Numeriese Metodes / *Numerical Methods* 324

Rekenaaropdrag 06 / *Computer Assignment* 06 - 2018

Sperdatum / *Deadline* - 18/05/2018

Let asseblief op:

- Opdragte word met die aanvang van die tutoriaalsessie op die sperdatum ingehandig. Laat inhandiging sal gepenaliseer word. Handig gedrukte weergawes in asb, geen elektroniese weergawes word aanvaar nie. **Jou opdrag moet beide jou kode en jou resultate insluit.**
- Alle werk ingehandig moet jou eie werk wees. Jy kan idees uitruil met ander studente, maar alle berekeninge en grafieke moet jou eie wees. **Stappe sal geneem word teen plagiaat.**

Please Note:

Computer assignments are handed in at the start of the tutorial session on the due date. There is a penalty if you hand in late. Please hand in hard copy, no electronic versions are accepted. **You should include both your code and your results.**

All work handed in must be your own work. You may exchange ideas with other students, but all calculations and graphs must be your own. **Action will be taken against plagiarism.**

Probleme:

1. (a) Los die DV hieronder analities op met TW244 metodes (of Wolfram-Alpha) en lei af dat die DV stram is:

   (b) Herskryf die vergelyking hierbo as 'n lineêre stelsel van die form $\boldsymbol{y}' = A\boldsymbol{y}$. Bepaal die stram verhouding van die matriks A en lei weer af dat die probleem stram is.

   (c) Herskryf die DV as 'n stelsel van eerste-orde vergelykings en los numeries op vir $0 \le t \le 1$ deur `ode45` (of NumPy ekwivalent) te gebruik.

   (d) Los weer op met 'n ingeboude oplosser ontwerp vir stram DVs. Vergelyk die aantal tydstappe deur die twee metodes benodig en bespreek.

2. Die kodes `ode324.m` en `ode324.py` op die kursus se webtuiste implementeer 'n aanpasbare RK skema soos beskryf in L32.

   (a) Voltooi die ontbrekende lyne kode vir `s1, s2, s3, p, yn1` en `err` sodat `ode324` die "embedded" Eksplisiete Trapesium en Simpson paar van L32 S10 implementeer. (Wenk: $p = 3$ en z_{i+1}.)

   (b) Wanneer jy 'n vuurhoutjie aansteek groei die bal vlamme vinnig tot dat dit 'n kritiese grote bereik. Dan bly dit daardie grote, omdat die hoeveelheid suurstof wat opgeneem is deur verbranding, in die binneste van die bal, die hoeveelheid beskikbaar deur die oppervlakte uit balanseer. Die eenvoudige model hiervoor is

   Die skalaar veranderlike $y(t)$ stel die radius van die bal voor en die terme y^2 en y^3 stem van die oppervlakte area en volume. Die kritiese parameter is die aanvanklike radius, $\delta = y(0)$, wat "klein" is.
   Gebruik jou `ode324` kode om die vergeyking hierbo optelos met $\delta = 0.001$ tot 'n toleransie van 10^{-6}.

   (c) Herhaal dele (a) en (b), maar gebruik nou die beroemde RKF45 paar (L32 S11–12). (Wenk: $p = 5$.) Lewer kommentaar op die verskille tussen die twee benaderings.

Problems:

1. (a) Solve the ODE below analytically using TW244 methods (or Wolfram-Alpha) and conclude that the ODE is stiff:

$$y'' + 1001y' + 1000y = 0, \qquad y(0) = 1, \; y'(0) = 0.$$

   (b) Rewrite the equation above as a linear system of the form $\boldsymbol{y}' = A\boldsymbol{y}$. Determine the stiffness ratio of the matrix A and again conclude that the problem is stiff.

   (c) Rewrite the ODE as a system of first-order equations and solve numerically for $0 \le t \le 1$ using `ode45` (or the NumPy equivalent).

   (d) Solve again using a built-in solver designed for stiff ODEs. Compare the number of timesteps required by the two solvers and discuss.

2. The codes `ode324.m` and `ode324.py` on the course website implement an adaptive RK scheme as described in L32.

   (a) Complete the missing lines of code for `s1, s2, s3, p, yn1` and `err` so that `ode324` implements the embedded Explicit Trapezium and Simpson pair from L32 S10. (Hint: $p = 3$ and z_{i+1}.)

   (b) When you light a match, the ball of flame grows rapidly until it reaches a critical size. Then it remains at that size because the amount of oxygen being consumed by the combustion in the interior of the ball balances the amount available through the surface. The simple model is for this is

$$y' = y^2 - y^3, \quad \text{for} \quad 0 \le t \le 2/\delta.$$

   The scalar variable $y(t)$ represents the radius of the ball and the y^2 and y^3 terms come from the surface area and the volume. The critical parameter is the initial radius, $\delta = y(0)$, which is "small".
   Use your `ode324` code to solve the equation above with $\delta = 0.001$ to a tolerance of 10^{-6}.

   (c) Repeat parts (a) and (b) but now use the famous RKF45 pair (L32 S11–12). (Hint: $p = 5$.) Comment on the differences between the two approaches.

3. Beskou die toetsprobleem opgelos in L30

Consider the test problem solved in L30

$$\frac{dy}{dt} = -ty, \quad y(1) = 4,$$

met eksakte oplossing

with exact solution

$$y = 4e^{\frac{1}{2}(1-t^2)}.$$

(a) Pas die Adams–Bashforth tweestap metode

(a) Apply the Adams–Bashforth two-step method

$$w_{j+1} = w_j + \tfrac{1}{2}h\big(3f_j - f_{j-1}\big)$$

toe om die oplossing by $t = 2$ te bereken. Gebruik Euler se metode om die vermiste beginwaarde te bereken. Maak 'n tabel van absolute foute by $t = 2$ soos in bladsy 11 van Lesing 30 en beraam die orde van konvergensie. Is jou beraming in ooreenstemming met die Taylor analise op p. 338–339 in die handboek?

to compute solution values at $t = 2$. Use Euler's method to compute the missing starting value. Make a table of absolute errors at $t = 2$ as in slide 11 of Lecture 30 and estimate the order of convergence. Does your estimate agree with the Taylor analysis on p. 338–339 in the text?

(b) Opsioneel: Herhaal deel (a) vir die meerstapmetode:

(b) Optional: Repeat part (a) for the multistep method:

$$w_{j+1} = -4w_j + 5w_{j-1} + h\big(4f_j + 2f_{j-1}\big)$$

In teorie moet hierdie metode se orde 3 wees, soos afgelei kan word uit (6.79) in die handboek. Jou resultate mag egter hiervan verskil. Om te sien wat verkeerd gaan, teken die oplossing.

In theory this method should have order 3, as can be deduced from (6.79) in the text. Your results may not agree, however. To see what goes wrong, plot the solution.

4. Die Simpson reël vir AWP's is gegee deur (L32, bladsy 10)

Simpson's rule for IVPs is given by (L32, slide 10)

$$\begin{aligned} s_1 &= f(t_i, w_i) \\ s_2 &= f(t_i + h, w_i + hs_1) \\ s_3 &= f(t_i + \tfrac{h}{2}, w_i + \tfrac{h}{2}\tfrac{s_1+s_2}{2})) \\ w_{n+1} &= w_n + \tfrac{h}{6}(s_1 + 4s_3 + s_2) \end{aligned}$$

(a) Pas hierdie reël toe op die standaard toets probleem vir A-stabiliteit en so lei die stabiliteits voorwaarde af:

(a) Apply this rule to the standard test problem for A-stability and hence derive the stability condition:

$$\left|1 + \overline{h} + \frac{1}{2}\overline{h}^2 + \frac{1}{6}\overline{h}^3\right| < 1.$$

(b) Gebruik die bostaande voorwaarde om numeries die maksimum staplengte te bereken wat hierdie metode mag gebruik om die volgende probleem op te los:

(b) Use the condition above to numerically determine the maximum step size this method may use to solve the problem

$$\frac{dy}{dt} = -10y, \qquad y(0) = 1.$$

Opsioneel / Optional:

5. Die Lorenz-stelsel is 'n stelsel van gewone DVs eerste bestudeer deur Edward Lorenz:

The Lorenz system is a system of ODEs first studied by Edward Lorenz:

$$\begin{aligned} \dot{x} &= -sx + sy \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned}$$

Dit is bekend vir chaotiese oplossings vir sekere parameterwaardes en beginvoorwaardes. In die besonder, die Lorenz "attractor" is 'n versameling van chaotiese oplossings van die stelsel wat soos 'n skoenlapper of die syfer agt lyk as dit geteken word.

It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions which, when plotted, resemble a butterfly or figure eight.

Gebruik 'n DV oplosser van jou keuse om die Lorenz-vergelykings op te los vir $0 \le t \le 100$ met die parameters en die beginvoorwaardes hieronder. Teken die oplossing met behulp van MATLAB se `plot3` om die skoenlapper te sien.

Use an ODE solver of your choice to solve the Lorenz equations for $0 \le t \le 100$ with the parameters and initial conditions given below. Plot the solution using MATLAB's `plot3` function to see the 'butterfly'.

$$s = 10, \quad r = 28, \quad b = 8/3, \qquad x(0) = -14, \quad y(0) = -15, \quad z(0) = 20.$$