Contents

Lower Algebraic K theory with a view towards Higher K theory

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This project focuses on a modern approach to lower K-theory, highlighting key theorems in classical K-theory while developing the essential theoretical foundations required to approach higher K-theory in the subsequent project.

Unless otherwise specified, all rings considered are assumed to be commutative with unity.

1 Abelian categories and homological algebra

We begin with some basic prerequisites of abelian categories and homological algebra, which provide a foundational framework. These concepts will be used extensively and implicitly throughout both the projects, forming an assumed background for the discussions. So for completeness sake we will begin by examining it in detail.

1.1 Abelian Categories

Abelian categories are essential to the understanding of homological algebra. It is motivated by the fact that it allows for using homological methods in a wide variety of applications and helps unify various (co)homology theories. They were first introduced by Grothendieck in his seminal Tohuku paper [Gro57].

There is a chain of conditions regarding 'abelian'-ness of categories which is roughly understood as follows.

$$\textbf{Abelian} \subseteq \textbf{Pre-Abelian} \subseteq \textbf{Additive} \subseteq \textbf{Ab-Enriched}$$

Ab-Enriched categories (sometimes referred to as pre-additive categories) are categories such that for $A, B \in \mathcal{C}$ the external hom set $\mathrm{Hom}(A, B)$ has the structure of an abelian group, furthermore it has a well defined notion of composition (which is bilinear due to the monoidal product in Ab), $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$.

We have chosen to omit the precise definitions of the coherence conditions for monoidal and monoidally enriched categories to make this section easier to read. Since we refrain from explicitly using them for computations anywhere, the basic background described above will suffice. For a more detailed overview of the definitions for monoidal and monoidally enriched categories refer to [Lan98] for a classical treatment or [Rie17] for an excellent modern exposition.

Example 1.1. *A ring is a single object* Ab-*Enriched category (In the same sense how a group is realized as a single object category with all arrows invertible).*

We cover a few basic results.

Proposition 1.2. *In Ab-Enriched categories intial and terminal objects coincide (it is often called the zero object)*

Proof. Let $\mathcal{C}$ be an Ab-Enriched category. Note that the Hom-sets between objects have 'zero morphisms', i.e. arrows in the Hom-set which behave like the additive identity in the Ab group induced by it. In particular for $0_{A,B} \in \mathrm{Hom}(A, B)$ we have the property that if $f : B \to C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \to D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathcal{C}$ is initial so there is a unique morphism $0 \to 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \to 0$ we can say that by the zero morphism property $f = 0$ so also 0 is terminal. □

Proposition 1.3. *In Ab-Enriched categories finite coproducts coincide with finite products (i.e. biproducts)* [1]

[1] This also holds over categories enriched over commutative monoids.

Proof. Let $\mathcal{C}$ be an Ab-enriched category and $A, B \in \mathcal{C}$ consider the product $A \times B$, which is determined by the following UMP,

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

Consider A and B in place of X in the diagram. By the UMP we have $q_1 : A \to A \times B, q_2 : B \to A \times B$

$$\begin{tikzcd}
A \arrow[dd, "1_A"'] \arrow[dr, "q_1"] & & B \arrow[dl, "q_2"'] \arrow[dd, "1_B"] \\
 & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

So $p_1 q_1 = 1_A$ and $p_2 q_2 = 1_B$ also $p_1 q_2 = p_2 q_1 = 0$.

Now note that $q_1 p_1 + q_2 p_2 = 1_{A \times B}$ as $p_1(q_1 p_1 + q_2 p_2) = p_1$ and $p_2(q_1 p_1 + q_2 p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following UMP holds for some arbitrary $C \in \mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "r_1"] \arrow[dd, "1_A"'] \arrow[dr, "q_1"] & C & B \arrow[l, "r_2"'] \arrow[dl, "q_2"'] \arrow[dd, "1_B"] \\
 & A \times B \arrow[u, dashed, "f"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

Define $f : A \times B \to C$ as $f = r_1 p_1 + r_2 p_2$. Now $f q_1 = r_1$ and $f q_2 = r_2$ if we show uniqueness of f we are done.

Say f' then $(f - f')1_{A \times B} = (f - f')(q_1 p_1 + q_2 p_2) = 0$. So $f = f'$.

$\square$

Definition 1.4 (Additive category)**.** *An Ab-Enriched category which has all finite (co)products (i.e. biproducts since they coincide).*

Example 1.5. *The category of vector bundles over a topology X is a additive category (but not an abelian category). We will see this in more detail in the following section.*

Functors between additive categories are called *additive functors*. And can be realized as functors which preserve additivity of homomorphisms between modules, $F(f+g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition 1.6 (Kernel)**.** *A kernel of a morphism $f : A \to B$ is the pullback along the unique morphism from $0 \to B$, i.e. it is $p : \ker f \to A$. Provided initials and pullbacks exist.*

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \to B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'. [2]

A cokernel is simply its dual.

Definition 1.7 (Pre-abelian categories)**.** *An additive category with all morphism having kernels and cokernels.*

The above definition is equivalent to saying a pre-abelian category is a Ab-Enriched category with all finite limits and colimits. This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Prop. 5.21]. And we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition 1.8 (Abelian category)**.** *Pre-abelian categories for which each monomorphism is a kernel and each epimorphism is a cokernel.*

Equivalently a category is abelian if its pre-abelian and $\bar{f}$ is an isomorphism in the canonical decomposition of $f : A \to B$ as

$$A \twoheadrightarrow \operatorname{coker} \ker f \xrightarrow{\bar{f}} \ker \operatorname{coker} f \hookrightarrow B.$$

[2] A minor point to note is that in the case of Ab-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a Ab-Enriched category has a initial 0 however this matches up with our intuition.

Example 1.9. *Some non examples are:*

1. *The category of hausdorff topological abelian groups is pre-abelian but not abelian. Since not every morphism which is a mono+epi is necessarily a isomorphism.*

   Consider a Hausdorff abelian topological group G with a non discrete topology and consider G' it's discretization. The inclusion map $G' \to G$ is a mono+epi but not isomorphic.

2. *The category of torsion free abelian groups (*TFAb*) is pre-abelian but not abelian as the mono $f : \mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}$ is not a kernel of some morphism. Say it were and there exists $A \in$ TFAb such that f is the kernel to $g : \mathbb{Z} \to A$, i.e.*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f(z)=2z"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ Z \arrow[r, "g"] & A \end{tikzcd}$$

   But this implies $1_{\mathbb{Z}}$ factors through f i.e. $1_{\mathbb{Z}} = f \circ h$ for some unique $h : \mathbb{Z} \to \mathbb{Z}$ which implies $h(1) = 1/2$ which is absurd.

Example 1.10. *Some examples of abelian categories:*

1. *The category of modules.*

2. *Category of representations of a group*

3. *Category of sheaves of abelian groups on some topological space.*

With this definition in mind we will now define a few important constructions we will use often. These are not restricted to abelian categories but we will use them very often in the case of abelian categories, so it is good to see it in action directly with the notion of an abelian category at hand.

Definition 1.11 (Subobject)**.** *A subobject for some $X \in \mathcal{C}$ is a monomorphisms into X.*

With slight abuse of notation we refer to $Y \leq X$ as a subobject of X where Y is just a representative of the codomain of a isomorphism class of monomorphisms into X. In particular for$X, Z \rightrightarrows X$ monics Z, X belong to the same subobject class if the morphisms are isomorphic, i.e. there exists an isomorphism between $Y \to Z$ making the triangle commute.

This is clearer when seen through the lens of a slice category. Note that arrows between subobjects of the same X are arrows in the slice category of X. So collection of subobjects form a category with a preorder (with inclusion). The reasoning behind such an definition for subobjects is motivated by the fact that we think of generalized elements in $\mathcal{C}$ as being not $X \in \mathrm{Ob}(\mathcal{C})$ but rather $\mathrm{Hom}_{\mathcal{C}}(-, X)$.

Definition 1.12 (Quotients in abelian categories)**.** *For $Y \leq X$ in an abelian category we can define X/Y as the cokernel of the monomorphism $Y \to X$.*

Definition 1.13 (Extension/short exact sequences in abelian categories)**.** *For $A, B \in \mathcal{A}$ an extension by A of B refers to some $E \in \mathcal{A}$ such that $0 \to A \to E \to B \to 0$ is a short exact sequence.*

A deep result on abelian categories is the Freyd-Mitchel embedding theorem which helps characterize all small abelian categories in terms of modules.

Theorem 1.14 (Freyd-Mitchell)**.** *Every small abelian category can be faithfully embedded as a full subcategory via an exact functor into $R-Mod$ for some ring R.*

The proof for the theorem is very extensive and as such is omitted. The canonical reference is Freyd's own book [FF64]. A proof sketch summarising Freyd's proof is given in an excellent MathOverflow post by the user Theo Buehler [Bue].

1.2 Chain complexes

In this section we define and prove the essential homological algebra results that we require. For further details refer to [Eis13, Wei94]. All of the results below stated for modules over rings apply for abelian categories. The proofs performed via diagram chases are well defined under the Freyd-Mitchel embedding on the full subcategory of the given diagram only.

Definition 1.15 (Chain complex)**.** *A chain complex $(A_\bullet, \varphi_\bullet)$ is a collection of modules over a commutative ring and homomorphisms $\varphi_i : A_i \to A_{i-1}$ such that $\varphi_i \varphi_{i+1} = 0$,*

$$\cdots \xrightarrow{\varphi_{i+2}} A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \xrightarrow{\varphi_{i-1}} \cdots$$

Definition 1.16 (Chain (Co)Homology)**.** *The homology of the complex at F_i is denoted as its i^{th} homology defined as follows,*

$$H_i A := \ker \varphi_i / \mathrm{im} \varphi_{i+1}$$

Reversing the arrows gives us the analogous definitions for cochain complexes and cohomology.

The homomorphisms are often called 'boundary operators' or 'differentials'. This nomenclature is motivated by de Rahm cohomology. Furthermore elements of $\ker \varphi_i$ are called 'cycles' and elements of $\mathrm{im}\varphi_{i+1}$ are called boundaries, this echoes the aphorism 'cycles modulo boundaries' often encountered in singular homology.

Definition 1.17 (Exact sequence)**.** *A chain complex is said to be exact if all its homologies are zero. In particular it is exact at one object if its homology there is zero.*

Definition 1.18 (Chain homotopy)**.** *If α, β are maps between differential modules $(A, \varphi), (B, \psi)$ then α is homotopically equivalent to β if there is a map $h : A \to B$ such that $\alpha - \beta = \psi h + h\varphi$. If grading is relevant the picture formed is as such, we require a family of maps $h_i : A_i \to B_{i+1}$* [3]

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "\varphi_i"] \arrow[dl, "h_i"'] \arrow[d, shift right, "\beta_i"'] \arrow[d, shift left, "\alpha_i"] & A_{i-1} \arrow[r] \arrow[dl, "h_{i-1}"'] \arrow[d, shift right, "\beta_{i-1}"'] \arrow[d, shift left, "\alpha_{i-1}"] & \cdots \arrow[dl, "h_{i-2}"'] \\
\cdots & B_i \arrow[l] \arrow[r, "\psi_i"'] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

The intution behind this particular choice of definition is that the map $\alpha - \beta$ maps all cycles to boundaries which have zero homology. So really $\alpha - \beta$ is null homotopic, as such this relation is an equivalence relation.

Definition 1.19 (Quasi-isomorphism)**.** *A chain map is called a quasi isomorphism if the induced map on the homologies constituents an isomorphism.*

The reason for 'quasi' is that the relation is reflexive and transitive but not symmetric.

[3] i.e. It has degree 1, sometimes the subscript is dropped and just treated as h

Definition 1.20 (Homotopy category of chain complexes)**.** *For a given category of chain complexes* $\mathrm{Ch}(\mathcal{A})$ *we can define* $\mathcal{K}(\mathcal{A})$ *to be the homotopy category of chain complexes with objects as objects of* $\mathrm{Ch}(\mathcal{A})$ *and arrows as chain homotopic maps as in Def.1.18.*

1.3 Projective modules

The category of finitely generated projective modules is the main object of study in algebraic K-theory. This is largely motivated by a theorem due to Swan's which we will prove in the next section.

Recall a the definition of a free module.

Definition 1.21 (Free module of rank n)**.** *A free module of of rank* n *is some is isomorphic to* n *direct sums of its underlying ring.*

In particular this means that there exists a linearly independent spanning set of the module with n elements.

And homomorphisms from free modules to other modules are determined by the image of their generators, i.e. free objects are left adjoints to forgetful functors. [4]

Definition 1.22 (Projective module)**.** *A module* P *is said to be* ***projective*** *if it satisfies the following lifting property, every morphism from* P *to* N *factors through an epi into* N*. Note that the lift need not be unique this is not an UMP*

$$\begin{tikzcd} & M \arrow[d, two heads] \\ P \arrow[ur, dashed] \arrow[r] & N \end{tikzcd}$$

Proposition 1.23 (Equivalent definitions of projectivity)**.** *TFAE,*

1. P *is projective.*

2. *For all epi's between* $M \twoheadrightarrow N$*, the induced map* $\mathrm{Hom}(P, g) : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$ *sending* $f \mapsto g \circ f$ *for* $g : M \to N$ *and* $f : P \to M$ *is an epi.*

[4] This holds in free monoids $\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$ where $F(X)$ denotes the free monoid generated by elements from the set X and $U(M)$ is the underlying set of a monoid M, refer to [Awo10, p. 208]

3. *For some epi from a free module F to P,* $\mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$ *is an epi.*

4. *There exists Q s.t. $P \oplus Q$ is free*

5. *Short exact sequences of the form* $0 \to A \to B \to P \to 0$ *split, i.e. isomorphic to another short exact where middle term is $A \oplus P$* [5]

Proof. $1 \iff 2$ is restatement of definitions.

$2 \implies 3$ is also just substitution.

$3 \implies 4$ consider a map in the preimage of identity in $\mathrm{Hom}(P, P)$ which is a splitting (inverse) of the epi F into P,

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, dashed, two heads, "\mathrm{Id}_P = g \circ f"] \arrow[d, phantom, "\Longrightarrow"] & \\ F \arrow[rr, two heads, "g"] & & P \end{tikzcd}$$

Now we have a short exact sequence $0 \to \ker g \to F \to P \to 0$, and also $f \circ g$ is idempotent so it naturally admits a decomposition $F = \mathrm{Im}(f \circ g) \oplus \mathrm{Ker}(f \circ g)$[6]$=\mathrm{Im}(g) \oplus \mathrm{Ker}(g)$ the first by the 1st isomorphism theorem and the second by f being a mono.

$4 \implies 2$ simply as $\mathrm{Hom}(P \oplus Q, -) = \mathrm{Hom}(P, -) \oplus \mathrm{Hom}(Q, -)$

$1 \iff 5$ To show that $0 \to A \to B \xrightarrow{\varphi} P$ splits we need to show that there exists a $\psi : P \to B$ such that $\varphi \circ \psi = 1_P$. But this is just obtained by the definition of P being projective.

$$\begin{tikzcd} & M \arrow[d, "\varphi"] \\ P \arrow[ur, dashed, "\psi"] \arrow[r, "="] & P \end{tikzcd}$$

□

Lemma 1.24 (Free modules are projective)**.**

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[5] In general any epis into projective objects split (i.e. have an inverse).

[6] For some idempotent e, $1 - e$ is also an idempotent and images under these two mappings decompose any module, furthermore image of $1 - e$ is just kernel of e

Proof. Consider the preimages of images of basis of P in N, that lie in M. Then map basis elements from P into these preimages. $\square$

Example 1.25 (Projectives are not always free)**.** *Let R, S be two nontrivial commutative rings with unity, consider $R \oplus S = M$ as a (free) module over itself. $p_1(M) = R \oplus \{0\}$ is projective as it is a direct summand of M. But it is clearly not free as for any supposed basis element b we have $(1,1)b = p_1(1,1)b = 1b = p_1(1,0)b = (1,0)b$ which is absurd due to the uniqueness of scalars in a free module.*

Theorem 1.26. *Proj. fin. generated modules over local rings are free*

Proof. Pick a minimal set of generators and see its residue classes in $M/\mathfrak{m}M$ as the basis of it as a vector space over $R/\mathfrak{m}$.

Now as for some free module $F, F = \varphi(M) \oplus K$ for some K and some homomorphism $\varphi : M \to F$, (by defn of projective module), we get

$$M/\mathfrak{m}M \cong F/\mathfrak{m}F = (R/\mathfrak{m})^n \cong R^n \otimes R/\mathfrak{m} \cong F \otimes R/\mathfrak{m} \cong (\varphi(M) \oplus K) \otimes R/\mathfrak{m}$$

Finally we get $M/\mathfrak{m}M \cong M/\mathfrak{m}M \oplus K/\mathfrak{m}K \implies K = \mathfrak{m}K \implies K = 0$ by Nakayama $\square$

This holds for not necessarily f.g. modules too refer to [Mat87, Th. 2.5] .

Using the convention of [Lam99] we define the rank of a projective module as such.

Definition 1.27 (Rank of a f.g. projective module)**.** *For any f.g. projective module P over commutative ring A the localization $P_\mathfrak{p} = P \otimes_A A_\mathfrak{p}$ is also a f.g. $A_\mathfrak{p}$ module. But $P_\mathfrak{p}$ being local is free by Th. 1.26. So the local rank of P is defined as the rank of the free $P_\mathfrak{p}$ module.*

This induces a map $\phi : \mathrm{Spec}(A) \to \mathbb{Z}$ sending each $\mathfrak{p}$ to the local rank of P. If ϕ is constant and the rank of P is the same for all localizations then we refer to that as the rank of P.

Proposition 1.28. *For a PID A a submodule M of a free module of finite rank say A^n is free, and the submodule has rank $\leq n$.*

Proof. We prove this by induction on n. When $n = 0$ there is nothing to prove. For $n = 1$ due to the fact that A is a PID the submodules of A (ideals) are one generated i.e. they are rank 1 free modules of A.

Proceed via induction. Now consider the case when $n = k$.

Let $M \subset A^k$ be non zero. Consider the componentwise projection maps $p_i : A^k \to A$ for each i. Then $\pi_i(M) \neq \{0\}$ for some i. Therefore $p_i(M)$ is a non-zero ideal in A, i.e. free with rank 1. Also, $\ker p_i \cap M$ is a submodule of $\ker p_i$ which is itself free of rank $n-1$. Therefore rank of $\ker p_i \cap M$ is $\leq n-1$. Let a be a generator for $p_i(M)$ consider some preimage of it as a_p.

Now $M = \ker p_i \cap M \oplus \langle a_p \rangle$. If $\{a_1, a_2, \cdots a_m\}$ is a basis of $\ker p_i \cap M$, then $\{a_1, a_2, \cdots a_m, a_p\}$ is a basis of M. Hence rank of M equals $m + 1 \leq n$. □

Proposition 1.29. *Projective f.g. modules over PIDs are free*

Proof. Every f.g. projective module P is a direct summand of a free module F meaning it is a submodule of F and by Prop. 1.28 it is free. □

Definition 1.30 (Stably isomorphic)**.** *Two $A-$modules M, N are said to be stably isomorphic if there exists r such that $M \oplus A^r \cong M \oplus A^r$.*

Definition 1.31 (Stably free module)**.** *An A module M is stably free if there exists a finitely generated free module F such that $M \oplus F$ is free, i.e. if M is stably isomorphic to a finitely generated free A module.*

We shall see an example of a stably free module that is not free in the later section. As we shall use the terminology of a unimodular row.

1.4 Long exact sequence of homologies

Consider $(A, \varphi), (B, \psi), (C, \chi)$ to be chain complexes we can define a short exact sequence of complexes as

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

For α, β maps of complexes as discussed above, and $\beta\alpha = 0$, if for all i the underlying sequence of modules is exact

$$0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$$

These maps also induce maps on the homologies $\alpha_i : H_iA \to H_iB, \beta_i : H_iB \to H_iC$. Furthermore there is a natural map

$$\delta_i : H_iC \to H_{i-1}A$$

which is called the **connecting homomorphism**

Before seeing how to construct this δ it is useful to have a complete picture of the data in front of us. This can be seen below,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_i \arrow[r, "\alpha_i"] \arrow[d, "\varphi_i"'] & B_i \arrow[r, "\beta_i"] \arrow[d, "\psi_i"'] & C_i \arrow[r] \arrow[d, "\chi_i"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d, "\varphi_{i-1}"'] & B_{i-1} \arrow[r, "\beta_{i-1}"] \arrow[d, "\psi_{i-1}"'] & C_{i-1} \arrow[r] \arrow[d, "\chi_{i-1}"'] & 0 \\
0 \arrow[r] & A_{i-2} \arrow[r, "\alpha_{i-2}"] \arrow[d] & B_{i-2} \arrow[r, "\beta_{i-2}"] \arrow[d] & C_{i-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We construct via a diagram chase. Suppose $h \in H_iC = \ker \chi_i / \mathrm{Im}\chi_{i+1}$ pick a cycle $x \in \ker \chi_i$. As β_i is surjective we know there exists $y \in B_i$ s.t. $\beta_i(y) = x$. Now also by the fact that $x \in \ker \chi_i$ and that we have maps between chain complexes so the squares commute. We have that $\beta_{i-1}(\psi_i(y)) = \chi_i(\beta_i(y)) = \chi_i(x) = 0$.

Now there is some $z \in A_{i-1}$ such that $\alpha_{i-1}(z) = \psi_i(y)$ (this is due to exactness of $i-1$ sequence hence the quotient isomorphism and the above condition).

As α_{i-2} is a monomorphism $\alpha_{i-2}\varphi_{i-1}(z) = \psi_{i-1}\alpha_{i-1}(z) = \psi_{i-1}\psi_i(y) = 0$ so $z \in \ker \alpha_{i-1}$. Just define $\delta_i(h)$ to be the image of z in $H_{i-1}A$.

The above definition is well defined as it is independent of the choice of lift x. Pick any other lift say x' now $\beta_i(x - x') = x - x = 0$. So it has a preimage in A_i and can be given as an embedding from $A_i \to B_i$ so $x - x' \in A_i$. $\phi_i(x - x') = \psi_i x - \psi_i x'$ which implies their images in $H_{i-1}A$ are homotopic.

The fact that δ_i is a group homomorphism is simply via linearity.

Proposition 1.32 (Induced long exact sequence of homology)**.** *For a given short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of chain complexes $(A,\varphi),(B,\psi),(C,\chi)$, then the connecting homomorphism $\delta_i : H_iC \to H_{i-1}A$ induces the following long exact sequence of homologies

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_iC \arrow[dll, "\delta_i"'] \\
H_{i-1}A \arrow[r] & H_{i-1}B \arrow[r] & H_{i-1}C \arrow[dll, "\delta_{i-1}"'] \\
H_{i-2}A \arrow[r] & \cdots &
\end{tikzcd}$$

Furthermore if the chain complexes are differential modules the following triangle commutes,

$$\begin{tikzcd}
HA \arrow[rr, "\alpha"] & & HB \arrow[dl, "\beta"] \\
 & HC \arrow[ul, "\delta"] &
\end{tikzcd}$$

Lemma 1.33 (Snake lemma)**.**

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

The above commutative diagram induces a exact sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \to \text{coker}\alpha \to \text{coker}\beta \to \text{coker}\gamma$$

Proof. The map $\ker\gamma \to \text{coker}\alpha$ is given by the connecting homomorphism. □

Lemma 1.34 (5-lemma)**.** *If we have a commutative diagram as such,*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

and if β, δ are isomorphisms with α epimorphism and ϵ a monomorphism implies that γ is an isomorphism.

1.5 Resolutions

Definition 1.35. *[Left resolution] Given a module M its left resolution is given by the data of a exact sequence $(A_\bullet, \varphi_\bullet)$ into M as such,*

$$\cdots \to A_1 \to A_0 \xrightarrow{\epsilon} M \to 0$$

where ϵ is called the augmentation map, if the exact sequence is free its a free resolution and such for projective.

If we have a cochain complex instead it forms a right resolution and if its elements are injective we call them injective resolutions.

Example 1.36. *A useful example of a resolution is that of the Koszul complex for a module. This will be defined and used in the proof of the Suslin factorial theorem at the end of this project.*

Proposition 1.37 (Horseshoe lemma)**.** *If there is a short exact sequence of modules,*

$$0 \to M \to N \to P \to 0$$

and both M, P have a projective resolutions A, C

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \\
 & & & N \arrow[d] & \\
\cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & P \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

as below then N also has a projective resolution B which forms a short exact sequence. Also the sequence splits due to C_i being projective so $B_i = A_i \oplus C_i$.

Proof. First note $\epsilon_P : C_0 \to P$ lifts due to projectivity to $C_0 \to N$ also $A_0 \to N$ via composition so simply define $B_0 = A_0 \oplus C_0$. This is an epi evidently via diagram chase. Also is projective as direct sum of projectives is projective. Now consider direct sum of kernel of $A_0 \to M, B_0 \to N, C_0 \to P$ and construct the direct sum again to get F_1. Now we get a 3×3. Exactness is due to the Snake lemma □

2 Vector bundles

In this section, we introduce the foundational concepts required to prove Swan's theorem, which demonstrates a key connection between projective modules and vector bundles over certain topological spaces. The purpose of proving Swan's theorem is to highlight why projective modules play a central role in algebraic K-theory, serving as essentially algebraic counterparts to vector bundles.

All results presented hold for both real and complex vector bundles. For simplicity, we use k to denote the underlying field.

More detailed exposition can be found in [MS74, Kar08].

Definition 2.1 (Vector bundle)**.** *A n dimensional vector bundle over k is a triple (E, p, X). Which consists of a continuous map $p : E \to X$ from the total space E to the base space X. Such that for all $x \in X$, $E_x = p^{-1}(x)$ the fibre of x has a k vector space structure. Along with the following property of local trivialization*

1. *For any $x \in X$ there exists a open open $U \subset X$ along with a homeomorphism*
$$h : X \times k^n \to p^{-1}(U)$$
such that for all $c \in U$ the map through h defines an isomorphism between E_c and k^n.

Example 2.2 (Trivial vector bundle)**.** *A trivial vector bundle is one in which the total space $E = X \times k^n$ with p just the trivial projection mapping.*

The local trivialization condition can now be understood as saying there exists an open neighbourhood for each point $x \in X$ such that $E|_U$ is a trivial vector bundle

Definition 2.3 (Quasi-vector bundle)**.** *A fibre bundle defined as above without the property of local trivialisation is called a quasi-vector bundle.*

Definition 2.4 (Rank of vector bundle)**.** *If each E_x has the same dimension n then n is referred to as the rank of the vector bundle.*

Example 2.5 (Tangent bundle on S^2)**.** *Let S^2 be the unit sphere in $\mathbb{R}^3$ associate to each point $x \in S^2$ the plane tangent to S^2 at x, label this T_xS^2 (this is the tangent space at point x) then the disjoint union of all the tangent spaces form a vector bundle $TS^2 = \sqcup_{x \in S^2} T_xS^2$.*

We don't use the concept of a tangent bundle proper so we refrain from defining it in its entirety.

Definition 2.6 (Vector bundle morphisms)**.** *Two vector bundles (E_1, p, X) and (E_2, p_2, X) are considered isomorphic if there exists a continuous map between their total spaces g such that the below diagram commutes*

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

and g induces a vector space homomorphism for each fibre.

If g is a homeomorphism and fibrewise isomorphism then it is a vector bundle isomorphism.

Definition 2.7 (Category of vector bundles)**.** *For a fixed base space B the collection of vector bundles over X with arrows as vector bundle homomorphisms forms a category which we denote as* $\mathrm{VB}(X)$*.*

Definition 2.8 (Whitney sums)**.** *For $(E_1, p_1, X), (E_2, p_2, X)$ their Whitney sum denoted as $E_1 \oplus E_2$ consists of total space as $E_1 \oplus E_2$ with fibrewise direct sums of vector spaces.*

A Whitney sum is a biproduct of vector bundles in $\mathrm{VB}(X)$. Which makes the category of vector bundles into an additive category, but not abelian. Since the kernel of a morphism of vector bundles need not be a vector bundle.

Along with a Whitney sum there also exists the expected notion of a tensor product of vector bundles this in turn allows us to view $\mathrm{VB}(X)$ as a symmetric monoidal category in the natural way.

In the particular case when $E_1 = X \times k^n, E_2 = X \times k^m$ are both trivial vector bundles over X. We can describe the vector bundle morphisms between them explicitly. This is a result we will use very often in the process of proving Swans theorem.

We have associated to the vector bundle morphism $g : E_1 \to E_2$ a natural linear map $g_x : k^n \to k^m$. Consider $\widetilde{g} : X \to \mathrm{Hom}_k(k^n, k^m)$ defined as $\widetilde{g}(x) = g_x$.

Theorem 2.9 (Characterization of trivial bundle morphisms)**.** *The map $\tilde{g} : X \to Hom_k(k^n, k^m)$ is continuous*[7]*, and for $h : X \to Hom_k(k^n, k^m)$ continuous, let $\tilde{h} : E \to E'$ be the map which induces $\tilde{h}(x)$ on each fiber. Then $\tilde{h}$ is a morphism of vector bundles.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a choice of basis for k^n and $\{e'_1, \ldots, e'_m\}$ for k^m.

Now $g_x \in \mathrm{Hom}_k(k^n, k^m)$ is naturally representable by a $m \times n$ matrix $(a_{ij}(x))$.

The function $x \mapsto a_{ij}(x)$ is obtained by composing the continuous maps:

$$X \xrightarrow{\beta_j} X \times k^n \xrightarrow{g} X \times k^m \xrightarrow{\gamma} k^m \xrightarrow{p_i} k$$

where $\beta_j(x) = (x, e_j)$, $\gamma(x, v) = v$, and p_i is the i-th projection onto k. Since each $a_{ij}(x)$ is continuous $\tilde{g}$ is continuous.

Now consider $h : X \to \mathrm{Hom}_k(k^n, k^m)$, continuous. By composing a sequence of continuous maps we obtain the required morphism of vector bundles.

$$X \times k^n \xrightarrow{\delta} X \times \mathrm{Hom}_k(k^n, k^m) \times k^n \xrightarrow{\epsilon} X \times k^m$$

where $\delta(x, v) = (x, h(x), v)$ and $\epsilon(x, u, v) = (x, u(v))$. $\square$

Proposition 2.10 (Sufficient condition for vector bundle isomorphism)**.** *Let E and F be two vector bundles over X, and let $g : E \to F$ be a morphism of vector bundles such that $g_x : E_x \to F_x$ is bijective for each point $x \in X$. Then g is an isomorphism of vector bundles.*

Proof. Let $h : F \to E$ be the map defined by $h(v) = g_x^{-1}(v)$ for $v \in F_x$. We need to prove h is continuous and so indeed a map in $\mathrm{VB}(X)$. The other conditions for isomorphisms are clearly met. If we prove continuity locally for every neighbourhood then we have the required continuity globally.

Consider a neighbourhood U of x and the local trivialization isomorphisms $\beta : E|_U \to U \times M$ and $\gamma : F|_U \to U \times N$. Let $g_1 = \gamma g_U \beta^{-1}$.

[7]Relative to the natural topology on $\mathrm{Hom}_k(k^n, k^m)$. The unique topological vector space topology making any $k^t \cong L(k^n, k^m)$ into a homeomorphism

Then $h_U = \beta^{-1}h_1^{-1}\gamma$, where h_1 is defined as $\widetilde{h_1}(x) = (\widetilde{g_1}(x))^{-1}$ (as in the above result Th. 2.9), h_1 is continuous. Therefore, h is continuous on a neighbourhood of each point of F, implying h is continuous on all of F. $\square$

2.1 Sections of a vector bundle

Definition 2.11 (Sections of a vector bundle)**.** *For a vector bundle (E, p, X) a section refers to a continuous map $s : X \to E$ such that $p \circ s = 1_X$, where 1_X denotes the identity map on X.*

These sections equivalently are a homomorphism of vector bundles from the trivial line bundle $(X \times \mathbb{R}, \pi_X, X) \to (E, p, X)$

We assume all sections to be continuous without necessarily specifying it.

Definition 2.12 (Linear independence of sections)**.** *A sequence $s_1, \cdots, s_n$ of sections on a vector bundle E is said to linearly independent if they are linearly independent for each point x.*

Furthermore the morphism $f : X \times k^n \to E$ given by $(x, c_1, \cdots, c_n) \mapsto \sum_{i=1}^n c_i s_i(x)$ induces an isomorphism of fibres if E has rank n due to Prop. 2.10.

Example 2.13 (Zero section)**.** *The map $s : X \to E$ sending every point x to the 0 vector in E_x.*

Definition 2.14 (Vector space of sections)**.** *For a vector bundle $p : E \to X$ the set of continuous sections of E is denoted as $\Gamma(X, E)$. It is a vector space with vector addition defined as $(s+t)(x) = s(x)+t(x)$ and scalar multiplication as $\lambda s(x) = (\lambda s)(x)$ for $\lambda \in k$.*

Proposition 2.15 (Section functor)**.** *$\Gamma(X, E)$ can be realized as a $C(X)$ module where $C(X)$ denotes the ring of continuous functions from X to k. With scalar multiplication defined as $(c \cdot s)(x) = c(x)s(x)$ for $s \in \Gamma(X, E), c \in C(X)$.*

For a trivial vector bundle $E = X \times k^n$ we have $\Gamma(X, E)$ corresponds to a free module $C(X)^n$. If E is a arbitrary vector bundle which is a direct summand of a trivial vector bundle then $\Gamma(X, E)$ corresponds to a projective $C(X)$ module.

In particular If $E \oplus E' \cong X \times k^n$ we have $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong C(X)^n$, i.e. $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ is an additive functor.

In fact this 'stably trivial' property of a vector bundle is always true when X is compact. This is an essential result in the proof of Swans theorem. In order to prove it we need a lemma for paracompact spaces first.

Definition 2.16 (Paracompact space)**.** *A Hausdorff topological space X is said to be paracompact if every open cover of X has a locally finite open refinement.*

(An open cover is locally finite if there exists a neighbourhood for every $x \in X$ which intersects only finitely many elements of the cover. A refinement of an open cover $\{U_i\}$ consists of a open cover $\{V_j\}$ such that for each $j, V_j \subset U_i$).

Lemma 2.17. *Let X be a paracompact space, and let E and F be vector bundles over X. Suppose $\alpha : E \to F$ is a morphism such that $\alpha_x : E_x \to F_x$ is surjective for each point $x \in X$. Then there exists a morphism $\beta : F \to E$ such that $\alpha \circ \beta = \mathrm{id}_F$.*

Proof. In this proof we very liberally use the various morphisms as seen in Th. 2.9.

Fix a point $x \in X$ by local trivialization we pick a neighborhood U of x such that E_U and F_U are trivial bundles. That is, $E|_U \cong M \times V$ and $F|_U \cong U \times N$.

Now with this representation we have $\alpha|_U : U \times M \to U \times N$, can be expressed as $\tilde{\theta}$ (as in 2.9) for the associated continuous map: $\theta : U \to \mathrm{Hom}_{\mathrm{VB}}(V, W)$.

Decomposing M as $N \oplus \ker(\theta(x))$ (which we can do thanks to surjectivity of α_x) allows us to choose a matrix representation for $\theta(y) : M \to N$ i.e. $\theta(y) : N \oplus \ker(\theta(x)) \to N$ as

$$\theta(y) = \begin{bmatrix} \theta_1(y) & \theta_2(y) \end{bmatrix},$$

the first component goes to 1 and the second to 0 continuously, i.e. an endomorphism of N. Realized as a topological vector space $\mathrm{Aut}(N)$ is an open subset (i.e. vector subspace) of $\mathrm{End}(N)$.

Therefore we can pick a neighbourhood of x say V_x such that $\theta_1(y)$ is an isomorphism for $y \in U_x$. Consider associated to this the map $\theta' : V_x \to \mathrm{Hom}_{\mathrm{VB}}(N, M)$ represented now by the matrix,

$$\theta'_x(y) = \begin{bmatrix} \theta_1(y)^{-1} \\ 0 \end{bmatrix}$$

This now induces the morphism $\tilde{\theta}'_x : F_{|}V_x \to E_{|}V_x$ so that $\alpha_{V_x}\tilde{\theta}'_x = 1$.

Varying the point x enables us to construct a locally finite open cover $\{V_i\}$ of X obeying the following properties, all morphisms $\beta_i : F_{V_i} \to E_{V_i}$ are right inverses of α_{V_i} as seen.

Now consider a 'partition of unity' $\{\eta_i\}$ associated with the cover [8] and $\beta : F \to E$ is defined using it as $\sum_i \eta_i(x)\beta_i(e)$ for $e \in E_x$. Continuity is maintained due to the fact that these sums are necessarily finite (see footnote). This gives us

$$\alpha\beta(e) = \sum_i \eta_i(x)(\alpha\beta_i)(e) = (\sum_i \eta_i(x))(\alpha\beta_i(e)) = e$$

Which completes the proof. □

Theorem 2.18. *If E is a vector bundle over a compact base space X then there exists a vector bundle E' such that $E \oplus E' \cong X \times k^n$*

Proof. Pick a finite open cover $\{U_i\}_{i=1}^r$. By local trivialisation we know that $E|_{U_i} \cong U_i \times k^{n_i}$.

Let $\{\eta_i\}$ be a 'partition of unity' associated to the cover similarly as before. So there exist n_i linearly independent sections $s_i^1, s_i^2, \ldots, s_i^{n_i}$ of $E|_{U_i}$ (as seen in Def. 2.12). By extending these local sections to zero outside U_i, we obtain globally defined sections $\eta_i s_i^1, \eta_i s_i^2, \ldots, \eta_i s_i^{n_i}$, which are linearly independent sections of $E|_{V_i}$, where $V_i = \eta_i^{-1}((0, 1])$.

Let σ_i^j denote the sections $\eta_i s_i^j$ for $1 \leq j \leq n_i$. These sections $\sigma_i^j(x)$ generate E_x as a vector space for each $x \in X$. As constructed in $Def.$2.12, there exists a morphism

$$\alpha : T = X \times k^n \to E,$$

where $n = \sum_{i=1}^r n_i$, such that $\alpha_x : T_x \to E_x$ is surjective for every $x \in X$.

Now by the above Lemma for paracompact spaces, there exists a morphism $\beta : E \to T$ such that $\alpha \circ \beta = 1_E$. Let E' be the kernel of the idempotent morphism $p = \beta \circ \alpha$.

Define a morphism from $E \oplus E'$ to T by taking the sum of $\beta : E \to T$ and the inclusion $i : E' \to T$. This morphism induces a fibrewise isomorphism since $E'_x \cong \ker(p_x)$ and $E_x \cong \ker(1 - p_x)$. Thus, the morphism is an isomorphism by Prop. 2.10 □

[8] A sequence of non negative real valued functions $\{\eta_i\}$ whose sum evaluated for every point $x \in X$ is 1. And such that every point $x \in X$ has an open neighbourhood not intersecting the support of η_i for all but finitely many i. Which implies the sums are finite and therefore well defined.

2.2 Karoubian categories and some results on vector bundles

In this section we cover an essential theorem which is applied in order to prove Swan's theorem. We follow the results as proven in Karoubi's book [Kar08].

Definition 2.19 (Karoubian category)**.** *A Karoubi category is an Ab-Enriched category such that every idempotent endomorphism (a morphism of the form $e : A \to A$ such that $e^2 = e$) has a kernel.*

These are sometimes referred to as pseudo-abelian categories (note that in an abelian category proper all morphisms have kernels and cokernels in this case its only the idempotents).

Example 2.20. *The category of vector bundles is Karoubian. We already know it is additive and so Ab-Enriched. In the conclusion of Th. 2.18 we have implicitly shown that the idempotent $p \in \mathrm{VB}(X)$ has a kernel.*

Example 2.21. *Hinting towards Swans theorem we can see that also* $\mathrm{Proj}(A)$ *for a ring A is Karoubian. We will see the relation between projective modules and idempotents in more detail in Sec. 3.2*

Idempotent elements occur ubiquitously in $K-$theory we will see another useful application of idempotents in computations of K_0.

The first theorem we cover in this section is well known by various different names. It involves showing the existence of a Karoubi envelope for an additive category. But this is often referred to as the Cauchy completion or idempotent completion.

Theorem 2.22. *[Existence of Karoubi envelope] Let $\mathcal{C}$ be an additive category. Then there exists a Karoubian category $\widetilde{\mathcal{C}}$ and a fully faithful additive functor $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ universal in the sense that for any other additive functor $\psi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Karoubian there exists a unique additive functor $\psi' : \widetilde{\mathcal{C}} \to \mathcal{D}$ such that the below diagram commutes.*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[dr, dashed, "\psi'"] & \\
\mathcal{C} \arrow[u, "\varphi"] \arrow[r, "\psi"'] & \mathcal{D}
\end{tikzcd}$$

Proof. We directly construct $\widetilde{\mathcal{C}}$ and φ. The objects of $\widetilde{\mathcal{C}}$ are of the form (E, p) for $E \in \mathrm{Ob}(\mathcal{C})$ and p idempotent endomorphism over E.

Arrows between two such objects (E, p) and (F, q) comprise of the arrows $(f : E \to F) \in \mathcal{C}$ such that $fp = qf = f$. Composition of morphisms is inherited from $\mathcal{C}$. The identity morphism comprises of $1_{(E,p)} = p$, and te sum of two objects $(E, p) \oplus (F, q)$ is defined naturally as $(E \oplus F, p \oplus q)$. This demonstrates that $\widetilde{\mathcal{C}}$ is indeed an additive category.

It remains to be verified that it is indeed Karoubian which is the main concern. Let f be an idempotent endomorphism of the object (E, p) in $\widetilde{\mathcal{C}}$ we wish to show that f indeed has a kernel.

Examining the below commutative diagram will help us conclude. The elements F, g, q, h are defined directly as seen below. Note that the condition $fp = pf = f$ is present in the righthand square

$$\begin{tikzcd}
& F \arrow[dl, dashed, "h"'] \arrow[dr, "g"] \arrow[ddd, bend left, "q"] & & \\
E \arrow[rr, "p-f"] \arrow[d, "(1-f)p"'] & & E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\
E \arrow[rr, "p-f"'] & & E \arrow[r, "f"'] & E \\
& F \arrow[ul, dashed, "h"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

Since $(1-f)p$ is itself an idempotent endomorphism on E we have $(E, (1-f)p)$ is an object in $\widetilde{\mathcal{C}}$ and $p - f$ defines an arrow from this object to (E, p). Since $(p - f)((1 - f)p) = (p)(p - f) = p - f$ as expected.

Now we claim that the object $(E, (1 - f)p)$ with the arrow $p - f$ is a kernel of $f : (E, p) \to (E, p)$. In particular it is a pullback of the below form. The diagram below drawn internally in $\widetilde{C}$ makes this image clearer.

$$\begin{tikzcd}
(F,q) \arrow[drr, bend left] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "g"'] & & \\
& (E,(1-f)p) \arrow[r] \arrow[d, "p-f"'] & 0 \arrow[d] \\
& (E,p) \arrow[r, "f"] & (E,p)
\end{tikzcd}$$

Compare this to the first diagram. $(F, q) \xrightarrow{g} (E, p)$ is picked such that $fg = 0$. And we obtain uniqueness of h is due to the fact that any such arrow h must obey the expression $h = (1-f)ph = p(1-f)h = pg = g$. Conversely if $h = g$ the diagram commutes naturally.

This shows that $\widetilde{\mathcal{C}}$ is Karoubian. Finally we construct $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ defined as sending $E \in \mathrm{Ob}(\mathcal{C})$ to $(E, 1_E)$ and $\varphi(f) = f$. This is naturally a full faithful functor by construction.

Also we can see that based on an analogous argument (E, p) is the kernel of $1 - p$ as an idempotent endomorphism over $\varphi(E) = (E, 1_E)$. Meaning that $\varphi(E) \cong (E, p) \oplus (E, 1-p)$, i.e. the functor is additive.

Finally to show that these constructions define ψ'.

If $\psi : \mathcal{C} \to \mathcal{D}$ (resp. $\psi' : \widetilde{\mathcal{C}} \to \mathcal{D}$) is an additive functor from $\mathcal{C}$ (resp. $\widetilde{\mathcal{C}}$) to another Karoubian category $\mathcal{D}$, such that $\psi'\varphi \cong \psi$. Then we have $\psi(\ker f) \cong \ker(\psi'(f))$ for every idempotent endomorphism f. Hence $\psi'(E, p) = \ker \psi(1 - p) : \psi(E) \to \psi(E)$ and $\psi'(f) = \psi(f)_{\ker \psi(1-p)}$ on the objects and morphisms respectively. Conversely, these formulas define ψ' (up to isomorphism). $\square$

Before proceeding for the last result in this section, recall the definition of equivalence of categories.

Definition 2.23 (Equivalence of categories)**.** *Two categories $\mathcal{C}, \mathcal{D}$ are said to be equivalent if there exist functors $E : \mathcal{C} \rightleftarrows \mathcal{D} : F$ and a pair of natural isomorphisms $\alpha : 1_{\mathbf{C}} \to F \circ E$ and $\beta : 1_{\mathbf{D}} \to E \circ F$.*

This is a weaker condition than isomorphism of categories in which we have an actual equality instead of natural isomorphism.

A more useful form of the definition is as such, the proof may be seen in [Awo10][7.7.25]

Proposition 2.24. *A functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories iff F has the following properties*

1. *F is full (The map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ defined as $f \mapsto F(f)$ is surjective for all $A, B \in \mathrm{Ob}(\mathcal{C})$).*

2. *F is faithful (The map $F_{A,B}$ as defined above is injective for all pairs A, B).*

3. *F is essentially surjective on objects (For every $D \in \mathrm{Ob}(\mathcal{D})$ there exists $C \in \mathrm{Ob}(\mathcal{C})$ such that $FC \cong D$).*

Corollary 2.25. *Let $\mathcal{C}$ be an additive category, $\mathcal{D}$ a Karoubian category, and $\psi : \mathcal{C} \to \mathcal{D}$ an additive functor which is fully faithful such that every object of $\mathcal{D}$ is a direct factor of an object in the image of ψ. Then the functor ψ' as defined in Th. 2.22 forms an equivalence between the categories $\mathcal{C}$ and $\mathcal{D}$.*

Proof. We will prove that ψ' is essentially surjective and fully faithful.

Let $G \in \mathcal{D}$. We seek $X \in \widetilde{\mathcal{C}}$ such that $\psi'(X) \cong G$.

By the hypothesis we have that for $G \in \mathcal{D}$, there exists $E \in \mathcal{C}$ and $G' \in \mathcal{D}$ with $\psi(E) \cong G \oplus G'$. Due to this we can choose an idempotent endomorphism $q : \psi(E) \to \psi(E)$ such that $\mathrm{Ker}(q) \cong G$ we are in a pseudo-abelian category.

Now as ψ is fully faithful, there's a idempotent $p : E \to E$ in $\mathcal{C}$ with $\psi(p) = q$. Then by the formulas in the end of Th. 2.22 we have $G \cong \varphi'(E, 1-p)$. This proves essential surjectivity.

Lastly to prove φ' is fully faithful consider two objects $H, H' \in \widetilde{\mathcal{C}}$ direct factors of $\varphi(E), \varphi(E')$. Then the following diagram shows that $\psi'_{H,H'}$ is an isomorphic function,

$$
\begin{tikzcd}
\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\varphi(E), \varphi(E')) \arrow[r, "\cong"] \arrow[dr, "\psi'_{\varphi(E),\varphi(E')}"'] & \mathrm{Hom}_{\mathcal{C}}(E, E') \arrow[r, shift left] \arrow[d, "\psi_{E,E'}"] & \mathrm{Hom}_{\widetilde{\mathcal{C}}}(H, H') \arrow[l, shift left] \arrow[d, "\psi'_{H,H'}"] \\
& \mathrm{Hom}_{\mathcal{D}}(\psi(E), \psi(E')) \arrow[r, shift right] & \mathrm{Hom}_{\mathcal{D}}(\psi'(H), \psi'(H')) \arrow[l, shift right]
\end{tikzcd}
$$

where the horizontal arrows are induced by the decompositions $\varphi(E) = H \oplus H_1$ and $\varphi(E') = H' \oplus H'_1$, since $\psi_{E,E'}$ is an isomorphism by hypothesis. □

2.3 Swan's theorem

We can now prove the celebrated Swan's theorem with the results we've built up so far.

Theorem 2.26 (Swan's theorem)**.** *Let X be a compact Hausdorff space, and let $A = C(X)$. Then the section functor Γ induces an equivalence of categories $\mathrm{VB}(X) \simeq \mathrm{Proj}(C(X))$.*

Proof. Since we assume that X is compact the section map Γ as seen in Prop. 2.15 is indeed a functor $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ due to Th. 2.18. Furthermore it induces a functor $\Gamma_T : \mathrm{VB}_T(X) \to \mathrm{Free}(C(X))$.

Where $\mathrm{VB}_T(X)$ refers to the full subcategory of $\mathrm{VB}(X)$ consisting of the trivial bundles over X, and $\mathrm{Free}(C(X))$ refers to finitely generated free modules over $C(X)$.

Since $C(X)^n \cong \Gamma_T(E)$ for $E = X \times k^n$, we have Γ_T is essentially surjective.

If $F : X \times k^p$ is some other trivial vector bundle and $f : E \to F$ is a morphism of vector bundles then as seen in Th. 2.9 we have full faithfullness of the functor Γ_T. Which shows that Γ_T induces a equivalence of categories between $\mathrm{VB}_T(X)$ and $\mathrm{Free}(C(X))$.

To extend this to our required case we make use of Th. 2.22 and Th. 2.25.

Comparing with Th. 2.22 since $\mathrm{VB}(X)$ being Karoubian itself is naturally the Karoubian envelope of its subcategory $\mathrm{VB}_T(X)$ and with the functor Γ realized as ψ' we see the below diagram commutes.

$$\begin{tikzcd}
VB(X) \arrow[drr, "\Gamma=\psi'"] & & \\
\mathrm{VB}_T(X) \arrow[u, "i"] \arrow[rr, "\psi"'] & & \mathrm{Proj}(C(X))
\end{tikzcd}$$

Finally due to Th. 2.25 we are done.

□

3 Grothendieck group K_0

The Grothendieck group K_0 arises from the natural idea of wanting to extending a commutative monoid (referred to as CMon) to a group in a universal way. This concept finds its roots in many naturally occurring mathematical structures, such as finitely generated projective modules or vector bundles.

3.1 Definitions and basic results

Proposition 3.1 (K_0 of a monoid (Group completion functor))**.** *Assign* $(A, +) \in \mathrm{CMon}$ *to*

$$K_0(A) \in \mathrm{Grp}$$

by taking the free group on symbols $[a]$ *for* $a \in A$ *and quotienting the monoidal relations* $[m + n] - [m] - [n]$.

The mapping is an injection iff the monoid is cancellative, i.e. ($a + b = a + c \implies b = c$) for all a, b, c in the monoid.

Definition 3.2 (Reduced K_0 groups)**.** *There is a canonical homomorphism $i : \mathbb{Z} \to K_0(A)$ given by $z \mapsto z[m]$ the reduced K group is defined as $\tilde{K}_0(A) := K_0(A)/\mathrm{Im}\, i$*

Definition 3.3 (K_0 for a ring A)**.** *Consider the stable isomorphism classes of finitely generated projective modules over A denoted as $\mathrm{Proj}(A)$. This forms a commutative monoid so $K_0(A)$ is defined as $K_0(\mathrm{Proj}(A))$.*

Definition 3.4 (G_0 for a ring A)**.** *The group completion of $M(A)$ the monoid of all finitely generated modules over A is denoted as $G_0(A)$*

There is a canonical inclusion map $K_0(A) \to G_0(A)$.

Proposition 3.5 (Eilenberg Swindle)**.** *K_0 for many abelian categories are trivial. If we consider R^∞ as a inf.g. free module over a ring R if $P \oplus Q \equiv R^n$ then*

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \equiv (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \equiv R^\infty$$

but this relation would imply $[P] = 0$ for all projectives.

This extends to higher K groups with an analogue that demonstrates the Quillen K space contracts, see V.1.9 in [WS13].

3.2 Computing K_0 groups with idempotent matrices and Morita invariance

We now see a few examples of computations of K_0. We must first prove a result about the invariant basis property. Recall that a ring A has the invariant basis property if $A^n \cong A^m \implies n = m$. A division ring (also called a skew field) is any nontrivial ring in which division by nonzero elements is defined. A commutative division ring is simply a field.

Proposition 3.6. *Any division ring A has the invariant basis property.*

Proof. Consider a free A-module M with two finite bases $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_m\}$, then if we prove $n = m$ we are done. We prove with induction on n.

If $n = 1$, then $B = \{b_1\}$. If $C = \{c_1, \ldots, c_m\}$ for some $m > 1$, we can express the c_i in terms of b_1: $c_1 = a_1 b_1$ and $c_2 = a_2 b_1$ where $a_1, a_2 \in A$ and $a_1, a_2 \neq 0$. Then $a_1^{-1} c_1 - a_2^{-1} c_2 = 0$, contradicting the linear independence of C. Therefore $m = 1 = n$.

Proceeding inductively assume the statement holds for $n = k$. Let M be a free A-module with a basis $B = \{b_1, \ldots, b_{k+1}\}$, and let $C = \{c_1, \ldots, c_m\}$ be another basis.

Since B spans M, we can write b_{k+1} as a linear combination of the c_i: $b_{k+1} = a_1 c_1 + \cdots + a_m c_m$ for some $a_i \in A$. Because $b_{k+1} \neq 0$, at least one $a_i \neq 0$. Without loss of generality, assume $a_m \neq 0$.

Consider the quotient module $M/(b_{k+1})$. The set $\{b_1, \ldots, b_k\}$ forms a basis for $M/(b_{k+1})$. Similarly, $\{c_1, \ldots, c_{m-1}\}$ is a basis for $M/(b_{k+1})$. Applying the inductive hypothesis to $M/(b_{k+1})$, we have $k = m-1$. Therefore $k+1 = m$. □

Corollary 3.7. *Nonzero commutative rings A have invariant basis property.*

Proof. For a free $A-$module M with basis $B = \{b_1, \cdots, b_n\}$, under the canonical surjection $A \to A/\mathfrak{m}$, $M/\mathfrak{m}M$ is also a free module over $A/\mathfrak{m}$ with basis $\{b_1 + \mathfrak{m}M, \cdots b_n + \mathfrak{m}M\}$. And since every basis for $M/\mathfrak{m}M$ has n elements due to the fact that $A/\mathfrak{m}$ is a field and Prop. 3.6 shows it has invariant basis property. This implies any basis of M also has only n elements, i.e. A itself has the invariant basis property. □

Proposition 3.8. *If A is a Field/division ring/local ring/PID then $K_0(A) \cong \mathbb{Z}$*

Proof. For fields and division rings this is true due to all finitely generated modules being free, i.e. having a basis. We prove this directly for division rings for simplicity.

The similar linear algebraic proof extends to division rings for M a module over division ring A. Pick a maximally linearly independent subset B by Zorn's. To show B is a generating set, the argument uses B's maximality. If $m \in M$ then. If $m \in B$ we are done. If $m \notin B$ then $B \cup m$ linearly dependent by maximality of B therefore there exists $a \in A$ such that $av \in \mathrm{span}(B)$ for some $a \neq 0$ and since a is invertible due to F being a division ring we have $v \in \mathrm{span}(B)$. Therefore, B must span M, making it a basis and so $M \cong F^n$

Similarly as seen in Th. 1.26 and Prop. 1.29 f.g. projective modules in a local ring/PID are free.

So in each case $\mathrm{Proj}(A) \cong \mathbb{N}$ so its group completion is $\mathbb{Z}$.

Throughout the proof we have implicitly assumed A has the invariant basis property which was proved above in Prop. 3.6 and Cor. 3.7 $\square$

Lemma 3.9. *For commutative ring A, $K_0(A) \cong \mathbb{Z} \implies$ projective modules over A are stably free.*

Proof. For a commutative ring A, $K_0(A) \cong \mathbb{Z} \implies \mathrm{Spec}(A)$ is connected. For if not then there exists a non trivial idempotent in A which results in a splitting of A as a product which would contradict $\mathrm{Spec}(A)$ being connected.

In light of Def. 1.27 we know that the rank of the projective modules must be constant due to the connectedness of $\mathrm{Spec}(A)$ and the fact that the only connected components in $\mathbb{Z}$ are singletons.

So the rank map $\phi : K_0(A) \to \mathbb{Z}$ defined as $P \mapsto \mathrm{Rank}(P)$ is well defined and surjective. A with rank 1 maps to 1, i.e. the generator of $\mathbb{Z}$. But by our assumption this is an isomorphism.

So any $[P] = n = [A]^n = [A^n]$ i.e. there exists Q such that $Q \oplus P \cong Q \oplus A^n$ if Q is projective add what's needed to make it projective. $\square$

We end this section with computing K_0 for semisimple rings. As it requires an instructive method of computing K_0 with idempotents. We first define what is means for a ring to be simple.

Definition 3.10 (Simple ring)**.** *A simple ring is a non-zero ring which have no non-trivial two-sided ideals.*

Example 3.11. *A commutative ring is simple iff it is a field.*

Example 3.12. *All division rings are simple rings.*

Example 3.13. *Not all division rings are simple consider $M_n(F)$ for some field F not all elements need be invertible.*

There are various equivalent definitions of a semisimple ring.

Definition 3.14 (Semisimple ring)**.** *A ring A is called semisimple if*

- *A is Artinian with trivial Jacobson ideal.*
- *A is a finite product of simple Artinian rings*
- *Every left/right $A-$module is projective*

A useful characteristic of semisimple rings is the Wedderburn-Artin theorem (see [Lam01][3.5] for a proof).

Theorem 3.15 (Wedderburn-Artin). *A ring A is semisimple iff it is isomorphic to a direct product of $n_i \times n_i$ matrix rings over division rings D_i, i.e. $A \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ where $D_i = \mathrm{Hom}_A(V_i, V_i), \dim_{D_i}(V_i) = n_i$ for V_i the simple A-modules components of A.*

We now discuss the role of idempotent matrices in computing K_0 following [WS13] this is presented in detail in [Ros95]. Recall that for a ring A and idempotent element e is one such that $e^2 = e$. We claim that idempotent elements of A are in a $1-1$ correspondence to decompositions of $A \cong P \oplus Q$.

If $e \in A$ then the ideal generated by e is a projective module. Since $A = \langle e \rangle \oplus \rangle 1 - e \rangle$. Conversely just pick $e \in P$ and $f \in Q$ such that $1 = e + f$. Then $e, f = 1 - e$ are the required idempotents.

In general for a f.g. projective module P over A, such that $P \oplus Q \cong A^n$ we can define a $R-$module homomorphism which is identity restricted to P and zero else. This is clearly an idempotent element in $M_n(A)$. Furthermore any idempotent $e \in M_n(A)$ determines a projective.

We must make a note of the fact that different idempotent matrices may induce projective modules in the same isomorphism class. This is made precise in the following result.

Proposition 3.16. *If e, f are idempotent matrices over A of possibly different sizes then the associated finitely generated projective modules are isomorphic iff e, f are conjugate over a larger common matrix group of order r (obtained by placing the matrices in the top left corner of a larger 0 matrix).*

Proof. Begin with the backward direction. If e and f are conjugate in some $GL_r(A)$, this means there exists an invertible matrix u such that $ueu^{-1} = f$. Let $P = \mathrm{Im}(e)$ and $Q = \mathrm{Im}(f)$ be the projective A-modules associated with e and f, respectively.

The map $\phi : Q \to P$ by $\phi(x) = (ueu^{-1})x$ for $x \in \mathrm{Im}(f) = Q$ defines the required isomorphism.

Now conversely, assume the projective modules corresponding to e and f are isomorphic. Let $e \in M_n(A)$ and $f \in M_m(A)$. This isomorphism $\mathrm{Im} e \cong \mathrm{Im} f$, i.e. $A^n e \cong A^m f$, extends to an A-module homomorphism $\alpha \colon A^n \to A^m$. Also α^{-1} to $\beta \colon A^m \to A^n$.

We can represent α and β by right multiplication with matrices $\gamma \in M_{n,m}(A)$ and $\delta \in M_{m,n}(A)$. These obey the following relations $\gamma\delta = e, \delta\gamma = f, \gamma = e\gamma = \gamma$ and $\delta = f\delta = \delta e$.

Choose $r = n + m$, claim that the following block matrix is invertible

$$\varphi = \begin{bmatrix} 1-e & \gamma \\ \delta & 1-f \end{bmatrix}$$

and is conjugate to $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$. This is true since $\varphi^2 = I_r$. A computation then shows that

$$\varphi \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \varphi = \varphi \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$$

A permutation matrix then conjugates $\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}$ to $\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$. Thus e and f are conjugate after appropriate embedding into $GL_r(A)$ and therefore represent isomorphic modules. □

Consider $GL_n(A) \subset GL_{n+1}(A)$ by placing the $n \times n$ matrix in the top right. In this manner we have a filtered system and we can define $GL(A) = \lim_{\rightarrow} GL_i(A)$ as the colimit. Similarly define $M(A)$. Denote the set of idempotent matrices in $M(A)$ as $\text{Idem}(A)$ so we have that the group $GL(A)$ acts on the set $\text{Idem}(A)$ by conjugation.

With the above discussion in mind we now have a alternate description for the monoid $\text{Proj}(A)$ in terms of idempotent matrices. In particular $\text{Proj}(A)$ corresponds to the conjugacy classes of the action of $GL(A)$ on $\text{Idem}(A)$. The monoid operation $e + f$ is the block matrix $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$.

Corollary 3.17 (Morita invariance of K_0). *Let A be a ring and $n \in \mathbb{N}$ arbitrary then $K_0(A) \cong K_0(M_n(A))$*

Proof. $\text{Idem}(M_n(A)) = \text{Idem}(A)$ and $GL(M_n(A)) = GL(A)$ so their monoid of f.g. protectives are the same meaning their group completions are isomorphic. □

Corollary 3.18. *For commutative ring A if $A \cong A_1 \times A_2$ for rings. Then $K_0(A) \cong K_0(A_1) \times K_0(A_2)$*

Proof. Using the fact that $GL(A) \cong GL(A_1 \times A_1) \cong GL(A_1) \times GL(A_2)$ and $\text{Idem}(A) \cong \text{Idem}(A_1 \times A_2) \cong \text{Idem}(A_1) \times \text{Idem}(A_2)$ we obtain the required result. $\square$

Corollary 3.19. *If A is the direct limit of rings, i.e. $A \cong \lim_{\to i \in I} A_i$ then $K_0(A) \cong \lim_{\to i \in I} K_0(A_i)$.*

The below result is a generalization of Prop. 3.8 (since every division ring is a simple ring)

Lemma 3.20. *Let A be a simple ring then $K_0(A) \cong \mathbb{Z}$.*

Proof. By Morita invariance 3.17 we know that $K_0(A) \cong K_0(M_n(A)) \cong K_0(\text{End}(A)) \cong K_0(D)$ for some division ring D. The last isomorphism is due to 'Schur's lemma' (see [Lam01][3.6]). Now applying Prop. 3.8 we are done. $\square$

Theorem 3.21. *If A is a semisimple ring then $K_0(A) \cong \mathbb{Z}^r$.*

Proof. Applying the Wedderburn-Artin theorem we obtain, $A \cong \prod_{i=1}^r M_{n_i}(D_i)$ now applying Morita invariance 3.17, Cor. 3.18 (the result for n direct sums is obtained via induction).

We get, $K_0(A) \cong K_0(\prod_{i=1}^r M_{n_i} D_i) \cong K_0(\prod_{i=1}^r D_i) \cong \prod_{i=1}^r K_0(D_i) \cong \mathbb{Z}^r$. $\square$

3.3 K_0 for exact and abelian categories

K_0 being the prototypical K group is easier to generalize. We will refer to Weibel for most of the definitions of K_0 [WS13]. The benefit of this approach will mainly be for building up an understanding for Higher K theory. Also this machinery allows for easier computations. In particular this will allow for a proof of Hilbert-Serre as a simple observation which we will use in the proof of Quillen-Suslin.

We begin with the definition of an exact category and then define K_0 for an exact category. This will subsume the definition of K_0 for an abelian category. Since every abelian category is exact over itself.

Definition 3.22 (Exact category)**.** *An exact category (sometimes referred to as a Quillen exact category) is a pair $(\mathcal{E}, E)$ for $\mathcal{E}$ an additive category* *which is a full subcategory of some abelian category $\mathcal{A}$. Along with a family of sequences E of the form*

$$0 \to A \to B \to C \to 0$$

which are short exact sequences in $\mathcal{A}$ and if in a sequence of the above form $A, C \in \mathcal{E}$ then B is isomorphic to some element which is in $\mathrm{Ob}(C)$

Example 3.23. *Every abelian category is trivially exact over itself.*

Example 3.24. *Torsion free abelian groups over the category of abelian groups is exact but not abelian. (Non abelian-ness was shown in Ex. 1.9.2).*

Definition 3.25 (K_0 for an exact category $\mathcal{E}$)**.** *$K(\mathcal{E})$ is generated by $[B]$ for each $B \in \mathrm{Ob}(\mathcal{E})$ and a relation of $[B] = [A] + [C]$ for all short exact sequences*

$$0 \to A \to B \to C \to 0$$

Naturally since every abelian category is exact this applies for abelian categories in particular.

3.4 Fundamental theorems for K_0, G_0

In this section we present here for reference the important 'fundamental' theorem for K_0 and G_0 of abelian categories. The proofs will be provided in the subsequent project.

The proofs are omitted here due to the need for a more in-depth discussion of concepts such as the localization of categories and multiplicative systems. Furthermore, the proofs for the K_0 case are readily extended to the higher K groups, especially in the context of Quillen-Q constructions, which are based on the calculus of fractions of categories. As such, it is more fitting to address these in detail in the next project, which will build on the work presented here.

Theorem 3.26 (Fundamental theorem for G_0 for noetherian rings A)**.** $G_0[A] \cong G_0(A[t]) \cong G_0(A[t, t^{-1}])$

Theorem 3.27 (Resolution theorem for K_0 for additive categories)**.** *Let $\mathcal{A}$ be abelian and $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ additive subcategories. If the following conditions hold,*

- *Every $B \in \mathrm{Ob}(\mathcal{B})$ has finite $\mathcal{C}$ dimension, in the sense that there exist a minimal finite $\mathcal{C}$ resolution of $\mathcal{B}$ (in the sense of Def. 1.35)*
- *$\mathcal{B}$ is closed under kernels of epis in $\mathcal{A}$.*

then the inclusion functor $i : \mathcal{C} \to \mathcal{B}$ induces an isomorphism $K_0(\mathcal{C}) \cong K_0(\mathcal{B})$

Definition 3.28 (Regular ring)**.** *A ring is called regular if every finitely generated ideal has finite projective dimension.*

Example 3.29. *Any Dedekind domain is a regular ring in particular a PID is a regular ring.*

Theorem 3.30. *[Fundamental theorem for K_0 of regular rings] For a regular ring A, $K_0(A) \cong G_0(A)$ and by Th. 3.26 we have*

$$K_0(A) \cong K_0(A[t]) \cong K_0(A[t, t^{-1}])$$

4 Quillen-Suslin Theorem

We will know move towards a detailed proof of Horrock's theorem which will give us a concise proof of the famous Quillen-Suslin theorem. We follow Lang's book for the first few results which recounts Vaserstein's proof of Quillen-Suslin [Lan02].

Theorem 4.1 (Hilbert-Serre)**.** *Every finitely generated module over $k[x_1, \dots x_n]$ is stably free where k is a PID.*

Proof. Apply Th 3.8 and Th 3.30 □

Definition 4.2 (Unimodular row)**.** *For a ring A, an element of A^n is said to be a unimodular row if its components generate A. We denote the set of all unimodular rows of length n in A as $\mathrm{Um}_n(A)$*

In particular $v = (v_1, \cdots, v_n) \in \mathrm{Um}_n(A)$ if there exists $a = (a_1, \cdots a_n) \in A^n$ such that $v \cdot a = v^t a = \sum_{i=1}^{n} v_i a_i = 1$.

Definition 4.3 (Unimodular matrix)**.** *In general we say an arbitrary matrix over A not necessarily square is unimodular if it is right invertible (i.e. a surjective map).*

Alternatively it can be useful to view a unimodular row as as element of $M_{1\times n}(A)$ as such it represents a surjective linear map $A^n \to A$, or even an element in $M_{n\times 1}$ in which case it represents a injection from $A \to A^n$.

Recall the definition of a stably free projective module (Def. 1.31). Based on these definitions we can see that the kernel of the surjective $1 \times n$ matrix $A^n \to A$ (i.e. of a unimodular row) is precisely a stably free projective of the form $\underbrace{P}_{\ker v} \times A \cong A^n$.

Definition 4.4 (Equivalence of unimodular rows)**.** *For unimodular rows $v, w \in A^n$ we say $v \sim w$ if there exists $\alpha \in GL_n(A)$ such that $\alpha v = w$.*

Definition 4.5 (Unimodular completion property)**.** *Given a unimodular row $v = (v_1, \dots v_n) \in A^n$ if we can construct a invertible $n \times n$ matrix with v in the first column we say v has the unimodular completion property.*

Lemma 4.6. *A unimodular row $v \in A^n$ has the unimodular completion property iff $v \sim (1, 0, \dots, 0)$*

Proof. If v can be extended to a invertible matrix $\alpha \in GL_n(A)$ then

$$\alpha^{-1} = (1, 0, \dots, 0).$$

Conversely if $\alpha' \in GL_n(A)$ s.t. $\alpha' v = (1, 0, \dots, 0)$ then α'^{-1} has v in the first column. □

Corollary 4.7. *Based on the above lemma we can see that naturally any row of a invertible matrix (and column realized as a row of its transpose) is a unimodular row.*

Corollary 4.8. *A projective module P is free iff the unimodular row $v : A^n \to A$ such that $P = \ker v$ is completable to a invertible matrix (since we can adjoin the basis of P).*

Example 4.9 (Stably free projective module which is not free)**.** *Consider the ring R of polynomial functions on the sphere S^2, $R = \mathbb{R}[x, y, z]/\langle x^2+y^2+z^2 = 1\rangle$. Consider the unimodular row $v = (x, y, z)$. The associated projective module is $P = \ker v = \ker\{(p, q, r) \mapsto xp+yq+zr\}$. By definition $P \oplus v \cong R^3$. We claim that v cannot be completed to an invertible 3x3 matrix.*

Every element (f, g, h) of R^3 yields a vector field in $\mathbb{R}^3$, (recall a vector field is a function $f : S^2 \to \mathbb{R}^3$ such that $\langle f(x, y, z), (x, y, z)\rangle = 0$.)

The unimodular row v is the vector field extending outward normal to the sphere. Therefore an element in P yields a vector field in tangent to the 2-sphere S^2.

If P were free, a basis of P would yield two tangent vector fields on S^2 which are linearly independent at every point of S^2.

This leads to a contradiction, note that the matrix

$$\begin{bmatrix} x & a & d \\ y & b & e \\ z & c & f \end{bmatrix}$$

must have a nonzero determinant in R. Since the determinant of this matrix is a unit in R, we could construct a nonvanishing vector field on S^2. But the 'Hairy ball theorem' [McG16] tells us that any continuous vector field on S^2 must have at least one zero. But then the determinant wouldn't be zero. This is the required contradiction. Therefore P cannot be free.

Proposition 4.10. *Over a PID A any two unimodular rows in A^n are equivalent.*

Proof. Let v be a unimodular row. So that we get a split sequence $0 \to A \xrightarrow{v} A^n \to P \to 0$ for some stably free P. But we have that $\mathrm{coker} v = A^n/\mathrm{imv}$ is free as submodules of free f.g. modules over a PID are free (Prop. 1.28). So there exists a basis for A^n containing v, i.e. $v \sim (1, 0, \cdots, 0)$

□

Using the fact that projective modules over local rings are free we obtain.

Proposition 4.11. *Over a local ring A any two unimodular rows are equivalent*

Theorem 4.12 (Horrocks' theorem)**.** *If $(A, \mathfrak{m})$ is a local ring then for any arbitrary unimodular row $v(x)$ in $A[x]^n$ such that one of its component elements has leading coefficient 1 implies that v has the unimodular extension property. Furthermore, any such v is equivalent to $v(0)$.*

Proof. Recall that for a local ring $x \notin \mathfrak{m} \iff x$ is a unit.

When $n = 1, 2$ there is nothing to prove. Assume $n \geq 3$.

Without loss of generality, we take $v_1(x)$ with degree d among components with leading coefficient 1 and $\deg v_i < d$, for $i \neq 1$. We shall induct on d.

By unimodularity we know there exists $w(x) \in A[x]^n$ such that,

$$\sum_{i=1}^{n} w_i v_i = 1$$

So we can say that not all of the coefficients of $v_2, \dots v_n$ can lie in $\mathfrak{m}$. For if it were the case, then reduced mod $\mathfrak{m}$ we arrive at a contradiction since we assumed v_1 has leading coefficient 1 and $w_1 v_1$ wouldn't have a constant residue.

Once again without loss of generality, assume some coefficient of $v_2(x)$ does not lie in $\mathfrak{m}$, and as such is a unit.

Now consider the ideal I generated by the leading coefficients of $w_1v_1+w_2v_2$ of degree $< d$.

I contains the coefficients of v_2 this can be inductively found when $w_1 = 0, w_2 = 1$ we get the coefficient of the x^m term where $\deg v_2 = m$. Using repeatedly different choices of polynomials we are done.

Since I has a unit which means it generates A. And consequently implies that there was some choice of polynomial $y_1v_1 + y_2v_2$ of degree $< d$ with leading coefficient 1.

The the appropriate row actions we can obtain this in some component of v. Repeating this process until we get $d = 0$ finishes the proof.

Now because of $\sum_{i=1}^{n} w_i v_i = 1$ there must be some constant term not in m and unital as such. So $v(0) \sim (1, 0, \dots, 0) \sim v$ as seen above. □

We now extend the idea of Horrock's theorem.

Lemma 4.13. *For an integral domain A and a multiplicative subset S if $v(x) \sim v(0)$ unimodular over $A_S[x]^n$ then there exists $b \in S$ such that $v(x + by) \sim v(x)$ over $A[x, y]^n$.*

Proof. By the equivalence $v(x) \sim v(0)$ we know there exists a matrix $\alpha(x) \in GL_n(A_S[x])$ such that $\alpha(x)v(x) = v(0)$ now consider

$$\beta(x, y) := \alpha(x)^{-1}\alpha(x + y)$$

Note that now $\beta(x, y)v(x + y) = v(x)$ and so also $y \mapsto by$ implies that $\beta(x, by)v(x + by) = v(x)$.

Now to show that indeed $\beta(x, by) \in A[x, y]$ for some choice of $b \in S$ but this is true since $\beta(x, 0) = I_n \implies \beta(x, y) = I + yP$ for some $P \in A_S[x, y]$ but this just means there is some appropriate choice of $b \in S$ that allow us to cancel out all the denominators in P so that $P[x, by] \in A[x, y]$. □

Lemma 4.14. *For an integral domain A and $v(x)$ unimodular row in $A[x]^n$ with at least one component having leading coefficient one implies $v(x) \sim v(0)$.*

Proof. Consider the set I containing all $b \in A$ such that $v(x + by) \sim v(x)$ as rows in $A[x, y]$ if the ideal contains 1 then sending $x \to 0$ would give us $v(y) \sim v(0)$ in $A[y]$.

We can achieve this by first showing I is an ideal and then showing that its not contained in any maximal ideal. To do this last step we will localize at the maximals and use the previous result.

First prove that I is an ideal.

1. $I \neq \emptyset$ as $0 \in I$

2. If $b, c \in I$ then $b-c \in I$ as $v(x+(b-c)y) = v(x+by-cy) \sim v(x+by) \sim v(x)$ by a substitution $x \mapsto x + by$

3. For $a \in A, b \in I$ then simply $v(x + bay) \sim v(x)$ by the $y \mapsto ay$

Now to show I isn't contained in any maximal ideal. Pick a maximal ideal $\mathfrak{m}$ and localize at it first due to Horrocks we know $v(x) \sim v(0)$ in $A_{\mathfrak{m}}[x]$ and then due to the previous lemma 4.13 we find some $b \in A \setminus \mathfrak{m}$ such that $v(x + by) \sim v(x) \sim v(0)$ but this just means that $b \in I$ and so $I \not\subset \mathfrak{m}$ this applies to any maximal and so we are done. □

Theorem 4.15. *For $A = k[x_1, \ldots, x_n]$ where k is a PID, then $v \sim (1, 0, \ldots, 0)$ for any unimodular row $v \in A^n$.*

Proof. Proceed with induction on n. We proved $n = 0$ above Prop. 4.10.

Assume $n \geq 1$ and that the result holds for $m - 1$.

Then $v \in k[x_1, \ldots, x_m] \cong k[x_1, \ldots, x_{m-1}][x_m]$ can be realized as $v(x_m)$ with coefficients in $k[x_1, \ldots, x_{m-1}]$. If $v(x_m)$ has some component with leading coefficient 1 then by Lemma 4.14 we now $v(x_m) \sim v(0) \in k[x_1, \ldots, x_{m-1}]$ and we can reduce by induction.

So if not by some appropriate change of variables as amongst $x_1, \ldots, x_{m-1}$ in the form of $x_i \mapsto x_i - x_m^{p_i}$ for very large p_i's this allows us obtain the leading coefficient in terms of x_m to be 1 as needed. □

Theorem 4.16 (Quillen-Suslin)**.** *Finitely generated projective modules over $A = k[x_1, \ldots, x_n]$ where k is a PID are free.*

Proof. We know such f.g. projective modules are stably free. And from above we know any unimodular row in A is equivalent to $(1, 0, \ldots, 0)$.

That is to say given a f.g. projective module P which is stably free, i.e. $P \oplus R^{m_1} \cong R^{m_2}$ then P is free.

When $m_1 = 1$ this is the split exact sequence (since P is projective see 1.23),

$$0 \to A \to A^{m_2} \to P \to 0$$

The injection $A \to A^{m_2}$ is precisely a unimodular row by definition which we know must correspond to the canonical embedding of $1 \mapsto (1, 0, \cdots, 0)$. So,

$$P = \operatorname{im}(A^{m_2} \to P) \cong A^{m_2} / \ker(A^{m_2} \to P) \cong A^{m_2} / \operatorname{im}(A \to A^{m_2}).$$

But $A^{m_2}/\operatorname{im}(A \to A^{m_2})$ is free since $\operatorname{im}(A \to A^{m_2})$ is naturally free due to the embedding.

When $m_1 \neq 1$ just take $(P \oplus A^{m_1 - 1}) \oplus A$. □

5 Whitehead group K_1

Definition 5.1 (Whitehead group for a ring)**.** *K_1 for a ring A is defined as the abelianization of its infinite general linear group.*

$$K_1 := \frac{GL(A)}{[GL(A) : GL(A)]}$$

Where $GL(A)$ the infinite general linear group is the colimit of $GL_n(A)$ with GL_n realized as a subgroup of GL_{n+1} by placing the matrix in the top left corner.

Definition 5.2 (Elementary matrices)**.** *We denote the $n \times n$ elementary matrices as $E_n(A)$ generated by standard elementary matrices of the form $e_{ij}(\lambda) := I_n + \lambda E_{ij}$ where E_{ij} is the matrix with 1 in the (i, j) entry and zero elsewhere.*

Lemma 5.3. *A nonsingular triangular matrix with 1's in the diagonal is a product of standard elementary matrices.*

Proof. Let $\alpha \in GL_n(A)$ then consider the following inductive procedure.

$$\alpha = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \alpha_{n-1} & \\ 0 & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \alpha_{n-1} & \\ 0 & & & \end{bmatrix} e_{12}(a_{12})e_{13}(a_{13})\cdots e_{1n}(a_{1n})$$

Repeat the procedure for α_{n-1} to obtain

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & 0 & & \alpha_{n-2} & \\ 0 & 0 & & & \end{bmatrix} \prod_{j=2}^{n} e_{2j}(a_{2j}) \prod_{i=1}^{n} e_{1i}(a_{1i})$$

Continuing this process we obtain the required result. □

Proposition 5.4. *Let A be a ring and u be a unit in A, i.e. $u \in A^\times$*

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \equiv I_2 \mod E_2(A)$$

Proof. $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} = e_{21}(u^{-1})e_{12}(1-u)e_{21}(-1)e_{12}(1-u^{-1})$. □

Lemma 5.5 (Whitehead)**.** *For $\alpha, \beta \in GL_n(A)$*

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \equiv \begin{bmatrix} \beta\alpha & 0 \\ 0 & I_n \end{bmatrix} \mod E_{2n}(A)$$

Proof. Let $A = M_n(A)$ and note $E_2(M_n(A)) \subset E_{2n}(A)$ in Prop. 5.4. □

Proposition 5.6.

$$[GL(A) : GL(A)] = E[A]$$

Proof. Using Lemma 5.5 we can see that

$$\begin{bmatrix} \alpha^{-1}\beta^{-1} & 0 \\ 0 & I_n \end{bmatrix} \equiv \begin{bmatrix} \beta^{-1}\alpha^{-1} & 0 \\ 0 & 1_n \end{bmatrix} \mod E_{2n}(A)$$

So the derived subgroup of $GL_n(A)$ is contained in $E_{2n}(A)$. □

Lemma 5.7. *For a Euclidean domain A we have $SL_n(A) = EL_n(A)$ for all n.*

Proof. With elementary row and column operations arrange the matrix so that the element with the smallest norm is in the top right position. And using elementary row operations reduce it to a matrix with a unit in the top left and 0s in the rest of the first column and first row. Proceeding similarly for the remaining $n-1 \times n-1$ matrix left we reduce it down to a matrix of the form.

$$\begin{bmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & u_n \end{bmatrix}$$

Now apply Whiteheads lemma □

Definition 5.8 (Relative K_1)**.** $SK_1(A) := \ker \det$

Where, $\det : K_1(A) \to A^{\times}$. *We have a split exact sequence*

$$0 \to SK_1(A) \to K_1(A) \to A^{\times} \to 0$$

6 Some results on linear groups

6.1 Suslin's Normality theorem

We now consider a result due to Suslin about the normality of $E_n(A)$ in $GL_n(A)$. The following Lemma due to Vaserstein will be useful.

Lemma 6.1 (Vaserstein)**.** *Let $\alpha \in M_{m,n}(A)$ and $\beta \in M_{n,m}(A)$ then $I_m + \alpha\beta \in GL_m(A)$ implies that $I_n + \beta\alpha \in GL_n(A)$ and*

$$\begin{bmatrix} I_m + \alpha\beta & 0 \\ 0 & (I_n + \beta\alpha)^{-1} \end{bmatrix} \in E_{m+n}(A)$$

Proof. Note that $(I_n + \beta\alpha)^{-1} = I_n - \beta(I_m + \alpha\beta)^{-1}\alpha$. Lem. 5.4 cannot be applied in this case since $n \neq m$ in general. But the idea is nearly the same.

$$\begin{bmatrix} I_m + \alpha\beta & 0 \\ 0 & (I_n + \beta\alpha)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I_m & 0 \\ (I_n + \beta\alpha)^{-1}\beta I_n & \end{bmatrix} \begin{bmatrix} I_m & -\alpha \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -\beta & I_n \end{bmatrix} \begin{bmatrix} I_m & (I_n + \alpha\beta)^{-1}\alpha \\ 0 & I_n \end{bmatrix} \in E_{m+n}(A)$$

We implicitly use Prop. 5.3 to justify that the triangular matrices here are indeed elementary. ☐

Corollary 6.2. *Let $v = (v_1, \ldots, v_n)^t$ and $w = (w_1, \ldots, w_n)^t$ be column vectors in R^n such that $w^t v = 0$, and suppose $w_i = 0$ for some $i \leq n$. Then $I_n + vw^t \in E_n(R)$.*

Proof. When $w_i = 0$ for $i \neq n$ we have $\alpha(I_n + vw^t)\alpha^{-1} = I_n + (\alpha v)(w^t\alpha^{-1})$for $\alpha = e_{in}(-1)e_{ni}(1)e_{in}(-1)$, which acts as a permutation matrix making the nth term i.e. $(w^t\alpha^{-1})^t_n = 0$.

Therefore, without loss of generality we may assume that $w_n = 0$. Now define $w' = (w_1, \cdots, w_{n-1})^t, v' = (v_1, \cdots, v_{n-1}) \in R^{n-1}$. Since $w_n = 0$ and $w^t v = 0 \implies w'^t v' = 0$ also.

Proceeding inductively on n. We can say that $I_{n-1} + v'w'^t \in E_{n-1}(R)$. Note that the base case $n = 1$ is given by Lem. 6.1.

Therefore we have

$$I_n + vw^t = \left[\begin{array}{c|c} & 0 \\ I_{n-1} + v'w'^t & 0 \\ & \vdots \\ \hline * & 1 \end{array}\right],$$

We can make the last row zero using appropriate column transformations using the last column (which are all elementary matrices). Therefore, $I_n + vw^t \in E_n(R)$ ☐

Lemma 6.3. *For v unimodular row in R^n, and $f : R^n \to R$ a $R-$ linear map determined by $e_i \mapsto v_i$. Where e_i is the standard basis element of R^n.*

$$\ker(f) = \left\{ w = (w_1, \cdots w_n)^t \mid \sum_i^n w_i v_i = 0 \right\}$$

and is generated by elements of the form $\{v_j e_i - v_i e_j\}$ for positive $i \leq n$.

Proof. We wish to show that $\ker(f)$ is generated by the elements $\{v_i e_i - v_i e_j \mid 1 \leq i \leq n\}$.

By definition of unimodularity of v, there exist elements $r_1, \ldots, r_n \in R$ such that $\sum_{i=1}^n r_i v_i = 1$. Consider the R-module homomorphism $g : R \to R^n$ given by $g(1) = (r_1, \ldots, r_n)^t$. This provides a splitting on the right of the exact sequence

$$0 \to \ker(f) \to R^n \xrightarrow{f} R \to 0$$

since $f(g(1)) = \sum_i r_i v_i = 1$. So the sequence is split exact and $R^n \cong \ker(f) \oplus R$.

Now, consider a map $h : R^n \to \ker f$ defined as $h(x) = x - g(f(x))$. This creates a splitting on the left side of the exact sequence, since $h|_{\ker(f)} = 1_{\ker(f)}$. Since h is surjective, the elements $h(e_i)$ generate $\ker(f)$.

$$\begin{aligned} h(e_i) = e_i - g(f(e_i)) = e_i - g(v_i) &= e_i - v_i \sum_j r_j e_j \\ = (\sum_j r_j v_j) e_i - \sum_j r_j v_i e_j &= \sum_j r_j (v_j e_i - v_i e_j) \end{aligned}$$

This shows that $\ker(f)$ is indeed generated by the claimed elements. $\square$

We finally generalize Cor. 6.2 to the following lemma which will be used in the proof of the normality theorem.

Proposition 6.4. *Let $n \geq 3$. If $v \in R^n$ is unimodular, and $w \in R^n$ such that $w^t v = 0$, then $I_n + vw^t \in E_n(R)$ and this is also true if w is unimodular and v is arbitrary by transposition.*

Proof. Consider the R-linear map $f : R^n \to R$ defined as $e_i \mapsto v_i$. The condition $w^t v = 0 \implies w^t \in Ker(f)$. By Lem. 6.3, there exists $r_{ij} \in R$ such that we can decompose w^t as such

$$w^t = \sum_{i<j} r_{ij}(v_i e_j - v_j e_i).$$

Label $w_{ij}^t = v_i e_j - v_j e_i$ and decompose $I_n + vw^t$

$$I_n + vw^t = I_n + v \sum_{i<j} w_{ij}^t = \prod_{i<j} (I_n + vw_{ij}^t).$$

We have $w_{ij} v = 0$ and since $n \geq 3$, there exists a zero component and so we have from Cor. 6.2 that $I_n + vw_{ij}^t \in E_n(R)$ for all $i < j$. This completes the proof. $\square$

Theorem 6.5 (Suslin's Normality theorem)**.** *For A, a commutative ring with unity, $E_n(A)$ normal in $GL_n(A)$ for $n \geq 3$.*

Proof. Since $E_n(R)$ is generated by $e_{ij}(\lambda)$ it suffices to check that $\alpha e_{ij}(\lambda)\alpha^{-1} \in E_n(R)$ for $\alpha \in GL_n(A)$.

Recall from 4.7 that the columns of α and the rows of α^{-1} are unimodular.

$$\alpha e_{ij}(\lambda)\alpha^{-1} = \alpha(I_n + \lambda E_{ij})\alpha^{-1} = I_n + \lambda c_i r_j$$

Where c_i is the i^{th} column of α and r_j is the j^{th} row of α^{-1}.

Furthermore since $\alpha^{-1}\alpha = I_n \implies r_j c_i = \delta_{ij} \implies$ using Prop. 6.4 that $\alpha e_{ij}(\lambda)\alpha^{-1} = I_n + \lambda c_i r_j \in E_n(A)$. □

6.2 Local-global principle for unimodular vectors

In this section, we establish an important result that plays a crucial role in proving Suslin's factorial theorem. Specifically, we prove a useful 'Local-global principle' for unimodular polynomial vectors.

Lemma 6.6. *Let S be a multiplicative set in A. For $f(x) \in GL_n(A_S[x])$ such that $f(0) = I_n$, there exists $\hat{f}(x) \in GL_n A[x])$ such that $\hat{f}(x)$ under the localization map maps to $f(sx)$ (for some $s \in S$), and $\hat{t}(0) = I_n$.*

Proof. Since $f(x) \in GL_n(A_S[x])$, there exists $g(x) \in GL_n(A_S[x])$ such that $f(x)g(x) = I_n$. The condition $f(0) = I_n$ implies $g(0) = I_n$.

In particular this means that the diagonal entries belong to $1 + xA_S[x]$ and off diagonal entries are of the form $xA_S[x]$.

Since only finitely many denominators appear in the entries of $f(x)$ and $g(x)$, there exists $s_1 \in S$ which is a common denominator. This allows us 'cancel' the denominators and to define matrices $f(s_1x)$ and $g(s_1x)$ with coefficients in A.

This means there exist $f_1(x)$ and $g_1(x)$ with polynomial entries in $A[x]$ with $f_1(0) = g_1(0) = I_n$ which map to $f(s_1x), g(s_1x)$ under the localization map.

Let $h(x) = f_1(x)g_1(x)$. Then $h(x)$ maps to $f(s_1x)g(s_1x) = I_n$ under the localization map. Since $h(0) = I_n$, there exists $s_2 \in S$ such that $h(s_2x) = I_n$. This implies that $f_1(s_2x)$ is invertible over $A[x]$ with inverse $g_1(s_2x)$.

Therefore, we define $\hat{f}(x) = f_1(s_2x)$. Then $\hat{f}(x) \in GL_n(A[x])$, $\hat{f}(0) = I_n$, and the image of $\hat{f}(x)$ under the localization map is $f(s_1s_2x)$. Thus, setting $s = s_1s_2$, the lemma is proved. □

The below result is a generalization of Lem. 4.13.

Proposition 6.7. *Let A be a commutative ring and S a multiplicative subset of A. For $v = (v_1, \ldots, v_n) \in \mathrm{Um}_n(A[x])$, the following statements are equivalent:*

1. *$v(x) \sim v(0)$ over $A_S[x]$;*

2. *There exists $b \in S$ such that $v(x + by) \sim v(x)$ over $A[x, y]$.*

Proof. Start with 2 $\implies$ 1. Consider a change of variable in $A_S[x, y]$ given by $x = 0, y = b^{-1}x$ so we have that under the localization map $v(0+bb^{-1}x) = v(x) \sim v(0)$ over $A_S[x]$ as required.

For the other direction. There exists $\alpha(x) \in GL_n(A_S[x])$ such that $v(x) \cdot \alpha(x) = v(0)$. Define β as such $\beta(x, y) = \alpha(x + y)\alpha(x)^{-1} \in GL_n(A_S[x, y])$.

$$\begin{aligned} v(x + y)\beta(x, y) &= v(x + y)\alpha(x + y)\alpha(x)^{-1} \\ &= v(0)\alpha(x)^{-1} = v(x) \in A_S[x, y]. \end{aligned}$$

This shows $v(x + y) \sim v(x)$ over $A_s[x, y]$ now we lift it to $A[x, y]$.

Since $\beta(x, 0) = \alpha(x) \cdot \alpha(x)^{-1} = I_n$, we can apply Lem. 6.6 over $A[x]$ to obtain $\hat{\beta}(x, y) \in GL_n(A[x, y])$ such that under the localization map it goes to $\beta(x, sy) \in GL_n(A_S[x, y])$ for some $s \in S$, and $\hat{\beta}(x, 0) = I_n$.

With $y \mapsto sy$ in the above relation lifted to $A[x, y]^n$, we have

$$v(x + sy)\hat{\beta}(x, y) - v(x) = yg(x, y)$$

for some $g(x, y)$ that localizes to 0, so there exists $s' \in S$ such that

$$v(x + ss'y)\hat{\beta}(x, s'y) - v(x) = ys'g(x, s'y) = 0.$$

Choose $n = ss'$ and we are done. □

Proposition 6.8. *Let $v(x) = (v_1(x), \ldots, v_n(x)) \in \mathrm{Um}_n(A[x])$. Define the ideals $\mathfrak{a}$ and $\mathfrak{b}$ as follows:*

$$\begin{aligned} \mathfrak{a} &= \{a \in A \mid v(x) \sim v(0) \text{ over } A_a[x]\} \\ \mathfrak{b} &= \{b \in A \mid v(x + by) \sim v(x) \text{ over } A[x, y]\} \end{aligned}$$

Then $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A, with $\mathfrak{a} = rad(\mathfrak{b})$.

Proof. If $b \in \mathfrak{b}$, then $v(x+by) \sim v(x)$. With a substitution of variables for any $r \in A$, we have $v(x+bry) \sim v(x)$, so $br \in \mathfrak{b}$. If $b, b' \in \mathfrak{b}$, then the substitution $x \mapsto x+b'y$ shows that $\mathfrak{b}$ is an ideal. Since, $v(x+(b'+b)y) \sim v(x+b'y) \sim v(x)$.

The equality $\mathfrak{a} = \mathrm{rad}(\mathfrak{b})$ follows from the above proposition. ☐

Theorem 6.9 (Local-global principle)**.** *Let* $v = (v_1, \ldots, v_n) \in \mathrm{Um}_n(A[x])$*. If* $v(x) \sim v(0)$ *over* $A_{\mathfrak{m}}[x]$ *for all maximal* $\mathfrak{m} \in A$*, then* $v(x) \sim v(0)$ *over* $A[x]$*.*

Proof. Define $\mathfrak{a}$ and $\mathfrak{b}$ as above. Suppose $v(x) \sim v(0)$ over $A_{\mathfrak{m}}[x]$ Prop. 6.7 implies that there exist $b \in A \setminus \mathfrak{m}$ such that $v(x+by) \sim v(x)$ over $A[x, y]$.

i.e $A \setminus \mathfrak{m}$ contains an element of $\mathfrak{b}$ this implies $\mathfrak{b} = A$. And since $\mathrm{rad}(\mathfrak{b}) = \mathrm{rad}(A) = A$, we have $\mathfrak{a} = A$, which implies $v(x) \sim v(0)$ over $A[x]$. ☐

6.3 Suslin's factorial theorem and Mennicke symbols

In this section we will prove a celebrated theorem due to Suslin.

Theorem 6.10 (Suslin's factorial theorem)**.** *Given* $(v_0, \ldots, v_n) \in \mathrm{Um}_{n+1}(A)$ *then*$n! | \prod_{i=0}^{n} m_i$ *implies* $(v_0^{m_1}, \ldots, v_n^{m_n}) \in \mathrm{Um}_{n+1}(A)$

The proof of the theorem in this section can be seen in detail in the papers by Suslin [Sus77] and the expository book by Lam [Lam10].

The converse of this result is also true as proved by Suslin in [Sus82]. But we do not cover that direction here.

Proposition 6.11. *If there exists* $v = (v_0, v_1, \cdots v_n) \in \mathrm{Um}_{n+1}(A)$ *such that* $\bar{v} = (\bar{v}_0, \cdots, \bar{v}_{n-1})$ *is completable over the ring* $\bar{A} = A/Av_n$ *then*

$$(v_0, \cdots, v_{n-1}, v_n^n) \in \mathrm{Um}_{n+1}(A)$$

and is completable in A*.*

Proof. Let $\alpha \in M_n(A)$ be some matrix with first row $(v_0 \cdots, v_{n-1})$ such that $\bar{\alpha} \in GL_n(\bar{A})$ (which we know exists due to v being completable) let $\beta \in M_n(A)$ be the lift of $\bar{\alpha}^{-1}$, i.e., $\bar{\alpha}\bar{\beta} = I_n$. Then

$$\alpha\beta = I_n + v_n\gamma, \quad \beta\alpha = I_n + v_n\delta$$

for some $\gamma, \delta \in M_n(A)$.

The matrix $\begin{bmatrix} \alpha & v_n I_n \\ \delta & \beta \end{bmatrix} \in GL_{2n}(A)$ since

$$\begin{bmatrix} \alpha & v_n I_n \\ \delta & \beta \end{bmatrix} \cdot \begin{bmatrix} \beta & -v_n I_n \\ -\gamma & \alpha \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ * & I_n \end{bmatrix} \in GL_{2n}(A).$$

Note that $\det(\alpha)$ is a unit in $\bar{A}$ using Lem. 5.5 on $\bar{A}$ and pulling back through the surjection $E_n(A) \to E_n(\bar{A})$ we have

$$\begin{bmatrix} v_n^n & 0 \\ 0 & I_{n-1} \end{bmatrix} = (v_n I_n)\epsilon + \det(a)\zeta$$

for some $\epsilon \in E_n(A)$ and $\zeta \in M_n(A)$.

Let $\alpha' = \mathrm{adj}(\alpha)$ be the adjoint of α. Recall that $\alpha \cdot \mathrm{adj}(\alpha) = \det(\alpha) I_n$, define a matrix φ as such

$$\begin{aligned} \varphi = \begin{bmatrix} \alpha & v_n I_n \\ \delta & \beta \end{bmatrix} \begin{bmatrix} I_n & \alpha' t \\ 0 & \epsilon \end{bmatrix} &= \begin{bmatrix} \alpha & \alpha\alpha'\zeta + v_n\epsilon \\ \delta & * \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \det(\alpha)\zeta + v_n\epsilon \\ \delta & * \end{bmatrix} \\ &= \left[\begin{array}{c|cc} \alpha & v_n^n & 0 \\ & 0 & I_{n-1} \\ \hline \delta & \multicolumn{2}{c}{*} \end{array}\right] \in GL_{2n}(R) \end{aligned}$$

We now rewrite φ in the following adjusted block form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}.$$

Where a_1 is $n \times (n+1)$ comprising of a along with the column $(v_n^n, 0, \ldots, 0)^t)$ adjoined at the right, $a_2 = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix}$.

With appropriate elementary transformations on the last n rows of φ, we obtain a matrix of the form

$$\varphi' = \begin{bmatrix} a_1 & a_2 \\ a_3' & 0 \end{bmatrix}$$

Now consider the submatrix of φ' formally complementary to I_{n-1} which we obtain by deleting the 2nd up to the nth rows and the last $n-1$ columns. This is the required invertible $(n+1) \times (n+1)$ matrix with top row $(v_0, \ldots, v_{n-1}, v_n^n)$. $\square$

By induction on n, we obtain the following corollary.

Corollary 6.12. *For* $(v_0, \cdots, v_n) \in \mathrm{Um}_{n+1}(A)$ *the row* $(v_0, v_1, v_2^2, \cdots, v_n^n)$ *is completable.*

We now prove a result involving moving powers of coefficients in a unimodular row. Which in conjunction with the above corollary will prove the forward direction of Th. 6.10.

Proposition 6.13. *For* $(v_0, \cdots, v_n) \in \mathrm{Um}_{n+1}(A)$ *and any* $r \in \mathbb{N}, i \neq j$

$$(v_0, \cdots, v_i^r, \cdots, v_n) \sim (v_0, \cdots, v_j^r, \cdots, v_n)$$

Proof. Begin with the case of $i = 0, j = 1$. Define $f(t) = (v_0^r, v_1 + v_0 t, \cdots, v_n) \in \mathrm{Um}_{n+1}(A[t])$.

Claim that $f(t) \sim f(0)$ over $A[t]$. Using Th. 6.9 Local-global principle to check this equivalence in $A_{\mathfrak{m}}$ for some arbitrary $\mathfrak{m}$ maximal in A.

Assume that $v_0, v_2, \cdots, v_n \in \mathfrak{m}$ else the claim is naturally true as they will be invertible if not. And so $v_1 \in A \setminus \mathfrak{m}$ is a unit.

Note that $(v_0^r, v_1 + v_0 t) \in \mathrm{Um}_2(A[t])$ since, which is naturally completable, unimodularity is due to the fact that not both v_0^r and $v_1 + v_0 t \in \mathfrak{m}$ for if that is the case then $v_0, v_1 \in \mathfrak{m}$ which is a contradiction. Since this is true for all $\mathfrak{m}$. We have by the local-global principle that $(v_0^r, v_1 + v_0 t) \in \mathrm{Um}_2(A[t])$ this is also completable as all length 2 unimodular rows are naturally completable. In particular if $v_0^r b_0 + (v_1 + v_0 t) b_1 = 1$ then consider the matrix $\alpha \in GL_2(A[t])$ defined as

$$\alpha = \begin{bmatrix} v_0^r & v_1 + v_0 t \\ -b_1 & b_0 \end{bmatrix}$$

Therefore over $A[t]$ we have $f(t)$ is also completable with the block matrix

$$\beta = \begin{bmatrix} \alpha & \beta \\ 0 & I_{n-2} \end{bmatrix} \in GL_n(A)$$

where the first row of β equals $(v_2, \cdots, v_n)$. Therefore we have

$$f(t) \sim (1, 0, \ldots, 0) \sim f(0)$$

as claimed.

Now note that $f(t) \sim f(0)$ over $A[t]$ implies that $f(-1) \sim f(0)$ over A

$$(v_0^r, v_1, v_2, \cdots, v_n) \sim (v_0^r, v_1 - v_0, v_2, \cdots, v_n).$$

With the substitution $v_0 \mapsto v_1$ and $v_1 \mapsto v_1 - v_0$

$$\begin{aligned} &\sim (v_1^r, -v_0, v_2, \ldots, v_n) \\ &\sim (v_0, v_1^r, v_2, \ldots, v_n) \end{aligned}$$

The last equivalence is true since the block matrix $\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$ is elementary (with column transformations it is equivalent to the block matrix with I in both diagonal places).

Repeating this procedure for other $i \neq j$ completes the proof. $\square$

Combining Cor. 6.12 and Prop. 6.13 we obtain the proof of the backward direction of Th. 6.10 as required.

References

[Awo10] Steve Awodey. *Category Theory, 2nd Edition*. Oxford University Press, Oxford, England, 2010.

[Bue] Theo. (https://mathoverflow.net/users/11081/theo-buehler) Buehler. Freyd-mitchell's embedding theorem. MathOverflow. URL:https://mathoverflow.net/q/47762 (version: 2012-12-26), Last Accessed November 18, 2024.

[Eis13] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2013.

[FF64] P. Freyd and P.J. Freyd. *Abelian Categories: An Introduction to the Theory of Functors*. A harper international edition. Harper & Row, 1964.

[Fos80] Robert M. Fossum. A characteristic class in algebraic k-theory. *Publications mathématiques et informatique de Rennes*, (S3):63–81, 1980.

[Gro57] Alexandre Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Mathematical Journal, Second Series*, 9(2):119–183, 1957.

[Kar08] M. Karoubi. *K-Theory: An Introduction*. Classics in Mathematics. Springer, 2008.

[Lam99] T.Y. Lam. *Lectures on Modules and Rings*. Graduate Texts in Mathematics. Springer New York, 1999.

[Lam01] T.Y. Lam. *A First Course in Noncommutative Rings*. Graduate Texts in Mathematics. Springer, 2001.

[Lam10] T.Y. Lam. *Serre's Problem on Projective Modules*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2010.

[Lan98] S.M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 1998.

[Lan02] Serge Lang. *Algebra*. Springer, New York, NY, 2002.

[Mat87] H. Matsumura. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1987.

[McG16] Peter McGrath. An extremely short proof of the hairy ball theorem. *The American Mathematical Monthly*, 123(5):502–503, 2016.

[MS74] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[Rie17] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017.

[Ros95] J. Rosenberg. *Algebraic K-Theory and Its Applications*. Graduate Texts in Mathematics. Springer New York, 1995.

[Sus77] A A Suslin. On stably free modules. *Mathematics of the USSR-Sbornik*, 31(4):479, apr 1977.

[Sus82] A. A. Suslin. Mennicke symbols and their applications in the k-theory of fields. In R. Keith Dennis, editor, *Algebraic K-Theory*, pages 334–356, Berlin, Heidelberg, 1982. Springer Berlin Heidelberg.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[WS13] C.A. Weibel and American Mathematical Society. *The K-book: An Introduction to Algebraic K-theory*. Graduate Studies in Mathematics. American Mathematical Society, 2013.