1.

a) $y(t) = 6\sin(5x(t)) + 8\cos(10x(t))$

Non-linear and time-invariant. Not LTI

b) $y(t) = t\sin(kt) + x(t)$

Linear and time-varying. Not LTI

c) $y(t) = ax(t) + b$

Linear time invarint if $b = 0$

d) $y(t) = kx(t-c)$

Linear. and time-invariant. It is an LTI

e) $y(t) = t^2 x(t)$

Linear and time-varying. Not an LTI

2 a) No transfer function because it non linear

b) No transfer function because time-varying

c) If $b = 0$

$H(s) = a$ is transfer function

d) $H(s) = ke^{(-cs)}$

e) No transfer function because its time varying

3. $m = 0.1kg,\ M_c = 1kg$

$L = 0.5m,\ g = 9.81m/s$

$\ddot{\theta} = F(\theta, \dot{\theta}) + G(\theta)u$

$$F(\theta, \dot{\theta}) = \frac{\sin\theta}{L(M_c + m\sin^2\theta)}\left\{-mL\dot{\theta}^2\cos\theta + (M_c+m)g\right\}$$

$$G(\theta) = \frac{-\cos\theta}{L(M_c + m\sin^2\theta)}$$

a) $\ddot{\theta} = 0,\ \dot{\theta} = 0$

$F(\theta_0, 0) + G(\theta_0)u = 0$

$$\frac{\sin\theta_0}{L(M_c + m\sin^2\theta_0)}(M_c+m)g + \cancel{G(\theta)u}\ \frac{-\cos\theta_0}{L(M_c+m\sin^2\theta_0)} = 0$$

$(M_c+m)g\sin\theta_0 = u\cos\theta_0 = 0$

$\sin\theta_0 = 0$

$\theta_0 = 0, \pi$

b) $\ddot{\theta} = F(\theta, \dot{\theta}) + G(\theta)u$

$$\ddot{\theta} = \frac{\sin\theta}{L(M_c + m\sin^2\theta)}\left[-mL\dot{\theta}^2\cos\theta + (M_c+m)g\right] + \frac{-\cos\theta}{L(M_c+m\sin^2\theta)}u$$

$$= f(\theta, \dot{\theta}, u)$$

$\cancel{\left.\frac{\partial f}{\partial \theta}\right|}$ At

$$\left.\frac{\partial f}{\partial \theta}\right|_{(0,0,0)} = \frac{(M_c+m)g}{LM_c}$$

$$\left.\frac{\partial f}{\partial \dot{\theta}}\right|_{(0,0,0)} = 0$$

$$\left.\frac{\partial f}{\partial u}\right|_{(0,0,0)} = -\frac{1}{LM_c}$$

$$\ddot{\theta} \approx f(0,0,0) + \left.\frac{\partial f}{\partial \theta}\right|_{(0,0,0)} \cdot (\theta - 0) + \left.\frac{\partial f}{\partial \dot{\theta}}\right|_{(0,0,0)} (\dot{\theta} - 0) + \left.\frac{\partial f}{\partial u}\right|_{(0,0,0)} (u-0)$$

$$\ddot{\theta} \approx \frac{(M_c+m)g}{LM_c}\theta - \frac{1}{LM_c}u$$

At $(\pi, 0)$

$$\left.\frac{\partial f}{\partial \theta}\right|_{(\pi,0,0)} = -\frac{(M_c+m)g}{LM_c}$$

$$\left.\frac{\partial f}{\partial \dot{\theta}}\right|_{(\pi,0,0)} = 0$$

$$\left.\frac{\partial f}{\partial u}\right|_{(\pi,0,0)} = -\frac{1}{L(M_c+m)}$$

$$\ddot{\theta} \approx -\frac{(M_c+m)g}{LM_c}(\theta-\pi) - \frac{1}{L(M_c+m)}u$$

$x_1 = \theta$.

$x_2 = \dot{\theta}$

$\theta = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ML} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\theta = \pi$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) At $\theta = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \end{bmatrix} u$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$sX(s) = AX(s) + BU(s)$$

$$\cancel{(sI - A)}\ X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = CX(s) = C(sI - A)^{-1} BU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{-1}{MLs^2 - (M+m)g}$$

Similarly for $\theta = \pi$

$$G(s) = \frac{-1}{MLs^2 + (M+m)g}$$

d) For $\theta = 0$

Characteristic eq $s^2 - g/L = 0$

s^2	1	$-g/L$
s^1	0	
s^0	$-g/L$	

It is unstable

For $\theta = \pi$

Characteristic eq $s^2 + g/L = 0$

s^2	1	g/L
s^1	0	
s^0	g/L	

It is stable

g) $\xi = \frac{\ln(10\%)}{\sqrt{\pi^2 + (\ln(10\%))^2}} = 0.591$

$\xi > 0.591$

Consider $\xi = 0.7$

$T_s = \frac{4}{\xi w_n} = 1$

$w_n = 5.71$ rad/s

Characteristic eq. is $s^2 + 2\xi w_n s + w_n^2 = 0$

$\cancel{ML s^2 + K_d}$

$\Rightarrow ML s^2 + K_d s + (K_p - (M+m)g) = 0$

$K_d = 2\xi w_n ML = 4$ Ns/rad

$K_p = w_n^2 ML + (M+m)g = 27.3$/rad

h) $\xi = \frac{\ln(15\%)}{\sqrt{\pi^2 + (\ln(15\%))^2}} = 0.517$

$T_s = \frac{4}{\xi w_n} = 0.5$

$w_n = 15.47$ rad/s

~~Characteristic eq. is~~ $\cancel{s^2 + 2\xi w_n s + w_n^2 = 0}$

$\Rightarrow \cancel{ML}$ $K_d = 2\xi w_n ML = 8$ Ns/rad

$K_p = \cancel{16.4}\ w_n^2 ML + (M+m)g = 130.8$ N/rad

i) $e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s\left(-\frac{1/s}{1+G(s)}\right) = \frac{1}{1+K_p}$

$= \cancel{\frac{1}{1+130.8}} = \frac{1}{1+130.8} =$

j) $0.5 \times \frac{\pi}{180} > \frac{1}{1+K_p}$

$0.00873 > \frac{1}{1+K_p}$

$K_p > 113.5$

k) Characteristic eq. is $s^2 + K_d s + K_p = 0$

For marginal stability => imaginary roots

$K_p > 0,\ K_d = 0$