Hierdie opdrag moet **15:50 26 September** in harde kopie ingehandig word. Geen elektroniese weergawes word aanvaar nie. Laat inhandiging sal gepenaliseer word. Samewerking word beperk tot die uitruil van enkele idees. **Die uitruil van kode, grafieke of besonderhede van wiskundige berekeninge word nie toegelaat nie.** Wat jy inhandig moet jou eie werk wees. **Plagiaat in opdragte sal ernstige gevolge hê.**

This assignment must be submitted on **15:50 26 September** in hard copy. No electronic versions will be accepted. Late submissions will be penalized. Cooperation is limited to the exchange of a few ideas. **The exchange of code, graphs or details of mathematical calculations are not allowed.** What you submit must be your own work. **Plagiarism in assignments will have severe consequences.**

Probleem 1: Hier leer ons oor MATLAB se `fzero` – 'n numeriese metode om vergelykings van die vorm $f(x) = 0$ op te los. Tik `>> help fzero` om meer te leer. (`scipy.optimize.fsolve` is die Python alternatief.)

(a) Die Lambert W function, $w = W(z)$, soos in Lesing 22 bespreek, is die oplossing w van $we^w = z$. Gebruik `fzero` om $W(1)$ te bereken.

(b) Kontroleer jou resultaat van (a) deur (i) te verifieer dat $W(1)e^{W(1)} = 1$ en (ii) gebruik die kode `lambertw.m` beskikbaar op die kursuswebblad (of `scipy.special.lambertw` in Python).

Problem 1: Here we learn about MATLAB's `fzero` – a numerical method for solving equations of the form $f(x) = 0$. Type `>> help fzero` to learn how to use it. (`scipy.optimize.fsolve` is the Python alternative.)

(a) The Lambert W function, $w = W(z)$, described in Lecture 22, is defined as the solution w of $we^w = z$. Use `fzero` to compute $W(1)$.

(b) Check your result from (a) by (i) verifying that $W(1)e^{W(1)} = 1$ and (ii) using the code `lambertw.m` available on the course webpage (or `scipy.special.lambertw` in Python).

Probleem 2: In die klas is gewys dat die trajek van 'n projektiel met **lineêre lugweerstand**, en aanvangswaardes $x(0) = y(0) = 0$, $x'(0) = v_0 \cos\theta$ en $y'(0) = v_0 \sin\theta$, gegee word deur

Problem 2: We showed in class that the trajectory of a projectile experiencing **linear air resistance** with initial conditions $x(0) = y(0) = 0$, $x'(0) = v_0 \cos\theta$ and $y'(0) = v_0 \sin\theta$, is given by

$$y(x) = \Big(\frac{g}{r} + v_0 \sin\theta\Big)\Big(\frac{x}{v_0 \cos\theta}\Big) + \frac{g}{r^2} \ln\Big(1 - \frac{rx}{v_0 \cos\theta}\Big),$$

(a) Skryf 'n uitdrukking van die vorm $f(R) = 0$ neer waaruit die reikafstand R bepaal kan word en gebruik `fzero` om R op te los, as $g = 9.81\,\mathrm{m/s^2}$, $r = 0.14$, $v_0 = 44.7m/s$ $(100mph)$, en $\theta = 45°$. Verifieer jou waarde van R deur gebruik te maak van die eksplisiete formule wat in Lesing 22 gegee word. Bepaal die trajek vir die parameters hierbo vir $0 \le x \le R$.

(b) Gebruik die formule in Lesing 22 om die optimale hoek θ_{max} vir die parameters in deel (a) te bepaal. Op dieselfde figuur as hierbo, plot hierdie optimale trajek.

(c) Projektielbeweging met **kwadratiese lugweerstand** word beskryf deur

(a) Write down an expression of the form $f(R) = 0$ from which the range R can be determined and use `fzero` to solve for R when $g = 9.81\,\mathrm{m/s^2}$, $r = 0.14$, $v_0 = 44.7m/s$ $(100mph)$, and $\theta = 45°$. Verify your value of R using the explicit formula given in Lecture 22. Plot the trajectory for the parameters above for $0 \le x \le R$.

(b) Use the formula in Lecture 22 to determine the optimal angle θ_{max} for the parameters in part (a). On the same figure as above plot this optimal trajectory.

(c) Projectile motion with **quadratic air resistance** is described by

$$\frac{dx'}{dt} = -rx'\sqrt{(x')^2 + (y')^2}, \quad \frac{dy'}{dt} = -g - ry'\sqrt{(x')^2 + (y')^2}, \quad \frac{dx}{dt} = x', \quad \frac{dy}{dt} = y',$$

met dieselfde aanvangswaardes as vroeër. Hierdie stelsel moet numeries opgelos word. Gebruik `ode45` om die stelsel met $r = 0.005$ en die ander aanvangswaardes soos in (a) op te los en stip y as 'n funksie van x. Raadpleeg dan jou grafiek om die reikafstand R te skat.

with the same initial conditions as above. This system has to be solved numerically. Use `ode45` to solve the system with $r = 0.005$ and the other initial conditions as in (a) and plot y as a function of x. Then consult your graph to estimate the range R.

Probleem 3: Oorweeg die volgende AVP-stelsel:

Problem 3 Consider the following IVP system:

$$\begin{aligned}\frac{dx_1}{dt} &= -\frac{1}{2}x_1 + 3x_2, \qquad x_1(0) = 2,\\ \frac{dx_2}{dt} &= \frac{3}{4}x_1 - 2x_2, \qquad x_2(0) = -\frac{1}{2}.\end{aligned}$$

(a) Skryf die bostaande probleem in Matriks-vektor vorm, m.a.w.,

(a) Write the problem above in Matrix-vector form, i.e.,

$$\frac{d\underline{x}}{dt} = A\underline{x}, \qquad \underline{x}(0) = \underline{x}_0.$$

Gebruik `eig` in MATLAB (of `scipy.linalg.eig` in Python) om die eiewaardes en eievektore van die matriks A te vind en gebruik dus die Metode van Eiewaardes om die probleem hierbo op te los. Teken die oplossing vir $0 \le t \le 2$.

Use `eig` in MATLAB (or `scipy.linalg.eig` in Python) to find the eigenvalues and eigenvectors of the matrix A and hence use the Method of Eigenvalues to solve the problem above. Plot the solution for $0 \le t \le 2$.

(b) Gebruik die matriks eksponensiaal (`expm` in MATLAB en `scipy.linalg.expm` in Python) om $\underline{x}(10) = e^{10A}\underline{x}_0$ te evalueer en voeg hierdie punt by jou plot van (a).[a]

[a]Let daarop dat hierdie nie 'n spelfout is nie. Gebruik `expm`, nie `exp` nie. Wat is die verskil? Sien Afdeling 8.4 in Z&W vir meer inligting.

(b) Use the matrix exponential (`expm` in MATLAB and `scipy.linalg.expm` in Python) to evaluate $\underline{x}(10) = e^{10A}\underline{x}_0$ and add this point to your plot from (a).[a]

[a]Note this is not a typo. Use `expm`, not `exp`. What is the difference? See Section 8.4 in Z&W for more info.

Probleem 4: Die drieliggaamprobleem neem die aanvanklike posisies en snelhede van drie puntmassas en los op vir die evolusie van hul beweging soos voorgeskryf deur Newton se wet van universele gravitasie ($F = G\frac{m_1 m_2}{r^2}$) en tweede wet van beweging ($F = ma$). In vektorvorm kan [a]

[a]Om dit te vereenvoudig normaliseer ons so dat $G = 1$.

Problem 4: The three-body problem takes the initial positions and velocities of three point masses and solves for the evolution of their motion as dictated by Newton's law of universal gravitation ($F = G\frac{m_1 m_2}{r^2}$) and second law of motion ($F = ma$). In vector form, we can express this as[a]

[a]For simplicity we normalise so that $G = 1$.

$$\underline{z}_1'' = m_2\frac{\underline{z}_2 - \underline{z}_1}{r_3^3} + m_3\frac{\underline{z}_3 - \underline{z}_1}{r_2^3}, \qquad \underline{z}_2'' = m_3\frac{\underline{z}_3 - \underline{z}_2}{r_1^3} + m_1\frac{\underline{z}_1 - \underline{z}_2}{r_3^3}, \qquad \underline{z}_3'' = m_1\frac{\underline{z}_1 - \underline{z}_3}{r_2^3} + m_2\frac{\underline{z}_2 - \underline{z}_3}{r_1^3},$$

waar $\underline{z}(t)_k = x_k(t)\underline{i} + y_k(t)\underline{j}$ die posisie van die $k$de voorwerp by tyd t wat massa m_k het en

where $\underline{z}(t)_k = x_k(t)\underline{i} + y_k(t)\underline{j}$ is the position of the kth object at time t, which has mass m_k, and

$$r_1 = |\underline{z}_2 - \underline{z}_3|, \qquad r_2 = |\underline{z}_1 - \underline{z}_3|, \qquad r_3 = |\underline{z}_1 - \underline{z}_2|.$$

(a) Skryf die vergelykings hierbo neer as 'n stelsel van 6 (komplekse-waarde) eerste orde DVs vir drie massas $m_1 = 5$, $m_2 = 3$, $m_3 = 4$ en begin van rus af by die aanvanklike posisies

(a) Write the equations above as a system of 6 (complex-valued) first-order DEs for three masses $m_1 = 5$, $m_2 = 3$, $m_3 = 4$ starting at rest from initial positions

$$\underline{z}_1(0) = (1, -1), \quad \underline{z}_2(0) = (1, 3), \quad \underline{z}_3(0) = (-2, -1).$$

Gebruik `ode45` of iets soortgelyk om die systeem hierbo op te los vir $0 \le t \le 10$. (Wenk: Ek stel voor dat jy toleransie na ten minste `1e-10` verhoog.) Skets die resulterende bane met 'n geskikte legende.

Use `ode45` or some equivalent to solve the system above for $0 \le t \le 10$. (Hint: I suggest you increase the default tolerance to at least Plot the resulting orbits with a suitable legend.

(b) Los die stelsel weer op, maar nou met $\underline{z}_3(0) = (-1, 1)$. Weereens, skets die wentelbane en wys dat een van die massas uit die stelsel uitgewis word.

(b) Solve the system again, but now with $\underline{z}_3(0) = (-1, 1)$. Again, plot the orbits, and show that one of the masses is ejected from the system.

(c) [Bonus] Los nog een keer op met $m_1 = m_2 = m_3 = 1$ en die aanvanklike posisies hieronder. Wat is interessant oor hierdie baan?

(c) [Bonus] Solve one more time with $m_1 = m_2 = m_3 = 1$ and the initial positions below. What is interesting about this orbit?

$$\underline{z}_1 = (0.5405, 0.3452), \quad \underline{z}_2(0) = (0.5405, -0.3452), \quad \underline{z}_3(0) = (-1.0810, 0)$$
$$\underline{z}_1'(0) = (-1.0971, -0.2336), \quad \underline{z}_2'(0) = (1.0971, -0.2336), \quad \underline{z}_3'(0) = (0, 0.4672)$$