Higher Algebraic K-Theory: A simplicial approach

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Certificate

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Abstract

This project presents an introduction to the ideas of algebraic K-theory through the lens of simplicial homotopy theory. We cover a large swath of mathematics surrounding simplicial homotopy theory, especially a section on model categories in general before tackling the main portion of this project focused on algebraic K-theory.

There is also a introductory chapter on homological algebra and topological K-theory where we cover the proof of the Swan theorem for the purposes of motivation.

Later we examine Quillen's Q construction and Waldhausen's $s_\bullet$ construction in detail through only simplicial methods. Demonstrating their equivalence and proving the additivity theorem. The additivity theorem is used extensively in proving the standard "fundamental theorems" of K-theory, namely the Dévissage and Resolution theorems.

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Chapter 1

Basic homological algebra

We begin with some basic prerequisite definitions and results on abelian categories and homological algebra, which provide a foundational framework. These concepts will be used extensively throughout the discussion, forming an assumed background. Abelian categories are essential to understanding homological algebra. It is motivated by the fact that they allow for the use of homological methods in a wide variety of applications and help unify various (co)homology theories. They were first introduced by Grothendieck in his seminal Tohoku paper [Gro57].

1.1 Abelian categories

There is a chain of conditions regarding 'abelian'-ness of categories, which can be roughly understood as follows.

$$\textbf{Abelian} \subseteq \textbf{Pre-Abelian} \subseteq \textbf{Additive} \subseteq \textbf{Ab-Enriched}$$

Ab-Enriched categories (sometimes referred to as pre-additive categories) are categories such that for any objects $A, B \in \mathcal{C}$, the external hom-set $\mathrm{Hom}(A, B)$ has the structure of an abelian group. Furthermore, it has a well-defined notion of composition, which is bilinear due to the monoidal product in **Ab**, $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) = \mathrm{Hom}(A, C)$. We shall be giving definitions of the remaining three types of categories in this section.

We have chosen to omit the precise definitions of the coherence conditions for monoidal and monoidally enriched categories to make this section more accessible. Since these conditions are not explicitly used for computations in this work, the basic background described above will suffice. For a more detailed overview of the definitions of monoidal and monoidally enriched categories, refer to [Lan98] for a classical treatment or [Rie17] for an excellent modern exposition.

Example 1.1.1. *A ring is a single object* Ab-*Enriched category (in the same sense how a group is realized as a single object category with all arrows invertible).*

Proposition 1.1.2. *In* **Ab**-*Enriched categories initial and terminal objects coincide (it is often called the zero object).*

Proof. Let $\mathcal{C}$ be an **Ab**-Enriched category. Note that the Hom-sets between objects have 'zero morphisms', i.e. arrows in the Hom-set which behave like the additive identity in the **Ab** group induced by it. In particular for $0_{A,B} \in \mathrm{Hom}(A, B)$ we have the property that if $f : B \to C$ then $f \circ 0_{A,B} = 0_{A,C}$ and $g : A \to D$ then $0_{A,B} \circ g = 0_{D,B}$.

Now suppose $0 \in \mathcal{C}$ is an initial object. There is a unique morphism $0 \to 0$ so in its Hom-set its both the additive inverse and the identity. So for any $f : X \to 0$ we can say that by the zero morphism property $f = 0$ so also 0 is a terminal object. $\square$

Proposition 1.1.3. *In* **Ab**-*Enriched categories finite coproducts coincide with finite products (i.e. biproducts)*[1].

Proof. Let $\mathcal{C}$ be an **Ab**-enriched category and objects $A, B \in \mathcal{C}$ consider the product $A \times B$, which is determined by the following universal property,

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Consider A and B in place of X in the diagram. By the universal property

[1]This also holds over categories enriched over commutative monoids.

we have $q_1 : A \to A \times B, q_2 : B \to A \times B$

$$\begin{tikzcd}
A \arrow[dd, "1_A"'] \arrow[dr, "q_1"] & & B \arrow[dd, "1_B"] \arrow[dl, "q_2"'] \\
& A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

So $p_1 q_1 = 1_A$ and $p_2 q_2 = 1_B$ also $p_1 q_2 = p_2 q_1 = 0$.

Now note that $q_1 p_1 + q_2 p_2 = 1_{A \times B}$ as $p_1(q_1 p_1 + q_2 p_2) = p_1$ and $p_2(q_1 p_1 + q_2 p_2) = p_2$. Claim this q_1, q_2 determine a coproduct $A + B$.

We wish to show the following universal property holds for some arbitrary $C \in \mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "r_1"] \arrow[dd, "1_A"'] \arrow[dr, "q_1"] & C & B \arrow[l, "r_2"'] \arrow[dd, "1_B"] \arrow[dl, "q_2"'] \\
& A \times B \arrow[u, dashed, "f"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B
\end{tikzcd}$$

Define $f : A \times B \to C$ as $f = r_1 p_1 + r_2 p_2$. Now $f q_1 = r_1$ and $f q_2 = r_2$. So if we show uniqueness of f we are done.

Say f' exists $f' : A \times B \to C$, then $(f - f')1_{A \times B} = (f - f')(q_1 p_1 + q_2 p_2) = 0$. So $f = f'$.

$\square$

Definition 1.1.4 (Additive category)**.** *An* **Ab***-Enriched category which has all finite (co)products (i.e. biproducts since they coincide).*

Example 1.1.5. *The category of vector bundles over a topology X is a additive category (but not an abelian category). We will see this in more detail in the following section.*

Functors between additive categories are called *additive functors*. They can be realized as functors which preserve additivity of homomorphisms between modules, $F(f + g) = F(f) + F(g)$.

Before proceeding further it is important to think about kernels and cokernels in the categorical sense.

Definition 1.1.6 (Equalizer)**.** *An equalizer of pair of parallel morphisms $f, g : A \to B$ consists of an object E and a morphism $e : E \to A$ universal such that $fe = ge$.*

Definition 1.1.7 (Kernel)**.** *A kernel of a morphism $f : A \to B$ is the pullback along the unique morphism from $0 \to B$, i.e. it is $p : \ker f \to A$, provided initials and pullbacks exist.*

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function $f : A \to B$ and the unique zero morphism $0_{A,B}$. The kernel object is the part of the domain that is 'going to zero'. [2]

A cokernel is simply its categorical dual.

Definition 1.1.8 (Cokernel)**.** *A cokernel of a morphism $f : A \to B$ is the pushout along the unique morphism $A \to 1$, where 1 is the terminal object.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{coker} f \end{tikzcd}$$

Definition 1.1.9 (Pre-abelian categories)**.** *An additive category with all morphism having kernels and cokernels is called a pre-abelian category.*

The above definition is equivalent to saying a pre-abelian category is a **Ab**-Enriched category with all finite limits and colimits.

This is a consequence to the fact that categories have finite limits iff it has finite products and equalizers [Awo10, Proposition 5.21], and we know equalizers exist because equalizers of two morphisms is just the kernel of $f - g$.

Definition 1.1.10 (Abelian category)**.** *Pre-abelian categories for which each monomorphism is a kernel and each epimorphism is a cokernel.*

[2] A minor point to note is that in the case of **Ab**-Enrichments the 'zero' in the Hom-sets isn't a terminal, its Hom-set specific. When you assume a **Ab**-Enriched category has a initial 0 however this matches up with our intuition.

Equivalently a category is abelian if its pre-abelian and the map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

Example 1.1.11. *Some non examples are:*

1. *The category of Hausdorff topological abelian groups is pre-abelian but not abelian. Since not every morphism which is a monomorphism + epimorphism is necessarily a isomorphism.*

   Consider a Hausdorff abelian topological group G with a non discrete topology and consider G' it's discretization. The inclusion map $G' \to G$ is a monomorphism + epimorphism but not isomorphism (bijective continuous group isomorphism).

2. *The category of torsion free abelian groups* **TFAb** *is pre-abelian but not abelian as the mono $f : \mathbb{Z} \xrightarrow{z \mapsto 2z} \mathbb{Z}$ is not a kernel of some morphism. Say it were and there exists $A \in$* **TFAb** *such that f is the kernel to $g : \mathbb{Z} \to A$, i.e.*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f(z)=2z"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ Z \arrow[r, "g"] & A \end{tikzcd}$$

   this implies $1_{\mathbb{Z}}$ factors through f i.e. $1_{\mathbb{Z}} = f \circ h$ for some unique $h : \mathbb{Z} \to \mathbb{Z}$ which implies $h(1) = 1/2$ which is absurd.

Example 1.1.12. *Some examples of abelian categories:*

1. *The category of modules.*
2. *Category of representations of a group.*
3. *Category of sheaves of abelian groups on some topological space.*

With this definition in mind we will now define a few important constructions we will use often. These are not restricted to abelian categories but we will use them very often in the case of abelian categories, so it is good to see it in action directly with the notion of an abelian category at hand.

Definition 1.1.13 (Subobject)**.** *A subobject Y for some $X \in \mathcal{C}$ is an isomorphism class of monomorphisms $i : Y \to X$.*

$(i_0 : Y_0 \to X) \sim (i_1 : Y_1 \to X) \iff i_1 = i_2 f$ *for an isomorphism* $f : Y_0 \to Y_1$.

With slight abuse of notation we refer to $Y \leq X$ as a subobject of X where Y is just a representative of the codomain of a isomorphism class of monomorphisms into X.

Definition 1.1.14 (Quotients in abelian categories)**.** *For $Y \leq X$ in an abelian category we can define X/Y as the cokernel of the monomorphism $Y \to X$.*

Definition 1.1.15 (Short exact sequences in abelian categories)**.** *A sequence of morphisms of the sort*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is refered to as short exact if i is a monomorphism, p is an epimorphism and $\mathrm{im}(i) = \ker(p)$.

Note the 0 object above is the 'zero' object which we proved exists.

A deep result on abelian categories is the Freyd-Mitchel embedding theorem which helps characterize all small abelian categories in terms of modules.

Theorem 1.1.16 (Freyd-Mitchell)**.** *Every small abelian category can be faithfully embedded as a full subcategory via an exact functor into $R-Mod$ for some ring R.*

The proof for the theorem is very extensive and as such is omitted. The canonical reference is Freyd's own book [FF64]. A proof sketch summarising Freyd's proof is given in an excellent MathOverflow post by the user Theo Buehler [Bue].

Chapter 2

Beginnings of K-theory: Vector bundles

In this section, we introduce the foundational concepts required to prove Swan's theorem, which demonstrates a key connection between projective modules and vector bundles over certain topological spaces. The purpose of proving Swan's theorem is to highlight why projective modules play a central role in algebraic K-theory, serving as essentially algebraic counterparts to vector bundles.

All results presented hold for both real and complex vector bundles. For simplicity, we use k to denote the underlying field.

We present the proof of Serre-Swan theorem as covered in Karoubi's text [Kar08]. This utilizes the notion of a idempotent completion of a category. These are referred to as pseudo-abelian categories in Karoubis text but have become better known as Karoubian categories. We will use the latter terminology.

Definition 2.0.1 (Vector bundle)**.** *A n dimensional vector bundle over k is a triple (E, p, X). Which consists of a continuous map $p : E \to X$ from the total space E to the base space X. Such that for all $x \in X$, $E_x = p^{-1}(x)$ the fibre of x has a k vector space structure. Along with the following property of local trivialization*

1. *For any $x \in X$ there exists a open open $U \subset X$ along with a homeomorphism*

$$h : X \times k^n \to p^{-1}(U)$$

such that for all $c \in U$ the map through h defines an isomorphism between E_c and k^n.

Example 2.0.2 (Trivial vector bundle)**.** *A trivial vector bundle is one in which the total space $E = X \times k^n$ with p just the trivial projection mapping.*

The local trivialization condition can now be understood as saying there exists an open neighbourhood for each point $x \in X$ such that $E|_U$ is a trivial vector bundle

Definition 2.0.3 (Quasi-vector bundle)**.** *A fibre bundle defined as above without the property of local trivialisation is called a quasi-vector bundle.*

Definition 2.0.4 (Rank of vector bundle)**.** *If each E_x has the same dimension n then n is referred to as the rank of the vector bundle.*

Example 2.0.5 (Tangent bundle on S^2)**.** *Let S^2 be the unit sphere in $\mathbb{R}^3$ associate to each point $x \in S^2$ the plane tangent to S^2 at x, label this T_xS^2 (this is the tangent space at point x) then the disjoint union of all the tangent spaces form a vector bundle $TS^2 = \sqcup_{x \in S^2} T_xS^2$.*

Definition 2.0.6 (Vector bundle morphisms)**.** *Two vector bundles (E_1, p, X) and (E_2, p_2, X) are considered isomorphic if there exists a continuous map between their total spaces g such that the below diagram commutes*

$$\begin{tikzcd}
E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

and g induces a vector space homomorphism for each fibre.

If g is a homeomorphism and fibrewise isomorphism then it is a vector bundle isomorphism.

Definition 2.0.7 (Category of vector bundles)**.** *For a fixed base space B the collection of vector bundles over X with arrows as vector bundle homomorphisms forms a category which we denote as $\mathrm{VB}(X)$.*

Definition 2.0.8 (Whitney sums)**.** *For $(E_1, p_1, X), (E_2, p_2, X)$ their Whitney sum denoted as $E_1 \oplus E_2$ consists of total space as $E_1 \oplus E_2$ with fibrewise direct sums of vector spaces.*

A Whitney sum is a biproduct of vector bundles in $\mathrm{VB}(X)$, which makes the category of vector bundles into an additive category, but not abelian. Since the kernel of a morphism of vector bundles need not be a vector bundle.

Along with a Whitney sum there also exists the expected notion of a tensor product of vector bundles this in turn allows us to view $\mathrm{VB}(X)$ as a symmetric monoidal category in the natural way.

In the particular case when $E_1 = X \times k^n, E_2 = X \times k^m$ are both trivial vector bundles over X. We can describe the vector bundle morphisms between them explicitly. This is a result we will use very often in the process of proving Swans theorem.

We have, associated to any vector bundle morphism $g : E_1 \to E_2$ a natural linear map $g_x : k^n \to k^m$ for each $x \in X$. Consider $\widetilde{g} : X \to \mathrm{Hom}_k(k^n, k^m)$ defined as $\widetilde{g}(x) = g_x$.

Theorem 2.0.9 (Characterization of quasi-vector bundle morphisms)**.** *The map $\tilde{g} : X \to Hom_k(k^n, k^m)$ as defined above is continuous*[1]*. If $h : X \to Hom_k(k^n, k^m)$ is continuous. Then the map $\tilde{h} : E_1 \to E_2$ which induces $h(x)$ on each fiber is a morphism of vector bundles.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a choice of basis for k^n and $\{e'_1, \ldots, e'_m\}$ for k^m.

Now the fibrewise maps $g_x \in \mathrm{Hom}_k(k^n, k^m)$ may be represented by a matrix $(a_{ij}(x))$.

The function $x \mapsto a_{ij}(x)$ is obtained by composing the below sequence of continuous maps:

$$X \xrightarrow{x \mapsto (x, e_j)} X \times k^n \xrightarrow{g} X \times k^m \xrightarrow{(x,v) \mapsto v} k^m \xrightarrow{\pi_i} k$$

Therefore, $\tilde{g}$ is continuous.

If h is continuous. Once again composing a sequence of continuous maps we obtain the required morphism of vector bundles.

$$X \times k^n \xrightarrow{(x,v) \mapsto (x, h(x), v)} X \times \mathrm{Hom}_k(k^n, k^m) \times k^n \xrightarrow{(x,f,v) \mapsto (x, f(v))} X \times k^m$$

where $h(x) = f$. $\square$

Proposition 2.0.10 (Sufficient condition for vector bundle isomorphism)**.** *Let E and F be two vector bundles over X, and let $g : E \to F$ be a morphism of vector bundles which are fibrewise bijective. Then g is an isomorphism of vector bundles.*

[1] With respect to the unique topological vector space topology on $\mathrm{Hom}_k(k^n, k^m)$ making any $k^t \cong L(k^n, k^m)$ into a homeomorphism.

Proof. Let $h : F \to E$ be the map defined by $h(v) = g_x^{-1}(v)$ for $v \in F_x$. We need to prove h is continuous and so indeed a map in $\mathrm{VB}(X)$. The other conditions for isomorphisms are clearly met. If we prove continuity locally for every neighbourhood then we have the required continuity globally.

Consider a neighbourhood U of x and the local trivialization isomorphisms $\beta : E|_U \to U \times M$ and $\gamma : F|_U \to U \times N$. Let $g_1 = \gamma g_U \beta^{-1}$.

Then $h_U = \beta^{-1} h_1^{-1} \gamma$, where h_1 is defined as $\widetilde{h_1}(x) = (\widetilde{g_1}(x))^{-1}$ (as in the above result Theorem 2.0.9), h_1 is continuous. Therefore, h is continuous on a neighbourhood of each point of F, implying h is continuous on all of F. $\square$

2.1 Sections of a vector bundle

Definition 2.1.1 (Sections of a vector bundle)**.** *For a vector bundle (E, p, X) a section refers to a continuous map $s : X \to E$ such that $ps = 1_X$, where 1_X denotes the identity map on X.*

These sections equivalently are a homomorphism of vector bundles from the trivial line bundle $(X \times \mathbb{R}, \pi_X, X) \to (E, p, X)$.

We assume all sections to be continuous without necessarily specifying it.

Definition 2.1.2 (Linear independence of sections)**.** *A sequence $s_1, \cdots, s_n$ of sections on a vector bundle E is said to linearly independent if they are linearly independent for each point x.*

Furthermore the morphism $f : X \times k^n \to E$ given by $(x, c_1, \cdots, c_n) \mapsto \sum_{i=1}^n c_i s_i(x)$ induces an isomorphism of fibres if E has rank n due to Proposition 2.0.10.

Example 2.1.3 (Zero section)**.** *The map $s : X \to E$ sending every point x to the 0 vector in E_x.*

Definition 2.1.4 (Vector space of sections)**.** *For a vector bundle $p : E \to X$ the set of continuous sections of E is denoted as $\Gamma(X, E)$. It is a vector space with vector addition defined as $(s+t)(x) = s(x)+t(x)$ and scalar multiplication as $\lambda s(x) = (\lambda s)(x)$ for $\lambda \in k$.*

Proposition 2.1.5 (Section functor)**.** *$\Gamma(X, E)$ can be realized as a $C(X)$ module where $C(X)$ denotes the ring of continuous functions from X to k. With scalar multiplication defined as $(c \cdot s)(x) = c(x)s(x)$ for $s \in \Gamma(X, E), c \in C(X)$.*

For a trivial vector bundle $E = X \times k^n$ we have $\Gamma(X, E)$ corresponds to a free module $C(X)^n$. If E is a arbitrary vector bundle which is a direct summand of a trivial vector bundle then $\Gamma(X, E)$ corresponds to a projective $C(X)$ module.

In particular If $E \oplus E' \cong X \times k^n$ we have $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong C(X)^n$, i.e. $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ is an additive functor.

In fact this 'stably trivial' property of a vector bundle is always true when X is compact. This is an essential result in the proof of Swans theorem. In order to prove it we need a lemma for paracompact spaces first.

Definition 2.1.6 (Paracompact space)**.** *A Hausdorff topological space X is said to be paracompact if every open cover of X has a locally finite open refinement.*

(An open cover is locally finite if there exists a neighbourhood for every $x \in X$ which intersects only finitely many elements of the cover. A refinement of an open cover $\{U_i\}$ consists of a open cover $\{V_j\}$ such that for each $j, V_j \subset U_i$).

Lemma 2.1.7. *Let X be a paracompact space, and let E and F be vector bundles over X. Suppose $\alpha : E \to F$ is a morphism such that $\alpha_x : E_x \to F_x$ is surjective for each point $x \in X$. Then there exists a morphism $\beta : F \to E$ such that $\alpha \circ \beta = \mathrm{id}_F$.*

Proof. In this proof we very liberally use the various morphisms as seen in Theorem 2.0.9.

Fix a point $x \in X$ by local trivialization we pick a neighborhood U of x such that E_U and F_U are trivial bundles. That is, $E|_U \cong M \times V$ and $F|_U \cong U \times N$.

Now with this representation we have $\alpha|_U : U \times M \to U \times N$, can be expressed as $\tilde{\theta}$ (as in 2.0.9) for the associated continuous map: $\theta : U \to \mathrm{Hom}_{\mathrm{VB}}(V, W)$.

Decomposing M as $N \oplus \ker(\theta(x))$ (which we can do thanks to surjectivity of α_x) allows us to choose a matrix representation for $\theta(y) : M \to N$ i.e. $\theta(y) : N \oplus \ker(\theta(x)) \to N$ as

$$\theta(y) = \begin{bmatrix} \theta_1(y) & \theta_2(y) \end{bmatrix},$$

the first component goes to 1 and the second to 0 continuously, i.e. an endomorphism of N. Realized as a topological vector space $\mathrm{Aut}(N)$ is an open subset (i.e. vector subspace) of $\mathrm{End}(N)$.

Therefore we can pick a neighbourhood of x say V_x such that $\theta_1(y)$ is an isomorphism for $y \in U_x$. Consider associated to this the map $\theta' : V_x \to \mathrm{Hom}_{\mathrm{VB}}(N, M)$ represented now by the matrix,

$$\theta'_x(y) = \begin{bmatrix} \theta_1(y)^{-1} \\ 0 \end{bmatrix}$$

This now induces the morphism $\tilde{\theta}'_x : F_| V_x \to E_| V_x$ so that $\alpha_{V_x}\tilde{\theta}'_x = 1$.

Varying the point x enables us to construct a locally finite open cover $\{V_i\}$ of X obeying the following properties, all morphisms $\beta_i : F_{V_i} \to E_{V_i}$ are right inverses of α_{V_i} as seen.

Now consider a 'partition of unity' $\{\eta_i\}$ associated with the cover [2] and $\beta : F \to E$ is defined using it as $\sum_i \eta_i(x)\beta_i(e)$ for $e \in E_x$. Continuity is maintained due to the fact that these sums are necessarily finite (see footnote). This gives us

$$\alpha\beta(e) = \sum_i \eta_i(x)(\alpha\beta_i)(e) = \left(\sum_i \eta_i(x)\right)(\alpha\beta_i(e)) = e,$$

which completes the proof. □

Theorem 2.1.8. *If E is a vector bundle over a compact base space X then there exists a vector bundle E' such that $E \oplus E' \cong X \times k^n$.*

Proof. Pick a finite open cover $\{U_i\}_{i=1}^r$. By local trivialisation we know that $E|_{U_i} \cong U_i \times k^{n_i}$.

Let $\{\eta_i\}$ be a 'partition of unity' associated to the cover similarly as before. So there exist n_i linearly independent sections $s_i^1, s_i^2, \ldots, s_i^{n_i}$ of $E|_{U_i}$ (as seen in Definition 2.1.2). By extending these local sections to zero outside U_i, we obtain globally defined sections $\eta_i s_i^1, \eta_i s_i^2, \ldots, \eta_i s_i^{n_i}$, which are linearly independent sections of $E|_{V_i}$, where $V_i = \eta_i^{-1}((0,1])$.

Let σ_i^j denote the sections $\eta_i s_i^j$ for $1 \leq j \leq n_i$. These sections $\sigma_i^j(x)$ generate E_x as a vector space for each $x \in X$. As constructed in $Definition$2.1.2, there exists a morphism

$$\alpha : T = X \times k^n \to E,$$

[2] A sequence of non negative real valued functions $\{\eta_i\}$ whose sum evaluated for every point $x \in X$ is 1, and such that every point $x \in X$ has an open neighbourhood not intersecting the support of η_i for all but finitely many i. This implies the sums are finite and therefore well defined.

where $n = \sum_{i=1}^{r} n_i$, such that $\alpha_x : T_x \to E_x$ is surjective for every $x \in X$.

Now by the above Lemma for paracompact spaces, there exists a morphism $\beta : E \to T$ such that $\alpha \circ \beta = 1_E$. Let E' be the kernel of the idempotent morphism $p = \beta \circ \alpha$.

Define a morphism from $E \oplus E'$ to T by taking the sum of $\beta : E \to T$ and the inclusion $i : E' \to T$. This morphism induces a fibrewise isomorphism since $E'_x \cong \ker(p_x)$ and $E_x \cong \ker(1 - p_x)$. Thus, the morphism is an isomorphism by Proposition 2.0.10. $\square$

2.2 Karoubian categories and some results on vector bundles

In this section we cover an essential theorem which is applied in order to prove Swan's theorem. We follow the results as proven in Karoubi's book [Kar08].

Definition 2.2.1 (Karoubian category)**.** *A Karoubi category is an* **Ab**-*Enriched category such that every idempotent endomorphism (a morphism of the form $e : A \to A$ such that $e^2 = e$) has a kernel.*

Note that in an abelian category proper all morphisms have kernels and cokernels in this case its only the idempotents.

Example 2.2.2. *The category of vector bundles is Karoubian. We already know it is additive and so* **Ab**-*Enriched. In the conclusion of Theorem 2.1.8 we have implicitly shown that the idempotent $p \in \mathrm{VB}(X)$ has a kernel.*

Example 2.2.3. *Hinting towards Swans theorem we can see that also* $\mathrm{Proj}(A)$ *for a ring A is Karoubian.*

Idempotent elements occur ubiquitously in $K-$theory we will see another useful application of idempotents in computations of K_0.

The first theorem we cover in this section is well known by various different names. It involves showing the existence of a Karoubi envelope for an additive category. But this is often referred to as the Cauchy completion or idempotent completion.

Theorem 2.2.4. *[Existence of Karoubi envelope] Let $\mathcal{C}$ be an additive category. Then there exists a Karoubian category $\widetilde{\mathcal{C}}$ and a fully faithful additive* *functor $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ universal in the sense that for any other additive functor $\psi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Karoubian there exists a unique additive functor $\psi' : \widetilde{\mathcal{C}} \to \mathcal{D}$ such that the below diagram commutes.*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[dr, dashed, "\psi'"] & \\
\mathcal{C} \arrow[u, "\varphi"] \arrow[r, "\psi"'] & \mathcal{D}
\end{tikzcd}$$

Proof. We directly construct $\widetilde{\mathcal{C}}$ and φ. The objects of $\widetilde{\mathcal{C}}$ are of the form (E, p) for $E \in \mathrm{Ob}(\mathcal{C})$ and p idempotent endomorphism over E.

Arrows between two such objects (E, p) and (F, q) comprise of the arrows $(f : E \to F) \in \mathcal{C}$ such that $fp = qf = f$. Composition of morphisms is inherited from $\mathcal{C}$. The identity morphism comprises of $1_{(E,p)} = p$, and te sum of two objects $(E, p) \oplus (F, q)$ is defined naturally as $(E \oplus F, p \oplus q)$. This demonstrates that $\widetilde{\mathcal{C}}$ is indeed an additive category.

It remains to be verified that it is indeed Karoubian which is the main concern. Let f be an idempotent endomorphism of the object (E, p) in $\widetilde{\mathcal{C}}$ we wish to show that f indeed has a kernel.

Examining the below commutative diagram will help us conclude. The elements F, g, q, h are defined directly as seen below. Note that the condition $fp = pf = f$ is present in the righthand square.

$$\begin{tikzcd}
& F \arrow[dl, dashed, "h"'] \arrow[dr, "g"] \arrow[ddd, bend left, "q"] & & \\
E \arrow[rr, "p-f"] \arrow[d, "(1-f)p"'] & & E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\
E \arrow[rr, "p-f"'] & & E \arrow[r, "f"'] & E \\
& F \arrow[ul, dashed, "h"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

Since $(1-f)p$ is itself an idempotent endomorphism on E we have $(E, (1-f)p)$ is an object in $\widetilde{\mathcal{C}}$ and $p - f$ defines an arrow from this object to (E, p). Since $(p - f)((1 - f)p) = (p)(p - f) = p - f$ as expected.

Now we claim that the object $(E,(1-f)p)$ with the arrow $p-f$ is a kernel of $f:(E,p)\to(E,p)$. In particular it is a pullback of the below form. The diagram below drawn internally in $\widetilde{C}$ makes this image clearer.

$$\begin{tikzcd}
(F,q) \arrow[dr, dashed, "h"] \arrow[drr, bend left] \arrow[ddr, bend right, "g"'] & & \\
& (E,(1-f)p) \arrow[r] \arrow[d, "p-f"'] & 0 \arrow[d] \\
& (E,p) \arrow[r, "f"] & (E,p)
\end{tikzcd}$$

Compare this to the first diagram. $(F,q)\xrightarrow{g}(E,p)$ is picked such that $fg=0$. And we obtain uniqueness of h is due to the fact that any such arrow h must obey the expression $h=(1-f)ph=p(1-f)h=pg=g$. Conversely if $h=g$ the diagram commutes naturally.

This shows that $\widetilde{\mathcal{C}}$ is Karoubian. Finally we construct $\varphi:\mathcal{C}\to\widetilde{\mathcal{C}}$ defined as sending $E\in\mathrm{Ob}(\mathcal{C})$ to $(E,1_E)$ and $\varphi(f)=f$. This is naturally a full faithful functor by construction.

Also we can see that based on an analogous argument (E,p) is the kernel of $1-p$ as an idempotent endomorphism over $\varphi(E)=(E,1_E)$. Meaning that $\varphi(E)\cong(E,p)\oplus(E,1-p)$, i.e. the functor is additive.

Finally to show that these constructions define ψ'.

If $\psi:\mathcal{C}\to\mathcal{D}$ (resp. $\psi':\widetilde{\mathcal{C}}\to\mathcal{D}$) is an additive functor from $\mathcal{C}$ (resp. $\widetilde{\mathcal{C}}$) to another Karoubian category $\mathcal{D}$, such that $\psi'\varphi\cong\psi$. Then we have $\psi(\ker f)\cong\ker(\psi'(f))$ for every idempotent endomorphism f. Hence $\psi'(E,p)=\ker\psi(1-p):\psi(E)\to\psi(E)$ and $\psi'(f)=\psi(f)_{\ker\psi(1-p)}$ on the objects and morphisms respectively. Conversely, these formulas define ψ' (up to isomorphism). $\square$

Before proceeding for the last result in this section, recall the definition of equivalence of categories.

Definition 2.2.5 (Equivalence of categories)**.** *Two categories $\mathcal{C},\mathcal{D}$ are said to be equivalent if there exist functors $E:\mathcal{C}\rightleftarrows\mathcal{D}:F$ and a pair of natural isomorphisms $\alpha:1_{\mathbf{C}}\to F\circ E$ and $\beta:1_{\mathbf{D}}\to E\circ F$.*

This is a weaker condition than isomorphism of categories in which we have an actual equality instead of natural isomorphism.

A more useful form of the definition is as such, the proof may be seen in [Awo10][7.7.25]

Proposition 2.2.6. *A functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories iff F has the following properties*

1. *F is full (The map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ defined as $f \mapsto F(f)$ is surjective for all $A, B \in \mathrm{Ob}(\mathcal{C})$).*

2. *F is faithful (The map $F_{A,B}$ as defined above is injective for all pairs A, B).*

3. *F is essentially surjective on objects (For every $D \in \mathrm{Ob}(\mathcal{D})$ there exists $C \in \mathrm{Ob}(\mathcal{C})$ such that $FC \cong D$).*

Corollary 2.2.7. *Let $\mathcal{C}$ be an additive category, $\mathcal{D}$ a Karoubian category, and $\psi : \mathcal{C} \to \mathcal{D}$ an additive functor which is fully faithful such that every object of $\mathcal{D}$ is a direct factor of an object in the image of ψ. Then the functor ψ' as defined in Theorem 2.2.4 forms an equivalence between the categories $\mathcal{C}$ and $\mathcal{D}$.*

Proof. We will prove that ψ' is essentially surjective and fully faithful.

Let $G \in \mathcal{D}$. We seek $X \in \widetilde{\mathcal{C}}$ such that $\psi'(X) \cong G$.

By the hypothesis we have that for $G \in \mathcal{D}$, there exists $E \in \mathcal{C}$ and $G' \in \mathcal{D}$ with $\psi(E) \cong G \oplus G'$. Due to this we can choose an idempotent endomorphism $q : \psi(E) \to \psi(E)$ such that $\mathrm{Ker}(q) \cong G$ we are in a pseudo-abelian category.

Now as ψ is fully faithful, there is an idempotent $p : E \to E$ in $\mathcal{C}$ with $\psi(p) = q$. Then by the formulas in the end of Theorem 2.2.4 we have $G \cong \varphi'(E, 1-p)$. This proves essential surjectivity.

Lastly to prove φ' is fully faithful consider two objects $H, H' \in \widetilde{\mathcal{C}}$ direct factors of $\varphi(E), \varphi(E')$. Then the following diagram shows that $\psi'_{H,H'}$ is an isomorphic function,

$$\begin{tikzcd}
\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\varphi(E), \varphi(E')) \arrow[r, "\cong"] \arrow[dr, "\psi'_{\varphi(E),\varphi(E')}"'] & \mathrm{Hom}_{\mathcal{C}}(E, E') \arrow[r, shift left] \arrow[d, "\psi_{E,E'}"] & \mathrm{Hom}_{\widetilde{\mathcal{C}}}(H, H') \arrow[l, shift left] \arrow[d, "\psi'_{H,H'}"] \\
& \mathrm{Hom}_{\mathcal{D}}(\psi(E), \psi(E')) \arrow[r, shift right] & \mathrm{Hom}_{\mathcal{D}}(\psi'(H), \psi'(H')) \arrow[l, shift right]
\end{tikzcd}$$

where the horizontal arrows are induced by the decompositions $\varphi(E) = H \oplus H_1$ and $\varphi(E') = H' \oplus H'_1$, since $\psi_{E,E'}$ is an isomorphism by hypothesis. $\square$

2.3 Swan's theorem

We can now prove the celebrated Swan's theorem with the results we've built up so far.

Theorem 2.3.1 (Swan's theorem)**.** *Let X be a compact Hausdorff space, and let $A = C(X)$. Then the section functor Γ induces an equivalence of categories* $\mathrm{VB}(X) \simeq \mathrm{Proj}(C(X))$*.*

Proof. Since we assume that X is compact the section map Γ as seen in Proposition 2.1.5 is indeed a functor $\Gamma : \mathrm{VB}(X) \to \mathrm{Proj}(C(X))$ due to Theorem 2.1.8. Furthermore it induces a functor $\Gamma_T : \mathrm{VB}_T(X) \to \mathrm{Free}(C(X))$.

Where $\mathrm{VB}_T(X)$ refers to the full subcategory of $\mathrm{VB}(X)$ consisting of the trivial bundles over X, and $\mathrm{Free}(C(X))$ refers to finitely generated free modules over $C(X)$.

Since $C(X)^n \cong \Gamma_T(E)$ for $E = X \times k^n$, we have Γ_T is essentially surjective.

If $F : X \times k^p$ is some other trivial vector bundle and $f : E \to F$ is a morphism of vector bundles then as seen in Theorem 2.0.9 we have full faithfullness of the functor Γ_T. Which shows that Γ_T induces a equivalence of categories between $\mathrm{VB}_T(X)$ and $\mathrm{Free}(C(X))$.

To extend this to our required case we make use of Theorem 2.2.4 and Theorem 2.2.7.

Comparing with Theorem 2.2.4 since $\mathrm{VB}(X)$ being Karoubian itself is naturally the Karoubian envelope of its subcategory $\mathrm{VB}_T(X)$ and with the functor Γ realized as ψ' we see the below diagram commutes.

$$\begin{tikzcd}
VB(X) \arrow[dr, "\Gamma=\psi'"] & \\
\mathrm{VB}_T(X) \arrow[u, "i"] \arrow[r, "\psi"'] & \mathrm{Proj}(C(X))
\end{tikzcd}$$

Finally due to Theorem 2.2.7 we are done. $\square$

Chapter 3

Basic homotopical algebra

In this section, we introduce some terminology and machinery necessary for the proceeding sections. These concepts are essential for understanding the fact that the homotopy theory of topological spaces and the homotopy theory of simplicial sets (which we will be working with) are essentially the same.

The concept of a model category was introduced by Quillen to characterize the general notion of a "homotopy theory." We provide a brief introduction to the concepts of model categories and Quillen equivalences. Most of the results and theorems mentioned here are as written in Quillen's seminal book [Qui67]. We also refer to an article on the nLab [nLa25] which follows Quillen's notations.

3.1 Closed model categories

There exist various formulations of the concept of a model category, some of which contain axioms that were later found to be redundant.

To present the material in the cleanest manner possible without spending too much time on the various existing definitions, we will follow the definitions given by Riehl [Rie].

We begin with the notion of a weak factorization system.

Definition 3.1.1 (Weak factorization system). *A weak factorization system on a category $\mathcal{C}$ consists of $(\mathcal{L}, \mathcal{R})$ two classes of morphisms such that,*

1. *Every morphism $f : X \to Y$ factors as a composition of maps $\lambda \in$*

$L, \rho \in R$

$$\begin{tikzcd} & Z \arrow[dr, "\rho\in\mathcal{R}"] & \\ X \arrow[ur, "\lambda\in\mathcal{L}"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

2. *$\mathcal{L}$ is the class of maps having the left lifting property with respect to $\mathcal{R}$, i.e., $\mathcal{L}$ consists of the morphisms $\lambda : A \to B$ such that for all $\rho \in \mathcal{R}$ and all morphisms in $\mathcal{C}$ making the outer square in the below diagram commute, there exists a lift h which makes the entire diagram commute.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\lambda"'] & X \arrow[d, "\rho"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

3. *$\mathcal{R}$ is the class of maps having the right lifting property with respect to $\mathcal{L}$, i.e., $\mathcal{R}$ consists of the morphisms $\rho : X \to Y$ such that for all $\lambda \in \mathcal{L}$, and all morphisms in $\mathcal{C}$ making the outer square in the above diagram commute, there exists a lift h which makes the entire diagram commute.*

Definition 3.1.2 ((Closed) Model category)**.** *A category $\mathcal{C}$ with 3 distinguished classes of morphisms $(\mathcal{W}, \mathcal{F}, \mathcal{C})$, and satisfying the following conditions:*

1. *$\mathcal{C}$ has all finite limits and colimits, (i.e. it is bicomplete).*

2. *$\mathcal{W}$ has the '2-out-of-3' property. If two out of $f, g, fg \in \mathcal{W}$ then so does the third.*

3. *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ both form weak factorization systems on $\mathcal{C}$.*

There exist common names for these distinguished classes of maps as such:

- Elements of $\mathcal{W}$ are weak equivalences.
- Elements of $\mathcal{C}$ are fibrations.
- Elements of $\mathcal{F}$ are cofibrations.
- Elements of $\mathcal{F} \cap \mathcal{W}$ are called acyclic fibrations.

- Elements of $\mathcal{C} \cap \mathcal{W}$ are called acyclic cofibrations.

It is often useful to talk about the weaker notion of a category with weak equivalences.

Definition 3.1.3 (Category with weak equivalences)**.** *A category $\mathcal{C}$ with a class of morphisms $\mathcal{W}$ such that $\mathcal{W}$ contains all isomorphisms and has the* 2*-out-of-*3 *property.*

Quillen's original definition of a model category contained a distinction to talk about a 'closed' model category which additionally had the axioms of closure under retraction, weak equivalences containing all isomorphisms and closure of $\mathcal{F}, \mathcal{C}$ under pullbacks/pushouts.

This distinction is largely unused when using the terminology of weak factorization systems due to the below theorem. Most authors refer to a closed model category to mean the same thing as just a model category.

Recall the definition of a retract in a category.

Definition 3.1.4 (Retract)**.** *Let $\mathcal{C}$ be a category. A retract of an object $B \in \mathcal{C}$ is an object A along with morphisms $s : A \to B, r : B \to A$ such that $ri = 1_A$. The map r is the retraction of B onto A.*

$$A \xrightarrow{s} B \xrightarrow{r} A$$

Definition 3.1.5 (Arrow category)**.** *Given any category $\mathcal{C}$ we may construct a category called the arrow category, denoted as $\mathcal{C}^2$ as such,*[1]

- *The objects in $\mathcal{C}^2$ are the morphisms of $\mathcal{C}$.*
- *An arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathcal{C}^2$ is the following commutative square*

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"'] & B' \end{tikzcd}$$

where g_1, g_2 are arrows in $\mathcal{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$

[1]Sometimes also denoted as $\mathrm{Arr}(\mathcal{C})$ or $\mathcal{C}^{\to}$

- *Composition is componentwise (gluing two squares).*

We will often only be concerned about retracts in $\mathcal{C}^2$. To be explicit a retract in $\mathcal{C}^2$ will be a commutative diagram of the following sort. Here f is a retract of g.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, equal, "1_A"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
D \arrow[r] \arrow[rr, bend right, equal, "1_D"'] & C \arrow[r] & D
\end{tikzcd}$$

Proposition 3.1.6. *Let $\mathcal{C}$ be a category and $(\mathcal{L}, \mathcal{R})$ a weak factorization system on it. Then,*

1. *Both contain all isomorphisms of $\mathcal{C}$,*
2. *Both are closed under compositions,*
3. *Both are closed under retractions in $\mathcal{C}^2$,*
4. *$\mathcal{L}$ is closed under pushouts, $\mathcal{R}$ is closed under pullbacks.*

Proof. 1. (Isomorphisms) Given an arbitrary commutative diagram of the sort, where i is an isomorphism.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "i"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

A lift of the form fi^{-1} always exists, therefore it exists when $g \in \mathcal{R}$ this implies $i \in \mathcal{L}$. Dually we also have when g is an isomorphism and $i \in \mathcal{L}$ this implies $g \in \mathcal{R}$.

2. (Compositions) Suppose $\rho_1, \rho_2 \in \mathcal{R}$ and $\lambda \in \mathcal{L}$ as in the below diagram. We wish to show there exists a lift $h : C \to X$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dd, bend right, "\lambda"'] & X \arrow[d, "\rho_1"] \\
B \arrow[d] & Y \arrow[d, "\rho_2"] \\
C \arrow[r] & Z
\end{tikzcd}$$

First paste A downwards along ρ_1, f to obtain, a lift $h_1 : C \to Y$.

$$\begin{tikzcd} A \arrow[r, "\rho_1 \circ f"] \arrow[d, "\lambda"'] & Y \arrow[d, "\rho_2"] \\ C \arrow[r] \arrow[ur, dashed, "h_1"] & Z \end{tikzcd}$$

Now paste C across this lift to obtain out required lift $h : C \to X$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda"'] & X \arrow[d, "\rho_1"] \\ C \arrow[r, "h_1"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

This gives us a right lift for $\rho_2 \circ \rho_1$ in our original diagram which implies it is in $\mathcal{R}$. Dually we also have $\mathcal{L}$ is closed under compositions.

3. (Retracts) Let f be a retract of $\lambda \in \mathcal{L}$

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, equal, "1_A"] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\lambda"] & A \arrow[d, "f"] \\ D \arrow[r] \arrow[rr, bend right, equal, "1_D"'] & C \arrow[r] & D \end{tikzcd}$$

For an arbitrary square of the form,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "\rho \in \mathcal{R}"] \\ D \arrow[r] & Y \end{tikzcd}$$

Paste it to the right of the retract diagram and obtain a lift $h : C \to X$. Then the sequence $A \xrightarrow{f} D \to C \xrightarrow{h} X$ gives a lift across f for every $X \xrightarrow{\rho} Y$. Which shows that $f \in \mathcal{L}$, and that $\mathcal{L}$ is closed under retracts. Dually we obtain $\mathcal{R}$ is closed under retracts.

4. (Pushouts/pullbacks) We show $\mathcal{R}$ is closed under pullbacks. Say $\rho \in \mathcal{R}$ in the below pullback diagram. We wish to show $r \in \mathcal{R}$ too.

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\rho"] \\
Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

For $\lambda \in \mathcal{L}$ consider the squares of the kind.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\lambda"'] & X \times_Z Y \arrow[d, "r"'] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

Paste this to the left of the pullback diagram and obtain a lift against ρ on the outer rectangle $h_1 : B \to X$. This lift allows us to utilize the universality of the pullback to obtain a unique map $h : B \to X \times_Z Y$ as such.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \times_Z Y \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\exists! h"'] \arrow[urr, bend left, "h_1"] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

We wish to show this h is indeed a lift against r. The bottom right triangle commutes due to the pullback. It remains to show the top left triangle commutes too.

Consider the two potential diagrams.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr] & X \times_Z \times Y \arrow[dr] & \\
B \arrow[dr] & X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
 & Y \arrow[r] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[d, "h"] & \\
A \arrow[ur] \arrow[d] \arrow[dr, dashed] & X \times_Z Y \arrow[dr] & \\
B \arrow[dr] & X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
 & Y \arrow[r] & Z
\end{tikzcd}$$

Due to uniqueness of the map $A \to X \times_Z Y$ due to it being a pullback, we obtain the required commutativity.

This finally gives us that h is indeed a lift across r giving us $r \in \mathcal{R}$. The dual of this argument gives us that $\mathcal{L}$ is closed under pushouts.

$\square$

Lastly the model category also often requires the weak equivalences to be closed under retracts. This is not immediately seen from the above theorem. But still is infact true. The following proof is from Riehl's handout [Rie] where she credits the original proof to Joyal.

Proposition 3.1.7. *Let $\mathcal{C}$ be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ as defined above then $\mathcal{W}$ is closed under retracts in $\mathcal{C}^2$.*

Proof. Let f be a retract of $w \in \mathcal{W}$

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, equal, "1_A"] \arrow[d, "f"] & B \arrow[r] \arrow[d, "w"] & A \arrow[d, "f"] \\
D \arrow[r] \arrow[rr, bend right, equal, "1_D"'] & C \arrow[r] & D
\end{tikzcd}$$

We consider the proof in two cases $f \in \mathcal{F}, f \notin \mathcal{F}$.

Case 1: $f \in \mathcal{F}$: Factor w as $w = \rho \circ \lambda$ using any of the two weak factorization systems. In either case by 2-out-of-3 we get $\lambda \in \mathcal{C} \cap \mathcal{W}, \rho \in \mathcal{F} \cap \mathcal{W}$.

We now have a diagram of the sort.

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "f"'] \arrow[dr, dashed, "s"] & B \arrow[r] \arrow[d, "\lambda"] & A \arrow[dd, "f"] \\
 & \bullet \arrow[d, "\rho"] \arrow[ur, dashed, "t"'] & \\
D \arrow[r] & C \arrow[r] & D
\end{tikzcd}$$

Here, s is obtained composition and t via the lift on square obtained on the right by pasting $\bullet \to D$, since $\lambda \in \mathcal{C} \cap W, f \in \mathcal{F}$.

Since the triangles commute we have $t \circ s = 1_A$ so f is a retract of ρ as well, but $\rho \in \mathcal{F} \cap \mathcal{W}$ this implies $f \in \mathcal{F} \cap \mathcal{W}$ both classes of any factorization system are closed under retracts by Proposition 3.1.6(3). Therefore $f \in \mathcal{W}$.

Case 2: No hypothesis on f can i saw $f \notin \mathcal{F}$.

Factorize $f = h \circ g$, where $g \in \mathcal{C} \cap \mathcal{W}$ and $h \in \mathcal{F}$. Construct the below pushout (which exists due to cocompleteness of $\mathcal{C}$).

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "c"'] \arrow[dd, bend left, "w"] & A \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[d, "h"'] & \bullet \arrow[d, dashed, "\exists! d"'] \arrow[ul, phantom, "\ulcorner", very near start] & \bullet \arrow[d, "h"] \\
D \arrow[r] & C \arrow[r] & D
\end{tikzcd}$$

We know by Proposition 3.1.6(4) that the left class of a factorization is closed under pushouts. This implies, $c \in \mathcal{C} \cap W$. Also uniqueness of d such that $d \circ c = w$ is true due to the pushout, via the maps into C as such, $B \xrightarrow{w} C$ and $\bullet \xrightarrow{h} D \to C$. Now via 2-out-of-3 we see that $d \in \mathcal{W}$.

Similarly via the maps $B \to A \xrightarrow{g} \bullet$ and the identity map $\bullet = \bullet$. We find a unique morphism y such that $yx = 1_\bullet$ as seen below.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & B \arrow[r] \arrow[d, "c"'] & A \arrow[d, "g"] \\
\bullet \arrow[r, "x"] \arrow[d, "h"'] & \bullet \arrow[r, dashed, "y"] \arrow[d, "d"'] \arrow[ul, phantom, "\ulcorner", very near start] & \bullet \arrow[d, "h"] \\
D \arrow[r] & C \arrow[r] & D
\end{tikzcd}$$

Therefore, h is found to be a retract of d. Recall that we had $h \in \mathcal{F}$ and $d \in \mathcal{W}$ therefore we have reduced our problem to the first case. This implies $h \in \mathcal{W}$. Now g by choice was in $\mathcal{W}$ so by the 2-out-of-3 we have $f = h \circ g \in \mathcal{W}$. $\square$

3.2 Homotopy in model categories

Definition 3.2.1 (Path space object)**.** *Let $\mathcal{C}$ be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. Then define the path object for $X \in \mathcal{C}$ as the object obtained in its factorization of the diagonal morphism*

$$\Delta : X \xrightarrow{(1_X, 1_X)} X \times X.$$

$$\begin{tikzcd}
& \mathrm{Path}(X) \arrow[dr, "{(r_0,r_1)\in\mathcal{F}}"] & \\
X \arrow[ur, "\lambda\in\mathcal{W}"] \arrow[rr, "\Delta"] & & X \times X
\end{tikzcd}$$

Note that such a factorization always exists due to the factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. However, the additional assumption that the weak equivalence is an acyclic cofibration is often dropped, as it is not necessary for the construction of homotopy. To be more permissive, we only require that $x \to \mathrm{Path}(X)$ be a weak equivalence.

In topological setting the path space is the set of all based continuous functions $\gamma : [0,1] \to X$.

Definition 3.2.2 (Cylinder object)**.** *Let $\mathcal{C}$ be a model category with the distinguished class of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. Then define the cylinder object for $X \in \mathcal{C}$ as the object obtained from the factorization of the codiagonal "folding" morphism $\nabla : X \sqcup X \xrightarrow{[1_X, 1_X]} X$.*

$$\begin{tikzcd}
& \mathrm{Cyl}(X) \arrow[dr, "\rho\in\mathcal{W}"] & \\
X \sqcup X \arrow[ur, "{(l_0,l_1)\in\mathcal{C}}"] \arrow[rr, "\nabla"'] & & X
\end{tikzcd}$$

Compare these to the definitions of cylinder objects discussed in Chapter 1, and the cones and cylinders of topology. In fact these are the same thing as we shall see when discussing examples of model structures.

Definition 3.2.3 (Fibrant objects)**.** *Let $\mathcal{C}$ be a model category with distinguished classes of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$. An object $A \in \mathcal{C}$ is said to be fibrant if the unique mapping into the terminal object $(f : A \to 1) \in \mathcal{F}$.*

Dually we have the notion of a cofibrant object.

Definition 3.2.4 (Cofibrant objects)**.** *As defined above, an object $B \in \mathcal{C}$ is said to be cofibrant if the unique mapping from the initial object $(g : 0 \to B) \in \mathcal{C}$.*

The requirement for defining these objects is that homotopies (as we define below) form equivalence relations only when the objects are (co)fibrant.

Example 3.2.5. *We shall see examples in more detail when we discuss examples of model categories.*

1. *The canonical example is that Kan complexes in the classical model structure of simplicial sets are fibrant.*

2. *All topological spaces in the classical model structure of topological spaces are fibrant.*

3. *In the projective model category of chain complexes the complexes with injective objects are cofibrant and the complexes with projective objects are fibrant.*

Proposition 3.2.6. *If $X \in \mathcal{C}$ a model category is fibrant then the map $\rho = (\rho_0, \rho_1) : \mathrm{Path}(X) \to X \times X$ is a fibration and also the induced maps ρ_0, ρ_1 are individually acyclic fibrations into X.*

Dually we have the map from $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration and the individual maps from $X \to \mathrm{Cyl}(X)$ are acyclic cofibrations provided X is a cofibrant object.

Proof. The map r is a fibration by definition. While the maps $\rho_0, \rho_1 : \mathrm{Path}(X) \to X \times X \xrightarrow{\pi_1 \text{ or } \pi_2} X$ are weak equivalences due to 2-out-of-3 since $X = X$ factors through $\mathrm{Path}(X)$.

Lastly to show ρ_0, ρ_1 are fibrations consider the pullback diagram,

$$\begin{tikzcd}
X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\in \mathcal{F}"] \\
X \arrow[r, "\in \mathcal{F}"'] & 1
\end{tikzcd}$$

□

Definition 3.2.7 (Left homotopy in a model category)**.** *Let $\mathcal{C}$ be a model category and objects $X, Y \in \mathcal{C}$.*

If we have $f, g : X \to Y$ then a left homotopy if it exists is a diagram of the following sort

$H_l : f \to g$ is a morphism $H_l : \mathrm{Cyl}(X) \to Y$ making the below diagram

commute.

$$\begin{tikzcd}
X \arrow[d, "l_1"] \arrow[dr, "f"] & \\
\mathrm{Cyl}(X) \arrow[r, "H_l"] & Y \\
X \arrow[u, "l_0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Definition 3.2.8 (Right homotopy in a model category)**.** *Let $\mathcal{C}$ be a model category and objects $X, Y \in \mathcal{C}$.*

If we have $f, g : X \to Y$ then a right homotopy if it exists is a diagram of the following sort

$H_r : f \to g$ is a morphism, $H_r : X \to \mathrm{Path}(Y)$ making the below diagram commute.

$$\begin{tikzcd}
& Y \\
X \arrow[ur, "f"] \arrow[r, "H_r"] \arrow[dr, "g"'] & \mathrm{Path}(Y) \arrow[u, "r_1"'] \arrow[d, "r_0"] \\
& Y
\end{tikzcd}$$

Lemma 3.2.9. *Let $\mathcal{C}$ be a model category and $f, g : X \to Y$ be morphisms. If X is cofibrant then a left homotopy implies existence of a right homotopy independent on choice of path space object.*

Proof. Consider a left homotopy $H_l : \mathrm{Cyl}(X) \to Y$ as above. Since X is cofibrant using Lemma 3.2.6 we have that the map $X \xrightarrow{l_0} \mathrm{Cyl}(X)$ is a acyclic cofibration. Therefore we have a lift $h : \mathrm{Cyl}(X) \to \mathrm{Path}(Y)$ as seen below.

$$\begin{tikzcd}
X \arrow[r, "\lambda f"] \arrow[d, "l_0"'] & \mathrm{Path}(Y) \arrow[d, "(r_0, r_1)"] \\
\mathrm{Cyl}(X) \arrow[r, "(f\rho, H_l)"'] \arrow[ur, dashed, "h"'] & Y \times Y
\end{tikzcd}$$

.

The map $H_r = h \circ l_1$ gives us the required right homotopy.

$$\begin{tikzcd}
 & & & Y \\
X \arrow[r, "l_1"] \arrow[urrr, bend left, "f"] \arrow[drrr, bend right, "g"'] & \mathrm{Cyl}(X) \arrow[r, "h"] & & \mathrm{Path}(Y) \arrow[u, "r_1"'] \arrow[d, "r_2"] \\
 & & & Y
\end{tikzcd}$$

□

Dually if Y is fibrant then a right homotopy implies existence of a left homotopy.

Corollary 3.2.10. *If as defined above X is cofibrant and Y is fibrant then left and right homotopies between them coincide and form an equivalence relation.*

3.3 Homotopy category of a model category

The proofs in this section are all fairly technical. Since we only require the big picture ideas and not so much the precise proofs we refer the reader to Quillen's book [Qui67] for all the results covered in this section.

A bifibrant object refers to an object which is both fibrant and cofibrant.

Definition 3.3.1 (Homotopy category of a model category)**.** *Given a model category $\mathcal{C}$ we can define its homotopy category* $\mathrm{Ho}(\mathcal{C})$ *as the category consisting of objects which are bifibrant and morphisms are homotopy classes of morphisms.*

This equivalently results in us just localizing the model category at weak equivalences to invert them [2].

This is a category universal in the sense that any functor which sends weak equivalences to isomorphisms factors through $\mathrm{Ho}(\mathcal{C})$.

As a demonstration of the utility and strength of all of the machinery built up so far we can prove a classical theorem in homotopy theory.

Theorem 3.3.2 (Whiteheads theorem)**.** *A weak equivalence between bifibrant objects is precisely a homotopy equivalence.*

[2]This is precisely the notion of a category of fractions in Gabriel-Zisman.

Proof. Let $f : X \to Y$ be a weak equivalence in a model category $\mathcal{C}$, where X and Y are both fibrant and cofibrant (bifibrant). By the factorization axioms and 2-out-of-3 we know that, f can be factored as $X \xrightarrow{j} Z \xrightarrow{p} Y$, where j is an acyclic cofibration and p is an acyclic fibration.

Now since X is cofibrant and j is an acyclic cofibration, Z is cofibrant. Since Y is fibrant and p is an acyclic fibration, Z is also fibrant. Thus Z is bifibrant. By the two-out-of-three property, j and p are also weak equivalences. It therefore suffices to show that an acyclic (co)fibration between bifibrant objects is a homotopy equivalence.

First, consider an acyclic fibration $f : X \to Y$ where X and Y are bifibrant. Since Y is cofibrant, the unique map from the initial object $\emptyset \to Y$ is a cofibration. We have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\in\mathcal{C}"'] & X \arrow[d, "f\in\mathcal{C}\cap\mathcal{W}"] \\ Y \arrow[r, equal, "1_Y"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

As f is an acyclic fibration and the map $0 \to Y$ is a cofibration, a lift $g : Y \to X$ exists such that $f \circ g = \mathrm{id}_Y$. Thus, g is a strict right inverse of f.

Next, we show that g is also a left homotopy inverse to f, i.e., $g \circ f \simeq \mathrm{id}_X$. Let $X \sqcup X \xrightarrow{i_X} \mathrm{Cyl}(X) \xrightarrow{p_X} X$ be a cylinder object factorization for X, where $i_X = (i_0, i_1)$ is a cofibration and p_X is an acyclic fibration (since X is fibrant), such that $p_X \circ i_0 = \mathrm{id}_X$ and $p_X \circ i_1 = \mathrm{id}_X$. Consider the commuting square:

$$\begin{tikzcd} X \sqcup X \arrow[r, "(g\circ f, \mathrm{id}_X)"] \arrow[d, "i_X"] & X \arrow[d, "f"] \\ \mathrm{Cyl}(X) \arrow[r, "f\circ p_X"'] \arrow[ur, dashed, "\eta"] & Y \end{tikzcd}$$

This square commutes because: $f \circ (g \circ f) = (f \circ g) \circ f = \mathrm{id}_Y \circ f = f$, and $f \circ \mathrm{id}_X = f$. So, $f \circ (g \circ f, \mathrm{id}_X) \circ \mathrm{incl}_0 = f \circ g \circ f = f$, and $(f \circ p_X) \circ i_0 = f \circ (p_X \circ i_0) = f \circ \mathrm{id}_X = f$. Similarly, $f \circ (g \circ f, \mathrm{id}_X) \circ \mathrm{incl}_1 = f \circ \mathrm{id}_X = f$, and $(f \circ p_X) \circ i_1 = f \circ (p_X \circ i_1) = f \circ \mathrm{id}_X = f$. Thus, the outer diagram commutes when precomposed with the inclusions from X into $X \sqcup X$. Since i_X is a cofibration and f is an acyclic fibration, a lift $\eta : \mathrm{Cyl}(X) \to X$ exists. This map η provides the left homotopy $\eta : g \circ f \simeq_L \mathrm{id}_X$, as $\eta \circ i_0 = g \circ f$ and $\eta \circ i_1 = \mathrm{id}_X$. Therefore, g is a homotopy inverse to f, and f is a homotopy equivalence.

The case where f is an acyclic cofibration between bifibrant objects is formally dual, using a path object for Y to construct the homotopy. $\square$

3.4 Examples of classical model structures

We exhibit without proof the two motivating examples of model categories. The classical model structure on Top and the classical model structure on positive degree chain complexes $\mathbf{Ch}_{\leq 0}(\mathcal{A})$.

Definition 3.4.1 (Serre fibration)**.** *A map $\rho : X \to Y$ is called a Serre fibration if for every finite CW complex A, the map ρ has the right lifting property with respect to the inclusion map $A \times 0 \to A \times [0, 1]$.*

Proposition 3.4.2 (Classical Quillen model structure on **Top**)**.** *Consider morphisms $f : X \to Y$ in* **Top**. *We can define a model structure on* **Top** *with the following distinguished classes of maps $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ as such,*

1. *$f \in \mathcal{W}$ if f is a weak homotopy equivalence in* **Top**, *i.e., $f : X \to Y$ is a map whose induced homomorphisms on homotopy groups (for every basepoint) are bijective.*

2. *$f \in \mathcal{F}$ if f is a Serre fibration.*

3. *$f \in \mathcal{C}$ if f is a retract of a relative cell complex [3].*

This is often denoted as $\mathbf{Top}_{\text{Quillen}}$. The fact that this construction indeed forms a model category is certainly not a trivial fact, for a proof see [Qui67][Section 2.3, Theorem. 2.1].

The key point to note is that this model structure gives rise to the usual notion of topological homotopy theory. Since the homotopy category of this model structure is the same as the classical homotopy category on CW complexes.

Example 3.4.3. *For simply connected topological spaces a weak homotopy equivalence exists if and only if the induced maps on singular homology groups are bijective [Hat02][Theorem 4.32]. This motivates the definition of quasi isomorphisms of chain complexes as being maps which preserve homologies.*

[3] A relative cell complex is just an arbitrary cell complex not necessarily countable like in the case of CW complexes.

Proposition 3.4.4 (Projective model structure on $\mathbf{Ch}_{\leq 0}(\mathcal{A})$)**.** *Let* $\mathbf{Ch}_{\leq 0}(\mathcal{A})$ *be the category of non negative chain complexes over some abelian category* $\mathcal{A}$*. We can define a model structure on* $\mathbf{Ch}_{\leq 0}(\mathcal{A})$ *as such,*

1. *Weak equivalences are quasi isomorphisms, i.e. maps which induce isomorphisms on homologies between chain complexes.*
2. *Fibrations are chain maps such that the underlying maps in* $\mathcal{A}$ *are epimorphisms.*
3. *Cofibrations similarly are underlying monomorphisms whose cokernels are projective objects* [4]*.*

For a proof see [Qui67][Section 2.4, Remark 5]. The reason for specifying 'classical' model structures is that there exist a huge multitude of different classes of maps one may take to define new and interesting model structures on these categories. However, the ones presented thus far are the canonical examples with historical significance.

3.5 Quillen adjunctions and equivalences

The main purpose of this section is to define the notion of a Quillen equivalence of model categories. Quillen equivalences are basically the 'correct' notion of equivalences in the context of homotopy theory. Put in informal terms it refers to an adjunction between model categories such that the adjunction induces an actual equivalence between their underlying homotopy categories.

This notion allows us to study the homotopy theory of a certain ambient model category through the lens a Quillen equivalent model category instead which is easier to work with.

Note that this is weaker than requiring two model categories to be actually equivalent.

Definition 3.5.1 (Quillen adjunction)**.** *A adjoint pair between two model categories* $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ *such that any of the following equivalent conditions hold,*

[4] It has a left lifting property with respect to epimorphisms. This is the same notion as projectivity as previously discussed over modules but we don't work with it over abelian categories often.

1. *L preserves cofibrations and acyclic cofibrations.*

2. *R preserves fibrations and acyclic fibrations.*

3. *L preserves cofibrations and R preserves fibrations.*

4. *L preserves acyclic cofibrations and R acyclic fibrations.*

We first define the notion of a derived functor. Recall the notion of a category with weak equivalences (Definition 3.1.3).

Definition 3.5.2 (Derived functor)**.** *For two categories $\mathcal{C}, \mathcal{D}$ with weak equivalences then given a functor $F : \mathcal{C} \to \mathcal{D}$ which sends weak equivalences to weak equivalences. It's derived functor is the induced functor $\mathrm{Ho}(F)$ between the homotopy categories of $\mathcal{C}, \mathcal{D}$.*

Definition 3.5.3 (Right/Left derived functor)**.** *For an arbitrary functor between model category $\mathcal{C}$ and a category with weak equivalences $\mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}$. If F restricted to the full subcategory of fibrant objects on $\mathcal{C}$ denoted as $\mathcal{C}_f$ sends weak equivalences to weak equivalences. Then the associated derived functor between their homotopy categories is called the right derived functor and is denoted as $\mathbb{R}F : \mathrm{Ho}(\mathcal{C}_f) \cong \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$.*

Replacing fibrant objects with cofibrant above gives the definition of a left derived functor which is denoted as $\mathbb{L}F$.

Definition 3.5.4 (Quillen equivalence)**.** *A Quillen adjunction between two model categories is said to be a Quillen equivalence if the induced right (or left) derived functor results in a equivalence of categories between the homotopy categories of the model categories.*

Chapter 4

Simplicial homotopy theory

In this section we will outline the basic machinery of simplicial sets the primary goal of this section is to appreciate the usefulness of the homotopy category of simplicial sets and how it gives us a combinatorial framework for studying homotopy without the need to resort to explicit topological approaches.

For a canonical reference to the materials we cover in this section refer to [GJ09], a rapid introduction to simplicial sets is also covered in [Rie11].

4.1 The simplex category

We begin with a notion of a purely combinatorial construction.

Definition 4.1.1 (Simplex/finite ordinal category)**.** *We refer to Δ as the simplex category. It is defined by the objects of finite non empty, totally ordered sets,*

$$[n] = \{0 \to 1 \to \cdots \to n\}$$

maps between these objects are order preserving, i.e. non decreasing maps between totally ordered sets. $f : [m] \to [n]$ is a map such that $f(0) \leq f(1) \leq \cdots \leq f(m)$.

The category formed by all such finite non empty, totally ordered sets and all the mappings between them is referred to as the simplex category Δ.

To be even more precise Δ is the full subcategory of Cat whose objects consist of finite non empty totally ordered sets with morphisms being poset maps.

Example 4.1.2. *$f : [1] \to [5]$, defined by $f(0 \to 1) = 2 \to 4$ $f : [2] \to [5]$, defined by $f(0 \to 1 \to 2) = 2 \to 3 \to 4$. $f : [3] \to [5]$, defined by $f(0 \to 1 \to 2 \to 3) = 3 \to 4 \to 4 \to 5$.*

Example 4.1.3. *$g : [4] \to [2]$ defined by $g(0 \to 1 \to 2 \to 3 \to 4) = 0 \to 0 \to 1 \to 1 \to 1$.*

Note that all morphisms in Δ are generated by a natural family of functions called coface and degeneracy maps defined as below.

Definition 4.1.4 (Coface maps)**.** *$d^i : [n-1] \to [n]$ the injection which misses the i^{th} element in $[n]$.*

Explicitly, for $0 \leq i \leq n$

$$d^i(k) = \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases}$$

Definition 4.1.5 (Codegeneracy maps)**.** *$s^i : [n+1] \to [n]$ the surjection which maps two elements to i.*

$$s^i(k) = \begin{cases} k, & k \leq i \\ k-1, & k > i \end{cases}$$

These obey the relations,

$$\begin{aligned}
d^j d^i &= d^i d^{j-1}, & i &< j \\
s^j s^i &= s^i s^{j+1}, & i &\leq j \\
s^j d^i &= 1, & i &= j, j+1 \\
s^j d^i &= d^i s^{j-1}, & i &< j \\
s^j d^i &= d^{i-1} s^j, & i &> j+1.
\end{aligned}$$

4.2 Simplicial sets and examples

Definition 4.2.1 (Simplicial set)**.** *A simplicial set is a functor $X : \Delta^{\text{op}} \to$ **Set**, i.e. presheaves [1] on Δ. It comprises of a collection of sets $X_n = X([n])$ which we to call the set of $n-$simplices of X with maps between them corresponding naturally with maps in Δ.*

[1] In the categorical sense.

Furthermore corresponding to coface maps from $[n-1] \to [n]$ in Δ we get a family of face maps between simplices $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$.

The degeneracy maps corresponding to codegeneracy maps $[n+1] \to n$ as a family of maps $s_i : X_n \to X_{n+1}$.

Defined as such,

$$\begin{aligned} d_i &= Xd^i : X_n \to X_{n-1} & 0 \leq i \leq n \\ s_i &= Xs^i : X_n \to X_{n+1} & 0 \leq i \leq n \end{aligned}$$

These obey the standard relations,

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ s_i s_j &= s_{j+1} s_i & i \leq j \\ d_i s_j &= 1, & i = j, j+1 \\ d_i s_j &= s_{j-1} d_i, & i < j \\ d_i s_j &= s_j d_{i-1}, & i > j+1 \end{aligned}$$

The face maps d_i can be understood as mapping each n-simplex $x \in X_n$ to $n+1$ many $n-1$ simplicies $d_i(x)$ $0 \leq i \leq n$ in X_{n-1}, the i^{th} face does not contain the i^{th} vertex of x.

Similarly for degeneracy maps s_i we can understand it as mapping $x \in X_n$ to $n+1$ many $n+1$ simplicies in X_{n+1} and $s_i(x)$ has x as its i^{th} and $i+1^{\text{th}}$ face.

Definition 4.2.2 (Degenerate simplex)**.** *We say $x \in X_n$ is degenerate if it is the image of a degeneracy map.*

We can equivalently define a simplicial set purely combinatorially in terms of a graded set.

Definition 4.2.3 ((Alternate definition) Simplicial set)**.** *A simplicial set X is a collection of sets X_n for each integer $n \geq 0$ together with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for all $0 \leq i \leq n$ and for each n satisfying the following relations:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ s_i s_j &= s_{j+1} s_i, & i \leq j \\ d_i s_j &= 1, & i = j, j+1 \\ d_i s_j &= s_{j-1} d_i, & i < j \\ d_i s_j &= s_j d_{i-1}, & i > j+1. \end{aligned}$$

A significant simplification in theory occurs when we choose to distinguish a particular type of a simplex. Namely, the standard n-simplex as $\Delta[n] = \mathrm{Hom}_\Delta(-,[n])$

Recall the Yoneda lemma [Awo10][Lemma 8.2],

Lemma 4.2.4 (Yoneda)**.** *For a locally small category $\mathcal{C}$ [2], an object $A \in \mathcal{C}$ and a functor F in the functor category $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ there exists an isomorphism,*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(yA, F) \cong FA.$$

Where $y : \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is the Yoneda embedding defined as,

$$y(A) : \mathrm{Hom}_{\mathcal{C}}(-, A).$$

For A an object of $\mathcal{C}$ and,

$$y(f : A \to B) : \mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, B).$$

This naturally leads to the definition of **sSet** the category of simplicial sets as the functor category $\mathbf{Set}^{\Delta^{\mathrm{op}}}$.

It is clear to see that placing $\mathcal{C} = \Delta$, $F = X \in \mathbf{sSet}$.

Proposition 4.2.5. *By Yoneda we have $X_n \cong \mathrm{Hom}(\Delta[n], X)$. In particular any n-simplex $x \in X_n$ can be thought of as a map $x : \Delta[n] \to X$.*

Definition 4.2.6 (Simplicial objects in arbitrary categories)**.** *For $\mathcal{C}$ an arbitrary category a simplical object in $\mathcal{C}$ is a functor $C : \Delta^{op} \to \mathcal{C}$.*

Example 4.2.7 (Simplicial abelian groups)**.** *A simplicial object in* **Ab**.

Furthermore there is a free forgetful adjoint pair between **sAb** and **sSet**.

Example 4.2.8 (Bisimplical sets)**.** $\mathbf{s}(\mathbf{sSets}) = \mathbf{s^2Sets}$.

Theorem 4.2.9 (Dold-Kan)**.** *There is a equivalence of categories*

$$N : \mathbf{sAb} \rightleftarrows \mathbf{Ch}_+ : \Gamma$$

where N denotes the normalized chain complex functor and Γ the simplicalization functor.

[2]Locally small implies each homset is indeed a small set (i.e. not a proper class). This is a weaker condition than just the category being small which means the collection of objects is a small set .

The Dold-Kan correspondence, serves as a bridge between simplicial homotopy theory and homological algebra. A proof can be seen in [GJ09].

In fact Dold-Kan can be thought of as a categorification of a theorem about divided differences (analogue to the fundamental theorem of calculus)! This insight was given by a nCafe post [Lei10] crediting it to Andre Joyal.

4.3 Geometric realisation

Definition 4.3.1 (Geometric realization of a standard n-simplical set)**.** *We define a functor* $|.| : \mathbf{sSet} \to \mathbf{Top}$ *as such. Send each standard n simplex* $\Delta[n]$ *to the standard n-toplogical simplex. In particular,*

$$|\Delta[n]| = \{(x_0, \dots, x_{n+1}) | 0 \leq x_i \leq 1, \sum x_i = 1\} \subset \mathbb{R}^{n+1}.$$

Since we defined simplicial sets as presheaves of the simplex category we have a lot of nice properties thanks to the well behavedness of presheaves, such as being complete and cocomplete. More than this however, one of the most useful results we can borrow is the below result often called the density theorem or the co-Yoneda lemma. Which essentially says any presheaf is a colimit of presentable presheaves.

This is a relatively technical proposition and will distract from its application in our case where we are considering only simplicial sets. So we state it without proof. For a proof see [Awo10][Proposition 8.10]

Definition 4.3.2 (Category of elements)**.** *Let* $\mathcal{C}$ *be a category and* $F \in \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *the category of elements of* F *is denoted as* $\int^{\mathcal{C}} F$ *and is defined as such.*

Objects are pairs (A, a) *for* A *an object of* $\mathcal{C}$ *and* $a \in FA$*.*

Morphisms between $(A, a) \to (B, b)$ *is a morphism* $f : A \to B$ *in* $\mathcal{C}$ *under which* $Ffb = a$*.*

The notation is suggestive of the fact that this is a coend.

The category of elements for a given simplicial set is called the category of simplicies.

Example 4.3.3 (Category of simplicies)**.** *Applying the above construction with* $\mathcal{C} = \Delta, F = X \in \mathbf{sSets}$ *we have. The objects in* $\int^{\Delta} X$ *are simplicies* $x \in X_n$ *and morphisms* $x \in X_n \to y \in X_m$ *is a map in* Δ*,* $f : [n] \to [m]$ *such that* $yf = x$*.*

Theorem 4.3.4 (Density theorem)**.** *Let $\mathcal{C}$ be a small category, every object $X \in \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is a colimit of representable functors for a index category J the category of elements of X,*

$$\operatorname*{colim}_{j \in J} yC_j \cong X.$$

Proposition 4.3.5. *Any simplicial set can be expressed as a colimit of standard n simplicies, where the indexing category is the category of simplices. In particular for $X \in \mathbf{sSet}$ we have,*

$$\operatorname*{colim}_{x \in X_n} \Delta[n] \cong X.$$

We can then define the geometric realization of a standard n-simplex as $|\Delta[n]| = \Delta_n$ where Δ_n the standard topological n-simplex.

The following nice visuals that are from [Fri23].

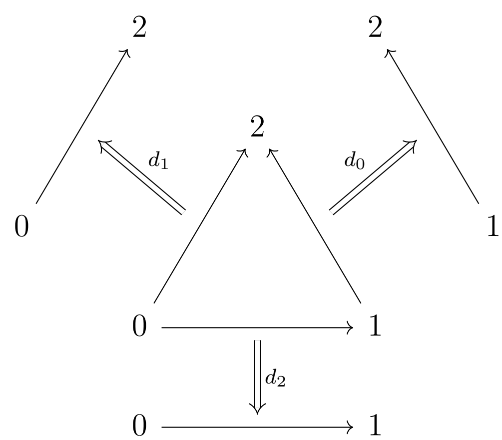

Figure 4.1: Face maps for 2-simplex

Recall that for some topological space X we define its singular simplicial set as $\mathrm{Hom}(\Delta[n], X)$, this infact forms an adjoint pair along with 'geometric realization' the functor $|.| : \mathbf{sSet} \to \mathbf{Top}$ sending standard $n-$simplicies to topological simplicies, in particular $|| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}, || \dashv \mathrm{Sing}$. We shall see a detailed treatment of these adjoints after some more discussions.

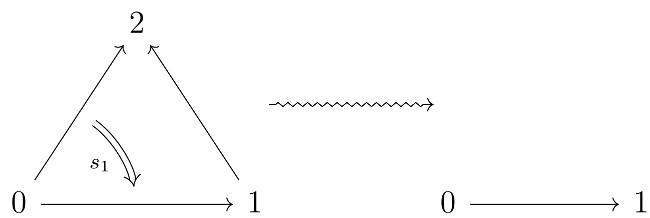

Figure 4.2: Degeneracy maps for 2-simplex

4.4 Kan complexes and Kan fibrations

Kan complexes also called fibrant simplicial sets are a special kind of simplicial set. They form the fibrant objects in the classical model structure of simplicial sets.

The motivation for presenting this definition is due to the fact that it allows us to define simplicial homotopy as an equivalence relation.

Definition 4.4.1 (Horn of a (topological) standard n-simplex)**.** *The k-th horn $|\Lambda_k[n]|$ on $|\Delta[n]|$ is a corresponds to the boundary of it with the k^{th} face removed. The associated simplicial set is denoted as $\Lambda_k[n]$.*

Definition 4.4.2 (Horn of a standard n-simplex)**.** *Given a standard n-simplex $\Delta[n]$ the k^{th} horn is denoted as $\Lambda_k[n]$ for $0 \leq k \leq n$. It is a subset of $\Delta[n]$ generated by all faces except the k^{th} face.*

Definition 4.4.3 (Kan complexes)**.** *$X \in$ **sSets** is a Kan complex if all horns on X can be filled. In particular this means that any map $\Lambda_k[n] \to X$ can be extended to a map $\Delta[n] \to X$.*

Definition 4.4.4 (Kan fibration)**.** *A simplicial map $f : X \to Y$ is said to be a Kan fibration if it has the right lifting property against all horn inclusions, i.e. the lift h below always exists.*

$$\begin{tikzcd}
\Lambda_k[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

$$
\begin{tikzcd}
 & 3 & & & & 3 & \\
0 \arrow[dr] & & 1 \arrow[dl] & \subset & 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 1 \arrow[ul] \arrow[dl] \\
 & 2 \arrow[uu] & & & & 2 \arrow[uu] & \\
 & \Lambda^2[3] & & & & \Delta[3] &
\end{tikzcd}
$$

Figure 4.3: Example of a horn of a 3-simplex

4.5 Nerve/Classifying spaces and the path category functor

Definition 4.5.1 (Nerve of a small category)**.** *Let $\mathcal{C}$ be a small category we define its nerve as the following simplicial set $N(\mathcal{C})_0 = \mathrm{Ob}(\mathcal{C}), \mathcal{C}$ and $N(\mathcal{C})_1 = \mathrm{Mor}(\mathcal{C})$ and $N(\mathcal{C})_k = \{(f_1, \dots, f_k) | f_i \in \mathrm{Mor}(\mathcal{C})\}$ consists of $k-$tuples of composable arrows, face maps defined as*

$$d_i(f_1, \dots, f_i, f_{i+1}, \dots, f_k) = (f_1, \dots f_i \circ f_{i+1}, \dots, f_k)$$

and degeneracy maps defined as

$$s_j = (f_1, \dots, f_k) = (f_1, \dots, f_{j-1}, 1, f_j, \dots, f_k).$$

In a concise manner the nerve is simply the simplicial set consisting of n-simplicies of the form $N(\mathcal{C}_n) := \mathrm{Hom}([n], \mathcal{C})$.

Definition 4.5.2 (Classifying space of a category (Usual definition))**.** *For a small category $\mathcal{C}$ we denote its classifying space as $B\mathcal{C} := |N(\mathcal{C})|$.*

However, in light of the homotopy theory on simplicial sets we will instead refer to a classifying space of a category $B\mathcal{C}$ as simply being its nerve. This is the convention as adopted in [GJ09].

Definition 4.5.3 ((Simplicial) Classifying space of a small category)**.** *For a small category $\mathcal{C}$ when we refer to the classifying space of X as BX we will be referring to its nerve as defined above.*

If we refer to a geometric realization in particular this will be made clear.

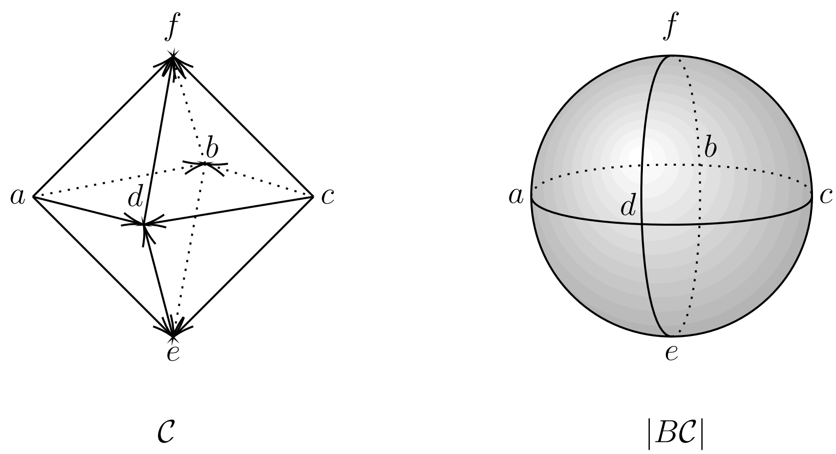

Figure 4.4: Example of a geometric realization of a simplex. Note here C is not a groupoid so the corresponding classifying space is not a Kan complex.

Proposition 4.5.4. *A small category $\mathcal{C}$ is a groupoid if and only if $B\mathcal{C}$ is a Kan complex.*

The proof for this proposition is fairly technical. A detailed proof may be seen in Kerodon [Lur25][Proposition 0037]

Definition 4.5.5 (Path category of a simplicial set)**.** *Let $X \in$ **sSets** define its path category P_*X to be a category with objects as X_0, i.e. the vertices of X and morphisms generated by 1simplices X_1 subject to the relation for $\sigma \in X_2$ the diagram below commutes*

$$\begin{tikzcd}
v_0 \arrow[rr, "d_2\sigma"] \arrow[dr, "d_1\sigma"'] & & v_1 \arrow[dl, "d_0\sigma"] \\
& v_2 &
\end{tikzcd}$$

Note that P_*X is a directed graph.

4.6 Free groupoid and fundamental groupoid of a simplicial set

Recall a groupoid is any category in which all morphisms are invertible.

Definition 4.6.1 (Free groupoid on a directed graph)**.** *Let X be a directed graph. Consider a category GX with objects as vertices of X and each edge of the directed graph inducing a morphism the concatenation of two edges gives a composition of morphisms. Finally introducing a formal inverse arrow for each morphism gives us a groupoid.*

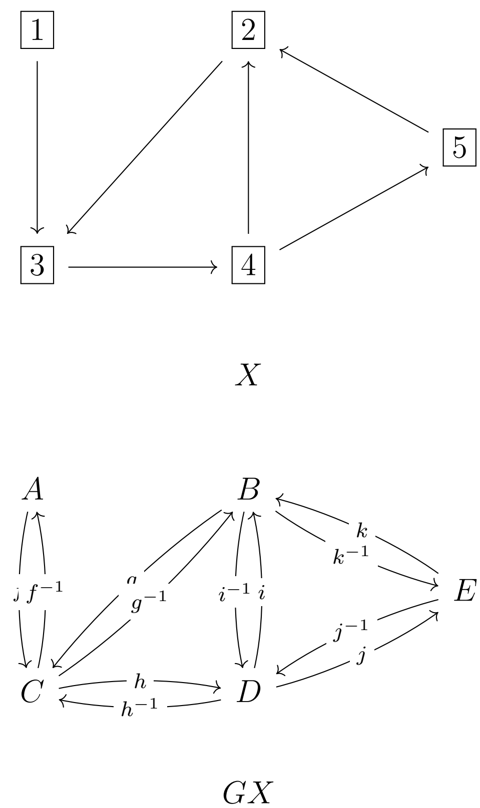

Figure 4.5: Example of a free groupoid of a directed graph.

Definition 4.6.2 (Fundamental gropupoid of a topological space)**.** *Denoted as ΠX it consists of all homotopy classes of paths in X.*

As defined in [GZ67] we can be motivated to say.

Definition 4.6.3 (Fundamental groupoid on a simplicial set)**.** *Defined as the fundamental groupoid of the P^*X i.e. GP_*X.*

Free groupoids can also be defined on small categories (which we see in the next section using the notion of a path category once again) or even on arbitrary categories in the sense of formally inverting all morphisms.

This latter notion involves a non-trivial amount of machinery which is not directly applied in our discussion on K-theory so we do not approach it in detail. Despite of the various definitional notions this is not super important. What is important however, is the following useful universal property that the free groupoid functor satisfies.

Proposition 4.6.4. *The construction of the free groupoid of some category $\mathcal{C}$ is universal in the sense that for any map inverting arrows out of $\mathcal{C}$ factors through $G\mathcal{C}$.*

Basically just the free-forgetful adjunction. For a more detailed perspective one may refer to [GZ67][1.5.4] where Gabriel-Zisman define the free groupoid as a category of fractions.

4.7 Adjunctions in simplices

In this section we summarize a few key adjunctions observed in **sSets**.

4.7.1 Path category-nerve adjunction

There exists an isomorphism between the free groupoid of a category and the fundamental groupoid of its classifying space.

The contents of this section can be examined in specific cases as covered in [GJ09] but we instead follow [Rie11] which provides a single unified presentation.

In conclusion we have the following.

Example 4.7.1 (Path category-nerve adjunction)**.** *The path category functor is left adjoint to the nerve functor, i.e.* $\mathrm{Hom}(P_*X, C) \cong \mathrm{Hom}(X, BC)$.

Proposition 4.7.2. *If $\mathcal{C}$ is a small category then the adjunct map $F : P_*B\mathcal{C} \to \mathcal{C}$ defines an isomorphism of categories.*

Proof. There exists a section map $G : \mathcal{C} \to P_*B\mathcal{C}$ which is identity on objects and sends each path from A to B (i.e. every chain of maps starting from A to B) to the same class as the composite $A \to B$. $\square$

Corollary 4.7.3. *There is a natural isomorphism between the free groupoid on a small category and the fundamental groupoid of its classifying space, i.e.* $G\mathcal{C} \cong GP_*B\mathcal{C}$.

Proof. This is found by applying the free groupoid functor to the map found in Proposition 4.7.2. $\square$

4.7.2 Barycentric subdivision-Ex adjunction

The importance of Kan's Ex^∞ functor is difficult to overstate. It allows us to always find a fibrant replacement for any simplicial set. As we shall soon see all simplicial sets are cofibrant so a procedure allowing us to make them fibrant indeed makes them bifibrant. Recall that we saw homotopy equivalences are indeed equivalences for bifrant objects.

Definition 4.7.4 (Barycentric subdivision of a standard simplicial set)**.** *For* $\Delta[n]$ *define its barycentric subdivison* $\mathrm{sd}\Delta[n]$ *as the nerve of the poset of non-degenerate simplicies* $\mathrm{nd}\Delta[n]$, *i.e.* $\mathrm{sd}\Delta[n] = B\mathrm{nd}\Delta[n]$.

For an arbitrary simplicial set X we proceed via the colimit of its representatives as one may expect,

$$\mathrm{sd}X = \mathrm{colim}_{x \in X_n} \mathrm{sd}(\Delta[n]).$$

This notion of a barycentric subdivision is exactly the same as that commonly encountered in standard algebraic topology.

$$\begin{tikzcd}
 & & 2 \arrow[dl] \arrow[d] \arrow[dr] & & \\
 & 02 \arrow[r] & 012 & 12 \arrow[l] & \\
0 \arrow[ur] \arrow[urr] \arrow[rr] & & 01 \arrow[u] & & 1 \arrow[ul] \arrow[ull] \arrow[ll]
\end{tikzcd}$$

Figure 4.6: Example of the barycentric subdivision of $\Delta[2]$.

Definition 4.7.5 (Ex functor)**.** *For* $X \in \mathbf{sSets}$ *define* Ex *levelwise as* $\mathrm{Ex}(X)_n = \mathrm{Hom}(\mathrm{sd}\Delta[n], X)$.

Since we know $X_n \cong \mathrm{Hom}(\Delta[n], X)$ we have by construction Ex is right adjoint to sd.

There is a natural map $X \to \mathrm{Ex}(X)$. Iterating this procedure repeatedly and taking a colimit of the diagram,

$$X \to \mathrm{Ex}(X) \to \mathrm{Ex}^2(X) \to \dots .$$

We obtain the simplicial set $\mathrm{Ex}^\infty X$.

The elements of $\mathrm{Ex}^\infty(X)_1$ consist of 'zig-zags' of morphisms in X.

$$\begin{tikzcd}
& \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] \arrow[r, dotted, no head] & {} \\
\bullet & & \bullet & & \bullet \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

For detailed proofs including the description of the map $X \to \mathrm{Ex}(X)$ refer to [Lur25][Section 00XF].

The most important thing to note is that.

Theorem 4.7.6. *For any simplicial set X, $\mathrm{Ex}^\infty(X)$ is a Kan complex.*

4.8 Simplicial homotopy

We shall soon define a model structure on simplicial sets. After which we can directly read off the definitions of homotopy and homotopy equivalence as covered in the chapter on model categories. However, for clarities sake we also write them out here in terms of simplicial sets only. Note the striking similarities with the classical algebraic topological definitions.

We consider the standard 1-simplex $\Delta[1]$ to represent the interval object. Homotopy between two morphisms of simplicial sets are defined in an analogous manner.

Definition 4.8.1 ((Left) Simplicial homotopy)**.** *Suppose $X, Y \in$ **sSets** and two morphisms $f, g : X \to Y$ are said to be homotopic if there exists a* *morphism $H : X \times \Delta[1] \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times \Delta[0] \arrow[r, equal] \arrow[d, "1_X \times d^1"'] & X \arrow[dr, "f"] & \\ X \times \Delta[1] \arrow[rr, Rightarrow, "H"] & & Y \\ X \times \Delta[0] \arrow[r, equal] \arrow[u, "1_X \times d^0"] & X \arrow[ur, "g"'] & \end{tikzcd}$$

This essentially just means $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ as we are used to in topology.

Definition 4.8.2 (Simplicial homotopy equivalence)**.** *A simplicial map $f : X \to Y$ is a homotopy equivalence if there exists a simplicial map $g : Y \to X$ such that there exists simplicial homotopies $1_X \to gf$ and $fg \to 1_Y$.*

4.9 Classical model structure on simplicial sets and a Quillen equivalence

The results in this section are proved in detail in [Qui67]. The classical definition of the notion of a weak equivalence in **sSets** is topologically motivated.

4.9.1 Simplicial weak equivalences

Definition 4.9.1 (Weak equivalences in simplicial sets (Classical))**.** *A map between simplicial sets is said to be a weak equivalence if the induced map between their geometric realizations is a topological weak equivalence.*

However, since we wish to have purely combinatorial constructions throughout this project we may instead define the notion of weak equivalences (motivated due to the above classical definition) in a purely combinatorial setting as well.

Proposition 4.9.2. *A simplicial map $f : X \to Y$ is a weak equivalence (classically) if and only if $\mathrm{Ex}^\infty(f) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ is a simplicial homotopy equivalence.*

This proposition can be used as a matter of definition.

4.9.2 $\mathbf{sSets}_{\text{Quillen}}$

Proposition 4.9.3 (Classical model structure on simplicial sets)**.** *Denoted as* $\mathbf{sSets}_{\text{Quillen}}$ *the classical model structure on simplicial sets consists of the following classes of morphisms*

1. *Weak equivalences are given as simplicial weak equivalences.*
2. *Fibrations are given as Kan fibrations.*
3. *Cofibrations are given by monomorphisms (levelwise injections).*

In this model structure the fibrant objects are precisely Kan complexes and every simplicial set is cofibrant. This doubly impresses upon us the usefulness of the Ex^∞ functor.

There exists a Quillen adjunction between the classical model structure on simplicial sets and the classical model structure on topological spaces. The Quillen adjunction is induced by none other than the singularisation-geometric realisation adjunction. The proof for this fact is nontrivial and is covered in [Qui67]. However, its implications for the usefulness of simplicial homotopy theory are abundant. It essentially states that the homotopy theories are equivalent and we are free to work entirely within the realms of simplicial sets.

4.10 Simplicial homotopy groups

Simplicial homotopy groups are defined only for fibrant objects, i.e. for Kan complexes. However, as we have seen when we wish to define it in general for some $X \in \mathbf{sSets}$ we instead find its fibrant replacement object Y. That is some fibrant object Y such that, $X \xrightarrow{\simeq} Y$ is a simplicial weak equivalence. There is more than one way to make this choice for Y. The most natural however is $Y = \mathrm{Ex}^\infty(X)$.

With this in mind we define the simplicial homotopy groups of Kan complexes as such.

We begin with a notion of a 'simplicial sphere'.

Definition 4.10.1 (Boundary of a standard simplical set)**.** *Let* $\Delta[n] \in \mathbf{sSets}$ *denote the standard* n *simplicial set. We denote its boundary as* $\partial\Delta[n]$ *and it is defined as the subsimplicial set of* $\Delta[n]$ *consisting of all non-degenerate* m

simplicies for $m < n$. That is to say all except its unique non-degenerate n simplex.

The way to visualize this is to think of the fact that the geometric realization of $\partial\Delta[n]$ is precisely homotopic to S^{n-1}.

Definition 4.10.2 (Simplicial homotopy groups)**.** *Let $X \in$ **sSets** be a Kan complex, choose some vertex $v \in X_0$.*

Define $\pi_0(X)$ as the set of simplicial homotopy classes of vertices of X.

Define the underlying set of $\pi_n(X, v)$ as the set of homotopy classes of morphisms $\alpha : \Delta[n] \to X$ such that these take the boundary of $\Delta[n]$ to the point x, i.e. there exists a commutative diagram as such.

$$\begin{tikzcd} \Delta[n] \arrow[r, "\alpha"] & X \\ \partial\Delta[n] \arrow[r] \arrow[u] & \Delta[0] \arrow[u, "v"'] \end{tikzcd}$$

The group operation is given as follows. Let f, g be distinct representatives in $\pi_n(X, v)$.

Consider the following n-simplices in X,

$$\begin{cases} v_i & = v, 0 \leq i \leq n-2, \\ v_{n-1} & = \alpha, \\ v_{n+1} & = \beta. \end{cases}$$

Note that $d_i v_j = d_{j-1} v_i$ for $i < j, i, j \neq n$. Therefore these v_i define a simplical map,

$$(v_0, \ldots, v_{n-1}, -, v_{n+1}) : \Lambda_n[n+1] \to X$$

which since X is fibrant gives us a lift θ.

$$\begin{tikzcd} \Delta_n[n+1] \arrow[r, "{(v_0,\ldots,v_{n-1},-,v_{n+1})}"] \arrow[d] & X \\ \Delta[n+1] \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

Note that,

$$\begin{aligned} \partial(d_n\theta) &= (d_0 d_n \theta, \ldots, d_{n-1} d_n \theta, d_n d_n \theta) \\ &= (d_{n-1} d_0 \theta, \ldots, d_{n-1} d_{n-1} \theta, d_n d_{n+1} \theta) \\ &= (v, \ldots, v), \end{aligned}$$

Therefore, $d_n\theta$ is an element of $\pi_n(X, v)$. We define the group product as $[f] \cdot [g] = [d_n\theta]$.

It still remains to show that the choice of $d_n\theta$ is independent of the representatives and the lift θ. Also that the product indeed defines a group. For this we refer the reader to [GJ09][7.1, 7.2].

4.11 Bisimplicial sets and homotopy (co)limits

The construction we give here is from [GJ09]. There are more general ways to construct the notion of a homotopy (co)limits but we do not need the abstract formulation here.

To motivate the need for this definition.

Let P denote the category $\bullet \leftarrow \bullet \rightarrow \bullet$, representing the shape of a pushout diagram. The ordinary colimit functor colim: $\mathbf{Top}^P \to \mathbf{Top}$, which assigns to such a diagram its pushout in the category of topological spaces, does not in general preserve weak equivalences.

Standard colimits and limits in **sSet** do not generally preserve weak equivalences, necessitating these modified versions.

For example, consider D^2 which is homotopy equivalent to the point $*$.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 4.11.1 (Bisimplicial set)**.** *A bisimplicial set X is a functor $X : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$. It assigns a set $X_{m,n}$ to each pair of objects $([m], [n])$ in Δ. The category of bisimplicial sets is $\mathbf{s^2Set}$.*

Definition 4.11.2 (Diagonal functor)**.** *The diagonal functor $d : \mathbf{s^2Set} \to \mathbf{sSet}$ takes a bisimplicial set X to the simplicial set $d(X)$ with k-simplices $d(X)_k = X_{k,k}$, equipped with diagonal structure maps.*

Homotopy colimits provide a way to form colimits "up to homotopy". One standard construction uses the diagonal of a specific bisimplicial set built from the diagram. This requires the notion of the translation category (or category of elements).

Definition 4.11.3 (Translation Category). *Given a functor $X : I \to \mathbf{Set}$ from a small category I, its translation category $E_I X$ has objects (i, x) for $i \in \mathrm{Ob}(I), x \in X(i)$, and morphisms $\alpha : (i, x) \to (j, y)$ given by morphisms $\alpha : i \to j$ in I such that $X(\alpha)(x) = y$.*

Definition 4.11.4 (Homotopy Colimit). *Let I be a small category and $X : I \to \mathbf{sSet}$ be an I-diagram. The homotopy colimit of X, denoted $\mathrm{hocolim}_I X$, is defined as the diagonal of the bisimplicial set $n \mapsto B(E_I(X_n))$, where $B(-)$ denotes the nerve:*

$$\mathrm{hocolim}_I X := d(n \mapsto B(E_I(X_n))).$$

Its k-simplices are $(\mathrm{hocolim}_I X)_k = \coprod_{i_0 \to \cdots \to i_k \text{ in } I} X_k(i_0)$.

Proposition 4.11.5. *A simplicial set is a homotopy colimit of its representative standard simplicial sets.*

Definition 4.11.6 (Classifying space BI). *The classifying space (nerve) of I, BI, is the homotopy colimit of the constant diagram $*$: $BI := \mathrm{hocolim}_I *$.*

Proposition 4.11.7 (Projection map). *There is a natural projection map $\pi : \mathrm{hocolim}_I X \to BI$ induced by the transformation $X \to *$.*

Lemma 4.11.8. *For a projective cofibrant I-diagram X, the map $\mathrm{hocolim}_I X \to \mathrm{colim}_I X$ is a weak equivalence.*

Dually, homotopy limits provide a homotopy-invariant notion of limit.

Definition 4.11.9 (Homotopy Limit). *Let I be a small category and $X : I^{op} \to \mathbf{sSet}$ be a diagram where each $X(i)$ is a Kan complex. The homotopy limit of X, denoted $\mathrm{holim}_I X$, is defined via the function complex:*

$$\mathrm{holim}_I X := \mathrm{hom}(B(I/-), X),$$

where $B(I/-) : I \to \mathbf{sSet}$ sends $i \mapsto B(I/i)$.

Specific homotopy limits are crucial.

Definition 4.11.10 (Homotopy Pullback and Fiber). *The homotopy pullback $X \times^h_Z Y$ of $X \xrightarrow{f} Z \xleftarrow{g} Y$ is a homotopy limit. The homotopy fiber of $p : E \to B$ over $b : * \to B$ is the homotopy pullback $E \times^h_B *$.*

Definition 4.11.11 (Homotopy Cartesian square). *A commutative square*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

is homotopy Cartesian if the natural map $A \to B \times_D^h C$ is a weak equivalence.

A foundational result connecting these ideas, crucial for relating diagrams to their homotopy colimits, is the following.

Theorem 4.11.12 (Quillen's Fiber Lemma). *Suppose $X : I \to \mathbf{sSet}$ is a diagram such that $X(\alpha)$ is a weak equivalence for all morphisms α in I. Then for each $i \in \mathrm{Ob}(I)$, the pullback square*

$$\begin{tikzcd} X(i) \arrow[r] \arrow[d] & \mathrm{hocolim}_I X \arrow[d, "\pi"] \\ \Delta[0] \arrow[r, "i"] & BI \end{tikzcd}$$

(where $X(i) \to \mathrm{hocolim}_I X$ is the canonical inclusion and $\Delta[0] \to BI$ picks out the object i) is homotopy Cartesian.

Chapter 5

Quillen Q construction

In this section we cover a perspective of Quillens Q construction through the lens of simplicial homotopy theory. We mainly refer to the book by Goerss and Jardine [GJ09] and Jardines lecture notes on K theory [Jar] which follow this simplicial methodology up towards presenting his alternate proof of Quillen's result on the higher K groups of finite fields. One may implicitly assume all the notions of homotopy being discussed below as being simplicial homotopies.

5.1 Exact categories and K_0 of an exact category

Definition 5.1.1 (Exact category)**.** *An exact category (sometimes referred to as a Quillen exact category) is a pair $(\mathcal{E}, E)$ for $\mathcal{E}$ an additive category which is a full subcategory of some abelian category $\mathcal{A}$. Along with a family of sequences E of the form,*

$$0 \to A \to B \to C \to 0.$$

Which are short exact sequences in $\mathcal{A}$ and if in a sequence of the above form $A, C \in \mathcal{E}$ then B is isomorphic to some element which is in $\mathrm{Ob}(C)$*, (i.e. it is closed under extensions).*

Example 5.1.2. *For A a commutative ring with unity, the collection of finitely generated projective A modules forms an exact category. Note that, it is a subcategory of the abelian category of all finitely generated A modules.*

Example 5.1.3. *Every abelian category is trivially exact over itself.*

Example 5.1.4. *Torsion free abelian groups over the category of abelian groups is exact but not abelian. (Non abelian-ness was shown in Example 1.1.11.2).*

Definition 5.1.5 (K_0 for an exact category $\mathcal{E}$)**.** *$K_0(\mathcal{E})$ is generated by the isomorphism classes $[B]$ for each $B \in \mathrm{Ob}(\mathcal{E})$ and a relation of $[B] = [A] + [C]$ for all short exact sequences,*

$$0 \to A \to B \to C \to 0.$$

Naturally since every abelian category is exact this applies for abelian categories in particular.

5.2 Results on exact categories

Recall from Definition 5.1.1 the definition of a Quillen exact category $(\mathcal{E}, E)$ as via embeddings. Where, $\mathcal{E}$ is a additive subcategory of some abelian category $\mathcal{A}$ closed under extensions along with a family E of exact sequences

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$.

This is an equivalent definition to the original axiomatic definition as it occurs in Quillen's paper [Qui73]. Which we recount here for its usefulness.

The monics that are part of the exact sequences in the family of exact sequences associated to the exact category are called admissible monics. Similarly, the epimorphisms which belong to exact sequences associated to the exact category are called admissible epis. Note in the below definition the map p is an admissible monic and the map j is an admissible epi. We will unconsciously stick to this convention throughout[1].

Definition 5.2.1 (Exact category (Axiomatic definition))**.** *A pair $(\mathcal{E}, E)$ where $\mathcal{E}$ is an additive category which is a subcategory of a abelian category closed under extensions, and a set of sequences E containing elements of the form $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ satisfying the following properties:*

[1]Some authors refer to admissible monics/epis as inflations and deflations respectively. A detailed disambiguation along with alternate proofs of the material covered here is provided in a survey paper by T. Bülher [Bü10].

(E1) *A sequence in $\mathcal{E}$ which is isomorphic to a sequence in E is in E. The split exact sequences*

$$0 \to A \to A \oplus B \to B \to 0$$

are in E. Every admissible monic is a kernel of a admissible epic and every admissible epi is a cokernel of a admissible monic.

(E2) *Admissible epis are closed under composition and pullbacks by maps in $\mathcal{E}$. Dually, admissible monics are closed under composition and under pushout.*

(E3) *If a morphism $A \to B$ having a kernel in $\mathcal{E}$ can factor a admissible epi $B \to C$ as $C \to A \to B$ then $A \to B$ is indeed an admissible epi. Dually, if a morphism $M \to N$ having a cokernel in $\mathcal{E}$ can factor a admissible monic $N \to P$ such as $M \to N \to P$ then the map $M \to N$ is indeed a admissible monic.*

When we wish to identify a morphism as being an admissible monics and epis we shall use the arrows $\rightarrowtail$ and $\twoheadrightarrow$ respectively.

Example 5.2.2. *The category of finitely generated A-modules for a noetherian ring A with all the exact sequences.*

Proposition 5.2.3. *The isomorphisms in an exact category are precisely those maps which are both admissible monics and admissible epis.*

Proof. Let $f : A \to B$ be an admissible monic epi. Since an exact category is also abelian this means it is an monic epi in some abelian category therefore under the Freyd Mitchell embedding we can say it is a monic epi in some category of modules but then it is an isomorphism.

If $f : A \to B$ is an isomorphism them note that 1_A is an admissible monic epi with the trivial identity maps $M \to A \oplus 0 \to 0$ and $0 \to A \oplus 0 \to M$, but admissible monics and epis are closed under isomorphisms therefore, with the below diagrams we find f is a admissible monic and an admissible epi.

$$\begin{tikzcd}
A \arrow[r, tail, "1_M"] \arrow[dr, "f"'] & A \arrow[r, "f"] \arrow[d, "f"] & B \\
& B \arrow[ur, two heads, "1_B"'] &
\end{tikzcd}$$

$\square$

Definition 5.2.4 (Exact functor)**.** *An exact functor is an additive functor between exact categories which takes exact sequences to exact sequences.*

Proposition 5.2.5 (2 out of 3 for exact sequences)**.** *Let $(\mathcal{E}, E)$ be an exact category, suppose we have a diagram of exact sequences in $\mathcal{E}$ such that all of the vertical sequences are in E.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "i_A"] \arrow[d, "j_1"'] & A \arrow[r, "p_A"] \arrow[d, "j_2"'] & A'' \arrow[r] \arrow[d, "j_3"'] & 0 \\
0 \arrow[r] & B' \arrow[r, "i_B"] \arrow[d, "q_1"'] & B \arrow[r, "p_B"] \arrow[d, "q_2"'] & B'' \arrow[r] \arrow[d, "q_3"'] & 0 \\
0 \arrow[r] & C' \arrow[r, "i_C"] \arrow[d] & C \arrow[r, "p_C"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If any of the two horizontal sequences are in E then so is the third.

Proof. Suppose the bottom two sequences are in E and all the vertical sequences are in E. Due to the fact that j_1 and i_B are admissible monics we have $i_B j_1 = j_2 i_A$ which is also admissible monic due to $(E2)$. Now since the A sequence is exact in the ambient abelian category $\mathcal{A}$ this implies i_A has a cokernel in $\mathcal{A}$ namely p_A. Now by $(E3)$ we have that i_A is an admissible monic finally by $(E1)$ this tells us p_A is an admissible epi too. Which implies the top sequence is in E

The case when the top two sequences are in E is the dual case.

If the top and bottom sequences are in E then we wish to show i_B is an admissible monic.

Construct the pushout of the diagram $B' \xleftarrow{j_1} A' \xrightarrow{i_A} A$ and note that due to $(E2)$ we have $h' : B' \to A \sqcup_{A'} B'$ is an admissible monic.

This gives us an exact sequence $0 \to B' \to A \sqcup_{A'} B' \to B'' \to 0$..

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r, "h'"] \arrow[d, equal] & A \sqcup_{A'} B' \arrow[r] \arrow[d, "h"] & A'' \arrow[r] \arrow[d, "j_3"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

Now by the 5-lemma and the fact that j_3 is monic we have that the map $h : A \sqcup_{A'} B' \to B$ (existence is due to the universal property) is monic. In fact it is admissible. Due to $(E2)$ we have $p_C q_2$ is an admissible epi which means the short exact sequence $0 \to A \sqcup_{A'} B' \xrightarrow{h} B \xrightarrow{p_C q_2} C'' \to 0$ is in E. Lastly we have that $i_B = hh'$ and it is therefore an admissible monic.

$\square$

5.3 Q construction of an exact category

Suppose that $\mathcal{E}$ is an exact category. Define an equivalence relation on all diagrams

$$A \twoheadleftarrow P \rightarrowtail B$$

by saying that the top and bottom pictures in the diagram

$$\begin{tikzcd}
 & P' \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, dashed, "\cong"] & \\
A & & B \\
 & P \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}$$

are equivalent if the displayed isomorphism exists, making the diagram commute.

The category $Q\mathcal{E}$ has for objects all objects of $\mathcal{E}$. The morphisms $A \to B$ are the equivalence classes of the pictures above. Composition is defined by pullback:

$$\begin{tikzcd}
 & & Q \times_M P \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
 & Q \arrow[dl, two heads] \arrow[dr, tail] & & P \arrow[dl, two heads] \arrow[dr, tail] & \\
A & & B & & C
\end{tikzcd}$$

In order for composition in $Q\mathcal{E}$ to be coherent we expect the morphism i and p as shown above to actually be admissible monics and epis respectively.

Proposition 5.3.1. *Admissible monics are closed under pullbacks along admissible epis, and admissible epis are closed under pushouts along admissible monics.*

Proof. Consider a admissible monic i in the below diagram.

$$\begin{tikzcd}
 & X \arrow[r, "i'"] \arrow[d, two heads, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "p_2"] & & \\
0 \arrow[r] & A \arrow[r, tail, "i"] & B \arrow[r, two heads, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

Given it has a pullback through admissible epis we wish to demonstrate that i' is also an admissible monic. This is true due to the fact that pp_2 is a composition of admissible epis therefore an admissible epi itself. Furthermore i forms as the kernel of pp_2 therefore i itself is an admissible monic. This can be proved with a simple diagram chase to verify $pp_2i' = pip_1 = (0_{AC}p_1) = 0_{XC}$. Universality is also due to universality of the given square which allows us to lift $Z \to A$ to $Z \to X$. $\square$

An exact functor between two exact categories induces a functor between their corresponding Q spaces.

Given an admissible monic $i : A \rightarrowtail B$ let $i_! : A \to B$ denote the morphism in $Q\mathcal{E}$ of the form

$$A \overset{1_A}{\twoheadleftarrow} A \overset{i}{\rightarrowtail} B$$

Similarly for an admissible epi $p : C \twoheadrightarrow D$ let $p^! : D \to C$ denote the morphism in $Q\mathcal{E}$ of the form

$$D \overset{p}{\twoheadleftarrow} C \overset{1_C}{\rightarrowtail} C$$

Let $\mathrm{Mon}(\mathcal{E})$ and $\mathrm{Epi}(\mathcal{E})$ denote the subcategories of admissible monics and epis of $\mathcal{E}$, i.e. the same objects but only morphisms are admissible monics and epis respectively. The map $i \mapsto i_!$ defines a functor $\mathrm{Mon}(\mathcal{E}) \to Q\mathcal{E}$ and the map $p \mapsto p^!$ defines a functor $\mathrm{Epi}(\mathcal{E})^{\mathrm{op}} \to Q\mathcal{E}$.

This is universal in the following sense.

Proposition 5.3.2. *Suppose for some category $\mathcal{C}$ there exists functors $m : \mathrm{Mon}(\mathcal{E}) \to \mathcal{C}$ and $e : \mathrm{Epi}(\mathcal{E}) \to \mathcal{C}$ such that m, e agree on objects and for* *pushout diagrams involving admissible monics and epis of the following type*

$$\begin{tikzcd} \bullet \arrow[r, tail, "i'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, two heads, "p"] \\ \bullet \arrow[r, tail, "i"] & \bullet \end{tikzcd}$$

We have the following coherence condition $e(p)m(i) = m(i')e(p')$.

Then there exists a unique functor $h : Q\mathcal{E} \to \mathcal{C}$.

Proof. We directly construct h. Let $h(A) = m(A) = e(A)$ for objects $A \in \mathcal{E}$. Since e, m agree on objects this is well defined.

For a morphism $f : A \to B$ represented by $A \stackrel{p}{\leftarrow} P \stackrel{i}{\to} B$, define $h(\theta) = m(i)e(p)$. We need to verify this definition is independent of the representative span (well-definedness) and that h is an additive functor.

For well-definedness, let $A \stackrel{p}{\leftarrow} P \stackrel{i}{\to} B$ and $A \stackrel{p'}{\leftarrow} P' \stackrel{i'}{\to} B$ be equivalent representatives for f. An isomorphism $\phi : P \to P'$ exists in $\mathcal{E}$ such that $i = i'\phi$ and $p = p'\phi$. We must show $m(i)e(p) = m(i')e(p')$. Using functoriality we have,

$$m(i)e(p) = m(i'\phi)e(p'\phi) = m(i')m(\phi)e(\phi)e(p').$$

It suffices to show that $m(\phi)e(\phi) = 1$. But this is easy, since ϕ is an isomorphism, it is both an admissible monic and an admissible epi by Proposition 5.2.3. Consider the pullback square,

$$\begin{tikzcd} P \arrow[r, tail, "1_P"] \arrow[d, two heads, "1_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, two heads, "\phi"] \\ P \arrow[r, tail, "\phi"] & P' \end{tikzcd}$$

Applying the coherence condition we get

$$e(\phi)m(\phi) = m(1_P)e(1_P) = 1.$$

Now $m(\phi)$ and $e(\phi)$ are isomorphisms so also $m(\phi)e(\phi) = 1$.

Therefore, a functor h with the specified properties exists.

Lastly we note this is unique. A morphism $f : A \to B$ in $Q\mathcal{E}$ is represented by a diagram $A \stackrel{p}{\leftarrow} P \stackrel{i}{\to} B$, where p is an admissible epi and i is an admissible monic. Note that based on our definitions of canonical admissible monics and epis we have $\theta = i_! \circ p^!$. Functoriality requires $h(\theta) = h(i_! \circ p^!) = h(i_!) \circ h(p^!)$. The given conditions specify $h(i_!) = m(i)$ and $h(p^!) = e(p)$. Therefore, $h(\theta)$ must be $m(i)e(p)$. Since h must also agree with m (and e) on objects, h is uniquely determined. $\square$

5.4 K-groups of an exact category

The below theorem is from [GJ09][III]

Theorem 5.4.1. *There is an equivalence of groupoids $GQ\mathcal{E} \cong K_0(\mathcal{E})$.*

Proof. First we construct a functor $Q\mathcal{E} \to K_0(\mathcal{E})$ using Proposition 5.3.2. We reuse the notations of e, m. First note that the notion of agreeing on objects is trivial in this case since $K_0(\mathcal{E}) \in \mathbf{Grp}$ and is therefore just a single object category, i.e. a single object groupoid.

Let i be an admissible monic, and let $P \xrightarrow{p} B$ be an admissible epi, it fits into a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail] & P \arrow[r, two heads] & B \arrow[r] & 0 \end{tikzcd}.$$

.

We define $e(p) = [A]$ and $m(i) = [0]$. Functorality of m is immediate but we need to check it for e. Essentially e maps each admissible epi to its kernel.

Suppose $B \xrightarrow{b} C$ is another admissible epi. We wish to show $e(bp) = e(b) + e(p)$.

Consider the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker p \arrow[r] \arrow[d] & \ker bp \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ker b \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker p \arrow[r] \arrow[d] & P \arrow[r, two heads, "p"] \arrow[d, "bp"] & B \arrow[r] \arrow[d, two heads, "b"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Recall the definition of addition in $K_0(\mathcal{E})$ from Definition 5.1.5. Since the top sequence is exact we have $[\ker bp] = [\ker p] + [\ker b]$, i.e. $e(bp) = e(b) + e(p)$ as we needed.

We now need to show the coherence condition. Consider a pullback diagram of the sort.

$$\begin{tikzcd}
P \arrow[r, tail, "i'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\
B \arrow[r, tail, "i"] & C
\end{tikzcd}$$

Then $e(p)+m(i) = e(p)+0 = e(p) = [\ker p] = [\ker p'] = e(p') = e(p')+m(i')$.

The key point to note is that the kernels of p and p' are isomorphic since they are parallel morphisms in a pullback square [2].

This defines a functor $\varphi^* : Q\mathcal{E} \to K_0(\mathcal{E})$. Now note that $K_0(\mathcal{E})$ is trivially a groupoid, i.e. its free groupoid is simply itself.. Therefore, by the universal property of the free groupoid (Proposition 4.6.4) we have that this functor extends to a map $\varphi : GQ\mathcal{E} \to K_0(\mathcal{E})$.

Now we construct a functor $\psi : K_0(\mathcal{E}) \to GQ\mathcal{E}$. This is more natural and sends $[B]$ to the object $0 \xrightarrow{p^!} B \xleftarrow{i_!} 0$ i.e., $\psi([B]) = i_{!B}^{-1} p^!_B$. To show functorality suppose Q fits into an exact sequence of the form,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "i"] & C \arrow[r, two heads, "p"] & B \arrow[r, tail] & 0 .
\end{tikzcd}$$

That is to say, we have the relation $[C]=[A]+[B]$ we wish to show this is preserved under ψ. Realize A as a pullback in the following sense.

$$\begin{tikzcd}
 & & C \arrow[dr, two heads, "p"] & & \\
 & A \arrow[ur, tail, "i"] \arrow[dr, two heads, "p^!_A"] \arrow[rr, phantom, "\rangle", very near start] & & B \arrow[dr, tail] & \\
0 \arrow[ur] & & 0 \arrow[ur, tail, "i_{!B}"] & & 0
\end{tikzcd}$$

This shows $\psi([C]) = \psi([A]) \circ \psi([B])$. Note that we are in $GQ\mathcal{E}$ so all the morphisms are reversible.

Lastly note that $\varphi\psi[P] = \varphi[i_{!P}^{-1} p^!_P] = \varphi(i_{!P}^{-1}) + \varphi(p_{!P}) = [P]$. $\square$

The above theorem along with Corollary 4.7.3 gives us the following famous result due to Quillen.

---

[2] Note that this is basically due to the fact that limits commute. We can think of the kernel of p' as $P \times_B 0$ and that of p as $A \times_C 0$. The notion of commuting limits is basically that $\ker p' \cong P \times_B 0 = (A \times_C B) \times_B 0 \cong A \times_C (B \times_B 0) \cong A \times_C 0 \cong \ker p$.

Corollary 5.4.2. *There is an isomorphism of groups* $K_0(\mathcal{E}) \cong \pi_1(BQ\mathcal{E}, 0)$

Proof. We have an equivalence of groupoids,

$$GP_*(BQ\mathcal{E}) \simeq K_0(\mathcal{E}).$$

Since the groupoid $K_0(\mathcal{E})$ has only one object, it is connected. The equivalence implies that the groupoid $GP_*(BQ\mathcal{E})$ is also connected.

The fundamental group $\pi_1(BQ\mathcal{E}, 0)$ is, by definition, the automorphism group of the object 0 in the fundamental groupoid. Therefore,

$$\pi_1(BQ\mathcal{E}, 0) \cong \mathrm{Aut}_{K_0(\mathcal{E})}(*) \cong K_0(\mathcal{E}).$$

□

This finally motivates the following definition of K-groups via the Q-construction.

Definition 5.4.3 (Higher K-groups). $K_n(\mathcal{E}) = \pi_{n+1}(BQ\mathcal{E}, 0)$

5.5 Quillen's Theorem B

We end with two theorems without proof. These are Quillens A and B theorem. Quillens A theorem follows from the B theorem. A full proof may be seen in [Qui73] or in [GJ09].

These theorems relate the homotopy theoretic properties of classifying spaces (nerves) of categories connected by a functor, based on the properties of the functor's homotopy fibers .

For a functor $f : \mathcal{C} \to \mathcal{D}$ and an object $d \in \mathrm{Ob}(\mathcal{D})$, recall the slice category $\mathcal{D}/d$ (objects of $\mathcal{D}$ over d). We also define the category f/d (sometimes written $f \downarrow d$), whose objects are pairs $(c, \alpha : f(c) \to d)$ where $c \in \mathrm{Ob}(\mathcal{C})$ and α is a morphism in $\mathcal{D}$. A morphism from (c, α) to (c', α') is a morphism $\gamma : c \to c'$ in $\mathcal{C}$ such that $\alpha' \circ f(\gamma) = \alpha$. There is a canonical projection functor $f/d \to \mathcal{C}$ given by $(c, \alpha) \mapsto c$, and another functor $f/d \to \mathcal{D}/d$ given by $(c, \alpha) \mapsto (\alpha : f(c) \to d)$.

Theorem 5.5.1 (Quillen's Theorem B). *Given a functor* $f : \mathcal{C} \to \mathcal{D}$ *between small categories. Suppose that for every morphism* $\alpha : d \to d'$ *in* $\mathcal{D}$*, the* *induced functor $f/d' \to f/d$ (given by composition with α) induces a weak equivalence on classifying spaces:*

$$B(f/d') \xrightarrow{\sim} B(f/d).$$

Then for every object $d \in \mathrm{Ob}(\mathcal{D})$, the induced commutative diagram

$$\begin{tikzcd} B(f/d) \arrow[r] \arrow[d] & B\mathcal{C} \arrow[d, "B(f)"] \\ B(\mathcal{D}/d) \arrow[r] & B\mathcal{D} \end{tikzcd}$$

(where the horizontal maps are induced by the projections $(c, \alpha) \mapsto c$ and $(\beta : x \to d) \mapsto x$, and the vertical maps are induced by $(c, \alpha) \mapsto (\alpha : f(c) \to d)$ and the functor $\mathcal{D}/d \to \mathcal{D}$) is homotopy Cartesian.

Theorem B relates the homotopy fiber $B(f/d)$ over a point d (represented by $B(\mathcal{D}/d)$ which is contractible) to the map $B(f)$. Theorem A is a direct consequence when the homotopy fibers are contractible.

Theorem 5.5.2 (Quillen's Theorem A)**.** *Given a functor $f : \mathcal{C} \to \mathcal{D}$ between small categories. If for every object $d \in \mathrm{Ob}(\mathcal{D})$, the classifying space $B(f/d)$ is weakly equivalent to a point (i.e., is contractible), then the induced map on classifying spaces*

$$B(f) : B\mathcal{C} \xrightarrow{\sim} B\mathcal{D}$$

is a weak equivalence.

Chapter 6

Waldhausen $s_\bullet$ construction

Waldhausen's $s_\bullet$ construction comes across as a more natural alternate to defining a K theory for spaces. The approach is infact incredibly general and is conducive to defining a K theory of ∞-categories. This perspective is not examined in this project. However, even in the case of exact categories Waldhausen's construction offers a very useful perspective. We examine only the case of the $s_\bullet$ construction of an exact category but refer the reader to Waldhausen's paper [Wal85] and the paper by Barwick [Bar16] for the construction over ∞-categories.

Recall the notion of a arrow category (Definition 3.1.5).

Example 6.0.1 (Arrow Category of an Ordinal)**.** *Let* $\mathbf{n} = \{0 \leq 1 \leq \cdots \leq n\}$ *be the category corresponding to the ordinal number n. A morphism $i \to j$ exists in* $\mathbf{n}$ *if and only if $i \leq j$. The objects of* $\mathrm{Arr}(\mathbf{n})$ *can be identified with pairs (i, j) such that $i \leq j$ in* $\mathbf{n}$. *A morphism from (i, j) to (k, l) in* $\mathrm{Arr}(\mathbf{n})$ *exists if and only if $i \leq k$ and $j \leq l$. This corresponds to a commutative square since composition in a poset is uniquely determined.*

Now we define a simplicial set using functors from these arrow categories into an exact category.

Definition 6.0.2 (Waldhausen $s_\bullet$-construction)**.** *Let $\mathcal{E}$ be an exact category with its distinguished class of exact sequences E. For each integer $n \geq 0$, let* $\mathrm{Arr}(\mathbf{n})$ *be the arrow category of the ordinal n. Define $s_n(\mathcal{E})$ to be the set of all functors*

$$P : \mathrm{Arr}(n) \to \mathcal{E}$$

such that the following conditions hold:

1. $P(i,i) \cong 0$ for all $0 \leq i \leq n$.

2. For any sequence $i \leq j \leq k$ in n, the sequence induced by the morphisms $(i,j) \to (i,k)$ and $(i,k) \to (j,k)$ in $\mathrm{Arr}(\mathbf{n})$, namely

$$0 \to P(i,j) \rightarrowtail P(i,k) \twoheadrightarrow P(j,k) \to 0$$

is an exact sequence in E.

A functor P satisfying these conditions is called an exact functor in this context.

These sets form a simplicial set $s_\bullet(\mathcal{E})$ whose n-simplices are the elements of $s_n(\mathcal{E})$.

This simplicial set $s_\bullet(\mathcal{E})$ is the Waldhausen $s_\bullet$-construction for the exact category $\mathcal{E}$.

Example 6.0.3. *The following is a picture of exact $P : \mathrm{Arr}(\mathbf{3}) \to \mathcal{E}$. Note that all the squares are pullback+pushout diagrams (bicartesian) since two parallel admissible epis share kernels.*

$$\begin{tikzcd}
 & & & P(0,3) \arrow[dr, two heads] & & & \\
 & & P(0,2) \arrow[ur, tail] \arrow[dr, two heads] & & P(1,3) \arrow[dr, two heads] & & \\
 & P(0,1) \arrow[ur, tail] \arrow[dr, two heads] & & P(1,2) \arrow[ur, tail] \arrow[dr, two heads] & & P(2,3) \arrow[dr, two heads] & \\
0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & 0
\end{tikzcd}$$

To recover the more familar definition of the Waldhausen construction note that the above diagram is generated by the string of monics

$$0 \rightarrowtail P(0,1) \rightarrowtail P(0,2) \rightarrowtail P(0,3)$$

by attaching all cokernels.

Recall the definition of a simplicial homotopy (Definition 4.8.1)

Lemma 6.0.4. *Let $\mathcal{C}, \mathcal{D}$ be two exact categories if there exist naturally isomorphic exact functors $F, G : \mathcal{C} \to \mathcal{D}$ then the induced maps $F_*, G_* : s_\bullet(\mathcal{C}) \to s_\bullet(\mathcal{D})$ are homotopic.*

Proof. Let h denote the natural isomorphism $F(C) = h(C, 0) \cong h(C, 1) = G(C)$.

We wish to find a simplicial map $H : s_\bullet(\mathcal{C}) \times \Delta[1] \to s_\bullet(\mathcal{D})$ which is a simplicial homotopy. Such a map H consists of a sequence of functions $H_t : s_n(\mathcal{C}) \to s_n(\mathcal{D})$ for each $t : \mathbf{n} \to \mathbf{1}$ resulting in the following diagrams commuting.

$$\begin{array}{ccc} \mathbf{n} & \qquad & \begin{tikzcd} s_n(\mathcal{C}) \arrow[r, "H_t"] \arrow[d, "f^*"'] & s_n(\mathcal{D}) \arrow[d, "f^*"] \\ s_m(\mathcal{C}) \arrow[r, "H_{tf}"'] & s_m(\mathcal{D}) \end{tikzcd} \\ {\scriptstyle f}\uparrow & & \\ \mathbf{m} & & \end{array}$$

Let $s : \mathrm{Arr}(\mathbf{1}) \to \mathbf{1}$ denote the map $(i, j) \mapsto i$ i.e., the source map.

Let $P : \mathrm{Arr}(\mathbf{n}) \to \mathcal{C}$ be exact consider $H_t(P)$ as the composite map,

$$\mathrm{Arr}(\mathbf{n}) \xrightarrow{(P, t_*)} \mathcal{C} \times \mathrm{Arr}(\mathbf{1}) \xrightarrow{(1_{\mathcal{C}}, s)} \mathcal{C} \times \mathbf{1} \xrightarrow{h} \mathcal{D}.$$

□

6.1 Segal edgewise subdivision

We now wish to draw a parallel between the Waldhausen $s_\bullet$ construction and Quillen's Q construction. This requires the notion of edgewise subdivision of a simplicial set. First recall the notion of the join of a poset.

Definition 6.1.1 (Join of elements in a poset)**.** *For two elements in a poset their join is just their coproduct.*

Example 6.1.2. *When* $\mathbf{n}$ *is a ordinal number consider* $\mathbf{n}^{\mathrm{op}}$ *then* $\mathbf{n}^{\mathrm{op}} \vee \mathbf{n} \cong \mathbf{2n+1}$*. This can be seen in the following diagram.*

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \\ 0 \arrow[u] & 1 \arrow[l] & 2 \arrow[l] & \ldots \arrow[l] & n \arrow[l] \end{tikzcd}$$

This defines a functor $e : \Delta \to \Delta$, as $e(\mathbf{n}) = \mathbf{n}^{\mathrm{op}} \vee \mathbf{n}$.

Definition 6.1.3 (Segal's edgewise subdivision of a simplicial set)**.** *For* $X \in$ **sSets** *consider the functor* $X^e = Xe^{\mathrm{op}}$*, i.e.*

$$X_n^e = X(\mathbf{n}^{\mathrm{op}} \vee \mathbf{n})$$

. The face and degeneracy maps are defined as such,

$$d_i^e = d_{n-i}d_{n+1+i} : X_{\mathbf{2n+1}} \to X_{\mathbf{2n-1}}$$
$$s_j^e = s_{n-j}s_{n+1+j} : X_{\mathbf{2n+1}} \to X_{\mathbf{2n+3}}$$

Example 6.1.4. *Consider $\Delta[2]$ the standard 2-simplex.*

$\Delta[2]^e_0 = \Delta[2]_1 = \{(0 \to 0), (0 \to 1), (0 \to 2), (1 \to 1), (1 \to 2), (2 \to 2)\}$ $\Delta[2]^e_1 = \Delta[2]_3 = \{(0 \to 0 \to 0 \to 0), (0 \to 0 \to 0 \to 1), (0 \to 0 \to 0 \to 2), (0 \to 0 \to 1 \to 1), (0 \to 0 \to 1 \to 2), (0 \to 0 \to 2 \to 2), (0 \to 1 \to 1 \to 1), (0 \to 1 \to 1 \to 2), (0 \to 1 \to 2 \to 2), (0 \to 2 \to 2 \to 2), (1 \to 1 \to 1 \to 1), (1 \to 1 \to 1 \to 2), (1 \to 1 \to 2 \to 2), (1 \to 2 \to 2 \to 2), (2 \to 2 \to 2 \to 2)\}$

With an appropriate abuse of notation we see a much clearer picture of the subdivision $\Delta[2]^e$ as such. Each chain (f_1, f_2, f_3, f_4) corresponds to $(f_2, f_3) \to (f_1, f_4)$

$$\begin{tikzcd}
 & & (1,1) \arrow[dl] \arrow[dr] \arrow[dd] & & \\
 & (0,1) \arrow[dr] & & (1,2) \arrow[dl] & \\
(0,0) \arrow[ur] \arrow[rr] & & (0,2) & & (2,2) \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Note the difference between this and barycentric subdivision which we saw while defining the Ex^∞ functor.

With this intuitive picture in mind the next theorem should come as no surprise. Recall the notion of a homotopy colimit (Definition 4.11.4).

Theorem 6.1.5 (A simplicial set is weakly equivalent to its Segal subdivision)**.** *The natural map $\omega : X^e \to X$ is a weak equivalence.*

Proof. Any simplicial set is weakly equivalent to the homotopy colimit of standard simplices indexed by its category of simplices. This gives us the maps in the commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_{x \in X^e_n} \Delta^e[n] \arrow[r, "\sim"] \arrow[d, "\omega_*"] & X^e \arrow[d, "\omega"] \\
\operatorname{hocolim}_{x \in X_n} \Delta[n] \arrow[r, "\sim"] & X
\end{tikzcd}$$

By 2-out-of-3 property if we show ω_* is a weak equivalence we are done.

The homotopy colimit functor preserves weak equivalences. Thus, ω_* is a weak equivalence if each map $\omega_n : \Delta[n]^e \to \Delta[n]$ is a weak equivalence for all $n \geq 0$.

We show that ω_n is a weak equivalence by demonstrating that both its domain and codomain are contractible. The standard n-simplex, $\Delta[n]$, is contractible. The simplicial set $\Delta[n]^e$ is the nerve of a poset P_n. The objects of P_n are pairs (i, j) such that $0 \leq i \leq j \leq n$. A morphism $(a, b) \leq_{P_n} (c, d)$ exists if and only if $c \leq a$ and $b \leq d$. In this poset P_n, the object $(0, n)$ is a terminal object: for any object $(i, j) \in P_n$, we have $0 \leq i$ and $j \leq n$, ensuring a unique morphism $(i, j) \to (0, n)$. The nerve of a poset with a terminal object is contractible. Therefore, $\Delta[n]^e = B(P_n)$ is contractible for all $n \geq 0$. □

6.2 $s_\bullet(\mathcal{E}) \cong BQ\mathcal{E}$

In this section we prove that the Waldhausen construction is weakly equivalent to Quillen's Q-construction. The goal with introducing the notion of edgewise subdivision will now become clear.

We begin by defining a simplicial set map $\pi : s_\bullet(\mathcal{E})^e \to BQ\mathcal{E}$. We do this by constructing levelwise $\pi_n : s_\bullet(\mathcal{E})^e_n \to BQ\mathcal{E}_n$.

Let $P \in (s_\bullet(\mathcal{E})^e)_n$, i.e., $P\colon \mathrm{Arr}(\mathbf{n}^o \vee \mathbf{n}) \to \mathcal{E}$. Define $\pi_n(P)$ as the functor $\mathbf{n} \to Q\mathcal{E}$ sending an object $k \in \mathbf{n}$ to $P(k^o, k) \in \mathrm{Ob}(Q\mathcal{E})$, and a morphism $i \leq j$ in $\mathbf{n}$ to $f^P_{i,j}\colon P(i^o, i) \to P(j^o, j)$. The $Q\mathcal{E}$-morphism $f^P_{i,j}$ is defined by the diagram of the following sort in $\mathcal{E}$ for $i \leq j$.

$$\begin{tikzcd} & P(j^o, i) \arrow[dl, two heads] \arrow[dr, tail] & \\ P(i^o, i) & & P(j^o, j) \end{tikzcd}$$

Functoriality of $\pi_n(P)$ is ensured by the bicartesian property of the diagram relating $P(k^o, i)$, $P(j^o, i)$, $P(k^o, j)$, and $P(j^o, j)$ for $i \leq j \leq k$. This construction yields $\pi_n(P) \in (BQ\mathcal{E})_n$. The functions π_n are natural in $\mathbf{n}$, thereby defining the simplicial map $\pi\colon s_\bullet(\mathcal{E})^e \to BQ\mathcal{E}$.

Before proceeding for the proof of the titular theorem of this section we need a lemma. This requires introducing a few definitions for making the machinery simpler.

Definition 6.2.1. *For an exact category $\mathcal{E}$ define $\mathrm{Iso}_n(\mathcal{E})$ as the category whose objects are all strings*

$$Q\colon Q_0 \xrightarrow{\cong} Q_1 \xrightarrow{\cong} \dots \xrightarrow{\cong} Q_n$$

of isomorphisms of length n. The morphisms are natural transformations.

This can be understood as a natural groupoid-ification of an exact category.

Lemma 6.2.2. *Let $\mathcal{E}$ be an exact category. Define functors $f : \mathcal{E} \to \mathrm{Iso}_n(\mathcal{E})$ by $P \mapsto (P \xrightarrow{1_P} P \xrightarrow{1_P} \dots \xrightarrow{1_P} P)$ and $g : \mathrm{Iso}_n(\mathcal{E}) \to \mathcal{E}$ by $(Q_0 \xrightarrow{q_0} Q_1 \to \dots \xrightarrow{q_{n-1}} Q_n) \mapsto Q_0$. Then f and g form an exact equivalence of categories.*

Proof. The functors f and g are exact. For $gf : \mathcal{E} \to \mathcal{E}$, we have $gf(P) = g(P \xrightarrow{1_P} \dots \xrightarrow{1_P} P) = P$. Thus $gf = 1_{\mathcal{E}}$, and the identity natural transformation $\eta : gf \Rightarrow 1_{\mathcal{E}}$ with $\eta_P = 1_P$ is a natural isomorphism.

For $fg : \mathrm{Iso}_n(\mathcal{E}) \to \mathrm{Iso}_n(\mathcal{E})$, let $Q = (Q_0 \xrightarrow{q_0} Q_1 \to \dots \xrightarrow{q_{n-1}} Q_n)$. Then $fg(Q) = f(Q_0) = (Q_0 \xrightarrow{1_{Q_0}} Q_0 \to \dots \xrightarrow{1_{Q_0}} Q_0)$. Define a natural transformation $\epsilon : fg \Rightarrow 1_{\mathrm{Iso}_n(\mathcal{E})}$. For $Q \in \mathrm{Iso}_n(\mathcal{E})$, $\epsilon_Q : fg(Q) \to Q$ is given by the tuple of isomorphisms $(\epsilon_{Q,i})_{i=0}^n$, where $\epsilon_{Q,i} : Q_0 \to Q_i$ is defined as $\epsilon_{Q,0} = 1_{Q_0}$ and $\epsilon_{Q,i} = q_{i-1} \circ \dots \circ q_0$ for $i > 0$. $\square$

Definition 6.2.3 (Simplicial exact category)**.** *Define $S_\bullet(\mathcal{E})$ as a category whose objects are exact functors $P : \mathrm{Arr}(\mathbf{n}) \to \mathcal{E}$ as previously defined. The morphisms in the category are natural transformations between functors.*

We consider the groupoidification of the exact categories $S_\bullet(\mathcal{E})^e$ and $BQ\mathcal{E}$.

The simplicial set map $\pi\colon s_\bullet(\mathcal{E})^e \to BQ\mathcal{E}$ is the object-level part of a map of simplicial groupoids, also denoted π:

$$\pi\colon \mathrm{Iso}(S_\bullet(\mathcal{E}))^e \to \mathrm{Iso}(BQ\mathcal{E})$$

As a map of simplicial groupoids, this π acts on objects as previously defined and also provides a consistent mapping for the natural isomorphisms (the morphisms within these groupoids).

Lemma 6.2.4. *The morphism of groupoids $\pi_n : \mathrm{Iso}(S_\bullet(\mathcal{E}))_n^e \to \mathrm{Iso}(BQ\mathcal{E})_n$ induces a weak equivalence between their nerves $B\mathrm{Iso}(S_\bullet(\mathcal{E}))_n^e \simeq B\mathrm{Iso}(BQ\mathcal{E})_n$*

Proof. We shall prove this by demonstrating that π_n forms an equivalence of categories. For this we show that π_n is fully faithful and essentially surjective (Proposition 2.2.6).

Let $P, Q \in \mathrm{Iso}(S_\bullet(\mathcal{E}))^e_n$. We wish to prove that the below map is a bijective set map

$$\mathrm{Hom}_{\mathrm{Iso}(S_\bullet(\mathcal{E}))^e_n}(P, Q) \longrightarrow \mathrm{Hom}_{\mathrm{Iso}(BQ\mathcal{E})_n}(\pi_n(P), \pi_n(Q)).$$

We will do this by demonstrating that for each natural isomorphism $\theta : \pi_n(P) \to \pi_n(Q)$ there exists a unique natural isomorphism $\Theta : P \to Q$ such that $\pi_n(\Theta) = \theta$. This will automatically take care of long chains of isomorphisms.

Unpack the definition of a natural isomorphism $\theta : \pi_n(P) \to \pi_n(Q)$.

A natural isomorphism $\theta : \pi_n(P) \to \pi_n(Q)$ consists of isomorphisms θ_k such that, the diagram commutes in $Q\mathcal{E}$.

$$\begin{tikzcd}
P(i^o, i) \arrow[r, "\alpha^P_{i,j}"] \arrow[d, "\theta_i"] & P(j^o, j) \arrow[d, "\theta_j"] \\
Q(i^o, i) \arrow[r, "\alpha^Q_{i,j}"] & Q(j^o, j)
\end{tikzcd}$$

We construct a unique natural isomorphism $\Theta : P \xrightarrow{\cong} Q$ such that for $A = (k^o \to k)$, $\Theta_A = \theta_k$.

The commutativity of the $Q\mathcal{E}$ diagram implies there's a unique isomorphism $\Theta(j^o, i) : P(j^o, i) \xrightarrow{\cong} Q(j^o, i)$ making the below diagram commute in $\mathcal{E}$.

$$\begin{tikzcd}
 & P(j^o, i) \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, "\Theta(j^0, i)"] & \\
P(i^o, i) \cong_{\Theta_i} Q(i^o, i) & & P(j^o, j) \cong_{\Theta_j} Q(j^o, j) \\
 & Q(j^o, i) \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}$$

Comparing this to the diagram of short exact sequences using the 5-Lemma we get that this determines $\Theta(n^o, i^o)$ uniquely for $i < n$.

$$
\begin{tikzcd}
0 \arrow[r] & P(n^o, i^o) \arrow[r, tail] \arrow[d, dashed] & P(n^o, 0) \arrow[r, two heads] \arrow[d, "\cong_{\Theta(n^0,0)}"] & P(i^o, 0) \arrow[r] \arrow[d, "\cong_{\Theta(i^0,0)}"] & 0 \\
0 \arrow[r] & Q(n^o, i^o) \arrow[r, tail] & Q(n^o, 0) \arrow[r, two heads] & Q(i^o, 0) \arrow[r] & 0
\end{tikzcd}
$$

Therefore it determines uniquely a natural isomorphism for the string of admissible monics.

This construction provides a unique Θ with $\pi_n(\Theta) = \theta$. Hence, π_n is fully faithful.

For essential surjectivity, given $\alpha : \mathbf{n} \to Q\mathcal{E}$, we construct $P : \mathrm{Ar}(\mathbf{n}^o * \mathbf{n}) \to \mathcal{E}$ such that $\pi_n(P) = \alpha$. Choose representatives $\alpha(i) \xleftarrow{q_{ij}} X_{ij} \xrightarrow{m_{ij}} \alpha(j)$ for each $\alpha(i \leq j)$.

Define $P(k^o, k) := \alpha(k)$ and $P(j^o, i) := X_{ij}$. The functoriality of α in $Q\mathcal{E}$ determines unique strings of admissible monics (e.g., $P(n^o, 0) \to \cdots \to P(n^o, n)$) and epis (e.g., $P(n^o, 0) \to \cdots \to P(0^o, 0)$). Objects like $P(n^o, i^o)$ are defined via exact sequences, e.g., $0 \to P(n^o, i^o) \to P(n^o, 0) \to P(i^o, 0) \to 0$. For general $n^o \leq r \leq s \leq n$, $P(r, s)$ is defined by the exact sequence $0 \to P(n^o, r) \to P(n^o, s) \to P(r, s) \to 0$. This construction ensures P is an exact functor and $\pi_n(P) = \alpha$. Thus, π_n is essentially surjective.

Since π_n is fully faithful and essentially surjective, it is an equivalence of groupoids.

□

We can now prove the main theorem of this section.

Theorem 6.2.5 (Equivalence between Waldhausen's $s_\bullet$ and Quillen Q construction)**.** *For an exact category $\mathcal{E}$, there exist weak equivalences $s_\bullet(\mathcal{E}) \simeq s_\bullet(\mathcal{E})^e \simeq BQ\mathcal{E}$.*

Proof. We will prove this by demonstrating the maps in the following diagram forms a weak equivalence.

$$
\begin{tikzcd}
& s_\bullet(\mathcal{E})^e \arrow[dl, "\omega"'] \arrow[dr, "\pi"] & \\
s_\bullet(\mathcal{E}) & & BQ\mathcal{E}
\end{tikzcd}
$$

The map $\omega : s_\bullet(\mathcal{E})^e \to s_\bullet(\mathcal{E})$ is a weak equivalence by Theorem 6.1.5. It remains to show that $\pi : s_\bullet(\mathcal{E})^e \to BQ\mathcal{E}$ is a weak equivalence.

By Lemma 6.2.4, the map of simplicial groupoids $\tilde{\pi} : \mathrm{Iso}(S_\bullet(\mathcal{E}))^e \to \mathrm{Iso}(BQ\mathcal{E})$ is an equivalence of groupoids, and thus induces a weak equivalence:

$$B(\tilde{\pi}) : B(\mathrm{Iso}(S_\bullet(\mathcal{E}))^e) \xrightarrow{\sim} B(\mathrm{Iso}(BQ\mathcal{E})).$$

We have a commutative diagram:

$$\begin{tikzcd}
s_\bullet(\mathcal{E})^e \arrow[r, "\eta", "\simeq"'] \arrow[d, "\pi"'] & B(\mathrm{Iso}(S_\bullet(\mathcal{E}))^e) \arrow[d, "\simeq"', "B(\tilde{\pi})"] \\
BQ\mathcal{E} \arrow[r, "\eta'"] & B(\mathrm{Iso}(BQ\mathcal{E}))
\end{tikzcd}$$

The map $\eta : s_\bullet(\mathcal{E})^e \to B(\mathrm{Iso}(S_\bullet(\mathcal{E}))^e)$ is induced by the inclusion of objects into the corresponding groupoid of isomorphisms in each simplicial degree.

Note that $\eta : s_\bullet(\mathcal{E})^e \to B(\mathrm{Iso}(S_\bullet(\mathcal{E}))^e)$ is also a weak equivalence. Indeed, this is due to applying Lemma 6.2.2 levelwise.

It remains to show that $\eta' : BQ\mathcal{E} \to B(\mathrm{Iso}(BQ\mathcal{E}))$ is a weak equivalence. Given the commutative diagram and that $B(\tilde{\pi})$ and η are weak equivalences, if η' is also a weak equivalence, then by the 2-out-of-3 property for weak equivalences (since $\eta' \circ \pi = B(\tilde{\pi}) \circ \eta$), π must be a weak equivalence.

The map η' is indeed a weak equivalence. For each $k \geq 0$, the category $(BQ\mathcal{E})_k$ (whose objects are functors $\mathbf{k} \to Q\mathcal{E}$) is an exact category. By Lemma 6.2.2, an exact category is equivalent to its category of n-strings of isomorphisms. Thus, $(BQ\mathcal{E})_k \simeq \mathrm{Iso}_n((BQ\mathcal{E})_k)$. Since the nerve of equivalent categories are weakly equivalent, and η' is induced by the inclusion of objects levelwise, it follows that η' is a weak equivalence, and we are done. $\square$

6.3 Additivity theorem

The additivity theorem is the most critical theorem of K-theory it is used in proving all of the "Fundamental" theorems of K-theory like Resolution and Dévissage.

Note that in the below results $\mathrm{Ex}(\mathcal{E})$ refers to the exact category of exact sequences in $\mathcal{E}$ with natural maps between them. This is not the same as the Kan Ex functor.

We begin by defining two natural functors from $\mathrm{Ex}(\mathcal{E}) \to \mathcal{E}$. For an object $0 \to A \to B \to C \to 0$ in $\mathrm{Ex}(\mathcal{E})$ consider $s, t : \mathrm{Ex}(\mathcal{E}) \to \mathcal{E}$ as the functors which sends the sequence to A and C respectively. This induces a natural map $s_* : s_\bullet(\mathrm{Ex}(\mathcal{E})) \to s_\bullet(\mathcal{E})$.

Lemma 6.3.1. *Let $\mathcal{E}$ be an exact category. The functor $\Delta/s_\bullet\mathcal{E} \to \mathrm{sSet}$ defined by $P \mapsto s^{-1}(P)$ is a diagram of equivalences in simplicial sets.*

Proof. It is enough to show that for each n-simplex $P : \mathrm{Ar}(\mathbf{n}) \to \mathcal{E}$ of $s_\bullet\mathcal{E}$, the composite map

$$s^{-1}(P) \longrightarrow s_\bullet\mathrm{Ex}(\mathcal{E}) \xrightarrow{t_*} s_\bullet\mathcal{E}$$

is a weak equivalence.

If $P = 0_*$ (the 0-simplex for the zero object in $\mathcal{E}$), then $s^{-1}(0_*) = s_\bullet\mathrm{Ex}_0(\mathcal{E})$, where $\mathrm{Ex}_0(\mathcal{E})$ is the category of exact sequences $0 \to A \to B \to 0 \to 0$. The functor $t : \mathrm{Ex}_0(\mathcal{E}) \to \mathcal{E}$ (sending to A) is an equivalence. Thus, the map $f : s^{-1}(0_*) \to s_\bullet\mathcal{E}$ is a weak equivalence. This map f has a section $g : s_\bullet\mathcal{E} \to s^{-1}(0_*)$ defined by $Q \mapsto (0 \to 0 \to Q \xrightarrow{1} Q \to 0)$.

Consider the diagram:

$$\begin{tikzcd}
s_\bullet\mathcal{E} \arrow[r, "g", "\cong"'] & s^{-1}(0_*) \arrow[r, "n_*"'] \arrow[d] & s^{-1}P \arrow[r] \arrow[d] & s_\bullet\mathrm{Ex}(\mathcal{E}) \arrow[d, shift right, "t_*"'] \arrow[d, shift left, "s_*"] \\
& \Delta[0] \arrow[ul, "0_*"] \arrow[r, "n"'] & \Delta[n] \arrow[r, "P"'] & s_\bullet\mathcal{E}
\end{tikzcd}$$

The composite $s^{-1}(0_*) \xrightarrow{n_*} s^{-1}(P) \to s_\bullet\mathrm{Ex}(\mathcal{E}) \xrightarrow{t_*} s_\bullet\mathcal{E}$ is f. It suffices to show that $\psi : s^{-1}(P) \xrightarrow{t_*} s_\bullet\mathcal{E} \xrightarrow{g} s^{-1}(0_*) \xrightarrow{n_*} s^{-1}(P)$ is homotopic to the identity.

An m-simplex of $s^{-1}(P)$ is a pair $(\theta : \mathbf{m} \to \mathbf{n}, \mathrm{seq})$, where seq denotes the data for an exact sequence of $S_m\mathcal{E}$ of the form $0 \to \theta^*(P) \to A \to B \to 0$. ψ maps this to $(\mathbf{n}_{\mathrm{const}} : \mathbf{m} \to \mathbf{n}, \mathrm{seq}')$ where seq' is $0 \to 0 \to B \xrightarrow{1} B \to 0$.

We now define a homotopy $H : s^{-1}(P) \times \Delta[1] \to s^{-1}(P)$. For an m-simplex given by (θ, A, B) and $\tau : \mathbf{m} \to \mathbf{1}$, H maps this to $(h_\tau(\theta), A_\tau, B)$. The map $h_\tau(\theta)$ is the composite $\mathbf{m} \xrightarrow{(\theta,\tau)} \mathbf{n} \times \mathbf{1} \xrightarrow{h} \mathbf{n}$. The term A_τ is constructed via pushouts. For each component, corresponding to a map $\alpha_{obj} : (i, j) \to (k, l)$ in $\mathrm{Ar}(\mathbf{m})$, the pushout diagram defining the new middle term (let's call it

$A_{\tau,obj}$) from the original middle term (A_{obj}) is,

$$\begin{tikzcd}
(\theta^*P)(i,j) \arrow[r, tail] \arrow[d] & A_{obj} \arrow[d] \\
(h_\tau(\theta)^*P)(i,j) \arrow[r, tail] & A_{\tau,obj}
\end{tikzcd}$$

The object B (the cokernel part) is unchanged in this step. The sequence $0 \to (h_\tau(\theta)^*P)(i,j) \to A_{\tau,obj} \to B_{obj} \to 0$ is exact. These diagrams are themselves induced by pushouts at the level of $\mathcal{E}$, as shown in diagram,

$$\begin{tikzcd}
P_{\mathrm{orig}}(i',j') \arrow[r, tail, "i"] \arrow[d, "\alpha_*"] & A'_{\mathrm{orig}} \arrow[d, "\gamma"] \\
P_{\mathrm{orig}}(k',l') \arrow[r, tail] & \overline{A'}
\end{tikzcd}$$

where $0 \to P_{\mathrm{orig}}(i',j') \to A'_{\mathrm{orig}} \to B'_{\mathrm{orig}} \to 0$ is an exact sequence in $\mathcal{E}$ corresponding to a part of $\theta^*(P)$, and $\alpha : (i',j') \to (k',l')$ is a morphism in $\mathrm{Ar}(\mathbf{n})$. The functions $h_\tau : (s^{-1}(P))_m \to (s^{-1}(P))_m$ given by $h_\tau(\theta, A, B) = (h_\tau(\theta), A_\tau, B)$ must satisfy:

$$\begin{tikzcd}
(s^{-1}(P))_m \arrow[r, "h_\tau"] \arrow[d, "\gamma_*"'] & (s^{-1}(P))_m \arrow[d, "\gamma_*"] \\
(s^{-1}(P))_k \arrow[r, "h_{\tau\gamma}"'] & (s^{-1}(P))_k
\end{tikzcd}$$

These h_τ define the homotopy from the identity map on $s^{-1}(P)$ to ψ. If τ corresponds to 0, $h_0(\theta) = \theta$, $A_\tau = A$, so H starts at the identity. If τ corresponds to 1, $h_1(\theta) = \mathbf{n}_{\mathrm{const}}$. Then $(h_1(\theta)^*P) = 0$. The pushout makes $A_\tau \cong A$. The exact sequence becomes $0 \to 0 \to A \to B \to 0$. This is incorrect according to the definition of ψ.

Revisiting the definition of ψ and the effect of H_1: $\psi(\theta, A, B) = (\mathbf{n}_{\mathrm{const}}, 0 \to B \to B)$. For H_1, the kernel $h_1(\theta)^*(P)$ is 0. The pushout

$$\begin{tikzcd}
\theta^*(P) \arrow[r] \arrow[d] & A \arrow[d] \\
0 \arrow[r] & A_\tau
\end{tikzcd}$$

and the sequence $0 \to 0 \to A_\tau \to B \to 0$ implies $A_\tau \cong B$. So $H_1(\theta, A, B) = (\mathbf{n}_{\text{const}}, 0 \to B \to B) = \psi(\theta, A, B)$. Thus, $\text{id} \simeq \psi$. Since ψ factors through $s^{-1}(0_*)$ and $f : s^{-1}(0_*) \to s_\bullet\mathcal{E}$ is a weak equivalence.

The original map $s^{-1}(P) \to s_\bullet\mathcal{E}$ is a homotopy equivalence, and thus a weak equivalence. $\square$

Theorem 6.3.2 (Additivity Theorem)**.** *Let $\mathcal{E}$ be an exact category. The simplicial set map*

$$(t_*, s_*) : s_\bullet\text{Ex}(\mathcal{E}) \to s_\bullet\mathcal{E} \times s_\bullet\mathcal{E}$$

is a weak equivalence.

Proof. Consider the square of simplicial sets

$$\begin{tikzcd} s_\bullet\mathcal{E} \arrow[r, "c_*"] \arrow[d, "\text{incl}"'] & s_\bullet\text{Ex}(\mathcal{E}) \arrow[d, "s_*"] \\ * \arrow[r, "0_*"'] & s_\bullet\mathcal{E} \end{tikzcd} \qquad (*)$$

Here $* = \Delta[0], 0_* : * \to s_\bullet\mathcal{E}$ picks out the zero object, and $c : \mathcal{E} \to \text{Ex}(\mathcal{E})$ sends $Q \mapsto (0 \to 0 \to Q \xrightarrow{\text{id}} Q \to 0)$. By Lemma 6.3.1, the map $t_* : s^{-1}(0_*) \to s_\bullet\mathcal{E}$ is a weak equivalence (where $s^{-1}(0_*)$ is the homotopy pullback of the right column and bottom row of $(*)$ over 0_*). This condition implies that the square $(*)$ is homotopy cartesian.

Since square $(*)$ is homotopy cartesian, we can compare the homotopy fiber sequences associated with the horizontal maps. Consider the diagram:

$$\begin{tikzcd} & s_\bullet\text{Ex}(\mathcal{E}) \arrow[dr, "s"] \arrow[dd, "{(t_*,s_*)}"] & \\ s_\bullet\mathcal{E} \arrow[ur, "c"] \arrow[dr, "in_L"'] & & s_\bullet\mathcal{E} \\ & s_\bullet\mathcal{E} \times s_\bullet\mathcal{E} \arrow[ur, "pr_R"'] & \end{tikzcd}$$

where $\text{in}_L(P) = (P, 0_*)$ and pr_R is the projection onto the right factor. The map $s_* \circ c_* = \text{id}_{s_\bullet\mathcal{E}}$, and $pr_R \circ \text{in}_L$ sends P to 0_*. The fact that $(*)$ is a homotopy cartesian square implies that $s_\bullet\mathcal{E}$ (top-left of $(*)$) is weakly equivalent to the homotopy pullback of $s_\bullet\text{Ex}(\mathcal{E}) \xrightarrow{s_*} s_\bullet\mathcal{E} \xleftarrow{0_*} *$. This homotopy pullback is $s^{-1}(0_*)$. The map $s_\bullet\mathcal{E} \xrightarrow{c_*} s_\bullet\text{Ex}(\mathcal{E})$ followed by $t_* : s_\bullet\text{Ex}(\mathcal{E}) \to s_\bullet\mathcal{E}$ gives

$t_*(c_*(P)) = t_*(0 \to 0 \to P \to P \to 0) = 0_*$. The map of fibrations (where the base map is $s_* : s_\bullet \mathrm{Ex}(\mathcal{E}) \to s_\bullet \mathcal{E}$ and $s_\bullet \mathcal{E} \times s_\bullet \mathcal{E} \xrightarrow{pr_R} s_\bullet \mathcal{E}$) induced by (t_*, s_*) must be a weak equivalence on total spaces because it is a weak equivalence on the base (s_* vs pr_R composed with s_*) and on the fibers. The fiber of $s_* : s_\bullet \mathrm{Ex}(\mathcal{E}) \to s_\bullet \mathcal{E}$ over $P'' \in s_\bullet \mathcal{E}$ is $s^{-1}(P'')$. The fiber of $pr_R : s_\bullet \mathcal{E} \times s_\bullet \mathcal{E} \to s_\bullet \mathcal{E}$ over $P'' \in s_\bullet \mathcal{E}$ is $s_\bullet \mathcal{E} \times \{P''\}$. The map $t_* : s^{-1}(P'') \to s_\bullet \mathcal{E}$ (which maps to the first component of $s_\bullet \mathcal{E} \times \{P''\}$) is a weak equivalence by Lemma 6.3.1. Therefore, the map (t_*, s_*) is a weak equivalence. $\square$

6.4 Spectra and Fundamental theorems of K theory

We list here a few important theorems in K-theory which are direct consequences of the additivity theorem and Quillen's Theorem A and B. We omit proofs for succinctness. the

Definition 6.4.1 (Symmetric Spectrum)**.** *A symmetric spectrum consists of the following data for $n \geq 0$.*

1. *A sequence of pointed simplicial sets X_n.*
2. *A continuous left action from the symmetric group $\Sigma_n \to X_n$ preserving the base point.*
3. *A sequence of based maps $\sigma_n : X \wedge S^1 \to X_{n+1}$.*

These components satisfy a coherence condition. Namely that for all $n, m \geq 0$ the composite map

$$X_n \wedge S^k \xrightarrow{\sigma_n \wedge id} X_{n+1} \wedge S^{k-1} \xrightarrow{\sigma_{n+1} \wedge id} \cdots \xrightarrow{\sigma_{n+k-1}} X_{n+k}$$

intertwines the action induced by $\Sigma_n \times \Sigma_k$.

Here $\wedge$ denotes the standard smash product of pointed simplicial sets (smash product of levelwise pointed sets) $X \wedge Y := \frac{X \times Y}{X \vee Y}$.

Definition 6.4.2 (K-theory Spectrum of an Exact Category $\mathcal{E}$)**.** *Let $\mathcal{E}$ be an exact category, pointed by a choice of zero object* 0*.*

1. *For $k \geq 0$, let $S^k_\bullet(\mathcal{E})$ be the k-fold simplicial exact category whose objects are functors $P : \mathrm{Ar}(\mathbf{n}_1) \times \cdots \times \mathrm{Ar}(\mathbf{n}_k) \to \mathcal{E}$ which are exact in each variable. Let $s^k_\bullet(\mathcal{E})$ denote the k-fold simplicial set of objects of $S^k_\bullet(\mathcal{E})$.*

2. *Let $s^k_\bullet(0)$ be the subcomplex of $s^k_\bullet(\mathcal{E})$ consisting of all such functors P that take values only in zero objects of $\mathcal{E}$. This is a contractible subcomplex.*

3. *The k-th space of the K-theory spectrum $\mathcal{K}(\mathcal{E})$ is defined as the pointed simplicial set:*
$$\mathcal{K}(\mathcal{E})^k = d\left(s^k_\bullet(\mathcal{E})/s^k_\bullet(0)\right)$$
where d is the multisimplicial diagonal functor. This functor takes a k-fold simplicial set X to a simplicial set $d(X)$ where $d(X)_n = X_{n,\ldots,n}$.

4. *There are canonical pointed simplicial set maps (referred to as σ maps in the context from which this is drawn):*
$$\sigma_k : \left(s^k_\bullet(\mathcal{E})/s^k_\bullet(0)\right) \wedge S^1 \to s^{k+1}_\bullet(\mathcal{E})/s^{k+1}_\bullet(0)$$
for each $k \geq 0$. The bonding maps for the spectrum $\mathcal{K}(\mathcal{E})$, denoted $\sigma^k_{\mathcal{E}} : S^1 \wedge \mathcal{K}(\mathcal{E})^k \to \mathcal{K}(\mathcal{E})^{k+1}$, are induced by these maps σ_k after applying the diagonal functor d.

The collection of spaces $\{\mathcal{K}(\mathcal{E})^k\}_{k\geq 0}$ together with the bonding maps $\{\sigma^k_{\mathcal{E}}\}$ defines the K-theory spectrum $\mathcal{K}(\mathcal{E})$ of the exact category $\mathcal{E}$. This construction yields a symmetric spectrum, and this structure is natural with respect to pointed exact categories $\mathcal{E}$.

Definition 6.4.3 (Stable Equivalence of K-theory Spectra)**.** *A map $\phi_* : K(\mathcal{E}_1) \to K(\mathcal{E}_2)$ of symmetric spectra, typically induced by an exact functor $f : \mathcal{E}_1 \to \mathcal{E}_2$, is a stable equivalence if it induces isomorphisms on all stable homotopy groups:*
$$\pi_n(\phi_*) : \pi_n(K(\mathcal{E}_1)) \xrightarrow{\cong} \pi_n(K(\mathcal{E}_2)) \quad \textit{for all } n \in \mathbb{Z}.$$
The stable homotopy groups $\pi_n(K(\mathcal{E}))$ are defined as $\mathrm{colim}_k \pi_{n+k}(K(\mathcal{E})^k)$. A map of symmetric spectra is a stable equivalence if it is an isomorphism in the stable homotopy category. For K-theory spectra, $\pi_n K(\mathcal{E})$ often corresponds to the classical K_n-groups of $\mathcal{E}$ for $n \geq 0$ and are zero for $n < 0$.

Theorem 6.4.4 (Resolution Theorem)**.** *Suppose that $\mathcal{P}$ is full and closed under extensions in the exact category $\mathcal{E}$, and that $\mathcal{P}$ and $\mathcal{E}$ satisfy the following conditions,*

1. *All admissible epis $P \twoheadrightarrow P'$ between objects of $\mathcal{P}$ in $\mathcal{E}$ are admissible epis of $\mathcal{P}$.*

2. *Given any admissible epi $f : Q \twoheadrightarrow P$ with $P \in \mathcal{P}$, there is a commutative diagram as such,*

$$\begin{tikzcd} P' \arrow[d] \arrow[dr, two heads] & \\ Q \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

Then the inclusions

$$\mathcal{P} \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_\infty$$

induce stable equivalences

$$K(\mathcal{P}) \simeq K(\mathcal{P}_1) \simeq K(\mathcal{P}_2) \simeq \cdots \simeq K(\mathcal{P}_\infty).$$

Theorem 6.4.5 (Dévissage Theorem)**.** *Suppose that $\mathcal{B}$ is a non-empty subcategory of a small abelian category $\mathcal{A}$ which is closed under taking finite direct sums, subobjects and quotients in $\mathcal{A}$. Suppose that every object Q of $\mathcal{A}$ has a finite filtration*

$$0 = F_{-1} \rightarrowtail F_0 \rightarrowtail F_1 \rightarrowtail \cdots \rightarrowtail F_n = Q$$

with all filtration quotients $F_i/F_{i-1} \in \mathcal{B}$. Then the inclusion $i : \mathcal{B} \to \mathcal{A}$ induces a stable equivalence $K(\mathcal{B}) \simeq K(\mathcal{A})$.

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