Classical Algebraic K-Theory (K_0, K_1)

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Bhoris Dhanjal

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Certificate

This is to certify the Semester III Research Project titled 'Classical Algebraic K-Theory (K_0, K_1)' in partial fulfilment of the requirements for the degree of M.Sc. Mathematics submitted by Mr. Bhoris Dhanjal is approved for submission.

Dr. Anuradha Garge
(Project Guide)

Prof. B. S. Desale
(Head of Department)

Date:

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Abstract

This project explores key results in classical algebraic K-theory, focusing on the Grothendieck groups K_0, Whitehead groups K_1, and its various applications. Topics include the Quillen–Suslin theorem, and Suslin's work on unimodular vectors linear groups.

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Chapter 1

Foundational prerequisites

We begin with some basic prerequisite definitions of abelian categories and homological algebra, which provide a foundational framework. These concepts will be used extensively forming an assumed background for the discussions.

1.1 Abelian Categories

Abelian categories are essential to the understanding of homological algebra. It is motivated by the fact that it allows for using homological methods in a wide variety of applications and helps unify various (co)homology theories. They were first introduced by Grothendieck in his seminal Tohuku paper [Gro57].

There is a chain of conditions regarding 'abelian'-ness of categories which is roughly understood as follows.

$$\textbf{Abelian} \subseteq \textbf{Pre-Abelian} \subseteq \textbf{Additive} \subseteq \textbf{Ab-Enriched}$$

Ab-Enriched categories (sometimes referred to as pre-additive categories) are categories such that for $A, B \in \mathcal{C}$ the external hom set $\mathrm{Hom}(A, B)$ has the structure of an abelian group, furthermore it has a well defined notion of composition (which is bilinear due to the monoidal product in Ab), $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$.

We have chosen to omit the precise definitions of the coherence conditions for monoidal and monoidally enriched categories to make this section easier to read. Since we refrain from explicitly using them for computations anywhere, the basic background described above will suffice. For a more detailed overview of the definitions for monoidal and monoidally enriched categories refer to [Lan98] for a classical treatment or [Rie17] for an excellent modern exposition.

Example 1.1.1. *A ring is a single object* Ab-*Enriched category (In the same sense how a group is realized as a single object category with all arrows invertible).*

We cover a few basic results.

Proposition 1.1.2. *In Ab-Enriched categories intial and terminal objects coincide (it is often called the zero object)*

Proof. Let $\mathcal{C}$ be an Ab-Enriched category. Note that the Hom-sets between objects have 'zero morphisms', i.e. arrows in the Hom-set which behave like the additive identity in the Ab group induced by it. In particular for $0_{A,B} \in \text{Hom}(A, B)$ we have the property that if $f : B \to C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \to D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathcal{C}$ is initial so there is a unique morphism $0 \to 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \to 0$ we can say that by the zero morphism property $f = 0$ so also 0 is terminal. $\square$

Proposition 1.1.3. *In Ab-Enriched categories finite coproducts coincide with finite products (i.e. biproducts)* [1]

Proof. Let $\mathcal{C}$ be an Ab-enriched category and $A, B \in \mathcal{C}$ consider the product $A \times B$, which is determined by the following UMP,

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

Consider A and B in place of X in the diagram. By the UMP we have $q_1 : A \to A \times B, q_2 : B \to A \times B$

$$\begin{tikzcd}
A \arrow[dd, "1_A"'] \arrow[dr, "q_1"'] & & B \arrow[dd, "1_B"] \arrow[dl, "q_2"'] \\
 & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

[1]This also holds over categories enriched over commutative monoids.

So $p_1q_1 = 1_A$ and $p_2q_2 = 1_B$ also $p_1q_2 = p_2q_1 = 0$.

Now note that $q_1p_1 + q_2p_2 = 1_{A\times B}$ as $p_1(q_1p_1 + q_2p_2) = p_1$ and $p_2(q_1p_1 + q_2p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following UMP holds for some arbitrary $C \in \mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "r_1"] \arrow[dr, "q_1"'] \arrow[dd, "1_A"'] & C & B \arrow[l, "r_2"'] \arrow[dl, "q_2"] \arrow[dd, "1_B"] \\
& A \times B \arrow[u, dashed, "f"'] \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
A & & B
\end{tikzcd}$$

Define $f : A \times B \to C$ as $f = r_1p_1 + r_2p_2$. Now $fq_1 = r_1$ and $fq_2 = r_2$ if we show uniqueness of f we are done.

Say f' then $(f - f')1_{A\times B} = (f - f')(q_1p_1 + q_2p_2) = 0$. So $f = f'$.

$\square$

Definition 1.1.4 (Additive category)**.** *An Ab-Enriched category which has all finite (co)products (i.e. biproducts since they coincide).*

Example 1.1.5. *The category of vector bundles over a topology X is a additive category (but not an abelian category). We will see this in more detail in the following section.*

Functors between additive categories are called *additive functors*. And can be realized as functors which preserve additivity of homomorphisms between modules, $F(f + g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition 1.1.6 (Kernel)**.** *A kernel of a morphism $f : A \to B$ is the pullback along the unique morphism from $0 \to B$, i.e. it is $p : \ker f \to A$. Provided initials and pullbacks exist.*

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \to B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'. [2]

A cokernel is simply its dual.

Definition 1.1.7 (Pre-abelian categories)**.** *An additive category with all morphism having kernels and cokernels.*

The above definition is equivalent to saying a pre-abelian category is a Ab-Enriched category with all finite limits and colimits. This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Proposition 5.21]. And we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition 1.1.8 (Abelian category)**.** *Pre-abelian categories for which each monomorphism is a kernel and each epimorphism is a cokernel.*

Equivalently a category is abelian if its pre-abelian and $\bar{f}$ is an isomorphism in the canonical decomposition of $f : A \to B$ as

$$A \twoheadrightarrow \operatorname{coker}\ker f \xrightarrow{\bar{f}} \ker \operatorname{coker} f \hookrightarrow B.$$

Example 1.1.9. *Some non examples are:*

1. *The category of hausdorff topological abelian groups is pre-abelian but not abelian. Since not every morphism which is a mono+epi is necessarily a isomorphism.*

   Consider a Hausdorff abelian topological group G with a non discrete topology and consider G' it's discretization. The inclusion map $G' \to G$ is a mono+epi but not isomorphic.

2. *The category of torsion free abelian groups (*TFAb*) is pre-abelian but not abelian as the mono $f : \mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}$ is not a kernel of some morphism. Say it were and there exists $A \in$ TFAb such that f is the kernel to $g : \mathbb{Z} \to A$, i.e.*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f(z)=2z"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ Z \arrow[r, "g"] & A \end{tikzcd}$$

[2] A minor point to note is that in the case of Ab-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a Ab-Enriched category has a initial 0 however this matches up with our intuition.

this implies $1_{\mathbb{Z}}$ factors through f i.e. $1_{\mathbb{Z}} = f \circ h$ for some unique $h : \mathbb{Z} \to \mathbb{Z}$ which implies $h(1) = 1/2$ which is absurd.

Example 1.1.10. *Some examples of abelian categories:*

1. *The category of modules.*

2. *Category of representations of a group*

3. *Category of sheaves of abelian groups on some topological space.*

With this definition in mind we will now define a few important constructions we will use often. These are not restricted to abelian categories but we will use them very often in the case of abelian categories, so it is good to see it in action directly with the notion of an abelian category at hand.

Definition 1.1.11 (Subobject)**.** *A subobject for some $X \in \mathcal{C}$ is a monomorphisms into X.*

With slight abuse of notation we refer to $Y \leq X$ as a subobject of X where Y is just a representative of the codomain of a isomorphism class of monomorphisms into X. In particular for$X, Z \rightrightarrows X$ monics Z, X belong to the same subobject class if the morphisms are isomorphic, i.e. there exists an isomorphism between $Y \to Z$ making the triangle commute.

This is clearer when seen through the lens of a slice category. Note that arrows between subobjects of the same X are arrows in the slice category of X. So collection of subobjects form a category with a preorder (with inclusion). The reasoning behind such an definition for subobjects is motivated by the fact that we think of generalized elements in $\mathcal{C}$ as being not $X \in \mathrm{Ob}(\mathcal{C})$ but rather $\mathrm{Hom}_{\mathcal{C}}(-, X)$.

Definition 1.1.12 (Quotients in abelian categories)**.** *For $Y \leq X$ in an abelian category we can define X/Y as the cokernel of the monomorphism $Y \to X$.*

Definition 1.1.13 (Extension/short exact sequences in abelian categories)**.** *For $A, B \in \mathcal{A}$ an extension by A of B refers to some $E \in \mathcal{A}$ such that $0 \to A \to E \to B \to 0$ is a short exact sequence.*

A deep result on abelian categories is the Freyd-Mitchel embedding theorem which helps characterize all small abelian categories in terms of modules.

Theorem 1.1.14 (Freyd-Mitchell). *Every small abelian category can be faithfully embedded as a full subcategory via an exact functor into $R-Mod$ for some ring R.*

The proof for the theorem is very extensive and as such is omitted. The canonical reference is Freyd's own book [FF64]. A proof sketch summarising Freyd's proof is given in an excellent MathOverflow post by the user Theo Buehler [Bue].

1.2 Resolutions

Definition 1.2.1. *[Left resolution] Given a module M its left resolution is given by the data of a exact sequence $(A_\bullet, \varphi_\bullet)$ into M as such,*

$$\cdots \to A_1 \to A_0 \xrightarrow{\epsilon} M \to 0$$

where ϵ is called the augmentation map, if the exact sequence is free its a free resolution and such for projective.

If we have a cochain complex instead it forms a right resolution and if its elements are injective we call them injective resolutions.

We cover a result about stably free modules. We will revisit this in greater detail in further sections.

Lemma 1.2.2. *A projective module is stably free iff if has a finite free resolution.*

Proof. Say $M \in A-\text{Mod}$ and is projective. Then M is stably free implies $M \oplus A^n \cong A^m$ for some n, m. So M has a finite free resolution given by

$$0 \to A^n \to A^m \to M \to 0$$

Conversely if M is a projective module with a finite free resolution.

$$0 \to F_n \to \cdots \to F_0 \to M \to 0$$

for F_i free modules over A.

We proceed via induction on n. If $n = 0$ then we are done as simply M is free. Let M_1 denote the kernel of $F_0 \to M$. Since M is projective, $F_0 = M \oplus M_1$. Now M_1 is projective and has a free resolution of length $< n$.

Thus, by the induction hypothesis, $M_1 \oplus F$ is free for some finite free module F, i.e. M_1 is stably free.

Consequently $F_0 \oplus F$ is trivially finitely free and so $F_0 \oplus F \cong (M \oplus M_1) \oplus F \cong M \oplus (M_1 \oplus F)$ is finite free, i.e. M is stably free. $\square$

Theorem 1.2.3. *Given a short exact sequence $0 \to A \to B \to C \to 0$. If any two of these modules have a finite free resolution then so does the third.*

Proposition 1.2.4 (Horseshoe lemma)**.** *If there is a short exact sequence of modules,*

$$0 \to M \to N \to P \to 0$$

and both M, P have a projective resolutions A, C

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \\
 & & & N \arrow[d] & \\
\cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & P \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

as below then N also has a projective resolution B which forms a short exact sequence. Also the sequence splits due to C_i being projective so $B_i = A_i \oplus C_i$.

Proof. First note $\epsilon_P : C_0 \to P$ lifts due to projectivity to $C_0 \to N$ also $A_0 \to N$ via composition so simply define $B_0 = A_0 \oplus C_0$. This is an epimorphism evidently via diagram chase. Also is projective as direct sum of projectives is projective. Now consider direct sum of kernel of $A_0 \to M, B_0 \to N, C_0 \to P$ and construct the direct sum again to get F_1. Now we get a 3×3. Exactness is due to the Snake lemma $\square$

1.3 Puppe/Homotopy cofiber sequence

The discussion in this section follows [Wei94].

TO DO: KOSZKUL COMPLEX AND HILBERT SYZYGY We previously saw the definition of a bilinear map when discussing tensor products. Now consider n- linear maps for some map between R-Modules M, P. Repeated application of the tensor product still provides us with universal such module, $M^n \to \otimes_{i=1}^n M$.

Also a map is called n-alternating if it vanishes when two of the arguments are the same. This also implies that sign changes when the arguments are interchanged. Also for a permutation the sign change changes to the sign of the permutation. Now when we require a notion of a universal n-linear alternating map we arrive to the definition of a wedge product.

In particular for a specific n. There is a universal alternating n-linear map sending $M \to \Lambda^n M$. If M is finitely generated by r elements then ΛM^n is a free module of rank $\binom{r}{n}$.

Definition 1.3.1 (Tensor algebra). *For a R-Module M and $T^n M = \otimes^n M$ the tensor algebra is defined to be*

$$T(M) = \oplus_{n=0}^{\infty} T^n M$$

Definition 1.3.2 (Exterior algebra and wedge product). *For an R-Module M. Consider the Tensor algebra $T(M) = \oplus_n \otimes^n M$. Consider the ideal I spanned by elements $v \otimes v$ for $v \in T(M)$.*

The quotient of $T(V)/I := \Lambda(M)$ is called the exterior algebra and its product the wedge product. It is universal with respect to multilinear alternating maps.

For $v\Lambda^n M, w \in \Lambda^m M$ multiplication is defined as such,

$$v\Lambda w = (-1)^{nm} w\Lambda v$$

Definition 1.3.3 (Koszkul complex). *For a R-module M there is an associated chain complex of $n^{\boldsymbol{th}}$ exterior algebras*

$$\Lambda^n M \to \Lambda^{n-1} M \to \cdots \to \Lambda^0 M \cong M$$

Definition 1.3.4 (Regular sequence). *A sequence of elements $r_1, \ldots, r_n$ is said to be a regular sequence in R if r_1 is not a zero divisor of R, r_2 is not a zero divisor of $R/\langle r_1 \rangle$, and so on*

Proposition 1.3.5. *For a finite regular sequence $\{r_i\}_{i=1}^n$ of a ring R the Koszkul complex forms the canonical free resolution of $R/\langle r_1, \ldots, r_n \rangle$ of the form*

$$0 \to R^{\binom{n}{n}} \to \cdots \to R^{\binom{n}{1}} \to R \to R/\langle r_1, \ldots, r_n \rangle \to 0$$

Proof. TO DO □

1.4 Simplicial objects and classifying spaces

The Dold-Kan correspondence provides a useful result, serving as a bridge between simplicial homotopy theory and homological algebra. We follow the proof as presented in [GJ09].

In fact Dold-Kan can be thought of as a categorification of a theorem about divided differences (fundamental theorem of calculus)! This insight was given by a nCafe post [Lei10] crediting it to Andrew Joyal.

For an exhaustive reference to this section refer to [GJ09], a rapid introduction to simplicial sets is also covered in [Rie11].

1.4.1 Basic idea of simplices

We begin with a notion of a purely combinatorial construction Δ denote the category with objects as non empty finite ordered sets and order preserving maps. Objects in Δ are denoted as $[n] = \{1, 2, \dots, n\}$, i.e. an $n-$simplex. This is sometimes defined as the full subcategory of Cat on free categories of finite directed graphs equivalently finite ordinals with order preserving maps.

This allows us to define the standard n-simplex as $\Delta[n] = \mathrm{Hom}(-, [n])$

There is the natural map from $\Delta \to$ Top sending $[n]$ to the standard $n-$simplex in $\mathbb{R}^{n+1}$.

1.4.2 Simplicial sets

Definition 1.4.1 (Simplicial set)**.** *A simplicial set is a functor $X : \Delta^{\mathrm{op}} \to$ **Set**, i.e. presheaves on Δ. It comprises of a collection of sets X_n which we to be the set of $n-$simplices of X with maps between them corresponding naturally with maps in Δ.*

Proposition 1.4.2. *By Yoneda we have $X_n \cong \mathrm{Hom}(\Delta[n], X)$*

As **sSet** is determined via presheaves it is (co)complete and all (co)limits over small diagrams can be computed objectwise.

Furthermore corresponding to injections from $[n-1] \to [n]$ in Δ we get a family of face maps between simplices $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$.

And degeneracy maps corresponding to surjections $[n+1] \to n$ as a family of maps $s_i : X_n \to X_{n+1}$.

These obey the standard relations,

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i, && i < j \\
s_i s_j &= s_{j+1} s_i m && i \leq j \\
d_i s_j &= 1, && i = j, j+1 \\
d_i s_j &= s_{j-1} d_i, && i < j \\
d_i s_j &= s_j d_{i-1}, && i > j+1
\end{aligned}
$$

The following nice visuals are from [Fri23].

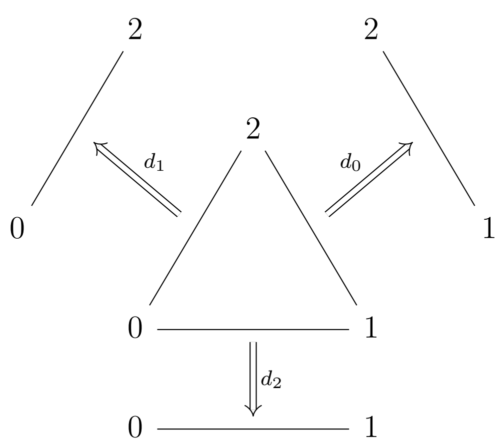

Figure 1.1: Face maps for 2-simplex

This naturally leads to the definition of **sSet** the category of simplicial sets as the functor category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$

Definition 1.4.3 (Simplicial objects in arbitrary categories)**.** *The same as above but over any category not just* **Set***.*

Definition 1.4.4 (Simplicial abelian groups)**.** *A simplicial object in* **Ab***.*

Further more there is a forgetful free forgetful adjoint pair between **sAb** and **sSet**.

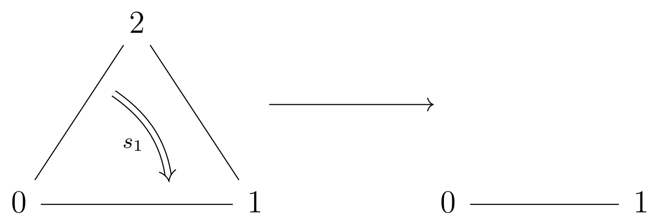

Figure 1.2: Degeneracy maps for 2-simplex

Recall that for some topological space X we define its singular simplicial set as $\mathrm{Hom}(\Delta_n, X)$, this infact forms an adjoint pair along with 'geometric realization' the functor $|| : \mathbf{sSet} \to \mathbf{Top}$ sending standard $n-$simplicies to topological simplicies, in particular $|| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}, || \dashv \mathrm{Sing}$

Since we defined simplicial sets as presheaves of the simplex category we have a lot of nice properties thanks to the well behavedness of presheaves.

Proposition 1.4.5. *Any simplicial set can be expressed as a colimit of standard n simplicies.*

This is a consequence of the fact that any presheaf is a colimit of representable presheaves (as a coend) given some $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}, F(-) = \int^{c \in \mathbf{C}} F(C) \times \mathrm{Hom}_{\mathbf{C}}(-, c)$ [3].

1.4.3 Moore complex and normalized chain complex

Prior to discussing the Dold-Kan correspondence proper there is the question of constructing a complex from a simplicial object. There are three different ways to approach this all of which end up being closely related to each other.

The first is the Moore complex.

Definition 1.4.6 (Moore complex)**.** *For a simplical abelian group $A_.$ associate a complex $A_\bullet$ whose n^{th} component is $A_n \in A_.$ with boundary maps $\partial : A_n \to A_{n-1}$ given by alternating sum of face maps $\sum_i^n (-1)^i d_i$*

Once again the singular chain complex is the Moore complex of the simplicial abelian group associated to the singular complex of a topological space.

[3]Compare this to Yoneda: $F(c) \cong \mathrm{Nat}(\mathrm{Hom}(-, c), F)$ and the fact that the Yoneda embedding is characterized via an exponential.

The second is a subcomplex of the Moore complex $DA_\bullet$ wherein the components are degenerate simplicies (i.e. it is in the image of some degeneracy map). The simplicial identities allow the construction to be a well defined complex.

The third is the Normalized chain complex of a simplicial abelian group $NA_\bullet$ where the n^{th} component is the subobject of A_n which vanishes under face maps with differentials being only $(-1)^n d_n$ once again well definedness is due to the simplicial identities.

Definition 1.4.7 (Normalized chain complex)**.** *$A \in \mathbf{sAb}$, the complex $NA_\bullet$ is defined as*

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n$$

With differentials as $\partial : NA_n \to NA_{n-1}$ defined as $(-1)^n d_n$ which forms a chain complex due to the simplicial identity $d_{n-1}d_n = d_{n-1}d_{n-1}$.

As it turns out the normalized chain complex and the subcomplex of the Moore complex we defined are isomorphic as chain complexes.

Theorem 1.4.8. *content*

1.4.4 Simplicialization functor

We have already seen some motivation for

1.4.5 Dold-Kan Correspondence for abelian groups

Recall that a equivalence of categories is a proposition that is stronger than adjoints but weaker than isomorphisms. It can be thought of as adjoints but with natural isomorphisms instead of transformations.

Definition 1.4.9 (Equivalence of categories)**.** *Two categories $\mathbf{C}, \mathbf{D}$ are said to be equivalent if $\exists E : C \rightleftarrows D : F$ and natural isomorphisms $\alpha : 1_{\mathbf{C}} \to F \circ E, 1_D : E \circ F$*

Theorem 1.4.10. *There is a equivalence of categories*

$$N : \mathbf{sAb} \rightleftarrows \mathbf{Ch}_+ : \Gamma$$

where N denotes the normalized chain complex functor and Γ the simplicalization functor.

Proof. We need to show a natural transformation between $\Gamma NA \cong A$ and $N\Gamma C \cong C$ $\square$

Selecta Mathematica (2021) 27:14 https://doi.org/10.1007/s00029-021-00618-5

1.4.6 Some applications of Dold-Kan

https://mathoverflow.net/questions/437382/applications-of-the-dold-kan-correspondence
Construct Eilenberg Maclane spaces

Derived functors of non additive functors

1.5 Category of chain complexes

1.5.1 Chain complexes are monoidal

This section consists mainly of definitions and terminologies that we will require in the section on spectral sequences.

Definition 1.5.1 (Category of chain complexes)**.** *For an abelian category $\mathcal{A}$, the collection of chain complexes in $\mathcal{A}$ with morphisms as chain maps form a category often denoted as* $\mathrm{Ch}(\mathcal{A})$*. It is again an abelian category.*

Interestingly enough the category of chain complexes is in fact also a closed monoidal category.

Definition 1.5.2 (Tensor product of chain complexes)**.** *For $A, B \in \mathrm{Ch}(\mathcal{A})$ define $A \otimes B$ component wise as*

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$$

with the differential as $\varphi_n(a, b) = (\varphi_A a \otimes b) + (-1)^{\deg(a)}(a \otimes \varphi_B b)$

This of course only makes sense provided $\mathcal{A}$ admits tensors. When discussing homological algebra in the context of $R-$Mod this distinction need not be made.

The mysterious sign rule (often referred to as the Koszul sign rule') can be motivated in one of many different ways.

The most important is that the monoidal product arises naturally when we consider the problem of reducing the double complex we obtain by tensoring piecewise (i.e. finding the total complex but we will see this later).

Perhaps another way to motivate it with the notion of a 'Day convolution'. The following treatment is due to a MathOverflow answer by Alexander Campbell [Cam20].

A Day convolution can be realized as a categorification of typical convolution product.

Definition 1.5.3 (Promonoidal category)**.** *content*

Definition 1.5.4 (Profunctor)**.** *content*

Definition 1.5.5 (Day Convolution as a coend)**.** *content...*

Finally it makes sense to consider that there is a promonoidal structure on $\mathrm{Ch}(\mathcal{A})$.

Definition 1.5.6 (Internal hom of chain complexes)**.** *For $A, B \in \mathrm{Ch}(\mathcal{A})$ we define an internal hom object as a chain complex with the following components*

$$[A, B]_n = \prod_i \mathrm{Hom}(A_i, Y_{i+n})$$

with differentials for $x \in [A, B]_n$

$$\varphi_n x = \varphi_B x - (-1)^n x \varphi_A$$

The above two definitions characterize $\mathrm{Ch}(\mathcal{A})$ as a closed monoidal category.

Definition 1.5.7 (Unital tensor)**.** *The unital tensor is considered as the unital tensor of $\mathcal{A}$ say $1_{\mathcal{A}}$ understood as a chain complex $\cdots \to 0 \to 0 \to 0 \to 1_{\mathcal{A}}$ (i.e. a chain complex concentrated at 0). In particular when $\mathcal{A} = R-Mod$ the unital tensor is just R.*

Definition 1.5.8 (Quasi-isomorphism)**.** *A chain map is called a quasi isomorphism if the induced map on the homologies constituents an isomorphism.*

The reason for 'quasi' is that the relation is reflexive and transitive but not symmetric.

Definition 1.5.9 (Homotopy category of chain complexes)**.** *For a given category of chain complexes $\mathrm{Ch}(\mathcal{A})$ we can define $\mathcal{K}(\mathcal{A})$ to be the homotopy category of chain complexes with objects as objects of $\mathrm{Ch}(\mathcal{A})$ and arrows as chain homotopic maps as introduced in Def.*??.

1.5.2 Cones, cylinders and homotopy cofiber (Puppe) sequences

The notion of mapping cones and cylinders are almost identical to those we first encounter in algebraic topology. We first see the constructions of these objects with their universal properties and then see what explicit cases later.

The universal properties elucidate what the construction is really saying better than viewing the explicit definition straight away which can seem confusing.

Sign conventions differ in the classical presentation of mapping cones to maintain consistency we shall follow the convention in Weibel [Wei94]

In order to discuss cylinders we must construct a notion of an 'interval' object similar to how we use $[0, 1]$ in topology.

The interval object for $\mathrm{Ch}(\mathcal{A})$ is defined as a chain complex as follows

Definition 1.5.10 (Interval object)**.** *For a unital tensor in $\mathcal{A}$ say 1 the interval complex denoted as I is defined as,*

$$\cdots 0 \to 0 \to 1 \to 1 \oplus 1$$

When $\mathcal{A} = R{-}Mod$ this is

$$\cdots 0 \to 0 \to R \to R \oplus R$$

With the differential simply being $(-i, i)$ the identity map.

This is motivated via the Dold-Kan correspodence as a sequence of maps starting from the standard 1 simplex to simplical sets to the free simplical abelian group, the normalized chain complex obtained at the end of this process is this interval object.

A cylinder is defined using the tensor unit $I \in \mathrm{Ch}(\mathcal{A})$.

Definition 1.5.11 (Cylinder object)**.** *For $A \in \mathrm{Ch}(\mathcal{A})$ its cylinder is defined as $\mathrm{Cyl}(A) = I \otimes A$.*

Definition 1.5.12 (Cone object)**.** *Defined as the following pushout the $1_{\mathcal{A}}$ denotes the unital tensor object,*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{cyl}(A) \arrow[d] \\
1_A \arrow[r] & \mathrm{cone}(A)
\end{tikzcd}$$

Similarly the mapping cylinder is also defined via a pushout.

Definition 1.5.13 (Mapping cylinder)**.**

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ \mathrm{cyl}(A) \arrow[r] & \mathrm{cyl}(\alpha) \end{tikzcd}$$

Proposition 1.5.14 (Components of the chain complex of a mapping cylinder)**.**

$$\mathrm{cyl}(\alpha)_n = A_n \oplus B_n \oplus A_{n-1}$$

Finally the mapping cone is the following pushout. As such

Definition 1.5.15 (Mapping cone)**.**

$$\begin{tikzcd} \mathrm{cyl}(A) \arrow[r] \arrow[d] & \mathrm{cyl}(\alpha) \arrow[d] \\ \mathrm{cone}(A) \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

Putting all the following definitions together we can instead view the mapping cone as the pushout for the following diagram,

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ A \arrow[r] \arrow[d] & \mathrm{cyl}(A) \arrow[r] \arrow[d] & \mathrm{cyl}(\alpha) \arrow[d] \\ 1_A \arrow[r] & \mathrm{cone}(A) \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

This is analogous to the topological definition of a mapping cone where we take a cylinder and identify together one side to a point.

Proposition 1.5.16 (Components of the chain complex of a mapping cone)**.**

$$\mathrm{cone}(\alpha)_n = B_n \oplus A_{n-1}$$

with the differential as

$$\begin{tikzcd}
B_{n+1} \arrow[r, "\varphi_B"] & B_n \\
\oplus & \oplus \\
A_n \arrow[r, "-\varphi_A"'] \arrow[uur, "\alpha"] & A_{n-1}
\end{tikzcd}$$

Definition 1.5.17 (Suspension of chain complex)**.** *For $A \in \mathrm{Ch}(\mathcal{A})$ the complex defined by shifting degrees down by one is defined as its suspension denoted as follows,*

$$A[-1]_n = A_{n-1}$$

with the differential shifted down but with a negative sign

$$\varphi_n = -\varphi^A_{n-1}$$

We can now identify a canonical short exact sequence

$$B \to \mathrm{cone}(\alpha) \to X[1]$$

with the mappings being inclusion and projection respectively. In particular it forms a pushout.

Infact this is universal even with respect to homotopy that means the universality of the pushout is upto chain homotopy. To summarize we have the following proposition.

Proposition 1.5.18 (Homotopy pushout induced by mapping cone)**.** *For $A, B \in \mathrm{Ch}(\mathcal{A})$ with $\alpha : A \to B$ some chain map. The following forms a short exact sequence of chain complexes with inclusion and projection mappings*

$$B \to \mathrm{cone}(\alpha) \to A[-1]$$

Which in turn induce a pushout square which is unique up to chain homotopy which characterizes $A[-1]$ as a cofiber of $i : B \to \mathrm{cone}(\alpha)$ over 0.

$$\begin{tikzcd}
B & & B \arrow[r] \arrow[d] & \mathrm{cone}(\alpha) \arrow[d] \\
A \arrow[u, "\alpha"'] & & 0 \arrow[r] & A[-1]
\end{tikzcd}$$

This induces what is called a homotopy cofiber sequence

Definition 1.5.19 (Homotopy cofiber sequence)**.** *For* $A, B \in \mathrm{Ch}(\mathcal{A})$ *with* $\alpha : A \to B$*,*

$$A \to B \to \mathrm{cone}(\alpha) \to A[-1] \to B \to \mathrm{cone}(\alpha) \to A[-2] \to \cdots$$

This gives rise to a more common triangle diagram

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & B \arrow[dl] \\ & \mathrm{cone}(\alpha) \arrow[ul] & \end{tikzcd}$$

This trianglular diagram will be useful in the next section of derived categories.

This notion of the homotopy cofiber sequence is in fact more fundamental than the long exact homology sequence we saw before. Taking a representative of the cone up to homotopy as another chain complex C and looking at the induced sequence of their homologies give us the long exact sequence of homology.

1.5.3 Double complex and total complex

A double complex is understood to be a 'chain complex of chain complexes'. In particular it is bigraded over $\mathbb{Z} \times \mathbb{Z}$.

Definition 1.5.20 (Double complex)**.** *A double complex* $A_{\bullet,\bullet} \in \mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ *is a chain complex of chain complexes in* $\mathrm{Ch}(\mathcal{A})$ *in particular we say it has both a vertical differential* $\varphi^v : A_{\bullet,m} \to A_{\bullet,m-1}$ *and a horizontal differential* $\varphi^h : A_{n,\bullet} \to A_{n-1,\bullet}$ *such that these differentials make all the squares commute, i.e.* $\varphi^v\varphi^h = \varphi^h\varphi^v$*,*

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & A_{n,m} \arrow[r, "\varphi^h_m"] \arrow[d, "\varphi^v_{n-1}"] & A_{n,m-1} \arrow[r] \arrow[d, "\varphi^v_{n-1}"] & \cdots \\ \cdots \arrow[r] & A_{n-1,m} \arrow[r, "\varphi^h_m"] \arrow[d] & A_{n-1,m-1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd}$$

Proposition 1.5.21 (Model for mapping cone in terms of double complexes)**.** *For $A, B \in \mathrm{Ch}(\mathcal{A})$ and chain map $\alpha : A \to B$ the double complex formed by stating $C_{1,n} = A_n$ and $C_{0,n} = B_n$ with the natural differential forms a homotopically equivalent model for* $\mathrm{cone}(f)$.

Definition 1.5.22 (Total complex)**.** *For a double complex $A_{\bullet,\bullet}$ there is associated to it a (regular $\mathbb{Z}$ graded) chain complex known as its total complex. The components of a total complex can be expressed as direct sums or direct products, i.e.*

$$\mathrm{Tot}^{\oplus}(A_{\bullet,\bullet})_n = \oplus_{n=p+q} A_{p,q}$$

or

$$\mathrm{Tot}^{\Pi}(A_{\bullet,\bullet})_n = \Pi_{n=p+q} A_{p,q}$$

with differentials in each case as

$$\varphi^{\mathrm{Tot}} = \varphi^v + (-1)^{\mathrm{verticaldeg}} \varphi^h$$

In fact the Total complex of a double complex given via objectiwise tensoring of two complexes X, Y is isomorphic to the tensor product of the chain complexes. This forms an alternative characterization of the chain complex tensor product.

It is the computation of the homology of a double complex which leads to equivalently computing the homology of the total complex which in turn leads to the formulation of the spectral sequence of a double complex.

1.6 Derived and triangulated category

Note that the algebraic notion of localization easily generalizes to categories proper. We can localize a category on some collection of arrows which sending the arrows to isomorphisms.

Definition 1.6.1 (Derived category (Universal property))**.** *For an abelian category $\mathcal{A}$ the localization of its category of chain complexes* $\mathrm{Ch}(\mathcal{A})$ *with respect to quasi-isomorphisms gives its associated derived category $D(\mathcal{A})$. It is universal in the sense that any functor from* $\mathrm{Ch}(\mathcal{A}) \to \mathcal{C}$ *which sends quasi-isomorphisms to isomorphisms factors through $D(A)$.*

Definition 1.6.2 (Derived category (With homotopy category))**.** *content...*

The usefulness of a derived category is ??????

Definition 1.6.3 (Triangulated categories)**.** *content...*

1.7 Spectral sequences

1.7.1 Spectral sequence of a filtered complex

1.7.2 Spectral sequence of a double complex

Chapter 2

Vector bundles

In this section, we introduce the foundational concepts required to prove Swan's theorem, which demonstrates a key connection between projective modules and vector bundles over certain topological spaces. The purpose of proving Swan's theorem is to highlight why projective modules play a central role in algebraic K-theory, serving as essentially algebraic counterparts to vector bundles.

All results presented hold for both real and complex vector bundles. For simplicity, we use k to denote the underlying field.

More detailed exposition can be found in [MS74, Kar08].

Definition 2.0.1 (Vector bundle). *A n dimensional vector bundle over k is a triple (E, p, X). Which consists of a continuous map $p : E \to X$ from the total space E to the base space X. Such that for all $x \in X$, $E_x = p^{-1}(x)$ the fibre of x has a k vector space structure. Along with the following property of local trivialization*

1. *For any $x \in X$ there exists a open open $U \subset X$ along with a homeomorphism*
$$h : X \times k^n \to p^{-1}(U)$$
such that for all $c \in U$ the map through h defines an isomorphism between E_c and k^n.

Example 2.0.2 (Trivial vector bundle). *A trivial vector bundle is one in which the total space $E = X \times k^n$ with p just the trivial projection mapping.*

The local trivialization condition can now be understood as saying there exists an open neighbourhood for each point $x \in X$ such that $E|_U$ is a trivial vector bundle

Definition 2.0.3 (Quasi-vector bundle). *A fibre bundle defined as above without the property of local trivialisation is called a quasi-vector bundle.*

Definition 2.0.4 (Rank of vector bundle). *If each E_x has the same dimension n then n is referred to as the rank of the vector bundle.*

Example 2.0.5 (Tangent bundle on S^2). *Let S^2 be the unit sphere in $\mathbb{R}^3$ associate to each point $x \in S^2$ the plane tangent to S^2 at x, label this T_xS^2 (this is the tangent space at point x) then the disjoint union of all the tangent spaces form a vector bundle $TS^2 = \sqcup_{x \in S^2} T_xS^2$.*

We don't use the concept of a tangent bundle proper so we refrain from defining it in its entirety.

Definition 2.0.6 (Vector bundle morphisms). *Two vector bundles (E_1, p, X) and (E_2, p_2, X) are considered isomorphic if there exists a continuous map between their total spaces g such that the below diagram commutes*

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

and g induces a vector space homomorphism for each fibre.

If g is a homeomorphism and fibrewise isomorphism then it is a vector bundle isomorphism.

Definition 2.0.7 (Category of vector bundles). *For a fixed base space B the collection of vector bundles over X with arrows as vector bundle homomorphisms forms a category which we denote as* $\mathrm{VB}(X)$*.*

Definition 2.0.8 (Whitney sums). *For $(E_1, p_1, X), (E_2, p_2, X)$ their Whitney sum denoted as $E_1 \oplus E_2$ consists of total space as $E_1 \oplus E_2$ with fibrewise direct sums of vector spaces.*

A Whitney sum is a biproduct of vector bundles in VB(X). Which makes the category of vector bundles into an additive category, but not abelian. Since the kernel of a morphism of vector bundles need not be a vector bundle.

Along with a Whitney sum there also exists the expected notion of a tensor product of vector bundles this in turn allows us to view VB(X) as a symmetric monoidal category in the natural way.

In the particular case when $E_1 = X \times k^n, E_2 = X \times k^m$ are both trivial vector bundles over X. We can describe the vector bundle morphisms between them explicitly. This is a result we will use very often in the process of proving Swans theorem.

We have associated to the vector bundle morphism $g : E_1 \to E_2$ a natural linear map $g_x : k^n \to k^m$. Consider $\widetilde{g} : X \to \mathrm{Hom}_k(k^n, k^m)$ defined as $\widetilde{g}(x) = g_x$.

Theorem 2.0.9 (Characterization of trivial bundle morphisms)**.** *The map $\tilde{g} : X \to Hom_k(k^n, k^m)$ is continuous*[1]*, and for $h : X \to Hom_k(k^n, k^m)$ continuous, let $\tilde{h} : E \to E'$ be the map which induces $\tilde{h}(x)$ on each fiber. Then $\tilde{h}$ is a morphism of vector bundles.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a choice of basis for k^n and $\{e'_1, \ldots, e'_m\}$ for k^m.

Now $g_x \in \mathrm{Hom}_k(k^n, k^m)$ is naturally representable by a $m \times n$ matrix $(a_{ij}(x))$.

The function $x \mapsto a_{ij}(x)$ is obtained by composing the continuous maps:

$$X \xrightarrow{\beta_j} X \times k^n \xrightarrow{g} X \times k^m \xrightarrow{\gamma} k^m \xrightarrow{p_i} k$$

where $\beta_j(x) = (x, e_j)$, $\gamma(x, v) = v$, and p_i is the i-th projection onto k. Since each $a_{ij}(x)$ is continuous $\tilde{g}$ is continuous.

Now consider $h : X \to \mathrm{Hom}_k(k^n, k^m)$, continuous. By composing a sequence of continuous maps we obtain the required morphism of vector bundles.

$$X \times k^n \xrightarrow{\delta} X \times \mathrm{Hom}_k(k^n, k^m) \times k^n \xrightarrow{\epsilon} X \times k^m$$

where $\delta(x, v) = (x, h(x), v)$ and $\epsilon(x, u, v) = (x, u(v))$. □

Proposition 2.0.10 (Sufficient condition for vector bundle isomorphism)**.** *Let E and F be two vector bundles over X, and let $g : E \to F$ be a morphism of vector bundles such that $g_x : E_x \to F_x$ is bijective for each point $x \in X$. Then g is an isomorphism of vector bundles.*

[1] Relative to the natural topology on $\mathrm{Hom}_k(k^n, k^m)$. The unique topological vector space topology making any $k^t \cong L(k^n, k^m)$ into a homeomorphism

Proof. Let $h : F \to E$ be the map defined by $h(v) = g_x^{-1}(v)$ for $v \in F_x$. We need to prove h is continuous and so indeed a map in $\mathrm{VB}(X)$. The other conditions for isomorphisms are clearly met. If we prove continuity locally for every neighbourhood then we have the required continuity globally.

Consider a neighbourhood U of x and the local trivialization isomorphisms $\beta : E|_U \to U \times M$ and $\gamma : F|_U \to U \times N$. Let $g_1 = \gamma g_U \beta^{-1}$.

Then $h_U = \beta^{-1} h_1^{-1} \gamma$, where h_1 is defined as $\widetilde{h_1}(x) = (\widetilde{g_1}(x))^{-1}$ (as in the above result Theorem 2.0.9), h_1 is continuous. Therefore, h is continuous on a neighbourhood of each point of F, implying h is continuous on all of F. □

2.1 Sections of a vector bundle

Definition 2.1.1 (Sections of a vector bundle)**.** *For a vector bundle (E, p, X) a section refers to a continuous map $s : X \to E$ such that $p \circ s = 1_X$, where 1_X denotes the identity map on X.*

These sections equivalently are a homomorphism of vector bundles from the trivial line bundle $(X \times \mathbb{R}, \pi_X, X) \to (E, p, X)$

We assume all sections to be continuous without necessarily specifying it.

Definition 2.1.2 (Linear independence of sections)**.** *A sequence $s_1, \cdots, s_n$ of sections on a vector bundle E is said to linearly independent if they are linearly independent for each point x.*

Furthermore the morphism $f : X \times k^n \to E$ given by $(x, c_1, \cdots, c_n) \mapsto \sum_{i=1}^n c_i s_i(x)$ induces an isomorphism of fibres if E has rank n due to Proposition 2.0.10.

Example 2.1.3 (Zero section)**.** *The map $s : X \to E$ sending every point x to the 0 vector in E_x.*

Definition 2.1.4 (Vector space of sections)**.** *For a vector bundle $p : E \to X$ the set of continuous sections of E is denoted as $\Gamma(X, E)$. It is a vector space with vector addition defined as $(s+t)(x) = s(x)+t(x)$ and scalar multiplication as $\lambda s(x) = (\lambda s)(x)$ for $\lambda \in k$.*

Proposition 2.1.5 (Section functor)**.** *$\Gamma(X, E)$ can be realized as a $C(X)$ module where $C(X)$ denotes the ring of continuous functions from X to k. With scalar multiplication defined as $(c \cdot s)(x) = c(x)s(x)$ for $s \in \Gamma(X, E), c \in C(X)$.*

For a trivial vector bundle $E = X \times k^n$ we have $\Gamma(X, E)$ corresponds to a free module $C(X)^n$. If E is a arbitrary vector bundle which is a direct summand of a trivial vector bundle then $\Gamma(X, E)$ corresponds to a projective $C(X)$ module.

In particular If $E \oplus E' \cong X \times k^n$ we have $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong C(X)^n$, i.e. $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ is an additive functor.

In fact this 'stably trivial' property of a vector bundle is always true when X is compact. This is an essential result in the proof of Swans theorem. In order to prove it we need a lemma for paracompact spaces first.

Definition 2.1.6 (Paracompact space)**.** *A Hausdorff topological space X is said to be paracompact if every open cover of X has a locally finite open refinement.*

(An open cover is locally finite if there exists a neighbourhood for every $x \in X$ which intersects only finitely many elements of the cover. A refinement of an open cover $\{U_i\}$ consists of a open cover $\{V_j\}$ such that for each $j, V_j \subset U_i$).

Lemma 2.1.7. *Let X be a paracompact space, and let E and F be vector bundles over X. Suppose $\alpha : E \to F$ is a morphism such that $\alpha_x : E_x \to F_x$ is surjective for each point $x \in X$. Then there exists a morphism $\beta : F \to E$ such that $\alpha \circ \beta = \mathrm{id}_F$.*

Proof. In this proof we very liberally use the various morphisms as seen in Theorem 2.0.9.

Fix a point $x \in X$ by local trivialization we pick a neighborhood U of x such that E_U and F_U are trivial bundles. That is, $E|_U \cong M \times V$ and $F|_U \cong U \times N$.

Now with this representation we have $\alpha|_U : U \times M \to U \times N$, can be expressed as $\tilde{\theta}$ (as in 2.0.9) for the associated continuous map: $\theta : U \to \mathrm{Hom}_{\mathrm{VB}}(V, W)$.

Decomposing M as $N \oplus \ker(\theta(x))$ (which we can do thanks to surjectivity of α_x) allows us to choose a matrix representation for $\theta(y) : M \to N$ i.e. $\theta(y) : N \oplus \ker(\theta(x)) \to N$ as

$$\theta(y) = \begin{bmatrix} \theta_1(y) & \theta_2(y) \end{bmatrix},$$

the first component goes to 1 and the second to 0 continuously, i.e. an endomorphism of N. Realized as a topological vector space $\mathrm{Aut}(N)$ is an open subset (i.e. vector subspace) of $\mathrm{End}(N)$.

Therefore we can pick a neighbourhood of x say V_x such that $\theta_1(y)$ is an isomorphism for $y \in U_x$. Consider associated to this the map $\theta' : V_x \to \mathrm{Hom}_{\mathrm{VB}}(N, M)$ represented now by the matrix,

$$\theta'_x(y) = \begin{bmatrix} \theta_1(y)^{-1} \\ 0 \end{bmatrix}$$

This now induces the morphism $\tilde{\theta}'_x : F_| V_x \to E_| V_x$ so that $\alpha_{V_x} \tilde{\theta}'_x = 1$.

Varying the point x enables us to construct a locally finite open cover $\{V_i\}$ of X obeying the following properties, all morphisms $\beta_i : F_{V_i} \to E_{V_i}$ are right inverses of α_{V_i} as seen.

Now consider a 'partition of unity' $\{\eta_i\}$ associated with the cover [2] and $\beta : F \to E$ is defined using it as $\sum_i \eta_i(x)\beta_i(e)$ for $e \in E_x$. Continuity is maintained due to the fact that these sums are necessarily finite (see footnote). This gives us

$$\alpha\beta(e) = \sum_i \eta_i(x)(\alpha\beta_i)(e) = (\sum_i \eta_i(x))(\alpha\beta_i(e)) = e$$

Which completes the proof. □

Theorem 2.1.8. *If E is a vector bundle over a compact base space X then there exists a vector bundle E' such that $E \oplus E' \cong X \times k^n$*

Proof. Pick a finite open cover $\{U_i\}_{i=1}^r$. By local trivialisation we know that $E|_{U_i} \cong U_i \times k^{n_i}$.

Let $\{\eta_i\}$ be a 'partition of unity' associated to the cover similarly as before. So there exist n_i linearly independent sections $s_i^1, s_i^2, \ldots, s_i^{n_i}$ of $E|_{U_i}$ (as seen in Definition 2.1.2). By extending these local sections to zero outside U_i, we obtain globally defined sections $\eta_i s_i^1, \eta_i s_i^2, \ldots, \eta_i s_i^{n_i}$, which are linearly independent sections of $E|_{V_i}$, where $V_i = \eta_i^{-1}((0, 1])$.

Let σ_i^j denote the sections $\eta_i s_i^j$ for $1 \leq j \leq n_i$. These sections $\sigma_i^j(x)$ generate E_x as a vector space for each $x \in X$. As constructed in $Definition 2.1.2$, there exists a morphism

$$\alpha : T = X \times k^n \to E,$$

[2] A sequence of non negative real valued functions $\{\eta_i\}$ whose sum evaluated for every point $x \in X$ is 1. And such that every point $x \in X$ has an open neighbourhood not intersecting the support of η_i for all but finitely many i. Which implies the sums are finite and therefore well defined.

where $n = \sum_{i=1}^{r} n_i$, such that $\alpha_x : T_x \to E_x$ is surjective for every $x \in X$.

Now by the above Lemma for paracompact spaces, there exists a morphism $\beta : E \to T$ such that $\alpha \circ \beta = 1_E$. Let E' be the kernel of the idempotent morphism $p = \beta \circ \alpha$.

Define a morphism from $E \oplus E'$ to T by taking the sum of $\beta : E \to T$ and the inclusion $i : E' \to T$. This morphism induces a fibrewise isomorphism since $E'_x \cong \ker(p_x)$ and $E_x \cong \ker(1 - p_x)$. Thus, the morphism is an isomorphism by Proposition 2.0.10 $\square$

2.2 Karoubian categories and some results on vector bundles

In this section we cover an essential theorem which is applied in order to prove Swan's theorem. We follow the results as proven in Karoubi's book [Kar08].

Definition 2.2.1 (Karoubian category)**.** *A Karoubi category is an Ab-Enriched category such that every idempotent endomorphism (a morphism of the form $e : A \to A$ such that $e^2 = e$) has a kernel.*

These are sometimes referred to as pseudo-abelian categories (note that in an abelian category proper all morphisms have kernels and cokernels in this case its only the idempotents).

Example 2.2.2. *The category of vector bundles is Karoubian. We already know it is additive and so Ab-Enriched. In the conclusion of Theorem 2.1.8 we have implicitly shown that the idempotent $p \in \mathrm{VB}(X)$ has a kernel.*

Example 2.2.3. *Hinting towards Swans theorem we can see that also* $\mathrm{Proj}(A)$ *for a ring A is Karoubian.*

Idempotent elements occur ubiquitously in $K-$theory we will see another useful application of idempotents in computations of K_0.

The first theorem we cover in this section is well known by various different names. It involves showing the existence of a Karoubi envelope for an additive category. But this is often referred to as the Cauchy completion or idempotent completion.

Theorem 2.2.4. *[Existence of Karoubi envelope] Let $\mathcal{C}$ be an additive category. Then there exists a Karoubian category $\widetilde{\mathcal{C}}$ and a fully faithful additive functor $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ universal in the sense that for any other additive functor $\psi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Karoubian there exists a unique additive functor $\psi' : \widetilde{\mathcal{C}} \to \mathcal{D}$ such that the below diagram commutes.*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[rd, dashed, "\psi'"] & \\
\mathcal{C} \arrow[u, "\varphi"] \arrow[r, "\psi"'] & \mathcal{D}
\end{tikzcd}$$

Proof. We directly construct $\widetilde{\mathcal{C}}$ and φ. The objects of $\widetilde{\mathcal{C}}$ are of the form (E, p) for $E \in \mathrm{Ob}(\mathcal{C})$ and p idempotent endomorphism over E.

Arrows between two such objects (E, p) and (F, q) comprise of the arrows $(f : E \to F) \in \mathcal{C}$ such that $fp = qf = f$. Composition of morphisms is inherited from $\mathcal{C}$. The identity morphism comprises of $1_{(E,p)} = p$, and te sum of two objects $(E, p) \oplus (F, q)$ is defined naturally as $(E \oplus F, p \oplus q)$. This demonstrates that $\widetilde{\mathcal{C}}$ is indeed an additive category.

It remains to be verified that it is indeed Karoubian which is the main concern. Let f be an idempotent endomorphism of the object (E, p) in $\widetilde{\mathcal{C}}$ we wish to show that f indeed has a kernel.

Examining the below commutative diagram will help us conclude. The elements F, g, q, h are defined directly as seen below. Note that the condition $fp = pf = f$ is present in the righthand square

$$\begin{tikzcd}
 & F \arrow[dl, dashed, "h"'] \arrow[dr, "g"] \arrow[ddd, bend left, "q"] & & \\
E \arrow[rr, "p-f"] \arrow[d, "(1-f)p"'] & & E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\
E \arrow[rr, "p-f"'] & & E \arrow[r, "f"'] & E \\
 & F \arrow[ul, dashed, "h"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

Since $(1-f)p$ is itself an idempotent endomorphism on E we have $(E, (1-f)p)$ is an object in $\widetilde{\mathcal{C}}$ and $p - f$ defines an arrow from this object to (E, p). Since $(p - f)((1 - f)p) = (p)(p - f) = p - f$ as expected.

Now we claim that the object $(E, (1-f)p)$ with the arrow $p - f$ is a kernel of $f : (E, p) \to (E, p)$. In particular it is a pullback of the below form. The diagram below drawn internally in $\widetilde{C}$ makes this image clearer.

$$\begin{tikzcd}
(F,q) \arrow[dr, dashed, "h"] \arrow[drr, bend left] \arrow[ddr, bend right, "g"'] & & \\
& (E,(1-f)p) \arrow[r] \arrow[d, "p-f"'] & 0 \arrow[d] \\
& (E,p) \arrow[r, "f"] & (E,p)
\end{tikzcd}$$

Compare this to the first diagram. $(F, q) \xrightarrow{g} (E, p)$ is picked such that $fg = 0$. And we obtain uniqueness of h is due to the fact that any such arrow h must obey the expression $h = (1-f)ph = p(1-f)h = pg = g$. Conversely if $h = g$ the diagram commutes naturally.

This shows that $\widetilde{\mathcal{C}}$ is Karoubian. Finally we construct $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ defined as sending $E \in \mathrm{Ob}(\mathcal{C})$ to $(E, 1_E)$ and $\varphi(f) = f$. This is naturally a full faithful functor by construction.

Also we can see that based on an analogous argument (E, p) is the kernel of $1-p$ as an idempotent endomorphism over $\varphi(E) = (E, 1_E)$. Meaning that $\varphi(E) \cong (E, p) \oplus (E, 1-p)$, i.e. the functor is additive.

Finally to show that these constructions define ψ'.

If $\psi : \mathcal{C} \to \mathcal{D}$ (resp. $\psi' : \widetilde{\mathcal{C}} \to \mathcal{D}$) is an additive functor from $\mathcal{C}$ (resp. $\widetilde{\mathcal{C}}$) to another Karoubian category $\mathcal{D}$, such that $\psi'\varphi \cong \psi$. Then we have $\psi(\ker f) \cong \ker(\psi'(f))$ for every idempotent endomorphism f. Hence $\psi'(E, p) = \ker \psi(1-p) : \psi(E) \to \psi(E)$ and $\psi'(f) = \psi(f)_{\ker \psi(1-p)}$ on the objects and morphisms respectively. Conversely, these formulas define ψ' (up to isomorphism). $\square$

Before proceeding for the last result in this section, recall the definition of equivalence of categories.

Definition 2.2.5 (Equivalence of categories)**.** *Two categories $\mathcal{C}, \mathcal{D}$ are said to be equivalent if there exist functors $E : \mathcal{C} \rightleftarrows \mathcal{D} : F$ and a pair of natural isomorphisms $\alpha : 1_{\mathbf{C}} \to F \circ E$ and $\beta : 1_{\mathbf{D}} \to E \circ F$.*

This is a weaker condition than isomorphism of categories in which we have an actual equality instead of natural isomorphism.

A more useful form of the definition is as such, the proof may be seen in [Awo10][7.7.25]

Proposition 2.2.6. *A functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories iff F has the following properties*

1. *F is full (The map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ defined as $f \mapsto F(f)$ is surjective for all $A, B \in \mathrm{Ob}(\mathcal{C})$).*

2. *F is faithful (The map $F_{A,B}$ as defined above is injective for all pairs A, B).*

3. *F is essentially surjective on objects (For every $D \in \mathrm{Ob}(\mathcal{D})$ there exists $C \in \mathrm{Ob}(\mathcal{C})$ such that $FC \cong D$).*

Corollary 2.2.7. *Let $\mathcal{C}$ be an additive category, $\mathcal{D}$ a Karoubian category, and $\psi : \mathcal{C} \to \mathcal{D}$ an additive functor which is fully faithful such that every object of $\mathcal{D}$ is a direct factor of an object in the image of ψ. Then the functor ψ' as defined in Theorem 2.2.4 forms an equivalence between the categories $\mathcal{C}$ and $\mathcal{D}$.*

Proof. We will prove that ψ' is essentially surjective and fully faithful.

Let $G \in \mathcal{D}$. We seek $X \in \widetilde{\mathcal{C}}$ such that $\psi'(X) \cong G$.

By the hypothesis we have that for $G \in \mathcal{D}$, there exists $E \in \mathcal{C}$ and $G' \in \mathcal{D}$ with $\psi(E) \cong G \oplus G'$. Due to this we can choose an idempotent endomorphism $q : \psi(E) \to \psi(E)$ such that $\mathrm{Ker}(q) \cong G$ we are in a pseudo-abelian category.

Now as ψ is fully faithful, there's a idempotent $p : E \to E$ in $\mathcal{C}$ with $\psi(p) = q$. Then by the formulas in the end of Theorem 2.2.4 we have $G \cong \varphi'(E, 1 - p)$. This proves essential surjectivity.

Lastly to prove φ' is fully faithful consider two objects $H, H' \in \widetilde{\mathcal{C}}$ direct factors of $\varphi(E), \varphi(E')$. Then the following diagram shows that $\psi'_{H,H'}$ is an isomorphic function,

$$\begin{tikzcd}
\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\varphi(E), \varphi(E')) \arrow[r, "\cong"] \arrow[dr, "\psi'_{\varphi(E),\varphi(E')}"'] & \mathrm{Hom}_{\mathcal{C}}(E, E') \arrow[r, shift left] \arrow[d, "\psi_{E,E'}"] & \mathrm{Hom}_{\widetilde{\mathcal{C}}}(H, H') \arrow[l, shift left] \arrow[d, "\psi'_{H,H'}"] \\
& \mathrm{Hom}_{\mathcal{D}}(\psi(E), \psi(E')) \arrow[r, shift right] & \mathrm{Hom}_{\mathcal{D}}(\psi'(H), \psi'(H')) \arrow[l, shift right]
\end{tikzcd}$$

where the horizontal arrows are induced by the decompositions $\varphi(E) = H \oplus H_1$ and $\varphi(E') = H' \oplus H'_1$, since $\psi_{E,E'}$ is an isomorphism by hypothesis. $\square$

2.3 Swan's theorem

We can now prove the celebrated Swan's theorem with the results we've built up so far.

Theorem 2.3.1 (Swan's theorem)**.** *Let X be a compact Hausdorff space, and let $A = C(X)$. Then the section functor Γ induces an equivalence of categories* $\mathrm{VB}(X) \simeq \mathrm{Proj}(C(X))$.

Proof. Since we assume that X is compact the section map Γ as seen in Proposition 2.1.5 is indeed a functor $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ due to Theorem 2.1.8. Furthermore it induces a functor $\Gamma_T : \mathrm{VB}_T(X) \to \mathrm{Free}(C(X))$.

Where $\mathrm{VB}_T(X)$ refers to the full subcategory of $\mathrm{VB}(X)$ consisting of the trivial bundles over X, and $\mathrm{Free}(C(X))$ refers to finitely generated free modules over $C(X)$.

Since $C(X)^n \cong \Gamma_T(E)$ for $E = X \times k^n$, we have Γ_T is essentially surjective.

If $F : X \times k^p$ is some other trivial vector bundle and $f : E \to F$ is a morphism of vector bundles then as seen in Theorem 2.0.9 we have full faithfullness of the functor Γ_T. Which shows that Γ_T induces a equivalence of categories between $\mathrm{VB}_T(X)$ and $\mathrm{Free}(C(X))$.

To extend this to our required case we make use of Theorem 2.2.4 and Theorem 2.2.7.

Comparing with Theorem 2.2.4 since $\mathrm{VB}(X)$ being Karoubian itself is naturally the Karoubian envelope of its subcategory $\mathrm{VB}_T(X)$ and with the functor Γ realized as ψ' we see the below diagram commutes.

$$\begin{tikzcd}
VB(X) \arrow[dr, "\Gamma=\psi'"] & \\
\mathrm{VB}_T(X) \arrow[u, "i"] \arrow[r, "\psi"'] & \mathrm{Proj}(C(X))
\end{tikzcd}$$

Finally due to Theorem 2.2.7 we are done.

using Theorem **??** which ensures that the trivial bundles suffice in building up to the construction. Theorems 2.2.4 and 2.2.7 combined guarantee that if a functor is fully faithful and satisfies the direct-factor property, then it's an equivalence. Since Theorem [reference needed for fully faithful proof] demonstrates that Γ is fully faithful, Γ is an equivalence of categories. Therefore, $\mathrm{VB}(X) \simeq \mathrm{Proj}(C(X))$. $\square$

Chapter 3

Fundamental theorems of K_0

3.1 K_0 for exact and abelian categories

K_0 being the prototypical K group is readily generalized to our new settings. We will refer to Weibel for most of the definitions of K_0 [WS13]. The benefit of this approach will mainly be for building up an understanding for Higher K theory. A

We begin with the definition of an exact category and then define K_0 for an exact category. This will subsume the definition of K_0 for an abelian category. Since every abelian category is exact over itself.

Definition 3.1.1 (Exact category)**.** *An exact category (sometimes referred to as a Quillen exact category) is a pair $(\mathcal{E}, E)$ for $\mathcal{E}$ an additive category which is a full subcategory of some abelian category $\mathcal{A}$. Along with a family of sequences E of the form*

$$0 \to A \to B \to C \to 0$$

which are short exact sequences in $\mathcal{A}$ and if in a sequence of the above form $A, C \in \mathcal{E}$ then B is isomorphic to some element which is in $\mathrm{Ob}(C)$

Example 3.1.2. *Every abelian category is trivially exact over itself.*

Example 3.1.3. *Torsion free abelian groups over the category of abelian groups is exact but not abelian. (Non abelian-ness was shown in Example 1.1.9.2).*

Definition 3.1.4 (K_0 for an exact category $\mathcal{E}$)**.** *$K(\mathcal{E})$ is generated by $[B]$ for each $B \in \mathrm{Ob}(\mathcal{E})$ and a relation of $[B] = [A] + [C]$ for all short exact sequences*

$$0 \to A \to B \to C \to 0$$

Naturally since every abelian category is exact this applies for abelian categories in particular.

3.2 Fundamental theorems for K_0, G_0

In this section we present here for reference the important 'fundamental' theorem for K_0 and G_0 of abelian categories. The proofs will be provided in the subsequent project.

The proofs are omitted here due to the need for a more in-depth discussion of concepts such as the localization of categories and multiplicative systems. Furthermore, the proofs for the K_0 case are readily extended to the higher K groups, especially in the context of Quillen-Q constructions, which are based on the calculus of fractions of categories. As such, it is more fitting to address these in detail in the next project, which will build on the work presented here.

3.3 Devissage

Theorem 3.3.1 (Devissage for K_0 in abelian categories)**.** *Let $\mathcal{B} \subset \mathcal{A}$ be abelian categories which are small (i.e. proper set of objects) then $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$ and the inclusion functor $i : \mathcal{B} \to \mathcal{A}$ is exact if the following conditions are met*

1. *$\mathcal{B}$ is a abelian exact subcategory of B closed under subobjects and quotients from $\mathcal{A}$.*

2. *Objects in $\mathcal{A}$ have finite filtrations*

$$A_n = 0 \subset A_{n-1} \subset \cdots \subset A_0 = A$$

with each of the quotients $A_i/A_{i+1} \in \mathcal{B}$

Proof. Since $i : \mathcal{B} \subseteq \mathcal{A}$ we denote by $\tilde{i} : K_0(\mathcal{B}) \to K_0(\mathcal{A})$ the natural induced homomorphism. This is naturally injective so we need to prove surjectivity and additivity.

Based on the hypothesis we can always find a finite filtration on $A \in \mathcal{A}$ of the form $\{A_i\}_{i=0}^n$ with the quotients in $\mathcal{B}$. We can then represent its consequent preimage in $K_0(\mathcal{B})$ as $[A] = \sum_i [A_i/A_{i+1}]$ in $K_0(\mathcal{A})$, i.e. $\phi^{-1}([A]) = \sum_i A_i/A_{i+1}$.

Note that such a preimage is also independent of the filtration due to the Schrier-Refinement theorem for abelian categories which states that we can always find a common refinement of filtrations. The proof is identical to the standard group theoretic proof.

To verify this claim look at a single refinement, say changing $A_i \supset A_{i+1}$ to $A_i \supset A' \supset A_{i+1}$.

$$0 \to A'/A_{i+1} \to A_i/A_{i+1} \to A_i/A' \to 0,$$

we see that $[A_i/A_{i+1}] = [A_i/A'] + [A'/A_{i+1}]$ in $K_0(\mathcal{B})$, as claimed. This is essentially

For additivity and exactness, note that given a short exact sequence $0 \to A \to B \to C \to 0$, we can construct a filtration $\{B_i\}$ for B by combining the filtration for A along with pullback of a filtration of C in B. For this filtration we have $\sum [A_i/A_{i+1}] = f(A') + f(A'')$. Therefore f is an additive function and defines a map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. By inspection, f is the inverse of the canonical map i_*. □

Corollary 3.3.2. *For a nilpotent ideal N in a noetherian ring A we have* $G_0(A/N) \cong G_0(A)$

Proof. Every finitely generated module has a natural filtration which goes to zero found by by multiplying a module with copies of N. □

Nil ideals (ideals consisting of all nilpotent elements) are in general not nilpotent ideal. However, due to a theorem by Levitzky (see [Lam01, Theorem 10.30]) which states that nil ideals of right noetherian rings are indeed nilpotent we can say that this holds for nil ideals as well.

3.4 Serre quotient

For an abelian category $\mathcal{A}$ a Serre subcategory of $\mathcal{A}$ is a specific kind of subcateogry which allows us to create a ‘quotient’ which we call a Serre quotient. In reality this is more akin to a localization than a quotient proper.

Definition 3.4.1 (Serre subcategory)**.** *Let $\mathcal{A}$ be an abelian category, a full subcategory $\mathcal{B} \subset \mathcal{A}$ is called a Serre subcategory of $\mathcal{A}$ if*

- *For exact sequence $0 \to A \to B \to C$ in $\mathcal{A}$, $B \in \mathcal{B} \iff A, C \in \mathcal{B}$*
- *Equivalently this means that $\mathcal{B}$ is closed under quotients, subobjects and extensions.*

Definition 3.4.2 (Serre quotient)**.** *Given $\mathcal{B} \subset \mathcal{A}$ a locally small Serre subcategory of an abelian category we can define its Serre question $\mathcal{A}/\mathcal{B}$ with the following construction.*

- $\mathrm{Ob}(\mathcal{A}/\mathcal{B})$ *consists of objects from $\mathcal{A}$*
- *Morphisms between $A \to B$ as $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B) = \lim_{\to} \mathrm{Hom}_{\mathcal{A}}(\tilde{A}, Y/\tilde{Y})$ where $\tilde{A} \leq A, \tilde{Y} \leq Y$ are subobjects.*

When $\mathcal{A}$ is small we can treat morphisms $A \to B$ as equivalence classes of diagrams of the form

$$A \xleftarrow{f} \bullet \xrightarrow{g} B$$

Where ker, coker of f are in $\mathcal{B}$. Equivalence with another diagram $A \leftarrow \circ \to B$ is defined with the existence of the below commuting diagram

$$\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[dr] & \\
A & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B \\
 & \bullet \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Where for $\bullet \leftarrow X \to \circ$ we have ker, coker of both the arrows in $\mathcal{B}$, making the below diagram commute.

Theorem 3.4.3. *$\mathcal{A}/\mathcal{B}$ is abelian and the inclusion functor $L : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact*

Proposition 3.4.4. *The Serre quotient $\mathcal{A}/\mathcal{B}$ universal in the following sense. Any exact functor $U : \mathcal{A} \to \mathcal{C}$ such that $U(B) \cong 0$ for $B \in \mathcal{B}$ will factor* *through L. i.e. the below diagram commutes*

$$\begin{tikzcd}
 & \mathcal{A}/\mathcal{B} \arrow[d, dashed] \\
\mathcal{A} \arrow[ur, "L"] \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

This result is the reason why the Serre subcategory is defined the way it is. The conditions required in the definitions are precisely those such that the above proposition may hold.

3.5 Localization theorem for K_0

Recall the definitions of Serre quotient.

Theorem 3.5.1 (Localization theorem for K_0)**.** *For a small abelian category $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory. The following sequence is exact*

$$K_0(\mathcal{B}) \xrightarrow{f} K_0(\mathcal{A}) \xrightarrow{L} K_0(\mathcal{A}/\mathcal{B}) \to 0$$

Proof. By construction we know that L is surjective. Note we already know that $\text{coker} f \to K_0(\mathcal{A}/\mathcal{B})$ is surjective due to the fact that $K_0(B)$ under composition through T goes to zero.

Consider the function $g : \mathcal{A}/\mathcal{B} \to \text{coker} f$ as $g(L(A)) := [A]$ as a natural candidate. If this is additive from we have found the required inverse.

We already know that T is bijective as a set function on objects of $\mathcal{A}$ by construction. Consider two isomorphic elements in the Serre quotient and claim their images under g in $\text{coker} f$ are also isomorphic. Consider $L(A) \cong L(B)$ by the definition of the morphisms this means we have a diagram representative as such.

$$A \xleftarrow{n} X \xrightarrow{m} B$$

with $\ker(n), \ker m, \text{coker}(n), \text{coker}(m)$ in $\mathcal{B}$ (since its an isomorphism).

As such in $K_0(\mathcal{A})$ we have,

$$[X] = [A] + [\ker n] - [\text{coker}(n)] = [B] + [\ker m] - [\text{coker}(n)]$$

so in $coker f$ we have $[X] = [A] = [B]$.

We have shown $L(A) \cong L(B) \implies [A] = [B]$ in coker$f$. Now to show additivity.

To see that g is an additive function, suppose we are given an exact sequence in $\mathcal{A}/\mathcal{B}$ of the form

$$0 \to L(A) \xrightarrow{u} L(B) \xrightarrow{v} L(C) \to 0;$$

we have to show that $[B] = [A] + [C]$ in F. Represent v by a diagram representative $B \xleftarrow{o} Y \xrightarrow{p} C$ with o with $\ker o, \text{coker} o \in \mathcal{B}$

Now since canonically

$$[Y] = [A] + [\ker(o)] - [\text{coker}(o)] \text{ in } K_0(A),$$

we have $[Y] = [A]$ in coker$f$. Since L is exact and we know the below sequence is canonically exact

$$0 \to \ker(p) \to Y \xrightarrow{p} B \to \text{coker}(p) \to 0$$

Applying L to above gives us that coker(p) is in $\mathcal{B}$ and that $L(\ker(p)) \cong L(A)$ in $\mathcal{A}/\mathcal{B}$. Hence, $[\ker(p)] = [A]$ in coker$f$, and in coker$f$ we have

$$[B] = [Y] = [C] + [\ker(p)] - [\text{coker}(p)] = [A] + [C],$$

$\square$

Example 3.5.2. *Consider the following non-example for why the above sequence need not be left exact. Consider $A = k[t]$ for a field k. Consider the subcategory of modules annhilated by a power of x. Then all the terms are isomorphic to $\mathbb{Z}$.*

Corollary 3.5.3. *For a multiplicative set S then the category of S torsion modules over A denoted as $M_S(A)$ form a Serre subcategory. And we have*

$$K_0(M_S(A)) \to G_0(A) \to G_0(S^{-1}A) \to 0$$

When $S = \{1, s, s^2, \cdots\}$ then just

$$K_0(A/sA) \to G_0(A) \to G_0[\frac{1}{s}] \to 0$$

The change is due to Devissage considering each module has a finite filtration given by multiplying by powers of s.

3.6 Fundamental theorems for G_0, K_0

We state a few important results here without proof their detailed proofs can be seen in [WS13]. These will be covered in the next project. As they lend themselves to quick generalizations and motivate the introduction of higher K groups.

Theorem 3.6.1 (Fundamental theorem for G_0 for noetherian rings A)**.** *$G_0[A] \cong G_0(A[t]) \cong G_0(A[t, t^{-1}])$.*

Theorem 3.6.2 (Resolution theorem for K_0 for additive categories)**.** *Let $\mathcal{A}$ be abelian and $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ additive subcategories. If the following conditions hold,*

- *Every $B \in \mathrm{Ob}(\mathcal{B})$ has finite $\mathcal{C}$ dimension, in the sense that there exist a minimal finite $\mathcal{C}$ resolution of $\mathcal{B}$ (in the sense of Definition 1.2.1)*
- *$\mathcal{B}$ is closed under kernels of epis in $\mathcal{A}$.*

then the inclusion functor $i : \mathcal{C} \to \mathcal{B}$ induces an isomorphism $K_0(\mathcal{C}) \cong K_0(\mathcal{B})$

Proof. This is a theorem due to Grothendieck. The evaluation morphism $e : A[t] \to A$ provides an inclusion $M(A) \subset M(A[t])$ and consequently a map $\tilde{e} : G_0(A) \to G_0(A[t])$ by Corollary 3.5.3. We obtain a exact sequence

$$G_0(A) \xrightarrow{\tilde{e}} G_0(A[t]) \to G_0(A[t, t^{-1}]) \to 0$$

Now consider the exact sequence in $G_0(A[t])$ given by

$$0 \to M[t] \to M[t] \to M \to 0$$

i.e. $\tilde{e}[M] = [M] = [M[t]] - M[[t]] = 0$ so $\tilde{e} = 0$ in $G_0[A[t]]$ so $\implies G_0[A[t]] \cong G_0[A[t, t^{-1}]]$.

Since $A = A[t]/tA[t]$ applying Serres formula we get $\tilde{e}[M] = [M/Mt] - \mathrm{Ann}_M(t)$ □

Definition 3.6.3 (Regular ring)**.** *A ring is called regular if every finitely generated ideal has finite projective dimension (minimal length of resolution by projective modules).*

Example 3.6.4. *Any Dedekind domain is a regular ring in particular a principal ideal domain is a regular ring.*

Theorem 3.6.5 (Fundamental theorem for K_0 of regular rings)**.** *For a regular ring A, $K_0(A) \cong G_0(A)$ and by Theorem 3.6.1 we have,*

$$K_0(A) \cong K_0(A[t]) \cong K_0(A[t, t^{-1}]).$$

Chapter 4

Quillen + Construction

4.1 Classifying space of a category

Definition 4.1.1 (Nerve of a small category)**.** *Let $\mathcal{C}$ be a small category we define its nerve as the following simplicial set $N(\mathcal{C})_0 = \mathrm{Ob}(\mathcal{C}), \mathcal{C}$ and $N(\mathcal{C})_1 =$ $\mathrm{Mor}(\mathcal{C})$ and $N(\mathcal{C})_k = \{(f_1, \dots, f_k)|f_i \in \mathrm{Mor}(\mathcal{C})\}$ consists of $k-$tuples of composable arrows, face maps defined as*

$$d_i(f_1, \dots, f_i, f_{i+1}, \dots, f_k) = (f_1, \dots f_i \circ f_{i+1}, \dots, f_k)$$

and degeneracy maps defined as

$$s_j = (f_1, \dots, f_k) = (f_1, \dots, f_{j-1}, 1, f_j, \dots, f_k).$$

The classifying space of a category is the geometric realization of its nerve.

4.2 + Construction

Recall the definition of $GL(A)$ for a ring A. Motivated by the definition of $K_1(A)$ as the quotient $GL(A)/E(A)$, Quillen defined the higher K groups of a ring as the homotopy groups of a particularly nice topological space which we denote as $BGL(A)^+$. Here, $BGL(A)$ represents the classifying space of $GL(A)$.

Definition 4.2.1 (Plus construction)**.** *$BGL(A)^+$ denotes a CW complex with a map $BGL(A) \to BGL(A)^+$ such that $\pi_1 BGL(A)^+ \cong K_1(A)$ with the* *induced map* $\pi_1 BGL(A) \to \pi_1 BGL(A)^+$ *being onto with kernel* $E(A)$*, and* $H_*(BGL(A); M) \cong H_*(BGL(A)^+; M)$ *for each* $K_1(A)$ *module* M*.*

Infact any complexes which model $BGL(A)^+$ are homotopy equivalent. Hence we can define $K_n(A) \cong \pi_n BGL(A)^+$ in a well defined manner. By construction this agrees with K_1.

Definition 4.2.2 (K-space of a ring)**.** *Define* $K(A) = K_0(A) \times BGL(A)^+$

Chapter 5

K theory of finite fields

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