Classical Algebraic K-Theory (K_0, K_1)

Bhoris Dhanjal

Department of Mathematics,
University of Mumbai

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Free modules

Recall the definition of a free module.

Definition (Free module of rank n)

A module over a ring A is said to be free with rank n if it is isomorphic to a module of the form A^n.

In particular this means that there exists a linearly independent spanning set of the module with n elements.

Projective modules

Definition (Projective module)

A module P is said to be projective if it satisfies the following lifting property, every morphism from P to N factors through an epimorphism into N. Note that the lift need not be unique.

$$\begin{tikzcd}
 & M \arrow[d, two heads] \\
P \arrow[ur, dashed] \arrow[r] & N
\end{tikzcd}$$

Equivalent definition

Proposition (Equivalent definitions of projectivity)

The following are equivalent,

1. P *is projective.*
2. *For all epimorphisms between* $M \twoheadrightarrow N$*, the induced map* $\mathrm{Hom}(P, g) : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$ *sending* $f \mapsto g \circ f$ *for* $g : M \to N$ *and* $f : P \to M$ *is an epimorphism.*
3. *For some epimorphism from a free module* F *to* P*,* $\mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$ *is an epimorphism.*
4. *There exists* Q *s.t.* $P \oplus Q$ *is free.*
5. *Short exact sequences of the form* $0 \to A \to B \to P \to 0$ *split, i.e. isomorphic to another short exact where middle term is* $A \oplus P$*.*

Properties of projective modules

A projective module is weaker than a free module.

Lemma (Free modules are projective)

Example (Projective modules are not always free)

Let R, S be two non-trivial commutative rings with unity, consider $R \oplus S$ as a (free) module over itself. Consider $R \oplus \{0\}$ as a submodule of $R \oplus S$, it is projective as it is a direct summand of $R \oplus S$. However, it cannot be free as $(R \oplus \{0\})^n \ncong R \oplus S$ for any n.

Definitions

Definition (Stably isomorphic)

Two $A-$modules M, N are said to be stably isomorphic if there exists r such that $M \oplus A^r \cong N \oplus A^r$.

Definition (Stably free module)

An A module M is stably free if there exists a finitely generated free module F such that $M \oplus F$ is free, i.e. if M is stably isomorphic to a finitely generated free A module.

- Projective modules over local rings are free.
- Projective finitely generated modules over principal ideal domains are free.

Definition of a monoid

Definition (Monoid)

A monoid is an algebraic object consisting of a set of symbols A with a associative binary operation $+$ and an identity element e (where $a + e = e + a = a$ for all $a \in A$).

Example (Commutative monoid)

The natural numbers $\mathbb{N}$ with usual addition $+$.

Example (Non-commutative monoid)

Square matrices of size n over some ring A along with matrix multiplication.

Group completion of a commutative monoid

We begin with a commutative monoid A to complete it into a group we formally add inverses for each symbol $[a] \in A$. Consider the free group on the set of symbols in the monoid labelled as $F(A)$. Now quotient away all the nontrivial monoidal relations $F(A)/\sim$ where $[a+b] \sim [a] + [b] \sim [b] + [a] \sim [b+a]$. This gives us the group completion of a monoid, i.e. the smallest group which has A as a submonoid.

Group competition of the naturals

The group completion of the natural numbers $\mathbb{N}$ is $\mathbb{Z}$. Following the group completion procedure as described above we obtain a formal inverse symbol $[b]$ for each symbol $[a] \in \mathbb{N}$, i.e. a symbol $[b]$ such that $[b] + [a] = [a] + [b] = [0]$ for all $[a] \in \mathbb{N}$, but note that this is naturally isomorphic to $\mathbb{Z}$ as $[b] \mapsto [-a]$.

Definition of K_0

Definition (K_0 of a monoid (Group completion functor))

For a commutative monoid A, the group completion of A is denoted as $K_0(A)$.

Definition (K_0 for a ring A)

Consider the isomorphism classes of finitely generated projective modules over A denoted as $\mathrm{Proj}(A)$. This forms a commutative monoid so $K_0(A)$ is defined as $K_0(\mathrm{Proj}(A))$.

Eilenberg swindle

The reason we require the caveat of finitely generated projective modules instead of simply considering the class of all projective modules is because for the non finitely generated case $K_0(A)$ becomes trivial as we see below.

Proposition (Eilenberg Swindle)

If we consider A^∞ as a non finitely generated free module over a ring A if $P \oplus Q \cong A^n$ then

$$P \oplus A^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \cong A^\infty$$

but this relation would imply $[P] = 0$ for all projectives.

K_0 for common algebraic objects I

Proposition

If A is a field/division ring/local ring/principal ideal domain then $K_0(A) \cong \mathbb{Z}$.

For fields and division rings this is true due to all finitely generated modules being free, i.e. having a basis. We prove this directly for division rings for simplicity.
The similar linear algebraic proof extends to division rings for M a module over division ring A. Pick a maximally linearly independent subset B by Zorn's lemma. To show B is a generating set, the argument uses B's maximality. If $m \in M$ then, if $m \in B$ we are done. If $m \notin B$ then $B \cup \{m\}$ is linearly dependent by maximality of B therefore there exists $a \in A$ such that $am \in \operatorname{span}(B)$ for

K_0 for common algebraic objects II

some $a \neq 0$ and since a is invertible due to F being a division ring we have $m \in \operatorname{span}(B)$. Therefore, B must span M, making it a basis and so $M \cong A^n$.
Similarly as seen before finitely generated projective modules in a local ring/principal ideal domain are free.
So in each case $\operatorname{Proj}(A) \cong \mathbb{N}$ so its group completion is $\mathbb{Z}$.

Computing K_0 using idempotents I

We claim that idempotent matrices over A are in a correspondence to finitely generated projective modules over A.
For a finitely generated projective module P over A, such that $P \oplus Q \cong A^n$ we can define a $R-$module homomorphism which is identity restricted to P and zero else. This is an idempotent element in $M_n(A)$, i.e. P is represented by a $n \times n$ matrix over A.
Conversely any idempotent matrix $e \in M_n(A)$ determines a projective. Simply consider the associated module morphism induced by the matrix e and then the image under e is projective, i.e. eA^n. This is true because $A \cong eA^n \oplus (1-e)A^n$.
We must make a note of the fact that different idempotent matrices may induce projective modules in the same isomorphism class. This is made precise in the following result.

Computing K_0 using idempotents II

Proposition

If e, f are idempotent matrices over A of possibly different sizes then the associated finitely generated projective modules are isomorphic iff e, f are conjugate over a larger common matrix group of order r (obtained by placing the matrices in the top left corner of a larger 0 matrix).

Consider $GL_n(A) \subset GL_{n+1}(A)$ by placing the $n \times n$ matrix in the top right. In this manner we have a filtered system and we can define $GL(A) = \lim_{\rightarrow} GL_i(A)$ as the colimit. Similarly define $M(A)$. Denote the set of idempotent matrices in $M(A)$ as $\mathrm{Idem}(A)$ so we have that the group $GL(A)$ acts on the set $\mathrm{Idem}(A)$ by conjugation.

Computing K_0 using idempotents III

With the above discussion in mind we now have a alternate description for the monoid $\mathrm{Proj}(A)$ in terms of idempotent matrices. In particular $\mathrm{Proj}(A)$ corresponds to the conjugacy classes of the action of $GL(A)$ on $\mathrm{Idem}(A)$. The monoid operation $e + f$ is the block matrix $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$.

Corollary (Morita invariance of K_0)

Let A be a ring and $n \in \mathbb{N}$ arbitrary. Then $K_0(A) \cong K_0(M_n(A))$.

Under the realisation $M_n(M_k(A)) = M_{nk}(A)$ we can note that the infinite general linear matrices over $M_n(A)$ and A are canonically equivalent, i.e. $GL(M_n(A)) = GL(A)$ in particular their infinite

Computing K_0 using idempotents IV

idempotent matrices are also equivalent $\mathrm{Idem}(M_n(A)) = \mathrm{Idem}(A)$. Consequently by the correspondence between idempotent matrices and projective modules their monoid of finitely generated protectives are the same meaning their group completions are isomorphic.

Applications of Morita invariance

Corollary

For commutative ring A if $A \cong A_1 \times A_2$ for rings. Then $K_0(A) \cong K_0(A_1) \times K_0(A_2)$.

Proof.

Notice that $GL(A) \cong GL(A_1 \times A_2) \cong GL(A_1) \times GL(A_2)$ and $\mathrm{Idem}(A) \cong \mathrm{Idem}(A_1 \times A_2) \cong \mathrm{Idem}(A_1) \times \mathrm{Idem}(A_2)$. □

Corollary

If A is the direct limit of rings, i.e. $A \cong \lim_{\rightarrow i \in I} A_i$ then $K_0(A) \cong \lim_{\rightarrow i \in I} K_0(A_i)$.

Simple rings I

Definition (Simple ring)

A simple ring is a non-zero ring which have no non-trivial two-sided ideals.

Example

A commutative ring is simple iff it is a field.

Example

All division rings are simple rings.

Simple rings II

Example

Not all division rings are fields consider $M_n(F)$ for some field F not all elements need be invertible.

A simple module is naturally now any module which is non-zero and has no non-trivial submodules.

Lemma (Schur's lemma)

If A is any ring and M is a simple $R-$module then $\operatorname{End}_A(M)$ is a division ring.

Semi-simple rings I

Definition (Semisimple ring)

A ring A is called semisimple if

- A is Artinian with trivial Jacobson ideal.
- A is a finite product of simple Artinian rings.
- Every left/right $A-$module is projective.

Wedderburn-Artin theorem

A useful characteristic of semisimple rings is the Wedderburn-Artin theorem.

Theorem (Wedderburn-Artin)

A ring A is semisimple iff it is isomorphic to a direct product of $n_i \times n_i$ matrix rings over division rings D_i, i.e. $A \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ where $D_i = \mathrm{Hom}_A(V_i, V_i), \dim_{D_i}(V_i) = n_i$ for V_i the simple A-modules components of A.

K_0 for a semi-simple ring I

Lemma

Let A be a simple ring then $K_0(A) \cong \mathbb{Z}$.

Proof.

By Morita invariance 12 we know that $K_0(A) \cong K_0(M_n(A)) \cong K_0(\mathrm{End}(A)) \cong K_0(D)$ for some division ring D. The last isomorphism is due to Schur's lemma 19. Now applying the fact that K_0 of a principal ideal domain is $\mathbb{Z}$ we are done. □

Theorem

If A is a semisimple ring then $K_0(A) \cong \mathbb{Z}^r$.

K_0 for a semi-simple ring II

Proof.

Due to Wedderburn-Artin 21, we know $A \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ now applying Morita invariance 12, Corollary 13 (the result for n direct sums is obtained via induction).

$$K_0(A) \cong K_0\left(\prod_{i=1}^{r} M_{n_i} D_i\right) \cong \prod_{i=1}^{r} K_0\left(M_{n_i} D_i\right) \cong \prod_{i=1}^{r} K_0(D_i) \cong \mathbb{Z}^r.$$

□

Stable freeness

Theorem (Hilbert-Serre)

Finitely generated module over $k[x_1, \ldots x_n]$ are stably free where k is a principal ideal domain.

A proof for this uses the fact that $K_0(A) \cong K_0(A[t])$. This is the fundamental theorem of K_0 which we prove in the next project on higher K-theory.

Unimodular rows I

We now introduce an important concept of a unimodular row. This perspective helps greatly simplify the proof of Quillen-Suslin.

Definition (Unimodular row)

For a ring A, an element of A^n is said to be a unimodular row if its components generate A. We denote the set of all unimodular rows of length n in A as $\mathrm{Um}_n(A)$.

In particular $v = (v_1, \cdots, v_n) \in \mathrm{Um}_n(A)$ if there exists $a = (a_1, \cdots a_n) \in A^n$ such that $v \cdot a = v^t a = \sum_{i=1}^n v_i a_i = 1$. Alternatively it can be useful to view a unimodular row as as element of $M_{1\times n}(A)$ as such it represents a surjective linear map

Unimodular rows II

$A^n \to A$, or even an element in $M_{n\times 1}$ in which case it represents a injection from $A \to A^n$.
Recall the definition of a stably free projective module. Based on these definitions we can see that the kernel of the surjective $1 \times n$ matrix $A^n \to A$ (i.e. of a unimodular row) is precisely a stably free projective of the form $\underbrace{P}_{\ker v} \times A \cong A^n$.

Definition (Equivalence of unimodular rows)

For unimodular rows $v, w \in A^n$ we say $v \sim w$ if there exists $\alpha \in GL_n(A)$ such that $v\alpha = w$.

Unimodular rows III

Definition (Unimodular completion property)

Given a unimodular row $v = (v_1, \dots v_n) \in A^n$ if we can construct an invertible $n \times n$ matrix with v in the first column we say v has the unimodular completion property.

Lemma

A unimodular row $v \in A^n$ has the unimodular completion property iff $v \sim (1, 0, \dots, 0)$.

Proof.

Unimodular rows IV

If v can be extended to an invertible matrix $\alpha \in GL_n(A)$ then

$$v\alpha^{-1} = (1, 0, \dots, 0).$$

Conversely if $\alpha' \in GL_n(A)$ s.t. $v\alpha' = (1, 0, \dots, 0)$ then α'^{-1} has v in the first column. $\square$

Corollary

Based on the above lemma we can see that naturally any row of an invertible matrix (and column realized as a row of its transpose) is a unimodular row.

Horrock's theorem I

Theorem (Horrocks' theorem)

If $(A, \mathfrak{m})$ is a local ring then for any arbitrary unimodular row $v(x)$ in $A[x]^n$ such that one of its component elements has leading coefficient one implies that v has the unimodular completion property. Furthermore, any such v is equivalent to $v(0)$.

Recall that for a local ring $x \notin \mathfrak{m}$ iff x is a unit.
When $n = 1$ there is nothing to prove. If $n = 2$ by unimodularity of $v(x)$ we have $v_1(x)w_1(x) + v_2(x)w_2(x) = 1$ simply consider the matrix

$$\begin{bmatrix} v_1(x) & -w_2(x) \\ v_2(x) & w_1(x) \end{bmatrix}.$$

Horrock's theorem II

We proceed with $n \geq 3$. Without loss of generality, we take $v_1(x)$ with degree d among components with leading coefficient 1 and $\deg v_i < d$, for $i \neq 1$ by repeated elementary row operations to move the components around. We proceed by inducting on d. Our goal to is show that we can choose polynomials z_1, z_2 such that $z_1 v_1 + z_2 v_2$ such that adding them onto v_3 gives us a polynomial of leading coefficient unit (then reduce to 1) of smaller degree $< d$. Repeating this procedure until $d = 0$ would give us a unit component allowing us to cancel out the rest and be left with $v \sim (1, 0, \ldots, 0)$ as expected.
The construction and existence of these z_1, z_2 are detailed in the project.

Sketch of proof of Quillen-Suslin I

We now extend the idea of Horrocks' theorem.

Lemma

For an integral domain A and a multiplicative subset S if $v(x) \sim v(0)$ unimodular over $A_S[x]^n$ then there exists $b \in S$ such that
$v(x + by) \sim v(x)$ *over* $A[x, y]^n$.

Lemma

For an integral domain A and $v(x)$ unimodular row in $A[x]^n$ with at least one component having leading coefficient one implies $v(x) \sim v(0)$.

Sketch of proof of Quillen-Suslin II

Theorem

For $A = k[x_1, \ldots, x_n]$ where k is a principal ideal domain, then $v \sim (1, 0, \ldots, 0)$ for any unimodular row $v \in A^n$.

Theorem (Quillen-Suslin)

Finitely generated projective modules over $A = k[x_1, \ldots, x_n]$ where k is a principal ideal domain are free.

We know such finitely generated projective modules are stably free, and from above we know any unimodular row in A is equivalent to $(1, 0, \ldots, 0)$.
That is to say we wish to prove given a finitely generated projective module P which is stably free, i.e. $P \oplus A^{m_1} \cong A^{m_2}$ then P is free.

Sketch of proof of Quillen-Suslin III

When $m_1 = 1$ this is the split exact sequence (since P is projective see 1.1),

$$0 \to A \to A^{m_2} \to P \to 0$$

The injection $A \to A^{m_2}$ is precisely a unimodular row by definition which we know must correspond to the canonical embedding of $1 \mapsto (1, 0, \cdots, 0)$. So,

$$P = \mathrm{im}(A^{m_2} \to P) \cong A^{m_2} / \ker(A^{m_2} \to P) \cong A^{m_2} / \mathrm{im}(A \to A^{m_2}).$$

Note $A^{m_2}/\mathrm{im}(A \to A^{m_2})$ is free since $\mathrm{im}(A \to A^{m_2})$ is naturally free due to the embedding of the unimodular vector as $v \sim e_1$.
When $m_1 \neq 1$ just take $(P \oplus A^{m_1 - 1}) \oplus A$.

Definition

Definition (Whitehead group for a ring)

K_1 for a ring A is defined as the abelianization of its infinite general linear group.

$$K_1 := \frac{GL(A)}{[GL(A):GL(A)]},$$

Where $GL(A)$ the infinite general linear group is the colimit of $GL_n(A)$ with GL_n realized as a subgroup of GL_{n+1} by placing the matrix in the top left corner.

Note that $[GL(A):GL(A)]$ denotes the derived/commutator subgroup of $GL(A)$, the subgroup generated by all commutators $[g:h] = g^{-1}h^{-1}gh$ for $g, h \in GL(A)$.

Elementary matrices

Definition (Elementary matrices)

We denote the $n \times n$ elementary matrices as $E_n(A)$ generated by standard elementary matrices of the form $e_{ij}(\lambda) := I_n + \lambda E_{ij}$ where E_{ij} is the matrix with 1 in the (i, j) entry and zero elsewhere.

Lemma

A nonsingular triangular matrix with 1's in the diagonal is a product of standard elementary matrices.

Properties of K_1 I

Proposition

Let A be a ring and u be a unit in A, i.e. $u \in A^\times$. Then,

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \equiv I_2 \mod E_2(A).$$

Proof.

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} = e_{21}(u^{-1})e_{12}(1-u)e_{21}(-1)e_{12}(1-u^{-1}).$$

□

Properties of K_1 II

Lemma (Whitehead)

For $\alpha, \beta \in GL_n(A)$,

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \equiv \begin{bmatrix} \beta\alpha & 0 \\ 0 & I_n \end{bmatrix} \mod E_{2n}(A).$$

Proof.

Let $A = M_n(A)$ and note $E_2(M_n(A)) \subset E_{2n}(A)$ in Proposition 4.1. □

Properties of K_1 III

Proposition

$[GL(A) : GL(A)] = E(A)$.

Proof.

Using Lemma 38 we can see that

$$\begin{bmatrix} \alpha^{-1}\beta^{-1} & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} \beta^{-1}\alpha^{-1} & 0 \\ 0 & 1_n \end{bmatrix} \mod E_{2n}(A)$$

So the derived subgroup of $GL_n(A)$ is contained in $E_{2n}(A)$. Furthermore, every elementary matrix $e_{ij}(\lambda)$ is realized as a commutator since, $e_{ij}(\lambda) = [e_{ik}(1), e_{kj}(\lambda)]$. □

Suslin's normality theorem I

We now consider a result due to Suslin about the normality of $E_n(A)$ in $GL_n(A)$. The following Lemma due to Vaserstein will be useful.

Lemma (Vaserstein)

Let $\alpha \in M_{m,n}(A)$ and $\beta \in M_{n,m}(A)$ then $I_m + \alpha\beta \in GL_m(A)$ implies that $I_n + \beta\alpha \in GL_n(A)$ and,

$$\begin{bmatrix} I_m + \alpha\beta & 0 \\ 0 & (I_n + \beta\alpha)^{-1} \end{bmatrix} \in E_{m+n}(A).$$

Suslin's normality theorem II

Corollary

Let $v = (v_1, \ldots, v_n)^t$ and $w = (w_1, \ldots, w_n)^t$ be column vectors in R^n such that $w^t v = 0$, and suppose $w_i = 0$ for some $i \leq n$. Then $I_n + vw^t \in E_n(R)$.

Suslin's normality theorem III

Lemma

For v unimodular row in R^n, and $f : R^n \to R$ a $R-$ linear map determined by $e_i \mapsto v_i$, where e_i is the standard basis element of R^n. We have,

$$\ker(f) = \left\{ w = (w_1, \cdots w_n)^t \mid \sum_i^n w_i v_i = 0 \right\}$$

and it is generated by elements of the form $\{v_j e_i - v_i e_j\}$ for positive $i \leq n$.

Suslin's normality theorem IV

Proposition

Let $n \geq 3$. If $v \in R^n$ is unimodular, and $w \in R^n$ such that $w^t v = 0$, then $I_n + vw^t \in E_n(R)$ and this is also true if w is unimodular and v is arbitrary by transposition.

Theorem (Suslin's Normality theorem)

For A, a commutative ring with unity, $E_n(A)$ normal in $GL_n(A)$ for $n \geq 3$.

Since $E_n(R)$ is generated by $e_{ij}(\lambda)$ it suffices to check that $\alpha e_{ij}(\lambda)\alpha^{-1} \in E_n(R)$ for $\alpha \in GL_n(A)$.

Suslin's normality theorem V

Recall from 29 that the columns of α and the rows of α^{-1} are unimodular.

$$\alpha e_{ij}(\lambda)\alpha^{-1} = \alpha(I_n + \lambda E_{ij})\alpha^{-1} = I_n + \lambda c_i r_j$$

Where c_i is the i^{th} column of α and r_j is the j^{th} row of α^{-1}. Furthermore since $\alpha^{-1}\alpha = I_n$ implies $r_j c_i = \delta_{ij}$ implies using Proposition 5.1 that $\alpha e_{ij}(\lambda)\alpha^{-1} = I_n + \lambda c_i r_j \in E_n(A)$.

Quilen's Local-global theorem and Suslins factorial theorem

Theorem (Local-global principle)

Let $v = (v_1, \ldots, v_n) \in \mathrm{Um}_n(A[x])$*. If* $v(x) \sim v(0)$ *over* $A_{\mathfrak{m}}[x]$ *for all maximal* $\mathfrak{m} \in A$*, then* $v(x) \sim v(0)$ *over* $A[x]$*.*

Theorem (Suslin's factorial theorem)

Given $(v_0, \ldots, v_n) \in \mathrm{Um}_{n+1}(A)$ *then* $n! | \prod_{i=0}^{n} m_i$*, then* $(v_0^{m_1}, \ldots, v_n^{m_n}) \in \mathrm{Um}_{n+1}(A)$.

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