Discrete Mathematics and Graph Theory (BMAT205L)

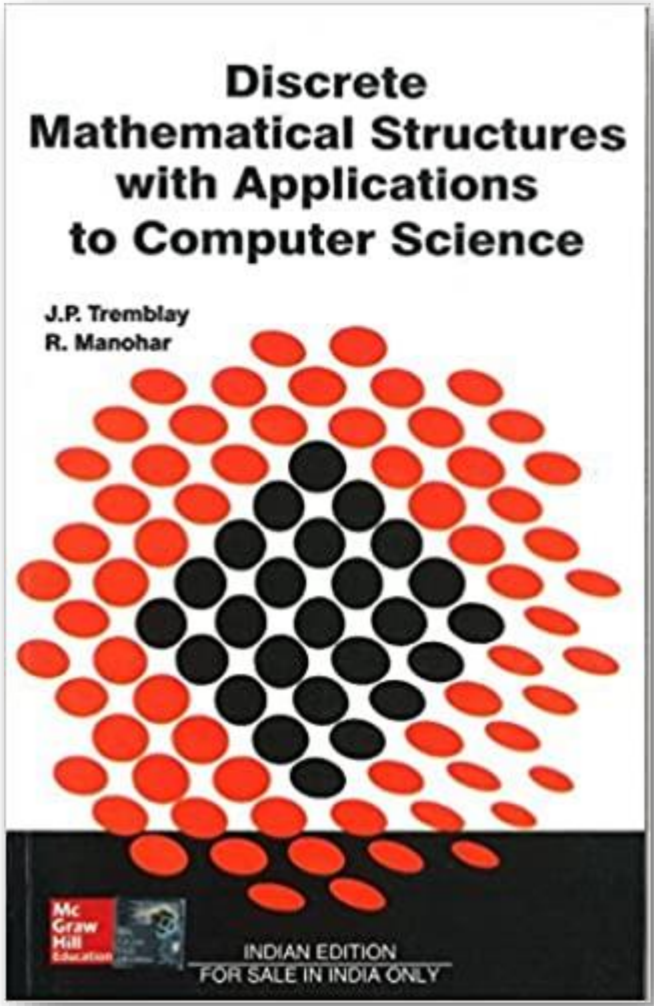
Discrete Mathematics

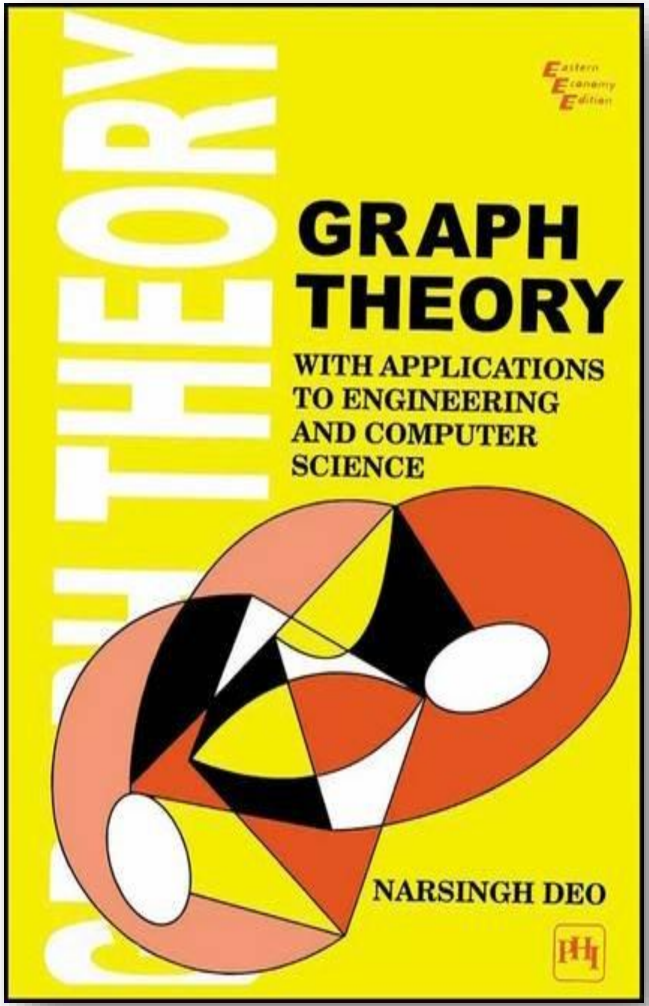
- It is the study of mathematical structures that are fundamentally discrete in nature and it does not require the notion of continuity.
- It is also called **Decision Mathematics** **or** **Finite Mathematics.**
- It is used in programming languages, software development, cryptography, algorithms etc. Due to its application in Computer Science, it has become popular in recent decades.
- Discrete Mathematics covers some important concepts such as **Set Theory, Logic**, **Permutation, Combination and Graph Theory.**

Module 1: Mathematical Logic

Statements and Notation-Connectives - Tautologies-Equivalence - Implications - Normal forms - The Theory of Inference for the Statement Calculus - Predicate Calculus - Inference Theory of the Predicate Calculus

Books for References





Mathematical Logic

Logic

Language for reasoning

Types of Logic

- Logic of sentences (Propositional logic)
- Logic of objects (Predicate logic) [Extension of Propositional logic]
- Logic involving uncertainities [applying the rules of probability]
- Logic dealing with fuzziness [uses the context of fuzzy sets]
- Temporal logic [Reasoning related to time]

Proposition logic

Proposition (or Statement)

is a declarative sentence which is either true or false but not both.

Example

- 6 is a prime number
- $1 + 1 = 2$
- The sun rises in North

Are these Propositions? Try !!!

- $X + Y = 2$
- Are these Propositions?

Note

Question, Command, Exclamation - are not propositions.

Elements of Propositional logic

Every proposition is represented by a variable denoted by capital letters. If a proposition is true, we say that its truth value is True; else we say it as False

Example

Let P denote the proposition 6 is a prime number.
The truth value of P is False

Connectives

Compound propositions are constructed from simple (atomic) propositions by combining them with connectives. The five basic connectives are: Negation ($\neg$), Conjunction ($\wedge$), Disjunction ($\vee$), Implication ($\rightarrow$), Biconditional ($\leftrightarrow$).

Connectives

Negation

The truth table for Not ($\neg$) of P is

P	$\neg P$
0	1
1	0

Table 1: Negation

Example

Let P denote "Today is Friday".
Then $\neg P$ is "Today is not Friday"
Or "It is not the case that today is Friday"

Connectives

Conjunction

Let P and Q be two propositional variables. Then
The truth table for And ($\wedge$) of P and Q is

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

Table 2: Conjunction

Example

- It is raining and it is warm
- $(2+3=5) \wedge (\sqrt{2} < 2)$

Connectives

Disjunction (Inclusive OR)

The truth table for Disjunction ($\vee$) of P and Q (Inclusive OR) is

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

Table 3: Disjunction

Example

- It is raining or this is my first online lecture.
- $(2+2=5) \vee (\sqrt{2} < 2)$

Connectives

XOR (Exclusive Or)

The truth table for Exclusive Or ($\oplus$) of P and Q is

P	Q	$P \oplus Q$
0	0	0
0	1	1
1	0	1
1	1	0

Table 4: $P \oplus Q$

Example

- The circuit is either on or off
- Let $ab < 0$. Then either $a < 0$ or $b < 0$ but not both

Connectives

Logical implication

The implication $P \to Q$ is the proposition that is false when P is true and Q is false and true otherwise.
P - "Premise" or "Hypothesis" or "Antecedent";
Q - "Conclusion" or "Consequence".
The truth table for $P \to Q$ is

P	Q	$P \to Q$
0	0	1
0	1	1
1	0	0
1	1	1

Table 1: Implication

Logical implication

The implication $P \to Q$ is equivalent to

- If P, then Q
- P only if Q
- Q when P
- Q whenever P
- Q if P
- Q unless not P
- Q follows from P
- P is a sufficient condition for Q
- Q is a necessary condition for P

Example

- If -4 is a positive number, then $2 + 2 = 5$
  Since the hypothesis is false and no matter what the conclusion is, the implication will evaluate to True
- If you buy your air ticket in advance, it is cheaper. True

Connectives

Biconditional

The bicondition $P \leftrightarrow Q$ is the proposition that is true when P and Q have the same truth values. It is false otherwise.
Note that it is equivalent to $(P \to Q) \wedge (Q \to P)$
The truth table for $P \leftrightarrow Q$ is

P	Q	$P \leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

Table 2: Biconditional

Biconditional

Equivalent statements for biconditional

$P \leftrightarrow Q$ is equivalent to

- P if and only if Q
- P is necessary and sufficient for Q
- If P then Q, and conversely
- P iff Q

Example

- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
- You may have pudding if and only if you eat breakfast

Converse and Contrapositive

Consider the implication $P \to Q$. Then

- the converse is $Q \to P$
- the contrapositive is $\neg Q \to \neg P$
- the inverse is $\neg P \to \neg Q$

Example

Write the contrapositive, the converse, and the inverse of the conditional statement The home team wins whenever it is raining

The given statement can be rewritten as If it is raining, then the home team wins

- Contrapositive - If the home team does not win, then it is not raining
- Converse - If the home team wins, then it is raining
- Inverse - If it is not raining, then the home team does not win

Order of preference

- Connectives in paranthesis
- $\neg$
- $\wedge$, $\vee$
- $\rightarrow$
- $\leftrightarrow$

English words related to the connectives

- Negation ($\neg$) - not, It is not true that, It is false that
- Or ($\vee$) - or
- And ($\wedge$) - and, but, In addition, Moreover, Also
- Biconditional ($\leftrightarrow$) - is necessary and sufficient, if and only if

Problems

1. Find the truth table for $(Q \wedge \neg P) \to P$

P	Q	$\neg P$	$Q \wedge \neg P$	$(Q \wedge \neg P) \to P$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	0	0	1

Problems

2. Let P and Q be propositions:
P : It is below freezing
Q : It is snowing
Write these propositions using P and Q and logical connectives

(a) It is below freezing and snowing $P \wedge Q$
(b) It is below freezing but not snowing $P \wedge \neg Q$
(c) It is not below freezing and it is not snowing $\neg P \wedge \neg Q$
(d) It is either snowing or below freezing (or both) $P \vee Q$
(e) If it is below freezing, it is also snowing $P \to Q$
(f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing $(P \oplus Q) \wedge (P \to \neg Q)$
(g) That it is below freezing is necessary and sufficient for it to be snowing $P \leftrightarrow Q$

Problems for Practice

3. Let P and Q be propositions: The election is decided and The votes have been counted. Express each of these compound propositions as an English sentence.
   - $\neg P$
   - $P \vee Q$
   - $\neg P \wedge Q$
   - $Q \rightarrow P$
   - $\neg Q \rightarrow \neg P$
   - $\neg P \rightarrow \neg Q$
   - $P \leftrightarrow Q$
   - $\neg Q \vee (\neg P \wedge Q)$
4. Write the negation of the following sentences:
   - The river is shallow or polluted
   - Jack is tall and thin
   - He swims iff the water is warm

Well formed formula

Well formed formula

- Every propositional variable is a Well formed formula
- Negation of a propositional variable is a Well formed formula
- If P and Q are Well formed formula then $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, $P \leftrightarrow Q$ are Well formed formula

Dual

Two formulas P and P^* are said to duals of each other if either one can be obtained by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$

Try this

Find the dual of $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$

Tautology

Tautology

is a compound proposition which is always true for every value of its propositional variables

Contradiction

is a compound proposition which is always false for every value of its propositional variables.

Contingency

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example:

P	Q	$P \vee Q$	$P \vee \neg P$	$P \wedge \neg P$
1	1	1	1	0
1	0	1	1	0
0	1	1	1	0
0	0	0	1	0

Contingency

Tautology

Contradiction

Valid, Satisfiable and Unsatisfiable

- A formula is valid, if for all combination of the truth values for the variables, the formula evaluates to true. In otherwords, the formula is a tautology
- A formula is satisfiable, if there exists at least one combination of truth values for the variables which evaluates to true. In otherwords, the formula is a contingency
- A formula is unsatisfiable, if there exists no combination of truth values for the variable which evaluates to true. In otherwords, the formula is a contradiction

Problem

Prove that $[(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a valid formula.

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee q$	$\neg(\neg p \wedge (\neg q \vee \neg r))$
1	1	1	0	0	0	1	$\neg(0 \wedge 0) = 1$
1	1	0	0	0	1	1	$\neg(0 \wedge 1) = 1$
1	0	1	0	1	0	1	$\neg(0 \wedge 1) = 1$
1	0	0	0	1	1	1	$\neg(0 \wedge 1) = 1$
0	1	1	1	0	0	1	$\neg(1 \wedge 0) = 1$
0	1	0	1	0	1	1	$\neg(1 \wedge 1) = 0$
0	0	1	1	1	0	0	$\neg(1 \wedge 1) = 0$
0	0	0	1	1	1	0	$\neg(1 \wedge 1) = 0$

Problem

Prove that $[(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a valid formula.

p	q	r	$\neg p$	$\neg q$	$\neg r$	$(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$
1	1	1	0	0	0	$(0) \vee (0) = 0$
1	1	0	0	0	1	$(0) \vee (0) = 0$
1	0	1	0	1	0	$(0) \vee (0) = 0$
1	0	0	0	1	1	$(0) \vee (0) = 0$
0	1	1	1	0	0	$(0) \vee (0) = 0$
0	1	0	1	0	1	$(0) \vee (1) = 1$
0	0	1	1	1	0	$(1) \vee (0) = 1$
0	0	0	1	1	1	$(1) \vee (1) = 1$

Problem

Prove that $[(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a valid formula.

$\neg(\neg p \wedge (\neg q \vee \neg r))$	$(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$	$(\quad) \vee (\quad)$
$\neg(0 \wedge 0) = 1$	$(0) \vee (0) = 0$	1
$\neg(0 \wedge 1) = 1$	$(0) \vee (0) = 0$	1
$\neg(0 \wedge 1) = 1$	$(0) \vee (0) = 0$	1
$\neg(0 \wedge 1) = 1$	$(0) \vee (0) = 0$	1
$\neg(1 \wedge 0) = 1$	$(0) \vee (0) = 0$	1
$\neg(1 \wedge 1) = 0$	$(0) \vee (1) = 1$	1
$\neg(1 \wedge 1) = 0$	$(1) \vee (0) = 1$	1
$\neg(1 \wedge 1) = 0$	$(1) \vee (1) = 1$	1

Tautology

Logical Equivalence ($P \equiv Q$ or $P \Leftrightarrow$ Q)

- ❖ The two compound propositions P and Q are called logically equivalent if and only if the truth values of P and Q are equal.
- ❖ Equivalently, two compound propositions P and Q are called logically equivalent if $P \leftrightarrow Q$ is a tautology.
- ❖ It is denoted by $P \equiv Q$ or $P \Leftrightarrow$ Q.

Note

Two ways of proving logical equivalence:

- Truth table
- Logical laws

Problem using Truth table method

Show that $P \to Q$; $\neg Q \to \neg P$; and $\neg P \vee Q$ are logically equivalent

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$	$\neg P \vee Q$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	1	0	0	1	1

Logical Laws

Idempotent Law	$p \vee p \Leftrightarrow p$
Idempotent Law	$p \wedge p \Leftrightarrow p$
Identity Law	$p \wedge T \Leftrightarrow p$
Identity Law	$p \vee F \Leftrightarrow p$
Domination Law	$p \vee T \Leftrightarrow T$
Domination Law	$p \wedge F \Leftrightarrow F$
Double negation Law	$\neg\neg p \Leftrightarrow p$
De Morgan's Law	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
De Morgan's Law	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
Distributive Law	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
Distributive Law	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Logical laws

	Absorption Law	$p \wedge (p \vee q) \Leftrightarrow p$
	Absorption Law	$p \vee (p \wedge q) \Leftrightarrow p$
	Commutative	$p \wedge q \Leftrightarrow q \wedge p$
	Commutative	$p \vee q \Leftrightarrow q \vee p$
	Associative Law	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
	Associative Law	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
Negation Law	Inverse Law	$p \wedge \neg p \Leftrightarrow F$
	Inverse Law	$p \vee \neg p \Leftrightarrow T$
	Conditional Law	$p \rightarrow q \Leftrightarrow \neg p \vee q$

TABLE **Logical Equivalences Involving Conditional Statements.**

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE **Logical Equivalences Involving Biconditional Statements.**

$\boxed{p \leftrightarrow q \equiv (p \to q) \land (q \to p)}$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$\boxed{p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)}$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Logical equivalence by laws

Problem 1

Without using truth table, prove that $p \to (q \to r)$, $p \to (\neg q \vee r)$, $(p \wedge q) \to r$ are logically equivalent.

Solution

Consider $p \to (q \to r)$

$\equiv p \to (\neg q \vee r)$ (Conditional law)

$\equiv \neg p \vee (\neg q \vee r)$ (Conditional law)

$\equiv (\neg p \vee \neg q) \vee r$ (Associative law)

$\equiv \neg(p \wedge q) \vee r$ (De Morgan's law)

$\equiv (p \wedge q) \to r$ (Conditional law)

Problem 2

Without using truth table, prove $\neg p \to (q \to r) \equiv q \to (p \vee r)$.

Solution

Consider $\neg p \to (q \to r)$

$\equiv \neg p \to (\neg q \vee r)$ (Conditional law)

$\equiv p \vee (\neg q \vee r)$ (Conditional law)

$\equiv (p \vee \neg q) \vee r$ (Associative law)

$\equiv (\neg q \vee p) \vee r$ (Commutative law)

$\equiv \neg q \vee (p \vee r)$ (Associative law)

$\equiv q \to (p \vee r)$ (Conditional law)

Problem 3

Without using truth table, prove $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$.

Solution

Consider $(\neg p \vee q) \wedge (p \wedge (p \wedge q))$

$\equiv (\neg p \wedge (p \wedge (p \wedge q)) \vee (q \wedge (p \wedge (p \wedge q)))$ (Distributive law)

$\equiv F \vee ((q \wedge p) \wedge (p \wedge q))$ (Inverse law)

$\equiv (p \wedge q)$ (Idempotent law).

More problems

Problem (Solve using Truth tables and using logical laws)

1. Show that $\neg(P \vee (\neg P \wedge Q))$ and $\neg P \wedge \neg Q$ are logically equivalent.
2. Show that $(P \rightarrow Q) \wedge (P \rightarrow R)$ and $P \rightarrow (Q \wedge R)$ are logically equivalent.

Normal Forms and Principal of Normal Forms

Normal Form

1. The problem of determining, in a finite number of steps, whether a given statement formula is tautology or a contradiction or contingency is known as **a *decision problem.***
2. The solution of the decision problem may not be simple in general. Also the construction of truth tables may not be practical, even with the aid of a computer.
3. Therefore consider other procedures known as reduction to ***normal forms***.

Normal forms

Conjunctive Normal Form (CNF)

A formula is in CNF if it is a conjunction of one or more clauses C_i where each clause C_i is a disjunction of literals (variables or its negation).

$$C_1 \wedge C_2 \wedge C_3 \wedge \dots$$

Example

$$(\neg p \vee q) \wedge (r \vee \neg t \vee \neg p) \wedge p$$

Here $C_1 = \neg p \vee q$, $C_2 = r \vee \neg t \vee \neg p$, $C_3 = p$

Normal forms

Disjunctive Normal Form (DNF)

A formula is in DNF if it is a disjunction of one or more clauses C_i where each clause C_i is a conjunction of literals (variables or its negation).

$$C_1 \vee C_2 \vee C_3 \vee \dots$$

Example

$$(p \wedge \neg q \wedge r) \vee (\neg p \wedge q) \vee (q \wedge r)$$

Here $C_1 = p \wedge \neg q \wedge r$, $C_2 = \neg p \wedge q$, $C_3 = q \wedge r$

Problems

1. Find the DNF of $(\neg p \to q) \wedge (q \leftrightarrow p)$

Solution:

$(\neg p \to q) \wedge (q \leftrightarrow p)$
$\equiv (p \vee q) \wedge [(p \wedge q) \vee (\neg p \wedge \neg q)]$ (conditional law)
$\equiv [(p \vee q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge (\neg(p \vee q))]$ (Distributive law, De Morgan's law)
$\equiv [(p \vee q) \wedge (p \wedge q)] \vee F$ (Inverse law)
$\equiv [(p \vee q) \wedge (p \wedge q)]$ (Identity law)
$\equiv [p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)]$ (Distributive law)
$\equiv [(p \wedge p) \wedge q] \vee [(q \wedge p) \wedge q]$ (Associative law)
$\equiv (p \wedge q) \vee [(p \wedge q) \wedge q]$ (Idempotent, commutative laws)
$\equiv (p \wedge q) \vee (p \wedge (q \wedge q))$ (Associative law)
$\equiv (p \wedge q) \vee (p \wedge q)$ (Idempotent law)
$\equiv (p \wedge q)$ (Idempotent law)

2. Find the CNF of $\neg(p \rightarrow (q \wedge r))$

Solution

$\neg(p \rightarrow (q \wedge r))$

$\equiv \neg[\neg p \vee (q \wedge r)]$ (Conditional law)

$\equiv \neg\neg p \wedge \neg(q \wedge r)$ (De Morgan's law)

$\equiv p \wedge \neg(q \wedge r)$ (Double negation law)

$\equiv p \wedge (\neg q \vee \neg r)$ (De Morgan's law)

Minterms and Maxterms

1. *Minterms* or *Boolean conjunctions of P and Q*:

$$P \wedge Q$$

$$P \wedge \neg Q$$

$$\neg P \wedge Q$$

$$\neg P \wedge \neg Q$$

2. *Maxterms* or *Boolean disjunctions of P and Q*:

$$P \vee Q$$

$$P \vee \neg Q$$

$$\neg P \vee Q$$

$$\neg P \vee \neg Q$$

PDNF and PCNF

Principal Conjunctive Normal Form

A CNF where each clause is a maxterm. If p and q are the variables involved in the formula then the maxterms are
$p \vee q, \neg p \vee \neg q, \neg p \vee q, p \vee \neg q$.

Example

$$(p \vee \neg q) \wedge (\neg p \vee \neg q)$$

Principal Disjunctive Normal Form

A DNF where each clause is a minterm. If p and q are the variables involved in the formula then the minterms are $p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q$.

Example

$$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$$

- Each *minterm* has the **truth value *T*** for exactly one combination of the truth values of the variables P and Q.
- If the truth table of any formula containing only the variables P and Q is known, then one can easily obtain an equivalent formula which consists of a disjunction of some of the minterms.

- Each *maxterm* has the **truth value *F*** for exactly one combination of the truth values of the variables P and Q.
- If the truth table of any formula containing only the variables P and Q is known, then one can easily obtain an equivalent formula which consists of a conjunction of some of the maxterms

Methods to find PDNF and PCNF

There are two methods of finding PDNF and PCNF

- Algebraic method
- Truth table method

PDNF and PCNF (using algebraic method)

1. Find the PDNF of $(p \vee q) \leftrightarrow (p \wedge q)$

Solution:

$(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv [(p \vee q) \wedge (p \wedge q)] \vee [\neg(p \vee q) \wedge \neg(p \wedge q)]$ (By definition)

$\equiv [(p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)] \vee [(\neg p \wedge \neg q) \wedge (\neg p \vee \neg q)]$
(Distributive law, De Morgan's law)

$\equiv [(p \wedge p) \wedge q] \vee [q \wedge (q \wedge p)] \vee [((\neg p \wedge \neg q) \wedge \neg p) \vee ((\neg p \wedge \neg q) \wedge \neg q)]$
(Associative, Commutative, Distributive law)

$\equiv (p \wedge q) \vee (q \wedge p) \vee (\neg p \wedge \neg q)$ (Associative, Idempotent, commutative law)

$\equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ (Commutative, Idempotent law)
which is a PDNF

2. Find the PDNF of $p \wedge (p \to q)$

Solution:

$p \wedge (p \to q)$

$\equiv p \wedge (\neg p \vee q)$ (Conditonal law)

$\equiv (p \wedge \neg p) \vee (p \wedge q)$ (Distributive law)

$\equiv F \vee (p \wedge q)$ (Inverse law)

$\equiv p \wedge q$ (Identity law)

PDNF and PCNF (using truth table method)

3. Find the PDNF and PCNF of $p \wedge (p \to q)$

Solution

p	q	$p \to q$	$p \wedge (p \to q)$	Rows
0	0	1	0	Row 1
0	1	1	0	Row 2
1	0	0	0	Row 3
1	1	1	1	Row 4

PDNF: $(p \wedge q)$ [by considering 1 in the column 4]

PCNF: $\neg Row1 \wedge \neg Row2 \wedge \neg Row3$

$\equiv \neg(\neg p \wedge \neg q) \wedge \neg(\neg p \wedge q) \wedge \neg(p \wedge \neg q)$

$\equiv (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q)$

4. Find the PDNF and PCNF of $(p \vee q) \leftrightarrow (p \wedge q)$

Solution:

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \leftrightarrow (p \wedge q)$	Rows
0	0	0	0	1	Row 1
0	1	1	0	0	Row 2
1	0	1	0	0	Row 3
1	1	1	1	1	Row 4

PDNF: $Row1 \vee Row4 \equiv (\neg p \wedge \neg q) \vee (p \wedge q)$

PCNF: $Row2 \wedge \ Row3 \equiv (p \vee \neg q) \wedge (\neg p \vee q)\ )$

Note:

1. No.of truth value T appears in the statement formula is same as the no.of minterms appears in its PDNF. Hence if all the minterms appears in its PDNF then the given statement formula is a tautology and is a valid formula.
2. No.of truth value F appears in the statement formula is same as the no.of maxterms appears in its PCNF. Hence if all the maxterms appears in its PCNF then the given statement formula is a contradiction and is an unsatisfiable formula.

Problems for Practice:

5. Find the PCNF and PDNF of the following:

   a. $(a \to b) \vee c$

   b. $a \wedge (b \leftrightarrow c)$

   c. $(a \leftrightarrow b) \leftrightarrow c$

   d. $(a \to b) \wedge (\neg a \to \neg b)$

   e. $(P \to (Q \wedge R)) \wedge (\neg P \to (\neg Q \wedge \neg R))$

Rules of Inference in Propositional Logic

Basic Terminologies

- **Premise** is a proposition on the basis of which we would able to draw a conclusion.
  - We can think of premise as an evidence or assumption. Therefore, initially we assume something is true and on the basis of that assumption we draw some conclusion.
- **Conclusion** is a proposition that is reached from the given set of premises.
  - We can think of it as the result of the assumptions that we made in an argument.

If **Premises** then **Conclusion**

Argument – Valid and Invalid

- **Argument** is sequence of statements that ends with a conclusion or it is a set of one(or more) premises and a conclusion.
- **Valid Argument:** An argument is said to be valid argument ***if and only if*** it is not possible to make all premises true and a conclusion false.

An **argument** in propositional logic is a sequence of propositions. An argument is **valid** if the premises imply the conclusion. That is, if all its premises are true, then the conclusion is true.

- **Invalid Argument:** An argument is said to be an invalid argument if it is not a valid argument.

Valid Argument

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \\ \hline \end{array}$$

or

$$((p \to q) \land p) \to q$$

✓

Invalid Argument

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \\ \hline \end{array}$$

or

$$((p \to q) \land q) \to p$$

✗

Inference and Rules of Inference

- **Inference** is a conclusion(s) derived on the basis of the evidence(s).
- **Rules of Inference** are the templates for constructing valid arguments.

Rules of inference is a step by step validation to arrive at conclusion q logically from the premises $p_1, p_2, \ldots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$$

Types of Inference Rules

Rule	Tautology	Name
p ——— $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ ——— $\therefore p$ or q	$(p \wedge q) \to p$ or $(p \wedge q) \to q$	Simplification
p q ——— $\therefore p \wedge q$	$(p) \wedge (q) \to (p \wedge q)$	Conjunction
p $p \to q$ ——— $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus Ponens

Rule	Tautology	Name
$\neg q$ $p \to q$ ——— $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus Tollens
$p \to q$ $q \to r$ ——— $\therefore p \to r$	$[(p \to q) \land (q \to r)] \to (p \to r)]$	Hypothetical Syllogism
$p \lor q$ $\neg p$ ——— $\therefore q$	$[(p \lor q) \land \neg p] \to q$	Disjunctive syllogism
$p \to q$ $\neg q \to r$ ——— $\therefore q \lor r$	$[(p \to q) \land (\neg q \to r) \to q \lor r]$	Resolution

P	Q	$P \to Q$	$P \wedge (P \to Q)$	$\big(P \wedge (P \to Q)\big) \to Q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

p $p \to q$ ——— $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus Ponens

Basic rules of inference

- **Rule P:** A premise may be introduced at any step in the derivation
- **Rule T:** A formula S may be introduced in the derivation, if S is tautologically implied by one or more preceding formulas in the derivation
- **Rule CP:** If a formula S can be derived from another formula R and a set of premises, then the statement $R \to S$ can be derived from the set of premises alone. This is also called as Conditional Proof.

Direct and Indirect Proof

Consider the implication

$$P \Rightarrow C$$

where

$$P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$$

is the conjunction of premises and C is the conclusion to be proven.
To prove that the argument is Valid

- direct proof (using P to directly show C)
- indirect proof
  - proof by contrapositive argument $(\neg C \Rightarrow \neg P)$
  - proof by contradiction

Step 1: Assume $\neg C$ is True

Step 2: Arrive at a stage that a set of premises is inconsistent

(i.e.) $p \wedge \neg p$ or $q \wedge \neg q$ or $r \wedge \neg r$ is False

Problems using rules of inference

1. Show that $t \wedge s$ can be derived from the premises $p \to q, q \to \neg r, r, p \vee (t \wedge s)$

Solution

Steps	Logical Statement	Reason
1	$q \to \neg r$	Rule P, Premise
2	r	Rule P, Premise
3	$\neg q$	Rule T, Modus tollens of 1 and 2
4	$p \to q$	Rule P, Premise
5	$\neg p$	Rule T, Modus tollens of 3 and 4
6	$p \vee (t \wedge s)$	Rule P, Premise
7	$t \wedge s$	Rule T, Disjunctive Syllogism of 5 and 6

Problems

2. Show that $r \to \neg q, r \vee s, s \to \neg q, p \to q \implies \neg p$

Solution

Steps	Logical Statement	Reason
1	$r \to \neg q$	Rule P, Premise
2	$s \to \neg q$	Rule P, Premise
3	$r \vee s \to \neg q$	Rule T, Logical equivalence using 1 and 2
4	$r \vee s$	Rule P, Premise
5	$\neg q$	Rule T, Modus ponens of 3 and 4
6	$p \to q$	Rule P, Premise
7	$\neg p$	Rule T, Modus tollens of 5 and 6

Problems

3. Prove the following using the rules of inferences:If it reains heavily, then travelling will be difficult. If students arrive on time, then travelling is not difficult. They arrived on time. Therefore, it didnot rain heavily

Solution:

Let P : It rains heavily, Q : Travelling is difficult, R : Students arrive on time.

$P \to Q, R \to \neg Q, R$ are the premises

Steps	Logical Statement	Reason
1	$R \to \neg Q$	Rule P, Premise
2	R	Rule P, Premise
3	$\neg Q$	Rule T, Modus ponens of 1 and 2
4	$P \to Q$	Rule P, Premise
5	$\neg P$	Rule T, Modus tollens of 3 and 4

Direct and Indirect Proof

Consider the implication

$$P \Rightarrow C$$

where

$$P = P_1 \wedge P_2 \wedge \ldots \wedge P_n$$

is the conjunction of premises and C is the conclusion to be proven.
To prove that the argument is Valid

- direct proof (using P to directly show C)
- indirect proof
  - proof by contrapositive argument ($\neg C \Rightarrow \neg P$)
  - proof by contradiction

Step 1: Assume $\neg C$ is True

Step 2: Arrive at a stage that a set of premises is inconsistent

(i.e.) $p \wedge \neg p$ or $q \wedge \neg q$ or $r \wedge \neg r$ is False

Indirect Proof

INCONSISTENT PREMISES

A set of premises (formulas) $H_1, H_2, \ldots H_n$ is said to be inconsistent, if their conjunction implies a contradiction.
viz. if $H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow R \wedge \neg R$, for some formula R.

A set of premises is said to be consistent, if it is not inconsistent.

INDIRECT METHOD OF PROOF

The notion of inconsistency is used to derive a proof at times. This procedure is called the indirect method of proof or proof by contradiction or reduction and absurdum.

In order to show that a conclusion C follows from the premises $H_1, H_2, \ldots H_n$ by this method, we assume that C is false and include $\neg C$ as an additional premise. If the new set of premises is inconsistent leading to a contradiction, then the assumption that $\neg C$ is true does not hold good. Hence C is true whenever $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is true. Thus C follows from $H_1, H_2, \ldots H_n$.

Example

For example, we prove that the premises $\neg q$, $p \rightarrow q$ result in the conclusion $\neg p$ by the indirect method of proof.

Sol:- We prove by contradiction:

↳ Assume, $\neg$(conclusion) is true.

(i.e.) $\neg(\neg p)$ is true.

Given:

$P_1 : \neg q$

$P_2 : p \rightarrow q$

Concl: $\neg p$.

Steps	Statement	Rules
1	$\neg(\neg p)$	Rule P
2	p	Rule T, Double negation of 1
3	$p \rightarrow q$	Rule P

4	$\neg q \rightarrow \neg p$	Rule T, contrapositive of 3
5	$\neg q$	Rule P
6	$\neg p$	Rule T, Modus ponens of 4 & 5
7	$p \wedge \neg p$	Rule T, Conjunction of 2 & 6

Here, $\neg$(conclusion) $= \neg(\neg p)$ leads to a contradiction. Therefore, $\neg p$ is true.

(i.e.) $\neg q,\ p \rightarrow q \Rightarrow \neg p$

Indirect method

4. Prove the following using indirect method:
$p \to q, q \to r, \neg(p \wedge r), p \vee r \Rightarrow r$

Solution

For indirect method, we consider the negation of the conclusion as an additional premise and arrive at a contradiction.
$\neg r$ is included as a premise.

Steps	Logical Statement	Reason
1	$\neg r$	Rule P (additional), Premise
2	$p \to q$	Rule P
3	$q \to r$	Rule P
4	$p \to r$	Rule T, Hypothetical Syllogism of (2) and (3)
5	$\neg p$	Rule T, Modus Tollens of (1) and (4)
6	$p \vee r$	Rule P
7	r	Rule T, Disjunctive Syllogism of (1) and (6)
8	$r \wedge \neg r$	Rule T, Conjunction of (1) and (7)

Here, $\neg$(Conclusion) = $\neg r$ leads to a contradiction.
Hence $p \to q, q \to r, \neg(p \wedge r), p \vee r \Rightarrow r$ is valid.

Practice

Contradiction:

1. $\neg$(conclusion) is True
2. we need to arrive at False.

Problem:

1. Show that b can be derived from the premises

$P_1: a \to b$

$P_2: c \to b$

$P_3: d \to (a \vee c)$

$P_4: d$

$\boxed{C: b}$

using the contradiction method

Step	Statement	Reason
1	$\neg(b)$	Rule P (additional)
2	$a \to b$	Rule P
3	$c \to b$	Rule P
4	$(a \vee c) \to b$	Rule T, Equivalence 2 & 3
5	$d \to (a \vee c)$	Rule P.
6	$d \to b$	Rule T, Hypoth. Syllogism.
7	d	Rule P
8	b	Rule T, Modus ponens 6 & 7
9	$b \wedge \neg b \equiv F$	Rule T, Conjunction 1 & 8

CP rule

5. Derive $p \to s$ using the CP-rule from the premises $\neg p \vee q, \neg q \vee r, r \to s$.

Solution

By CP-rule, we assume an additional premise p along with the given premises and prove s.

Steps	Logical Statement	Reason
1	p	Rule P (additional)
2	$\neg p \vee q$	Rule P
3	q	Rule T, Disjunctive Syllogism of (1) and (2)
4	$\neg q \vee r$	Rule P
5	r	Rule T, Disjunctive Syllogism of (3) and (4)
6	$r \to s$	Rule P
7	s	Rule T, Modus Ponens of (5) and (6)

Problems

1. Show that the following argument is valid. If today is Tuesday, I have a test in Mathematics or Economics. If my Economics professor is sick, I will not have a test in Economics. Today is Tuesday, and my Economics professor is sick. Therefore, I will have a test in Mathematics

2. Show that $p \to q, r \to s, q \to t, s \to u, \neg(t \wedge u), p \to r \implies \neg p$

3. Check the validity of the argument.

If John gets the office position and works hard, then he will get bonus. If he gets bonus, then he will go on a trip . He didnot go on a trip. Therefore, either he did not get the office position or he didnot work hard.

4. Show that the following set of premises is inconsistent.

If Rama gets her degree, she will go for a job. If she goes for a job, she will get married soon. If she goes for higher studies, she will not get married. Rama gets her degree and goes for higher studies.

5. Check whether the following set of premises is consistent.
If the contract is valid then John is liable for penalty. If John is liable for penalty he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact,the contract is valid and the bank will loan him money

6. Determine the validity of the following argument.
If my father praises me, then I can be proud of myself. Either I do well in sports or I can't be proud of myself. If I study hard, then I cannot do well in sports. Therefore, If my father praises me then I donot study well.

7. Show that $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \implies r$ [Hint: use logical laws]

8. Show that $(p \to (q \to r)) \wedge (q \to (r \to s)) \implies p \to (q \to s)$

9. Prove the following: (a) $p \to q, q \to r, \neg(p \wedge q), p \vee r \implies r$, (b) $c \vee d, (c \vee d) \to \neg h, \neg h \to (a \wedge \neg b), (a \wedge \neg b) \to (r \vee s) \implies r \vee s$, (c) $p \to (q \to s), \neg r \vee p, q \implies r \to s$

10. Use indirect method of proof to verify the following: (i) $\neg(p \wedge q)$ follows from $\neg p \vee \neg q$, (ii) $\neg q$ follows from $p \to q, \neg p$

11. Write the inverse, converse, and contrapositive of "Unless you report to the exam cell before 9 am, you will not be permitted to write the exam"

12. Consider a situation where 3 men (A,C,E) and 2 women (B,D) are travelling in a train. The train passes through a tunnel and when it emerges, it is found that E is murdered. An inquiry is held where A, B,C,D made the following statements. Each person makes two statements.
A says : I am innocent. B was talking to E when the train was passing through the tunnel.
B says: I am innocent. I was not talking to E when the train was passing through the tunnel.
C says: I am innocent. D committed the murder.
D says: I am innocent. One of the men committed the murder.
Out of the 8 statements given above, 4 are true and 4 are false. Who is the murderer. Support your answer with a precise and concise justification.

Predicate Calculus

Predicate
&
Quantifiers

Statement: "x is greater than 10"

What is the truth value of this statement?

Is it True or False for all x values?

Is it True or False for some x values?

Is this a Proposition ?
If not, then how to make it as a Proposition?

Predicates and Quantifiers

Predicate:-

Grammar:- The part of a sentence or clause containing a verb and stating something about the subject.

Logic:- Something which is affirmed or denied concerning an argument of a proposition.

Eg:- The statement

"x is greater than 10"

Subject — x

Predicate — is greater than 10

Now, let $P(x) \rightarrow x$ is greater than 10.

~~Here $P \rightarrow$ Predicate~~

Here, $P \rightarrow$ Predicate

$x \rightarrow$ variable

value has been

⇒ Once a ______________ assigned to the variable; then the statement $P(x)$ becomes a proposition and has a truth value.

Predicate logic:

Predicate Logic is a logical extension of propositional logic. Also known as First order logic.

Domain or Universe of discourse:

UOD or Domain of a predicate variable is the collection of all possible values that the variable may take.

Problem:-

(i) Given: "x is greater than 10"

$x \rightarrow$ variable

$P \rightarrow$ Peridicate "is greater than 10".

$P(x) \rightarrow x$ is greater than 10.

Now, $P(x)$ can not have a truth value T or F, since x is not known to us.

(2) Now, we can assign a value to the variable x, so that it becomes a propositional.

Let, $x = 5$ and $x = 15$

(3) $P(x) \equiv$ "$x > 10$"

$P(5) \equiv$ "$5 > 10$" is False.

$P(15) \equiv$ "$15 > 10$" is True.

(4) We need to convert the given statement into a proposition by assigning a value to the variable(s) involved in it. Then, we can go for the truth value for the proposition.

Eg:- $P(x, y) =$ " $x + y = 20$ "

Now, $P(10, 10)$ is True.

$P(2, 7)$ is False.

$P(13, 7)$ is True.

Page

for all   there exist

Logic

- **Predicate**
- **Quantifiers** ($\forall$ and $\exists$)

$P(x)$ or $P(x,y)$ $P(x)$ or $P(x,y)$.

→ Universal quantifier ($\forall$)

→ Existential quantifier ($\exists$)

Universal Quantifiers
vs
Existential Quantifiers

Universal quantifier	Existential quantifiers
① $P(x) \rightarrow$ Propositional function at 'x'	① $P(x) \rightarrow$ Propositional function at 'x'.
② $P(x)$ will be true for all values of x.	② There exists at least one x such that $P(x)$ is true.
③ Denoted :- $\forall x\ P(x)$	③ Denoted :- $\exists x\ P(x)$
④ Here, $\forall$ is called the universal quantifier.	④ Here, $\exists$ is called the existential quantifier.

(5) Eg:- $P(x) \equiv "x^2 > 10"$ and this is true for all values of x more than 4.

$\therefore \forall x \, P(x)$ is true

— x —

Eg:- $P(x) \equiv "x^2 > 10"$ and $x < 5$.

$\therefore \forall x \, P(x)$ is false

Eg:- $P(x) \equiv "x^2 > 10"$ and $x > 4$.

$\therefore \exists x \, P(x)$ is true.

— x —

Eg:- $P(x) \equiv "x^2 > 10"$ and $x < 5$

$\therefore \exists x \, P(x)$ is true since $P(4) \equiv "16 > 10"$ is true.

(b) In other words,

$$\forall x\, P(x) = \overbrace{P(x_1) \land P(x_2) \land \cdots \land P(x_n)}^{\text{All has to be true.}}$$
$$= T \land T \land \cdots \land T$$
$$\forall x\, P(x) = T, \quad \text{if } x_1 \le x \le x_n$$

In other words,

$$\exists x\, P(x) = \overbrace{P(x_1) \lor P(x_2) \lor \cdots \lor P(x_n)}^{\text{Any one has to be true}}$$
$$= T \lor F \lor \cdots \lor F$$
$$\exists x\, P(x) = T, \quad \text{if } x_1 \le x \le x_n$$

Universal Quantifiers

Examples

1. If $P(x) \equiv$ {(-x)^2 = x^2 } where the universe consists of all integers, then the truth value of $\forall x((-x)^2 = x^2)$ is T.
2. If $Q(x) \equiv$ "$2x > x$", where the universe consists of all real numbers, then the truth value of $\forall x\ Q(x)$ is F.
3. If $P(x) \equiv$ "$x^2 < 10$", where the universe consists of the positive integers 1, 2, 3 and 4, then $\forall x\ P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ and so the truth value of $\forall x\ P(x) = \text{T} \wedge \text{T} \wedge \text{T} \wedge \text{F} = \text{F}$.
4. Let $P(x) \equiv x$ is an integer and $Q(x) \equiv x$ is either positive or negative. Then $P(x) \to Q(x)$ is a compound propositional function. Obviously $\forall x(P(x) \to Q(x))$, where the universe of discourse consists of integers.

Existential Quantifiers

Examples

1. When $P(x)$ denotes the propositional function "$x > 3$", the truth value of $\exists x P(x)$ is T, where the universe of discourse consists of all real numbers, since "$x > 3$" is true for $x = 4$.

   Note When the elements of the universe of discourse is finitely many, viz., consists of $x_1, x_2, \ldots x_n$, then $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$, since this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

2. When $P(x)$ denotes "$x^2 > 10$", where the universe of discourse consists of the positive integers not exceeding 4, then the truth value of $\exists x P(x)$ is T, since $P(1) \vee P(2) \vee P(3) \vee P(4)$ is true as $P(4)$ [viz., $4^2 > 10$] is true.

Negation of a quantified expression.

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg(P(x))$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg(P(x)).$$

NEGATION OF A QUANTIFIED EXPRESSION

If $P(x)$ is the statement "x has studied computer programming", then $\forall\, xP(x)$ means that "every student (in the class) has studied computer programming". The negation of this statement is "It is not the case that every student in the class has studied computer programming" or equivalently "There is a student in the class who has not studied computer programming" which is denoted by $\exists x \neg P(x)$. Thus we see that $\neg\, \forall x P(x) \equiv \exists x \neg P(x)$.

Similarly, $\exists x P(x)$ means that "there is a student in the class who has studied computer programming "The negation of this statement is "Every student in this class has not studied computer programming", which is denoted by $\forall\, x\, \neg P(x)$. Thus we get

$$\neg\, \exists x P(x) \equiv \forall\, x\, \neg P(x)$$

Further we note that $\neg\, \forall\, xP(x)$ is true, when there is an x for which $P(x)$ is false and false when $P(x)$ is true for every x, since

$$\begin{aligned} \neg\, \forall\, xP(x) &\equiv \exists x \neg P(x) \\ &\equiv \neg P(x_1) \vee \neg P(x_2) \ldots \vee \neg P(x_n) \end{aligned}$$

$\neg \exists x P(x)$ is true, when $P(x)$ is false for every x and false when there is an x for which $P(x)$ is true,

since

$$\begin{aligned} \neg \exists x P(x) &\equiv \forall\, x\, \neg P(x) \\ &\equiv \neg P(x_1) \wedge \neg P(x_2) \ldots \wedge \neg P(x_n) \end{aligned}$$

NESTED (MORE THAN ONE) QUANTIFIERS

There are situations when quantifiers occur in combinations in respect of 1-place or n-place predicate formulas (i.e., propositional functions containing 1 or n variables). For example let us consider a 2-place predicate formula $P(x, y)$.

Now
$$\forall x \, \forall y \, P(x, y) \equiv \forall x[\forall y \, P(x, y)] \equiv \forall y[\forall x \, P(x, y)] \qquad (1)$$

and
$$\exists x \, \exists y \, P(x, y) \equiv \exists x[\exists y \, P(x, y)] \equiv \exists y[\exists x \, P(x, y)] \qquad (2)$$

From the meaning of quantifiers and by (1) and (2) the following simplifications hold good:

$$\forall x \, \forall y \, P(x, y) \Rightarrow (\exists y) \, \forall x \, P(x, y) \Rightarrow \forall x \, \exists y \, P(x, y)$$
$$\forall y \, \forall x \, P(x, y) \Rightarrow (\exists x) \, \forall y \, P(x, y) \Rightarrow \forall y \, \exists x \, P(x, y)$$

Note The negation of multiply quantified predicate formulas may be obtained by applying the rules for negation (given earlier) from left to right.

Thus
$$\neg[\forall x \, \exists y \, P(x, y)] \equiv \exists x[\neg \exists y \, P(x, y)]$$
$$\equiv \exists x \, \forall y[\neg P(x, y)]$$

Eg:- (Nested quantifiers).

Let $P(x, y)$: x is taller than y.

If x is taller than y, then y is not taller than x.

$$P(x, y) \rightarrow \neg P(y, x)$$

Note that this will be true for every x and y.

$$\therefore \forall x \forall y \, P(x, y) \rightarrow \neg P(y, x).$$

→ Free and bound variables

- Bound variable. (quantifiers)
- Scope. (Logical expression)
- Free variable.
  (variable which is not quantified).

FREE AND BOUND VARIABLES

When a quantifier is used on a variable x or when we have to assign a value to this variable to get a proposition, the occurrence of the variable is said to be *bound* or the variable is said to be a bound variable. An occurrence of a variable that is not bound by a quantifier or that is set equal to a particular value is said to be *free*.

The part of the logical expression or predicate formula to which a quantifier is applied is called the *scope* of the quantifier.

Examples

Table 1.36

Predicate formula	*Bound variable and scope*	*Free variable*
$\forall x\, P(x, y)$	x; $P(x, y)$	y
$\forall x\, (P(x) \rightarrow Q(x))$	x; $P(x) \rightarrow Q(x)$	—
$\forall x\, (P(x) \wedge Q(x)) \vee \forall y\, R(y)$	x; $P(x) \wedge Q(x)$ y; $R(y)$	—
$\exists x\, P(x) \wedge Q(x)$	First x; $P(x)$	Second x

Valid Formulas and Equivalences

- Let A and B be any two predicate formulas defined over a particular domain D. When each of the variables appearing in A and B is replaced by any element of D, if the resulting statements have the same truth values, then A and B are said to be ***equivalent*** to each other over D.
- It is denoted by $A \equiv B$ or $A \Leftrightarrow B$ over D

Eg: $\boldsymbol{P(x) \to Q(x) \equiv \neg P(x) \lor Q(x)}$

Predicate calculus from Statement calculus

Logically valid formulas in predicate calculus can be obtained from tautologies of propositional calculus by replacing primary propositions such as p, q, r by propositional functions $P(x), Q(x), R(x)$.

Eg: $\boldsymbol{p} \vee \neg \boldsymbol{p} \equiv \boldsymbol{T}$

Now we replace $\boldsymbol{p}$ by a propositional function $\forall \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x})$

Therefore we have, $\forall \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x}) \vee \neg(\forall \boldsymbol{x}\, \boldsymbol{P}(\boldsymbol{x})) \equiv \boldsymbol{T}$

Inference Theory of Predicate Calculus

- Three basic rules **Rule P, Rule T and Rule CP** of inference used in statement calculus can also be used in predicate calculus.
- The indirect method can also be used in predicate calculus.
- We need some additional rules for predicate calculus such as
  - **Rule US**
  - **Rule ES**

    Used to eliminate quantifiers ($\forall$ *and* $\exists$)

  - **Rule UG**
  - **Rule EG**

    Used to include quantifiers ($\forall$ *and* $\exists$)

Note:

- US stands for Universal Specification
- ES stands for Existential Specification

Note:

- UG stands for Universal Generalization
- EG stands for Existential Generalization

Rule US and Rule ES

Rule US: If $\forall x\, P(x)$ is true, then we can conclude that $P(c)$ is true where c is an arbitrary element of the domain. This is also called as universal instantiation.

Rule ES: If $\exists x\, P(x)$ is true, then we can conclude that there is a particular element c in the domain for which $P(c)$ is true. This is also called as existential instantiation.

Rule UG and Rule EG

Rule UG: If c is an arbitrary element in the domain for which $P(c)$ is true, then we can conclude that $\forall x\, P(x)$ is true.

Rule EG: If there is a particular element c in the domain for which $P(c)$ is true then we can conclude that $\exists x\, P(x)$ is true.

Example *Consider these statements. The first two are called premises and the third is called conclusion. The entire set is called an argument.*

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

(we will discuss the issue of determining whether the conclusion is a valid consequence of premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion," "x is fierce," and drinks coffee," respectively. Assuming that the domain consists of all creatures, express the statements in argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

Solution We can express these statements as:

$$\forall x(p\,(x) \rightarrow Q(x)).$$
$$\exists x(p\,(x) \wedge \neg R(x)).$$
$$\exists x(Q(x) \wedge \neg R(x)).$$

Problem 1:

1. Let us consider the following "Famous Socrates argument" which is given by:

   All men are mortal.

   Socrates is a man.

   Therefore Socrates is a mortal.

   Let us use the notations $H(x)$: x is a man

   $M(x)$: x is a mortal

   s: Socrates

   With these symbolic notations, the problem becomes

$$\forall x(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$$

The derivation of the proof is as follows:

Step No.	*Statement*	*Reason*
1.	$\forall x\, (H(x) \rightarrow M(x))$	P
2.	$H(s) \rightarrow M(s)$	US, and 1
3.	$H(s)$	P
4.	$M(s)$	T, 2, 3, Modus ponens

Problem 2:

Let UOD is set of integers

$N(x)$: x is a non-negative integer

$E(x)$: x is even

$O(x)$: x is odd

$P(x)$: x is prime

Express the following sentences in logical expressions:

(a) Every integer is even or odd

$\forall x[E(x) \vee O(x)]$

(b) There exists an even integer

$\exists x E(x)$

(c) Not all integers are odd

$\neg\forall x O(x)$ or $\exists x \neg O(x)$

(d) All prime integers are non negative

$\forall x[P(x) \rightarrow N(x)]$

(e) The only even prime is two

$\forall x[(E(x) \wedge P(x)) \rightarrow (x = 2)]$

Problems using Rules

Problems:

① Show that the premises "One student in this class knows how to write programs in JAVA" and "Everyone who knows how to write programs in JAVA can get a high-paying job" imply the conclusion "Someone in this class can get a high-paying job".

Sol:-

$C(x)$: x is in this class

$J(x)$: x knows JAVA

$H(x)$: x can get a high-paying Job

where, $x \in$ Set of students

✓ $P_1 : \exists x\,(C(x) \wedge J(x))$

✓ $P_2 : \forall x\,(J(x) \longrightarrow H(x))$

$C : \exists x\,(\underline{\underline{C(x) \wedge H(x)}})$

Logical Expression

St. No	Statement	Reason
1	$\exists x\,(C(x) \wedge J(x))$	Rule P
2	$C(a) \wedge J(a)$	Rule ES, Rule T
3	$C(a)$	Rule T, 2, Simplification
4	$J(a)$	Rule T, 2, Simplification
5	$\forall x\,(J(x) \longrightarrow H(x))$	Rule P
6	$J(a) \longrightarrow H(a)$	Rule US, Rule T
7	$H(a)$	Rule T, Modus ponens 4 & 6
8	$C(a) \wedge H(a)$	Rule T, conjunction 3 & 7
9	$\exists x\,(C(x) \wedge H(x))$	Rule T, Rule EG - 8

Problems using Rules

Problem 2: Prove the implication

$$\forall x\,(P(x) \rightarrow Q(x)),\ \forall x\,(R(x) \rightarrow \neg Q(x)) \Rightarrow \forall x\,(R(x) \rightarrow \neg P(x)).$$

Step No.	*Statement*	*Reason*
1.	$\forall x\,(P(x) \rightarrow Q(x))$	P
2.	$P(a) \rightarrow Q(a)$	US and 1
3.	$\forall x\,(R(x) \rightarrow \neg Q(x))$	P
4.	$R(a) \rightarrow \neg Q(a)$	US and 2
5.	$Q(a) \rightarrow \neg R(a)$	T, 4 and equivalence
6.	$P(a) \rightarrow \neg R(a)$	T, 2, 5 and hypothetical syllogism
7.	$R(a) \rightarrow \neg P(a)$	T, 6 and equivalence
8.	$\forall x\, R(x) \rightarrow \neg P(x))$	UG and 7

$$\boxed{P(x) \rightarrow Q(x) \equiv \neg Q(x) \rightarrow \neg P(x)}$$

Hypothetical Syllogism

$$\left.\begin{array}{l} P(x) \rightarrow Q(x) \\ Q(x) \rightarrow \neg R(x) \end{array}\right\}\ P(x) \rightarrow \neg R(x)$$

Problem 3:

Every living thing is a plant or an animal. Rama's dog is alive and it is not a plant. All animals have hearts. Therefore, Rama's dog has a heart.

To check the validity of the above argument, let

$L(x)$: x is alive

$P(x)$: x is a plant

$A(x)$: x is an animal

$H(x)$: x has a heart

Premises are:

$\forall x[L(x) \to (P(x) \vee A(x))]$

$L(\text{Rd}) \wedge \neg P(\text{ Rd})$

$\forall x[A(x) \to H(x)]$

Conclusion is:

$H(\text{ Rd })$ where Rd denotes Rama's dog

Solution:

Logical Statement	Reason
1. $L(\text{ Rd }) \wedge \neg P(\text{ Rd })$	Rule P
2. $L(\text{ Rd })$	Rule T, Simplification of (1)
3. $\neg P(\text{ Rd })$	Rule T, Simplification of (1)
4. $\forall x[L(x) \rightarrow (P(x) \vee A(x))]$	Rule P
5. $L(\text{ Rd}) \rightarrow (P(\text{ Rd }) \vee A(\text{ Rd }))$	Rule T, US of (4)
6. $(P(\text{ Rd }) \vee A(\text{ Rd }))$	Rule T, Modus Ponens of (2) and (5)
7. $A(\text{ Rd})$	Rule T, Disjuctive syllogism of (3) and (6)
8. $\forall x[A(x) \rightarrow H(x)]$	Rule P
9. $A(\text{ Rd}) \rightarrow H(\text{ Rd })$	Rule T, US of (8)
10. $H(\text{ Rd })$	Rule T, Modus Ponens of (7) and (9)

Problem 4: Use the indirect method to prove that the conclusion $\exists z\, Q(z)$ follows from the premises $\forall x\,(P(x) \to Q(x))$ and $\exists y\, P(y)$.

Let us assume the additional premise $\urcorner(\exists z\, Q(z))$ and prove a contradiction

Step No.	*Statement*	*Reason*
1.	$\urcorner(\exists z\, Q(z))$	P (additional)
2.	$\forall z\,(\urcorner Q(z))$	T, 1 and negation equivalence
3.	$\urcorner Q(a)$	US and 2
4.	$\exists y\, P(y)$	P
5.	$P(a)$	ES and 4
6.	$P(a) \wedge \urcorner Q(a)$	T, 3, 5 and conjunction
7.	$\urcorner(\urcorner P(a) \vee Q(a))$	T, 6 and equivalence
8.	$\urcorner(P(a) \to Q(a))$	T, 7 and equivalence
9.	$\forall x\,(P(x) \to Q(x))$	P
10.	$P(a) \to Q(a)$	US and 9
11.	$(P(a) \to Q(a)) \wedge \urcorner(P(a) \to Q(a))$	T, 8, 10 and conjunction
12.	F	T, 11 and negative law

Practice Problems

Check the validity of the following arguments:

1. All integers are rational numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2.
2. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.
3. No junior or senior is enrolled in a physical education class. Sai has enrolled in a physical education class. Therefore Sai is not a senior.
4. All squares have four sides. Quadrilateral ABCD has four sides. Therefore quadrilateral ABCD is a square.
5. Every computer science student takes discrete mathematics. Geethu is taking discrete mathematics. Therefore, Geethu is a computer science student.
6. A student in this class has not read the book. Everyone in this class passed the first exam. Therefore, someone who passed the first exam has not read the book.

Thank You