HW-1

Q1 $p(x|\omega_1) = \begin{cases} 1/2 & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$

$p(x|\omega_2) = \begin{cases} \frac{3x^3}{8} & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$

$P(\omega_1) = \frac{3}{7}$ $P(\omega_2) = \frac{4}{7}$

a] The minimum-error-rate classification decides class label ω_1 if $P(\omega_1|x) > P(\omega_2|x)$.
This equivalent to:

$$p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{3}{7} > \frac{3x^2}{8} \cdot \frac{4}{7}$$

$$\Rightarrow \frac{3}{14} > \frac{3x^2}{14}$$

$\Rightarrow 1 > x^2$

Since $x \in [0,2] \Rightarrow 1 > x$

$\boxed{\text{Therefore } a=1}$

b] From part a] $x^2 < 1$, By Bayes Decision Rule we can decide ω_1 if $p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2)$

Since $x^2 < 1$, $\Rightarrow \frac{3}{14} > \frac{3x^2}{14}$

$P(\text{error}|x) = P(\omega_2|x)$ when $0 \leq x < 1$

And $P(\text{error}|x) = P(\omega_1|x)$ when $1 \leq x \leq 2$. $\rightarrow$

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$$\cancel{P(\omega_2|x) = p(x|\omega_2)P(\omega_2}$$

$$P(\omega_2|x) = \frac{p(x|\omega_2)P(\omega_2)}{p(x|\omega_1)P(\omega_1) + p(x|\omega_2)P(\omega_2)}$$

$$= \frac{\frac{3x^2}{14}}{\frac{3x^2+3}{14}}$$

$$P(\omega_2|x) = \frac{3x^2}{3x^2+3} = P(\text{error}|x) \quad \text{when } 0 \le x < 1$$

And $P(\text{error}|x) = P(\omega_1|x)$ when $1 \le x \le 2$

$$P(\omega_1|x) = \frac{p(x|\omega_1)P(\omega_1)}{p(x|\omega_1)P(\omega_1) + p(x|\omega_2)P(\omega_2)}$$

$$= \frac{\frac{3}{14}}{\frac{3x^2+3}{14}} = \frac{3}{3x^2+3}$$

$$P(\text{error}) = \int_0^2 P(\text{error}|x)\,p(x)\,dx = \int_0^1 P(\text{error}|x)p(x)\,dx + \int_1^2 P(\text{error}|x)\,p(x)\,dx$$

$$\boxed{\therefore P(\text{error}) = 2/7}$$

$$p(x) = \frac{3x^2+3}{14}$$

$$P(\text{error}) = \int_0^1 \frac{3x^2}{14}\,dx + \int_1^2 \frac{3}{14}\,dx = \frac{1}{14} + \frac{3}{14} = \boxed{\frac{2}{7}}$$

Q2

$\lambda_{11} = \lambda_{22} = 0$; $\lambda_{21} = \frac{1}{4}$, $\lambda_{12} = \frac{3}{4}$

Based on Decision rule, if ω_1 then:

$$\lambda_{21} P(\omega_1)\, p(x|\omega_1) > \lambda_{12} P(\omega_2)\, p(x|\omega_2)$$

$$= \frac{1}{4} \times \frac{1}{3} \times N(\mu_1, \sigma^2) > \frac{3}{4} \times \frac{2}{3} \times N(\mu_2, \sigma^2)$$

$$= \frac{1}{12} \times N(\mu_1, \sigma^2) > \frac{1}{2} \times N(\mu_2, \sigma^2)$$

$$= N(\mu_1, \sigma^2) > 6 \times N(\mu_2, \sigma^2)$$

$$= \frac{N(\mu_1, \sigma^2)}{N(\mu_2, \sigma^2)} > 6$$

$$= \frac{e^{\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right)}}{e^{\left(-\frac{(x-\mu_2)^2}{2\sigma^2}\right)}} > 6$$

$$= e^{\left(\frac{(x-\mu_2)^2 - (x-\mu_1)^2}{2\sigma^2}\right)} > 6$$

Take ln i.e natural log on both sides:

$$= \frac{x^2 + \mu_2^2 - 2x\mu_2 - x^2 - \mu_1^2 + 2x\mu_1}{2\sigma^2} > \ln(6)$$

$$= (\mu_2^2 - \mu_1^2) + 2x(\mu_1 - \mu_2) > 2\sigma^2 \ln(6)$$

$$\cancel{2x(\mu_1 - \mu_2) > \frac{2\sigma^2 \ln(6)}{\mu_2^2 - \mu_1^2}}$$

$$2x(\mu_1 - \mu_2) > (\mu_1^2 - \mu_2^2) + \sigma^2 \ln(6) \cdot 2$$

$\cancel{x(\mu_1 - \mu_2) \geq}$ Dividing both sides by 2.

$$x(\mu_1 - \mu_2) > \frac{(\mu_1^2 - \mu_2^2)}{2} + \sigma^2 \ln(6)$$

$$x > \frac{(\mu_1 + \mu_2)}{2} + \frac{\sigma^2 \ln(6)}{\mu_1 - \mu_2}$$

$\therefore$ The decision rule is to choose ω_1 if $x >$ threshold.

Otherwise, if $\mu_1 < \mu_2$:

$$x < \frac{(\mu_1 + \mu_2)}{2} + \frac{\sigma^2 \ln(6)}{\mu_1 - \mu_2}$$

Q3 a] $P(N=Yes \mid D=Yes, W=No) = P(N=Yes \mid B=Yes, W=No)P(B=Yes \mid D=Yes)$
$+$
$P(N=Yes \mid B=NO, W=No)P(B=No \mid D=Yes)$

$= 0.8 \times 0.8 + 0.1 \times 0.2$
$= 0.64 + 0.02$
$= 0.66$

b] $D = Yes : D$, $D = No : D'$, same for all letters

$$P(D \mid N) = \frac{P(N \mid D)\, P(D)}{P(N)}$$

$P(B) = P(B \mid D)\,P(D) + P(B \mid D')\,P(D')$
$= 0.8 \times 0.2 + 0.1 \times 0.8$
$= 0.16 + 0.08$
$= 0.24$
$P(B') = 0.76$

$P(N) = P(N \mid B', W')P(B')P(W') + P(N \mid B', W)P(B')P(W)$
$+ P(N \mid B, W')P(B)P(W') + P(N \mid B, W)P(B)P(W)$

$= (0.1 \times 0.76 * 0.6) + (0.3 \times 0.76 \times 0.4) + (0.8 \times 0.24 \times 0.6) +$
$(0.9 \times 0.24 \times 0.4)$
$= 0.0456 + 0.0912 + 0.1152 + 0.0864$
$= 0.3384$

~~$P(N \mid D) = P(N \mid B', W')P(D')P(W') + P(N \mid B', W)P(D')P(W)$~~
~~$+ P(N \mid B, W')P(D')P($~~

$$P(N|D) = P(N|B,W)P(B|D)P(W) + P(N|B,W')P(B|D)P(W') + P(N|B',W)P(B'|D)P(W) + P(N|B',W')P(B'|D)P(W')$$

$$= (0.9 \times 0.8 \times 0.4) + (0.8 \times 0.8 \times 0.6) + (0.3 \times 0.2 \times 0.4) + (0.1 \times 0.2 \times 0.6)$$

$$= 0.288 + 0.384 + 0.024 + 0.012$$

$$= 0.708$$

$$\boxed{P(D|N) = \frac{0.708 \times 0.2}{0.3384} = 0.41844}$$