



HIGHER ALGEBRAIC K-THEORY: A COMBINATORIAL APPROACH

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CERTIFICATE

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A combinatorial approach' in partial fulfilment of the requirements for the degree of M.Sc. Mathematics submitted by Mr. Bhoris Dhanjal is approved for submission.

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TODO NEW

ABSTRACT

THINGS TO DO

1. Homotopy in model categories and quillen equivalences
2. Kan complexes as fibrant objects
3. classical model structure on simplicial sets
4. Quillen equivalence between homotopy category of simplicial sets and homotopy category of topological spaces.
5. Spectral sequences in homological algebra in order to introduce Serre spectral sequences. Needed for proof of Quillen theorem B.
6. Proof of Quillen theorem B (implies theorem A)
7. Proofs of Fundamental theorems of higher K theory requires quillen theorem B
- 8.

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Chapter 1

Basic homological algebra

We begin with some basic prerequisite definitions and results on abelian categories and homological algebra, which provide a foundational framework. These concepts will be used extensively throughout the discussion, forming an assumed background. Abelian categories are essential to understanding homological algebra. It is motivated by the fact that they allow for the use of homological methods in a wide variety of applications and help unify various (co)homology theories. They were first introduced by Grothendieck in his seminal Tohoku paper [Gro57].

1.1 Abelian categories

There is a chain of conditions regarding ‘abelian’-ness of categories, which can be roughly understood as follows.

$$\mathbf{Abelian} \subseteq \mathbf{Pre-Abelian} \subseteq \mathbf{Additive} \subseteq \mathbf{Ab-Enriched}$$

Ab-Enriched categories (sometimes referred to as pre-additive categories) are categories such that for any objects $A, B \in \mathcal{C}$, the external hom-set $\mathrm{Hom}(A, B)$ has the structure of an abelian group. Furthermore, it has a well-defined notion of composition. Which is bilinear due to the monoidal product in \mathbf{Ab} , $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$.

We have chosen to omit the precise definitions of the coherence conditions for monoidal and monoidally enriched categories to make this section more accessible. Since these conditions are not explicitly used for computations

in this work, the basic background described above will suffice. For a more detailed overview of the definitions of monoidal and monoidally enriched categories, refer to [Lan98] for a classical treatment or [Rie17] for an excellent modern exposition.

Example 1.1.1. *A ring is a single object **Ab**-Enriched category (In the same sense how a group is realized as a single object category with all arrows invertible).*

Proposition 1.1.2. *In **Ab**-Enriched categories initial and terminal objects coincide (it is often called the zero object).*

Proof. Let \mathcal{C} be an **Ab**-Enriched category. Note that the Hom-sets between objects have ‘zero morphisms’, i.e. arrows in the Hom-set which behave like the additive identity in the **Ab** group induced by it. In particular for $0_{A,B} \in \text{Hom}(A, B)$ we have the property that if $f : B \rightarrow C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \rightarrow D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathcal{C}$ is initial so there is a unique morphism $0 \rightarrow 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \rightarrow 0$ we can say that by the zero morphism property $f = 0$ so also 0 is terminal. \square

Proposition 1.1.3. *In **Ab**-Enriched categories finite coproducts coincide with finite products (i.e. biproducts)¹.*

Proof. Let \mathcal{C} be an **Ab**-enriched category and objects $A, B \in \mathcal{C}$ consider the product $A \times B$, which is determined by the following universal property,

$$\begin{array}{ccccc} & & X & & \\ & \swarrow x_1 & \downarrow u & \searrow x_2 & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array}$$

Consider A and B in place of X in the diagram. By the universal property we have $q_1 : A \rightarrow A \times B$, $q_2 : B \rightarrow A \times B$

$$\begin{array}{ccccc} A & & & & B \\ & \searrow q_1 & & \swarrow q_2 & \\ & & A \times B & & \\ & \swarrow p_1 & & \searrow p_2 & \\ A & & & & B \end{array}$$

¹This also holds over categories enriched over commutative monoids.

So $p_1q_1 = 1_A$ and $p_2q_2 = 1_B$ also $p_1q_2 = p_2q_1 = 0$.

Now note that $q_1p_1 + q_2p_2 = 1_{A \times B}$ as $p_1(q_1p_1 + q_2p_2) = p_1$ and $p_2(q_1p_1 + q_2p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following universal property holds for some arbitrary $C \in \mathcal{C}$

$$\begin{array}{ccccc}
 A & \xrightarrow{r_1} & C & \xleftarrow{r_2} & B \\
 & \searrow q_1 & \uparrow f & \swarrow q_2 & \\
 1_A \downarrow & & A \times B & & \downarrow 1_B \\
 & \swarrow p_1 & & \searrow p_2 & \\
 A & & & & B
 \end{array}$$

Define $f : A \times B \rightarrow C$ as $f = r_1p_1 + r_2p_2$. Now $f q_1 = r_1$ and $f q_2 = r_2$ if we show uniqueness of f we are done.

Say f' then $(f - f')1_{A \times B} = (f - f')(q_1p_1 + q_2p_2) = 0$. So $f = f'$. □

Definition 1.1.4 (Additive category). An **Ab**-Enriched category which has all finite (co)products (i.e. biproducts since they coincide).

Example 1.1.5. The category of vector bundles over a topology X is a additive category (but not an abelian category). We will see this in more detail in the following section.

Functors between additive categories are called *additive functors*. And can be realized as functors which preserve additivity of homomorphisms between modules, $F(f + g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition 1.1.6 (Kernel). A kernel of a morphism $f : A \rightarrow B$ is the pullback along the unique morphism from $0 \rightarrow B$, i.e. it is $p : \ker f \rightarrow A$. Provided initials and pullbacks exist.

$$\begin{array}{ccc}
 \ker f & \xrightarrow{\quad} & 0 \\
 \downarrow p & \lrcorner & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \rightarrow B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'.²

A cokernel is simply its dual.

Definition 1.1.7 (Pre-abelian categories). *An additive category with all morphism having kernels and cokernels.*

The above definition is equivalent to saying a pre-abelian category is a **Ab**-Enriched category with all finite limits and colimits. This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Proposition 5.21]. And we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition 1.1.8 (Abelian category). *Pre-abelian categories for which each monomorphism is a kernel and each epimorphism is a cokernel.*

Equivalently a category is abelian if its pre-abelian and \bar{f} is an isomorphism in the canonical decomposition of $f : A \rightarrow B$ as

$$A \twoheadrightarrow \text{coker ker } f \xrightarrow{\bar{f}} \text{ker coker } f \hookrightarrow B.$$

Example 1.1.9. *Some non examples are:*

1. *The category of Hausdorff topological abelian groups is pre-abelian but not abelian. Since not every morphism which is a mono+epi is necessarily a isomorphism.*

Consider a Hausdorff abelian topological group G with a non discrete topology and consider G' it's discretization. The inclusion map $G' \rightarrow G$ is a mono+epi but not isomorphic.

2. *The category of torsion free abelian groups **TFAb** is pre-abelian but not abelian as the mono $f : \mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}$ is not a kernel of some morphism. Say it were and there exists $A \in \mathbf{TFAb}$ such that f is the kernel to $g : \mathbb{Z} \rightarrow A$, i.e.*

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 0 \\ f(z)=2z \downarrow & \lrcorner & \downarrow \\ \mathbb{Z} & \xrightarrow{g} & A \end{array}$$

²A minor point to note is that in the case of **Ab**-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a **Ab**-Enriched category has a initial 0 however this matches up with our intuition.

this implies $1_{\mathbb{Z}}$ factors through f i.e. $1_{\mathbb{Z}} = f \circ h$ for some unique $h : \mathbb{Z} \rightarrow \mathbb{Z}$ which implies $h(1) = 1/2$ which is absurd.

Example 1.1.10. *Some examples of abelian categories:*

1. *The category of modules.*
2. *Category of representations of a group*
3. *Category of sheaves of abelian groups on some topological space.*

With this definition in mind we will now define a few important constructions we will use often. These are not restricted to abelian categories but we will use them very often in the case of abelian categories, so it is good to see it in action directly with the notion of an abelian category at hand.

Definition 1.1.11 (Subobject). *A subobject for some $X \in \mathcal{C}$ is a monomorphism into X .*

With slight abuse of notation we refer to $Y \leq X$ as a subobject of X where Y is just a representative of the codomain of a isomorphism class of monomorphisms into X . In particular for $X, Z \rightrightarrows X$ monics Z, X belong to the same subobject class if the morphisms are isomorphic, i.e. there exists an isomorphism between $Y \rightarrow Z$ making the triangle commute.

This is clearer when seen through the lens of a slice category. Note that arrows between subobjects of the same X are arrows in the slice category of X . So collection of subobjects form a category with a preorder (with inclusion). The reasoning behind such an definition for subobjects is motivated by the fact that we think of generalized elements in \mathcal{C} as being not $X \in \text{Ob}(\mathcal{C})$ but rather $\text{Hom}_{\mathcal{C}}(-, X)$.

Definition 1.1.12 (Quotients in abelian categories). *For $Y \leq X$ in an abelian category we can define X/Y as the cokernel of the monomorphism $Y \rightarrow X$.*

Definition 1.1.13 (Extension/short exact sequences in abelian categories). *For $A, B \in \mathcal{A}$ an extension by A of B refers to some $E \in \mathcal{A}$ such that $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ is a short exact sequence.*

A deep result on abelian categories is the Freyd-Mitchel embedding theorem which helps characterize all small abelian categories in terms of modules.

Theorem 1.1.14 (Freyd-Mitchell). *Every small abelian category can be faithfully embedded as a full subcategory via an exact functor into $R\text{-Mod}$ for some ring R .*

The proof for the theorem is very extensive and as such is omitted. The canonical reference is Freyd’s own book [FF64]. A proof sketch summarising Freyd’s proof is given in an excellent MathOverflow post by the user Theo Buehler [Bue].

1.2 Resolutions

Let \mathcal{A} be an abelian category.

Definition 1.2.1. *[Left resolution] Given $M \in \mathcal{A}$ its left resolution is given by the data of a exact sequence $(A_\bullet, \varphi_\bullet)$ into M as such,*

$$\cdots \rightarrow A_1 \rightarrow A_0 \xrightarrow{\epsilon} M \rightarrow 0$$

where ϵ is called the augmentation map, if the exact sequence is free its a free resolution and such for projective.

If we have a cochain complex instead it forms a right resolution and if its elements are injective we call them injective resolutions.

We cover a result about stably free modules. We will revisit this in greater detail in further sections.

Theorem 1.2.2. *Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. If any two of these objects have a finite free resolution then so does the third.*

Proposition 1.2.3 (Horseshoe lemma). *If there is a short exact sequence,*

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

and both M, P have a projective resolutions A, C

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 \cdots & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & M \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & N & & \\
 & & & & \downarrow & & \\
 \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & P \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

as below then N also has a projective resolution B which forms a short exact sequence. Also the sequence splits due to C_i being projective so $B_i = A_i \oplus C_i$.

Proof. First note $\epsilon_P : C_0 \rightarrow P$ lifts due to projectivity to $C_0 \rightarrow N$ also $A_0 \rightarrow N$ via composition so simply define $B_0 = A_0 \oplus C_0$. This is an epimorphism evidently via diagram chase. Also is projective as direct sum of projectives is projective. Now consider direct sum of kernel of $A_0 \rightarrow M, B_0 \rightarrow N, C_0 \rightarrow P$ and construct the direct sum again to get F_1 . Now we get a 3×3 . Exactness is due to the Snake lemma \square

A standard example of a resolution is the Koszul complex.

Definition 1.2.4 (Tensor algebra). *For a R -Module M and $T^n M = \otimes^n M$ the tensor algebra is defined to be*

$$T(M) = \oplus_{n=0}^{\infty} T^n M$$

Definition 1.2.5 (Exterior algebra and wedge product). *For an R -Module M . Consider the Tensor algebra $T(M) = \oplus_n \otimes^n M$. Consider the ideal I spanned by elements $v \otimes v$ for $v \in T(M)$.*

The quotient of $T(V)/I := \Lambda(M)$ is called the exterior algebra and its product the wedge product. It is universal with respect to multilinear alternating maps.

For $v \in \Lambda^n M, w \in \Lambda^m M$ multiplication is defined as such,

$$v \wedge w = (-1)^{nm} w \wedge v$$

Example 1.2.6 (Koszul complex). *For a R -module M there is an associated chain complex of n^{th} exterior algebras*

$$\Lambda^n M \rightarrow \Lambda^{n-1} M \rightarrow \cdots \rightarrow \Lambda^0 M \cong M$$

1.3 Category of chain complexes

The discussion in this section follows [Wei94].

Definition 1.3.1 (Category of chain complexes). *For an abelian category \mathcal{A} , the collection of chain complexes in \mathcal{A} with morphisms as chain maps form a category often denoted as $\mathbf{Ch}(\mathcal{A})$. It is again an abelian category.*

Interestingly enough the category of chain complexes is in fact also a closed monoidal category.

Definition 1.3.2 (Tensor product of chain complexes). *For $A, B \in \mathbf{Ch}(\mathcal{A})$ define $A \otimes B$ component wise as,*

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j.$$

With the differential as $\varphi_n(a, b) = (\varphi_A a \otimes b) + (-1)^{\deg(a)}(a \otimes \varphi_B b)$

This of course only makes sense provided \mathcal{A} admits tensors. When discussing homological algebra in the context of $R\text{-Mod}$ this distinction need not be made.

The mysterious sign rule (often referred to as the Koszul sign rule) can be motivated in one of many different ways.

The most important is that the monoidal product arises naturally when we consider the problem of reducing the double complex we obtain by tensoring piecewise (i.e. finding the total complex but we will see this later).

Perhaps another interesting way to motivate it with the notion of a ‘Day convolution’. The following treatment is due to a MathOverflow answer by Alexander Campbell [Cam20].

A Day convolution can be realized as a categorification of typical convolution product. We do not delve into this in any more detail.

Definition 1.3.3 (Internal hom of chain complexes). *For $A, B \in \mathbf{Ch}(\mathcal{A})$ we define an internal hom object as a chain complex with the following components*

$$[A, B]_n = \prod_i \mathrm{Hom}(A_i, Y_{i+n})$$

with differentials for $x \in [A, B]_n$

$$\varphi_n x = \varphi_B x - (-1)^n x \varphi_A$$

The above two definitions characterize $\mathbf{Ch}(\mathcal{A})$ as a closed monoidal category.

Definition 1.3.4 (Unital tensor). *The unital tensor is considered as the unital tensor of \mathcal{A} say $1_{\mathcal{A}}$ understood as a chain complex $\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 1_{\mathcal{A}} \rightarrow 0 \rightarrow \cdots$ (i.e. a chain complex concentrated at 0). In particular when $\mathcal{A} = R\text{-Mod}$ the unital tensor is just R .*

1.4 Homotopy category of a chain complex

We don't immediately use the contents of this section. However, before diving straight into the abstract homotopy theory in the further chapters it is a good idea to first work with the more natural setting of category of chain complexes.

The notion of a homotopy category and derived category (i.e. localisation at weak equivalences) will all be made more clear in the further chapters where we will realize the notion of a chain complex as a “abelianised” simplicial set³.

Definition 1.4.1 (Quasi isomorphism). *A chain map is called a quasi isomorphism if the induced map on the homologies constituents an isomorphism.*

The reason for ‘quasi’ is that the relation is reflexive and transitive but not symmetric.

Definition 1.4.2 (Chain homotopy). *If α, β are maps between differential modules $(A, \varphi), (B, \psi)$ then α is homotopically equivalent to β if there is a*

³This is the Dold-Kan correspondence.

map $h : A \rightarrow B$ such that $\alpha - \beta = \psi h + h \varphi$. If grading is relevant the picture formed is as such, we require a family of maps $h_i : A_i \rightarrow B_{i+1}$ ⁴.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & A_i & \xrightarrow{\varphi_i} & A_{i-1} & \longrightarrow & \cdots \\
 & \searrow h_i & \downarrow \beta_i & & \downarrow \beta_{i-1} & \swarrow h_{i-1} & \\
 & & \alpha_i & & \alpha_{i-1} & & \\
 & \swarrow h_{i-1} & \downarrow \beta_{i-1} & & \downarrow \beta_{i-2} & \swarrow h_{i-2} & \\
 \cdots & \longleftarrow & B_i & \xrightarrow{\psi_i} & B_{i-1} & \longrightarrow & \cdots
 \end{array}$$

The intuition behind this particular choice of definition is that the map $\alpha - \beta$ maps all cycles to boundaries which have zero homology. So really $\alpha - \beta$ is null homotopic, as such this relation is an equivalence relation.

Definition 1.4.3 (Homotopy category of chain complexes). *For a given category of chain complexes $\mathbf{Ch}(\mathcal{A})$ we can define $\mathcal{K}(\mathcal{A})$ to be the homotopy category of chain complexes with objects as objects of $\mathbf{Ch}(\mathcal{A})$ and arrows as chain homotopic maps.*

Compare this with the notion of the naive homotopy category of topological spaces \mathbf{hTop} in which objects are topological spaces and maps are homotopy classes of maps between spaces.

1.5 Cones, cylinders and homotopy cofiber (Puppe) sequences

The notion of mapping cones and cylinders are almost identical to those we first encounter in algebraic topology. We first see the constructions of these objects with their universal properties and then see what explicit cases later.

The universal properties elucidate what the construction is really saying better than viewing the explicit definition straight away which can seem confusing.

Sign conventions differ in the classical presentation of mapping cones to maintain consistency we shall follow the convention in Weibel [Wei94]

In order to discuss cylinders we must construct a notion of an ‘interval’ object similar to how we use $[0, 1]$ in topology.

The interval object for $\mathbf{Ch}(\mathcal{A})$ is defined as a chain complex as follows

⁴i.e. It has degree 1, sometimes the subscript is dropped and just treated as h

Definition 1.5.1 (Interval object). *For a unital tensor in \mathcal{A} say 1 the interval complex denoted as I is defined as,*

$$\cdots 0 \rightarrow 0 \rightarrow 1 \rightarrow 1 \oplus 1$$

When $\mathcal{A} = R\text{-Mod}$ this is

$$\cdots 0 \rightarrow 0 \rightarrow R \rightarrow R \oplus R$$

With the differential simply being $(-i, i)$ the identity map.

This is motivated via the Dold-Kan correspondence as a sequence of maps starting from the standard 1 simplex to simplicial sets to the free simplicial abelian group, the normalized chain complex obtained at the end of this process is this interval object.

A cylinder is defined using the tensor unit $I \in \mathbf{Ch}(\mathcal{A})$.

Definition 1.5.2 (Cylinder object). *For $A \in \mathbf{Ch}(\mathcal{A})$ its cylinder is defined as $\text{Cyl}(A) = I \otimes A$.*

Definition 1.5.3 (Cone object). *Defined as the following pushout, \bullet denotes the trivial terminal point object (here the zero complex),*

$$\begin{array}{ccc} A & \longrightarrow & \text{cyl}(A) \\ \downarrow & \lrcorner & \downarrow \\ \bullet & \longrightarrow & \text{cone}(A) \end{array}$$

Similarly the mapping cylinder is also defined via a pushout.

Definition 1.5.4 (Mapping cylinder).

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow & \lrcorner & \downarrow \\ \text{cyl}(A) & \longrightarrow & \text{cyl}(\alpha) \end{array}$$

Proposition 1.5.5 (Components of the chain complex of a mapping cylinder).

$$\text{cyl}(\alpha)_n = A_n \oplus B_n \oplus A_{n-1}$$

Finally the mapping cone is the following pushout. As such

Definition 1.5.6 (Mapping cone).

$$\begin{array}{ccc} \text{cyl}(A) & \longrightarrow & \text{cyl}(\alpha) \\ \downarrow & \lrcorner & \downarrow \\ \text{cone}(A) & \longrightarrow & \text{cone}(\alpha) \end{array}$$

Putting all the following definitions together we can instead view the mapping cone as the pushout for the following diagram,

$$\begin{array}{ccccc} & & A & \xrightarrow{\alpha} & B \\ & & \downarrow & & \downarrow \\ A & \longrightarrow & \text{cyl}(A) & \longrightarrow & \text{cyl}(\alpha) \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \text{cone}(A) & \longrightarrow & \text{cone}(\alpha) \end{array}$$

This is analogous to the topological definition of a mapping cone where we take a cylinder and identify together one side to a point.

Proposition 1.5.7 (Components of the chain complex of a mapping cone).

$$\text{cone}(\alpha)_n = B_n \oplus A_{n-1}$$

with the differential as

$$\begin{array}{ccc} B_{n+1} & \xrightarrow{\varphi_B} & B_n \\ \oplus & \nearrow \alpha & \oplus \\ A_n & \xrightarrow{-\varphi_A} & A_{n-1} \end{array}$$

Definition 1.5.8 (Suspension of chain complex). *For $A \in \mathbf{Ch}(\mathcal{A})$ the complex defined by shifting degrees down by one is defined as its suspension denoted as follows,*

$$A[-1]_n = A_{n-1}$$

with the differential shifted down but with a negative sign

$$\varphi_n = -\varphi_{n-1}^A$$

We can now identify a canonical short exact sequence

$$B \rightarrow \text{cone}(\alpha) \rightarrow X[1]$$

with the mappings being inclusion and projection respectively. In particular it forms a pushout.

Infact this is universal even with respect to homotopy that means the universality of the pushout is upto chain homotopy. To summarize we have the following proposition.

Proposition 1.5.9 (Homotopy pushout induced by mapping cone). *For $A, B \in \mathbf{Ch}(\mathcal{A})$ with $\alpha : A \rightarrow B$ some chain map. The following forms a short exact sequence of chain complexes with inclusion and projection mappings*

$$B \rightarrow \text{cone}(\alpha) \rightarrow A[-1]$$

Which in turn induce a pushout square which is unique up to chain homotopy which characterizes $A[-1]$ as a cofiber of $i : B \rightarrow \text{cone}(\alpha)$ over 0.

$$\begin{array}{ccc} B & \longrightarrow & \text{cone}(\alpha) \\ \uparrow \alpha & & \downarrow \\ A & \longrightarrow & 0 \longrightarrow A[-1] \end{array}$$

This induces what is called a homotopy cofiber sequence

Definition 1.5.10 (Homotopy cofiber sequence). *For $A, B \in \mathbf{Ch}(\mathcal{A})$ with $\alpha : A \rightarrow B$,*

$$A \rightarrow B \rightarrow \text{cone}(\alpha) \rightarrow A[-1] \rightarrow B \rightarrow \text{cone}(\alpha) \rightarrow A[-2] \rightarrow \dots$$

This gives rise to a more common triangle diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & \searrow & \swarrow \\ & \text{cone}(\alpha) & \end{array}$$

This triangular diagram will be useful in the next section of derived categories.

This notion of the homotopy cofiber sequence is in fact more fundamental than the long exact homology sequence we saw before. Taking a representative of the cone up to homotopy as another chain complex C and looking at the induced sequence of their homologies give us the long exact sequence of homology.

1.6 Double complex and total complex

A double complex is understood to be a ‘chain complex of chain complexes’. In particular it is bigraded over $\mathbb{Z} \times \mathbb{Z}$.

Definition 1.6.1 (Double complex). *A double complex $A_{\bullet,\bullet} \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ is a chain complex of chain complexes in $\mathbf{Ch}(\mathcal{A})$ in particular we say it has both a vertical differential $\varphi^v : A_{\bullet,m} \rightarrow A_{\bullet,m-1}$ and a horizontal differential $\varphi^h : A_{n,\bullet} \rightarrow A_{n-1,\bullet}$ such that these differentials make all the squares commute, i.e. $\varphi^v \varphi^h = \varphi^h \varphi^v$,*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & A_{n,m} & \xrightarrow{\varphi_m^h} & A_{n,m-1} & \longrightarrow & \cdots \\
 & & \downarrow \varphi_{n-1}^v & & \downarrow \varphi_{n-1}^v & & \\
 \cdots & \longrightarrow & A_{n-1,m} & \xrightarrow{\varphi_m^h} & A_{n-1,m-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array}$$

Proposition 1.6.2 (Model for mapping cone in terms of double complexes). *For $A, B \in \mathbf{Ch}(\mathcal{A})$ and chain map $\alpha : A \rightarrow B$ the double complex formed by stating $C_{1,n} = A_n$ and $C_{0,n} = B_n$ with the natural differential forms a homotopically equivalent model for $\text{cone}(f)$.*

Definition 1.6.3 (Total complex). *For a double complex $A_{\bullet,\bullet}$ there is associated to it a (regular \mathbb{Z} graded) chain complex known as its total complex. The components of a total complex can be expressed as direct sums or direct products, i.e.*

$$\text{Tot}^\oplus(A_{\bullet,\bullet})_n = \bigoplus_{n=p+q} A_{p,q}$$

or

$$\mathrm{Tot}^\Pi(A_{\bullet,\bullet})_n = \Pi_{n=p+q} A_{p,q}$$

with differentials in each case as

$$\varphi^{\mathrm{Tot}} = \varphi^v + (-1)^{\mathrm{verticaldeg}} \varphi^h$$

In fact the Total complex of a double complex given via objectwise tensoring of two complexes X, Y is isomorphic to the tensor product of the chain complexes. This forms an alternative characterization of the chain complex tensor product.

It is the computation of the homology of a double complex which leads to equivalently computing the homology of the total complex which in turn leads to the formulation of the spectral sequence of a double complex.

1.7 Derived and triangulated category

Note that the algebraic notion of localization easily generalizes to categories proper. We can localize a category on some collection of arrows which sending the arrows to isomorphisms.

Definition 1.7.1 (Derived category (Universal property)). *For an abelian category \mathcal{A} the localization of its category of chain complexes $\mathbf{Ch}(\mathcal{A})$ with respect to quasi-isomorphisms gives its associated derived category $D(\mathcal{A})$. It is universal in the sense that any functor from $\mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{C}$ which sends quasi-isomorphisms to isomorphisms factors through $D(\mathcal{A})$.*

1.8 Spectral sequences

1.8.1 Serre spectral sequence

Chapter 2

Prehistory of K -theory: Vector bundles

In this section, we introduce the foundational concepts required to prove Swan's theorem, which demonstrates a key connection between projective modules and vector bundles over certain topological spaces. The purpose of proving Swan's theorem is to highlight why projective modules play a central role in algebraic K -theory, serving as essentially algebraic counterparts to vector bundles.

All results presented hold for both real and complex vector bundles. For simplicity, we use k to denote the underlying field.

More detailed exposition can be found in [MS74, Kar08].

Definition 2.0.1 (Vector bundle). *A n dimensional vector bundle over k is a triple (E, p, X) . Which consists of a continuous map $p : E \rightarrow X$ from the total space E to the base space X . Such that for all $x \in X$, $E_x = p^{-1}(x)$ the fibre of x has a k vector space structure. Along with the following property of local trivialization*

1. *For any $x \in X$ there exists a open open $U \subset X$ along with a homeomorphism*

$$h : U \times k^n \rightarrow p^{-1}(U)$$

such that for all $c \in U$ the map through h defines an isomorphism between E_c and k^n .

Example 2.0.2 (Trivial vector bundle). *A trivial vector bundle is one in which the total space $E = X \times k^n$ with p just the trivial projection mapping.*

The local trivialization condition can now be understood as saying there exists an open neighbourhood for each point $x \in X$ such that $E|_U$ is a trivial vector bundle

Definition 2.0.3 (Quasi-vector bundle). *A fibre bundle defined as above without the property of local trivialisation is called a quasi-vector bundle.*

Definition 2.0.4 (Rank of vector bundle). *If each E_x has the same dimension n then n is referred to as the rank of the vector bundle.*

Example 2.0.5 (Tangent bundle on S^2). *Let S^2 be the unit sphere in \mathbb{R}^3 associate to each point $x \in S^2$ the plane tangent to S^2 at x , label this $T_x S^2$ (this is the tangent space at point x) then the disjoint union of all the tangent spaces form a vector bundle $TS^2 = \sqcup_{x \in S^2} T_x S^2$.*

We don't use the concept of a tangent bundle proper so we refrain from defining it in its entirety.

Definition 2.0.6 (Vector bundle morphisms). *Two vector bundles (E_1, p, X) and (E_2, p_2, X) are considered isomorphic if there exists a continuous map between their total spaces g such that the below diagram commutes*

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

and g induces a vector space homomorphism for each fibre.

If g is a homeomorphism and fibrewise isomorphism then it is a vector bundle isomorphism.

Definition 2.0.7 (Category of vector bundles). *For a fixed base space B the collection of vector bundles over X with arrows as vector bundle homomorphisms forms a category which we denote as $\text{VB}(X)$.*

Definition 2.0.8 (Whitney sums). *For $(E_1, p_1, X), (E_2, p_2, X)$ their Whitney sum denoted as $E_1 \oplus E_2$ consists of total space as $E_1 \oplus E_2$ with fibrewise direct sums of vector spaces.*

A Whitney sum is a biproduct of vector bundles in $\text{VB}(X)$. Which makes the category of vector bundles into an additive category, but not abelian. Since the kernel of a morphism of vector bundles need not be a vector bundle.

Along with a Whitney sum there also exists the expected notion of a tensor product of vector bundles this in turn allows us to view $\text{VB}(X)$ as a symmetric monoidal category in the natural way.

In the particular case when $E_1 = X \times k^n, E_2 = X \times k^m$ are both trivial vector bundles over X . We can describe the vector bundle morphisms between them explicitly. This is a result we will use very often in the process of proving Swans theorem.

We have associated to the vector bundle morphism $g : E_1 \rightarrow E_2$ a natural linear map $g_x : k^n \rightarrow k^m$. Consider $\tilde{g} : X \rightarrow \text{Hom}_k(k^n, k^m)$ defined as $\tilde{g}(x) = g_x$.

Theorem 2.0.9 (Characterization of trivial bundle morphisms). *The map $\tilde{g} : X \rightarrow \text{Hom}_k(k^n, k^m)$ is continuous¹, and for $h : X \rightarrow \text{Hom}_k(k^n, k^m)$ continuous, let $\tilde{h} : E \rightarrow E'$ be the map which induces $\tilde{h}(x)$ on each fiber. Then \tilde{h} is a morphism of vector bundles.*

Proof. Let $\{e_1, \dots, e_n\}$ be a choice of basis for k^n and $\{e'_1, \dots, e'_m\}$ for k^m .

Now $g_x \in \text{Hom}_k(k^n, k^m)$ is naturally representable by a $m \times n$ matrix $(a_{ij}(x))$.

The function $x \mapsto a_{ij}(x)$ is obtained by composing the continuous maps:

$$X \xrightarrow{\beta_j} X \times k^n \xrightarrow{g} X \times k^m \xrightarrow{\gamma} k^m \xrightarrow{p_i} k$$

where $\beta_j(x) = (x, e_j)$, $\gamma(x, v) = v$, and p_i is the i -th projection onto k . Since each $a_{ij}(x)$ is continuous \tilde{g} is continuous.

Now consider $h : X \rightarrow \text{Hom}_k(k^n, k^m)$, continuous. By composing a sequence of continuous maps we obtain the required morphism of vector bundles.

$$X \times k^n \xrightarrow{\delta} X \times \text{Hom}_k(k^n, k^m) \times k^n \xrightarrow{\epsilon} X \times k^m$$

where $\delta(x, v) = (x, h(x), v)$ and $\epsilon(x, u, v) = (x, u(v))$. □

Proposition 2.0.10 (Sufficient condition for vector bundle isomorphism). *Let E and F be two vector bundles over X , and let $g : E \rightarrow F$ be a morphism of vector bundles such that $g_x : E_x \rightarrow F_x$ is bijective for each point $x \in X$. Then g is an isomorphism of vector bundles.*

¹Relative to the natural topology on $\text{Hom}_k(k^n, k^m)$. The unique topological vector space topology making any $k^t \cong L(k^n, k^m)$ into a homeomorphism

Proof. Let $h : F \rightarrow E$ be the map defined by $h(v) = g_x^{-1}(v)$ for $v \in F_x$. We need to prove h is continuous and so indeed a map in $\text{VB}(X)$. The other conditions for isomorphisms are clearly met. If we prove continuity locally for every neighbourhood then we have the required continuity globally.

Consider a neighbourhood U of x and the local trivialization isomorphisms $\beta : E|_U \rightarrow U \times M$ and $\gamma : F|_U \rightarrow U \times N$. Let $g_1 = \gamma g_U \beta^{-1}$.

Then $h_U = \beta^{-1} h_1^{-1} \gamma$, where h_1 is defined as $\widetilde{h}_1(x) = (\widetilde{g}_1(x))^{-1}$ (as in the above result Theorem 2.0.9), h_1 is continuous. Therefore, h is continuous on a neighbourhood of each point of F , implying h is continuous on all of F . \square

2.1 Sections of a vector bundle

Definition 2.1.1 (Sections of a vector bundle). *For a vector bundle (E, p, X) a section refers to a continuous map $s : X \rightarrow E$ such that $p \circ s = 1_X$, where 1_X denotes the identity map on X .*

These sections equivalently are a homomorphism of vector bundles from the trivial line bundle $(X \times \mathbb{R}, \pi_X, X) \rightarrow (E, p, X)$.

We assume all sections to be continuous without necessarily specifying it.

Definition 2.1.2 (Linear independence of sections). *A sequence s_1, \dots, s_n of sections on a vector bundle E is said to linearly independent if they are linearly independent for each point x .*

Furthermore the morphism $f : X \times k^n \rightarrow E$ given by $(x, c_1, \dots, c_n) \mapsto \sum_{i=1}^n c_i s_i(x)$ induces an isomorphism of fibres if E has rank n due to Proposition 2.0.10.

Example 2.1.3 (Zero section). *The map $s : X \rightarrow E$ sending every point x to the 0 vector in E_x .*

Definition 2.1.4 (Vector space of sections). *For a vector bundle $p : E \rightarrow X$ the set of continuous sections of E is denoted as $\Gamma(X, E)$. It is a vector space with vector addition defined as $(s+t)(x) = s(x)+t(x)$ and scalar multiplication as $\lambda s(x) = (\lambda s)(x)$ for $\lambda \in k$.*

Proposition 2.1.5 (Section functor). *$\Gamma(X, E)$ can be realized as a $C(X)$ module where $C(X)$ denotes the ring of continuous functions from X to k . With scalar multiplication defined as $(c \cdot s)(x) = c(x)s(x)$ for $s \in \Gamma(X, E), c \in C(X)$.*

For a trivial vector bundle $E = X \times k^n$ we have $\Gamma(X, E)$ corresponds to a free module $C(X)^n$. If E is a arbitrary vector bundle which is a direct summand of a trivial vector bundle then $\Gamma(X, E)$ corresponds to a projective $C(X)$ module.

In particular If $E \oplus E' \cong X \times k^n$ we have $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong C(X)^n$, i.e. $\Gamma : \text{VB}(X) \rightarrow \text{Proj}(C(X))$ is an additive functor.

In fact this ‘stably trivial’ property of a vector bundle is always true when X is compact. This is an essential result in the proof of Swans theorem. In order to prove it we need a lemma for paracompact spaces first.

Definition 2.1.6 (Paracompact space). A Hausdorff topological space X is said to be paracompact if every open cover of X has a locally finite open refinement.

(An open cover is locally finite if there exists a neighbourhood for every $x \in X$ which intersects only finitely many elements of the cover. A refinement of an open cover $\{U_i\}$ consists of a open cover $\{V_j\}$ such that for each $j, V_j \subset U_i$).

Lemma 2.1.7. Let X be a paracompact space, and let E and F be vector bundles over X . Suppose $\alpha : E \rightarrow F$ is a morphism such that $\alpha_x : E_x \rightarrow F_x$ is surjective for each point $x \in X$. Then there exists a morphism $\beta : F \rightarrow E$ such that $\alpha \circ \beta = \text{id}_F$.

Proof. In this proof we very liberally use the various morphisms as seen in Theorem 2.0.9.

Fix a point $x \in X$ by local trivialization we pick a neighborhood U of x such that $E|_U$ and $F|_U$ are trivial bundles. That is, $E|_U \cong M \times V$ and $F|_U \cong U \times N$.

Now with this representation we have $\alpha|_U : U \times M \rightarrow U \times N$, can be expressed as $\tilde{\theta}$ (as in 2.0.9) for the associated continuous map: $\theta : U \rightarrow \text{Hom}_{\text{VB}}(V, W)$.

Decomposing M as $N \oplus \ker(\theta(x))$ (which we can do thanks to surjectivity of α_x) allows us to choose a matrix representation for $\theta(y) : M \rightarrow N$ i.e. $\theta(y) : N \oplus \ker(\theta(x)) \rightarrow N$ as

$$\theta(y) = \begin{bmatrix} \theta_1(y) & \theta_2(y) \end{bmatrix},$$

the first component goes to 1 and the second to 0 continuously, i.e. an endomorphism of N . Realized as a topological vector space $\text{Aut}(N)$ is an open subset (i.e. vector subspace) of $\text{End}(N)$.

Therefore we can pick a neighbourhood of x say V_x such that $\theta_1(y)$ is an isomorphism for $y \in U_x$. Consider associated to this the map $\theta' : V_x \rightarrow \text{Hom}_{\text{VB}}(N, M)$ represented now by the matrix,

$$\theta'_x(y) = \begin{bmatrix} \theta_1(y)^{-1} \\ 0 \end{bmatrix}$$

This now induces the morphism $\tilde{\theta}'_x : F|_{V_x} \rightarrow E|_{V_x}$ so that $\alpha_{V_x} \tilde{\theta}'_x = 1$.

Varying the point x enables us to construct a locally finite open cover $\{V_i\}$ of X obeying the following properties, all morphisms $\beta_i : F|_{V_i} \rightarrow E|_{V_i}$ are right inverses of α_{V_i} as seen.

Now consider a ‘partition of unity’ $\{\eta_i\}$ associated with the cover ² and $\beta : F \rightarrow E$ is defined using it as $\sum_i \eta_i(x) \beta_i(e)$ for $e \in E_x$. Continuity is maintained due to the fact that these sums are necessarily finite (see footnote). This gives us

$$\alpha\beta(e) = \sum_i \eta_i(x) (\alpha\beta_i)(e) = \left(\sum_i \eta_i(x) \right) (\alpha\beta_i(e)) = e$$

Which completes the proof. \square

Theorem 2.1.8. *If E is a vector bundle over a compact base space X then there exists a vector bundle E' such that $E \oplus E' \cong X \times k^n$.*

Proof. Pick a finite open cover $\{U_i\}_{i=1}^r$. By local trivialisation we know that $E|_{U_i} \cong U_i \times k^{n_i}$.

Let $\{\eta_i\}$ be a ‘partition of unity’ associated to the cover similarly as before. So there exist n_i linearly independent sections $s_i^1, s_i^2, \dots, s_i^{n_i}$ of $E|_{U_i}$ (as seen in Definition 2.1.2). By extending these local sections to zero outside U_i , we obtain globally defined sections $\eta_i s_i^1, \eta_i s_i^2, \dots, \eta_i s_i^{n_i}$, which are linearly independent sections of $E|_{V_i}$, where $V_i = \eta_i^{-1}((0, 1])$.

Let σ_i^j denote the sections $\eta_i s_i^j$ for $1 \leq j \leq n_i$. These sections $\sigma_i^j(x)$ generate E_x as a vector space for each $x \in X$. As constructed in Definition 2.1.2, there exists a morphism

$$\alpha : T = X \times k^n \rightarrow E,$$

²A sequence of non negative real valued functions $\{\eta_i\}$ whose sum evaluated for every point $x \in X$ is 1. And such that every point $x \in X$ has an open neighbourhood not intersecting the support of η_i for all but finitely many i . Which implies the sums are finite and therefore well defined.

where $n = \sum_{i=1}^r n_i$, such that $\alpha_x : T_x \rightarrow E_x$ is surjective for every $x \in X$.

Now by the above Lemma for paracompact spaces, there exists a morphism $\beta : E \rightarrow T$ such that $\alpha \circ \beta = 1_E$. Let E' be the kernel of the idempotent morphism $p = \beta \circ \alpha$.

Define a morphism from $E \oplus E'$ to T by taking the sum of $\beta : E \rightarrow T$ and the inclusion $i : E' \rightarrow T$. This morphism induces a fibrewise isomorphism since $E'_x \cong \ker(p_x)$ and $E_x \cong \ker(1 - p_x)$. Thus, the morphism is an isomorphism by Proposition 2.0.10. \square

2.2 Karoubian categories and some results on vector bundles

In this section we cover an essential theorem which is applied in order to prove Swan's theorem. We follow the results as proven in Karoubi's book [Kar08].

Definition 2.2.1 (Karoubian category). *A Karoubi category is an **Ab**-Enriched category such that every idempotent endomorphism (a morphism of the form $e : A \rightarrow A$ such that $e^2 = e$) has a kernel.*

These are sometimes referred to as pseudo-abelian categories (note that in an abelian category proper all morphisms have kernels and cokernels in this case its only the idempotents).

Example 2.2.2. *The category of vector bundles is Karoubian. We already know it is additive and so **Ab**-Enriched. In the conclusion of Theorem 2.1.8 we have implicitly shown that the idempotent $p \in \text{VB}(X)$ has a kernel.*

Example 2.2.3. *Hinting towards Swans theorem we can see that also $\text{Proj}(A)$ for a ring A is Karoubian.*

Idempotent elements occur ubiquitously in K -theory we will see another useful application of idempotents in computations of K_0 .

The first theorem we cover in this section is well known by various different names. It involves showing the existence of a Karoubi envelope for an additive category. But this is often referred to as the Cauchy completion or idempotent completion.

Theorem 2.2.4. *[Existence of Karoubi envelope] Let \mathcal{C} be an additive category. Then there exists a Karoubian category $\tilde{\mathcal{C}}$ and a fully faithful additive functor $\varphi : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ universal in the sense that for any other additive functor $\psi : \mathcal{C} \rightarrow \mathcal{D}$ where \mathcal{D} is Karoubian there exists a unique additive functor $\psi' : \tilde{\mathcal{C}} \rightarrow \mathcal{D}$ such that the below diagram commutes.*

$$\begin{array}{ccc}
 & \tilde{\mathcal{C}} & \\
 \varphi \uparrow & \text{---} \psi' \text{---} & \\
 \mathcal{C} & \xrightarrow{\psi} & \mathcal{D}
 \end{array}$$

Proof. We directly construct $\tilde{\mathcal{C}}$ and φ . The objects of $\tilde{\mathcal{C}}$ are of the form (E, p) for $E \in \text{Ob}(\mathcal{C})$ and p idempotent endomorphism over E .

Arrows between two such objects (E, p) and (F, q) comprise of the arrows $(f : E \rightarrow F) \in \mathcal{C}$ such that $fp = qf = f$. Composition of morphisms is inherited from \mathcal{C} . The identity morphism comprises of $1_{(E,p)} = p$, and the sum of two objects $(E, p) \oplus (F, q)$ is defined naturally as $(E \oplus F, p \oplus q)$. This demonstrates that $\tilde{\mathcal{C}}$ is indeed an additive category.

It remains to be verified that it is indeed Karoubian which is the main concern. Let f be an idempotent endomorphism of the object (E, p) in $\tilde{\mathcal{C}}$ we wish to show that f indeed has a kernel.

Examining the below commutative diagram will help us conclude. The elements F, g, q, h are defined directly as seen below. Note that the condition $fp = pf = f$ is present in the righthand square

$$\begin{array}{ccccc}
 & & F & & \\
 & \swarrow h & \searrow g & & \\
 E & \xrightarrow{p-f} & E & \xrightarrow{f} & E \\
 (1-f)p \downarrow & & \downarrow p & & \downarrow p \\
 E & \xrightarrow{p-f} & E & \xrightarrow{f} & E \\
 & \nwarrow h & \nearrow g & & \\
 & & F & &
 \end{array}$$

Since $(1-f)p$ is itself an idempotent endomorphism on E we have $(E, (1-f)p)$ is an object in $\tilde{\mathcal{C}}$ and $p-f$ defines an arrow from this object to (E, p) . Since $(p-f)((1-f)p) = (p)(p-f) = p-f$ as expected.

Now we claim that the object $(E, (1 - f)p)$ with the arrow $p - f$ is a kernel of $f : (E, p) \rightarrow (E, p)$. In particular it is a pullback of the below form. The diagram below drawn internally in $\tilde{\mathcal{C}}$ makes this image clearer.

$$\begin{array}{ccccc}
 (F, q) & & & & \\
 \swarrow \scriptstyle h & \searrow & & \searrow & \\
 & (E, (1 - f)p) & \xrightarrow{\quad} & 0 & \\
 & \downarrow \scriptstyle p-f & & \downarrow & \\
 & (E, p) & \xrightarrow{\quad f \quad} & (E, p) & \\
 \swarrow \scriptstyle g & & & &
 \end{array}$$

Compare this to the first diagram. $(F, q) \xrightarrow{g} (E, p)$ is picked such that $fg = 0$. And we obtain uniqueness of h is due to the fact that any such arrow h must obey the expression $h = (1 - f)ph = p(1 - f)h = pg = g$. Conversely if $h = g$ the diagram commutes naturally.

This shows that $\tilde{\mathcal{C}}$ is Karoubian. Finally we construct $\varphi : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ defined as sending $E \in \text{Ob}(\mathcal{C})$ to $(E, 1_E)$ and $\varphi(f) = f$. This is naturally a full faithful functor by construction.

Also we can see that based on an analogous argument (E, p) is the kernel of $1 - p$ as an idempotent endomorphism over $\varphi(E) = (E, 1_E)$. Meaning that $\varphi(E) \cong (E, p) \oplus (E, 1 - p)$, i.e. the functor is additive.

Finally to show that these constructions define ψ' .

If $\psi : \mathcal{C} \rightarrow \mathcal{D}$ (resp. $\psi' : \tilde{\mathcal{C}} \rightarrow \mathcal{D}$) is an additive functor from \mathcal{C} (resp. $\tilde{\mathcal{C}}$) to another Karoubian category \mathcal{D} , such that $\psi'\varphi \cong \psi$. Then we have $\psi(\ker f) \cong \ker(\psi'(f))$ for every idempotent endomorphism f . Hence $\psi'(E, p) = \ker \psi(1 - p) : \psi(E) \rightarrow \psi(E)$ and $\psi'(f) = \psi(f)_{\ker \psi(1 - p)}$ on the objects and morphisms respectively. Conversely, these formulas define ψ' (up to isomorphism). \square

Before proceeding for the last result in this section, recall the definition of equivalence of categories.

Definition 2.2.5 (Equivalence of categories). *Two categories \mathcal{C}, \mathcal{D} are said to be equivalent if there exist functors $E : \mathcal{C} \rightleftarrows \mathcal{D} : F$ and a pair of natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow F \circ E$ and $\beta : 1_{\mathcal{D}} \rightarrow E \circ F$.*

This is a weaker condition than isomorphism of categories in which we have an actual equality instead of natural isomorphism.

A more useful form of the definition is as such, the proof may be seen in [Awo10][7.7.25]

Proposition 2.2.6. *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces an equivalence of categories iff F has the following properties*

1. *F is full (The map $F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(FA, FB)$ defined as $f \mapsto F(f)$ is surjective for all $A, B \in \text{Ob}(\mathcal{C})$).*
2. *F is faithful (The map $F_{A,B}$ as defined above is injective for all pairs A, B).*
3. *F is essentially surjective on objects (For every $D \in \text{Ob}(\mathcal{D})$ there exists $C \in \text{Ob}(\mathcal{C})$ such that $FC \cong D$).*

Corollary 2.2.7. *Let \mathcal{C} be an additive category, \mathcal{D} a Karoubian category, and $\psi : \mathcal{C} \rightarrow \mathcal{D}$ an additive functor which is fully faithful such that every object of \mathcal{D} is a direct factor of an object in the image of ψ . Then the functor ψ' as defined in Theorem 2.2.4 forms an equivalence between the categories \mathcal{C} and \mathcal{D} .*

Proof. We will prove that ψ' is essentially surjective and fully faithful.

Let $G \in \mathcal{D}$. We seek $X \in \tilde{\mathcal{C}}$ such that $\psi'(X) \cong G$.

By the hypothesis we have that for $G \in \mathcal{D}$, there exists $E \in \mathcal{C}$ and $G' \in \mathcal{D}$ with $\psi(E) \cong G \oplus G'$. Due to this we can choose an idempotent endomorphism $q : \psi(E) \rightarrow \psi(E)$ such that $\text{Ker}(q) \cong G$ we are in a pseudo-abelian category.

Now as ψ is fully faithful, there's a idempotent $p : E \rightarrow E$ in \mathcal{C} with $\psi(p) = q$. Then by the formulas in the end of Theorem 2.2.4 we have $G \cong \varphi'(E, 1 - p)$. This proves essential surjectivity.

Lastly to prove φ' is fully faithful consider two objects $H, H' \in \tilde{\mathcal{C}}$ direct factors of $\varphi(E), \varphi(E')$. Then the following diagram shows that $\psi'_{H,H'}$ is an isomorphic function,

$$\begin{array}{ccccc}
 \text{Hom}_{\tilde{\mathcal{C}}}(\varphi(E), \varphi(E')) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(E, E') & \xrightleftharpoons{\quad} & \text{Hom}_{\tilde{\mathcal{C}}}(H, H') \\
 & \searrow \psi'_{\varphi(E), \varphi(E')} & \downarrow \psi_{E, E'} & & \downarrow \psi'_{H, H'} \\
 & & \text{Hom}_{\mathcal{D}}(\psi(E), \psi(E')) & \xrightleftharpoons{\quad} & \text{Hom}_{\mathcal{D}}(\psi'(H), \psi'(H'))
 \end{array}$$

where the horizontal arrows are induced by the decompositions $\varphi(E) = H \oplus H_1$ and $\varphi(E') = H' \oplus H'_1$, since $\psi_{E, E'}$ is an isomorphism by hypothesis. \square

2.3 Swan's theorem

We can now prove the celebrated Swan's theorem with the results we've built up so far.

Theorem 2.3.1 (Swan's theorem). *Let X be a compact Hausdorff space, and let $A = C(X)$. Then the section functor Γ induces an equivalence of categories $\text{VB}(X) \simeq \text{Proj}(C(X))$.*

Proof. Since we assume that X is compact the section map Γ as seen in Proposition 2.1.5 is indeed a functor $\Gamma : \text{VB}(X) \rightarrow \text{Proj}(C(X))$ due to Theorem 2.1.8. Furthermore it induces a functor $\Gamma_T : \text{VB}_T(X) \rightarrow \text{Free}(C(X))$.

Where $\text{VB}_T(X)$ refers to the full subcategory of $\text{VB}(X)$ consisting of the trivial bundles over X , and $\text{Free}(C(X))$ refers to finitely generated free modules over $C(X)$.

Since $C(X)^n \cong \Gamma_T(E)$ for $E = X \times k^n$, we have Γ_T is essentially surjective.

If $F : X \times k^p$ is some other trivial vector bundle and $f : E \rightarrow F$ is a morphism of vector bundles then as seen in Theorem 2.0.9 we have full faithfulness of the functor Γ_T . Which shows that Γ_T induces an equivalence of categories between $\text{VB}_T(X)$ and $\text{Free}(C(X))$.

To extend this to our required case we make use of Theorem 2.2.4 and Theorem 2.2.7.

Comparing with Theorem 2.2.4 since $\text{VB}(X)$ being Karoubian itself is naturally the Karoubian envelope of its subcategory $\text{VB}_T(X)$ and with the functor Γ realized as ψ' we see the below diagram commutes.

$$\begin{array}{ccc}
 \text{VB}(X) & & \\
 \uparrow i & \searrow \Gamma = \psi' & \\
 \text{VB}_T(X) & \xrightarrow{\psi} & \text{Proj}(C(X))
 \end{array}$$

Finally due to Theorem 2.2.7 we are done.

Using Theorem ?? which ensures that the trivial bundles suffice in building up to the construction. Theorems 2.2.4 and 2.2.7 combined guarantee that if a functor is fully faithful and satisfies the direct-factor property, then it's an equivalence. Since Theorem [reference needed for fully faithful proof] demonstrates that Γ is fully faithful, Γ is an equivalence of categories. Therefore, $\text{VB}(X) \simeq \text{Proj}(C(X))$. \square

Chapter 3

Fundamental theorems of K_0 theory

3.1 K_0 for exact and abelian categories

K_0 being the prototypical K group is readily generalized to our new settings. We will refer to Weibel for most of the definitions of K_0 [WS13]. The benefit of this approach will mainly be for building up an understanding for Higher K theory. A

We begin with the definition of an exact category and then define K_0 for an exact category. This will subsume the definition of K_0 for an abelian category. Since every abelian category is exact over itself.

Definition 3.1.1 (Exact category). *An exact category (sometimes referred to as a Quillen exact category) is a pair (\mathcal{E}, E) for \mathcal{E} an additive category which is a full subcategory of some abelian category \mathcal{A} . Along with a family of sequences E of the form,*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Which are short exact sequences in \mathcal{A} and if in a sequence of the above form $A, C \in \mathcal{E}$ then B is isomorphic to some element which is in $\text{Ob}(C)$, (i.e. it is closed under extensions).

Example 3.1.2. *Every abelian category is trivially exact over itself.*

Example 3.1.3. *Torsion free abelian groups over the category of abelian groups is exact but not abelian. (Non abelian-ness was shown in Example 1.1.9.2).*

Definition 3.1.4 (K_0 for an exact category \mathcal{E}). $K(\mathcal{E})$ is generated by $[B]$ for each $B \in \text{Ob}(\mathcal{E})$ and a relation of $[B] = [A] + [C]$ for all short exact sequences,

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Naturally since every abelian category is exact this applies for abelian categories in particular.

3.2 Fundamental theorems for K_0, G_0

In this section we present here for reference the important 'fundamental' theorem for K_0 and G_0 of abelian categories. The proofs will be provided in the subsequent project.

The proofs are omitted here due to the need for a more in-depth discussion of concepts such as the localization of categories and multiplicative systems. Furthermore, the proofs for the K_0 case are readily extended to the higher K groups, especially in the context of Quillen-Q constructions, which are based on the calculus of fractions of categories. As such, it is more fitting to address these in detail in the next project, which will build on the work presented here.

3.3 Devissage

Theorem 3.3.1 (Devissage for K_0 in abelian categories). *Let $\mathcal{B} \subset \mathcal{A}$ be abelian categories which are small (i.e. proper set of objects) then $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$ and the inclusion functor $i : \mathcal{B} \rightarrow \mathcal{A}$ is exact if the following conditions are met:*

1. \mathcal{B} is a abelian exact subcategory of \mathcal{A} closed under subobjects and quotients from \mathcal{A} .
2. Objects in \mathcal{A} have finite filtrations

$$A_n = 0 \subset A_{n-1} \subset \cdots \subset A_0 = A$$

with each of the quotients $A_i/A_{i+1} \in \mathcal{B}$.

Proof. Since $i : \mathcal{B} \subseteq \mathcal{A}$ we denote by $\tilde{i} : K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A})$ the natural induced homomorphism. This is naturally injective so we need to prove surjectivity and additivity.

Based on the hypothesis we can always find a finite filtration on $A \in \mathcal{A}$ of the form $\{A_i\}_{i=0}^n$ with the quotients in \mathcal{B} . We can then represent its consequent preimage in $K_0(\mathcal{B})$ as $[A] = \sum_i [A_i/A_{i+1}]$ in $K_0(\mathcal{A})$, i.e. $\phi^{-1}([A]) = \sum_i A_i/A_{i+1}$.

Note that such a preimage is also independent of the filtration due to the Schrier-Refinement theorem for abelian categories which states that we can always find a common refinement of filtrations. The proof is identical to the standard group theoretic proof.

To verify this claim look at a single refinement, say changing $A_i \supset A_{i+1}$ to $A_i \supset A' \supset A_{i+1}$.

$$0 \rightarrow A'/A_{i+1} \rightarrow A_i/A_{i+1} \rightarrow A_i/A' \rightarrow 0,$$

we see that $[A_i/A_{i+1}] = [A_i/A'] + [A'/A_{i+1}]$ in $K_0(\mathcal{B})$, as claimed. This is essentially

For additivity and exactness, note that given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we can construct a filtration $\{B_i\}$ for B by combining the filtration for A along with pullback of a filtration of C in B . For this filtration we have $\sum [A_i/A_{i+1}] = f(A') + f(A'')$. Therefore f is an additive function and defines a map $K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$. By inspection, f is the inverse of the canonical map i_* . \square

Corollary 3.3.2. *For a nilpotent ideal N in a noetherian ring A we have $G_0(A/N) \cong G_0(A)$.*

Proof. Every finitely generated module has a natural filtration which goes to zero found by multiplying a module with copies of N . \square

Nil ideals (ideals consisting of all nilpotent elements) are in general not nilpotent ideal. However, due to a theorem by Levitzky (see [Lam01, Theorem 10.30]) which states that nil ideals of right noetherian rings are indeed nilpotent we can say that this holds for nil ideals as well.

3.4 Serre quotient

For an abelian category \mathcal{A} a Serre subcategory of \mathcal{A} is a specific kind of subcategory which allows us to create a ‘quotient’ which we call a Serre quotient. In reality this is more akin to a localization than a quotient proper.

Definition 3.4.1 (Serre subcategory). *Let \mathcal{A} be an abelian category, a full subcategory $\mathcal{B} \subset \mathcal{A}$ is called a Serre subcategory of \mathcal{A} if*

- *For exact sequence $0 \rightarrow A \rightarrow B \rightarrow C$ in \mathcal{A} , $B \in \mathcal{B} \iff A, C \in \mathcal{B}$*
- *Equivalently this means that \mathcal{B} is closed under quotients, subobjects and extensions.*

Definition 3.4.2 (Serre quotient). *Given $\mathcal{B} \subset \mathcal{A}$ a locally small Serre subcategory of an abelian category we can define its Serre quotient \mathcal{A}/\mathcal{B} with the following construction.*

- *$\text{Ob}(\mathcal{A}/\mathcal{B})$ consists of objects from \mathcal{A}*
- *Morphisms between $A \rightarrow B$ as $\text{Hom}_{\mathcal{A}/\mathcal{B}}(A, B) = \lim_{\rightarrow} \text{Hom}_{\mathcal{A}}(\tilde{A}, Y/\tilde{Y})$ where $\tilde{A} \leq A, \tilde{Y} \leq Y$ are subobjects.*

When \mathcal{A} is small we can treat morphisms $A \rightarrow B$ as equivalence classes of diagrams of the form

$$A \xleftarrow{f} \bullet \xrightarrow{g} B$$

Where \ker, coker of f are in \mathcal{B} . Equivalence with another diagram $A \leftarrow \circ \rightarrow B$ is defined with the existence of the below commuting diagram

$$\begin{array}{ccccc} & & \bullet & & \\ & \swarrow & \uparrow & \searrow & \\ A & \xleftarrow{\quad} & X & \xrightarrow{\quad} & B \\ & \swarrow & \downarrow & \searrow & \\ & & \bullet & & \end{array}$$

Where for $\bullet \leftarrow X \rightarrow \circ$ we have \ker, coker of both the arrows in \mathcal{B} , making the below diagram commute.

Theorem 3.4.3. *\mathcal{A}/\mathcal{B} is abelian and the inclusion functor $L : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ is exact*

Proposition 3.4.4. *The Serre quotient \mathcal{A}/\mathcal{B} universal in the following sense. Any exact functor $U : \mathcal{A} \rightarrow \mathcal{C}$ such that $U(B) \cong 0$ for $B \in \mathcal{B}$ will factor*

through L . i.e. the below diagram commutes

$$\begin{array}{ccc} & & \mathcal{A}/\mathcal{B} \\ & \nearrow L & \downarrow \text{dashed} \\ \mathcal{A} & \xrightarrow{U} & \mathcal{C} \end{array}$$

This result is the reason why the Serre subcategory is defined the way it is. The conditions required in the definitions are precisely those such that the above proposition may hold.

3.5 Localization theorem for K_0

Recall the definitions of Serre quotient.

Theorem 3.5.1 (Localization theorem for K_0). *For a small abelian category \mathcal{A} and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory. The following sequence is exact*

$$K_0(\mathcal{B}) \xrightarrow{f} K_0(\mathcal{A}) \xrightarrow{L} K_0(\mathcal{A}/\mathcal{B}) \rightarrow 0$$

Proof. By construction we know that L is surjective. Note we already know that $\text{coker } f \rightarrow K_0(\mathcal{A}/\mathcal{B})$ is surjective due to the fact that $K_0(B)$ under composition through T goes to zero.

Consider the function $g : \mathcal{A}/\mathcal{B} \rightarrow \text{coker } f$ as $g(L(A)) := [A]$ as a natural candidate. If this is additive from we have found the required inverse.

We already know that T is bijective as a set function on objects of \mathcal{A} by construction. Consider two isomorphic elements in the Serre quotient and claim their images under g in $\text{coker } f$ are also isomorphic. Consider $L(A) \cong L(B)$ by the definition of the morphisms this means we have a diagram representative as such.

$$A \xleftarrow{n} X \xrightarrow{m} B$$

with $\ker(n), \ker m, \text{coker}(n), \text{coker}(m)$ in \mathcal{B} (since its an isomorphism).

As such in $K_0(\mathcal{A})$ we have,

$$[X] = [A] + [\ker n] - [\text{coker}(n)] = [B] + [\ker m] - [\text{coker}(n)]$$

so in $\text{coker } f$ we have $[X] = [A] = [B]$.

We have shown $L(A) \cong L(B) \implies [A] = [B]$ in $\text{coker } f$. Now to show additivity.

To see that g is an additive function, suppose we are given an exact sequence in \mathcal{A}/\mathcal{B} of the form

$$0 \rightarrow L(A) \xrightarrow{u} L(B) \xrightarrow{v} L(C) \rightarrow 0;$$

we have to show that $[B] = [A] + [C]$ in F . Represent v by a diagram representative $B \xleftarrow{o} Y \xrightarrow{p} C$ with o with $\ker o, \text{coker } o \in \mathcal{B}$

Now since canonically

$$[Y] = [A] + [\ker(o)] - [\text{coker}(o)] \text{ in } K_0(A),$$

we have $[Y] = [A]$ in $\text{coker } f$. Since L is exact and we know the below sequence is canonically exact

$$0 \rightarrow \ker(p) \rightarrow Y \xrightarrow{p} B \rightarrow \text{coker}(p) \rightarrow 0$$

Applying L to above gives us that $\text{coker}(p)$ is in \mathcal{B} and that $L(\ker(p)) \cong L(A)$ in \mathcal{A}/\mathcal{B} . Hence, $[\ker(p)] = [A]$ in $\text{coker } f$, and in $\text{coker } f$ we have

$$[B] = [Y] = [C] + [\ker(p)] - [\text{coker}(p)] = [A] + [C],$$

□

Example 3.5.2. Consider the following non-example for why the above sequence need not be left exact. Consider $A = k[t]$ for a field k . Consider the subcategory of modules annihilated by a power of x . Then all the terms are isomorphic to \mathbb{Z} .

Corollary 3.5.3. For a multiplicative set S then the category of S torsion modules over A denoted as $M_S(A)$ form a Serre subcategory. And we have

$$K_0(M_S(A)) \rightarrow G_0(A) \rightarrow G_0(S^{-1}A) \rightarrow 0$$

When $S = \{1, s, s^2, \dots\}$ then just

$$K_0(A/sA) \rightarrow G_0(A) \rightarrow G_0\left[\frac{1}{s}\right] \rightarrow 0$$

The change is due to Devissage considering each module has a finite filtration given by multiplying by powers of s .

3.6 Fundamental theorems for G_0, K_0

We state a few important results here without proof their detailed proofs can be seen in [WS13]. These will be covered in the next project. As they lend themselves to quick generalizations and motivate the introduction of higher K groups.

Theorem 3.6.1 (Fundamental theorem for G_0 for noetherian rings A). $G_0[A] \cong G_0(A[t]) \cong G_0(A[t, t^{-1}])$.

Theorem 3.6.2 (Resolution theorem for K_0 for additive categories). *Let \mathcal{A} be abelian and $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ additive subcategories. If the following conditions hold,*

- *Every $B \in \text{Ob}(\mathcal{B})$ has finite \mathcal{C} dimension, in the sense that there exist a minimal finite \mathcal{C} resolution of B (in the sense of Definition 1.2.1)*
- *\mathcal{B} is closed under kernels of epis in \mathcal{A} .*

then the inclusion functor $i : \mathcal{C} \rightarrow \mathcal{B}$ induces an isomorphism $K_0(\mathcal{C}) \cong K_0(\mathcal{B})$

Proof. This is a theorem due to Grothendieck. The evaluation morphism $e : A[t] \rightarrow A$ provides an inclusion $M(A) \subset M(A[t])$ and consequently a map $\tilde{e} : G_0(A) \rightarrow G_0(A[t])$ by Corollary 3.5.3. We obtain an exact sequence

$$G_0(A) \xrightarrow{\tilde{e}} G_0(A[t]) \rightarrow G_0(A[t, t^{-1}]) \rightarrow 0$$

Now consider the exact sequence in $G_0(A[t])$ given by

$$0 \rightarrow M[t] \rightarrow M[t] \rightarrow M \rightarrow 0$$

i.e. $\tilde{e}[M] = [M] = [M[t]] - M[[t]] = 0$ so $\tilde{e} = 0$ in $G_0(A[t])$ so $\implies G_0(A[t]) \cong G_0(A[t, t^{-1}])$.

Since $A = A[t]/tA[t]$ applying Serres formula we get $\tilde{e}[M] = [M/Mt] - \text{Ann}_M(t)$ \square

Definition 3.6.3 (Regular ring). *A ring is called regular if every finitely generated ideal has finite projective dimension (minimal length of resolution by projective modules).*

Example 3.6.4. *Any Dedekind domain is a regular ring in particular a principal ideal domain is a regular ring.*

Theorem 3.6.5 (Fundamental theorem for K_0 of regular rings). *For a regular ring A , $K_0(A) \cong G_0(A)$ and by Theorem 3.6.1 we have,*

$$K_0(A) \cong K_0(A[t]) \cong K_0(A[t, t^{-1}]).$$

Chapter 4

Basic homotopical algebra

In this section, we introduce some terminology and machinery necessary for the proceeding sections. These concepts are essential for understanding the fact that the homotopy theory of topological spaces and the homotopy theory of simplicial sets (which we will be working with) are essentially the same.

The concept of a model category was introduced by Quillen to characterize the general notion of a “homotopy theory.” We provide a brief introduction to the concepts of model categories and Quillen equivalences. Most of the results and theorems mentioned here can be found in Quillen’s seminal book [Qui67]. However, for a canonical modern reference along with proofs for all the results discussed, we primarily refer to [JT99].

4.1 Closed model categories

There exist various formulations of the concept of a model category, some of which contain axioms that were later found to be redundant.

To present the material in the cleanest manner possible without spending too much time on the various existing definitions, we will follow the definitions given by Riehl [Rie].

We begin with the notion of a weak factorization system.

Definition 4.1.1 (Weak factorization system). *A weak factorization system on a category \mathcal{C} consists of $(\mathcal{L}, \mathcal{R})$ two classes of morphisms such that,*

1. *Every morphism $f : X \rightarrow Y$ factors as a composition of maps $\lambda \in$*

$L, \rho \in R$

$$\begin{array}{ccc} & Z & \\ \lambda \in \mathcal{L} \nearrow & & \searrow \rho \in \mathcal{R} \\ X & \xrightarrow{f} & Y \end{array}$$

2. \mathcal{L} is the class of maps having the left lifting property with respect to \mathcal{R} , i.e., \mathcal{L} consists of the morphisms $\lambda : A \rightarrow B$ such that for all $\rho \in \mathcal{R}$ and all morphisms in \mathcal{C} making the outer square in the below diagram commute, there exists a lift h which makes the entire diagram commute.

$$\begin{array}{ccc} A & \longrightarrow & X \\ \lambda \downarrow & \nearrow h & \downarrow \rho \\ B & \longrightarrow & Y \end{array}$$

3. \mathcal{R} is the class of maps having the right lifting property with respect to \mathcal{L} , i.e., \mathcal{R} consists of the morphisms $\rho : X \rightarrow Y$ such that for all $\lambda \in \mathcal{L}$, and all morphisms in \mathcal{C} making the outer square in the above diagram commute, there exists a lift h which makes the entire diagram commute.

Definition 4.1.2 ((Closed) Model category). A category \mathcal{C} with 3 distinguished classes of morphisms $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. Satisfying the following conditions:

1. \mathcal{C} has all finite limits and colimits, (i.e. it is bicomplete).
2. \mathcal{W} has the ‘2-out-of-3’ property. If two out of $f, g, fg \in \mathcal{W}$ then so does the third.
3. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ both form weak factorization systems on \mathcal{C} .

There exist common names for these distinguished classes of maps as such:

- Elements of \mathcal{W} are weak equivalences.
- Elements of \mathcal{C} are fibrations.
- Elements of \mathcal{F} are cofibrations.
- Elements of $\mathcal{F} \cap \mathcal{W}$ are called acyclic fibrations.
- Elements of $\mathcal{C} \cap \mathcal{W}$ are called acyclic cofibrations.

Quillen's original definition of a model category contained a distinction to talk about a 'closed' model category which additionally had the axioms of closure under retraction, weak equivalences containing all isomorphisms and closure of \mathcal{F}, \mathcal{C} under pullbacks/pushouts.

This distinction is largely unused when using the terminology of weak factorization systems due to the below theorem. Most authors refer to a closed model category to mean the same thing as just a model category.

Recall the definition of a retract in a category.

Definition 4.1.3 (Retract). *Let \mathcal{C} be a category. A retract of an object $B \in \mathcal{C}$ is an object A along with morphisms $s : A \rightarrow B, r : B \rightarrow A$ such that $ri = 1_A$. The map r is the retraction of B onto A .*

$$A \xrightarrow{s} B \xrightarrow{r} A$$

Definition 4.1.4 (Arrow category). *Given any category \mathcal{C} we may construct a category called the arrow category, denoted as \mathcal{C}^2 as such,¹*

- *The objects in \mathcal{C}^2 are the morphisms of \mathcal{C} .*
- *An arrow g from $f : A \rightarrow B$ to $f' : A' \rightarrow B'$ in \mathcal{C}^2 is the following commutative square*

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A' \\ f \downarrow & & \downarrow f' \\ B & \xrightarrow{g_2} & B' \end{array}$$

where g_1, g_2 are arrows in \mathcal{C} , i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \rightarrow B$ is the pair $(1_A, 1_B)$

- *Composition is componentwise (gluing two squares).*

We will often only be concerned about retracts in \mathcal{C}^2 . To be explicit a retract in \mathcal{C}^2 will be a commutative diagram of the following sort. Here f is

¹Sometimes also denoted as $\text{Arr}(\mathcal{C})$ or $\mathcal{C}^{\rightarrow}$

a retract of g .

$$\begin{array}{ccccc}
 A & \xRightarrow{1_A} & B & \xrightarrow{\quad} & A \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 D & \xrightarrow{\quad} & C & \xrightarrow{\quad} & D \\
 & \xRightarrow{1_D} & & &
 \end{array}$$

Proposition 4.1.5. *Let \mathcal{C} be a category and $(\mathcal{L}, \mathcal{R})$ a weak factorization system on it. Then,*

1. *Both contain all isomorphisms of \mathcal{C} ,*
2. *Both are closed under compositions,*
3. *Both are closed under retractions in \mathcal{C}^2 ,*
4. *\mathcal{L} is closed under pushouts, \mathcal{R} is closed under pullbacks.*

Proof. 1. (Isomorphisms) Given an arbitrary commutative diagram of the sort, where i is an isomorphism.

$$\begin{array}{ccc}
 \bullet & \xrightarrow{f} & \bullet \\
 i \downarrow & \nearrow & \downarrow g \\
 \bullet & \longrightarrow & \bullet
 \end{array}$$

A lift of the form fi^{-1} always exists, therefore it exists when $g \in \mathcal{R} \implies i \in \mathcal{L}$. Dually we also have when g is an isomorphism and $i \in \mathcal{L} \implies g \in \mathcal{R}$.

2. (Compositions) Suppose $\rho_1, \rho_2 \in \mathcal{R}$ and $\lambda \in \mathcal{L}$ as in the below diagram. We wish to show there exists a lift $h : C \rightarrow X$.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \downarrow & & \downarrow \rho_1 \\
 B & & Y \\
 \downarrow & & \downarrow \rho_2 \\
 C & \longrightarrow & Z
 \end{array}
 \quad \lambda \left(\begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right)$$

First paste A downwards along ρ_1, f to obtain, a lift $h_1 : C \rightarrow Y$.

$$\begin{array}{ccc} A & \xrightarrow{\rho_1 \circ f} & Y \\ \lambda \downarrow & \nearrow h_1 & \downarrow \rho_2 \\ C & \longrightarrow & Z \end{array}$$

Now paste C across this lift to obtain out required lift $h : C \rightarrow X$.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \lambda \downarrow & \nearrow h & \downarrow \rho_1 \\ C & \xrightarrow{h_1} & Y \end{array}$$

This gives us a right lift for $\rho_2 \circ \rho_1$ in our original diagram which implies it is in \mathcal{R} . Dually we also have \mathcal{L} is closed under compositions.

3. (Retracts) Let f be a retract of $\lambda \in \mathcal{L}$

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\ \downarrow f & & \downarrow \lambda & & \downarrow f \\ D & \xrightarrow{\quad} & C & \xrightarrow{\quad} & D \end{array}$$

1_A (top curved arrow), 1_D (bottom curved arrow)

For an arbitrary square of the form,

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & & \downarrow \rho \in \mathcal{R} \\ D & \longrightarrow & Y \end{array}$$

Paste it to the right of the retract diagram and obtain a lift $h : C \rightarrow X$. Then the sequence $A \xrightarrow{f} D \rightarrow C \xrightarrow{h} X$ gives a lift across f for every $X \xrightarrow{\rho} Y$. Which shows that $f \in \mathcal{L}$, and that \mathcal{L} is closed under retracts. Dually we obtain \mathcal{R} is closed under retracts.

4. (Pushouts/pullbacks) We show \mathcal{R} is closed under pullbacks. Say $\rho \in \mathcal{R}$ in the below pullback diagram. We wish to show $r \in \mathcal{R}$ too.

$$\begin{array}{ccc}
X \times_Z Y & \longrightarrow & X \\
r \downarrow & \lrcorner & \downarrow \rho \\
Y & \xrightarrow{f} & Z
\end{array}$$

For $\lambda \in \mathcal{L}$ consider the squares of the kind.

$$\begin{array}{ccc}
A & \longrightarrow & X \times_Z Y \\
\lambda \downarrow & & \downarrow r \\
B & \xrightarrow{g} & Y
\end{array}$$

Paste this to the left of the pullback diagram and obtain a lift against ρ on the outer rectangle $h_1 : B \rightarrow X$. This lift allows us to utilize the universality of the pullback to obtain a unique map $h : B \rightarrow X \times_Z Y$ as such.

$$\begin{array}{ccccc}
& & h_1 & \xrightarrow{\quad} & \\
A & \xrightarrow{\quad} & X \times_Z Y & \longrightarrow & X \\
\downarrow & \nearrow \text{dashed } \exists! h & \downarrow r & \lrcorner & \downarrow \rho \\
B & \xrightarrow{g} & Y & \xrightarrow{f} & Z
\end{array}$$

We wish to show this h is indeed a lift against r . The bottom right triangle commutes due to the pullback. It remains to show the top left triangle commutes too.

Consider the two potential diagrams.

$$\begin{array}{ccc}
A \longrightarrow X \times_Z Y & & B \\
\downarrow & \searrow & \downarrow h \\
B & \xrightarrow{\quad} & X \times_Z Y \\
& \searrow & \downarrow \\
& & Y \longrightarrow Z
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\quad} & X \times_Z Y \\
\downarrow & \searrow & \downarrow \\
B & \xrightarrow{\quad} & Y \longrightarrow Z
\end{array}$$

Due to uniqueness of the map $A \rightarrow X \times_Z Y$ due to it being a pullback, we obtain the required commutativity.

This finally gives us that h is indeed a lift across r giving us $r \in \mathcal{R}$. The dual of this argument gives us that \mathcal{L} is closed under pushouts. \square

Lastly the model category also often requires the weak equivalences to be closed under retracts. This is not immediately seen from the above theorem. But still is infact true.

Proposition 4.1.6. *Let \mathcal{C} be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ as defined above then \mathcal{W} is closed under retracts in \mathcal{C}^2 .*

Proof. Let f be a retract of $w \in \mathcal{W}$

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\
 \downarrow f & & \downarrow w & & \downarrow f \\
 D & \xrightarrow{\quad} & C & \xrightarrow{\quad} & D
 \end{array}$$

$\xrightarrow{1_A}$ (top curved arrow) $\xrightarrow{1_D}$ (bottom curved arrow)

We consider the proof in two cases $f \in \mathcal{F}, f \notin \mathcal{F}$.

Case 1: $f \in \mathcal{F}$: Factor w as $w = \rho \circ \lambda$ using any of the two weak factorization systems. In either case by 2-out-of-3 we get $\lambda \in \mathcal{C} \cap \mathcal{W}, \rho \in \mathcal{F} \cap \mathcal{W}$.

We now have a diagram of the sort.

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\
 \downarrow f & \searrow s & \downarrow \lambda & \nearrow t & \downarrow f \\
 & & \bullet & & \\
 & & \downarrow \rho & & \\
 D & \xrightarrow{\quad} & C & \xrightarrow{\quad} & D
 \end{array}$$

Here, s is obtained composition and t via the lift on square obtained on the right by pasting $\bullet \rightarrow D$, since $\lambda \in \mathcal{C} \cap \mathcal{W}, f \in \mathcal{F}$.

Since the triangles commute we have $t \circ s = 1_A$ so f is a retract of ρ as well, but $\rho \in \mathcal{F} \cap \mathcal{W}$ this implies $f \in \mathcal{F} \cap \mathcal{W}$ both classes of any factorization system are closed under retracts by Proposition 4.1.5(3). Therefore $f \in \mathcal{W}$.

Case 2: No hypothesis on f can i saw $f \notin \mathcal{F}$.

Factorize $f = h \circ g$, where $g \in \mathcal{C} \cap \mathcal{W}$ and $h \in \mathcal{F}$. Construct the below pushout (which exists due to cocompleteness of \mathcal{C}).

$$\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & A \\
g \downarrow & & \downarrow c & & \downarrow g \\
\bullet & \longrightarrow & \bullet & \xrightarrow{w} & \bullet \\
h \downarrow & & \exists! d \downarrow & & \downarrow h \\
D & \longrightarrow & C & \longrightarrow & D
\end{array}$$

We know by Proposition 4.1.5(4) that the left class of a factorization is closed under pushouts. This implies, $c \in \mathcal{C} \cap \mathcal{W}$. Also uniqueness of d such that $d \circ c = w$ is true due to the pushout, via the maps into C as such, $B \xrightarrow{w} C$ and $\bullet \xrightarrow{h} D \rightarrow C$. Now via 2-out-of-3 we see that $d \in \mathcal{W}$.

Similarly via the maps $B \rightarrow A \xrightarrow{g} \bullet$ and the identity map $\bullet = \bullet$. We find a unique morphism y such that $yx = 1_\bullet$ as seen below.

$$\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & A \\
g \downarrow & & \downarrow c & & \downarrow g \\
\bullet & \xrightarrow{x} & \bullet & \xrightarrow{y} & \bullet \\
h \downarrow & & \downarrow d & & \downarrow h \\
D & \longrightarrow & C & \longrightarrow & D
\end{array}$$

Therefore, h is found to be a retract of d . Recall that we had $h \in \mathcal{F}$ and $d \in \mathcal{W}$ therefore we have reduced our problem to the first case. This implies $h \in \mathcal{W}$. Now g by choice was in \mathcal{W} so by the 2-out-of-3 we have $f = h \circ g \in \mathcal{W}$. \square

4.2 Homotopy in model categories

Definition 4.2.1 (Path space object). *Let \mathcal{C} be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. Then define the path object for $X \in \mathcal{C}$ as the object obtained in its factorization of the diagonal morphism $\Delta : X \xrightarrow{(1_X, 1_X)} X \times X$.*

$$\begin{array}{ccc}
& \text{Path}(X) & \\
(r_0, r_1) \in \mathcal{W} \nearrow & & \searrow \lambda \in \mathcal{F} \\
X & \xrightarrow{\Delta} & X \times X
\end{array}$$

Note that such a factorization always exists due to the factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. However, the additional assumption that the weak equivalence is an acyclic cofibration is often dropped, as it is not necessary for the construction of homotopy. To be more permissive, we only require that $x \rightarrow \text{Path}(X)$ be a weak equivalence.

In topological setting the path space is the set of all based continuous functions $\gamma : [0, 1] \rightarrow X$.

Definition 4.2.2 (Cylinder object). *Let \mathcal{C} be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. Then define the cylinder object for $X \in \mathcal{C}$ as the object obtained from the factorization of the codiagonal “folding” morphism $\nabla : X \sqcup X \xrightarrow{[1_X, 1_X]} X$.*

$$\begin{array}{ccc} & \text{Cyl}(X) & \\ (l_0, l_1) \in \mathcal{C} \nearrow & & \searrow \rho \in \mathcal{W} \\ X \sqcup X & \xrightarrow{\quad \nabla \quad} & X \end{array}$$

Compare these to the definitions of cylinder objects discussed in Chapter 1, and the cones and cylinders of topology. In fact these are the same thing as we shall see when discussing examples of model structures.

Definition 4.2.3 (Fibrant objects). *Let \mathcal{C} be a model category with distinguished classes of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. An object $A \in \mathcal{C}$ is said to be fibrant if the unique mapping into the terminal object $(f : A \rightarrow 1) \in \mathcal{F}$.*

Dually we have the notion of a cofibrant object.

Definition 4.2.4 (Cofibrant objects). *As defined above, an object $B \in \mathcal{C}$ is said to be cofibrant if the unique mapping from the initial object $(g : 0 \rightarrow B) \in \mathcal{C}$.*

The requirement for defining these objects is that homotopies (as we define below) form equivalence relations only when the objects are (co)fibrant.

Example 4.2.5. *We shall see examples in more detail when we discuss examples of model categories.*

1. *The canonical example is that Kan complexes in the classical model structure of simplicial sets are fibrant.*

2. All topological spaces in the classical model structure of topological spaces are fibrant.
3. In the projective model category of chain complexes the complexes with injective objects are cofibrant and the complexes with projective objects are fibrant.

Proposition 4.2.6. *If $X \in \mathcal{C}$ a model category is fibrant then the map $r = (r_0, r_1) : \text{Path}(X) \rightarrow X \times X$ is a fibration and the induced maps r_0, r_1 are individually acyclic fibrations into X .*

Proof. **TODO** □

Dually we have the map from $X \sqcup X \rightarrow \text{Cyl}(X)$ is a cofibration and the individual maps from $X \rightarrow \text{Cyl}(X)$ are acyclic cofibrations provided X is a cofibrant object.

Definition 4.2.7 (Left homotopy in a model category). *Let \mathcal{C} be a model category and objects $X, Y \in \mathcal{C}$.*

If we have $f, g : X \rightarrow Y$ then a left homotopy if it exists is a diagram of the following sort

$H_l : f \Rightarrow g$ is a morphism $H_l : \text{Cyl}(X) \rightarrow Y$ making the below diagram commute.

$$\begin{array}{ccc}
 X & & \\
 \downarrow l_1 & \searrow f & \\
 \text{Cyl}(X) & \xrightarrow{H_l} & Y \\
 \uparrow l_0 & \nearrow g & \\
 X & &
 \end{array}$$

Definition 4.2.8 (Right homotopy in a model category). *Let \mathcal{C} be a model category and objects $X, Y \in \mathcal{C}$.*

If we have $f, g : X \rightarrow Y$ then a right homotopy if it exists is a diagram of the following sort

$H_r : f \Rightarrow g$ is a morphism, $H_r : X \rightarrow \text{Path}(Y)$ making the below diagram commute.

$$\begin{array}{ccc}
& & Y \\
& \nearrow f & \downarrow r_1 \\
X & \xrightarrow{H_r} & \text{Path}(Y) \\
& \searrow g & \uparrow r_0 \\
& & Y
\end{array}$$

Lemma 4.2.9. *Let \mathcal{C} be a model category and $f, g : X \rightarrow Y$ be morphisms. If X is cofibrant then a left homotopy implies existence of a right homotopy.*

Proof. TO DO □

Dually if Y is fibrant then a right homotopy implies existence of a left homotopy.

Theorem 4.2.10. *If as defined above X is cofibrant and Y is fibrant then left and right homotopies between them coincide and form an equivalence relation.*

4.3 Homotopy category of a model category

TODO Basically just localizing the model category at weak equivalences to invert them.

Theorem 4.3.1 (Whitehead in model categories). *content*

4.4 Examples of classical model structures

We exhibit without proof the two motivating examples of model categories. The classical model structure on \mathbf{Top} and the classical model structure on positive degree chain complexes $\mathbf{Ch}_{\leq 0}(\mathcal{A})$.

Definition 4.4.1 (Serre fibration). *A map $\rho : X \rightarrow Y$ is called a Serre fibration if for every CW complex A , the map ρ has the right lifting property with respect to the inclusion map $A \times 0 \rightarrow A \times [0, 1]$.*

Proposition 4.4.2 (Classical model structure on \mathbf{Top}). *Consider morphisms $f : X \rightarrow Y$ in \mathbf{Top} . We can define a model structure on \mathbf{Top} with the following distinguished classes of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ as such,*

1. $f \in \mathcal{W}$ if f is a weak homotopy equivalence in **Top**, i.e., $f : X \rightarrow Y$ is a map whose induced homomorphisms on homotopy groups (for every basepoint) are bijective.
2. $f \in \mathcal{F}$ if f is a Serre fibration.
3. $f \in \mathcal{C}$ if f is a retract of a map $f : X \rightarrow Y'$ in which Y' is formed by attaching cells to give a relative CW complex **apparently not necessarily countable like cw complexes so transfinite attachments allowed???** *TODO more accurately, describe cofibrations in more detail or say how two classes determine the third*).

For a proof see [Qui67][Section 2.3, Theorem. 2.1].

The key point to note is that this model structure gives rise to the usual notion of topological homotopy theory. Since the homotopy category of this model structure is the same as the classical homotopy category on CW complexes.

Example 4.4.3. *For simply connected topological spaces a weak homotopy equivalence exists if and only if the induced maps on singular homology groups are bijective [Hat02][Theorem 4.32]. This motivates the definition of quasi isomorphisms of chain complexes as being maps which preserve homologies.*

Proposition 4.4.4 (Projective model structure on $\mathbf{Ch}_{\leq 0}(\mathcal{A})$). *Let $\mathbf{Ch}_{\leq 0}(\mathcal{A})$ be the category of non negative chain complexes over some abelian category \mathcal{A} . We can define a model structure on $\mathbf{Ch}_{\leq 0}(\mathcal{A})$ as such,*

1. *Weak equivalences are quasi isomorphisms, i.e. maps which induce isomorphisms on homologies between chain complexes.*
2. *Fibrations are chain maps such that the underlying maps in \mathcal{A} are epimorphisms.*
3. *Cofibrations similarly are underlying monomorphisms whose cokernels are projective objects ².*

²It has a left lifting property with respect to epimorphisms. This is the same notion as projectivity as previously discussed over modules but we don't work with it over abelian categories often.

For a proof see [Qui67][Section 2.4, Remark 5]. The reason for specifying ‘classical’ model structures is that there exist a huge multitude of different classes of maps one may take to define new and interesting model structures on these categories. However, the ones presented thus far are the canonical examples with historical significance.

4.5 Quillen adjunction and equivalences

TODO just to state that the homotopy category of simplicial sets is equivalent to that of topological spaces.

Basically just ordinary adjunctions preserving (acyclic) (co)fibrations

Equivalences are just the derived functors from adjunctions in the homotopy category forming a equivalence of categories. Just some bookkeeping todo.

Chapter 5

Simplicial homotopy theory

In this section we will outline the basic machinery of simplicial sets the primary goal of this section is to appreciate the usefulness of the homotopy category of simplicial sets and how it gives us a combinatorial framework for studying homotopy without the need to resort to explicit topological approaches.

The preceeding section and this one are deeply linked by virtue of a result known as the Dold-Kan correspondence. We will not examine this in too much detail but mention it without proof.

For a canonical reference to the materials we cover in this section refer to [GJ09], a rapid introduction to simplicial sets is also covered in [Rie11].

5.1 The simplex category

We begin with a notion of a purely combinatorial construction.

Definition 5.1.1 (Simplex/finite ordinal category). *We refer to Δ as the simplex category. It is defined by the objects of finite non empty, totally ordered sets,*

$$[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$$

maps between these objects are order preserving, i.e. non decreasing maps between totally ordered sets. $f : [m] \rightarrow [n]$ is a map such that $f(0) \leq f(1) \leq \cdots \leq f(m)$.

The category formed by all such finite non empty, totally ordered sets and all the mappings between them is referred to as the simplex category Δ .

To be even more precise Δ is the full subcategory of \mathbf{Cat} whose objects consist of finite non empty totally ordered sets with morphisms being poset maps.

Example 5.1.2. $f : [2] \rightarrow [5]$, defined by $f(0 \rightarrow 1 \rightarrow 2) = 2 \rightarrow 3 \rightarrow 4$.

Example 5.1.3. $g : [4] \rightarrow [2]$ defined by $g[0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4] = 0 \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 1$.

Note that all morphisms in Δ are generated by a natural family of functions called coface and degeneracy maps defined as below.

Definition 5.1.4 (Coface maps). $d^i : [n-1] \rightarrow [n]$ the injection which misses the i^{th} element in $[n]$.

Explicitly, for $0 \leq i \leq n$

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases}$$

Definition 5.1.5 (Codegeneracy maps). $s^i : [n+1] \rightarrow [n]$ the surjection which maps two elements to i .

$$s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases}$$

These obey the relations,

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, & i < j \\ s^j s^i &= s^i s^{j+1}, & i \leq j \\ s^j d^i &= 1, & i = j, j+1 \\ s^j d^i &= d^i s^{j-1}, & i < j \\ s^j d^i &= d^{i-1} s^j, & i > j+1. \end{aligned}$$

5.2 Simplicial sets and examples

Definition 5.2.1 (Simplicial set). A simplicial set is a functor $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$, i.e. presheaves¹ on Δ . It comprises of a collection of sets $X_n = X([n])$ which we call the set of n -simplices of X with maps between them corresponding naturally with maps in Δ .

¹In the categorical sense.

Furthermore corresponding to coface maps from $[n-1] \rightarrow [n]$ in Δ we get a family of face maps between simplices $d_i : X_n \rightarrow X_{n-1}$, $0 \leq i \leq n$.

And degeneracy maps corresponding to codegeneracy maps $[n+1] \rightarrow [n]$ as a family of maps $s_i : X_n \rightarrow X_{n+1}$.

Defined as such,

$$\begin{aligned} d_i &= Xd^i : X_n \rightarrow X_{n-1} & 0 \leq i \leq n \\ s_i &= Xs^i : X_n \rightarrow X_{n+1} & 0 \leq i \leq n \end{aligned}$$

These obey the standard relations,

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ s_i s_j &= s_{j+1} s_i & i \leq j \\ d_i s_j &= 1, & i = j, j+1 \\ d_i s_j &= s_{j-1} d_i, & i < j \\ d_i s_j &= s_j d_{i-1}, & i > j+1 \end{aligned}$$

The face maps d_i can be understood as mapping each n -simplex $x \in X_n$ to $n+1$ many $n-1$ simplices $d_i(x)$ $0 \leq i \leq n$ in X_{n-1} , the i^{th} face does not contain the i^{th} vertex of x .

Similarly for degeneracy maps s_i we can understand it as mapping $x \in X_n$ to $n+1$ many $n+1$ simplices in X_{n+1} and $s_i(x)$ has x as its i^{th} and $i+1^{\text{th}}$ face.

Definition 5.2.2 (Degenerate simplex). *We say $x \in X_n$ is degenerate if it is the image of a degeneracy map.*

We can equivalently define a simplicial set purely combinatorially in terms of a graded set.

Definition 5.2.3 ((Alternate definition) Simplicial set). *A simplicial set X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i : X_n \rightarrow X_{n-1}$ and $s_i : X_n \rightarrow X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ s_i s_j &= s_{j+1} s_i, & i \leq j \\ d_i s_j &= 1, & i = j, j+1 \\ d_i s_j &= s_{j-1} d_i, & i < j \\ d_i s_j &= s_j d_{i-1}, & i > j+1. \end{aligned}$$

A significant simplification in theory occurs when we choose to distinguish a particular type of a simplex. Namely, the standard n -simplex as $\Delta[n] = \text{Hom}_\Delta(-, [n])$

Recall the Yoneda lemma [Awo10][Lemma 8.2],

Lemma 5.2.4 (Yoneda). *For a locally small category \mathcal{C} ², an object $A \in \mathcal{C}$ and a functor F in the functor category $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ there exists an isomorphism,*

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(yA, F) \cong FA.$$

Where $y : \mathcal{C} \rightarrow \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ is the Yoneda embedding defined as,

$$y(A) : \text{Hom}_{\mathcal{C}}(-, A).$$

For A an object of \mathcal{C} and,

$$y(f : A \rightarrow B) : \text{Hom}_{\mathcal{C}}(-, f) : \text{Hom}_{\mathcal{C}}(-, A) \rightarrow \text{Hom}_{\mathcal{C}}(-, B).$$

This naturally leads to the definition of **sSet** the category of simplicial sets as the functor category $\mathbf{Set}^{\Delta^{\text{op}}}$.

It is clear to see that placing $\mathcal{C} = \Delta$, $F = X \in \mathbf{sSet}$.

Proposition 5.2.5. *By Yoneda we have $X_n \cong \text{Hom}(\Delta[n], X)$. In particular any n -simplex $x \in X_n$ can be thought of as a map $x : \Delta[n] \rightarrow X$.*

Definition 5.2.6 (Simplicial objects in arbitrary categories). *For \mathcal{C} an arbitrary category a simplicial object in \mathcal{C} is a functor $C : \Delta^{\text{op}} \rightarrow \mathcal{C}$.*

Example 5.2.7 (Simplicial abelian groups). *A simplicial object in **Ab**.*

Furthermore there is a free forgetful adjoint pair between **sAb** and **sSet**.

Example 5.2.8 (Bisimplicial sets). $\mathbf{s}(\mathbf{sSets}) = \mathbf{s}^2\mathbf{Sets}$.

Theorem 5.2.9 (Dold-Kan). *There is a equivalence of categories*

$$N : \mathbf{sAb} \rightleftarrows \mathbf{Ch}_+ : \Gamma$$

where N denotes the normalized chain complex functor and Γ the simplicialization functor.

²Locally small implies each homset is indeed a small set (i.e. not a proper class). This is a weaker condition than just the category being small which means the collection of objects is a small set .

The Dold-Kan correspondence, serves as a bridge between simplicial homotopy theory and homological algebra. A proof can be seen in [GJ09].

In fact Dold-Kan can be thought of as a categorification of a theorem about divided differences (analogue to the fundamental theorem of calculus)! This insight was given by a nCafe post [Lei10] crediting it to Andre Joyal.

5.3 Geometric realisation

Definition 5.3.1 (Geometric realization of a standard n -simplicial set). *We define a functor $|\cdot| : \mathbf{sSet} \rightarrow \mathbf{Top}$ as such. Send each standard n simplex $\Delta[n]$ to the standard n -topological simplex. In particular,*

$$|\Delta[n]| = \{(x_0, \dots, x_{n+1} | 0 \leq x_i \leq 1, \sum_i x_i = 1\} \subset \mathbb{R}^{n+1}$$

Since we defined simplicial sets as presheaves of the simplex category we have a lot of nice properties thanks to the well behavedness of presheaves, such as being complete and cocomplete. More than this however, one of the most useful results we can borrow is the below result often called the density theorem or the co-Yoneda lemma. Which essentially says any presheaf is a colimit of presentable presheaves.

This is a relatively technical proposition and will distract from its application in our case where we are considering only simplicial sets. So we state it without proof. For a proof see [Awo10][Proposition 8.10]

Definition 5.3.2 (Category of elements). *Let \mathcal{C} be a category and $F \in \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ the category of elements of F is denoted as $\int_{\mathcal{C}} F$ and is defined as such.*

Objects are pairs (A, a) for A an object of \mathcal{C} and $a \in FA$.

Morphisms between $(A, a) \rightarrow (B, b)$ is a morphism $f : A \rightarrow B$ in \mathcal{C} under which $Ffb = a$.

The notation is suggestive of the fact that this is simply an end.

The category of elements for a given simplicial set is confusingly called the category of simplices.

Example 5.3.3 (Category of simplicies). *Applying the above construction with $\mathcal{C} = \Delta, F = X \in \mathbf{sSets}$ we have. The objects in $\int_{\Delta} X$ are simplicies $x \in X_n$ and morphisms $x \in X_n \rightarrow y \in X_m$ is a map in Δ , $f : [n] \rightarrow [m]$ such that $yf = x$.*

Theorem 5.3.4 (Density theorem). *Let \mathcal{C} be a small category, every object $X \in \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ is a colimit of representable functors for a index category J the category of elements of X ,*

$$\text{colim}_{j \in J} C_j \cong X$$

Proposition 5.3.5. *Any simplicial set can be expressed as a colimit of standard n simplicies. Where the indexing category is the category of simplices. In particular for $X \in \mathbf{sSet}$ we have,*

$$\text{colim}_{x \in X_n} \Delta[n] \cong X$$

We can then define the geometric realization of a standard n -simplex as $|\Delta[n]| = \Delta_n$ where Δ_n the standard topological n -simplex.

The following nice visuals that are from [Fri23].

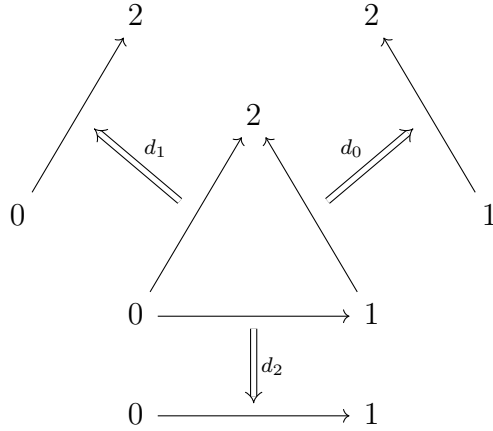


Figure 5.1: Face maps for 2-simplex

Recall that for some topological space X we define its singular simplicial set as $\text{Hom}(\Delta[n], X)$, this infact forms an adjoint pair along with ‘geometric realization’ the functor $|\cdot| : \mathbf{sSet} \rightarrow \mathbf{Top}$ sending standard n -simplices to topological simplices, in particular $|| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \text{Sing}, || \dashv \text{Sing}$. We shall see a detailed treatment of these adjoints after some more discussions.

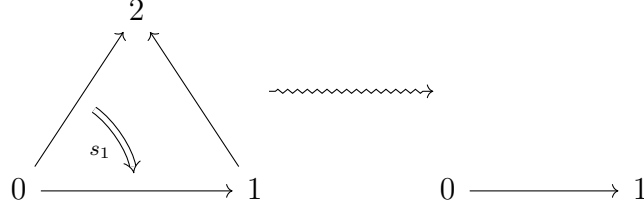


Figure 5.2: Degeneracy maps for 2-simplex

5.4 Adjunctions in simplices

TODO This will avoid any separate discussion of nerve realization, nerve path category adjoints as they are all kind of the same adjoint. Given any cocomplete category E and a functor $F : \Delta \rightarrow E$ we can completely characterize a large group of adjunctions which have right

5.5 Kan complex

Kan complexes, also called Kan fibrations or fibrant simplicial sets are a special kind of simplicial set. They form the fibrant objects in the model structure of simplicial sets, which we will define later.

The motivation for presenting this definition is due to the fact that it allows us to define simplicial homotopy as an equivalence relation.

Definition 5.5.1 (Horn of a standard n -simplex). *The k -th horn $|\Lambda_k[n]|$ on $|\Delta[n]|$ corresponds to the boundary of it with the k^{th} face removed. The associated simplicial set is denoted as $\Lambda_k[n]$.*

Definition 5.5.2 (Kan complexes (Usual)). *$X \in \mathbf{sSets}$ is a Kan complex if all horns on X can be filled. In particular this means that any map $\Lambda_k[n] \rightarrow X$ can be extended to a map $\Delta[n] \rightarrow X$.*

We also say X is a fibrant object in \mathbf{sSet} .

5.6 Nerve/Classifying spaces and the path category functor

Definition 5.6.1 (Nerve of a small category). *Let \mathcal{C} be a small category we define its nerve as the following simplicial set $N(\mathcal{C})_0 = \text{Ob}(\mathcal{C})$, \mathcal{C} and $N(\mathcal{C})_1 = \text{Mor}(\mathcal{C})$ and $N(\mathcal{C})_k = \{(f_1, \dots, f_k) | f_i \in \text{Mor}(\mathcal{C})\}$ consists of k -tuples of composable arrows, face maps defined as*

$$d_i(f_1, \dots, f_i, f_{i+1}, \dots, f_k) = (f_1, \dots, f_i \circ f_{i+1}, \dots, f_k)$$

and degeneracy maps defined as

$$s_j = (f_1, \dots, f_k) = (f_1, \dots, f_{j-1}, 1, f_j, \dots, f_k).$$

In a concise manner the nerve is simply the simplicial set consisting of n -simplices of the form $N(\mathcal{C}_n) := \text{Hom}([n], \mathcal{C})$.

Definition 5.6.2 (Classifying space of a category (Usual definition)). *For a small category \mathcal{C} we denote its classifying space as $BC := |N(\mathcal{C})|$.*

However, in light of the homotopy theory on simplicial sets we will instead refer to a classifying space of a category BC as simply being its nerve. This is the convention as adopted in [GJ09].

Definition 5.6.3 ((Simplicial) Classifying space of a small category). *For a small category \mathcal{C} when we refer to the classifying space of X as BX we will be referring to its nerve as defined above.*

If we refer to a geometric realization in particular this will be made clear.

Proposition 5.6.4. *A small category \mathcal{C} is a groupoid if and only if BC is a Kan complex.*

Proof. TODO □

Definition 5.6.5 (Path category of a simplicial set). *Let $X \in \mathbf{sSets}$ define its path category P_*X to be a category with objects as X_0 , i.e. the vertices of X and morphisms generated by 1simplices X_1 subject to the relation for $\sigma \in X_2$ the diagram below commutes*

$$\begin{array}{ccc} v_0 & \xrightarrow{d_2\sigma} & v_1 \\ & \searrow d_1\sigma & \swarrow d_0\sigma \\ & v_2 & \end{array}$$

Note that P_*X is a directed graph.

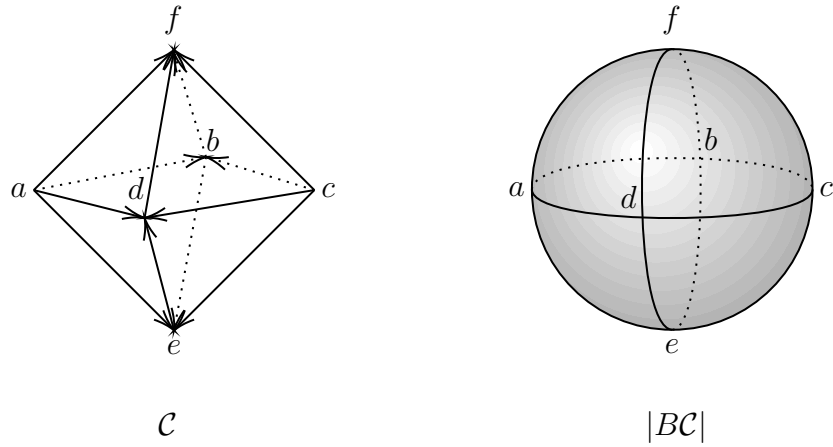


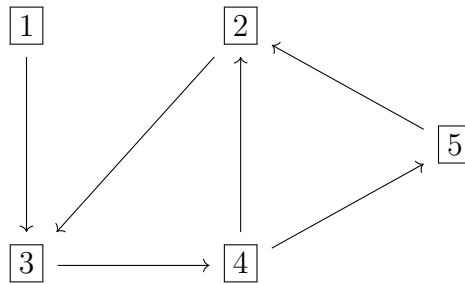
Figure 5.3: Example of a geometric realization of a simplex. Note here C is not a groupoid so the corresponding classifying space is not a Kan complex.

5.7 Free groupoid and fundamental groupoid of a simplicial set

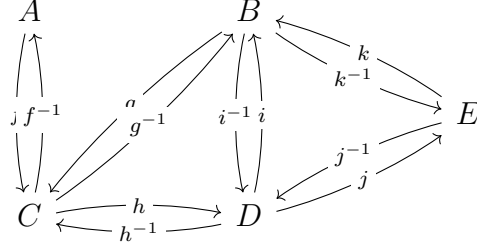
Recall a groupoid is any category in which all morphisms are invertible.

Definition 5.7.1 (Free groupoid on a directed graph). *Let X be a directed graph. Consider a category GX with objects as vertices of X and each edge of the directed graph inducing a morphism the concatenation of two edges gives a composition of morphisms. Finally introducing a formal inverse arrow for each morphism gives us a groupoid.*

Example 5.7.2.



X



GX

Definition 5.7.3 (Fundamental groupoid of a topological space). *Denoted as ΠX it consists of all homotopy classes of paths in X .*

As defined in [GZ67] we can be motivated to say.

Definition 5.7.4 (Fundamental groupoid on a simplicial set). *Defined as the fundamental groupoid of the P^*X i.e. GP_*X .*

5.8 Simplicial homotopy

We consider the standard 1-simplex $\Delta[1]$ to represent the interval object. Homotopy between two morphisms of simplicial sets are defined in an analogous manner.

Definition 5.8.1 ((Left) Simplicial homotopy). *replace with capital H Suppose $X, Y \in \mathbf{sSets}$ and two morphisms $f, g : X \rightarrow Y$ are said to be homotopic if there exists a morphism $h : X \times \Delta[1] \rightarrow Y$ such that the following diagram commutes*

$$\begin{array}{ccc}
 X & \begin{array}{c} \xrightarrow{f} \\ \Downarrow h \\ \xrightarrow{g} \end{array} & Y \\
 & & \\
 X \times \Delta[0] & \xlongequal{\quad} & X \\
 \downarrow 1_X \times d^1 & & \searrow f \\
 X \times \Delta[1] & \xlongequal{\quad h \quad} & Y \\
 \uparrow 1_X \times d^0 & & \nearrow g \\
 X \times \Delta[0] & \xlongequal{\quad} & X
 \end{array}$$

This essentially just means $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ as we are used to in topology.

Definition 5.8.2 (Homotopy equivalence). *content*

5.9 Simplicial homotopy groups

5.10 Classical model structure on simplicial sets and a Quillen equivalence

There exists a Quillen adjunction.

Chapter 6

Quillen Q construction

We begin with two theorems without proof. These are Quillens A and B theorem. Quillens A theorem follows from the B theorem. A full proof may be seen in [Qui73] or in [GJ09].

MAYBE MOVE THIS DOWN AFTER DEFINING K GROUPS AS SIMPLICIAL HOMOTOPY GRPS AND GIVE A PROOF

Theorem 6.0.1 (Theorem A).

Theorem 6.0.2 (Theorem B). *Given a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ between small categories such that every morphism $d \rightarrow d'$ in \mathcal{D} induces a weak equivalence $B(f/d) \rightarrow B(f/d')$ in \mathcal{C} . Then all induced pullback diagrams of the sort ??? are homotopy cartesian.*

Recall from Definition 3.1.1 the definition of a Quillen exact category (\mathcal{E}, E) as via embeddings. Where, \mathcal{E} is a additive subcategory of some abelian category \mathcal{A} closed under extensions along with a family E of exact sequences

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{A} .

This is an equivalent definition to the original axiomatic definition as it occurs in Quillen's paper [Qui73]. Which we recount here for its usefulness.

The monics that are part of the exact sequences in the family of exact sequences associated to the exact category are called admissible monics. Similarly, the epimorphisms which belong to exact sequences associated to the exact category are called admissible epics. Note in the below definition the map i is an admissible monic and the map j is an admissible epi.

Definition 6.0.3 (Exact category (Axiomatic definition)). *A pair (\mathcal{E}, E) where \mathcal{E} is an additive category which is a subcategory of a abelian category closed under extensions, and a set of sequences E containing elements of the form $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ satisfying the following properties:*

1. *A sequence in \mathcal{E} which is isomoprhic to a sequence in E is in E . The split exact sequences*

$$0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$$

are in E .

2. *Admissible epis are closed under composition and pullbacks by maps in \mathcal{E} . Dually, admissible monics are closed under composition and under pushout.*
3. *If a morphism $A \rightarrow B$ having a kernel in \mathcal{E} can fact a admissible epi $B \rightarrow C$ as $C \rightarrow A \rightarrow B$ then $A \rightarrow B$ is indeed an admissible epi. Dually, if a morphism $M \rightarrow N$ having a cokernel in \mathcal{E} can factor a admissible monic $N \rightarrow P$ such as $M \rightarrow N \rightarrow P$ then the map $M \rightarrow N$ is indeed a admissible monic.*

Example 6.0.4. *The category of finitely generated A -modules for a noetherian ring A with all the exact sequences.*

Proof. content...

□

Proposition 6.0.5. *The isomorphisms in an exact category are precisely those maps which are both admissible monics and admissible epis.*

Proof. content

□

Before proving the main result in this section we introduce some notation.

Definition 6.0.6 (Canonical admissible monic). *content*

Definition 6.0.7 (Canonical admissible epi). *content*

Theorem 6.0.8 (2 out of 3 for exact categories). *Let (\mathcal{E}, E) be an exact category, suppose we have a diagram of exact sequences in \mathcal{E} such that all of*

the vertical sequences are in E .

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A' & \xrightarrow{i_A} & A & \xrightarrow{p_A} & A'' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B & \xrightarrow{i_B} & B & \xrightarrow{p_B} & B'' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C & \xrightarrow{i_C} & C & \xrightarrow{p_C} & C'' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

If any of the two horizontal sequences are in E then so is the third.

Proof. content...

□

Example 6.0.9 (Add examples of usecases of above theorem). *content*

6.1 Q construction of a category

This construction, which can be viewed as a form of localization.

Suppose that \mathcal{E} is an exact category. Define an equivalence relation on all diagrams

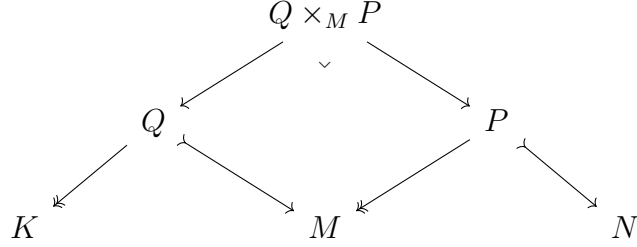
$$M \leftarrow P \rightarrow N$$

by saying that the top and bottom pictures in the diagram

$$\begin{array}{ccccc}
& & P & & \\
& \swarrow & \downarrow \theta & \searrow & \\
M & \xleftarrow{\sim} & P' & \xrightarrow{\sim} & N
\end{array}$$

are equivalent if the displayed isomorphism exists, making the diagram commute.

The category $Q\mathcal{E}$ has for objects all objects of \mathcal{E} . The morphisms $M \rightarrow M''$ are the equivalence classes of the pictures above. Composition is defined by pullback:



Definition 6.1.1 (Exact functor). *content*

An exact functor between two exact categories induces a functor between their corresponding Q spaces.

6.2 K-groups of an exact category

The below theorem is from [GJ09][III. Theorem 1.1]

Theorem 6.2.1. *There is an equivalence of groupoids $G(\Delta \downarrow X)$, GP_* and $\Pi|X|$.*

Proof. TODO □

The above theorem leads to a easy proof of the below major result due to Quillen.

Theorem 6.2.2. *There is an equivalence of groupoids $GQ\mathcal{E}$ and $K_0(\mathcal{E})$*

Corollary 6.2.3.

Definition 6.2.4 (Higher K -groups). $K_n(\mathcal{E}) = \pi_n(BQE)$

Chapter 7

Quillen + Construction

7.1 + Construction

Recall the definition of $GL(A)$ for a ring A . Motivated by the definition of $K_1(A)$ as the quotient $GL(A)/E(A)$, Quillen defined the higher K groups of a ring as the homotopy groups of a particularly nice topological space which we denote as $BGL(A)^+$. Here, $BGL(A)$ represents the classifying space of $GL(A)$.

Definition 7.1.1 (Plus construction). *$BGL(A)^+$ denotes a CW complex with a map $BGL(A) \rightarrow BGL(A)^+$ such that $\pi_1 BGL(A)^+ \cong K_1(A)$ with the induced map $\pi_1 BGL(A) \rightarrow \pi_1 BGL(A)^+$ being onto with kernel $E(A)$, and $H_*(BGL(A); M) \cong H_*(BGL(A)^+; M)$ for each $K_1(A)$ module M .*

In fact any complexes which model $BGL(A)^+$ are homotopy equivalent. Hence we can define $K_n(A) \cong \pi_n BGL(A)^+$ in a well defined manner. By construction this agrees with K_1 .

Definition 7.1.2 (K-space of a ring). *Define $K(A) = K_0(A) \times BGL(A)^+$*

Bibliography

- [Awo10] Steve Awodey. *Category Theory, 2nd Edition*. Oxford University Press, Oxford, England, 2010.
- [Bue] Theo. (<https://mathoverflow.net/users/11081/theo-buehler>) Buehler. Freyd-mitchell's embedding theorem. MathOverflow. <https://mathoverflow.net/q/47762> (version: 2012-12-26), Last Accessed February 16, 2025.
- [Cam20] Alexander Campbell. Is the tensor product of chain complexes a day convolution? MathOverflow <https://mathoverflow.net/q/359316>(version:2020-05-04), 2020.
- [FF64] P. Freyd and P.J. Freyd. *Abelian Categories: An Introduction to the Theory of Functors*. A harper international edition. Harper & Row, 1964.
- [Fri23] Greg Friedman. An elementary illustrated introduction to simplicial sets, 2023.
- [GJ09] Paul Gregory Goerss and J. F. Jardine. *Simplicial homotopy theory*. Birkhäuser, 2009.
- [Gro57] Alexandre Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Mathematical Journal, Second Series*, 9(2):119–183, 1957.
- [GZ67] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3Island Press, 1967.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

- [JT99] A. Joyal and M. Tierney. *An Introduction to Simplicial Homotopy Theory*. 1999.
- [Kar08] M. Karoubi. *K-Theory: An Introduction*. Classics in Mathematics. Springer, 2008.
- [Lam01] T.Y. Lam. *A First Course in Noncommutative Rings*. Graduate Texts in Mathematics. Springer, 2001.
- [Lan98] S.M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 1998.
- [Lei10] Tom Leinster. The dold–kan theorem: Two questions. n-Category Cafe, 2010.
- [MS74] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.
- [Qui67] D.G. Quillen. *Homotopical Algebra*. Lecture notes in mathematics. Springer-Verlag, 1967.
- [Qui73] Daniel Quillen. Higher algebraic k-theory: I. In H. Bass, editor, *Higher K-Theories*, pages 85–147, Berlin, Heidelberg, 1973. Springer Berlin Heidelberg.
- [Rie] Emily Riehl. A concise definition of a model category. <https://math.jhu.edu/~eriehl/modelcat.pdf>. Last accessed February 16, 2025.
- [Rie11] Emily Riehl. A leisurely introduction to simplicial sets. <https://math.jhu.edu/~eriehl/ssets.pdf>, 2011. Last accessed February 16, 2025.
- [Rie17] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017.
- [Str20] Neil Strickland. The model structure for chain complexes, 2020.
- [Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

- [WS13] C.A. Weibel and American Mathematical Society. *The K-book: An Introduction to Algebraic K-theory*. Graduate Studies in Mathematics. American Mathematical Society, 2013.

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