Contents

Algebraic K-Theory

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1 Small K groups

The category of finitely generated projective modules is the main object of study in algebraic K-theory. This is largely motivated by the following theorem due to Swan [Swa62] which relates algebraic K-theory to topological K-theory.

Theorem 1.1 (Swan's theorem). *There exists an equivalence of categories between* $\mathrm{Vect}(X)$ *the category of vector bundles over a compact, Hausdorff space* X *and finitely generated projective* $C(X)$ *modules. With the cross section functor.*

Proof. content... □

1.1 Grothendieck group K_0

The big picture idea that Grothendieck had was that of a free completion of a commutative monoid. Commutative monoids occured in nature very often as finitely generated projective modules/vector bundles.

This is a fairly natural approach which results in a Free-Forgetful adjoint pair between CMon and Ab. We will refer to K book for most of the definitions [WS13]

Proposition 1.1 (Group completion functor). *Assign* $(A, +) \in \mathrm{CMon}$ *to*

$$K_0(A)$$

as

Proposition 1.2 (Eilenberg Swindle). K_0 *for many abelian categories are trivial.*

1.2 Whitehead group K_1

Definition 1.1 (Whitehead group for a ring). $K_1 = \frac{GL(A)}{[GL(A):GL(A)]}$ *Where* $GL(A)$ *denotes the colimit of* $GL_n(A)$ *with* GL_n *realized as a subgroup of* GL_{n+1} *by placing the matrix in the top left corner.*

Theorem 1.2 (Horrocks' theorem). *A vector bundle* E *over* $\mathbf{P}^n$ *splits as a direct sum of line bundles when*

$$H^i(\mathbb{P}^n, E(k)) = 0$$

for all $i = 1, \ldots, n-1$ *and for all* $k \in \mathbb{Z}$.

Appendices

A Projective modules

Recall a **free module** of rank n is one that is isomorphic to n direct sums of its underlying ring. And homomorphisms from free modules to other modules are determined by the image of their generators, i.e. free objects are left adjoints to forgetful functors. [1]

A module P is said to be **projective** if it satisfies the following lifting property, every morphism from P to N factors through an epi into N. Note that the lift need not be unique this is *not* an UMP

$$\begin{tikzcd}
 & M \arrow[d, two heads] \\
P \arrow[ur, dashed] \arrow[r] & N
\end{tikzcd}$$

Lemma A.1 (Free modules are projective).

Proof. Consider the preimages of images of basis of P in N, that lie in M. Then map basis elements from P into these preimages. □

[1]This holds in free monoids $\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$ where $F(X)$ denotes the free monoid generated by elements from the set X and $U(M)$ is the underlying set of a monoid M, refer to [Awo10, p. 208]

Proposition A.1 (Equivalent definitions of projectivity)**.** *TFAE,*

1. *P is projective.*
2. *For all epi's between $M \twoheadrightarrow N$, the induced map $\mathrm{Hom}(P,g) : \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$ sending $f \mapsto g \circ f$ for $g : M \to N$ and $f : P \to M$ is an epi.*
3. *For some epi from a free module F to P, $\mathrm{Hom}(P,F) \to \mathrm{Hom}(P,P)$ is an epi.*
4. *There exists Q s.t. $P \oplus Q$ is free*
5. *Short exact sequences of the form $0 \to A \to B \to P \to 0$ split, i.e. isomorphic to another short exact where middle term is $A \oplus P$* [2]

Proof. $1 \iff 2$ is restatement of definitions.

$2 \implies 3$ also just substitution.

$3 \implies 4$ consider a map in the preimage of identity in $\mathrm{Hom}(P,P)$ which is a splitting (inverse) of the epi F into P,

$$\begin{tikzcd}
& P \arrow[dl, "f"'] \arrow[dr, dashed, "\mathrm{Id}_P = g \circ f"] \arrow[d, phantom, "\Longrightarrow"] & \\
F \arrow[rr, two heads, "g"] & & P
\end{tikzcd}$$

Now we have a short exact sequence $0 \to \ker g \to F \to P \to 0$, and also $f \circ g$ is idempotent so it naturally admits a decomposition $F = \mathrm{Im}(f \circ g) \oplus \mathrm{Ker}(f \circ g)$[3]$=\mathrm{Im}(g) \oplus \mathrm{Ker}(g)$ the first by the 1st isomorphism theorem and the second by f being a mono.

$4 \implies 2$ simply as $\hom(P \oplus Q, -) = \hom(P, -) \oplus \hom(Q, -)$

$1 \iff 5$ should be clear from above.

□

Theorem A.2 (Proj. fin. generated modules over local rings are free)**.**

[2] In general any epis into projective objects split (i.e. have an inverse).

[3] For some idempotent e, $1 - e$ is also an idempotent and images under these two mappings decompose any module, furthermore image of $1 - e$ is just kernel of e

Proof. pick a minimal set of generators and see its residue classes in $M/\mathfrak{m}M$ as the basis of it as a vector space over $R/\mathfrak{m}$.

Now as for some free module $F, F = \varphi(M) \oplus K$ for some K and some homomorphism $\varphi : M \to F$, (by defn of projective module), we get

$$M/\mathfrak{m}M \cong F/\mathfrak{m}F = (R/\mathfrak{m})^n \cong R^n \otimes R/\mathfrak{m} \cong F \otimes R/\mathfrak{m} \cong (\varphi(M) \oplus K) \otimes R/\mathfrak{m}$$

Finally we get $M/\mathfrak{m}M \cong M/\mathfrak{m}M \oplus K/\mathfrak{m}K \implies K = \mathfrak{m}K \implies K = 0$ by Nakayama □

This holds for not necessarily finitely generated modules too refer to [Mat87, Th. 2.5] .

Proposition A.2. *If M is a finitely presented module over a Noetherian ring R (prime ideals fin gen) then TFAE*

1. *M is projective.*
2. *M localized at maximal ideals is free.*
3. *A finite set of elements $\{x_i\}^n$ in R generate R such that $M[x_i^{-1}]$ is free over $R[x_i^{-1}]$.*

This proceeds just from the previous result.

Theorem A.3 (Quillen–Suslin)**.** *Every finitely generated projective module over a polynomial algebra is free.*

This was an open problem for a long time as such the proof is very involved. Refer to [nLa23] or to a condensed proof in [Lan02, p. 848]

B Vector bundles

Definition B.1 (Vector bundle)**.** *A real n dimensional vector bundle is a triple (E, p, B). Which consists of a map $p : E \to B$. Such that for all $b \in B$, $p^{-1}(b)$ has a real vector space structure. Along with the following properties,*

content...

Definition B.2 (Vector bundle mappings)**.** *content...*

Definition B.3 (Sections of a vector bundle)**.** *content...*

Definition B.4 (Pullbacks of bundles)**.** *content...*

Definition B.5 (Whitney sums)**.** *content...*

Definition B.6 (Steifel Whitney Class)**.** *content...*

C Categories

C.1 Abelian Categories

There is a chain of conditions regarding 'abelian'-ness of categories which is roughly understood as follows,

$$\textbf{Abelian} \subseteq \textbf{Pre-Abelian} \subseteq \textbf{Additive} \subseteq \textbf{Ab-Enriched}$$

The motivation behind them is to have categories which resemble algebras.

Ab-Enriched categories are categories such that for objects $A, B \in \mathbf{C}$ the external hom set $\mathrm{Hom}(A, B)$ has the structure of an abelian group, furthermore it has a well defined notion of composition (which is bilinear due to the monoidal product in Ab), $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$.

Proposition C.1. *In Ab-Enriched categories intial and terminal objects coincide (it is often called the zero object)*

Proof. Let $\mathbf{C}$ be an Ab-Enriched category. Note that the Hom-sets between objects have 'zero morphisms', i.e. arrows in the Hom-set which behave like the additive identity in the Ab group induced by it. In particular for $0_{A,B} \in \mathrm{Hom}(A, B)$ we have the property that if $f : B \to C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \to D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathbf{C}$ is initial so there is a unique morphism $0 \to 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \to 0$ we can say that by the zero morphism property $f = 0$ so also 0 is terminal. $\square$

Proposition C.2. *In Ab-Enriched categories finite coproducts coincide with finite products (i.e. biproducts)* [4]

Proof. Let $\mathbf{C}$ be an Ab-enriched category and $A, B \in \mathbf{C}$ consider the product $A \times B$, which is determined by the following UMP,

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

[4]This also holds over categories enriched over commutative monoids.

Consider A and B in place of X in the diagram. By the UMP we have $q_1 : A \to A \times B, q_2 : B \to A \times B$

$$\begin{tikzcd}
A \arrow[dr, "q_1"] \arrow[dd, "1_A"'] & & B \arrow[dl, "q_2"'] \arrow[dd, "1_B"] \\
& A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

So $p_1 q_1 = 1_A$ and $p_2 q_2 = 1_B$ also $p_1 q_2 = p_2 q_1 = 0$.

Now note that $q_1 p_1 + q_2 p_2 = 1_{A \times B}$ as $p_1(q_1 p_1 + q_2 p_2) = p_1$ and $p_2(q_1 p_1 + q_2 p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following UMP holds for some arbitrary $C \in \mathbf{C}$

$$\begin{tikzcd}
A \arrow[r, "r_1"] \arrow[dr, "q_1"'] \arrow[dd, "1_A"'] & C & B \arrow[l, "r_2"'] \arrow[dl, "q_2"] \arrow[dd, "1_B"] \\
& A \times B \arrow[u, dashed, "f"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

Define $f : A \times B \to C$ as $f = r_1 p_1 + r_2 p_2$. Now $f q_1 = r_1$ and $f q_2 = r_2$ if we show uniqueness of f we are done.

Say f' then $(f - f')1_{A \times B} = (f - f')(q_1 p_1 + q_2 p_2) = 0$. So $f = f'$.

$\square$

Definition C.1 (Additive category)**.** *An Ab-Enriched category which has all finite coproducts.*

Functors between additive categories are called *additive functors*. And can be realized as functors which preserve additivity of homomorphisms between modules, $F(f + g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition C.2 (Kernel)**.** *A kernel is a pullback of a morphism $f : A \to B$ and the unique morphism from $0 \to B$. Provided initials and pullbacks exist.*

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \to B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'. [5]

Definition C.3 (Pre-abelian categories)**.** *An additive category with all morphism having kernels and cokernels.*

The above definition is equivalent to saying a pre-abelian category is a Ab-Enriched category with all finite limits and colimits. This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Prop. 5.21]. And we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition C.4 (Abelian category)**.** *Pre-additive categories for which each mono is a kernel and each epic is a cokernel.*

Largely the purpose of abelian categories were motivated by wanting to generalize homological methods and to unify various (co)homology theories. It was defined in the modern formulation by Grothendieck in his Tohuku paper [**?**]. We never directly reference this paper for its mathematical content but it is interesting from a historical perspective.

C.1.1 Examples

Some examples of abelian categories are as follows,

1. **The category of modules.**

2. **A very unique example is that the category of representations of a group is abelian!** A group representation is a group action of

[5]A minor point to note is that in the case of Ab-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a Ab-Enriched category has a initial 0 however this matches up with our intuition.

a group G on some vector space V via invertable maps, alternatively just a group homomorphism $G \to GL(V) = \text{Aut}(V)$. When posed categorically it becomes a littly silly. This is nothing but a functor between 'abelian' objects. The category of representations really is just all such representations between G and all automorphisms of vector spaces.

To show the category of representations is abelian (i.e. the functor category). Note the morphisms in between representations $G \to \text{Aut}(V)$ themselves form a vector space. Direct sums are also easy to define naturally. The only difficult notion is to show all monos are some kernel and all epics are some cokernel. Consider some representations $f : A \to \text{Aut}(V), g : A \to \text{Aut}(W)$. If we assume there is a natural transformations $\alpha : f \to g$. It is monic iff its kernel is trivial. (Left cancellation between composition of function with inclusion and zero map.)

3. **Category of sheaves of abelian groups on some topological space.**

   To recall the definition of a sheaf. The first example we naturally see is the sheaf of continuous real valued scalar functions of n variables or n times differentiable functions on some open U set of $\mathbb{R}^n$. The following properties form useful motivations,

   (a) The original set of functions have restrictions down to any other open $V \subset V$, namesly $f \mapsto f|_V$.

   (b)

   This gives sufficient motivation for the typical definition of a pre-sheaf and a sheaf,

   Definition C.5 (Presheaf)**.** *For a category C a presheaf is any functor $F : \mathbf{C}^{\text{op}} \to \mathbf{Sets}$.*

   In particular in the case for a topological space X a presheaf of groups (or any algebraic object) on X (in truth the set of the lattice of open sets of X ordered by inclusion) is a some contravariant functor F which sends open sets $U \subseteq X$ to some $F(U)$, it respects inclusions (i.e. there for open sets $V \subseteq V$ is a natural transformation $\rho_{UV} : F(U) \to F(V)$

in the form of a restriction). Furthermore, function composition, unitals and empty sets going to empty sets hold (to make it a category). Note that all these notions of presheaves are really just a special case of the categorical definition where the sheaf of groups is really just a group object in the categorical presheaf.

Definition C.6 (Sheaf of sets on a topology)**.** *A sheaf of a topology X is a presheaf which satisfies two additional properties, for open sets $U \in X$ and open covers U_i of U*

(a) (Locality) ***A section****, i.e. an element $s \in F(U)$ goes to zero restricted at U_i for all i implies $s = 0$.*

(b) (Gluing) If there is a collection of sections $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j then there is some $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

These two conditions can be written is short as just saying we require $F(U)$ to be the equalizer for the following diagram

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Now finally we get back to the original example. The category of sheaves of abelian groups on a topological space form a abelian category. Additivity is natural due to the functorial nature of F. A slightly unsatisfying proof is due to 'sheafification', i.e. the left adjoint to the inclusion functor from sheaves into presheaves. Presheves of abelian groups can be understood to have all the required properties to be an Abelian category due the functorial representation. Now due to the following result [**?**] we can extend this notion to the sheaves via sheafification.

C.1.2 Important results

There are a few concepts and definitions relevant in the conversation of abelian categories which we will list out here for completeness. Firstly is the notion of **exact functors** the typical notion of a functor carrying forward exact sequences. With the prefix of left/right added to determine it carrying forward only left or right sides of the exact sequence.

Proposition C.3. *Given a pair of adjoint functors $F \dashv U$ between abelian categories $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ if the left adjoint F is exact, faithful and if $\mathbf{D}$ has enough injectives also $\mathbf{C}$ has enough injectives.*

Proof. content... $\square$

References

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