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Calculus IV

Lecture Notes
for SMAT401

Contents

1	Functions of several variables	1
1.1	Examples of functions of several variables	1
1.2	Non-existence of limit by 2 path test	1
1.3	Existence of limit with ε, δ definition	2
1.4	Continuity	3
1.5	Polar Coordinates	4
1.5.1	Limits in Polar coordinates	5
1.5.2	Epsilon-delta with polar coordinates	5
1.6	Algebra of limits	5
1.7	General multivariate limit	6
1.8	Iterated (Repeated) limits	6
1.9	Limits in 3 variables	9
1.9.1	Two path test for non-existence of limit	9
1.10	Partial derivatives	10
2	Differentiation	12
3	Applications	13

Chapter 1

Functions of several variables

1.1 Examples of functions of several variables

$$\begin{array}{lll} f(x, y) = x + y \log x & f : \mathbb{R}^2 \rightarrow \mathbb{R} & \text{Scalar valued function} \\ f(x, y) = (x^2 y, \cos x, e^x - 9) & f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 & \text{Vector valued function} \end{array}$$

Clearly, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a particular case of scalar valued function.

1.2 Non-existence of limit by 2 path test

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the limit exists if limit value is the same along all possible paths, i.e. left hand and right hand limit are equivalent.

For a multivariate function the 2 path test can be used to show non existence of a limit.

Example 1.1. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2+y^2}$ doesn't exist.

Proof. Consider $x = my^2$ and let $y \rightarrow 0$, then

$$\lim_{y \rightarrow 0} f(my^2, y) = \lim_{y \rightarrow 0} \frac{2my^4}{(m^2 + 1)y^2} = \frac{2m}{1 + m^2}$$

.

Therefore, the limit value varies for different values of m . □

Example 1.2. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x-y}$ doesn't exist.

Proof. Consider first along x axis (i.e. $y = 0$)

$$\lim_{x \rightarrow 0} \frac{x}{x} = 1$$

Consider now along y axis (i.e. $x = 0$)

$$\lim_{y \rightarrow 0} \frac{y}{-y} = -1$$

Since the limit is not path independent we can say the limit does not exist. \square

Example 1.3. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$ doesn't exist.

Proof. Along x and y axis the limits are both zero. Consider instead the path $y = x^2$

$$\lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limit is not path independent it does not exist. \square

Example 1.4. Show that the $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2 - 2x}$ doesn't exist.

Proof. Along x, y axis the limit is 0. Consider the path $y = \sqrt{2x}$

$$\lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1$$

Since the limit is not path independent it does not exist. \square

1.3 Existence of limit with ε, δ definition

Recall the single variable definition of a limit,

Definition 1.5 (Limit of a single valued function). For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $\lim_{x \rightarrow a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ such that $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$

Definition 1.6 (Limit of a multivariate function). For a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\lim_{(x,y) \rightarrow (a,b)} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ such that

$$0 < \|(x, y) - (a, b)\|_2 < \delta \implies |f(x, y) - L| < \varepsilon$$

, alternatively

$$\sqrt{(x - a)^2 + (y - b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Example 1.7. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{1+x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$, consider

$$\begin{aligned} |f(x, y) - L| &= |f(x, y)| = \left| \frac{x-y}{1+x^2+y^2} \right| \\ &= \frac{|x-y|}{1+x^2+y^2} \end{aligned}$$

since $1+x^2+y^2 \geq 1$

$$\begin{aligned} &\leq |x-y| \\ &\leq |x| + |y| \\ &\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\sqrt{x^2+y^2} \end{aligned}$$

Therefore, if $2\sqrt{x^2+y^2} < \varepsilon \implies |f(x, y) - L| < \varepsilon$ so take $\delta = \varepsilon/2$. \square

Example 1.8 (H.W). Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$, consider

$$\begin{aligned} |f(x, y) - L| &= \left| \frac{xy^2}{x^2+y^2} - 0 \right| = \frac{|x|y^2}{x^2+y^2} \\ &= \frac{|x|}{\frac{x^2}{y^2} + 1} \\ &\leq |x| \\ &\leq \sqrt{x^2+y^2} < \varepsilon \implies |f(x, y) - L| < \varepsilon \end{aligned}$$

So we can just pick $\delta = \varepsilon$. \square

1.4 Continuity

Definition 1.9 (Continuity). A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be continuous at a point (a, b) if $\forall \varepsilon > 0, \exists \delta > 0$ such that,

$$0 < \|(x, y) - (a, b)\|_2 < \delta \implies |f(x, y) - f(a, b)| < \varepsilon$$

provided $f(a, b)$ exists. Alternatively,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

Note that, we can show the function is discontinuous if

1. $f(a, b)$ doesn't exist.
2. $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ doesn't exist.
3. Both exist but are not equal to each other.

Example 1.10. Show that the given function is continuous at $(0, 0)$ where,

$$f(x, y) = \begin{cases} xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Proof. Here, $f(0, 0) = 0$. Clearly we have that $|x^2 - y^2| \leq |x^2 + y^2|$.
Let $\varepsilon > 0$,

$$\begin{aligned} |f(x, y) - L| &= \left| xy \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| \\ &= |x||y| \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \\ &\leq |x||y| \\ &\leq \sqrt{x^2 + y^2} \sqrt{x^2 + y^2} = x^2 + y^2 \end{aligned}$$

So when $x^2 + y^2 < \varepsilon \implies |f(x, y) - f(0, 0)| < \varepsilon$ so we take $\delta = \sqrt{\varepsilon}$. \square

Example 1.11. Show that the given function is discontinuous at $(0, 0)$ where,

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Proof. content... \square

1.5 Polar Coordinates

The polar coordinates r (the radial coordinate) and θ (the angular coordinate), are defined in terms of cartesian coordinates as below.

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta \\ r &= \sqrt{x^2 + y^2}, \theta = \arctan \left(\frac{y}{x} \right) \end{aligned}$$

1.5.1 Limits in Polar coordinates

Use polar coordinates when you are over $(0,0)$

Example 1.12. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2+y^2}$ doesn't exist.

Proof. Put $x = r \cos \theta$ and $y = r \sin \theta$

$$f(x, y) = \frac{2xy}{x^2 + y^2} \iff f(r, \theta) = \frac{2r^2 \cos \theta \sin \theta}{r^2} = 2 \cos \theta \sin \theta$$

$$\lim_{r \rightarrow 0} f(r, \theta) = \lim_{r \rightarrow 0} 2 \cos \theta \sin \theta = 2 \cos \theta \sin \theta$$

Which depends on θ . □

1.5.2 Epsilon-delta with polar coordinates

Definition 1.13. $\lim_{r \rightarrow 0} f(r, \theta) = L \iff \forall \varepsilon > 0 \exists \delta > 0$ s.t.

$$0 < |r| < \delta \implies |f(r, \theta) - L| < \varepsilon$$

Example 1.14. Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2+y^2}$

Proof.

$$f(r, \theta) = \frac{r^3 \cos^3 \theta}{r^2} = r(\cos \theta)^3 \tag{1.1}$$

Let $\varepsilon > 0$, consider $|f(r, \theta) - L| = |r| |\cos \theta|^3 \leq |r|$. So we can set $\delta = \varepsilon$ □

Example 1.15. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Proof. The sqrt interior must be positive so take $x^2 + y^2 \leq 9$, so its a circle of radius 3 centred at 0. So the domain is the circle. The range is $\{z \mid 0 \leq z \leq 3\} = [0, 3]$ □

1.6 Algebra of limits

Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}, p \in \mathbb{R}^n$ and $k_1, k_2 \in \mathbb{R}$.

Theorem 1.16. If $\lim_{x \rightarrow p} f(x) = L_1, \lim_{x \rightarrow p} g(x) = L_2$, then

- $\lim_{x \rightarrow p} (k_1 f(x) + k_2 g(x)) = k_1 L_1 + k_2 L_2$
- $\lim_{x \rightarrow p} (f(x)g(x)) = L_1 L_2$
- For non-zero L_2 , $\lim_{x \rightarrow p} (f(x)/g(x)) = L_1/L_2$

1.7 General multivariate limit

Theorem 1.17 (Limit of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$). *For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\lim_{x \rightarrow a} f(x) = L$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$0 < \|x - a\|_n < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.18 (ε - neighbourhood). $B(a, \varepsilon)$ open ball of radius ε around a .

$$0 \leq \|x - a\|_n < \varepsilon$$

Definition 1.19 (Deleted ε neighbourhood). $B(a, \varepsilon) - \{a\}$

Definition 1.20 (Alternate definition of a limit). *For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\lim_{x \rightarrow a} f(x) = L$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$x \in B * (a, \delta) \implies |f(x) - L| < \varepsilon$$

Definition 1.21 (Bounded function). *Let E be a non-empty subset of \mathbb{R}^n . The function $f : E \rightarrow \mathbb{R}$ is said to be bounded in some δ -neighbourhood of point $p \in \mathbb{R}^n$ if there exists $M > 0$ in \mathbb{R} such that*

$$|f(x)| \leq M \forall x \in B(p, \delta)$$

Definition 1.22 (Relation between bounded function and limit of a function in \mathbb{R}^n). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $p \in \mathbb{R}^n$. Let $f(p)$ be defined. If $\lim_{x \rightarrow p} f(x)$ exists then f is bounded in some neighbourhood of point p .*

The converse of 1.22 isn't true.

Theorem 1.23 (Uniqueness of limit in \mathbb{R}^n). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $p \in \mathbb{R}^n$. If $\lim_{x \rightarrow p} f(x)$ exists then it is unique.*

1.8 Iterated (Repeated) limits

Let $(a, b) \in E$ and $f : E \rightarrow \mathbb{R}$ be a function where $E \subseteq \mathbb{R}^2$,

1. Suppose there exists $\delta > 0$ such that $\forall x$ with $0 < |x - a| < \delta$, we have $\lim_{y \rightarrow b} f(x, y)$ exists.
Define a new function $g : \mathbb{R} \rightarrow \mathbb{R}$ as $g(x) = \lim_{y \rightarrow b} f(x, y)$. If $\lim_{x \rightarrow a} g(x)$ exists then this limit is called **iterated limit** which is given by $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y)$.

2. Suppose there exists $\delta > 0$ such that $\forall y$ with $0 < |y - b| < \delta$, we have $\lim_{x \rightarrow a} f(x, y)$ exists

Theorem 1.24. *Existence of double limit does not imply existence of iterated limit.*

Proof. Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as,

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases}$$

We show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ i.e. double limit exists.
Let $\varepsilon > 0$. Consider then

$$\begin{aligned} |f(x, y) - L| &= |x \sin(1/y) - 0| = |x| |\sin(1/y)| \leq |x| \\ &\leq \sqrt{x^2} \\ &\leq \sqrt{x^2 + y^2} \end{aligned}$$

so $\sqrt{x^2 + y^2} < \varepsilon \implies |f(x, y) - L| < \varepsilon$. So choose $\delta = \varepsilon$.
We will now check its iterated limit.

$$\begin{aligned} \lim_x \lim_y f(x, y) &= \lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} x \sin 1/y \right] \\ &= \lim_{x \rightarrow 0} x \left[\lim_{y \rightarrow 0} \sin 1/y \right] \end{aligned}$$

The limit inside doesn't exist.

Claim that $\lim_y \phi(y) = \lim_y \sin 1/y$ doesn't exist, Take $a_n = \frac{1}{(4n+1)\pi/2}$, $b_n = \frac{1}{(4n-1)\pi/2}$. The sequences converge to zero but their sequences $\phi a_n, \phi b_n$ don't converge to the same limit. \square

Example 1.25 (Both iterated limits exist but double limit doesn't exist).
consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined as

$$f(x, y) = \begin{cases} \frac{x^2}{x^2 + y^2 - y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Proof. Begin with two part test to show that the double limit does not exist.
Consider first the path $x = y$ the limit is 0,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{y^2 + y^2 - y} = \lim_{(x,y) \rightarrow (0,0)} \frac{y}{2y - 1} = 0$$

Then consider the path $x = y^2$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{y^4 + y^2 - y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{y^4} = 1$$

So double limit does not exist.

Now consider the iterated limits,

$$\begin{aligned} \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) &= \lim_{x \rightarrow 0} \left[\lim_{y \rightarrow 0} \frac{x^2}{x^2 + y^2 - x} \right] \\ &= \lim_{x \rightarrow 0} \frac{x^2}{x^2 - x} = 0 \end{aligned}$$

Now consider

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0$$

□

Example 1.26 (Both iterated limits exist (not equal) but double limit doesn't exist). Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ define as,

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Proof. First show that the double limit does not exist.

Consider the bath $x = 0$ the limit is equal to 1. Consider the path $x = y$ we will have the limit equal to 0.

Consider now the iterated limits,

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \rightarrow 0} 1 = 1$$

And now the other direction,

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y \rightarrow 0} -1 = -1$$

□

Theorem 1.27. Suppose $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exists and is equal to L . If both iterated limits exist then, the iterated limits are both equal to L .

Proof. Given that $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$. Let $\lim_{y \rightarrow b} f(x,y) = g(x)$. Let $\varepsilon > 0$, then there exists $\delta_1 > 0$ s.t.

$$0 < \|(x,y) - (a,b)\|_2 < \delta_1 \implies |f(x,y) - L| < \frac{\varepsilon}{2}$$

and there exists $\delta_2 > 0$ such that

$$0 < |y - b| < \delta_2 \implies |f(x,y) - g(x)| < \frac{\varepsilon}{2}$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta \leq \delta_1, \delta \leq \delta_2$ which gives

$$0 < |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta \implies |f(x,y) - L| < \frac{\varepsilon}{2}$$

$$0 < |y - b| < \delta \implies |f(x,y) - g(x)| < \varepsilon/2$$

With respect to $0 < |x - a| < \delta$ consider

$$\begin{aligned} |g(x) - L| &= |g(x) + f(x,y) - f(x,y) - L| = |f(x,y) - L - (f(x,y) - g(x))| \\ &\leq |f(x,y) - L| + |f(x,y) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Thus $\forall \varepsilon > 0, \exists \delta > 0$ s.t. **Complete this later...**

Do the same thing for $h(y)$. □

1.9 Limits in 3 variables

1.9.1 Two path test for non-existence of limit

Two path can be used for non-existence of a limit in 3 variables. However a single equation is not enough to define a path in \mathbb{R}^3 two Cartesian equations are required for a path in \mathbb{R}^3 .

Example 1.28. *Show that*

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2}$$

Proof. Take $y = x, z = x$ then

$$\lim_{x \rightarrow 0} \frac{x^2}{3x^2} = \frac{1}{3}$$

Take other path $y = x, z = 0$

$$\lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = 1$$

□

Definition 1.29 (Limit of a function $\mathbb{R}^3 \rightarrow \mathbb{R}$). For a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\lim_{(x,y,z) \rightarrow (a,b,c)} f(x,y,z) = L$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < \|(x,y,z) - (a,b,c)\|_3 < \delta \implies |f(x,y,z) - L| < \varepsilon$$

i.e.

$$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta \implies |f(x,y,z) - L| < \varepsilon$$

Definition 1.30 (Continuity of a function $\mathbb{R}^3 \rightarrow \mathbb{R}$). Replace L with $f(a,b,c)$.

Example 1.31. Show that

$$\lim_{(x,y,z) \rightarrow (1,2,3)} 4x + 2y + z = 11$$

using epsilon delta

Proof. Let $\varepsilon > 0$ consider,

$$\begin{aligned} |f(x,y,z) - L| &= |4x + 2y + z - 11| \\ &= |(4x - 4) + (2y - 4) + (z - 3)| \\ &\leq 4|x - 1| + 2|y - 2| + |z - 3| \\ &\leq 4\sqrt{(x-1)^2} + 2\sqrt{(y-2)^2} + \sqrt{(z-3)^2} \\ &\leq 7\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} \end{aligned}$$

So take $\delta = \varepsilon/7$ □

Example 1.32. Evaluate $\lim_{(x,y) \rightarrow (3,3)} \frac{x^2 + xy - 2y^2}{x^2 - y^2}$

Proof. Factorize $(x - y)$ on numerator and denominator then just plug and chug. □

1.10 Partial derivatives

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. For a function f in n variables x_1, x_2, \dots, x_n we can define the m^{th} partial derivative as,

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_m + h, \dots, x_n) - f(x_1, \dots, x_m, \dots, x_n)}{h}$$

Partial derivatives can be taken with respect to multiple variables and are denoted as follows,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= f_{xx} \\ \frac{\partial^2 f}{\partial x \partial y} &= f_{xy} \\ \frac{\partial^3 f}{\partial x^2 \partial y} &= f_{xxy}\end{aligned}$$

Chapter 2

Differentiation

Chapter 3

Applications