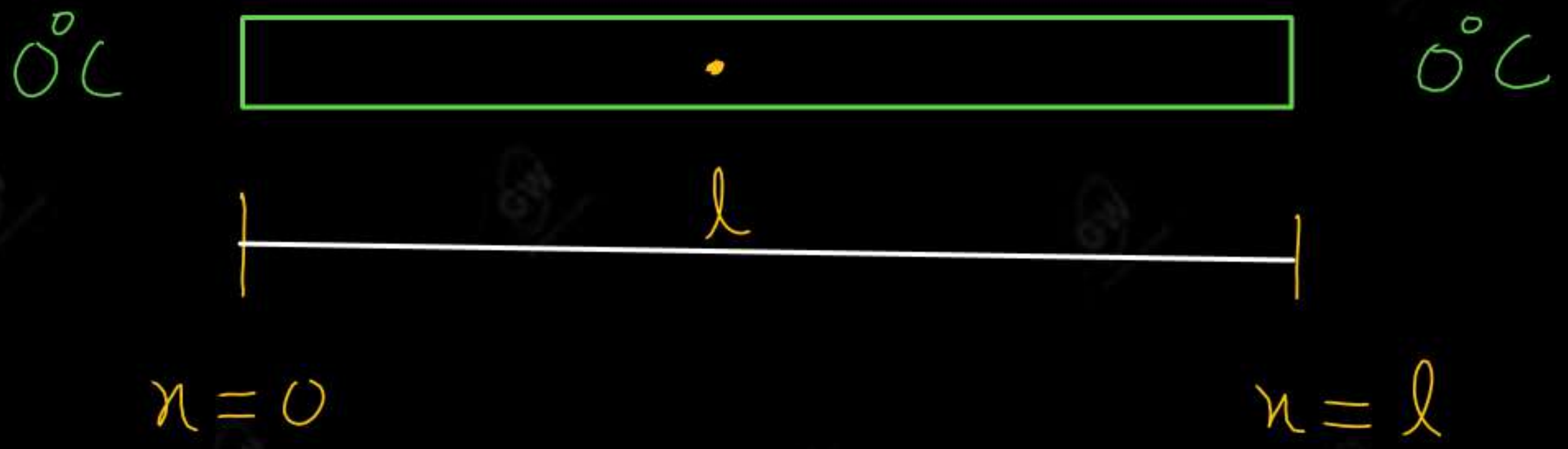
On the Basis of Boundary conditions

Type -1: When both ends of rod are at $0^o C$

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Type -1 : When both ends of rod are at 0^oC



$$u(x,t) = u_s(x) + u_t(x,t)$$

$u_s(x)$ → Steady state

$u_t(x,t)$ → Transient state

Boundary Conditions

At $x = 0$, $u = 0$ $\Rightarrow$ $u(0,t) = 0^oC$ BC − 1

At $x = l$, $u = 0$ $\Rightarrow$ $u(l,t) = 0^oC$ BC − 2

$u_s(x) = a + bx$

Apply BC−1

$0 = a + 0$

$a = 0$

$u_s(x) = bx$

$0 = bl$

$b = 0$

$\Rightarrow \boxed{u_s(x) = 0}$

$u(x,t) = u_t(x,t)$

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t}}$$

Q. 1:- *Find the temperature in a bar of length 2 m whose ends are kept at zero and lateral surface insulated if the initial temperature is* $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$. (U.P.T.U. 2015)

One-Dimensional Heat equation

$$\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——— (1)}$$

Length of rod $(l) = 2\,m$

Boundary conditions

At $x = 0$, $u = 0$ BC-1

At $x = 2$, $u = 0$ BC-2

Initial condition

At $t = 0$, $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Most Suitable solution

$$u(x,t) = (c_1 \cos px + c_2 \sin px)c_3 e^{-c^2p^2t} \quad \text{— (8)}$$

Apply BC-1

$$0 = c_1 c_3 e^{-c^2p^2t}$$

$\Rightarrow c_1 = 0$

Updated solution

$$u(x,t) = c_2 \sin px \times c_3 e^{-c^2p^2t} \quad \text{--- (9)}$$

Apply BC-2

$$0 = c_2 \sin 2p \times c_3 e^{-c^2p^2t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = c_2 \sin \frac{n\pi}{2}x \times c_3 e^{-\frac{c^2n^2\pi^2t}{4}}$$

$$u(x,t) = c_2 c_3 \sin \frac{n\pi}{2}x \times e^{-\frac{c^2n^2\pi^2}{4}t} \quad \text{--- (10)}$$

Most general solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2}x \times e^{-\frac{c^2n^2\pi^2t}{4}} \quad \text{--- (11)}$$

Apply IC

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = b_1 \sin\frac{\pi x}{2} + b_2 \sin\pi x + b_3 \sin\frac{3\pi x}{2} + b_4 \sin 2\pi x + b_5 \sin\frac{5\pi x}{2} + -$$

on comparison

$b_1 = 1$ $\quad$ $b_4 = 0$

$b_2 = 0$ $\quad$ $b_5 = 3$

$b_3 = 0$

From 11

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x \times e^{-\frac{c^2 n^2 \pi^2 t}{4}}$$

$$u(x,t) = b_1 \sin\frac{\pi x}{2} \times e^{-\frac{c^2\pi^2 t}{4}} + b_5 \sin\frac{5\pi x}{2}\, e^{-\frac{c^2 25\pi^2 t}{4}}$$

$$\boxed{u(x,t) = e^{-\frac{c^2\pi^2 t}{4}} \sin\frac{\pi x}{2} + 3e^{-\frac{25c^2\pi^2 t}{4}} \sin\frac{5\pi x}{2}}$$

Q. 2:- Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$. (G.B.T.U. 2012)

One-Dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

length of rod = 2m

Boundary conditions

At $x = 0$, $u = 0$ BC-1

At $x = 2m$, $u = 0$ BC-2

Initial condition

At $t = 0$, $u = f(x) = 100x$

Most suitable solution

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2p^2t} \quad \text{——} \quad (8)$$

Apply BC-1

$$0 = c_1 c_3 e^{-c^2p^2t}$$

$\Rightarrow c_1 = 0$

Updated Solution

$$u(x,t) = c_2 \sin px \times c_3 e^{-c^2p^2t}$$ (9)

Apply BC-2

$$0 = c_2 \sin 2p \times c_3 e^{-c^2p^2t}$$

$\Rightarrow \sin 2p = 0$

$\sin 2p = \sin n\pi$

$2p = n\pi$

$p = \frac{n\pi}{2}$

Updated solution

$$u(x,t) = c_2 \sin \frac{n\pi}{2} x \times c_3 e^{-c^2p^2t}$$ (10)

Most General Solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x \times e^{-c^2p^2t}$$ (11)

Apply I C

$$100x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

Using Half-Range sine-series

$$b_n = \frac{\cancel{2}}{\cancel{2}}\int_0^2 100x \times \sin\frac{n\pi x}{2}\,dx$$

$$b_n = 100\int_0^2 \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{\sin\frac{n\pi x}{2}}\,dx$$

$$b_n = 100\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2$$

$$b_n = 100\left[\frac{-2\cos n\pi}{\frac{n\pi}{2}} - 0 - (0-0)\right]$$

$$b_n = 100\left(\frac{-4\cos n\pi}{n\pi}\right)$$

$$b_n = -\frac{400}{\pi}\left(\frac{\cos n\pi}{n}\right)$$

Put b_n in (11)

$$u(x,t) = \sum_{n=1}^{\infty} \left(-\frac{400}{\pi}\right)\left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2 p^2 t}$$

$$\boxed{u(x,t) = -\frac{400}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{2} \times e^{-c^2 p^2 t}}$$

where $\cos n\pi = (-1)^n$

DPP- 6

Topic : One Dimensional Heat Equation

Q.1. *A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function $u(x,t)$.* (U.P.T.U. 2015)

Q.2. Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0,t) = 0, u(l,t) = 0\ (t > 0)$ and the initial condition $u(x,0) = x$; l being the length of the bar. (U.P.T.U. 2015)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 2)
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional Heat Equation : $\boxed{\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$

On the Basis of Boundary conditions, there are three types of question

Type -1: When both ends of rod are at $0^o C$

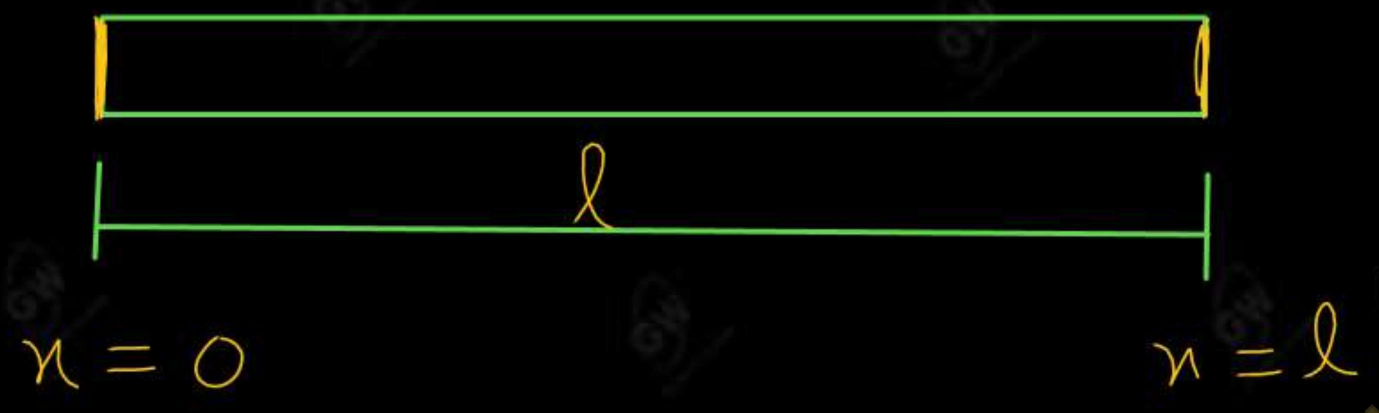
Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Type -2 : When both ends of rod are perfectly insulated

Since ends of rod are insulated, no heat can transfer from either sides.

$$\frac{\partial u}{\partial x}$$



$$u(x,t) = u_s(x) + u_t(x,t)$$

$u_s(x)$ → ***Steady state***

$u_t(x,t)$ → ***Transient state***

Boundary Conditions

At $x = 0$, $\frac{\partial u}{\partial x} = 0$

At $x = l$, $\frac{\partial u}{\partial x} = 0$

$u_s = a + bx$

$\frac{\partial u}{\partial x} = 0 + b$

Apply BC-1

$0 = b$

$$\boxed{u_s = a}$$

$$u(x,t) = a + u_t(x,t)$$

$$\boxed{u(x,t) = a + (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2 p^2 t}}$$

Q. 1:- **Find the temperature distribution in a rod of length π which is totally insulated including the ends and the initial temperature distribution is $100 \cos x$.** (U.P.T.U. 2015)

one-dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \quad (1)$$

length of rod $(l) = \pi$

Boundary conditions

At $x = 0$, $\frac{\partial u}{\partial x} = 0$ BC-1

At $x = \pi$, $\frac{\partial u}{\partial x} = 0$ BC-2

Initial condition

At $t = 0$, $u = 100 \cos x$ IC

complete solution of (1)

$$u(x,t) = a + (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} \quad \text{——} \quad (8)$$

$$\frac{\partial u}{\partial x} = 0 + (-c_1 p \sin px + c_2 p \cos px) c_3 e^{-c^2 p^2 t}$$

Apply BC-1

$$0 = c_2 p \times c_3 e^{-c^2 p^2 t}$$

$\Rightarrow c_2 = 0$

Updated solution

$$u(x,t) = a + (c_1 \cos px)\, c_3 e^{-c^2p^2t} \quad \text{—— (9)}$$

$$\frac{\partial u}{\partial x} = 0 + (-c_1 p \sin px)\, c_3 e^{-c^2p^2t}$$

Apply BC-2

$$0 = -c_1 p \sin \pi p \times c_3 e^{-c^2p^2t}$$

$$\Rightarrow \sin p\pi = 0$$

$$\sin p\pi = \sin n\pi$$

$$p\cancel{\pi} = n\cancel{\pi}$$

$$p = n$$

Updated Solution

$$u = a + (c_1 \cos nx)\, c_3 e^{-c^2n^2t} \quad \text{—— (10)}$$

Most-General solution

$$u = a + \sum_{n=1}^{\infty} b_n \cos nx \times e^{-c^2n^2t} \quad \text{—— (11)}$$

Apply IC

$$100\cos x = a + \sum_{n=1}^{\infty} b_n \cos nx$$

$$100\cos x = a + b_1\cos x + b_2\cos 2x + \ldots$$

By comparison

$a = 0$

$b_1 = 100$

$b_2 = 0$

$b_3 = 0$

$\vdots$

From (11)

$$u(x,t) = a + \sum_{n=1}^{\infty} b_n \cos nx \times e^{-c^2n^2t}$$

$$= a + b_1\cos x \times e^{-c^2t} + b_2\cos 2x \times e^{-c^2\times 2^2\times t}$$

$$u(x,t) = 0 + 100\cos x \times e^{-c^2t}$$

$$\boxed{u(x,t) = 100e^{-c^2t}\cos x}$$

DPP- 7

Topic : One Dimensional Heat Equation

Q.1. The temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Assuming the initial temperature distribution as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time. (M.T.U. 2011)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- One Dimensional Heat Equation (Part – 3)
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation
2. Solution of one Dimensional Heat Equation
3. Solution of two Dimensional heat Equation

One Dimensional Heat Equation : $\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$

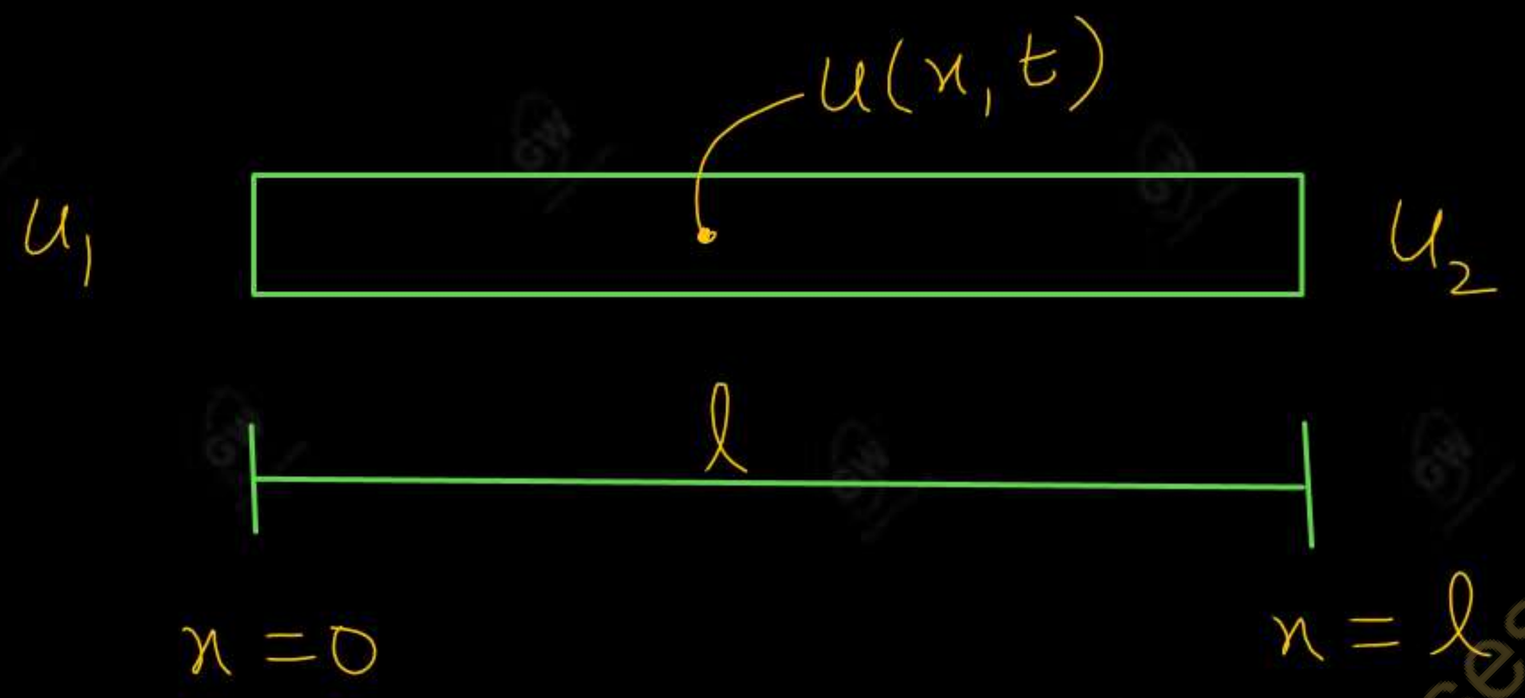
On the Basis of Boundary conditions, there are three types of question

Type -1: When both ends of rod are at $0^o C$

Type -2: When both ends of rod are perfectly insulated

Type -3: When both ends of rod are kept at different temperature.

Type -3 : When both ends of rod are kept at different temperature.



$$\boxed{u_s = a + bx}$$

$$\boxed{u_t = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2 p^2 t}}$$

complete solution

$$u(x,t) = u_s(x) + u_t(x,t)$$

$u_s(x)$ → *Steady state*

$u_t(x,t)$ → *Transient state*

Q. 1: A bar of 10 cm length with insulated sides A and B are kept and 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly varied to 50°C and the same instant at B, lowered at 10°C. Find the subsequent temperature at any point of the bar at any time. *(A.K.T.U. 2017)*

one-dimensional heat equation

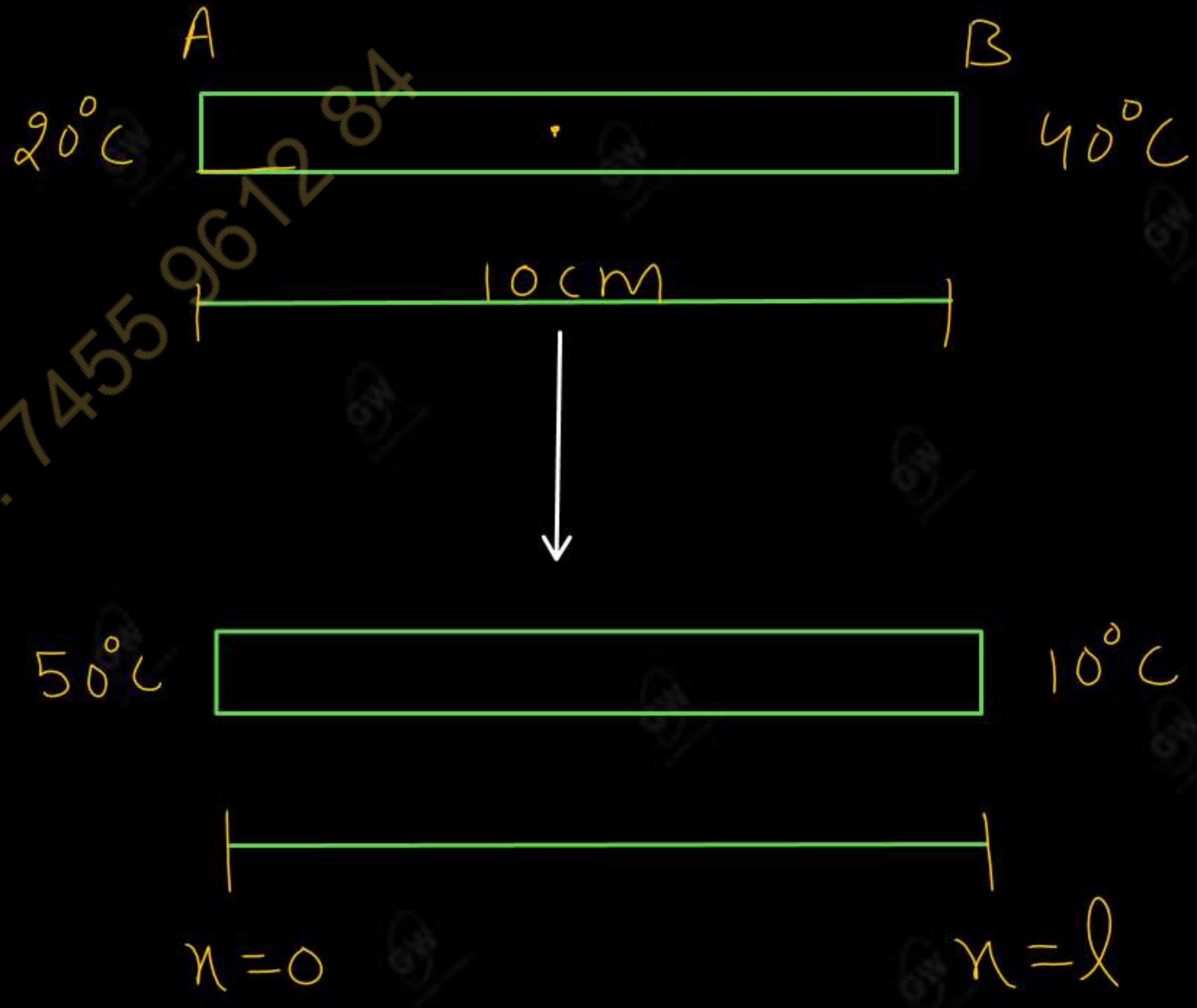
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

length of bar $(l) = 10\,cm$

Boundary conditions

At $x = 0$, $u = 50^\circ$ BC-1

At $x = 10\,cm$, $u = 10^\circ C$ BC-2



Initial condition

$$u = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$u = 20 + \left(\frac{40-20}{10}\right)x$$

$$u = 20 + 2x$$

At $t = 0$, $u = 20 + 2x$

comple solution of (1)

$$u(x,t) = u_s(x) + u_t(x,t)$$

$$u(x,t) = (a + bx) + (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} \quad \text{—— (8)}$$

Steady state solution

$$u_s(x) = T_A + \left(\frac{T_B - T_A}{10}\right)x$$

$$= 50 + \frac{(10-50)}{10}x$$

$$\boxed{u_s(x) = 50 - 4x}$$ Put in (8)

$$u(x,t) = (50-4x) + (c_1\cos px + c_2\sin px)c_3 e^{-c^2p^2t} \quad \text{(9)}$$

Apply BC-1

$$\cancel{50} = \cancel{50} + c_1 \times c_3 e^{-c^2p^2t}$$

$$0 = c_1 c_3 e^{-c^2p^2t}$$

$$\Rightarrow c_1 = 0$$

Updated solution

$$u(x,t) = (50-4x) + c_2\sin px \times c_3 e^{-c^2p^2t} \quad \text{(10)}$$

Apply BC-2

$$10 = 50-40 + c_2 \sin 10p \times c_3 e^{-c^2p^2t}$$

$$\cancel{10} = \cancel{10} + c_2 \sin 10p \times c_3 e^{-c^2p^2t}$$

$$0 = c_2 \sin 10p \times c_3 e^{-c^2p^2t}$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Updated solution

$$u(x,t) = (50-4x) + c_2 \sin\frac{n\pi}{10}x \times c_3 e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{——} \quad (11)$$

Most general solution

$$u(x,t) = (50-4x) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times e^{-\frac{c^2 n^2 \pi^2}{100}t} \quad \text{——} \quad (12)$$

Apply initial condition

$$20+2x = 50-4x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \times 1$$

$$-30+6x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half range sine series

Formula

$$f(x) = \sum b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{\cancel{2}}{\cancel{10}_{5}}\int_0^{10} \underset{\mathrm{I}}{(6x-30)}\,\underset{\mathrm{II}}{\sin\frac{n\pi x}{10}}\,dx$$

$$b_n = \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$b_n = \frac{1}{5}\left[\frac{-30\cos n\pi}{\frac{n\pi}{10}} - 0 - \left(\frac{-30(-1)}{\frac{n\pi}{10}} - 0\right)\right]$$

$$b_n = \frac{1}{5}\left[-\frac{300\cos n\pi}{n\pi} - \frac{300}{n\pi}\right]$$

$$b_n = \frac{-\cancel{300}^{\,60}}{\cancel{5}\,n\pi}\left(\cos n\pi + 1\right)$$

$$b_n = -\frac{60}{n\pi}\left(1+\cos n\pi\right)$$

Put b_n in (12)

$$u(x,t) = (50-4x) + \sum_{n=1}^{\infty} \frac{-60}{n\pi}\left(1+\cos n\pi\right) \times \sin\frac{n\pi x}{10} \times e^{-\frac{c^2 n^2 \pi^2}{100}t}$$

$$u(x,t) = (50-4x) - \frac{60}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(1+\cos n\pi)\sin\frac{n\pi x}{10}\, e^{-\frac{c^2 n^2 \pi^2}{100}t}$$

DPP- 8

Topic : One Dimensional Heat Equation

Q.1. (i) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t. [G.B.T.U. (A.G.) 2011, U.K.T.U. 2011]

(ii) Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-1
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3 L
2. Solution of one Dimensional Heat Equation 3 L
3. Solution of two Dimensional heat Equation 2 L

$\boxed{\text{Method of Separation of Variables}}$ 2 L

Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\longrightarrow \frac{\partial u}{\partial t} = 0$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Known as Laplace equation

Solution of Laplace equation in two dimension by using Separation of Variables Method.

GW GATEWAY CLASSES

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——} \quad (1)$$

Let $u(x,y)=XY$ —— (2)

$$\frac{\partial^2 u}{\partial x^2}=Y\frac{d^2X}{dx^2}=YX''$$

$$\frac{\partial^2 u}{\partial y^2}=X\frac{d^2Y}{dy^2}=XY''$$

Put these values in (1)

$$YX''+XY''=0$$

Divide by XY

$$\frac{\cancel{Y}X''}{X\cancel{Y}}+\frac{\cancel{X}Y''}{\cancel{X}Y}=0$$

$$\frac{X''}{X}+\frac{Y''}{Y}=0$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=K\text{(say)}$$

$$\frac{X''}{X}=K$$

$$X''=KX$$

$$X''-KX=0$$

$$D^2X-KX=0$$

$$(D^2-K)X=0 \quad \text{——} \quad (3)$$

$$-\frac{Y''}{Y}=K$$

$$-Y'' = KY$$

$$Y'' + KY = 0$$

$$D^2Y + KY = 0$$

$$(D^2 + K)Y = 0 \quad \text{—— (4)}$$

CASE-1 : When $K = 0$

$$(D^2 - K)X = 0$$

$$D^2X = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$X = c_1 + c_2 x$$

$$(D^2 + K)Y = 0$$

$$D^2Y = 0$$

$$D^2 = 0$$

$$D = 0, 0$$

$$Y = c_3 + c_4 y$$

Put x and y in (2)

$$\boxed{u(x, y) = (c_1 + c_2 x)(c_3 + c_4 y)} \quad \text{—— (5)}$$

CASE-2: When $k = p^2$

$$(D^2 - k)X = 0$$

$$(D^2 - p^2)X = 0$$

$$D^2 - p^2 = 0$$

$$D^2 = p^2$$

$$D = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$(D^2 + k)Y = 0$$

$$(D^2 + p^2)Y = 0$$

$$D^2 + p^2 = 0$$

$$D = \pm ip$$

$$D = 0 \pm ip$$

$$Y = c_3 \cos py + c_4 \sin py$$

Put X and Y in (2)

$$\boxed{u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)} \quad (6)$$

CASE-3: When $k = -p^2$

$$(D^2 - k)X = 0$$

$$(D^2 + p^2)X = 0$$

$$D^2 + p^2 = 0$$

$$D^2 = -p^2$$

$D = \sqrt{-p^2}$

$D = 0 \pm ip$

$X = c_1 \cos px + c_2 \sin px$

$(D^2 + K)Y = 0$

$(D^2 - p^2)Y = 0$

$D^2 - p^2 = 0$

$D^2 = p^2$

$D = \pm p$

$Y = c_3 e^{py} + c_4 e^{-py}$

Put these values in (2)

$$\boxed{u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})}$$

(7)

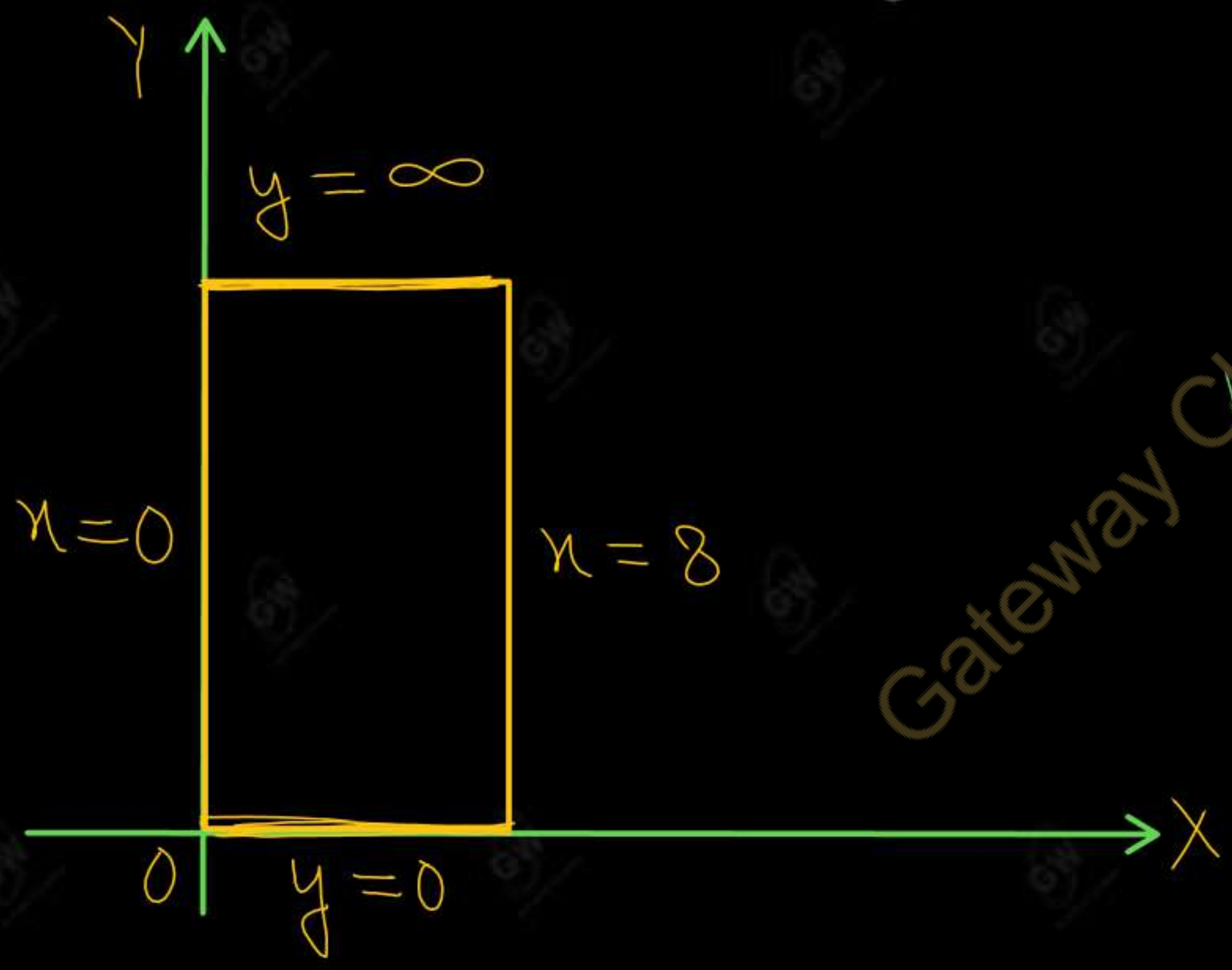
Solution of Laplace Equation

$$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y) \quad \text{—— (5)} \qquad X \qquad K = 0$$

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{—— (6)} \qquad K = p^2$$

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{—— (7)} \qquad K = -p^2$$

Q. 1: A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}, 0 < x < 8$ *while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0 ⁰ C, show that the steady state temperature at any point of the plate is given by* $u(x, y) = 100\, e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}$.



Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Where $u(x, y)$ is steady state temp of plate

Under the Boundary condition

$u(0,y) = 0$ BC-1

$u(8,y) = 0$ BC-2

$u(x,\infty) = 0 \Rightarrow \lim_{y\to\infty} u(x,y) = 0$ BC-3

$u(x,0) = 100 \sin\frac{\pi x}{8}$ BC-4

Solution of Laplace equation

$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$ — (5)

$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$ — (6)

$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ — (7)

Most suitable solution

$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ — (8)

Apply BC-1

$$0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow c_1 = 0$$

Updated solution

$$u(x,y) = c_2 \sin px \, (c_3 e^{py} + c_4 e^{-py})$$

9

Apply BC-2

$$0 = c_2 \sin 8p \, (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \sin 8p = 0$$

$$\sin 8p = n\pi$$

$$8p = n\pi$$

$$p = \frac{n\pi}{8}$$

Updated solution

$$u(x,y) = c_2 \sin \frac{n\pi}{8} x \left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}} \right)$$

10

Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} c_2 \sin\frac{n\pi x}{8}\left(c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}}\right)$$

$$0 = c_2 \sin\frac{n\pi x}{8}\left[\lim_{y\to\infty} c_3 e^{\frac{n\pi y}{8}} + \lim_{y\to\infty} c_4 e^{-\frac{n\pi y}{8}}\right]$$

$$\Rightarrow c_3 = 0$$

Updated solution

$$u(x,y) = c_2 \sin\frac{n\pi x}{8} \times c_4 e^{-\frac{n\pi y}{8}} \quad \text{--- } (11)$$

Most General solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}\, e^{-\frac{n\pi y}{8}}$$

Apply BC-4

$$100\sin\frac{\pi x}{8} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}$$

$$100\sin\frac{\pi x}{8} = b_1 \sin\frac{\pi x}{8} + b_2 \sin\frac{2\pi x}{8} + \cdots$$

on comparison

$$b_1 = 100$$

$$n = 1$$

$$u(x,y) = b_1 \sin\frac{\pi x}{8}\, e^{-\frac{\pi y}{8}}$$

$$u(x,y) = 100\sin\frac{\pi x}{8}\, e^{-\frac{\pi y}{8}}$$

$$\boxed{u(x,y) = 100\, e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}}$$

DPP- 9

Topic : Two Dimensional Heat Equation (Laplace Equation)

Part-1

Q.1. Solve temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < \pi, 0 < y < \pi,$$ which satisfies the conditions :

$u(0,y) = u(\pi, y) = u(x, \pi) = 0$ and $u(x, 0) = sin^2 x$ (U.K.T.U. 2011)

ENGINEERING MATHEMATICS

UNIT : Application of partial differential equations and Fourier Transform

Today's Target

- Two Dimensional Heat Equation (Laplace Equation) Part-2
- PYQ
- DPP

Application of partial differential equations

1. Solution of one Dimensional Wave Equation 3L
2. Solution of one Dimensional Heat Equation 3L
3. Solution of two Dimensional heat Equation 2L

$\boxed{\text{Method of separation of variables}}$ 2L

Two Dimensional Heat Equation (Laplace Equation)

(i) Two Dimensional Heat Equation

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

(ii) Two Dimensional Heat Equation in steady state (Laplace Equation)

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$ Known as Laplace equation

Solution of Laplace equation in two dimension by using Separation of Variables Method.

1. $$\boxed{u(x,y) = (c_1 + c_2 x)(c_3 e^{py} + c_4 e^{-py})} \quad \text{--- (5)}$$

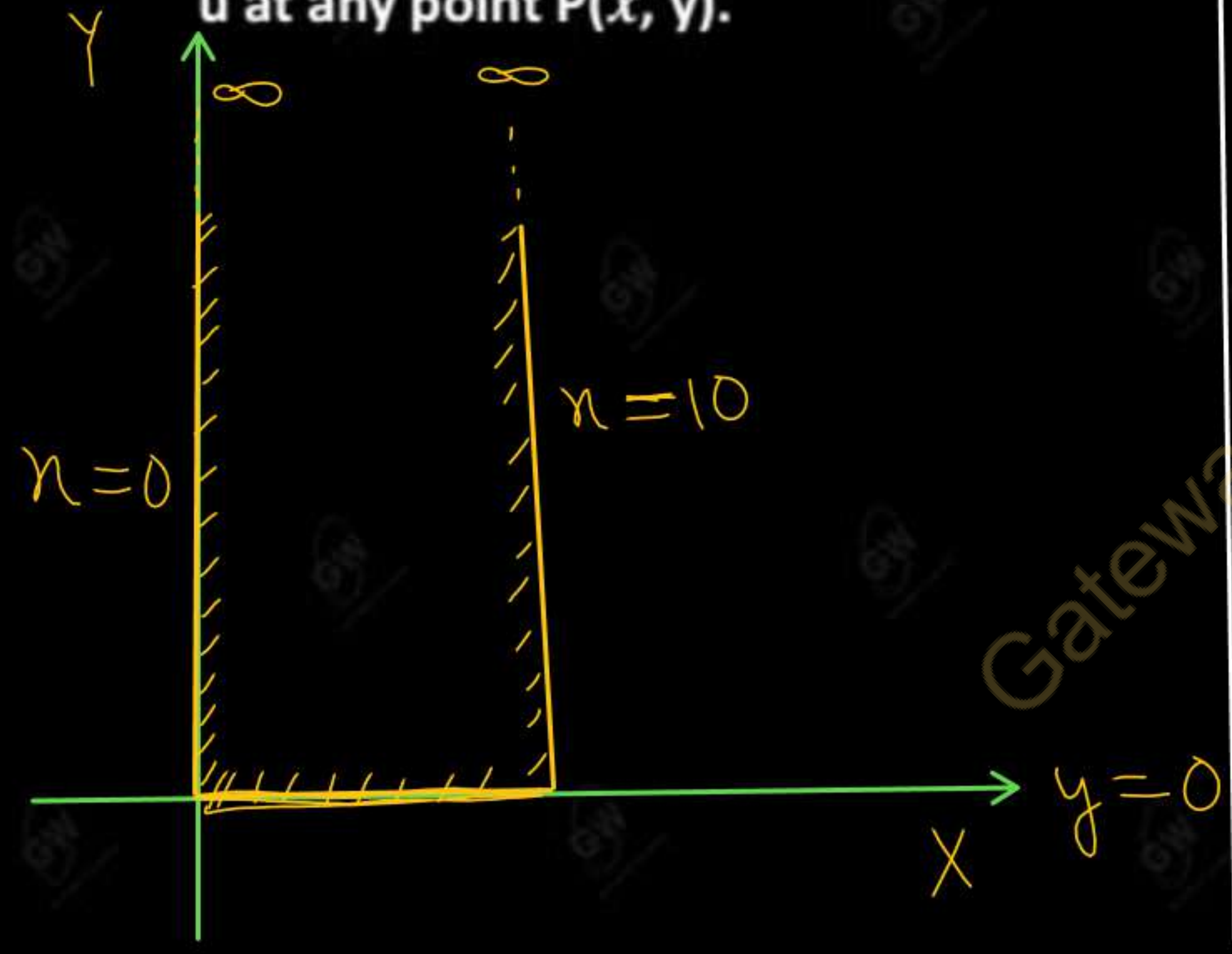
2. $$\boxed{u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)} \quad \text{--- (6)}$$

3. $$\boxed{u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})} \quad \text{--- (7)}$$

Q.1 A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. IF the temperature along the short edge y = 0 is given by

And $$u(x, y) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$$

And the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y). **(A.K.T.U. 2018)**



Let u(x, y) be the steady state temp. given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

$u(0,y) = 0$ BC-1

$u(10,y) = 0$ BC-2

$\lim_{y\to\infty} u(x,y) = 0$ BC-3

$$u(x,0) = \begin{cases} 20x & 0<x\le 5 \\ 20(10-x) & 5<x<10 \end{cases}$$ BC-4

Solution of eqn 1

$$u(x,y) = (c_1\cos px + c_2\sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{— 7}$$

<u>Apply BC-1</u>

$$0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow c_1 = 0$$

Updated solution

$$u(x,y) = c_2 \sin px \,(c_3 e^{py} + c_4 e^{-py}) \quad \text{— 8}$$

<u>Apply BC-2</u>

$$0 = c_2 \sin 10p\,(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

Updated solution

$$u(x,y) = c_2 \sin\frac{n\pi x}{10}\left(c_3 e^{\frac{n\pi y}{10}} + c_4 e^{\frac{-n\pi y}{10}}\right) \quad (9)$$

Apply BC-3

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} c_2 \sin\frac{n\pi x}{10}\left(c_3 e^{\frac{n\pi y}{10}} + c_4 e^{\frac{-n\pi y}{10}}\right)$$

$$0 = c_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} c_3 e^{\frac{n\pi y}{10}} + \lim_{y\to\infty} c_4 e^{\frac{-n\pi y}{10}}\right]$$

$$0 = c_2 \sin\frac{n\pi x}{10}\left[\lim_{y\to\infty} c_3 e^{\frac{n\pi y}{10}}\right]$$

$$\Rightarrow c_3 = 0$$

Updated solution

$$u(x,y) = c_2 \sin\frac{n\pi x}{10} \times c_4 e^{\frac{-n\pi y}{10}} \quad (10)$$

$$u(x,y) = c_2 c_4 \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}} \quad \text{--- (11)}$$

Most General solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \times e^{-\frac{n\pi y}{10}} \quad \text{--- (12)}$$

Apply BC-4

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Apply Half Range sine series

$$b_n = \frac{2}{l}\int_0^l u(x,0) \sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{10}\int_0^{10} u(x,0) \sin\frac{n\pi x}{10}\,dx$$

$$b_n = \frac{1}{5}\left[\int_0^5 u(x,0) \sin\frac{n\pi x}{10}\,dx + \int_5^{10} u(x,0) \sin\frac{n\pi x}{10}\,dx\right.$$

$$b_n = \frac{1}{5}\int_0^5 20x \sin\frac{n\pi x}{10}\,dx + \frac{1}{5}\int_5^{10} 20(10-x) \sin\frac{n\pi x}{10}\,dx$$

$$b_n = \frac{20}{5}\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \frac{20}{5}\int_5^{10} (10-x)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = 4\left[\frac{x\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - 1\times\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + 4\left[\frac{(10-x)\left(-\cos\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - (-1)\frac{\left(-\sin\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}$$

$$b_n = 4\left[\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} + \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right] + 4\left[0 + 0 - \left(\frac{-5\cos\frac{n\pi}{2}}{\frac{n\pi}{10}} - \frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]$$

$$b_n = 4\left(-\frac{50}{n\pi}\cos n\pi + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + 4\left(\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$b_n = \cancel{-\frac{200}{n\pi}\cos\frac{n\pi}{2}} + \frac{400}{\pi^2 n^2}\sin\frac{n\pi}{2} + \cancel{\frac{200}{n\pi}\cos\frac{n\pi}{2}} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

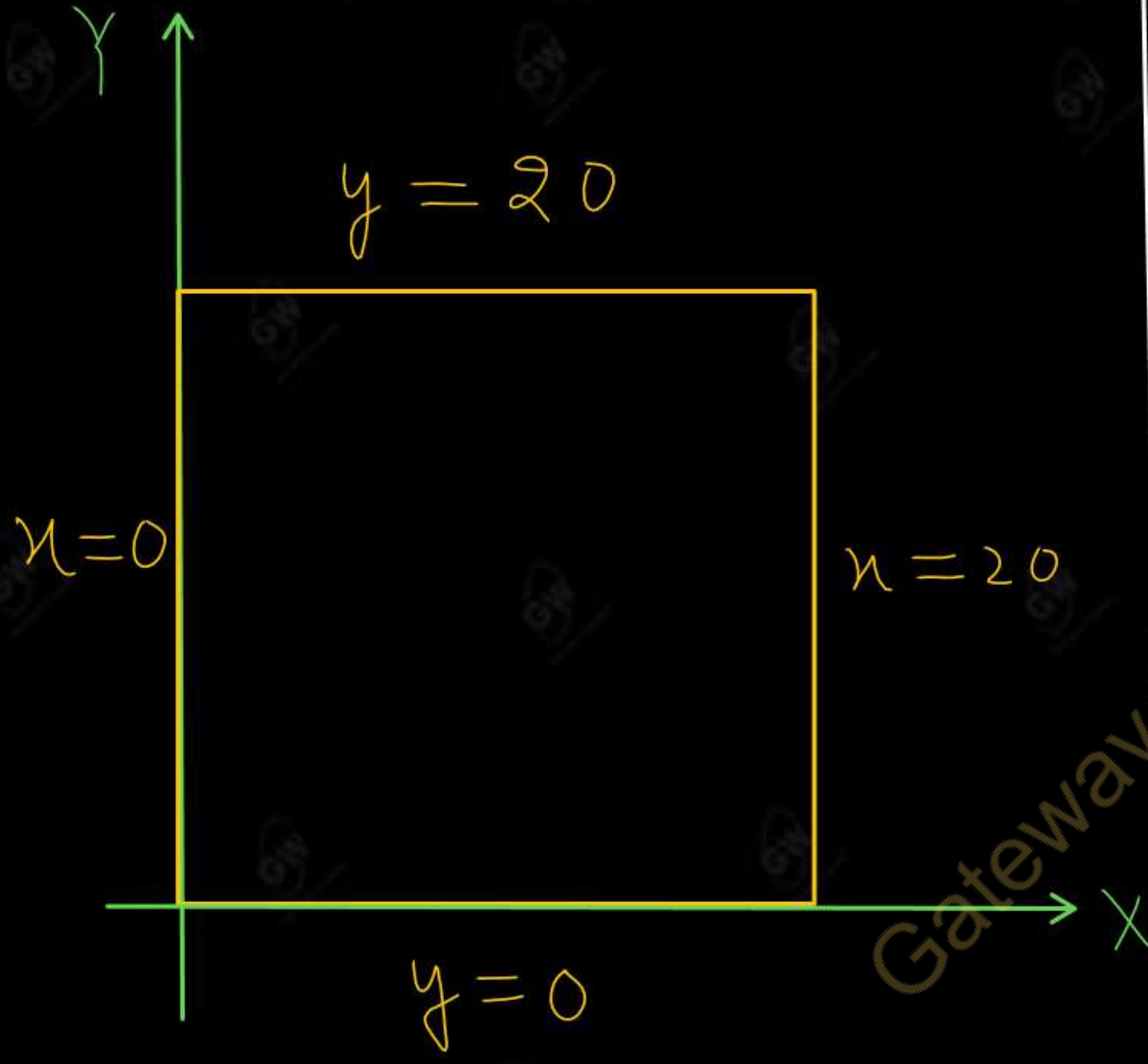
$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Put b_n in (12)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

$$\boxed{u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}}$$

Q.2:- A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept 0°C. Find the steady state temperature in the plate. **(A.K.T.U. 2014, 2017)**



Let $u(x,y)$ be the steady state temp given by Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Under the Boundary conditions

$$u(0,y) = 0 \quad BC\text{-}1$$

$$u(x,0) = 0 \quad BC\text{-}2$$

$u(20, y) = 0$ BC-3

$u(x, 20) = x(20-x)$ BC-4

Solution of eqn (1)

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{--- (7)}$$

Apply BC-1

$$0 = c_1 (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow c_1 = 0$$

Updated Solution

$$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \text{--- (8)}$$

Apply BC-2

$$0 = c_2 \sin px (c_3 + c_4)$$

$$\Rightarrow c_3 + c_4 = 0$$

$$c_4 = -c_3$$

Updated Solution

$$u(x,y) = c_2 \sin px (c_3 e^{py} - c_3 e^{-py})$$

$$u(x,y) = c_2 c_3 \sin px \left(e^{py} - e^{-py}\right) \quad \text{—— (9)}$$

Apply BC - 3

$$0 = c_2 c_3 \sin 20p \left(e^{py} - e^{-py}\right)$$

$$\Rightarrow \sin 20p = 0$$

$$\sin 20p = \sin n\pi$$

$$20p = n\pi$$

$$p = \frac{n\pi}{20}$$

Updated solution

$$u(x,y) = c_2 c_3 \sin \frac{n\pi}{20} x \left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right) \quad \text{—— (10)}$$

Most General solution

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}\right) \quad \text{—— (11)}$$

Apply BC - 4

$$x(20-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \left(e^{n\pi} - e^{-n\pi}\right)$$

$$(20x - x^2) = \sum_{n=1}^{\infty} b'_n \sin\frac{n\pi x}{20}$$

Apply Half range sine-series

$$b'_n = \frac{2}{l}\int_0^l u(x,20)\sin\frac{n\pi x}{l}\,dx$$

$$b'_n = \frac{\cancel{2}}{\cancel{20}}\int_0^{20} \underset{\mathrm{I}}{(20x-x^2)}\,\underset{\mathrm{II}}{\sin\frac{n\pi x}{20}}\,dx$$

$$b'_n = \frac{1}{10}\left[(20x-x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - (20-2x)\left(\frac{-\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}\right) + (-2)\left(\frac{+\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}\right)\right]_0^{20}$$

$$b_n' = \frac{1}{10}\left[0+0-\frac{2\cos n\pi}{\left(\frac{n\pi}{20}\right)^3} - \left(0+0-\frac{2}{\left(\frac{n\pi}{20}\right)^3}\right)\right]$$

$$b_n' = \frac{1}{10}\left[-\frac{2\times 8000}{n^3\pi^3}\cos n\pi + \frac{2\times 8000}{n^3\pi^3}\right]$$

$$b_n' = \frac{2\times 800\cancel{0}}{\cancel{10}\times n^3\pi^3}\left[1-\cos n\pi\right]$$

$$b_n' = \frac{1600}{n^3\pi^3}\left[1-(-1)^n\right]$$

When n is even

$$b_n' = 0$$

When n is odd

$$b_n' = \frac{3200}{n^3\pi^3}$$

$$b_n\left(e^{n\pi}-e^{-n\pi}\right) = \frac{3200}{n^3\pi^3}$$

$$2b_n\left(\frac{e^{n\pi}-e^{-n\pi}}{2}\right) = \frac{3200}{n^3\pi^3}$$

$$2 b_n \sinh n\pi = \frac{3200}{n^3 \pi^3}$$

$$\boxed{u(x,y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{20}}{n^3} \frac{\sinh\left(\frac{n\pi y}{20}\right)}{\sinh n\pi}}$$

where n is odd

$$b_n = \frac{1 \times 3200}{2 \sinh n\pi \times n^3 \pi^3}$$

$$\boxed{u(x,y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{20} \times \sinh \frac{(2n-1)\pi y n}{20}}{(2n-1)^3 \sinh (2n-1)\pi}}$$

Put b_n in (11)

$$u(x,y) = \sum_{n=1}^{\infty} \frac{3200}{2 \sinh n\pi \times n^3 \pi^3} \sin \frac{n\pi x}{20} \left(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}} \right)$$

$$u(x,y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{20} \times \cancel{2}}{\cancel{2} n^3 \sinh n\pi} \left(\frac{e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}}}{2} \right)$$

DPP- 10

Topic : Two Dimensional Heat Equation (Laplace Equation)

Part-2

Q.1. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy – plane with u(x, 0) = 0, u(x, b) = 0, u(0,y) = 0 and u(a, y) = f(y) parallel to y-axis. (U.K.T.U. 2016)

Q.2 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions u(0, y) = 0, u(a, y) = 0, u(x, 0) = 0 and u(x, b) = x. [U.P.T.U. 2015]

ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Complex Fourier transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(p)$ and defined as

F(s)

$$\boxed{F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx}$$

$$\boxed{i = \sqrt{-1}}$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$\boxed{F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dp}$$

Some Important Points and Formulae

1. Even and Odd Function :

(i) Even Function : A function is said be even if

$$\boxed{f(-x) = f(x)}$$

Example

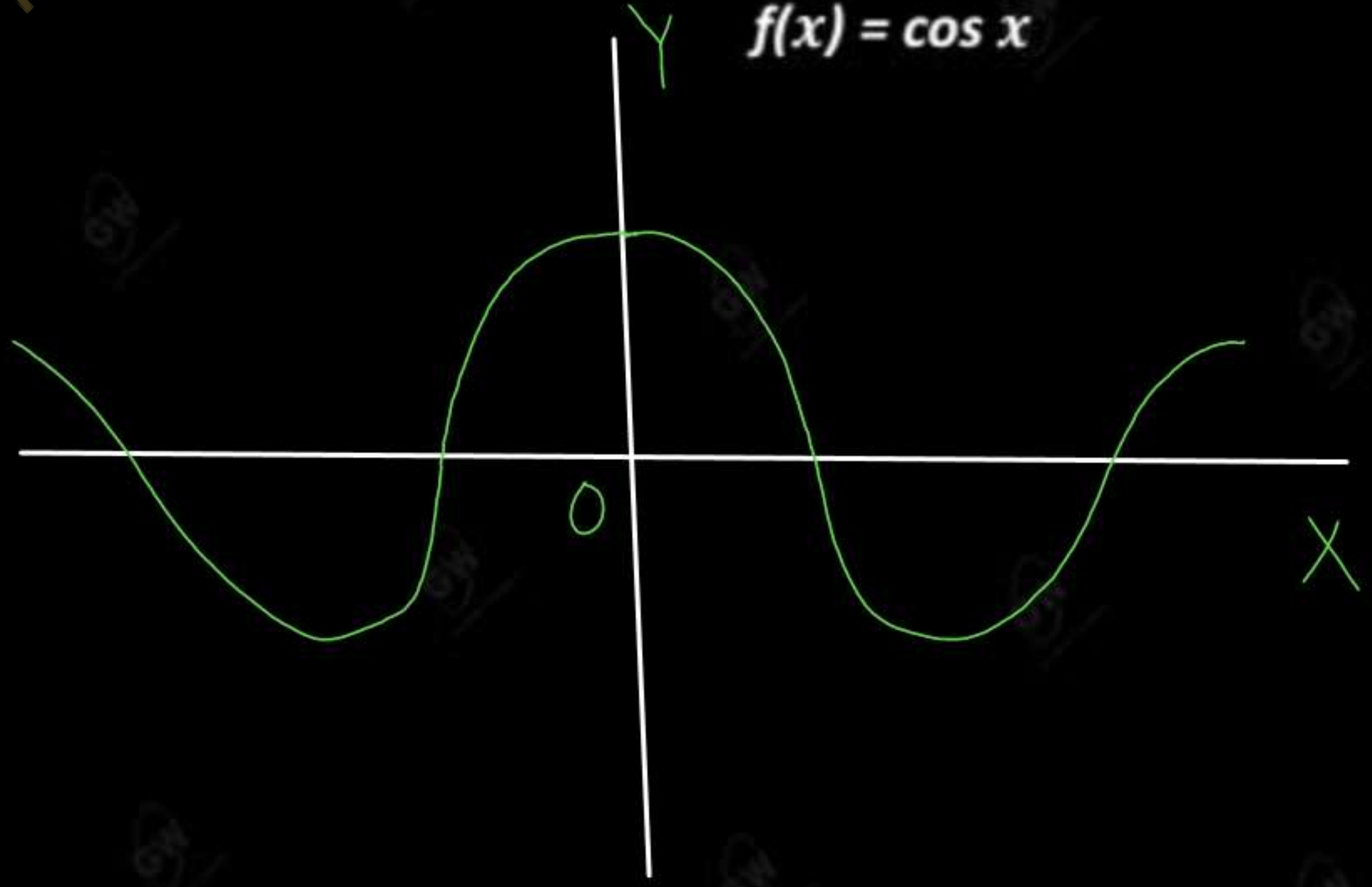
(1) $f(x) = \cos x$

(2) $f(x) = x^2$

Note: Graph of even function is symmetrical about Y-axis

$f(x) = x^2$

$f(x) = \cos x$



(ii) Odd Function : *A function is said be Odd if*

$$\boxed{f(-x) = -f(x)}$$

Example

(a) $f(x) = \sin x$

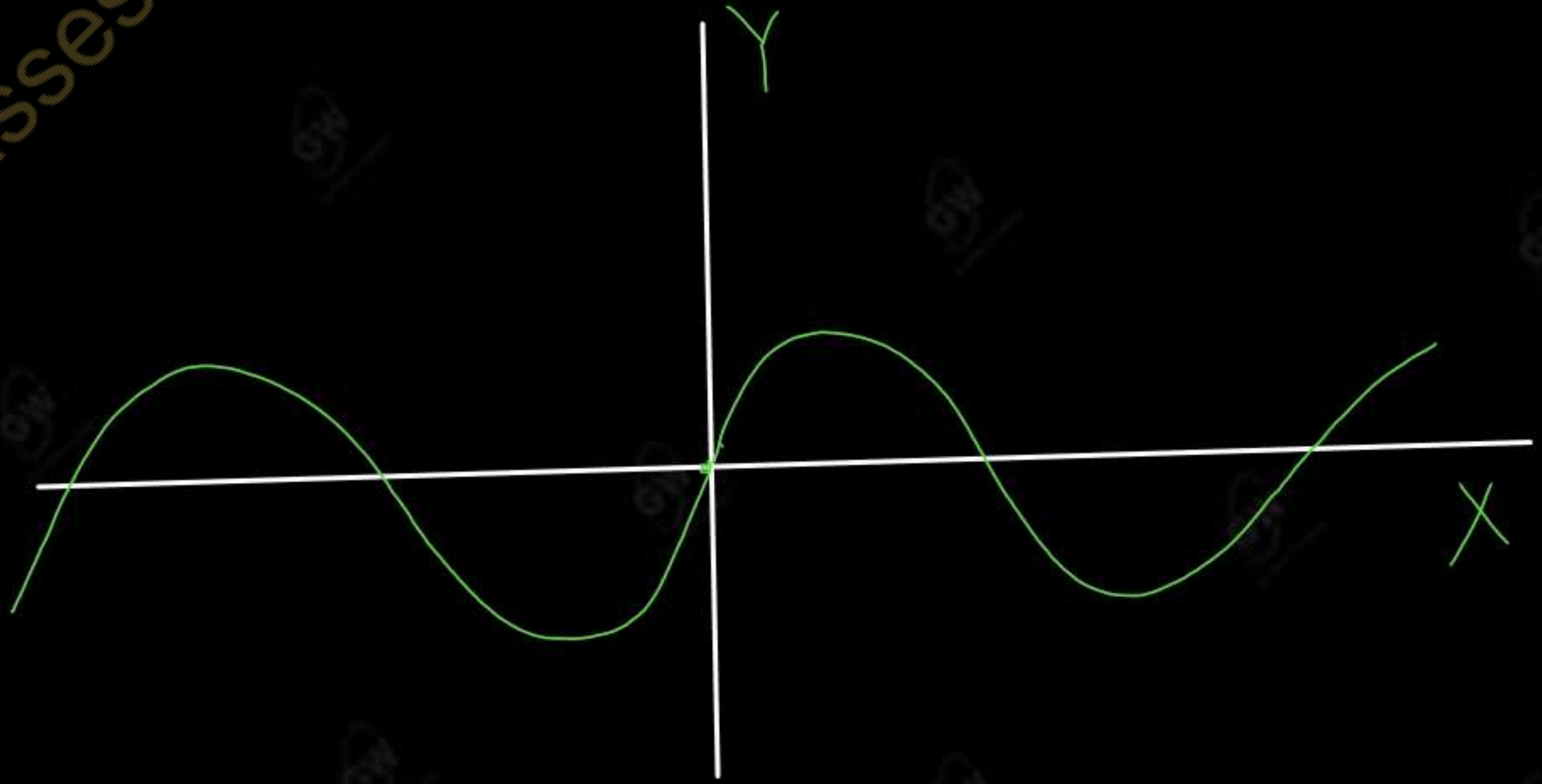
(b) $f(x) = \sin^3 x$

(c) $f(x) = x$

Note: *The graph $f(x)$ of an odd function is symmetrical about origin and in opposite quadrant.*

$f(x) = x$

$f(x) = \sin x$



Also Remember

(i) Even function × Even function = Even function

(ii) Odd function×Odd function = Even function

(iii) Even function × Odd function = Odd function

(iv) Odd function×Even function = Odd function

2 **An important Property of Definite integral**

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \qquad \text{If } f(x) \text{ is an even function}$$

$$= 0 \qquad \text{If } f(x) \text{ is an odd function}$$

3 Integration By-Parts

$$\int \underset{\mathrm{I}}{u}\,\underset{\mathrm{II}}{v}\,dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \cdots$$

4

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

5

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

6

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

NOTE:

① $|x| \leq a$

$-a \leq x \leq a$

② $|x| < a$

$-a < x < a$

③ $|x| \geq a$

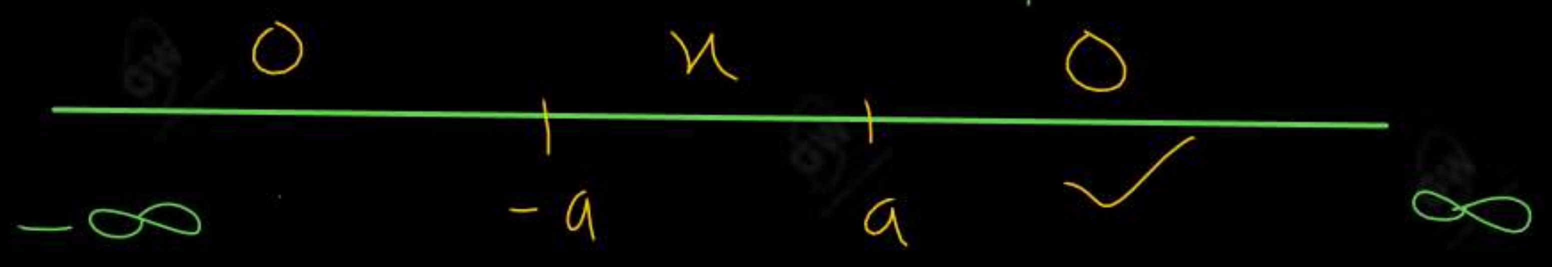
$x \geq a$

$x \leq -a$

④ $|x| > a$

$x > a$

$x < -a$



Q.1. *Find the Fourier Transform of* $F(x) = \begin{cases} x, & if \quad |x| < a \\ 0, & if \quad |x| > a \end{cases}$

$$F(x) = \begin{cases} x & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} x\, e^{ipx}\, dx + 0$$

$$f(p) = \int_{-a}^{a} \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^{ipx}}\, dx$$

$$f(p) = \left[x\left(\frac{e^{ipx}}{ip}\right) - 1\left(\frac{e^{ipx}}{i^2 p^2}\right) \right]_{-a}^{a}$$

$$f(p) = \frac{a e^{ipa}}{ip} - \frac{e^{ipa}}{i^2 p^2} - \left(-\frac{a e^{-ipa}}{ip} - \frac{e^{-ipa}}{i^2 p^2}\right)$$

$$f(p) = \frac{a e^{ipa}}{ip} + \frac{e^{ipa}}{p^2} + \frac{a e^{-ipa}}{ip} - \frac{e^{-ipa}}{p^2}$$

$$f(p) = \frac{a}{ip}\left(e^{ipa} + e^{-ipa}\right) + \frac{1}{p^2}\left(e^{ipa} - e^{-ipa}\right)$$

$$f(p) = \frac{2a}{ip}\left(\frac{e^{ipa} + e^{-ipa}}{2}\right) + \frac{2i}{p^2}\left(\frac{e^{ipa} - e^{-ipa}}{2i}\right)$$

$$f(p) = \frac{2a}{ip}\cos ap + \frac{2i}{p^2}\sin ap$$

$$f(p) = \frac{2i}{p^2}\left(\frac{ap\cos ap}{i^2} + \sin ap\right)$$

$$\boxed{f(p) = \frac{2i}{p^2}\left(\sin ap - ap\cos ap\right)}$$

Q.2 . Final the complex Fourier transform of $f(x) = e^{-ax^2}$, $(a > 0)$. Hence find the Fourier transform of

(i) $F(x) = e^{-x^2}$ (ii) $F(x) = e^{-x^2/2}$ **(AKTU-2017)**

$$F(x) = e^{-ax^2}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2} \times e^{ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-ax^2 + ipx}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x^2 - \frac{ipx}{a}\right)}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\underbrace{x^2 - \frac{ip}{a}x + \left(\frac{ip}{2a}\right)^2} - \left(\frac{ip}{2a}\right)^2\right]}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 - \frac{i^2p^2}{4a^2}\right]}$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left[\left(x - \frac{ip}{2a}\right)^2 + \frac{p^2}{4a^2}\right]}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2 - \frac{\cancel{a}p^2}{4a^{\cancel{2}}}}\, dx$$

$$f(p) = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{ip}{2a}\right)^2} \times e^{\frac{-p^2}{4a}}\, dx$$

$$f(p) = e^{-p^2/4a} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\left(x - \frac{ip}{2a}\right)\right]^2} dx$$

Put $\sqrt{a}\left(x - \frac{ip}{2a}\right) = t$

$$\sqrt{a}\, dx = dt$$

$$dx = \frac{dt}{\sqrt{a}}$$

$$f(p) = e^{-\frac{p^2}{4a}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{a}}$$

$$f(p) = \frac{e^{-\frac{p^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$\boxed{f(p) = \frac{e^{-\frac{p^2}{4a}} \times \sqrt{\pi}}{\sqrt{a}}} \quad \text{—— (1)}$$

(i) $F(x) = e^{-x^2}$

Here $a = 1$

Put $a =$ in ①

$$\boxed{f(p) = e^{-\frac{p^2}{4}} \times \sqrt{\pi}}$$

(ii) $F(x) = e^{-\frac{x^2}{2}}$

Here $a = \frac{1}{2}$

Put $a = \frac{1}{2}$ in ①

$$f(p) = \underline{e^{-\frac{p^2}{2}} \times \sqrt{\pi}} \times \sqrt{2}$$

$$\boxed{f(p) = e^{-\frac{p^2}{2}} \times \sqrt{2\pi}}$$

Q.3 . Find the Fourier transform of

$$F(x) = \begin{cases} 1, & if \quad |x| < a \\ 0, & if \quad |x| > a \end{cases}$$

(UPTU-2016, 2014 & 2010)

Hence evaluate (i) $\int_{-\infty}^{\infty} \left(\frac{\sin as \cos xs}{s}\right) ds.$ (ii) $\int_{0}^{\infty} \left(\frac{\sin s}{s}\right) ds.$

$$F(x) = \begin{cases} 1 & -a < x < a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

By Fourier transform

$$f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = \int_{-\infty}^{-a} F(x) e^{ipx} dx + \int_{-a}^{a} F(x) e^{ipx} dx + \int_{a}^{\infty} F(x) e^{ipx} dx$$

$$f(p) = 0 + \int_{-a}^{a} 1 \times e^{ipx} dx + 0$$

$$f(p) = \int_{-a}^{a} e^{ipx} dx$$

$$f(p) = \left(\frac{e^{ipx}}{ip}\right)_{-a}^{a}$$

$$f(p) = \frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}$$

$$f(p) = \frac{1}{ip}\left(e^{ipa} - e^{-ipa}\right)$$

$$f(p) = \frac{2\cancel{i}}{\cancel{i}p}\left(\frac{e^{ipa} - e^{-ipa}}{2i}\right)$$

$$\boxed{f(p) = \frac{2}{p}\sin ap}$$

(i) By Inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin ap}{p}\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap\; e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{p}\sin ap(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin ap\cos px}{p} - i\,\frac{\sin ap\sin px}{p}\right)dp$$

even odd

By property of Definite integral

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap\cos px}{p}\,dp$$

Put $p = s$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds$$

$$\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds = \pi F(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as\cos sx}{s}\,ds = \pi F(x)$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \end{cases}}$$

(ii) Put $a = 1$ and $x = 0$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi F(0)$$

$$\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi \cdot 1$$

even

By the property of Definite integral

$$2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\boxed{\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}}$$

GW GATEWAY CLASSES

Q.4 *Find the Fourier transform of* $\begin{cases} 1-x^2, & if\ |x|<1 \\ 0, & if\ |x|>1 \end{cases}$ *and use it to Evaluate* $\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx.$

(UPTU-2015)

$$F(x)=\begin{cases} 1-x^2 & -1<x<1 \\ 0 & x>1 \text{ or } x<-1 \end{cases}$$

By Fourier transform

$$f(p)=\int_{-\infty}^{\infty} F(x)\,e^{ipx}\,dx$$

$$f(p)=\int_{-\infty}^{-1} F(x)e^{ipx}dx+\int_{-1}^{1} F(x)e^{ipx}dx+\int_{1}^{\infty} F(x)e^{ipx}dx$$

$$f(p)=0+\int_{-1}^{1}(1-x^2)e^{ipx}dx+0$$

$$f(p)=\int_{-1}^{1}\underset{\mathrm{I}}{(1-x^2)}\,\underset{\mathrm{II}}{e^{ipx}}\,dx$$

$$f(p)=\left[(1-x^2)\left(\frac{e^{ipx}}{ip}\right)-(-2x)\left(\frac{e^{ipx}}{i^2p^2}\right)+(-2)\left(\frac{e^{ipx}}{i^3p^3}\right)\right]_{-1}^{1}$$

$$f(p) = \left[(1-x^2)\frac{e^{ipx}}{ip} - \frac{2x\,e^{ipx}}{p^2} + \frac{2\,e^{ipx}}{ip^3} \right]_{-1}^{1}$$

$$f(p) = \left[0 - \frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \left(0 + \frac{2e^{-ip}}{p^2} + \frac{2e^{-ip}}{ip^3} \right) \right]$$

$$f(p) = -\frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \frac{2e^{-ip}}{p^2} - \frac{2e^{-ip}}{ip^3}$$

$$f(p) = -\frac{2}{p^2}\left(e^{ip} + e^{-ip}\right) + \frac{2}{ip^3}\left(e^{-ip} - e^{-ip}\right)$$

$$f(p) = -\frac{4}{p^2}\left(\frac{e^{ip}+e^{-ip}}{2}\right) + \frac{4}{p^3}\left(\frac{e^{ip}-e^{-ip}}{2i}\right)$$

$$f(p) = -\frac{4}{p^2}\cos p + \frac{4}{p^3}\sin p$$

$$\boxed{f(p) = \frac{4}{p^3}\left(\sin p - p\cos p\right)}$$

By inverse Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\, e^{-ipx}\, dp$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}\left(\sin p - p\cos p\right) e^{-ipx}\, dp$$

$$F(x) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{4}^{2}}{p^3}(\sin p - p\cos p)(\cos px - i\sin px)\,dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p\cos px}{p^3} - \frac{i\sin p\sin px}{p^3} - \frac{\cancel{p}\cos p\cos px}{p^{\cancel{3}\,2}} + \frac{i\cancel{p}\cos p\sin px}{p^{\cancel{3}\,2}}\right)dp$$

even odd even odd

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p\cos px}{p^3} - \frac{p\cos p\cos px}{p^3}\right)dp$$

$$F(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos px\,dx$$

Put $x = \frac{1}{2}$

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\frac{p}{2}\,dp$$

Replace p by x

$$F\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx$$

$$\int_{-\infty}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{\pi}{2}F\left(\frac{1}{2}\right)$$

$$\int_{-\infty}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{\pi}{2}F\left(\frac{1}{2}\right) \quad \text{—— (1)}$$

$$F(x) = 1 - x^2$$

$$F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2$$

$$F\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$\boxed{\int_{-\infty}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx=-\frac{3\pi}{8}}$$

DPP- 11

Topic : *Complex Fourier transform*

Q.1. *Find the Fourier Transform of* $F(x) = \begin{cases} x^2, & if\ |x| < a \\ 0, & if\ |x| > a \end{cases}$

Q.2. *Find the Fourier Transform of* $f(x) = e^{-|x|}$

Q.3. *Find the Fourier transform of* $F(x) = \begin{Bmatrix} 1, & |x| < 1 \\ 0, & |x| > 1 \end{Bmatrix}$. *hence avaluate* $\int_0^\infty \frac{\sin x}{x} dx.$ (GBTU-2011)

Q. 4. *Find the Fourier transform of Block function* $f(t)$ *of height 1 and duration a defined by* (AKTU-2016)

$$f(t) = \begin{Bmatrix} 1, & for\ |t| \le \frac{a}{2} \\ 0, & otherwise \end{Bmatrix}$$

ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier sine transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(p)$ and defined as

OR

$F(s)$

$$\boxed{F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x) e^{ipx} dx}$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$\boxed{F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) e^{-ipx} dp}$$

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$\boxed{F_s\{F(x)\} = f_s(p) = \int_0^\infty F(x) \sin px \, dx}$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$\boxed{F(x) = F_s^{-1}\{f_s(p)\} = \frac{2}{\pi} \int_0^\infty f_s(p) \sin px \, dp}$$

Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

Q.1. *Find the Fourier sine transform of* $F(x) = e^{-3x} + e^{-4x}$

$$F(x) = e^{-3x} + e^{-4x}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x)\sin px\,dx$$

$$f_s(p) = \int_0^\infty \left(e^{-3x} + e^{-4x}\right)\sin px\,dx$$

$$f_s(p) = \int_0^\infty e^{-3x}\sin px\,dx + \int_0^\infty e^{-4x}\sin px\,dx$$

$$f_s(p) = \left[\frac{e^{-3x}}{(-3)^2+p^2}\left(-3\sin px - p\cos px\right)\right]_0^\infty$$

$$+\left[\frac{e^{-4x}}{(-4)^2+p^2}\left(-4\sin px - p\cos px\right)\right]_0^\infty$$

$$f(p) = \left[0 - \frac{1}{9+p^2}(0-p)\right] +$$

$$\left[0 - \frac{1}{16+p^2}(0-p)\right]$$

$$f_s(p) = \left[\frac{-1}{9+p^2}(-p)\right] + \left[\frac{-1}{16+p^2}(-p)\right]$$

$$\boxed{f_s(p) = \frac{p}{p^2+9} + \frac{p}{p^2+16}}$$

Q.2 . *Find the Fourier sine transform of* $e^{-|x|}$. *Hence evaluate* $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$.

$$F(x) = e^{-|x|}$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

For Fourier sine transform

$$x > 0$$

$$\Rightarrow |x| = x$$

$$\Rightarrow F(x) = e^{-x}$$

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$= \int_0^\infty e^{-x} \sin px \, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2 + p^2} \left(-\sin px - p\cos px \right) \right]_0^\infty$$

$$= \left[0 - \left(\frac{1}{1+p^2} (-p) \right) \right]$$

$$f_s(p) = -\left(\frac{-p}{1+p^2}\right)$$

$$\boxed{f_s(p) = \frac{p}{p^2+1}}$$

By Inverse Fourier sine Transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin px\,dp$$

Put $x = m$

$$F(m) = \frac{2}{\pi}\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp$$

$$\int_0^\infty \frac{p}{p^2+1}\sin pm\,dp = \frac{\pi}{2}F(m)$$

Replace p by x

$$\boxed{\int_0^\infty \frac{x}{x^2+1}\sin mx\,dx = \frac{\pi}{2}e^{-m}}$$

Q.3 . Find Fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$. Hence find the Fourier sine transform of $\frac{1}{x}$

(GBTU-2011; UPTU-2015)

$$F(x) = \frac{e^{-ax}}{x}$$

By Fourier sine transform

$$f_s(p) = \int_0^\infty F(x) \sin px \, dx$$

$$f_s(p) = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx$$

Let $f_s(p) = I$

$$I = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx \quad \text{——} (1)$$

differentiate w.r.t. p

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{dp}(\sin px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \times \cancel{x} \cos px \, dx$$

$$\frac{dI}{dp} = \int_0^\infty e^{-ax} \cos px \, dx$$

$$\frac{dI}{dp} = \left[\frac{e^{-ax}}{(-a)^2 + p^2}\left(-a\cos px + p\sin px\right)\right]_0^\infty$$

$$\frac{dI}{dp} = \left[0 - \left\{\frac{1}{a^2+p^2}(-a)\right\}\right]$$

$$\frac{dI}{dp} = \frac{a}{a^2+p^2}$$

$$dI = \frac{a}{a^2+p^2}\, dp$$

Integrate w r t p

$$\int dI = \int \frac{a}{a^2+p^2}\, dp$$

$$I = \tan^{-1}\left(\frac{p}{a}\right) + c \quad \text{—} \; (2)$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right) + c$$

Let initially $p = 0$

Put $p = 0$ in ①

$I = 0$

Put $p = 0$ and $I = 0$ in ②

$0 = 0 + c$

$c = 0$

$$\boxed{f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)}$$

$$f_s(p) = \tan^{-1}\left(\frac{p}{a}\right)$$

$$\int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx = \tan^{-1}\left(\frac{p}{a}\right)$$

Put $a = 0$

$$\int_0^\infty \frac{1}{x}\sin px\,dx = \tan^{-1}\left(\frac{p}{0}\right) = \tan^{-1}(\infty)$$

$$\boxed{\int_0^\infty \frac{1}{x}\sin px\,dx = \frac{\pi}{2}}$$

Q.4. *Find $F(x)$ if its Fourier sine transform is e^{-ap}*

$$f_s(p) = e^{-ap}$$

By inverse Fourier sine transform

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(p)\sin px\,dp$$

$$= \frac{2}{\pi}\int_0^\infty e^{-ap}\sin px\,dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-ap}}{(-a)^2+x^2}\left(-a\sin px - x\cos px\right)\right]_0^\infty$$

$$F(x) = \frac{2}{\pi}\left[0 - \left\{\frac{1}{a^2+x^2}(-x)\right\}\right]$$

$$F(x) = \frac{2}{\pi}\left(\frac{x}{a^2+x^2}\right)$$

$$\boxed{F(x) = \frac{2x}{\pi(a^2+x^2)}}$$

DPP- 12

Topic : *Fourier sine transform*

Q.1. *Find Fourier sine transform of* $\frac{1}{x}$

Q. 2. *Find the Fourier sine transform of Block function* $f(t)$ *of height 1 and duration a defined by*

$$f(t) = \begin{cases} sinx, & 0 < x < a \\ 0, & x > a \end{cases}$$

Q.3. *Find* $F_s^{-1}\left(\frac{e^{-ap}}{p}\right)$ *and hence evaluate* $F_s^{-1}\left(\frac{1}{p}\right)$. (AKTU-2016)

ENGINEERING MATHEMATICS

UNIT-2 : Application of partial differential equations and Fourier Transform

Today's Target

- ***Fourier cosine transform***
- PYQ
- DPP

Fourier Transform

1. Complex Fourier Transform
2. Fourier sine Transform
3. Fourier cosine Transform
4. Convolution theorem
5. Application of Fourier transform to solve partial differential equation

Complex Fourier transform

Let $F(x)$ defined for all real values, then complex Fourier transform of $F(x)$ is denoted by $F\{F(x)\}$ or $f(p)$ and defined as

$$\boxed{F\{F(x)\} = f(p) = \int_{-\infty}^{\infty} F(x)e^{ipx}dx}$$

Where p is a parameter and $p > 0$

Inverse Fourier transform

If $F\{F(x)\} = f(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier transform and defined as

$$\boxed{F^{-1}\{f(p)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)e^{-ipx}dx}$$

Fourier sine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier sine transform of $F(x)$ is denoted by $F_s\{F(x)\}$ or $f_s(p)$ and defined as

$$\boxed{F\{F(x)\} = f_s(p) = \int_0^{\infty} F(x) \sin px \, dx}$$

Where p is a parameter and $p > 0$

Inverse Fourier sine transform

If $F_s\{F(x)\} = f_s(p)$ is the Fourier transform of $F(x)$ then $F(x)$ is called inverse Fourier sine transform and defined as

$$\boxed{F_s^{-1}\{f_s(p)\} = F(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} f_s(p) \sin px \, dp}$$

Fourier cosine transform

Let $F(x)$ defined in the interval $(0, \infty)$, then Fourier cosine transform of $F(x)$ is denoted by $F_c\{F(x)\}$ or $f_c(p)$ and defined as

$$F_c\{F(x)\} = f_c(p) = \int_0^\infty F(x) \cos px\, dx$$

Where p is a parameter and $p > 0$

Inverse Fourier cosine transform

If $F_c\{F(x)\} = f_c(p)$ is the Fourier cosine transform of $F(x)$, then $F(x)$ is called inverse Fourier cosine transform and defined as

$$F_c^{-1}\{f_c(p)\} = F(x) = \frac{2}{\pi} \int_0^\infty f_c(p) \cos px\, dp$$

Two Important results

(i) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

(ii) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

Trigonometric Identities

(1) $2 \sin A \cos A = \sin 2A$

(2) $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

(3) $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

(4) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

(5) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

Q.1. *Find the Fourier cosine transform of* $F(x) = e^{-2x} + 4e^{-3x}$

$$F(x) = e^{-2x} + 4e^{-3x}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x)\cos px\,dx$$

$$f_c(p) = \int_0^\infty \left(e^{-2x} + 4e^{-3x}\right)\cos px\,dx$$

$$f_c(p) = \int_0^\infty e^{-2x}\cos px\,dx + 4\int_0^\infty e^{-3x}\cos px\,dx$$

$$f_c(p) = \left[\frac{e^{-2x}}{(-2)^2 + p^2}\left(-2\cos px + p\sin px\right)\right]_0^\infty$$

$$+4\left[\frac{e^{-3x}}{(-3)^2 + p^2}\left(-3\cos px + p\sin px\right)\right]_0^\infty$$

$$f_c(p) = \left[0 - \left\{\frac{1}{4+p^2}(-2)\right\}\right]$$

$$+4\left[0 - \left\{\frac{1}{9+p^2}(-3)\right\}\right]$$

$$\boxed{f_c(p) = \frac{2}{p^2+4} + \frac{12}{p^2+9}}$$

Q.2 . Find Fourier cosine transform of $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$F(x) = \begin{cases} \cos x & , 0 < x < a \\ 0 & x > a \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a F(x) \cos px \, dx + \int_a^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^a \cos x \cos px \, dx + 0$$

$$f_c(p) = \frac{1}{2} \int_0^a 2 \cos x \cos px \, dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{ \cos(x + px) + \cos(x - px) \} dx$$

$$f_c(p) = \frac{1}{2} \int_0^a \{ \cos(1+p)x + \cos(1-p)x \} dx$$

$$f_p(c) = \frac{1}{2} \left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p} \right]_0^a$$

$$f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p} - (0+0)\right]$$

$$\boxed{f_c(p) = \frac{1}{2}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]}$$

Q.3 . *Obtain Fourier cosine transform of* $F(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

$$F(x) = \begin{cases} x & , \quad 0 < x < 1 \\ 2-x & , \quad 1 < x < 2 \\ 0 & , \quad x > 2 \end{cases}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 F(x) \cos px \, dx + \int_1^2 F(x) \cos px \, dx + \int_2^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^1 \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\cos px} \, dx + \int_1^2 \underset{\mathrm{I}}{(2-x)} \underset{\mathrm{II}}{\cos px} \, dx + 0$$

$$f_c(p) = \left[x \left(\frac{\sin px}{p} \right) - 1 \left(\frac{-\cos px}{p^2} \right) \right]_0^1$$

$$+ \left[(2-x) \left(\frac{\sin px}{p} \right) - (-1) \left(\frac{-\cos px}{p^2} \right) \right]_1^2$$

$$b_c(p) = \left[\frac{x\sin px}{p} + \frac{\cos px}{p^2}\right]_0^1 + \left[(2-x)\frac{\sin px}{p} - \frac{\cos px}{p^2}\right]_1^2$$

$$b_c(p) = \frac{\sin p}{p} + \frac{\cos p}{p^2} - \left(0 + \frac{1}{p^2}\right) + 0 - \frac{\cos 2p}{p^2} - \left(\frac{\sin p}{p} - \frac{\cos p}{p^2}\right)$$

$$b_c(p) = \cancel{\frac{\sin p}{p}} + \frac{\cos p}{p^2} - \frac{1}{p^2} - \frac{\cos 2p}{p^2} - \cancel{\frac{\sin p}{p}} + \frac{\cos p}{p^2}$$

$$\boxed{b_c(p) = \frac{2\cos p}{p^2} - \frac{\cos 2p}{p^2} - \frac{1}{p^2}}$$

$$f_c(p) = \frac{2\cos p}{p^2} - \left(\frac{2\cos^2 p - 1}{p^2}\right) - \frac{1}{p^2} \quad \left\{ \because \cos 2\theta = 2\cos^2\theta - 1 \right\}$$

$$f_c(p) = \frac{2\cos p}{p^2} - \frac{2\cos^2 p}{p^2} + \cancel{\frac{1}{p^2}} - \cancel{\frac{1}{p^2}}$$

$$f_c(p) = \frac{2\cos p - 2\cos^2 p}{p^2}$$

$$\boxed{f_c(p) = \frac{2\cos p(1-\cos p)}{p^2}}$$

Q.4. *Find Fourier cosine transform of* $\frac{1}{1+x^2}$ *and hence find Fourier sine transform of* $\frac{x}{1+x^2}$.

(AKTU-2018, 2017)

$$F(x) = \frac{1}{1+x^2}$$

By Fourier cosine transform

$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$f_c(p) = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

Let $f_c(p) = I$

$$I = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx \quad \text{---} \quad (1)$$

Differentiate w r t p

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \frac{d}{dp}(\cos px) \, dx$$

$$\frac{dI}{dp} = \int_0^\infty \frac{1}{1+x^2} \times x(-\sin px) \, dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{x \sin px}{1+x^2} \, dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{x^2 \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2-1)\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{(1+x^2)\sin px - \sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\left[\int_0^\infty \left\{\frac{\cancel{(1+x^2)}\sin px}{x\cancel{(1+x^2)}} - \frac{\sin px}{x(1+x^2)}\right\}dx\right.$$

$$\frac{dI}{dp} = -\int_0^\infty \frac{\sin px}{x}\,dx + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx$$

$$\frac{dI}{dp} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)}\,dx \quad \text{— (2)}$$

Differentiate wrt p

$$\frac{d^2I}{dp^2} = 0 + \int_0^\infty \frac{\cancel{x}\cos px}{\cancel{x}(1+x^2)}\,dx$$

$$\frac{d^2I}{dp^2} = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx$$

$$\frac{d^2I}{dp^2} = I$$

$$\frac{d^2I}{dp^2} - I = 0$$

solution of above Differential equation is

$$I = c_1 e^{p} + c_2 e^{-p}$$

$$I = c_1 e^{p} + c_2 e^{-p} \quad \text{---} \; (3)$$

diff wrt p

$$\frac{dI}{dp} = c_1 e^{p} - c_2 e^{-p} \quad \text{---} \; (4)$$

When $p = 0$

From (3)

$$I = c_1 + c_2 \quad \text{---} \; (5)$$

From (4)

$$\frac{dI}{dp} = c_1 - c_2 \quad \text{---} \; (6)$$

From (1)

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

$$I = \left(\tan^{-1}x\right)_0^\infty$$

$$I = \tan^{-1}\infty - \tan^{-1}0$$

$$I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{2}$$

From (2)

$$\frac{dI}{dp} = -\frac{\pi}{2}$$

Put I and $\frac{dI}{dp}$ in (5) and (6)

$$c_1 + c_2 = \frac{\pi}{2} \quad \text{—— (7)}$$

$$c_1 - c_2 = -\frac{\pi}{2} \quad \text{—— (8)}$$

Solve (7) and (8)

$$c_1 + \cancel{c_2} = \cancel{\frac{\pi}{2}}$$

$$c_1 - \cancel{c_2} = -\cancel{\frac{\pi}{2}}$$

$$2c_1 = 0$$

$$\boxed{c_1 = 0}$$

$$\boxed{c_2 = \frac{\pi}{2}}$$

Put c_1 and c_2 in (3)

$$I = \frac{\pi}{2}e^{-p}$$

$$\boxed{\int_0^\infty \frac{1}{1+x^2}\cos px\, dx = \frac{\pi}{2}e^{-p}}$$

$$\boxed{\int_0^\infty \frac{x}{1+x^2}\sin px\, dx = \frac{\pi}{2}e^{-p}}$$

diff w r t p

$$\int_0^\infty \frac{1}{1+x^2}\frac{d}{dp}(\cos px)\, dx = \frac{\pi}{2}\frac{d}{dp}\left(e^{-p}\right)$$

$$\int_0^\infty \frac{-x\sin px}{1+x^2}\, dx = -\frac{\pi}{2}e^{-p}$$

DPP- 13

Topic : *Fourier cosine transform*

Q.1. *Find Fourier cosine transform* $F(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases}$

Q. 2. *Find the Fourier cosine transform of* e^{-x^2}

Q.3. *Find Fourier cosine transform of* $F(x) = 2e^{-5x} + 2e^{-2x}$

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Thank You