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Sketches on homology

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The material is presented mostly module theoretically with appropriate categorical abstractions when appropriate (and pedagogically relevant) to better elucidate the concepts in play. We give a proof sketch of the Freyd-Mitchell embedding at the end and demonstrate usage of abelian categories. The overall structure of the presented material is influenced by the appendix on homological algebra in Eisenbud's excellent book [Eisenbud, 2013].

1 Modules and tensor products

Definition 1.1 (Module). *Recall that for a commutative ring with unity R a **module** over R is a set M along with operations of addition $+: M \times M \rightarrow M$ and scalar multiplication $R \times M \rightarrow M$ such that*

- $(M, +)$ is an abelian group,
- Scalar multiplication distributes over addition,
- Scalar multiplication respects unity and associativity.

Definition 1.2 (Bilinear maps between modules). *It is a map in two variable map which is linear over its arguments individually (when the other is held fixed).*

Definition 1.3 (Tensor product). *The **tensor product** for R -modules M, N denoted as $M \otimes_R N$ ¹ is an R module along with a universal A -bilinear map $\varphi: M \times N \rightarrow M \otimes_R N$. That is to say all R -bilinear maps factor through*

¹When the choice of ring is obvious we will drop the subscript.

the tensor product, this is elucidated by saying the following diagram commutes for all R -modules P and f some bilinear map from $M \times N$ into P

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes N \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & P \end{array}$$

In order to appreciate this universality first lets see the explicit construction of the tensor product.

1.1 Explicit construction of tensor products

The universality is the key point via which we aim to define a explicit construction. With it in mind the typical construction goes as such, consider the free module generated by the basis set $M \times N$.

Quotienting out the ideal which is generated by the bilinearity conditions we require will give us the explicit module we require. The required bilinearity conditions are,

- $(m + m', n) - (m, n) - (m', n')$ and $(m, n + n') - (m, n) - (m, n')$
- $(xm, n) - x(m, n)$ and $(m, xn) - x(m, n)$

For all $m, m' \in M, n, n' \in N, x \in R$. It should be clear from the definitions that identifying these points together makes the quotient bilinear as required and it is the smallest such module as there we have added no extraneous conditions. This is easy to verify any maps $f : M \times N \rightarrow P$ as above will extend naturally to $F : R^{M \times N} \rightarrow P$ and is factoring through the tensor product mapping say $\varphi : M \times N \rightarrow M \otimes N$

COMPLETE THIS PROOF CITE THIS SOMEWHERE

1.2 Properties of tensor products

Let R be some commutative ring with unity and let M, N, P be R -modules.

Proposition 1.1 (Commutativity). $M \otimes N = N \otimes M$

Proof. Note that the map $(m, n) \rightarrow (m, n)$ induces an isomorphism between the free groups generated by the products. The conditions for bilinearity aren't affected so the quotients are the same. \square

Proposition 1.2 (Identity). $R \otimes M = M$

Proof. Consider the canonical bilinear map $f : R \times M \rightarrow P$ given by $(r, m) \mapsto rm$. M satisfies the UMP for this. \square

Proposition 1.3 (Associativity). $(M \otimes N) \otimes X = M \otimes (N \otimes X)$

Proof. Direct consequence of the ump simply draw the diagrams first starting with $M \times N \rightarrow P$ and then $N \times X \rightarrow P$ \square

1.3 Tensor-Internal Hom adjunction

As covered in the appendix we have a notion of an internal hom functor as a bifunctor $[-, -] : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$ for some symmetric monoidal category \mathbf{C} . We construct this as an internal hom by considering for all $A \in \mathbf{C}$ adjoint pairs between $(- \otimes A) \dashv [A, -] : \mathbf{C} \rightarrow \mathbf{C}$, if they exist. If all these adjoint pairs exist the category is further called a closed monoidal category.

To see why this is the natural definition and not simply meaningless abstract nonsense it is useful to think about the construction over **Sets**.

Consider the idea of set exponents (which internal homs are a special case of). Given sets A, B , we can consider the set $[A, B]$ of functions from $A \rightarrow B$ often denoted in terms of exponentiation $B^A = \prod_{a \in A} B$. This choice of notation stems from the fact that exponentiation gives the cardinality of the function set. Furthermore this leads to a well defined notion of exponential objects which we won't discuss here.

But if we have any functions from a set C into B^A its naturally bijective to the functions from $C \times A \rightarrow B$ (note that cartesian products are also symmetric monoidal products so this is just a special case). Since we have the condition $(B^A)^C = B^{(A \times C)}$!

The adjoint definition is simply a rephrasing of this.

Proposition 1.4 (Tensor-Internal Hom adjunction). *For a closed monoidal category \mathbf{C} and objects $A, B, C \in \mathbf{C}$. The tensor product is left adjoint to the internal hom functor, in particular*

$$[A \otimes B, C] \cong [A, [B, C]]$$

We state a categorical fact here instead of the appendix due to its huge importance, the proof is as in [Awodey, 2010]

Proposition 1.5 (Right adjoints preserve limits).

Proof. Say there is an adjunction pair between categories \mathbf{C}, \mathbf{D} given as $F \dashv U$, in particular $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$, given some diagram $D : J \rightarrow \mathbf{D}$ where J is some indexable category over which we are defining a diagram to find a limiting cone. Suppose the limit exists say $\lim_{\leftarrow} D_j$, note this is just an object in \mathbf{D} . Then for $A \in \mathbf{C}$ we have the following string of natural isomorphisms,

$$\begin{aligned} \text{Hom}_{\mathbf{C}}(A, U(\lim_{\leftarrow} D_j)) &\cong \text{Hom}_{\mathbf{D}}(F(A), \lim_{\leftarrow} D_j) \text{ ,by definition of adjunction} \\ &\cong \lim_{\leftarrow} \text{Hom}_{\mathbf{D}}(F(A), D_j) \text{ ,as Hom preserves limits} \\ &\cong \lim_{\mathbf{C}}(A, U(D_j)) \text{ ,by adjunction} \\ &\cong \text{Hom}_{\mathbf{C}}(A, \lim_{\leftarrow} U(D_j)) \end{aligned}$$

Now by Yoneda we can say $U(\lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} U(D_j)$ which is the required result!

Note that the dual construction holds too to state that right adjoints preserve colimits. \square

This was proved in the appendix.

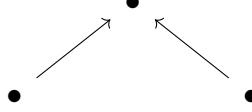
Now we claim that this adjunction between Tensor product and internal homs is all we need to say that Tensor is left adjoint to the typical Hom functor in Mod.

First note that Mod is a symmetric monoidal category with the typical tensor product. Now note that the typical Hom functor takes two modules M, N to the set of module homomorphisms between them say $L(M, N)$ but this is a module itself! So the external Hom can be recovered from the internal hom. This is covered in the appendix in greater detail.

We state these very quickly here Let R be a commutative ring and M, N modules over R .

Proposition 1.6 (Distribution over direct sums). $M \otimes (\oplus_{i \in I} N_i) = \oplus_{i \in I} (M \otimes N_i)$

Proof. Direct sums in modules are by definition a coproduct of modules, whose universality condition gives that is is just a colimit of the following diagram.



Of course this means the direct sum distributes due to left adjoints preserving colimits. \square

The notions of exact sequence will be seen again in the next section, but for completeness sake we state it here.

Definition 1.4 (Exact sequence of modules). *A collection of modules A_i and module homomorphisms $\alpha_i : A_i \rightarrow A_{i+1}$ is called exact if $\text{Im}(\alpha_i) = \text{Ker}(\alpha_{i+1})$*

Definition 1.5 (Short exact sequence of modules). *If a sequence of modules and maps as such is exact, $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. The map between $A \rightarrow B$ is injective and the map between $B \rightarrow C$ is surjective, furthermore $C \cong B/A$ due to the fact that A can be considered an ideal of B due to the exactness conditions.*

Definition 1.6 (Split exact sequence of modules). *If $B \cong A \oplus C$ in the above definition.*

All long exact sequences canonically decompose into short exact sequences. Assume $(A_\bullet, \alpha_\bullet)$ as in the first definition, it can be decomposed into the following collection of short exact sequences,

$$\begin{aligned}
 0 &\rightarrow \text{Im}\alpha_1 \rightarrow A_1 \rightarrow \text{Im}\alpha_2 \rightarrow 0 \\
 0 &\rightarrow \text{Im}\alpha_2 \rightarrow A_2 \rightarrow \text{Im}\alpha_3 \rightarrow 0 \\
 &\vdots
 \end{aligned}$$

Definition 1.7 (Covariant exact functor). *A covariant exact functor is a functor that preserves short exact sequences. If it preserves exactness of $0 \rightarrow A \rightarrow B \rightarrow C$ its called left exact and if it preserves exactness of $A \rightarrow B \rightarrow C \rightarrow 0$ it is called right exact. Simply switch left and right for contravariant case.*

Proposition 1.7 (Right exactness of tensor product).

Proof. This is simply a consequence of left adjoints preserving colimits. \square

Corollary 1.0.1. $R/I \otimes M = M/IM$ where I an ideal of R

Proof. Consider the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ as tensor product is right exact apply $-\otimes M$ we get $I \otimes M \rightarrow R \otimes M \rightarrow (R/I) \otimes M \rightarrow 0$ but we know from above that $R \otimes M = M$ so we get $I \otimes M \rightarrow M \rightarrow (R/I) \otimes M \rightarrow 0$.

But by exactness we get $M/IM \cong (R/I) \otimes M/IM$ \square

Corollary 1.0.2. $R/I \otimes R/J = R/(I + J)$ for I, J ideals of R .

2 Chain complexes

A **chain complex** $(A_\bullet, \varphi_\bullet)$ is a collection of modules over a commutative ring and homomorphisms $\varphi_i : A_i \rightarrow A_{i-1}$ such that $\varphi_i \varphi_{i+1} = 0$,

$$\cdots \xrightarrow{\varphi_{i+2}} A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \xrightarrow{\varphi_{i-1}} \cdots$$

The **homology** of the complex at A_i is denoted as its i^{th} homology defined as follows,

$$H_i := \ker \varphi_i / \text{im} \varphi_{i+1}$$

Reversing the arrows gives us the analogous definitions for cochain complexes and cohomology.

The homomorphisms are often called ‘**boundary operators**’ or ‘**differentials**’. This nomenclature is motivated by the de Rham cohomology. Furthermore elements of $\ker \varphi_i$ are called ‘**cycles**’ and elements of $\text{im} \varphi_{i+1}$ are called **boundaries**, this echoes the aphorism ‘cycles modulo boundaries’ often encountered in singular homology.

A chain complex is said to be **exact** if all its homologies are zero. In particular it is exact at one object if its homology there is zero.

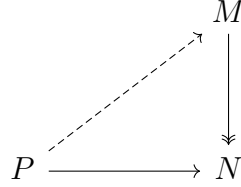
Note that these definitions can be easily viewed in terms of objects of Abelian categories, in particular for such a given enumerable chain it is called a \mathbb{Z} -graded chain complex. (WRITE THIS PROPERLY)

3 Free and Projective modules

Recall a **free module** of rank n is one that is isomorphic to n direct sums of its underlying ring. And homomorphisms from free modules to other modules

are determined by the image of their generators, i.e. free objects are left adjoints to forgetful functors.²

A module P is said to be **projective** if it satisfies the following lifting property, i.e. every morphism from P to N factors through an epi into N , note that the lift need not be unique this is *not* an UMP



Lemma 3.1 (Free modules are projective).

Proof. Consider the preimages of images of basis of P in N , that lie in M . Then map basis elements from P into these preimages. \square

Proposition 3.1 (Equivalent definitions of projectivity). *TFAE*,

1. P is projective.
2. For all epi's between $M \twoheadrightarrow N$, the induced map $\text{Hom}(P, g) : \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$ sending $f \mapsto g \circ f$ for $g : M \rightarrow N$ and $f : P \rightarrow M$ is an epi.
3. For some epi from a free module F to P , $\text{Hom}(P, F) \rightarrow \text{Hom}(P, P)$ is an epi.
4. There exists Q s.t. $P \oplus Q$ is free
5. Short exact sequences of the form $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ split, i.e. isomorphic to another short exact where middle term is $A \oplus P$ ³

Proof. 1 \iff 2 is restatement of definitions.

2 \implies 3 also just substitution.

²This holds in free monoids $\text{Hom}_{\mathbf{Mon}}(F(X), M) \cong \text{Hom}_{\mathbf{Sets}}(X, U(M))$ where $F(X)$ denotes the free monoid generated by elements from the set X and $U(M)$ is the underlying set of a monoid M . Recall the hom-set definition of adjunctions. Refer to [Awodey, 2010, p. 208]

³In general any epis into projective objects split (i.e. have an inverse).

3 \implies 4 consider a map in the preimage of identity in $\text{Hom}(P, P)$ which is a splitting (inverse) of the epi F into P ,

$$\begin{array}{ccc}
 & P & \\
 f \swarrow & & \searrow \text{Id}_P = g \circ f \\
 F & \xrightarrow{\quad g \quad} & P
 \end{array}$$

Now we have a short exact sequence $0 \rightarrow \ker g \rightarrow F \rightarrow P \rightarrow 0$, and also $f \circ g$ is idempotent so it naturally admits a decomposition $F = \text{Im}(f \circ g) \oplus \text{Ker}(f \circ g)$ ⁴ $= \text{Im}(g) \oplus \text{Ker}(g)$ the first by the 1st isomorphism theorem and the second by f being a mono.

4 \implies 2 simply as $\text{hom}(P \oplus Q, -) = \text{hom}(P, -) \oplus \text{hom}(Q, -)$

1 \iff 5 should be clear from above.

□

Theorem 3.2 (Proj. fin. generated modules over local rings are free).

Proof. pick a minimal set of generators and see its residue classes in $M/\mathfrak{m}M$ as the basis of it as a vector space over R/\mathfrak{m} .

Now as for some free module F , $F = \varphi(M) \oplus K$ for some K and some homomorphism $\varphi : M \rightarrow F$, (by defn of projective module), we get

$$M/\mathfrak{m}M \cong F/\mathfrak{m}F = (R/\mathfrak{m})^n \cong R^n \otimes R/\mathfrak{m} \cong F \otimes R/\mathfrak{m} \cong (\varphi(M) \oplus K) \otimes R/\mathfrak{m}$$

Finally we get $M/\mathfrak{m}M \cong M/\mathfrak{m}M \oplus K/\mathfrak{m}K \implies K = \mathfrak{m}K \implies K = 0$ by Nakayama □

This holds for not necessarily finitely generated modules too refer to [Matsumura, 1987, Th. 2.5] .

Proposition 3.2. *If M is a finitely presented module over a Noetherian ring R (prime ideals fin gen) then TFAE*

1. M is projective.
2. M localized at maximal ideals is free.

⁴For some idempotent e , $1 - e$ is also an idempotent and images under these two mappings decompose any module, furthermore image of $1 - e$ is just kernel of e

3. A finite set of elements $\{x_i\}^n$ in R generate R such that $M[x_i^{-1}]$ is free over $R[x_i^{-1}]$.

This proceeds just from the previous result.

Theorem 3.3 (Quillen–Suslin). *Every finitely generated projective module over a polynomial algebra is free.*

This was an open problem for a long time as such the proof is very involved. Refer to [nLab authors, 2023c] or to a condensed proof in [Lang, 2002, p. 848]

4 Resolutions

Given a module M its **left resolution** is given by the data of a exact sequence $(A_\bullet, \varphi_\bullet)$ into M as such,

$$\cdots \rightarrow A_1 \rightarrow A_0 \xrightarrow{\alpha} M \rightarrow 0$$

where α is called the **augmentation map**, if the exact sequence is free its a free resolution and such for projective. This will come up again in injective resolutions.

WRITE EXAMPLE OF KOSZKUL COMPLEX

5 Injective modules

It is the dual notion to projective modules. For a commutative ring R a R –module Q is said to be injective if for every mono of R –modules $M \hookrightarrow N$ and every homomorphism $M \rightarrow Q$ lifts through the mono, as seen below,

$$\begin{array}{ccc} M & \hookrightarrow & N \\ \downarrow & & \swarrow \exists \\ Q & & \end{array}$$

Lemma 5.1. *Product of injective modules is injective*

Proposition 5.1 (Baer's criterion: Ideal inclusions are enough). *If for every ideal $I \subseteq R$, homomorphisms $I \rightarrow Q$ lift through R then Q is injective, i.e.*

$$\begin{array}{ccc} I & \xrightarrow{\subseteq} & R \\ \downarrow & \nearrow \text{dashed} & \\ Q & & \end{array}$$

Proof. First consider two R -modules M, N which have a mono between them $M \hookrightarrow N$ and say $f : M \rightarrow Q$ is some homomorphism. We have to find some lift $h : N \rightarrow Q$.

Now consider a submodule M' lying between M, N and consider canonical extensions of f (i.e. restriction on N is just f).

$$\begin{array}{ccccc} M & \longrightarrow & M' & \longrightarrow & N \\ \downarrow f & & \nearrow f' & & \\ Q & & & & \end{array}$$

Arrange all possible submodules along with their morphisms into Q in a poset, sorted via inclusion. Now assuming choice, by Zorn's lemma we have a maximal element say M' along with morphism $f' : M' \rightarrow Q$ now if $M' = N$ we are done.

The proof proceeds via contradiction. Assume $M' \neq N$ now consider $x \in N - M'$ and consider the module $\langle x \rangle + M'$. If we show there is a mapping from $\langle x \rangle + M' \rightarrow Q$ we contradict the choice of maximal element by Zorn's and we are done.

Consider the ideal $I = \{r \in R \mid rx \in M'\}$ of R there is a morphism into Q given by $g(r) = f'(rx)$ now by assumption this lifts over R to some $k : R \rightarrow Q$. So for some arbitrary $z \in \langle x \rangle + M'$ which takes the form $rx + m$ we have the required extension of $\langle x \rangle + M' \rightarrow Q$ as $f''(rx + m) = k(r) + f'(m)$ this contradicts maximality of the pair of M' and f' and so we are done.

The assumption was wrong and $N = M'$.

□

Baer's criterion has a few useful applications which we will now see. Note as Baer's relies on choice so do its corollaries.

Corollary 5.1.1. *Abelian groups Q regarded as \mathbb{Z} -modules are injective iff each element can be written as a product of nonzero $n \in \mathbb{Z}$ and some $q \in Q$ (i.e. Q is divisible)*

Proof. (\implies) If Q is injective fix some arbitrary $q \in Q$. Then consider the map $f : \mathbb{Z} \rightarrow Q, f(1) = q$ and consider $g : \mathbb{Z} \rightarrow n\mathbb{Z}$ for some $n \neq 0$, now via injectivity there's a map from $h : n\mathbb{Z} \rightarrow Q$ such that $hg = f$ so $hg(1) = h(n) = nh(1) = f(1) = q$ so we got $nh(1) = q$ as required.

(\impliedby) Say $\forall q \in Q, q = nq', n \in \mathbb{Z} - \{0\}$. Consider $n\mathbb{Z} \subseteq \mathbb{Z}$ an ideal. And consider the canonical map $n\mathbb{Z} \rightarrow Q$ taking $n \rightarrow q$, but $q = nq'$ so the map $\mathbb{Z} \rightarrow Q$ taking $1 \rightarrow q'$ is the extension we require and by Baer's criterion we are done. \square

The above result also holds to show that any subgroup of injective abelian group is abelian as if the abelian group is injective (i.e. divisible) so are its quotients, as such we have the particular case that \mathbb{Q}/\mathbb{Z} is divisible. This forms an 'almost' dual notion to Neilsen-Schierer.

Proposition 5.2 (R -Mod has enough injectives!). *Every module has a mono into an injective module*

Proof. The proposition is proved in two parts. First we prove it for abelian groups (i.e. \mathbb{Z} -modules).

For the first part note that \mathbb{Q} and there is a canonical embedding from $\mathbb{Z} \rightarrow \mathbb{Q}$. Now every abelian group A can be written as a quotient $F(A)/\ker f$ where $f : A \rightarrow F(A)$ sends A to its free group $F(A)$, which itself is a direct sum of copies of \mathbb{Z} so it embeds into direct sums of copies of \mathbb{Q} which is divisible too.

For the second part let $M \in R\text{-Mod}$ now consider the homomorphisms $M \xrightarrow{\cong} \text{Hom}_R(R, M) \rightarrow \text{Hom}_{\mathbb{Z}}(R, M) \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ where D is some divisible group as the above statement, and we consider M as an abelian group forgetting structure in the last arrow.

We know D is injective, now claim $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective R -module. Let $M \in R\text{-Mod}$. Now $\text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, D)) \cong \text{Hom}_{\mathbb{Z}}(M, D)$

(Prove this categorically via free-forgetful adjunction)

\square

Alternatively consider the following alternative proof for the second part

Proposition 5.3. *Given Ab has enough injectives then so does $R\text{-Mod}$.*

Proof. The proof proceeds categorically and relies on the notion of the free-forgetful functor adjunction. In particular in this case we have the adjunction as

$$(F \dashv U), F : \mathbf{Ab} \rightleftarrows R\text{-Mod} : U$$

Where U denotes the forgetful functor and F denotes the ‘free’ functor sending $A \in \mathbf{Ab}$ to $A \otimes_{\mathbb{Z}} R$.

Now from Prop. and Prop. we are done if we show forgetful functor is additive and exact. \square

Proposition 5.4 (Bass-Papp). *A ring is Noetherian iff every direct sum of injective modules over it are injective*

Proof. Let R be a Noetherian ring. First note that due to right adjointness of the Hom functor we have $\text{Hom}_R(M, \oplus_i N_i) \cong \oplus_i \text{Hom}_R(M, N_i)$ for M, N R -modules. \square

This result is also true for arbitrary direct limits as shown in [Lam, 2012, Th. 3.46].

Definition 5.1 (Indecomposable module). *An injective module E is indecomposable if (cannot be written as a direct sum of two nonzero modules).*

Proposition 5.5. *If R is Noetherian, and E is some indecomposable injective R -module we have,*

1. $E \cong E(R/P)$ for some prime ideal $P \subset R$.
2. $E(R/P) \cong E(R/Q) \iff P = Q$ so there is a 1-1 correspondence between indecomposable injectives and primes.

Proof. Use primary decomposition. \square

5.1 Injective resolutions

Definition 5.2 (Injective resolution). *We can build an injective resolution inductively, starting from $M \in R\text{-Mod}$ we may imbed it into an injective module say Q_0 , $f : M \hookrightarrow Q_0$ now $\text{coker } f = Q_0/f(M)$ into an injective module Q_1 and so on, giving us an **injective resolution**.*

$$0 \rightarrow M \rightarrow Q_0 \rightarrow Q_1 \rightarrow \dots$$

The prototypical example is that of

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

The key point is that any module over arbitrary rings has a unique minimal injective resolution. This theory is developed with the notions of **injective envelopes**.

5.2 Injective envelopes

Definition 5.3 (Essential submodules/extensions). *For a ring R and $M, E \in R\text{-Mod}$ such that $M \subset N$ we say M is an **essential submodule** of E or E is an **essential extension** of M if every non-zero submodule of E has nonzero intersection with M , i.e. $M \cap L = 0 \implies L = 0$*

The following proposition provides the definition of an **injective envelope**

Proposition 5.6. *For a ring R*

1. *For $M, F \in R\text{-Mod}$ and $M \subset F$ there is a maximal submodule E of F containing M such that $M \subset E$ is essential*
2. *If F is injective so is E*
3. *There is, up to isomorphism, a unique essential extension E of M that is an injective R -module, this E is called the **injective envelope** of M written as $E(M)$.*

Proof. We proof the propositions in order of statement,

1. Consider preorder of essential submodules E_i of F . Any submodule N of $\cup_i E_i$ meets some E_i nontrivially, and thus meets M nontrivially. So M is essential in $\cup_i E_i$. It follows via Zorn's that there is a maximal essential extension of M contained in F .
2. Suppose F is injective and $M \subset E \subset F$ with E being the maximal essential extension of M . If E' was an essential extension of E in F then it would be one of M and so implies $E' = E$.

Let N be a submodule of F maximal along those not meeting E . Since they dont meet $E \oplus N \cong E + N \subset F$. To show that $F = E + N$ from which it follows $E \oplus N \cong F$ so E is injective.

Consider $\alpha : E \subset F \rightarrow F/N$ an epi. As N does not meet E , this epi is also a inclusion.

(Now if a submodule of F/N failed to meet E then its preimage in F would be submodule larger than N not meeting E which is contradiction.)

Since F is injective we may find a map $\beta : F/N \rightarrow F$ extending α . Since $\ker \beta \cap E = \ker \alpha = 0$ and E essential in F/N we see even $\ker \beta = 0$. So $\beta(F/N)$ essential extension of E by maximality it means its equal to E so $F/N = E$. So $E + N = F$. And direct summand of injective is injective.

3. By the notion of enough injectives we can say there is a mono from M to E by previous construction. To show uniqueness suppose there are two such essential maximal inclusions. So there are monos and an extension (due to essentiality of E_2) as below,

$$\begin{array}{ccc} & & E_1 \\ & \nearrow f_1 & \downarrow g \\ M & \xrightarrow{f_2} & E_2 \end{array}$$

We get $\ker g = 0$ as $\ker f_1 = 0$ and chasing M around kernel of g restricted to M is also 0. So we get $g(E_1)$ injective submodule of E_2 and its a direct summand say $E_2 = g(E_1) \oplus K$ but also $f_2(M) \subset g(E_1)$ so by essentiality of M in E_2 again $K = 0$ so they are the same.

□

To recap,

Definition 5.4 (Injective envelope/hull). *For R -modules $M \subset F$ the unique essential extension of M which is injective and as an intermediate submodule is called the injective envelope of M .*

Proposition 5.7. *If $N \subset M$ is an essential submodule then any map $M \rightarrow E$ that restricts to a monomorphism on N is a monomorphism.*

Proof. We have that for $L \subset M$ an submodule $N \cap L = 0 \implies L = 0$. □

6 Flat modules

7 Morphisms and homotopies of complexes

Recall the definition of a chain complex. A **chain complex** $(A_\bullet, \varphi_\bullet)$ is a collection of R -Modules and homomorphisms $\varphi_i : A_i \rightarrow A_{i-1}$ such that $\varphi_i \varphi_{i+1} = 0$,

$$\cdots \xrightarrow{\varphi_{i+2}} A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \xrightarrow{\varphi_{i-1}} \cdots$$

It is useful to denote the entire chain complex as A and think of it as a graded R -Module⁵.

7.1 Diagram chasing lemmas

8 Derived functors

8.1 Tor functor

8.2 Ext functor

8.3 Local cohomology

OVER HERE DO THE NEXT APPENDIX TOO

⁵Where R is just trivially graded.

9 Mapping cones and double complex

10 Spectral sequences

11 Spectral sequence of double complex

12 Derived categories

13 Freyd-Mitchell Embedding

Theorem 13.1 (Freyd-Mitchell Embedding). *If A is a small abelian category there is an exact full and faithful embedding functor from A into the category of modules over some ring R which embeds A as a full subcategory.*

Proof. The inclusion from A to its Ind-completion is full faithful, exact, compact, preserves generators and projectives.

Locally finite presentable categories preserve injective cogenerators.

Cocomplete abelian category with compact projective generator is equivalent to $\text{End}(P)$ modules \square

A Categories

A.1 Preliminary notions

Definition A.1 (Category). *A **category** consists of the following,*

- *Objects: A, B, C, \dots*
- *Arrows/Morphisms: f, g, h, \dots*
- *For each f there exists, $\text{dom}(f), \text{cod}(f)$ called domain and codomain of f . We write $f : A \rightarrow B$ to indicate $A = \text{dom}(f)$ and $B = \text{cod}(f)$.*
- *Given $f : A \rightarrow B$ and $g : B \rightarrow C$ there exists, $g \circ f : A \rightarrow C$ called the composite of f and g .*
- *For each A , there exists $1_A : A \rightarrow A$ called the identity arrow of A .*

- Arrows should also satisfy the following,
 - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$.
 - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \rightarrow B$.

Definition A.2 (Functor). For categories \mathbf{C}, \mathbf{D} we define a **functor** $F : \mathbf{C} \rightarrow \mathbf{D}$ to be a mapping of objects and arrows to objects and arrows, such that

- $F(f : A \rightarrow B) = F(f) : F(A) \rightarrow F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Definition A.3 (Generalized elements). For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \rightarrow A$ are called the **generalized elements** of A with stage of definition given by X .

Definition A.4 (Bifunctor). A **bifunctor** is any functor of two variables i.e. domain in terms of a product category.

Lemma A.1 (Bifunctor lemma). A mapping $F : \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$ is a bifunctor if its functorial in each component and the following square commutes,

$$\begin{array}{ccc}
 A' & B' & \\
 \uparrow & \uparrow & \\
 f & g & \\
 \downarrow & \downarrow & \\
 A & B &
 \end{array}
 \quad
 \begin{array}{ccc}
 F(A, B) & \xrightarrow{F(A, g)} & F(A, B') \\
 \downarrow F(f, B) & & \downarrow F(f, B') \\
 F(A', B) & \xrightarrow{F(A', g)} & F(A', B')
 \end{array}$$

Definition A.5 (Natural transformations). A **natural transformation** is a map between functors.

For functors $F, G : \mathbf{C} \rightarrow \mathbf{D}$ a natural transformation $\eta : F \rightarrow G$ is a family of morphisms (in \mathbf{D}) which consist of **components** η_X which associates for every object $C \in \mathbf{C}$ a morphism between objects in \mathbf{D} , $\eta_C : F(C) \rightarrow G(C)$.

Also components must commute naturally, in particular for $f : C \rightarrow C'$ we have $\eta_{C'} \circ F(f) = G(f) \circ \eta_C$, i.e. below diagram commutes,

$$\begin{array}{ccccc}
 C & & F(C) & \xrightarrow{\eta_C} & G(C) \\
 \downarrow f & & \downarrow F(f) & & \downarrow G(f) \\
 C' & & F(C') & \xrightarrow{\eta_{C'}} & G(C')
 \end{array}$$

Definition A.6 (Yoneda embedding). The **Yoneda embedding** is a functor $y : \mathbf{C} \rightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$ mapping objects to their contravariant representable functor (i.e. presheaves). In particular $y(C) = \text{Hom}_{\mathbf{C}}(-, C)$ it takes arrows to the natural transformation $yf = \text{Hom}_{\mathbf{C}}(-, f) : \text{Hom}_{\mathbf{C}}(-, C) \rightarrow \text{Hom}_{\mathbf{C}}(-, D)$

Proposition A.1 (Yoneda Lemma). For a locally small category (i.e. small hom-sets) \mathbf{C} we have $\text{Hom}(yC, F) \cong FC$ for $C \in \mathbf{C}$ and a functor $F \in \mathbf{Sets}^{\mathbf{C}^{op}}$. The isomorphism is natural in both C and F .

For locally small categories

- The Yoneda embedding $y : \mathbf{C} \rightarrow \mathbf{Sets}^{\mathbf{C}^{op}}$ is full and faithful.
- $yC \cong yC' \implies C \cong C'$ for objects C, C'
- All objects in presheaves are colimits of some representable functors (in particular the end).
- So Yoneda embedding is the free cocompletion of any category. In particular there is a UMP for maps from \mathbf{C} to any cocomplete categories factoring through presheaves wrt Yoneda embedding

An adjoint categorifies the idea of a very weak notion of equivalence sometimes even called a pseudo inverse.

Definition A.7 (Classical definition of adjoint). The functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ form an **adjunction** between categories if there exists a natural transformation $\eta : 1_{\mathbf{C}} \rightarrow U \circ F$ with this unit having the following UMP,

$$\begin{array}{ccc}
 D & & U(F(C)) \xrightarrow{U(g)} U(D) \\
 \uparrow \text{---} & & \uparrow \eta_C \quad \nearrow f \\
 F(C) & & C
 \end{array}$$

F is called the left adjoint of U and vice versa, and is denoted as $F \dashv U$.

Definition A.8 (Hom-set definition of adjoint). *Alternatively we also have the following formulation, $F \dashv U$ if for $C \in \mathbf{C}, D \in \mathbf{D}$ there exist natural isomorphisms $\phi : \text{Hom}_{\mathbf{D}}(FC, D) \cong \text{Hom}_{\mathbf{C}}(C, UD) : \psi$*

Definition A.9 (Unit-counit definition). *For $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ and natural transformations $\eta : 1_{\mathbf{C}} \rightarrow U \circ F$ and $\epsilon : F \circ U \rightarrow 1_{\mathbf{D}}$ namely unit and counit.*

We say $F \dashv U$ iff the following triangle identities hold,

$$\begin{array}{ccc}
 UD & \xrightarrow{1_{UD}} & UD \\
 \searrow \eta_{UD} & & \nearrow U_{\epsilon_D} \\
 & UFUD &
 \end{array}$$

$$\begin{array}{ccc}
 FC & \xrightarrow{1_{FC}} & FC \\
 \searrow F_{\eta_C} & & \nearrow \epsilon_{FC} \\
 & FUF C &
 \end{array}$$

The unit-counit characterization is useful theoretically but we typically only use the Hom-set characterization for all practical purposes.

A.1.1 Examples of adjoints

free-forgetful adjunction. Typically the forgetful functor usually has a left adjoint which is vaguely ‘free’. Obviously neither ‘free’-ness or ‘forgetful’-ness have concrete categorical definitions but differ from case to case. We will use it for proving injectives are enough.

Tensor-Internal Hom adjunction, which we shall see shortly has a special case over topologies where it is called the smash functor hom adjunction. A special case of this (over S^1) is the suspension-loop adjunction which yields an example of the Eckmann-Hilton duality.

For arbitrary diagonal functor over some diagram, its right adjoint is limits and left adjoint is its colimit.

A.2 Monoidal categories

A **monoidal category** is a category \mathbf{C} with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, a ‘unit’ element I , and natural isomorphisms that make the functor associative

and unital with I as expected. It is a generalization of the notion of a ‘tensor product’. Its used to define enriched categories.

This can be formalized as follows, there exists $I \in \mathbf{C}$ and natural isomorphisms,

- $\alpha_{ABC} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$
- $\lambda_A : I \otimes A \rightarrow A$ (read as left)
- $\rho_A : A \otimes I \rightarrow A$ (read as right)

And the following diagrams commute,

$$\begin{array}{ccccc}
 & & (A \otimes B) \otimes (C \otimes D) & & \\
 & \nearrow & & \searrow & \\
 A \otimes (B \otimes (C \otimes D)) & & & & ((A \otimes B) \otimes C) \otimes D \\
 \downarrow & & & & \uparrow \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\hspace{10em}} & & & (A \otimes (B \otimes C)) \otimes D \\
 & & & & \\
 A \otimes (I \otimes B) & \xrightarrow{\hspace{10em}} & (A \otimes I) \otimes B & & \\
 & \searrow & & \swarrow & \\
 & A \otimes B & & &
 \end{array}$$

These coherence conditions were first introduced by MacLane and a textbook presentation for it is there in his book [Lane, 1998, Sec. VII Monoids]. The informal reasoning behind the coherence conditions is that all diagrams made with the given natural isomorphisms commute.

We have a specific notion of a **symmetric monoidal category** wherein there exists another natural isomorphism called **braiding** which just ‘switches’ the order of the tensor product, i.e. $B_{A,B} : A \otimes B \rightarrow B \otimes A$ furthermore $B_{A,B}B_{B,A} = 1_{A \otimes B}$, i.e. the tensor product is commutative!

The coherence conditions for this includes two hexagon conditions which essentially say that braiding is associative across three objects as expected. This is excluded for brevity but can be seen in detail in [nLab authors, 2023a]

A.3 External Hom and internal hom functor

The usual notion of the Hom functor involves simply considering the Hom-sets between two objects in a locally small category as a set. This is sometimes called the **external** Hom functor to differentiate it from the internal hom.

Definition A.10 (Covariant Hom-functor). $\text{Hom}(X, -) : \mathbf{C} \rightarrow \mathbf{Sets}$ sending each object $Y \in \mathbf{C}$ to the set of morphisms between X, Y and mapping morphisms naturally as composition, i.e., $\text{Hom}(X, -)$ sending $f : Y \rightarrow Z$ to $\text{Hom}(X, f) : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$ sending $g \in \text{Hom}(X, Y)$ to $g \circ f$. This has the natural covariant formulation.

Perhaps more importantly, $\text{Hom}(-, -)$ can be considered as a bifunctor in the natural way from $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Sets}$ [nLab authors, 2023b].

The notion of an **internal hom** is essentially a functor that behaves the same as a Hom functor but takes values into the category itself instead of into **Sets**. Internal homs are well defined for categories which are symmetric monoidal category as follows.

Definition A.11 (Internal hom). For a symmetric monoidal category (\mathbf{C}, \otimes) , the internal hom is a functor

$$[X, -] : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$$

characterized by a pair of adjoint functors $X \in \mathbf{C}$

$$- \otimes X \dashv [X, -] : \mathbf{C} \rightarrow \mathbf{C}$$

If these exist it is called a symmetric closed monoidal category

Proposition A.2 (Internal hom bifunctor). For a symmetric monoidal category (\mathbf{C}, \otimes) , the internal hom is a unique bifunctor

$$[-, -] : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$$

corresponds to the previous definition, and also

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}[A, [B, C]]$$

where this is the external hom

Proof. The natural isomorphisms exist due to the homset definition of adjunction keeping the components fixed at B and B, C . Furthermore bifactorality is a consequence of bifunctor lemma A.1. \square

Theorem A.2 (Tensor-Internal Hom adjunction).

$$[A \otimes B, C] \cong [A, [B, C]]$$

Proof. requires full faithfull ness of yoneda complete this later \square

Proposition A.3 (Recovery of external hom from internal hom).

$$\text{Hom}[A, B] \cong \text{Hom}(I, [A, B])$$

A.3.1 Examples of internal homs

In general internal homs and external homs need not correspond, intuitively the internal hom contains a lot more information than just the external hom. And for monoidal unit I maps from $I \rightarrow [A, B]$ correspond to the external homs $A \rightarrow B$.

Heyting algebras correspond to intuitionistic propositional logic. And its internal hom corresponds to its preorder that is to say $p \leq q$ iff $p \implies q$. This is quite interesting as external homs in Heyting algebras are simply sets with at most one element while internal homs are very illuminating. Furthermore extending this analogy to type theory the idea of the tensor internal hom adjunction is just the notion of ‘currying’. **(CITE WHY I FORGOT)** .

Example In the category of chain complexes and I being the monoidal unit as before. Maps from $I \rightarrow A$ for A a **(COMPLETE THIS)**

A.4 Enriched categories

Very often the hom-set of a category instead of just being a set may have additional structure (or we may want it to have additional structure). In order to formalize this notion we replace hom-sets with hom-*objects* which themselves are objects in another category say K . We say its enriched over K . This choice of K is typically taken to be monoidal in order to define composition of hom-sets via the monoidal product ⁶. A category \mathbf{C} is said

⁶This can further be dropped and work over bicategories [Garner and Shulman, 2015]. But this is out of the scope of this article.

to be enriched over some monoidal category K if ordered pairs in \mathbf{C} can be represented as objects in K with composition defined via the monoidal product. The coherence conditions required to define this are excluded for brevity but can be seen in [Riehl, 2014, Chp. 3] which presents it without proof.

The actual definition isn't important here, we only mention it as the main object of study (abelian categories) is a special case of an enrichment over category of abelian groups.

A.5 Abelian Categories

We first define **pre-additive categories** as a category enriched over \mathbf{Ab} with finite biproducts⁷ such that every morphism has a kernel and cokernel.

But what are kernels and cokernels in the categorical sense? A **kernel** is a pullback of a morphism $f : A \rightarrow B$ and the unique morphism from $0 \rightarrow B$. Provided initials and pullbacks exist.

$$\begin{array}{ccc} \ker f & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

The intuition behind this definition is that alternatively it is seen as an equalizer of a function f and the unique morphism between $A \rightarrow 0 \rightarrow B$. The kernel object is the part of the domain that is 'going to zero'.

Now we can define **abelian categories** as pre-additive categories wherein each mono is a kernel and each epic is a cokernel of some morphism and. Largely the purpose of abelian categories were motivated by wanting to generalize homological methods and to unify various (co)homology theories.

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⁷Same as a direct sum, simultaneously a product and coproduct.

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