Contents

Algebraic K-Theory

Bhoris Dhanjal

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1 Classical K-Theory

The category of f.g. projective modules is the main object of study in algebraic K-theory. This is largely motivated by the following theorem due to Swan [Swa62] which relates algebraic K-theory to topological K-theory.

Theorem 1.1 (Swan's theorem)**.** *There exists an equivalence of categories between* $\mathrm{Vect}(X)$ *the category of vector bundles over a compact, Hausdorff space* X *and f.g. projective* $C(X)$ *modules. With the cross section functor.*

1.1 Grothendieck group K_0

The big picture idea that Grothendieck had was that of a free completion of a commutative monoid. Commutative monoids occurred in nature very often as f.g. projective modules/vector bundles.

This is a fairly natural approach which results in a Free-Forgetful adjoint pair between CMon and Ab.

Proposition 1.2 (K_0 of a monoid (Group completion functor))**.** *Assign* $(A, +) \in \mathrm{CMon}$ *to*

$$K_0(A) \in \mathrm{Grp}$$

by taking the free group on symbols $[a]$ *for* $a \in A$ *and quotienting the monoidal relations* $[m + n] - [m] - [n]$*.*

The mapping is an injection iff the monoid is cancellative.

Definition 1.3 (Reduced K_0 groups)**.** *There is a canonical homomorphism* $i : \mathbb{Z} \to K_0(A)$ *given by* $z \mapsto z[m]$ *the reduced* K *group is defined as* $\tilde{K}_0(A) := K_0(A)/\mathrm{Im}\, i$

Definition 1.4 (K_0 for a ring A)**.** *Consider the isomorphism classes of f.g. projective modules over A. This forms a commutative monoid so consider its group completion $K_0(A)$*

Definition 1.5 (G_0 for a ring A)**.** *The group completion of $M(A)$ the monoid of all f.g. modules over A is denoted as $G_0(A)$*

There is a canonical inclusion map $K_0(A) \to G_0(A)$.

Definition 1.6 (G_0 for a noetherian scheme)**.** *content (relevance?)*

Proposition 1.7 (Eilenberg Swindle)**.** *K_0 for many abelian categories are trivial. If we consider R^∞ as a inf.g. free module over a ring R if $P \oplus Q \equiv R^n$ then*

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \equiv (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \equiv R^\infty$$

but this relation would imply $[P] = 0$ for all projectives.

This extends to higher K groups with an analogue that demonstrates the Quillen K space contracts, see V.1.9 in [WS13].

Proposition 1.8. *If A is a Field/local ring/PID then $K_0(A) \cong \mathbb{Z}$*

Proof. For fields and division rings its just due to all f.g. modules being equal to some A^n. Similarly as seen in A.3 and A.7 f.g. projective modules in a local ring/PID are free.

So in each case $\mathrm{Proj}(A) \cong \mathbb{N}$ so its group completion is $\mathbb{Z}$. □

Lemma 1.9. *For commutative ring A, $K_0(A) \cong \mathbb{Z} \implies$ projective modules over A are stably free.*

Proof. For a commutative ring A, $K_0(A) \cong \mathbb{Z} \implies \mathrm{Spec}(A)$ is connected. For if not then there exists a non trivial idempotent in A which results in a splitting of A as a product which would imply $K_0(A)$ would contain an element of the form $\mathbb{Z} \oplus \mathbb{Z}$.

In light of Def. A.4 we know that the rank of the projective modules must be constant due to the connectedness of $\mathrm{Spec}(A)$ and the fact that the only connected components in $\mathbb{Z}$ are singletons.

So the rank map $\phi : K_0(A) \to \mathbb{Z}$ defined as $P \mapsto \mathrm{Rank}(P)$ is well defined and trivially surjective mapping A to 1. But by our assumption this is an isomorphism so since $\phi(A)$ generates $\mathbb{Z}$ we have $\phi(P) = n \cdot \phi(A) = \phi(A^n)$. □

Theorem 1.10 (Devissage for K_0 in abelian categories)**.** *Let $\mathcal{B} \subset \mathcal{A}$ be abelian categories which are small (i.e.* Ob *finite) then $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$ if the following conditions are met*

1. *$\mathcal{B}$ is a abelian exact subcategory of B* [1]

Theorem 1.11 (Localization theorem for G_0)**.**

Theorem 1.12 (Grothendieck)**.** $G_0[A] \cong G_0(A[t]) \cong G_0(A[t, t^{-1}])$

Proof. pg 134 □

Definition 1.13 (Regular ring)**.** *A ring is called regular if*

Example 1.14. *content...*

Theorem 1.15. *For a regular ring A the map $A \to A[x]$ induces an isomorphism $K_0(A) \cong K_0(A[x])$*

Proof. pg 149 □

K_0 being the prototypical K group is easier to generalize. We will refer to Weibel for most of the definitions of K_0 leading upwards to its definition for a Waldhausen category [WS13]. The benefit of this approach will mainly be for building up an understanding for Higher K theory. Also this machinery allows for easier computations.

Definition 1.16 (K_0 for an exact category $\mathcal{E}$)**.** *$K_(\mathcal{E})$ is generated by $[A]$ for each $A \in \mathrm{Ob}(\mathcal{E})$ and a relation of $[A] = [A'] + [A'']$ for all short exact sequences*

$$0 \to A' \to A \to A'' \to 0$$

Naturally since every abelian category is exact this applies for abelian categories in particular.

Definition 1.17 (K_0 for Waldhausen categories)**.**

[1] It is important to mention B is abelian again. Since a subcategory of an abelian category need not be abelian. Consider the category of torsion free abelian groups.

1.2 Quillen-Suslin Theorem

We will know move towards Horrock's theorem which will enable a short proof of the famous Quillen-Suslin theorem. We follow Lang's book for the first few results which recounts Vaserstein's proof of Quillen-Suslin [Lan02].

Definition 1.18. *An A module M is stably free if there exists a f.g. free module F such that $M \oplus F$ is free.*

Lemma 1.19. *A proj. module is stably free iff if has a finite free resolution.*

Theorem 1.20 (Hilbert-Serre)**.** *Every f.g. module over $k[x_1, \dots x_n]$ is stably free where k is a PID.*

Proof. Apply Th 1.8 and Th 1.15 □

Example 1.21 (Stably free module that is not free)**.** *Consider S^2 described as the ring $R = \mathbb{R}[x, y, z]/\langle x^2 + y^2 + z^2 - 1\rangle$. Then the tangent space $T = \{(a, b, c) \mid ax + by + cz = 0\}$ is stably free since $T \oplus R \equiv R^3$ but $T \neq R^2$*

Proof. content... □

Definition 1.22 (Unimodular row)**.** *For a ring A, an element of A^n is said to be a unimodular row if its components generate A. We denote the set of all unimodular rows as $U_n(A)$*

Definition 1.23 (Unimodular matrix)**.** *In general we say an arbitrary matrix over A not nessecarily square is unimodular if it is right invertible.*

Alternatively it can be useful to view a unimodular row as as element of $M_{1\times n}(A)$ as such it represents a surjective linear map $A^n \to A$, or even an element in $M_{n\times 1}$ in which case it represents a injection from $A \to A^n$.

Definition 1.24 (Equivalence of unimodular rows)**.** *For unimodular rows $v, w \in A^n$ we say $v \sim w$ if $\exists M \in GL_n(A)$ such that $Mv = w$.*

Definition 1.25 (Unimodular extension property)**.** *Given a unimodular row $v = (v_1, \dots v_n) \in A^n$ if we can construct a invertible $n \times n$ matrix with v in the first column we say v has the unimodular extension property.*

We don't use the above terminology in the light of the following fact.

Lemma 1.26. *A unimodular row $v \in A^n$ has the unimodular extension property iff $v \sim (1, 0, \dots, 0)$*

Proof. If v can be extended to a invertible matrix $M \in GL_n(A)$ then

$$M^-1 = (1, 0, \dots, 0)$$

. Conversely if $M' \in GL_n(A)$ s.t. $M'v = (1, 0, \dots, 0)$ then M'^-1 has v in the first column. □

Corollary 1.27. *Based on the above lemma we can see that trivially any row of a invertible matrix (and column realized as a row of its transpose) is a unimodular row.*

Proposition 1.28. *Over a PID A any two unimodular rows in A^n are equivalent.*

Proof. We will instead show that any unimodular row is equivalent to $(1, 0, \dots, 0)$. Since its a PID one element must generate A move this to the first coordinate by elementary row operations then make it equivalent to 1 by another row operation. □

Proposition 1.29. *Over a local ring A any two unimodular rows are equivalent*

Proof. Use the fact that projective modules over local rings are free. □

Theorem 1.30 (Horrocks' theorem)**.** *If $(A, \mathfrak{m})$ is a local ring then for any arbitrary unimodular row $v(x)$ in $A[x]^n$ such that one of its component elements has leading coefficient 1 implies that v has the unimodular extension property. Furthermore, any such v is equivalent to $v(0)$.*

Proof. Recall that for a local ring $x \notin \mathfrak{m} \iff x$ is a unit.

When $n = 1, 2$ there is nothing to prove. Assume $n \geq 3$.

Without loss of generality, we take $v_1(x)$ with degree d among components with leading coefficient 1 and $\deg v_i < d$, for $i \neq 1$. We shall induct on d.

By unimodularity we know there exists $w(x) \in A[x]^n$ such that,

$$\sum_{i=1}^{n} w_i v_i = 1$$

So we can say that not all of the coefficients of $v_2, \dots v_n$ can lie in $\mathfrak{m}$. For if it were the case, then reduced mod $\mathfrak{m}$ we arrive at a contradiction since we assumed v_1 has leading coefficient 1 and $w_1 v_1$ wouldn't have a constant residue.

Once again without loss of generality, assume some coefficient of $v_2(x)$ does not lie in $\mathfrak{m}$, and as such is a unit.

Now consider the ideal I generated by the leading coefficients of $w_1v_1+w_2v_2$ of degree $< d$.

I contains the coefficients of v_2 this can be inductively found when $w_1 = 0, w_2 = 1$ we get the coefficient of the x^m term where $\deg v_2 = m$. Using repeatedly different choices of polynomials we are done.

Since I has a unit which means it generates A. And consequently implies that there was some choice of polynomial $y_1v_1 + y_2v_2$ of degree $< d$ with leading coefficient 1.

The the appropriate row actions we can obtain this in some component of v. Repeating this process until we get $d = 0$ finishes the proof.

Now because of $\sum_{i=1}^n w_iv_i = 1$ there must be some constant term not in m and unital as such. So $v(0) \sim (1, 0, \dots, 0) \sim v$ as seen above. □

We now extend the idea of Horrock's theorem.

Lemma 1.31. *For an integral domain A and a multiplicative subset S if $v(x) \sim v(0)$ over $A_S[x]^n$ then there exists $c \in S$ such that $v(x + cy) \sim v(x)$ over $A[x, y]^n$*

Proof. By the equivalence $v(x) \sim v(0)$ we know there exists a matrix $M \in GL_n(R_S[x])$ such that $M(x)v(x) = v(0)$ now consider

$$N(x, y) = M(x)^{-1}M(x + y)$$

Note that now $N(x, y)v(x + y) = v(x)$ and so also $y \mapsto cy$ implies that $N(x, cy)v(x + cy) = v(x)$.

Now to show that indeed $N(x, cy) \in R[x, y]$ for some choice of $c \in S$ but this is true since $N(x, 0) = I_N \implies N(x, y) = I + yP$ for some $P \in R_S[x, y]$ but this just means there is some appropriate choice of $c \in S$ that allow us to cancel out all the denominators in P so that $P[x, cy] \in R[x, y]$. □

Lemma 1.32. *For an ID A and $v(x)$ unimodular row in $A[x]^n$ with at least one component having leading coefficient one implies $v(x) \sim v(0)$.*

Proof. Consider the set I containing all $c \in A$ such that $v(x + cy) \sim v(x)$ as rows in $A[x, y]$ if the ideal contains 1 then sending $x \to 0$ would give us $v(y) \sim v(0)$ in $A[y]$.

We can achieve this by first showing I is an ideal and then showing that its not contained in any maximal ideal. To do this last step we will localize at the maximals and use the previous result.

First prove that I is an ideal.

1. $I \neq \emptyset$ as $0 \in I$
2. If $c, d \in I$ then $c-d \in I$ as $v(x+(c-d)y) = v(x+cy-dy) \sim v(x+cy) \sim v(x)$ by a substitution $x \mapsto x + cy$
3. For $a \in A, c \in I$ then simply $v(x + cay) \sim v(x)$ by the $y \mapsto ay$

Now to show I isn't contained in any maximal ideal. Pick a maximal ideal $\mathfrak{m}$ and localize at it first due to Horrocks we know $v(x) \sim v(0)$ in $A_{\mathfrak{m}}[x]$ and then due to the previous lemma 1.31 we find some $c \in A \setminus \mathfrak{m}$ such that $v(x + cy) \sim v(x) \sim v(0)$ but this just means that $c \in I$ and so $I \not\subset \mathfrak{m}$ this applies to any maximal and so we are done. $\square$

Theorem 1.33. *For $A = k[x_1, \ldots, x_n]$ where k is a PID, then $v \sim (1, 0, \ldots, 0)$ for any unimodular row $v \in A^n$.*

Proof. Proceed with induction on n. We proved $n = 0$ above Prop. 1.28.

Assume $n \geq 1$ and that the result holds for $m - 1$.

Then $v \in k[x_1, \ldots, x_m] \cong k[x_1, \ldots, x_{m-1}][x_m]$ can be realized as $v(x_m)$ with coefficients in $k[x_1, \ldots, x_{m-1}]$. If $v(x_m)$ has some component with leading coefficient 1 then by Lemma 1.32 we now $v(x_m) \sim v(0) \in k[x_1, \ldots, x_{m-1}]$ and we can reduce by induction.

So if not by some appropriate change of variables as amongst $x_1, \ldots, x_{m-1}$ in the form of $x_i \mapsto x_i - x_m^{p_i}$ for very large p_i's this allows us obtain the leading coefficient in terms of x_m to be 1 as needed. $\square$

Theorem 1.34 (Quillen-Suslin)**.** *Finitely generated projective modules over $A = k[x_1, \ldots, x_n]$ where k is a PID are free.*

Proof. We know such f.g. proj. modules are stably free. And from above we know any unimodular row in A is equivalent to $(1, 0, \ldots, 0)$.

That is to say given a f.g. proj. module P which is stably free, i.e. $P \oplus R^{m_1} \cong R^{m_2}$ then P is free.

When $m_1 = 1$ this is the split exact sequence (since P is projective see A.2),

$$0 \to A \hookrightarrow A^{m_2} \twoheadrightarrow P \to 0$$

The injection $A \to A^{m_2}$ is precisely a unimodular row by definition which we know must correspond to the canonical embedding of $1 \mapsto (1, 0, \cdots, 0)$. So,

$$P = \mathrm{im}(A^{m_2} \to P) \cong A^{m_2}/\ker(A^{m_2} \to P) \cong A^{m_2}/\mathrm{im}(A \to A^{m_2}).$$

But $A^{m_2}/\mathrm{im}(A \to A^{m_2})$ is free since $\mathrm{im}(A \to A^{m_2})$ is naturally free due to the embedding.

When $m_1 \neq 1$ just take $(P \oplus A^{m_1-1}) \oplus A$. $\square$

1.3 Mennicke symbols

Definition 1.35 (Mennicke symbol)**.** *A Mennicke symbol is a map $\phi : \mathrm{Un}_n(A) \to G$ where G is a group such that,*

1. $\phi(1, 0, \ldots, 0) = 1, \phi(v) = \phi(vM)$ *for* $M \in E_n(A)$
2. $\phi(a, a_2, \ldots, a_n) \cdot \phi(b, a_2, \ldots a_n) = \phi(ab, a_2, \ldots, a_n)$ *if* $(a, a_2, \ldots, a_n), (b, a_2, \ldots, b_n) \in \mathrm{Um}_n(A)$.

1.4 Whitehead group K_1

Definition 1.36 (Whitehead group for a ring)**.** *K_1 for a ring A is defined as the abelianization of its infinite general linear group.*

$$K_1 := \frac{GL(A)}{[GL(A) : GL(A)]}$$

Where $GL(A)$ the infinite general linear group is the colimit of $GL_n(A)$ with GL_n realized as a subgroup of GL_{n+1} by placing the matrix in the top left corner.

Proposition 1.37.

$$[GL(A) : GL(A)] = E[A]$$

Proof. Using Lemma 1.42 we can see that

$$\begin{bmatrix} a^{-1}b^{-1} & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} b^{-1}a^{-1} & 0 \\ 0 & 1_n \end{bmatrix} \mod E_{2n}(A)$$

So the derived subgroup of $GL_n(A)$ is contained in $E_{2n}(A)$. $\square$

Definition 1.38 (Relative K_1)**.** $SK_1(A) := \ker \det$

Where, $\det : K_1(A) \to A^\times$*. We have a split exact sequence*

$$0 \to SK_1(A) \to K_1(A) \to A^\times \to 0$$

1.5 Some results on linear groups

Definition 1.39 (Elementary matrices). *We denote the elementary matrices as $E_n(A)$ generated by standard elementary matrices of the form $I_n + \lambda E_{ij}$ where E_{ij} is the matrix with 1 in the (i,j) entry and zero elsewhere. In shorthand notation we will write it as $e_{ij}(\lambda)$.*

Lemma 1.40. *A nonsingular triangular matrix with 1's in the diagonal is a product of standard elementary matrices.*

Proof. Let $A \in GL_n(A)$ then consider the following inductive procedure.

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & A_{n-1} & \\ 0 & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{n-1} & \\ 0 & & & \end{bmatrix} e_{12}(a_{12})e_{13}(a_{13}) \cdots e_{1n}(a_{1n})$$

Repeat the procedure for A_{n-1} to obtain

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & 0 & & A_{n-2} & \\ 0 & 0 & & & \end{bmatrix} \prod_{j=2}^{n} e_{2j}(a_{2j}) \prod_{i=1}^{n} e_{1i}(a_{1i})$$

□

Proposition 1.41. *Let A be a ring and $u \in A^\times$*

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \equiv I_2 \mod E_2(A)$$

Proof. $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} = e_{21}(u^{-1})e_{12}(1-u)e_{21}(-1)e_{12}(1-u^{-1})$. □

Lemma 1.42 (Whitehead)**.** *For $a, b \in GL_n(A)$*

$$\begin{bmatrix} ab & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \equiv \begin{bmatrix} ba & 0 \\ 0 & I_n \end{bmatrix} \mod E_{2n}(A)$$

Proof. Let $A = M_n(A)$ and note $E_2(M_n(A)) \subset E_{2n}(A)$ in Prop. 1.41. □

Lemma 1.43. *For E.D. A we have $SL_n(A) = EL_n(A)$ for all n.*

Proof. With elementary row and column operations arrange the matrix so that the element with the smallest norm is in the top right position. And using elementary row operations reduce it to a matrix with a unit in the top left and 0s in the rest of the first column and first row. Proceeding similarly for the remaining $n-1 \times n-1$ matrix left we reduce it down to a matrix of the form.

$$\begin{bmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & u_n \end{bmatrix}$$

Now apply Whiteheads lemma □

We now consider a result due to Suslin about the normality of $E_n(A)$ in $GL_n(A)$. The following Lemma due to Vaserstein will be useful.

Lemma 1.44 (Vaserstein)**.** *Let $a \in M_{m,n}(A)$ and $b \in M_{n,m}(A)$ then if $I_m + ab \in GL_m(A) \implies I_n + ba \in GL_n(A)$ and*

$$\begin{bmatrix} I_m + ab & 0 \\ 0 & (I_n + ba)^{-1} \end{bmatrix} \in E_{m+n}(A)$$

Proof. Note that $(I_n + ba)^{-1} = I_n - b(I_m + ab)^{-1}a$. Lem. 1.41 cannot be applied in this case since $n \neq m$ in general. But the idea is nearly the same.

$$\begin{bmatrix} I_m + ab & 0 \\ 0 & (I_n + ba)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I_m & 0 \\ (I_n + ba)^{-1} 1bI_n & \end{bmatrix} \begin{bmatrix} I_m & -a \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -b & I_n \end{bmatrix} \begin{bmatrix} I_m & (I_n + ab)^{-1}a \\ 0 & I_n \end{bmatrix} \in E_{m+n}(A)$$

We implicitly use Prop. 1.40 to justify that the triangular matrices there are indeed elementary. □

Theorem 1.45 (Suslin's Normality theorem)**.** *For A a commutative ring with unity, $E_n(A)$ normal in $GL_n(A)$ for $n \geq 3$.*

Proof. Let $a \in GL_n(A)$ consider $e_{ij}(\lambda) \in E_n(A)$ arbitrary. Recall from 1.27 that the columns of a and the rows of a^{-1} are unimodular.

$$ae_{ij}(\lambda)a^{-1} = I_n + \lambda c_i r_j$$

Where c_i is the i^{th} column of a and r_j is the j^{th} row of a^{-1}.

Furthermore since $a^{-1}a = I_n \implies b_j a_i = \delta_{ij} \implies$ using Prop. that $ae_{ij}(\lambda)a^{-1} = I_n + \lambda c_i r_j \in E_n(A)$ and since $E_n(A)$ is generated by matrices of the form $e_{ij}(\lambda)$ we are done. $\square$

Proposition 1.46 (Cohn)**.** *If $n = 2$ $E_2(A)$ need not be normal in $SL_2(A)$ content*

Theorem 1.47 (Suslin)**.** *Given $(x_1, \ldots, x_n) \in U_n(A)$ then $(x_1^{m_1}, \ldots, x_n^{m_n}) \in U_n(A)$ iff $(n-1)! | m_1 \dot{m}_2 \cdots m_n$*

1.6 Relationship between K_0 and K_1

Theorem 1.48 (Mayer-Vietoris)**.**

Theorem 1.49. *Let A be a ring and S denote a multiplicatively closed set of central elements in A. We obtain the following exact sequence*

$$K_1(A) \to K_1(S^{-1}A) \to K_0(A \text{ on } S) \to K_0(A) \to K_0(S^{-1}A)$$

2 Higher K theory

2.1 Quillen + Construction

2.2 Quillen Q construction

2.3 $+ = Q$

2.4 Waldhausen construction

Appendices

A Homological algebra

A.1 Projective modules

Recall a **free module** of rank n is one that is isomorphic to n direct sums of its underlying ring. And homomorphisms from free modules to other modules are determined by the image of their generators, i.e. free objects are left adjoints to forgetful functors. [2]

A module P is said to be **projective** if it satisfies the following lifting property, every morphism from P to N factors through an epi into N. Note that the lift need not be unique this is *not* an UMP

$$\begin{tikzcd} & M \arrow[d, two heads] \\ P \arrow[ur, dashed] \arrow[r] & N \end{tikzcd}$$

Lemma A.1 (Free modules are projective)**.**

Proof. Consider the preimages of images of basis of P in N, that lie in M. Then map basis elements from P into these preimages. $\square$

[2]This holds in free monoids $\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$ where $F(X)$ denotes the free monoid generated by elements from the set X and $U(M)$ is the underlying set of a monoid M, refer to [Awo10, p. 208]

Proposition A.2 (Equivalent definitions of projectivity)**.** *TFAE,*

1. *P is projective.*
2. *For all epi's between $M \twoheadrightarrow N$, the induced map $\mathrm{Hom}(P, g) : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$ sending $f \mapsto g \circ f$ for $g : M \to N$ and $f : P \to M$ is an epi.*
3. *For some epi from a free module F to P, $\mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$ is an epi.*
4. *There exists Q s.t. $P \oplus Q$ is free*
5. *Short exact sequences of the form $0 \to A \to B \to P \to 0$ split, i.e. isomorphic to another short exact where middle term is $A \oplus P$* [3]

Proof. $1 \iff 2$ is restatement of definitions.

$2 \implies 3$ also just substitution.

$3 \implies 4$ consider a map in the preimage of identity in $\mathrm{Hom}(P, P)$ which is a splitting (inverse) of the epi F into P,

$$\begin{tikzcd}
& P \arrow[dl, "f"'] \arrow[dr, dashed, two heads, "\mathrm{Id}_P = g \circ f"] \arrow[d, phantom, "\Longrightarrow"] & \\
F \arrow[rr, two heads, "g"] & & P
\end{tikzcd}$$

Now we have a short exact sequence $0 \to \ker g \to F \to P \to 0$, and also $f \circ g$ is idempotent so it naturally admits a decomposition $F = \mathrm{Im}(f \circ g) \oplus \mathrm{Ker}(f \circ g)$[4]$=\mathrm{Im}(g) \oplus \mathrm{Ker}(g)$ the first by the 1st isomorphism theorem and the second by f being a mono.

$4 \implies 2$ simply as $\hom(P \oplus Q, -) = \hom(P, -) \oplus \hom(Q, -)$

$1 \iff 5$ should be clear from above.

□

Theorem A.3 (Proj. fin. generated modules over local rings are free)**.**

[3] In general any epis into projective objects split (i.e. have an inverse).

[4] For some idempotent e, $1 - e$ is also an idempotent and images under these two mappings decompose any module, furthermore image of $1 - e$ is just kernel of e

Proof. pick a minimal set of generators and see its residue classes in $M/\mathfrak{m}M$ as the basis of it as a vector space over $R/\mathfrak{m}$.

Now as for some free module $F, F = \varphi(M) \oplus K$ for some K and some homomorphism $\varphi : M \to F$, (by defn of projective module), we get

$$M/\mathfrak{m}M \cong F/\mathfrak{m}F = (R/\mathfrak{m})^n \cong R^n \otimes R/\mathfrak{m} \cong F \otimes R/\mathfrak{m} \cong (\varphi(M) \oplus K) \otimes R/\mathfrak{m}$$

Finally we get $M/\mathfrak{m}M \cong M/\mathfrak{m}M \oplus K/\mathfrak{m}K \implies K = \mathfrak{m}K \implies K = 0$ by Nakayama ☐

This holds for not necessarily f.g. modules too refer to [Mat87, Th. 2.5] .

Using the convention of [Lam99] we define the rank of a projective module as such.

Definition A.4 (Rank of a f.g. projective module)**.** *For any f.g. projective module P over commutative ring A the localization $P_\mathfrak{p} = P \otimes_A A_\mathfrak{p}$ is also a f.g. $A_\mathfrak{p}$ module. But $P_\mathfrak{p}$ being local is free by Th. A.3. So the local rank of P is defined as the rank of the free $P_\mathfrak{p}$ module.*

This induces a map $\phi : \mathrm{Spec}(A) \to \mathbb{Z}$ sending each $\mathfrak{p}$ to the local rank of P. If ϕ is constant and the rank of P is the same for all localizations then we refer to that as the rank of P.

Proposition A.5. *If M is a finitely presented module over a Noetherian ring R (prime ideals fin gen) then TFAE*

1. *M is projective.*
2. *M localized at maximal ideals is free.*
3. *A finite set of elements $\{x_i\}^n$ in R generate R such that $M[x_i^{-1}]$ is free over $R[x_i^{-1}]$.*

This proceeds just from the previous result.

Lemma A.6. *A is Noetherian iff all submodules of f.g. A modules are f.g..*

Proposition A.7. *For a PID A a submodule of a free module of finite rank is free.*

A.2 Resolutions

Given a module M its **left resolution** is given by the data of a exact sequence $(A_\bullet, \varphi_\bullet)$ into M as such,

$$\cdots \to A_1 \to A_0 \xrightarrow{\epsilon} M \to 0$$

where ϵ is called the **augmentation map**, if the exact sequence is free its a free resolution and such for projective.

If we have a cochain complex instead it forms a **right resolution** and if its elements are injective we call them injective resolutions.

Proposition A.8 (Horseshoe lemma). *If there is a short exact sequence of modules,*

$$0 \to M \to N \to P \to 0$$

and both M, P have a projective resolutions A, C

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \\
 & & & N \arrow[d] & \\
\cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & P \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

as below then N also has a projective resolution B which forms a short exact sequence. Also the sequence splits due to C_i being projective so $B_i = A_i \oplus C_i$.

Proof. First note $\epsilon_P : C_0 \to P$ lifts due to projectivity to $C_0 \to N$ also $A_0 \to N$ via composition so simply define $B_0 = A_0 \oplus C_0$. This is an epi evidently via diagram chase. Also is projective as direct sum of projectives is projective. Now consider direct sum of kernel of $A_0 \to M, B_0 \to N, C_0 \to P$ and construct the direct sum again to get F_1. Now we get a 3×3. Exactness is due to the Snake lemma $\square$

A.3 Puppe/Homotopy cofiber sequence

B Vector bundles

More detailed exposition can be found in [MS74]. We define the basics as needed for Swam's theorem. We understand all maps as continuous functions.

Definition B.1 (Vector bundle)**.** *A real n dimensional vector bundle is a triple (E, p, B). Which consists of a continuous map $p : E \to B$ from the total space E to the base space B. Such that for all $b \in B$, $F_b = p^{-1}(b)$ the fibre of b has a real/complex vector space structure. Along with the following property of local trivialization*

1. *For any $b \in B$ there exists a open open $U \subset B$ along with a homeomorphism*
$$h : U \times \mathbb{R}^n \to p^{-1}(U)$$
such that for all $c \in U$ the map through h defines an isomorphism between F_c and $\mathbb{R}^n$.

A trivial vector bundle is one in which the total space $E = B \times \mathbb{R}^n$ with p just the trivial projection mapping.

Definition B.2 (Vector bundle isomorphisms)**.** *Two vector bundles (E_1, p, B) and (E_2, p_2, B) are considered isomorphic if there exists a homeomorphism between their total spaces h such that the below diagram commutes*

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

and also if h induces a vector space isomorphism for each fibre.

Definition B.3 (Sections of a vector bundle)**.** *For a topological vector bundle (E, p, B) a section refers to a map $s : B \to E$ such that $p \circ s = 1_B$ where 1_B denotes the identity map on B.*

These sections equivalently are a homomorphism of vector bundles from the trivial line bundle $(B \times \mathbb{R}, \pi_B, B) \to (E, p, B)$

C Categories

C.1 Abelian Categories (shorten this too many example)

There is a chain of conditions regarding 'abelian'-ness of categories which is roughly understood as follows,

$$\textbf{Abelian} \subseteq \textbf{Pre-Abelian} \subseteq \textbf{Additive} \subseteq \textbf{Ab-Enriched}$$

The motivation behind them is to have categories which resemble algebras.

Ab-Enriched categories are categories such that for objects $A, B \in \mathbf{C}$ the external hom set $\mathrm{Hom}(A, B)$ has the structure of an abelian group, furthermore it has a well defined notion of composition (which is bilinear due to the monoidal product in Ab), $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$.

Proposition C.1. *In Ab-Enriched categories intial and terminal objects coincide (it is often called the zero object)*

Proof. Let $\mathbf{C}$ be an Ab-Enriched category. Note that the Hom-sets between objects have 'zero morphisms', i.e. arrows in the Hom-set which behave like the additive identity in the Ab group induced by it. In particular for $0_{A,B} \in \mathrm{Hom}(A, B)$ we have the property that if $f : B \to C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \to D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathbf{C}$ is initial so there is a unique morphism $0 \to 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \to 0$ we can say that by the zero morphism property $f = 0$ so also 0 is terminal. □

Proposition C.2. *In Ab-Enriched categories finite coproducts coincide with finite products (i.e. biproducts)* [5]

Proof. Let $\mathbf{C}$ be an Ab-enriched category and $A, B \in \mathbf{C}$ consider the product $A \times B$, which is determined by the following UMP,

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \end{tikzcd}$$

[5] This also holds over categories enriched over commutative monoids.

Consider A and B in place of X in the diagram. By the UMP we have $q_1 : A \to A \times B, q_2 : B \to A \times B$

$$\begin{tikzcd}
A \arrow[dd, "1_A"'] \arrow[dr, "q_1"'] & & B \arrow[dl, "q_2"'] \arrow[dd, "1_B"] \\
& A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

So $p_1 q_1 = 1_A$ and $p_2 q_2 = 1_B$ also $p_1 q_2 = p_2 q_1 = 0$.

Now note that $q_1 p_1 + q_2 p_2 = 1_{A \times B}$ as $p_1(q_1 p_1 + q_2 p_2) = p_1$ and $p_2(q_1 p_1 + q_2 p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following UMP holds for some arbitrary $C \in \mathbf{C}$

$$\begin{tikzcd}
A \arrow[r, "r_1"] \arrow[dd, "1_A"'] \arrow[dr, "q_1"'] & C & B \arrow[l, "r_2"'] \arrow[dl, "q_2"] \arrow[dd, "1_B"'] \\
& A \times B \arrow[u, dashed, "f"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

Define $f : A \times B \to C$ as $f = r_1 p_1 + r_2 p_2$. Now $f q_1 = r_1$ and $f q_2 = r_2$ if we show uniqueness of f we are done.

Say f' then $(f - f')1_{A \times B} = (f - f')(q_1 p_1 + q_2 p_2) = 0$. So $f = f'$.

□

Definition C.3 (Additive category)**.** *An Ab-Enriched category which has all finite coproducts.*

Functors between additive categories are called *additive functors*. And can be realized as functors which preserve additivity of homomorphisms between modules, $F(f + g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition C.4 (Kernel)**.** *A kernel is a pullback of a morphism* $f : A \to B$ *and the unique morphism from* $0 \to B$*. Provided initials and pullbacks exist.*

$$\begin{array}{ccc} \ker f & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \to B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'. [6]

Definition C.5 (Pre-abelian categories). *An additive category with all morphism having kernels and cokernels.*

The above definition is equivalent to saying a pre-abelian category is a Ab-Enriched category with all finite limits and colimits. This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Prop. 5.21]. And we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition C.6 (Abelian category). *Pre-abelian categories for which each mono is a kernel and each epic is a cokernel.*

Largely the purpose of abelian categories were motivated by wanting to generalize homological methods and to unify various (co)homology theories. It was defined in the modern formulation by Grothendieck in his Tohuku paper [**?**]. We never directly reference this paper for its mathematical content but it is interesting from a historical perspective.

C.1.1 Examples

Some examples of abelian categories are as follows,

1. **The category of modules.**

2. **Category of representations of a group**

[6] A minor point to note is that in the case of Ab-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a Ab-Enriched category has a initial 0 however this matches up with our intuition.

3. **Category of sheaves of abelian groups on some topological space.**

   Definition C.7 (Presheaf)**.** *For a category C a presheaf is any functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$.*

   In particular in the case for a topological space X a presheaf of groups (or any algebraic object) on X (in truth the set of the lattice of open sets of X ordered by inclusion) is a some contravariant functor F which sends open sets $U \subseteq X$ to some $F(U)$, it respects inclusions (i.e. there for open sets $V \subseteq V$ is a natural transformation $\rho_{UV} : F(U) \to F(V)$ in the form of a restriction). Furthermore, function composition, unitals and empty sets going to empty sets hold (to make it a category). Note that all these notions of presheaves are really just a special case of the categorical definition where the sheaf of groups is really just a group object in the categorical presheaf.

   Definition C.8 (Sheaf of sets on a topology)**.** *A sheaf of a topology X is a presheaf which satisfies two additional properties, for open sets $U \in X$ and open covers U_i of U*

   (a) (Locality) ***A section****, i.e. an element $s \in F(U)$ goes to zero restricted at U_i for all i implies $s = 0$.*

   (b) (Gluing) If there is a collection of sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j then there is some $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

   These two conditions can be written is short as just saying we require $F(U)$ to be the equalizer for the following diagram

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

   Now finally we get back to the original example. The category of sheaves of abelian groups on a topological space form a abelian category. Additivity is natural due to the functorial nature of F. A slightly unsatisfying proof is due to 'sheafification', i.e. the left adjoint to the inclusion functor from sheaves into presheaves. Presheves of abelian groups can be understood to have all the required properties to be an Abelian category due the functorial representation. Now due to the following result [Sta23] we can extend this notion to the sheaves via sheafification.

C.2 Derived categories

C.3 Exact categories

An exact category (sometimes referred to as a Quillen exact category) is a pair (C, E) for C an additive category which is a full subcategory of some abelian category $\mathcal{A}$. Along with a family of sequences E of the form

$$0 \to X \to Y \to Z \to 0$$

which are short exact sequences in $\mathcal{A}$ and if in a sequence of the above form $X, Z \in C$ then $Y \cong Y_C \in \mathrm{Ob}(C)$

Example C.9. *Every abelian category is trivially exact over itself.*

Example C.10. *The vector bundles on a scheme X form a exact category.*

C.4 Factorization systems, Model categories

C.5 Triangulated categories

Add examples from Puppe sequence discussion from homological alg notes they form the triangulation in the case of the stable homotopy category.

Also include quillen model cats somewhere in between.

C.6 Waldhausen categories

Every exact category has a Waldhausen structure.

D Stable homotopy and Spectra (relevant or not? in top k yes in alg k ?)

The definitions in this section are mainly using the convention in Adam's blue book [Ada74].

Definition D.1 (CW-Spectrum)**.** *A sequence of based CW-complexes $\{E_n\}_{n\in\mathbb{Z}}$ with structure maps $\sum E_n \to E_{n+1}$*

The suggestive notation is pointing in the direction of the canonical spectrum known as the suspension spectrum when the structure maps are indeed suspensions.

A natural question to ask is why can we not simply define the morphism as an obvious map translating between spectra (up to a degree shift) that respects the suspension functors through a obvious commuting square. Such a construction would fail due to the existence of nontrivial maps such as the Hopf fibration.

Consider the suspension spectra of spheres denoted as $\mathbf{S}$ the sphere spectrum.

Theorem 2.2 (Brown representability)**.** *Some nice contravariant functors on homotopy category are representable.*

Theorem 2.3. *Extraordinary (co)homology theories are representable by a spectrum.*

K theory spectrum

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