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Calculus IV

Lecture Notes
for SMAT401

Contents

Chapter 1

Functions of several variables

1.1 Examples of functions of several variables

$f(x,y) = x + y \log x$	$f : \mathbb{R}^2 \to \mathbb{R}$	Scalar valued function
$f(x,y) = (x^2 y, \cos x, e^x - 9)$	$f : \mathbb{R}^2 \to \mathbb{R}^3$	Vector valued function

Clearly, $f : \mathbb{R} \to \mathbb{R}$ is a particular case of scalar valued function.

1.2 Non-existence of limit by 2 path test

For a function $f : \mathbb{R} \to \mathbb{R}$ the limit exists if limit value is the same along all possible paths, i.e. left hand and right hand limit are equivalent.

For a multivariate function the 2 path test can be used to show non existence of a limit.

Example 1.1. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$ *doesn't exist.*

Proof. Consider $x = my^2$ and let $y \to 0$, then

$$\lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^2} = \frac{2m}{1+m^2}$$

.

Therefore, the limit value varies for different values of m. □

Example 1.2. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ *doesn't exist.*

Proof. Consider first along x axis (i.e. $y = 0$)

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Consider now along y axis (i.e. $x = 0$)

$$\lim_{y \to 0} \frac{y}{-y} = -1$$

Since the limit is not path independent we can say the limit does not exist. $\square$

Example 1.3. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ *doesn't exist.*

Proof. Along x and y axis the limits are both zero. Consider instead the path $y = x^2$

$$\lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limit is not path independent it does not exist. $\square$

Example 1.4. *Show that the* $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2-2x}$ *doesn't exist.*

Proof. Along x, y axis the limit is 0. Consider the path $y = \sqrt{2x}$

$$\lim_{x \to 0} \frac{x^2}{x^2} = 1$$

Since the limit is not path independent it does not exist. $\square$

1.3 Existence of limit with ε, δ definition

Recall the single variable definition of a limit,

Definition 1.5 (Limit of a single valued function)**.** *For a function* $f : \mathbb{R} \to \mathbb{R}$, $\lim_{x\to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ *such that* $0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon$

Definition 1.6 (Limit of a multivariate function)**.** *For a function* $f : \mathbb{R}^2 \to \mathbb{R}$, $\lim_{(x,y)\to(a,b)} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ *such that*

$$0 < ||(x, y) - (a, b)||_2 < \delta \implies |f(x, y) - L| < \varepsilon$$

, alternatively

$$\sqrt{(x-a)^2 + (x-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Example 1.7. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x-y}{1+x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$, consider

$$\begin{aligned}|f(x,y) - L| = |f(x,y)| &= \left|\frac{x-y}{1+x^2+y^2}\right| \\ &= \frac{|x-y|}{1+x^2+y^2}\end{aligned}$$

since $1 + x^2 + y^2 \geq 1$

$$\begin{aligned}&\leq |x-y| \\ &\leq |x| + |y| \\ &\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\sqrt{x^2+y^2}\end{aligned}$$

Therefore, if $2\sqrt{x^2+y^2} < \varepsilon \implies |f(x,y) - L| < \varepsilon$ so take $\delta = \varepsilon/2$. □

Example 1.8 (H.W)**.** *Show that* $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$,consider

$$\begin{aligned}|f(x,y) - L| &= \left|\frac{xy^2}{x^2+y^2} - 0\right| = \frac{|x|y^2}{x^2+y^2} \\ &= \frac{|x|}{\frac{x^2}{y^2} + 1} \\ &\leq |x| \\ &\leq \sqrt{x^2+y^2} < \varepsilon \implies |f(x,y) - L| < \varepsilon\end{aligned}$$

So we can just pick $\delta = \varepsilon$. □

1.4 Continuity

Definition 1.9 (Continuity)**.** *A function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is said to be continuous at a point* (a,b) *if* $\forall\varepsilon > 0, \exists\delta > 0$ *such that,*

$$0 < ||(x,y) - (a,b)||_2 < \delta \implies |f(x,y) - f(a,b)| < \varepsilon$$

provided $f(a,b)$ *exists. Alternatively,*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note that, we can show the function is discontinuous if

1. $f(a, b)$ doesn't exist.
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ doesn't exist.
3. Both exist but are not equal to each other.

Example 1.10. *Show that the given function is continous at* $(0, 0)$ *where,*

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. Here, $f(0, 0) = 0$. Clearly we have that $|x^2 - y^2| \leq |x^2 + y^2|$.
Let $\varepsilon > 0$,

$$\begin{aligned} |f(x,y) - L| &= \left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| \\ &= |x||y| \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \\ &\leq |x||y| \\ \\ &\leq \sqrt{x^2 + y^2}\sqrt{x^2 + y^2} = x^2 + y^2 \end{aligned}$$

So when $x^2 + y^2 < \varepsilon \implies |f(x, y) = f(0, 0)| < \varepsilon$ so we take $\delta = \sqrt{\varepsilon}$. □

Example 1.11. *Show that the given function is discontinuous at* $(0, 0)$ *where,*

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. content... □

1.5 Polar Coordinates

The polar coordinates r(the radial coordinate) and θ(the angular coordinate), are defined in terms of cartesian coordinates as below.

$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \theta = \arctan\left(\frac{y}{x}\right)$$

1.5.1 Limits in Polar coordinates

Use polar coordinates when you are over (0,0)

Example 1.12. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ *doesn't exist.*

Proof. Put $x = r\cos\theta$ and $y = r\sin\theta$

$$f(x,y) = \frac{2xy}{x^2+y^2} \iff f(r,\theta) = \frac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta$$
$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} 2\cos\theta\sin\theta = 2\cos\theta\sin\theta$$

Which depends on θ. □

1.5.2 Epsilon-delta with polar coordinates

Definition 1.13. $\lim_{r\to 0} f(r,\theta) = L \iff \forall\varepsilon > 0 \exists\delta > 0 s.t.$

$$0 < |r| < \delta \implies |f(r,\theta) - L| < \varepsilon$$

Example 1.14. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

Proof.

$$f(r,\theta) = \frac{r^3\cos^3\theta}{r^2} = r(\cos\theta)^3 \tag{1.1}$$

Let $\varepsilon > 0$, consider $|f(r,\theta) - L| = |r||\cos\theta|^3 \leq |r|$. So we can set $\delta = \varepsilon$ □

Example 1.15. *Find the domain and range of* $g(x,y) = \sqrt{9 - x^2 - y^2}$

Proof. The sqrt interior must be positive so take $x^2 + y^2 \leq 9$, so its a circle of radius 3 centred at 0. So the domain is the circle. The range is $\{z \mid 0 \leq z \leq 3\} = [0,3]$ □

1.6 Algebra of limits

Let $f, g : \mathbb{R}^n \to \mathbb{R}, p \in \mathbb{R}^n$ and $k_1, k_2 \in \mathbb{R}$.

Theorem 1.16. *If* $\lim_{x\to p} f(x) = L_1, \lim_{x\to p} g(x) = L_2$, *then*

- $\lim\limits_{x\to p}(k_1 f(x) + k_2 g(x)) = k_1 L_1 + k_2 L_2$
- $\lim\limits_{x\to p}(f(x)g(x)) = L_1 L_2$

- *For non-zero* L_2, $\lim\limits_{x\to p}(f(x)/g(x)) = L_1/L_2$

Theorem 1.17. *Let* $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}$ *and* $p \in \mathbb{R}^n$. *If* $\lim_{x\to p} f(x) = L_1$, *and* $\lim_{x\to p} g(x) = L_2$ *then* $\lim_{x\to p}[f(x) + g(x)] = L_1 + L_2$.

Proof. Let $\varepsilon > 0$.

Then there exists $\delta_1 > 0$ such that

$$x \in B^*(p, \delta_1) \implies |f(x) - L_1| < \varepsilon/2$$

also there exists $\delta_2 > 0$ such that

$$x \in B^*(p, \delta_2) \implies |g(x) - L_2| < \varepsilon/2$$

Define $\delta = \min \delta_1, \delta_2$. Then $\delta > 0$ and $\delta \leq \delta_1, \delta \leq \delta_2$.

So we have

$$x \in B^*(p, \delta) \implies |f(x) - L_1| < \varepsilon/2 \text{ and } |g(x) - L_2| < \varepsilon/2$$

Consider that $|[f(x) + g(x)] - [L_1 + L_2]| < \varepsilon/2 + \varepsilon/2$ if $x \in B^*(p, \delta)$.

Thus $\forall\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in B^*(p, \delta) \implies |[f(x) + g(x)] - [L_1 + L_2]| < \varepsilon$$

Hence, $\lim_{x\to p}[f(x) + g(x)] = L_1 + L_2 = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$ □

1.7 General multivariate limit

Theorem 1.18 (Limit of a function $f : \mathbb{R}^n \to \mathbb{R}$)**.** *For a function* $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ *if and only if* $\forall\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$0 < ||x - a||_n < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.19 (ε- neighbourhood)**.** $B(a, \varepsilon)$ *open ball of radius* ε *around* α.

$$0 \leq ||x - a||_n < \varepsilon$$

Definition 1.20 (Deleted ε neighbourhood)**.** $B(a, \varepsilon) - \{a\}$

Definition 1.21 (Alternate definition of a limit)**.** *For a function* $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ *if and only if* $\forall\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in B^*(a, \delta) \implies |f(x) - L| < \varepsilon$$

Definition 1.22 (Bounded function)**.** *Let E be a non-empty subset of $\mathbb{R}^n$. The function $f : E \to \mathbb{R}$ is said to be bounded in some δ-neighbourhood of point $p \in \mathbb{R}^n$ if there exists $M > 0$ in $\mathbb{R}$ such that*

$$|f(x)| \leq M \forall x \in B(p, \delta)$$

Theorem 1.23 (Relation between bounded function and limit of a function in $\mathbb{R}^n$)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$. Let $f(p)$ be defined. If $\lim_{x \to p} f(x)$ exists then f is bounded in some neighbourhood of point p.*

Proof. As $\lim_{x \to p} f(x)$ exists, let $\lim_{x \to p} f(x) = L \in \mathbb{R}$.

Then for $\varepsilon = 1 > 0$ there exists $\delta > 0$ such that

$$x \in B^*(p, \delta) \implies |f(x) - L| < \varepsilon = 1$$

Consider $|f(x)| = |(f(x) - L) + L| \leq |f(x) - L| + |L|$ so we have $|f(x)| < 1 + |L|$ for $x \in B^*(p, \delta)$. Define $M = \max\{|f(p)|, 1 + |L|\}$. Then $M > 0, M \geq |f(p)|, M \geq 1 + |L|$. Thus in any case we have $|f(x)| \leq M \forall x \in B(p, \delta)$. □

The converse of 1.23 isn't true.

Example 1.24.

$$f(x, y) = \frac{xy}{x^2 + y^2}, \textit{ for non zero and equal to 0 for 0}$$

Theorem 1.25 (Uniqueness of limit in $\mathbb{R}^n$)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$. If $\lim_{x \to p} f(x)$ exists then it is unique.*

Proof. Assume $\lim_{x \to p} f(x)$ is not unique and let it have two limits L_1, L_2.

Take $\varepsilon = \frac{1}{2}|L_1 - L_2|$.

Then we have the following,

$$\begin{cases} x \in B^*(p, \delta_1) \implies |f(x) - L_1| < \varepsilon \\ x \in B^*(p, \delta_2) \implies |f(x) - L_2| < \varepsilon \end{cases}$$

Chose $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and $\delta \leq \delta_1, \delta \leq \delta_2$.

So $x \in B^*(p, \delta) \implies |f(x) - L_1| < \varepsilon$ and $|f(x) - L_2| < \varepsilon$.

Consider that $|L_1 - L_2| = |f(x) - L_2 - f(x) - L_1| \leq |f(x) - L_2| + |f(x) - L_1|$ so $|L_1 - L_2| < \varepsilon + \varepsilon$ so we have $|L_1 - L_2| < |L_1 - L_2|$ a contradiction, so our initial assumption is wrong and limit must be unique if it exists. □

Theorem 1.26. *If function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $p \in \mathbb{R}^n$ then $|f|$ is continuous at p.*

Proof. Let $\varepsilon > 0$. Then as f is continuous at p we know the following.

$$\exists \delta > 0 s.t. x \in B(p, \delta) \implies |f(x) - f(p)| < \varepsilon$$

Then consider the fact that,

$$||f(x)| - |f(p)|| \leq |f(x) - f(p)| < \varepsilon$$

So just use the same epsilon for the continuity for the absolute valued function.

□

1.8 Iterated (Repeated) limits

Let $(a, b) \in E$ and $f : E \to \mathbb{R}$ be a function where $E \subseteq \mathbb{R}^2$,

1. Suppose there exists $\delta > 0$ such that $\forall x$ with $0 < |x - a| < \delta$,we have $\lim_{y \to b} f(x, y)$ exists.
   Define a new function $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = \lim_{y \to b} f(x, y)$. If $\lim_{x \to a} g(x)$ exists then this limit is called **iterated limit** which is given by $\lim_{x \to a} g(x) = \lim_{x \to a} \lim_{y \to b} f(x, y)$.

2. Suppose there exists $\delta > 0$ such that $\forall y$ with $0 < |y - b| < \delta$, we have $\lim_{x \to a} f(x, y)$ exists

Theorem 1.27. *Existence of double limit does not imply existence of iterated limit.*

Proof. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as,

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases}$$

We show that $\lim_{(x,y) \to (0,0)} f(x, y) = 0$ i.e. double limit exists.
Let $\varepsilon > 0$. Consider then

$$|f(x, y) - L| = |x \sin(1/y) - 0| = |x| \, |\sin(1/y)| \leq |x|$$

$$\leq \sqrt{x^2}$$

$$\leq \sqrt{x^2 + y^2}$$

so $\sqrt{x^2+y^2} < \varepsilon \implies |f(x,y) - L| < \varepsilon$. So choose $\delta = \varepsilon$.
We will now check its iterated limit.

$$\begin{aligned} \lim_x \lim_y f(x,y) &= \lim_{x\to 0}\left[\lim_{y\to 0} x \sin 1/y\right] \\ &= \lim_{x\to 0} x \left[\lim_{y\to 0} \sin 1/y\right] \end{aligned}$$

The limit inside doesn't exist.
Claim that $\lim_y \phi(y) = \lim_y \sin 1/y$ doesn't exist, Take $a_n = \frac{1}{(4n+1)\pi/2}, b_n = \frac{1}{(4n-1)\pi/2}$. The sequences converge to zero but their sequences $\phi a_n, \phi b_n$ dont converge to the same limit. $\square$

Example 1.28 (Both iterated limits exist but double limit doesn't exist)**.** *consider* $f : \mathbb{R}^2 \to \mathbb{R}$, *defined as*

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2-x} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. Begin with two part test to show that the double limit does not exist. Consider first the path $x = y$ the limit is 0,

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{y^2+y^2-y} = \lim_{(x,y)\to(0,0)} \frac{y}{2y-1} = 0$$

Then consider the path $x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{y^4+y^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4} = 1$$

So double limit does not exist.
Now consider the iterated limits,

$$\begin{aligned} \lim_{x\to 0}\lim_{y\to 0} f(x,y) &= \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2}{x^2+y^2-x}\right] \\ &= \lim_{x\to 0} \frac{x^2}{x^2-x} = 0 \end{aligned}$$

Now consider

$$\lim_{y\to 0}\lim_{x\to 0} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

$\square$

Example 1.29 (Both iterated limits exist (not equal) but double limit doesn't exist)**.** *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *definde as,*

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. First show that the double limit does not exist.
Consider the bath $x = 0$ the limit is equal to 1. Consider the path $x = y$ we will have the limit equal to 0.
Consider now the iterated limits,

$$\lim_{x\to 0} \lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} 1 = 1$$

And now the other direction,

$$\lim_{y\to 0} \lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} -1 = -1$$

□

Theorem 1.30. *Suppose* $\lim_{(x,y)\to(a,b)} f(x,y)$ *exists and is equal to* L*. If both iterated limits exist then, the iterated limits are both equal to* L*.*

Proof. Since

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \forall \varepsilon > 0$$

we know there exists $\delta > 0$ such that if $|x-a| < \delta$ and $|y-b| < \delta$ then

$$|f(x,y) - L| < \varepsilon$$

Let $L_a(y) = \lim_{x\to a} f(x,y)$
Then with $|y-b| < \delta$ □

1.9 Limits in 3 variables

1.9.1 Two path test for non-existence of limit

Two path can be used for non-existence of a limit in 3 variables. However a single equation is not enough to define a path in $\mathbb{R}^3$ two Cartesian equations are required for a path in $\mathbb{R}^3$.

Example 1.31. *Show that*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2}$$

Proof. Take $y = x, z = x$ then

$$\lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}$$

Take other path $y = x, z = 0$

$$\lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

$\square$

Definition 1.32 (Limit of a function $\mathbb{R}^3 \to \mathbb{R}$)**.** *For a function* $f : \mathbb{R}^3 \to \mathbb{R}, \lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$ *if and only if* $\forall \varepsilon > 0, \exists \delta > 0$ *such that*

$$0 < ||(x,y,z) - (a,b,c)||_3 < \delta \implies |f(x,y,z) - L| < \varepsilon$$

i.e.

$$0 < \sqrt{(x-a)^2 + (y-b)^2(z-c)^2} < \delta \implies |f(x,y,z) - L| < \varepsilon$$

Definition 1.33 (Continuity of a function $\mathbb{R}^3 \to \mathbb{R}$)**.** *Replace* L *with* $f(a,b,c)$.

Example 1.34. *Show that*

$$\lim_{(x,y,z)\to(1,2,3)} 4x + 2y + z = 11$$

using epsilon delta

Proof. Let $\varepsilon > 0$ consider,

$$\begin{aligned} |f(x,y,z) - L| &= |4x + 2y + z - 11| \\ &= |(4x-4) + (2y-4) + (z-3)| \\ &\le 4|x-1| + 2|y-2| + |z-3| \\ &\le 4\sqrt{(x-1)^2} + 2\sqrt{(y-2)^2} + \sqrt{(z-3)^2} \\ &\le 7\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^3} \end{aligned}$$

So take $\delta = \varepsilon/7$ $\square$

Example 1.35. *Evaluate* $\lim_{(x,y)\to(3,3)} \frac{x^2+xy-2y^2}{x^2-y^2}$

Proof. Factorize $(x-y)$ on numerator and denominator then just plug and chug. $\square$

Chapter 2

Differentiation

2.1 Partial derivatives

Partial derivates are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. For a function f in n variables $x_1, x_2, \ldots, x_n$ we can define the m^{th} partial derivative as,

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}$$

Partial derivatives can be taken with respect to multiple variables and are denoted as follows,

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

Lemma 2.1. *Existence of partial derivatives $f(x, y)$ at a point (a, b) does not imply continuity of f at that point.*

Definition 2.2 (Directional derivatives)**.** *Consider he function $f : \mathbb{R}^2 \to \mathbb{R}$ the direction derivative of $f(x, y)$ along unit vector $u = u_1 i + u_2 j$ at $p = (a, b)$ is*

$$D_u f(p) = \lim_{s \to 0} \frac{f(a + su_1, b + su_2) - f(a, b)}{s}$$

DD along $u = i$ is partial derivative w.r.t x similar for $u = j$.

Corollary 2.3. *Existence of directional derivative of $f(x, y)$ at a point $P \implies$ Existence of partial derivative of $f(x, y)$ at point P. But converse need not be true.*

Theorem 2.4 (Mixed partial derivatives are equal if they are continuous)**.** *$E \subseteq \mathbb{R}^2$ Let f_x, f_y, f_{xy}, f_{yx} exist. If f_{xy}, f_{yx} are continuous at (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$*

Example 2.5. *Find unit vector normal to level curve $x^2 + y^2 = a^2$ at point $P = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$*

2.2 Gradient

Definition 2.6 (Gradient)**.** *For $f : \mathbb{R}^3 \to \mathbb{R}$*

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Example 2.7. *If $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ find ∇f_p where $p = (\sqrt{2}, \sqrt{2}, -3)$*

Proof.

$$f_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

□

Example 2.8. *Find ∇f at $p = (0, \pi/2)$ if $f(x, y) = \sin(xy)$ and its norm at p.*

Theorem 2.9 (Chain rule for two variables)**.** *If $w = f(x, y)$ has continuous p.d. f_x, f_y and if $x = x(t), y = y(t)$ are differentiable functions of t then the composite function $w \circ f(x(t), y(t))$ is a differentiable function of t and*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 2.10. *If $u = x^2 + y^2$ and $x = at^2$ and $y = 2at$ find $\frac{du}{dt}$*

Proof.

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Consider the two partial derivatives first,

$$f_x = 2x, f_y = 2y$$

Now $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$ So we have $\frac{du}{dt} = 2x(2at) + 2y(2a) = 4a^2(t^3 + 2t)$ □

2.3 Level curves

Definition 2.11. *The level curves of a function f of two variables are curves with equations $f(x, y) = k$ where k is a constant (in the range of f).*

Theorem 2.12. *The vector $\nabla f(x, y)$ is normal (perpendicular to tangent) to level curve of f.*

2.4 Total derivative

Definition 2.13 (Total derivative)**.** *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at point $a = (a_1, a_2)$ if $\exists \alpha = (\alpha_1, \alpha_2)$ such that,*

$$\lim_{h \to 0} \frac{|f(a + h) - f(a) - \alpha h|}{||h||} = 0$$

$$i.e., \lim_{(h_1,h_1) \to (0,0)} \frac{|f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) - (\alpha_1, \alpha_2) \cdot (h_1, h_2)|}{\sqrt{h_1^2 + h_2^2}} = 0$$

Corollary 2.14. *If a function is differentiable then directional derivative along any unit vector exists.*

Corollary 2.15. *If function is differentiable then $D_u f(a) = \langle \nabla f_a, u \rangle$*

Corollary 2.16. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^2$ then it is continuous at a.*

2.5 How to show not differentiable, maximizing directional derivative

Example 2.17. *If $x = e^u \cos v, y = e^u \sin v$ then prove that*

$$y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} = e^{2u} \frac{dz}{???}$$

Example 2.18. *Find D.D. of $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ in direction of $i + 2j + 2k$. Also find direction and magnitude of greater D.D. at that point.*

Proof. Find $(\nabla\phi)_= i - 3j - 3k$.

Then $(D_u\phi)_p = (\nabla\phi)_p \hat{u} = (i - 3j - 3k) \cdot (i/3 + 2j/3 + 2k/3) = -\frac{11}{3}$

Greatest D.D. is normal to the curve and its magnitude is norm of the gradient. $||(\nabla\phi)_p|| = \sqrt{19}$ □

Example 2.19. *Find acute angle between surfaces at* $(2,-1,2), x^2+y^2+z^2=9$ *and* $z=x^2+y^2+3$

Proof. Acute angle between the surfaces is equal to the acute angles between its normals.

$f=x^2+y^2+z^2, g=x^2+y^2-z$

$$(\nabla f)_p = 4i-2j+4k = u$$
$$(\nabla g)_p = 4i-2j-k = v$$

We require the angle between u, v so,

$$\begin{aligned}\cos\theta &= \frac{u\cdot v}{||u||||v||} = \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{16+4+16}\sqrt{16+4+1}} \\ &= \frac{16+4-4}{\sqrt{36}\sqrt{21}} \\ &= \frac{16}{6\sqrt{21}} \\ &= \frac{8}{3\sqrt{21}}\end{aligned}$$

So $\theta = \arccos\left(\frac{8}{3\sqrt{21}}\right)$ □

To find the eq. of the line/tangent to the curve find its gradient and dot product with p and equate to zero. that gives you equation of tangent line/tangent.

Equation of line L through A parallel to $\overline{V}$. As $L||\overline{V}$ we have $\overline{AB}||\overline{V}$

$$(x-a_1)i+(y-a_2)j+(z-a_3)k||v_1i+v_2j+v_3j$$

so we get

$$t=\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

In view of this equation of normal at p is

$$\frac{x-x_0}{(f_x)p}=\frac{y-y_0}{(f_y)p}=\frac{z-z_0}{(f_z)_p]}$$

Example 2.20. *Find the equation of tangent plane and normal line to surface* $f(x,y,z)=f(p)$ *at* $p=(1,2,3)$ *???*

Proof. Equation of tangent is

Equation of normal line at p is given by

$$\frac{x-1}{-24} = \frac{y-2}{-1} = \frac{z-3}{1}$$

□

2.6 Lagrange mean value theorem in $\mathbb{R}^n$

Theorem 2.21. *Let E be an open set in $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be differentiable. If $a, b \in E$ then $\exists \theta \in (0,1)$ such that,*

$$f(b) - f(a) = \langle \nabla f(a + \theta(b-a)), (b-a) \rangle$$

Proof. Consider a unit vector $u = \frac{b-a}{||b-a||}$ let $||b-a|| = r \in \mathbb{R}$.

Then we have $||b-a|| = r \in \mathbb{R}$.

Define a function $g : [0,r] \to \mathbb{R}$ as $g(t) = f(a + tu), \forall t \in [0,r]$.

Then g is continuous on $[0,r]$ and differentiable on $(0,r)$. Applying LMVT (in $\mathbb{R}$) to this function g.

Therefore, there exists $c \in (0,r)$ such that

$$g'(c) = \frac{g(r) - g(0)}{r - 0}$$

$$\begin{aligned}
\lim_{h \to 0} \frac{content...}{den} &= \\
&\vdots \\
D_u f(a + cu) &= \frac{1}{r}(f(b) - f(a)) \\
\langle \nabla f(a + cu), u \rangle &= \frac{1}{r}(f(b) - f(a))
\end{aligned}$$

Let $\theta = \frac{c}{r}$ so $\theta \in (0,1)$

$$\frac{1}{r}(f(b) - f(a)) = \langle \nabla f \left(a + c \left(\frac{b-a}{r} \right) \right), \frac{b-a}{r} \rangle$$

Since $\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$ So we get the 1/r out and cancel from both sides giving the desired result. □

Example 2.22. *Find $\theta \in (0,1)$ in MVT for the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined as*

$$f(x,y,z) = xy + yz + zx$$

take $a = (0,0,0), b = (2,1,1)$

Proof.

$$\begin{aligned}
f(b) - f(a) &= \langle \nabla f(a + \theta(b-a)), (b-a) \rangle \\
f(2,1,1) - f(0,0,0) &= \langle \nabla f(\theta(2,1,1)), (2,1,1) \rangle \\
5 &= \langle \nabla f(2\theta, \theta, \theta), (2,1,1) \rangle
\end{aligned}$$

Gradient is given as $\nabla f = (y+z)i + (x+z)j + (x+y)k$

$$\begin{aligned}
5 &= \langle (2\theta, 3\theta, 3\theta), (2,1,1) \rangle \\
5 &= 4\theta + 3\theta + 3\theta \\
\theta &= \frac{1}{2}
\end{aligned}$$

□

2.7 Sequences in $\mathbb{R}^n$

Theorem 2.23 (Sequential definition of limit)**.** *Prove that $f : \mathbb{R}^n \to \mathbb{R}$ has limit l as $x \to p$ iff for every sequence $\{x_k\} \in \mathbb{R}^n$ conversing to p, sequence $\{f(x_k)\}$ converges to l.*

Theorem 2.24. *$f : \mathbb{R}^n \to \mathbb{R}$ is continuous at p iff for every sequence $\{x_k\} \in \mathbb{R}^n$ converging to p the sequence $\{f(x_k)\}$ converges to $f(p)$.*

2.8 Chain rule for vector value function

2.9 Taylor series in two variables

Theorem 2.25. $g(t) = f(a_1, a_2) - t\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a_1, a_2) + ???$

Example 2.26. $f(x,y) = \sin xy + \log(x+y)$ *about the point* $(1,0)$

Proof.

$$f(1+h, 0+k) = f(1,0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(1,0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(0,1)$$

□

2.10 Second order partial derivative test

`https://en.wikipedia.org/wiki/Second_partial_derivative_test` Let $f_x(a,b) = 0, f_y(a,b) = 0$ for twice differentiable functions.

Example 2.27. *A rectangular box without a top with a volume* 108*cubic units is to constructed from a sheet of metal. Find the dimensions of the bo if least amount of material is to be used in its manufacturing.*

Proof. Let the dimensions be x, y, z. Volume is $= xyz$. So $z = \frac{108}{xy}$.

So to minimize surface area $S = xy + 2xz + 2yz$

Sub z

$$S = 214\left(\frac{1}{y} + \frac{1}{x}\right) + xy$$

$$\begin{aligned} f_x &= -216/x^2 + y \\ f_y &= -216/y^2 + x \\ f_{xx} &= 432/x^3 \\ f_{yy} &= 432/y^3 \\ f_{xy} &= 1 \end{aligned}$$

To get stationary points of f solve $f_x = 0$ and $f_y = 0$

$f_x = 0 \implies y = 216/x^2 \implies y = 216/216^2y^4$

$f_y = 0 \implies x = 216/y^2 \implies y^3 = 216 \implies y = 6, x = 6$

So here $A = 2, B = 1, C = 2$ so $AC - B^2 = 3 > 0$ and $A > 0$ so $f(x,y)$ has minimum at $(6,6)$ also $z = 108/xy = 3$ dimensions are then $6 \times 6 \times 3$. □

Example 2.28. *Find shortest distance from* $(1,0,-2)$ *to the plane* $x + 2y + z = 4$.

Proof. The distance from any point to $(1,0,-2)$ is $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ if it lies on that plane then $z = 4 - x - 2y$ so the distance is instead $d = \sqrt{(x-1)^2 + y^2 + (4 - x - 2y + 2)^2}$ □

2.11 Method of Lagrange multiplier

To find maximum and minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z) = k$.

- Find all values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, yz)$ and $g(x, y, z) = k$

If we write the vector equation $\nabla = \lambda \nabla g$ in terms of its components.

$$f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g(x, y, z) = k$$

Example 2.29. $2xz + 2yz + xy = 12$

$xyz = \lambda(2xz + xy)$
$xyz = \lambda(2yz + xy)$
$xyz = \lambda(2xz + 2yz)$
$xz = yz, 2xz = xy$ and $x = y$ $y = 2z$ so $x = y = 2z$ so we get $4z^2 + 4z^2 + 4z^2 = 12$ so we have $z = 1, x = 2, y = 2$

Example 2.30. *find extreme values for $f(x, y) = x^2+2y^2$ on circle $x^2+y^2 = 1$*

Proof. $g(x, y) = x^2 + y^2 = 1$ solve the equations $\nabla f = \lambda \nabla g$ adn $g(x, y) = 1$ which can be written as

$$f_x = \lambda g_x, f_y = \lambda g_y, g(x, y) = 1$$

$$\begin{aligned} 2x &= 2x\lambda \\ 4y &= 2y\lambda \\ x^2 + y^2 &= 1 \end{aligned}$$

x cannot be cancelled in the top cause we dont know if its non zero. We get either $x = 0$ or $\lambda = 1$ from eq. 1. So if $x = 0, y = \pm 1$ so we got $(0, 1), (0, -1)$ but if $\lambda = 1$ then $y = 0$ and we get $x = \pm 1$ so f has possible extreme vaues at $(0, 1), (0, -1), (1, 0), (-1, 0)$.

Compute value of f at each of these points and the maximum among them is the maximum which is $f((0, \pm 1)) = 2$ and miinimum at $f((\pm 1, 0)) = 1$. □

2.11.1 Two constraints

If we have two constraints $g(x, y, z), h(x, y, z)$ we consider

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Example 2.31. $f(x, y, z) = x + 2y + 3z$ *subject to constraints*

2.12 Limits and continuity of vector values function

Definition 2.32. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a vector function.*

Definition 2.33 (Limit of vector valued function)**.** *Let S be a non empty open subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}^m$ be a vector field. Let $a \in S$.*

Then an element $\ell \in \mathbb{R}^m$ is said to be the limit of f at $x = a$ if for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$||x - a||_n < \delta \implies ||f(x) - \ell||_m < \varepsilon$$

and we write

$$\lim_{x \to a} f(x) = \ell$$

Example 2.34. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x^2, y^2, xy)$. Find* $\lim_{(x,y)\to(0,0)} f(x, y)$

Proof. Let $\varepsilon > 0$

□

Theorem 2.35 (Relation between limit of ector field and limit of component functions)**.** *Let S be non empty open subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}^m$ be a vector field given by*

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

Let $\ell = (\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{R}^m$. The limit at f at $x = a$ is ℓ iff the limit of the coordinates is,

Proof. Suppose $\lim_{x\to a} f(x) = \ell$ Let $\varepsilon > 0, \exists \delta > 0$ such that

$$||x - a||_n < \delta \implies ||f(x) - \ell||_m < \varepsilon$$

$$||f(x) - \ell||_m = \sqrt{\sum_{i=1}^{m} (f_i(x) - \ell_i)^2} < \varepsilon$$

But we have,

$$|f_i(x) - \ell_i| = \sqrt{(f_i(x) - \ell_i)^2} \leq \sqrt{\sum_{i=1}^{m} (f_i(x) - \ell_i)^2} < \varepsilon$$

Thus,

$$||x - a||_n < \delta \implies |f_i(x) - \ell_i| < \varepsilon$$

□

Definition 2.36 (Continuity of vector field)**.** *Let S be a non empty open subset of $\mathbb{R}^n$. Let $f : S \to \mathbb{R}^m$ be a vector field. Let $a \in S$, then f is said to be continuous at a if given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$||x - a||_n < \delta \implies ||f(x) - f(a)||_p$$

2.12.1 Algebra of limit for vector valued functions

Let $\alpha, \beta \in \mathbb{R}$ and $f, g : \mathbb{R}^n \to \mathbb{R}^m$ and $L, M \in \mathbb{R}^m, p \in \mathbb{R}^n$ If $\lim_{x \to p} f(x) = L, \lim_{x \to p} g(x) = M$ then

$$\lim_{x \to p} [\alpha f(x) + \beta(x)] = \alpha L + \beta M$$

Proof. content... □

Theorem 2.37 (Continuity)**.** *content...*

Definition 2.38 (Differentiability of vector valued functions)**.** *content...*

Definition 2.39 (Jacobian)**.** *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function given by.*

$$f \equiv (f_1, f_2, \ldots, f_m)$$

Let $a \in S$ and $\frac{\partial f_i}{\partial x_i}(a)$ exist for $i = 1, 2, \ldots, n$ then

$$Jf(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \ldots & \frac{\partial f_1}{x_n}(a) \end{bmatrix}$$

denoted as $\frac{\partial(f_1, f_2, \ldots f_m)}{\partial(x_1, x_2, \ldots, x_n)}$

Example 2.40. *Find jacobian for $f(x, y) = (2x^2 + 3y, 4x - 2y, x^3 + y^3)$ at* $(1, -1)$

Proof.

$$\begin{bmatrix} 4 & 3 \\ 4 & -2 \\ 3 & 3 \end{bmatrix}$$

□

Theorem 2.41 (Important)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $p \in \mathbb{R}^n$ is defined by its Jacobian f at p*

Proof. As f is differentiate there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{||f(p+h) - f(p) - T(h)||_m}{||h||_n} = 0$$

Where $T(h) = D(f(p)) \cdot h$ put $h = te_j$ where $t \in \mathbb{R}$ We get,

$$\lim_{t\to 0} \frac{||f(p+te_j) - f(p) - tT(e_j)||_m}{|t|} = 0$$

$$\lim_{t\to 0} \left|\left| \frac{f(p+te_j) - f(p)}{t} - T(e_j) \right|\right|$$

$$T(e_j) = D_{e_j} f(p)$$

So partial derivatives exist

Now $D_{e_j} f(p) = \frac{\partial f(p)}{\partial x_j}$

So $T(e_j) = \left(\frac{\partial f_1(p)}{\partial x_j}, \cdots, \frac{\partial f_m(p)}{\partial x_j} \right)$

$T(e_j) = a_{1j}e_1^n + a_{2j}e_2^n + \cdots + a_{mj}e_m^n$ □

Definition 2.42 (Hessian matrix)**.** *For $f : \mathbb{R}^n \to \mathbb{R}$*

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \end{bmatrix}$$

Example 2.43. *Find hessian for $f(x,y,z) = x^2 + 2xyz + y^2 z$*

Theorem 2.44 (IMPORTANT Differentiability implies continuity for vector fields)**.** *content...*

Chapter 3

Applications