Simulations

STAT 133

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How do we use a computer to simulate a chance process?

Case Study 1

Simulation Probability examples

For instance

- Simulate flipping a coin
- Simulate rolling a die
- Simulate drawing a card from a deck
- Simulate a probability experiment with balls in an urn
- Simulate the "Monty Hall Problem"

Flipping a Coin

Simulating a Coin

How to simulate tossing a coin?

Simulating a coin

One way to simulate a coin

```r
coin <- c("heads", "tails")
```

Simulating a coin

One way to simulate a coin

```r
coin <- c("heads", "tails")
```

One way to simulate flipping a coin

```r
sample(coin, size = 1)

## [1] "heads"
```

Probability

Probability allows us to quantify statements about the chance of an event taking place

For example, flip a fair coin

- What's the chance it lands heads?
- Flip it 4 times, what proportion of heads do you expect?
- Will you get exactly that proportion?
- What happens when you flip the coin 1000 times?

Simulating a Coin

When you flip a coin

- it may land heads
- it may land tails
- with what probability it lands heads?
- If it is a fair coin: $p = 0.5$

Simulating a Coin

Tossing a coin can be modeled with a random variable following a Bernoulli distribution:

- heads ($X = 1$) with probability p
- tails ($X = 0$) with probability $q = 1 - p$

The Bernoulli distribution is a special case of the Binomial distribution: $B(1, p)$

Simulating tossing a coin

Tossing a coin simulated with a **binomial** distribution:

```
# binomial distribution generator
rbinom(n = 1, size = 1, prob = 0.5)

## [1] 1
```

Flipping a Coin function

Function that simulates flipping a coin

```
# flipping coin function
coin <- function(prob = 0.5) {
  rbinom(n = 1, size = 1, prob = prob)
}

coin()

## [1] 0
```

Flipping a Coin function

It's better if we assign labels

```
# flipping coin function
coin <- function(prob = 0.5) {
  out <- rbinom(n = 1, size = 1, prob = prob)
  ifelse(out, "heads", "tails")
}

coin()

## [1] "heads"
```

Flipping a Coin function

```r
# 10 flips
for (i in 1:10) {
  print(coin())
}

## [1] "tails"
## [1] "tails"
## [1] "tails"
## [1] "tails"
## [1] "heads"
## [1] "tails"
## [1] "heads"
## [1] "heads"
## [1] "heads"
## [1] "tails"
```

4 Flips

In 4 flips

- Possible outputs:
  - HHHH, THHH, HTHH, HHTH, HHHT, ...
- we can get 0, 1, 2, 3, 4 heads
- so the proportion of heads can be: 0, 0.25, 0.5, 0.75, 1
- we expect the proportion to be 0.5
- but a proportion of 0.25 is also possible

4 Flips

- we can think of the proportion of Heads in 4 flips as a **statistic** because it summarizes data
- this proportion is a random quantity: it takes on 5 possible values, each with some probability
  - $0 \to 1/16$
  - $0.25 \to 4/16$
  - $0.50 \to 8/16$
  - $0.75 \to 4/16$
  - $1.0 \to 1/16$

Simulating flipping n coins

Function that simulates flipping a coin n times (i.e. flipping n coins)

```r
# generic function
flip_coins <- function(n = 1, prob = 0.5) {
  out <- rbinom(n = n, size = 1, prob = prob)
  ifelse(out, "heads", "tails")
}

flip_coins(5)

## [1] "heads" "heads" "heads" "heads" "tails"
```

Proportion of Heads

```r
# number of heads
num_heads <- function(x) {
  sum(x == "heads")
}


# proportion of heads
prop_heads <- function(x) {
  num_heads(x) / length(x)
}
```

1000 Flips

- when we flip the coin 1000 times, we can get many different possible proportions of Heads
- 0, 0.001, 0.002, 0.003, ..., 0.999, 1.000
- It's highly unlikely that we would get 0 for the proportion—how unlikely?
- what does the distribution of the porpotion of heads in 1000 flips look like?

1000 Flips

- With some probability theory and math tools we can figure this out
- But we can also get a good idea using simulation
- In our simulation we'll assume that the chance of Heads is 0.5 (i.e. fair coin)
- we can find out what the possible values for the proportion of heads in 1000 flips look like

Flipping coins

```r
set.seed(99900)
flips <- flip_coins(1000)

num_heads(flips)

## [1] 494

prop_heads(flips)

## [1] 0.494
```

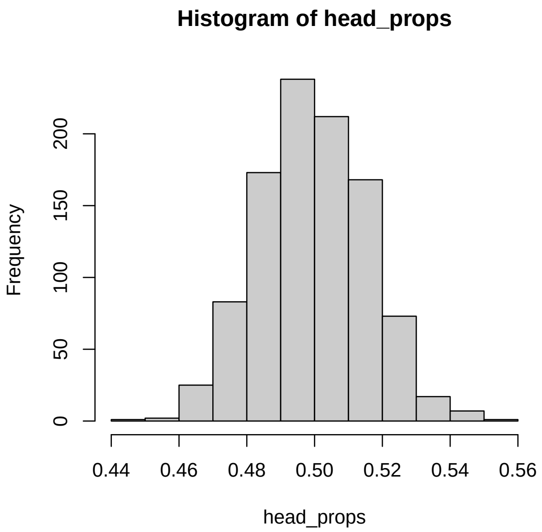
Flipping coins

```
set.seed(76547)
a_flips <- flip_coins(1000)
b_flips <- flip_coins(1000)

num_heads(a_flips)

## [1] 493

num_heads(b_flips)

## [1] 507
```

1000 Flips 1000 times

```r
times <- 1000

head_props <- numeric(times)

for (s in 1:times) {
  flips <- flip_coins(1000)
  head_props[s] <- prop_heads(flips)
}
```

Empirical distribution of 1000 flips



Flipping coins

Experiment: flipping a coin 100 times and counting number of heads

You flip a coin 100 times and you get 65 heads. Is it a fair coin?

Flipping coins

Experiment: flipping a coin 100 times and counting number of heads

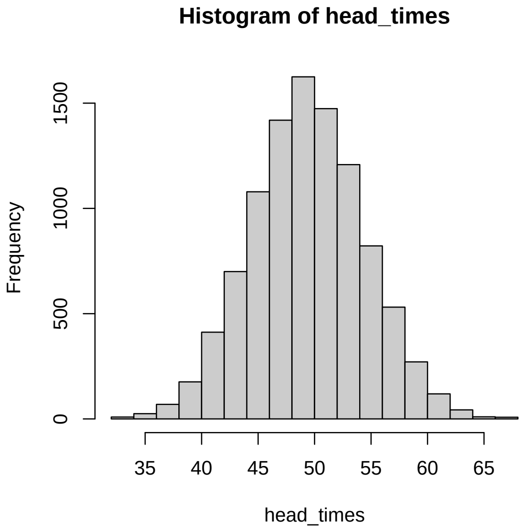
You flip a coin 100 times and you get 65 heads. Is it a fair coin?

We could perform a hypothesis test, or we could perform resampling

Flipping coins

```
# repeat experiment 100 times
times <- 10000
head_times <- numeric(times)
for (s in 1:times) {
  flips <- flip_coins(100)
  head_times[s] <- num_heads(flips)
}

sum(head_times >= 65)

## [1] 18

sum(head_times >= 65) / times

## [1] 0.0018
```

Flipping coins



Case Study 2

Chevalier de Mere

- Antoine Gombaud (1607 - 1684)
- *Nom de plume* “Chevalier de Mere”
- French writer and gambler, but not a nobleman
- Amateur mathematician

De Mere's games

Game 1

- One die
- Four rolls
- Win: at least one 6

De Mere's games

Game 1

- One die
- Four rolls
- Win: at least one 6

Game 2

- Two dice
- 24 rolls
- Win: at least one double 6

De Mere's games

Game 1

- One die
- Four rolls
- Win: at least one 6

Game 2

- Two dice
- 24 rolls
- Win: at least one double 6

De Mere was making money with game 1, but losing money with game 2. He turned to Blaise Pascal for help.

De Mere's faulty reasoning

Game 1

- The chance of getting a six in one roll of a die is $\frac{1}{6}$.
- In four rolls of a die, the chance of getting one six would be $\frac{4}{6} = \frac{2}{3}$.

De Mere's faulty reasoning

Game 2

- The chance of getting a double six in rolling a pair of dice is $\frac{1}{36}$.
- In 24 rolls of a pair of dice, the chance of getting one double six would be $\frac{24}{36} = \frac{2}{3}$.

Pascal and Fermat

- De Mere turned into Blaise Pascal
- Pascal consulted with Pierre de Fermat
- Beginning of Pascal's and Fermat's famous correspondence (1650's)
- Origin of combinatorial probability

Blaise Pascal
(1623 - 1662)
French mathematician

Pierre de Fermat
(1601 - 1665)
French lawyer and amateur mathematician

Letter from Pascal to Fermat

"If one undertakes to throw a six with one die, the advatange of undertaking it in 4 throws is as 671 to 625. If one undertakes to throw a double-six with two dice, there is a disadvantage of undertaking it in 24 throws. And nevertheless 24 is to 36 (which is the number of faces of two dice) as 4 to 6 (which is the number of faces of one die)."

De Mere's Game 1

Probability of no six in four rolls:

$$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296} = 0.482253$$

De Mere's Game 1

Probability of no six in four rolls:

$$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296} = 0.482253$$

P(at least one six) = 1 - Probability of no six in four rolls:

$$1 - \frac{625}{1296} = \frac{671}{1296} = 0.517747$$

De Mere's Game 2

Probability of no double six in 24 rolls:

$$P(\text{no double six in 24 rolls}) = \left(\frac{35}{36}\right)^{24} = 0.5086$$

De Mere's Game 2

Probability of no double six in 24 rolls:

$$P(\text{no double six in 24 rolls}) = \left(\frac{35}{36}\right)^{24} = 0.5086$$

P(at least one double six in 24 rolls) $=$

$$1 - 0.5086 = 0.4914$$

Using a computer to simulate a chance process

Some Questions

- Rather than solve the problem analytically, we can simulate 4 rolls of a die and count the number of 6's
- If we simulate rolling 4 dice many times, then the proportion of times we get 0, 1, 2, 3, or 4 sixes should be close to the chance of that many sixes on any 4 rolls

Some Questions

- What is the chance of getting one 6 when rolling a die?
- What is the chance of getting one 6 when rolling two dice?
- What is the chance of getting at least one 6 when rolling 4 dice?

What are the steps?

- Simulate rolling one die
- Simulate rolling a pair of dice
- Simulate rolling four dice
- Count the number of sixes

Simulating one die

- Let's start with one die
- What features?
- How to create a `die` object?

One die

```r
die <- 1:6

die

## [1] 1 2 3 4 5 6
```

Your turn

Which option would produce an invalid die:

```
# A
die <- 1L:6L

# B
die <- seq(from = 1, to = 6)

# C
die <- c(1, 2, 3, 4, 5, 6)

# D
die <- seq(from = 1, to = 6, by = 6)

# E
die <- seq_len(6)
```

How to simulate the roll of a die?

`sample()`

Very useful function for selecting from a discrete set (vector) of possibilities.

`sample()` arguments

- `x`
- `size`
- `replace`
- `prob`

Roll a die

```
# use of sample() to simulate the roll of a die
sample(die, size = 1)

## [1] 4
```

Roll a die

```r
# use of sample() to simulate the roll of a die
sample(die, size = 1)

## [1] 4
```

```r
sample(die, size = 1)

## [1] 2

sample(die, size = 1)

## [1] 5

sample(die, size = 1)

## [1] 6
```

Function: Roll one die

Write a function to make it more convenient

```
# function
rolldie <- function() {


}
```

Function: Roll one die

Write a function to make it more convenient

```
# function
rolldie <- function() {
  die <- 1:6
  sample(die, size = 1)
}
```

Function: Roll one die

Write a function to make it more convenient

```r
# function
rolldie <- function() {
  die <- 1:6
  sample(die, size = 1)
}
```

```r
rolldie()

## [1] 2
```

Rolling one die

Let's simulate 10 rolls of a die

```
# roll 10 times
for (i in 1:10) {
  rolldie()
}
```

Rolling one die

Let's simulate 10 rolls of a die

```
# roll 10 times
for (i in 1:10) {
  rolldie()
}
```

What's *wrong*?

Why nothing is shown on screen?

Rolling one die

```r
# roll 10 times
for (i in 1:10) {
  print(rolldie())
}
```

```
## [1] 2
## [1] 3
## [1] 4
## [1] 6
## [1] 2
## [1] 6
## [1] 6
## [1] 4
## [1] 4
## [1] 1
```

Rolling one die

```r
# roll until first 5
repeat {
  rol <- rolldie()
  print(rol)
  if (rol == 5) break
}

## [1] 4
## [1] 2
## [1] 6
## [1] 4
## [1] 3
## [1] 6
## [1] 5
```

Rolling one die

```r
# roll until first 5
rol <- 1

while (rol != 5) {
  rol <- rolldie()
  print(rol)
}

## [1] 4
## [1] 2
## [1] 6
## [1] 4
## [1] 3
## [1] 6
## [1] 5
```

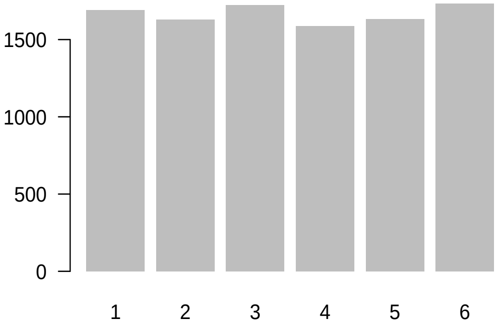
Rolling one die

```r
# roll 10,000 times
results <- numeric(10000)

for (i in 1:10000) {
  results[i] <- rolldie()
}
```

Rolling one die

```r
# frequencies
table(results)

## results
##    1    2    3    4    5    6
## 1692 1631 1724 1586 1633 1734

# relative frequencies
table(results) / 10000

## results
##      1      2      3      4      5      6
## 0.1692 0.1631 0.1724 0.1586 0.1633 0.1734
```

Distribution of results



How to simulate a **loaded** die?

Loaded Die

Changing argument `prob` to create a loaded die

```
probs <- c(1/21, 2/21,3/21, 4/21, 5/21, 6/21)

# function
loaded <- function() {
  die <- 1:6
  sample(die, size = 1, prob = probs)
}
```

Loaded Die

Changing argument `prob` to create a loaded die

```
probs <- c(1/21, 2/21,3/21, 4/21, 5/21, 6/21)

# function
loaded <- function() {
  die <- 1:6
  sample(die, size = 1, prob = probs)
}
```

```
loaded()

## [1] 6
```

Rolling a loaded die

```
# roll 10 times
for (i in 1:10) {
  print(loaded())
}

## [1] 6
## [1] 5
## [1] 4
## [1] 2
## [1] 6
## [1] 2
## [1] 2
## [1] 4
## [1] 4
## [1] 6
```

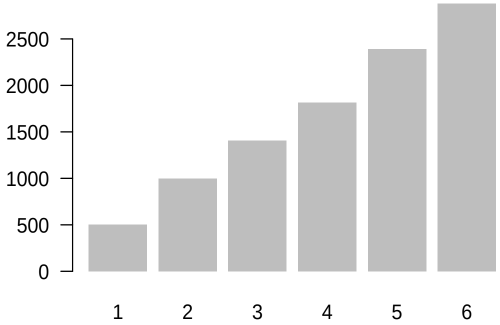
Rolling a loaded die

```r
# roll 10,000 times
results <- numeric(10000)

for (i in 1:10000) {
  results[i] <- loaded()
}
```

Rolling a lodade die

```
# frequencies
table(results)

## results
##    1    2    3    4    5    6
##  505 1000 1405 1817 2391 2882

# relative frequencies
table(results) / 10000

## results
##      1      2      3      4      5      6
## 0.0505 0.1000 0.1405 0.1817 0.2391 0.2882
```

Loaded Distribution



Simulating rolling a pair of dice

Roll a pair of dice

```
# 1st die
sample(die, size = 1)

## [1] 1

# 2nd die
sample(die, size = 1)

## [1] 5
```

Function: roll a pair of dice

```
# pair of dice function
roll2 <- function() {




}
```

Function: roll a pair of dice

```r
# pair of dice function
roll2 <- function() {
  die <- 1:6
  rol1 <- sample(die, size = 1)
  rol2 <- sample(die, size = 1)
  c(rol1, rol2)
}
```

Various rolls

```r
roll2()

## [1] 5 1

roll2()

## [1] 1 1

roll2()

## [1] 6 6
```

Function

```r
# pair of dice function
roll2 <- function() {
  die <- 1:6
  rol1 <- sample(die, size = 1)
  rol2 <- sample(die, size = 1)  # repeated command!
  c(rol1, rol2)
}
```

Roll a pair of dice

```r
die <- 1:6

# avoid repetition with one call of 'sample()'
sample(die, size = 2)

## [1] 6 1
```

Roll a pair dice

```
for (i in 1:15) {
  print(sample(die, size = 2))
}
```

```
## [1] 1 3
## [1] 1 4
## [1] 5 4
## [1] 4 3
## [1] 3 1
## [1] 2 6
## [1] 4 6
## [1] 3 4
## [1] 4 6
## [1] 6 3
## [1] 2 5
## [1] 2 5
## [1] 2 3
## [1] 3 2
## [1] 2 5
```

```
# Can you spot a problem?
```

Roll a pair of dice

```r
die <- 1:6

# sample with replacement
sample(die, size = 2, replace = TRUE)

## [1] 4 1
```

Roll a pair of dice

```r
for (i in 1:15) {
  print(sample(die, size = 2, replace = TRUE))
}
```

```
## [1] 6 6
## [1] 6 5
## [1] 5 4
## [1] 3 6
## [1] 1 2
## [1] 1 5
## [1] 1 2
## [1] 3 1
## [1] 6 1
## [1] 1 2
## [1] 1 6
## [1] 3 2
## [1] 3 1
## [1] 2 2
## [1] 1 1
```

Roll a pair of dice

```r
# rewrite roll2()
roll2 <- function() {
  die <- 1:6
  sample(die, size = 2, replace = TRUE)
}
```

Roll a pair of dice

```r
# rewrite roll2()
roll2 <- function() {
  die <- 1:6
  sample(die, size = 2, replace = TRUE)
}
```

```r
roll2()

## [1] 5 4
```

Rolling a die any number of times

Rolling several dice

More general function to roll a die any number of times

```
roll <- function(times = 1) {
  die <- 1:6
  sample(die, size = times, replace = TRUE)
}
```

Rolling several dice

More general function to roll a die any number of times

```
roll <- function(times = 1) {
  die <- 1:6
  sample(die, size = times, replace = TRUE)
}
```

```
roll()

## [1] 4
```

Rolling several dice

```r
# default (one roll)
roll()

## [1] 4

# two rolls
roll(2)

## [1] 1 5

# 4 rolls
roll(4)

## [1] 6 4 6 5
```

De Mere's Game 1

De Mere's Game 1

Game 1

- One die
- Four rolls
- Win: at least one 6

De Mere's Game 1

```
# play 100 times
results <- matrix(0, nrow = 100, ncol = 4)

for (i in 1:100) {
  results[i, ] <- roll(times = 4)
}

head(results)

##      [,1] [,2] [,3] [,4]
## [1,]    2    5    4    2
## [2,]    6    6    1    6
## [3,]    3    4    4    2
## [4,]    5    2    3    6
## [5,]    6    2    3    1
## [6,]    4    3    6    1
```

De Mere's Game 1

```
counts <- 0

for (i in 1:100) {
  if (any(results[i, ] == 6))
    counts <- counts + 1
}

# proportion of wins
counts / 100

## [1] 0.56
```

De Mere's Game 1

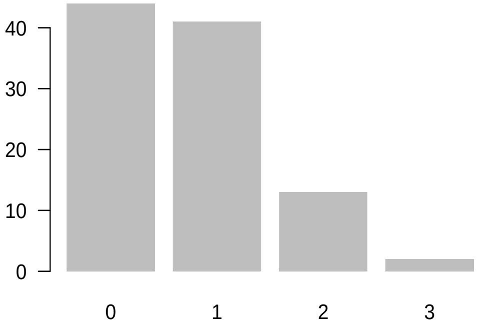
```
sixes <- apply(results, 1, function(x) sum(x == 6))

table(sixes)

## sixes
##  0  1  2  3
## 44 41 13  2
```

De Mere's Game 1

```r
sixes <- apply(results, 1, function(x) sum(x == 6))

table(sixes)

## sixes
##  0  1  2  3
## 44 41 13  2
```

```r
# rolls with at least one six
sum(table(sixes)[-1])

## [1] 56
```

De Mere's Game 1



De Mere's Game 1

Considerations

- How would you make the code more flexible?
- What type of "parameters"?
- Avoid Repetition

De Mere's Game 1

```r
games <- 10000
results <- matrix(0, nrow = games, ncol = 4)

for (i in 1:games) {
  results[i, ] <- roll(times = 4)
}

counts <- 0
for (i in 1:games) {
  if (any(results[i, ] == 6))
    counts <- counts + 1
}

counts / games  # proportion of wins

## [1] 0.5204
```

De Mere's Game 1

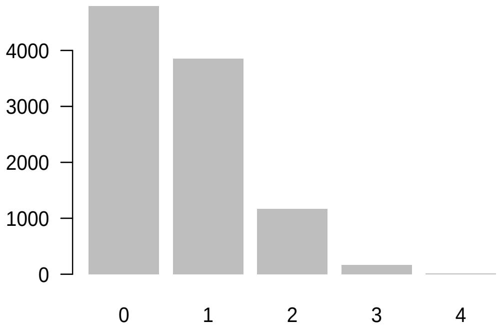
```r
sixes <- apply(results, 1, function(x) sum(x == 6))

table(sixes)

## sixes
##    0    1    2    3    4
## 4796 3858 1171  166    9
```

De Mere's Game 1

```r
sixes <- apply(results, 1, function(x) sum(x == 6))

table(sixes)

## sixes
##    0    1    2    3    4
## 4796 3858 1171  166    9
```

```r
# rolls with at least one six
sum(table(sixes)[-1])

## [1] 5204
```

De Mere's Game 1



De Mere's Game 2

De Mere's Game 2

Game 2

- Two dice
- 24 rolls
- Win: at least one double 6

De Mere's Game 2

```r
roll2 <- function(times = 1) {
  dice2 <- unlist(lapply(1:6, function(x) x + 1:6))
  sample(dice2, size = times, replace = TRUE)
}

roll2()

## [1] 10

roll2(24)

##  [1]  8  4  4  3  8  6  7  6  9  9  5  4  5  7  3 11  9 11 10
```

De Mere's Game 2

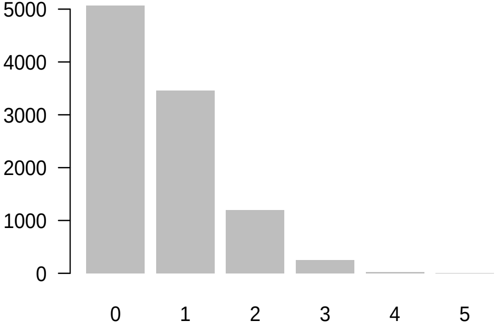
Game 2

- It's better if we sum the points of rolling two dice
- Possible outcomes: $\{2, 3, 4, \ldots, 10, 11, 12\}$
- Double six is equivalent to 12 points

De Mere's Game 2

```r
games <- 10000
results <- matrix(0, nrow = games, ncol = 24)

for (i in 1:games) {
  results[i, ] <- roll2(24)
}

doublesix <- apply(results, 1, function(x) sum(x == 12))

sum(table(doublesix)[-1])

## [1] 4930
```

De Mere's Game 2



De Mere's Game 2

```
counts <- 0
for (i in 1:games) {
  if (any(results[i, ] == 12))
    counts <- counts + 1
}

counts / games  # proportion of wins

## [1] 0.493
```