Hierdie opdrag moet **15:50 05 Sept** in harde kopie ingehandig word. Geen elektroniese weergawes word aanvaar nie. Laat inhandiging sal gepenaliseer word. Samewerking word beperk tot die uitruil van enkele idees. **Die uitruil van kode, grafieke of besonderhede van wiskundige berekeninge word nie toegelaat nie.** Wat jy inhandig moet jou eie werk wees. **Plagiaat in opdragte sal ernstige gevolge hê.**

Let daarop: `ode45` is 'n MATLAB funksie wat 1ste-orde aanvangswaardeprobleme numeries oplos. Lees die onderstaande dokument of tik `doc ode45` in MATLAB om meer van sy gebruik te leer

This assignment must be submitted on **15:50 05 Sept** in hard copy. No electronic versions will be accepted. Late submissions will be penalized. Cooperation is limited to the exchange of a few ideas. **The exchange of code, graphs or details of mathematical calculations are not allowed.** What you submit must be your own work. **Plagiarism in assignments will have severe consequences.**

Note: `ode45` is a function in MATLAB that solves 1st-order initial value problems numerically. Read the document below or type `doc ode45` in MATLAB to learn more about its usage.

http://appliedmaths.sun.ac.za/TW244/files/ode45quickstart.pdf

(`scipy.integrate.odeint` is die Python ekwivalent.)

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Probleem 1: **(a)** Voeg 'n geskikte funksie `f` en aanvangsvoorwaarde `x0` by die onderstaande MATLAB kode (links) om met Euler se metode die onderstaande Lotka-Volterra (roofdier-prooi) stelsel van vergelykings (regs) vir $0 \le t \le 10$ jaar op te los met staplengte $h = 0.01$.[a][b] Wys die resulterende grafiek van die oplossing en jou kode vir `f` en `x0`. Watter een van x en y verteenwoordig die aantal (in duisende) roofdiere, en watter een die aantal prooi? Benoem die grafiek, gee 'n titel en geskikte name vir die asse.

[a] Jy mag jou eie Python kode skryf indien jy wil.

[b] Let daarop dat Euler se kode vir stelsels van DVs in MATLAB presies dieselfde is as wat ons in RO 01 vir skalaarvergelykings gesien het!

Problem 1: **(a)** Add a suitable function `f` and initial condition `x0` to the MATLAB code below (left) to solve the Lotka–Volterra (predator-prey) system of equations below (right) for $0 \le t \le 10$ years using Euler's method with step size $h = 0.01$.[a][b] Show the resulting plot and your code for `f` and `x0`. Which one of x and y represents the number (in thousands) of predators, and which one the number of prey? Add an appropriate legend (and title, axis labels, etc) to your plot.

[a] You may write your own Python code if you prefer.

[b] Observe that Euler for systems of DEs in MATLAB is exactly the same as in CA01 for scalar equations!

```
f =
x0 =
% Start/stop time and step size:
t0 = 0; tend = 10; h = 1/100;
% Initialise t0 and x0:
t = t0; x = x0;
% Loop until t = tend:
while ( t <= tend )
    x = x + h*f(t, x);
    t = t + h;
    plot(t, x(1), '.b', t, x(2), '.r');
    hold on
end
hold off
```

$$
\begin{aligned}
\frac{dx}{dt} &= -3x + 3xy, \\
\frac{dy}{dt} &= y - 2xy, \\
x(0) &= 0.3, \\
y(0) &= 1.
\end{aligned}
$$

(b) Deur dieselfde funksie `f` as in deel **(a)** te gebruik, los dieselfde stelsel op deur `ode45` te gebruik. Stip die oplossing (met 'n kontinue lyn) op dieselfde Figuur as hierbo deur `hold on` te gebruik. Wys die skets en die kode wat jy gebruik het om `ode45` te roep. Vergelyk dit met die skets in (a). Watter een is meer akkuraat?

(b) Using the same function `f` as in part **(a)**, solve the same system using `ode45`. Plot the solution (using a continuous line) on the same Figure as above using `hold on`. Show the plot and the code you used to call `ode45`. Compare with the plot from (a). Which do you think is more accurate?

(c) Vind benaderings vir die minimum en maksimum aantal roofdiere wat teenwoordig sal wees.

(c) Find approximations for the minimum and maximum number of predators that will be present.

(d) [Opsioneel] Herhaal deel (a) deur die aangepaste Euler metode van RO01 te gebruik. Voeg die resulterende skets tot jou figuur van deel (b). Vergelyk die akkuraatheid van Euler en aangepaste Euler.

(d) [Optional] Repeat part (a) using the modified Euler method from CA01. Add the resulting plot to your figure from part (b). Comment on the accuracy of Euler and modified Euler.

(e) Gestel nou in die afwesigheid van roofdiere dat die beperkingskapasity van die prooi 10 (duisend) is. Voeg 'n geskikte logistieke term by die bostaande model, los die model op met behulp van `ode45` vir 'n lang genoeg tyd om die gedrag van die twee bevolkings te bereken as $t \to \infty$. Skets die resultaat op 'n nuwe figuur.

(e) Suppose now that in the absense of predators, the limiting capacity of the environment for the prey is 10 (thousand). Add a suitable logistic term to the model above, solve the model using `ode45` for a long enough time to estimate the behaviour of the two populations as $t \to \infty$. Plot the result on a new figure.

Probleem 2: Die SIR model
Van Wikipedia:

Problem 2: The SIR model
From Wikipedia:

The SIR model [divides the population in to] three compartments S = number susceptible, I =number infectious, and R =number recovered (immune) [to/with/from a disease]. This is a good and simple model for many infectious diseases including measles, mumps and rubella. The dynamics of an epidemic, for example the flu, are often much faster than the dynamics of birth and death, therefore, birth and death are often omitted in simple compartmental models. The SIR system without so-called vital dynamics (birth and death, sometimes called demography) described above can be expressed by the following set of ordinary differential equations:

$$\frac{dS}{dt} = -\beta IS, \qquad \frac{dI}{dt} = \beta IS - \gamma I, \qquad \frac{dR}{dt} = \gamma I.$$

(a) Hierdie deel van die vraag tel nie punte nie (en dis nie nodig om iets te skryf nie), maar jy moet hierdie vergelykings bestudeer en probeer verstaan hoe en waarom hulle 'n besmetting modelleer. Byvoorbeeld, wat verteenwoordig die term $\frac{dR}{dt} = \gamma I$? Waarom? Oortuig ook jouself dat $S(t) + R(t) + I(t) =$ konstant.

Ietwat verneder besef jy dat die zombie apokalips van Lesing 11 eintlik niks meer as 'n kombinasie van 'n slegte griep-uitbraak en jou ooraktiewe verbeelding was nie. Los die SIR model op met behulp van `ode45` (of enige alternatief) met $\beta = 0.00083$ per dag, $\gamma = 0.05$ per dag, en aanvangswaardes $S(0) = 999$, $I(0) = 1$, $R(0) = 0$ om hierdie griep-uitbraak in jou koshuis te modelleer.

(b) Stip 'n grafiek van S, I, en R teenoor t (almal op dieselfde Figuur) vir die eerste 4 weke van die besmetting.

(c) Vanaf jou grafiek, skat **(i)** die maksimum aantal besmette studente op enige tydstip en **(ii)** die aantal huidige besmette studente na vier weke (m.a.w., *28 Dae later*).

(d) Deur probeer-en-fout (of 'n meer gesofistikeerde metode), bepaal wat die waarde van γ in die bostaande model sou moes wees sodat slegs die helfde van die studente in jou koshuis ooit griep kry. (Wenk: Jy sal jou model vir ten minste 60 dae moet hardloop om dit te sien.)

(e) Veronderstel dat dit vir 'n klein aantal pasiënte onmoontlik is om immuniteit teen die griep op te bou, en in plaas daarvan om te herstel beweeg hul na 'n vatbare toestand. Hoe sal jy voorstel moet die SIR model aangepas word om hierdie in berekening te bring? (Dis nie nodig om hierdie nuwe model op te los nie.)

(a) This part of the question is worth no marks (and there is no need to write anything), but you should study these equations and try to understand how and why they are modelling an infection. For example, what does the term $\frac{dR}{dt} = \gamma I$ represent? Why? Also convince yourself that $S(t) + R(t) + I(t) =$ constant.

Now, rather embarrassingly, you've realised the zombie apocalypse from Lecture 11 was actually nothing more than the combination of a bad flu outbreak and your overactive imagination. Solve the SIR model using `ode45` (or some alternative) with $\beta = 0.00083$ per day, $\gamma = 0.05$ per day, and initial values $S(0) = 999$, $I(0) = 1$, $R(0) = 0$ to model this flu outbreak in your res.

(b) Plot a graph of S, I, and R against t (all on the same Figure) for the first 4 weeks of the infection.

(c) From your graph or otherwise, estimate **(i)** the maximum number of infectious students at any one time and **(ii)** the number of currently infectious after four weeks (i.e., *28 Days Later*).

(d) By trial and error (or some more sophisticated means) determine what value of γ would be necessary in the model above so that only half on the students in your res ever catch the flu. (Hint: You will need to run your model for at least 60 days to see this.)

(e) Suppose that a small number of patients are unable to build an immunity to the flu and instead of moving from infectious to recovering instead revert to a susceptible state. Suggest how you would modify the SIR model to account for this. (No need to solve this new model.)

Probleem 3: Beskou die tweede-orde AWP:

Problem 3: Consider the second-order IVP:

$$y'' + 2y' + y = \log(x), \qquad y(1) = 0, \quad y'(1) = 1.$$

(a) Met behulp van 'n nuwe veranderlike $z = y'$, herskryf die AWP vergelyking as 'n eerste-orde stelsel en los vir $1 \leq x \leq 4$ met behulp van `ode45`. Wys jou kode en 'n skets van die oplossing $(x, y(x))$ deur 'n soliede lyn te gebruik.

(a) By introducing a new variable $z = y'$, rewrite the IVP as a first-order system and solve using `ode45` for $1 \leq x \leq 4$. Show your code and a plot of the solution $(x, y(x))$ using a solid line.

(b) [Opsioneel en baie moeilik = bonuspunte.] Gebruik die metode van variasie van parameters wat in lesings bespreek is om aan te toon dat die oplossing van hierdie AWP as volg geskryf kan word

(b) [Optional and very hard = bonus marks.] Use the method of variation of parameters discussed in lectures to show the solution to of this IVP may be written as

$$e^{-x}\Big((ei(1) - ei(x))(1 + x) + (x - 2)e\Big) + 1 + \log(x),$$

waar ei die eksponensiele integraal funksie is en word gegee deur

where ei is the exponential integral function given by

$$ei(x) = \int_{-\infty}^{x} \frac{e^{-t}}{t}\, dt.$$

(c) Skets die analitiese oplossing hierbo op dieselfde figuur as jou numeriese oplossing deur gebruik te maak van 'n streeplyn. (Wenk. Gebruik `ei = @(x) -expint(-x)` in MATLAB en `scipy.special.expi` in Python).

(c) Plot the analytical solution above on the same figure as your numerical solution using a dashed line. (Hint: Use `ei = @(x) -expint(-x)` in MATLAB and `scipy.special.expi` in Python).

Comments (non-examinable, but maybe interesting):
Problem 3 demonstrates that the method of variation of parameters discussed in class can often lead to solutions defined in terms of integrals that we may not be able to explicitly compute (such as the exponential integral above). The course textbook gives more examples of such integrals in Appendix A.

One can argue (and people do) whether such forms of the solution count as being in 'closed form', since these integrals still need to be approximated numerically to obtain a solution. However, since these integrals and other special functions arise so often, people put a lot of effort in to developing efficient and convenient codes to compute them accurately (such as `expint()` in MATLAB).