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Differential equations

Lecture Notes
for SMAT403

Contents

Introduction

> *"This course is justly viewed as the most unpleasant undergraduate course in mathematics, by both teachers and students. Some of my colleagues have publicly announced that they would rather resign from MIT than lecture in sophomore differential equations"*
>
> – Gian-Carlo Rota

These are lecture notes for the SMAT403 Differential Equations course. If you spot mistakes just message me and let me know. There probably might be some typos.

Differential equations might well be the most boring topic in maths I think I will ever study.

Hopefully these notes make it more tolerable. If I remember to do it, there might be an appendix with common derivates and integrals at the end.

1

First order ordinary linear differential equations

"Just use Mathematica bro."

– Isaac Newton

1.1 Homogenous equations

The differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is said to be homogenous if M and N are of the same degree.

Substitute $y = vx$ to solve a homogenous ODE.

Example 1.1. *Solve* $(x + y)\ dx - (x - y)\ dy = 0$

Proof.

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

Let $y = vx$

$$\begin{aligned} \frac{dy}{dx} &= v + x\frac{dv}{dx} \\ v + x\frac{dv}{dx} &= \frac{x + vx}{x - vx} \\ \frac{(1 - v)}{1 + v^2}\, dv &= \frac{1}{x}\, dx \end{aligned}$$

Integrate both sides

$$\arctan v - \frac{1}{2}\log(1+v^2) = \log(x) + c$$
$$\arctan\left(\frac{y}{x}\right) = \log(\sqrt{x^2+y^2}) + c$$

□

1.2 Exact differential equation

IF you have $M\,dx + N\,dy = 0$ and if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then the differential equation is exact.

The solution is $f(x,y) = c$ where $\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$

Example 1.2. $e^y\,dx + (xe^y + 2y)\,dy = 0$

Proof. Here we have, $M = e^y, N = xe^y + 2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(e^y) = e^y$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xe^y + 2y) = e^y$$

Therefore, it is exact.

$$\frac{\partial f}{\partial x} = e^y$$
$$f(x,y) = \int e^y\,dx = xe^y + g(y)$$
$$\frac{\partial f}{\partial y} = xe^y + \frac{dg}{dy} = N = xe^y + 2y$$
$$\frac{dg}{dy} = 2y$$

□

2

Higher order ordinary linear differential equations

"Damn this is boring."
– Euclid, *When a time traveller showed him differential equations*

2.1 Second order linear differential equations

2.1.1 Definition

One dependent variable y and one independent variable x.
The general second order linear differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

or

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$

2.1.2 Existence and uniqueness theorem

Theorem 2.1. *Let $P(x), Q(x), R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if y_0, y_0' are any numbers. Then eq. 2.1 has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$*

Proof. Not covered in class. Just trust me bro. $\square$

Example 2.2. *Find the largest interval where* $(x^2-1)y''+3xy'+(\cos x)y = e^x, y(0) = 4, y'(0) = 5$ *is guaranteed to have a unique solution.*

Proof. We need to divide by (x^2-1) but this automatically implies that from $\mathbb{R}$ we cannot include $-1, 1$.
$P(x) = \frac{3x}{x^2-1}$, it is continuous over $\mathbb{R} \setminus \{-1, 1\}$, $Q(x) = \frac{\cos(x)}{x^2-1}$ same case, $R(x) = \frac{e^x}{x^2-1}$ same.
Therefore the interval is simply, $(-\infty, -1) \bigcap (-1, 1) \bigcap (1, \infty)$ we will choose just the middle interval since we need 0.
Therefore the largest interval in which the DE is guaranteed to have a unique solution is (-1,1). □

Example 2.3. $(x+2)y''+xy'+(\cot x)y = x^2+1, y(2) = 11, y'(2) = -2$

Proof. Divide by $(x+2)$ so $x = -2$ cannot be included. Also $\cot x$ isnt defined for $x = n\pi$.
So the required largest interval would be $(0, \pi)$ □

Example 2.4. *Find the largest interval where* $(x^2-4x)y''+3xy'+4y = 2, y(3) = 0, y'(3) = -1$

Proof. We need to divide by (x^2-4x) so we cannot include the points $0, 4$ so the interval is $(0, 4)$. □

Example 2.5. $(x-3)y''+xy'+\log|x|y = 0, y(1) = 0, y'(1) = 1$

Proof. We need to divide by $(x-3)$ so $x \neq 3$. Also $\log x$ is not defined for $x = 0$. So the required interval is just $(0, 3)$ □

2.1.3 Homogenous equation

Definition 2.6. *The equation*

$$y''+P(x)y'+Q(x) = 0$$

is called a homogenous equation.

Theorem 2.7. *If* $y_p(x)$ *is a fixed particular solution of eq 2.1 and* $y(x)$ *is any general solution of eq. 2.1, then* $y(x) - y_p(x)$ *is a solution of 2.6*

Proof. Let y_1 be some solution to 2.1 and y_2 be some other solution to 2.1. Then $y_1 - y_2$ will be a solution to 2.6.??? (*Check this later it makes no sense*) □

Theorem 2.8. *If $y_1(x)$ and $y_2(x)$ are any two solutions of eq. 2.6, then $c_1y_1(x) + c_2y_2(x)$ is also a solution for any constants c_1 and c_2.*

Proof. The proof is trivial and is left as an exercise to the next person who reads this.

□

Example 2.9. *Verify that $y = c_1 \cos x + c_2 \sin x$ is a solution of $y'' + y = 0$ Find the solution which satisfies (A)(?wtf is A) and*

- $y(0) = 0, y'(0) = 1$
- $y(0) = 1, y'(0) = 0$

Proof. Consider case i,

$$\begin{aligned} y(0) &= c_1 \cos(0) + c_2 \sin(0) \\ &= c_1 \end{aligned}$$

Therefore, $c_1 = 0$ but c_2 is undecided.... *finish this later* □

Example 2.10. *Solve $y'' + y' = 0$*

Proof.

$$\begin{aligned} y'' + y' &= 0 \\ y' = t(x) \ \rightarrow y'' &= t'(x) \\ t' + tz &= 0 \\ \frac{dt}{t} + dx &= 0 \\ \ln|t| + x &= C \\ \ln|t| + \ln e^x &= \ln e^C \\ t \cdot e^x &= C_1 \\ y' \cdot e^x &= C_1 \\ dy =& C_1 \frac{dx}{e^x} \\ \int dy =& C_1 \int \frac{dx}{e^x} \\ &\ldots \end{aligned}$$

General solution is $y = c_1 e^{-x} + c_2$ □

Example 2.11. *Solve* $x^2y'' + 2xy' - 2y = 0$

Proof. Let $y = x^m$ so $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

$$\begin{aligned} x^2m(m-1)x^{m-2} + 2x(mx^{m-1}) - 2x^m &= 0 \\ x^m(m^2 - m + 2m - 2) &= 0 \end{aligned}$$

$m = 1$ or $m = -2$

$$\begin{aligned} \therefore y_1(x) = x' = x \text{ and } y_2(x) &= x^{-2} \\ \text{General solution } y(x) &= c_1x + c_2x^{-2} \end{aligned}$$

□

Example 2.12. *Verify that* $y_1 = 1, y_2 = x^2$ *are solutions of* $xy'' - y' = 0$ *and write down the general solution.*

Proof. We will not insult the intelligence of the reader by verifying the solution.
For the general solution consider that $1, x^2$ are Linearly independent, so the general solution is $y(x) = c_1(1) + c_2(x^2)$ □

2.2 General solution of the homogenous system

Definition 2.13. *If two functions* $f(x), g(x)$ *are defined on an interval* I *and have the property that one is a constant multiple of the other, then they are said to be linearly dependent on* I*. Otherwise they are called linearly independent.*

Note that, if $f(x) \equiv 0$ and $g(x)$ are linearly dependent for every function $g(x)$.

Definition 2.14. *Let* $y_1(x), y_2(x)$ *be linearly independent solutions of the homogenous equations* $y'' + P(x)y' + Q(x)y = 0$*. Then the function* $W(y_1, y_2) = y_1y_2' - y_1'y_2$ *is called the Wronskian of* y_1, y_2*.*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If two functions are dependent then their Wronskian is identically zero.

Lemma 2.15. *If* $y_1(x), y_2(x)$ *are any two solutions to eq. 2.6 on interval* I *then their Wronskian is either identically zero or never zero on* I*.*

Proof. I trust my professor. □

Example 2.16. *By eliminating c_1 and c_2, find the differential equation for the family of the curves $y = c_1e^x + c_2e^{-3x}$. Then use Abel's formula to find the Wronskian.*

Proof.

$$y' = c_1e^x - 3c_2e^{-3x} \tag{2.1}$$
$$y'' = c_1e^x + 9c_2e^{-3x} \tag{2.2}$$

Now do $y - y'$

$$y - y'' = 4c_2e^{3x} \implies c_2 = \frac{y - y'}{4e^{-3x}} \tag{2.3}$$

Now do $y'' - y'$

$$y'' - y' = 12c_2e^{-3x} \implies c_2 = \frac{y'' - y'}{12e^{-3x}} \tag{2.4}$$

But now we have the following equality

$$y'' - y' = 3(y - y') \tag{2.5}$$

Expanding this gives us the differential equation required as,

$$y'' + 2y' - 3y = 0 \tag{2.6}$$

□

Lemma 2.17. *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 2.6 on I, then they are linearly dependent on this interval $\iff$ their Wronskian is identically 0.*

Proof. **Case 1:** If the function is linearly dependent then its Wronskian is equal to 0.

$$\begin{aligned} W(y_1, y_2) &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ (cy_2)y_2' - (cy_2')y_2 &= 0 \end{aligned}$$

Case 2: If the Wronskian is identically equal to 0 then we have to prove the function is linearly dependent.

Case 2a: If $y_1 \equiv 0 \to y_1$ is the zero function then y_1, y_2 are L.D.
Case 2b: If $y_1 \not\equiv 0 \implies y_1(x_0) \neq 0$, for some $x_0 \in I$. This implies $\exists [c,d] \subseteq I$ s.t. $y_1(x_0) \neq 0 \forall x_0 \in [c,d]$.
Also $W = 0$ on $[c,d] \implies y_1 y_2' - y_1' y_2 = 0 \implies \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \implies \left(\frac{y_2}{y_1}\right)'$.
And we get $y_2(x) = k y_1(x)$ for some $[c,d] \in I$. We need to extend this to all I. $y_2(x_0) = k y_1(x_0) = y_0 \forall x_0 \in [c,d]$ $y_2'(x_0) = k y_1'(x_0) = y_0' \forall x_0 \in [c,d]$, then use existence and uniqueness theorem. □

Lemma 2.18. *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 2.6 on I, then they are linearly independent on this interval iff their Wronskian is never zero on I.*

Proof. This proof is very boring and is left as an exercise to the reader. Use lemma 2.17.
Assume that y_1, y_2 are L.I. then we have to show $W \neq 0$. Assume $W = 0$ then use lemma 2.17.
If W is never zero then show that y_1, y_2 are L.I. □

Theorem 2.19. *Let $y_1(x)$ and $y_2(x)$ be linearly indepedent solutions of the homogenous equation 2.6 on I. Then $c_1 y_1(x) + c_2 y_2(x)$ is the general soltuion of equation 2.6 on I.*

Proof. Show $c_1 y_1 + c_2 y_2$ be a solution of eq 2.6. Next, let $y(x)$ be any other solution of 2.6 then show that there exists $c_1, c_2 \in \mathbb{R}$ such taht $y(x) = c_1 y_1 + c_2 y_2$. That is, to show that for some $x_0 \in I$ we can find c_1, c_2 s.t. $c_1 y_1(x_0) + c_2' y_2(x_0) = y(x_0)$ and $c_1 y_1'(x_0) + c_2 y_2'(x_0)$. That is we have to show,

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \end{bmatrix}$$

Complete this later its fucking boring □

Example 2.20. *Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval, and find the particular solution for which $y(0) = 2$ and $y'(0) = 3$.*

Proof.

$$y_1 = \sin x, y_2 = \cos x$$

It is easy to see that it a solution. Now we just have to show that it is linearly independent and them by the previous theorem we can say its linear combination is a general solution. Using the initial conditions we then solve and find the values for c_1 and c_2 □

Example 2.21 (H.W)**.** *Show that the space of solutions for the homogenous equation $y'' + P(x)y' + Q(x)y = 0$ is a vector space over $\mathbb{R}$.*

Example 2.22 (Review)**.** *Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ in any interval.*

Proof. Verify that it is a solution. Consider then its Wronskian. □

2.2.1 Using a known solution to find another

Theorem 2.23 (Use of a known solution to find another)**.** [1] *Consider $y'' + P(x)y' + Q(x)y = 0 \to (1)$. Assume $y_1(x)$ is a known non-zero solution of (1). Assume $y_2 = vy_1$ is another solution to (1), where $v = v(x)$. Then $y_2'' + P(x)y_2' + Q(x)y_2 = 0 \implies [vy_1]'' + P(x)[vy_1]' + Q(x)[vy_1] = 0$.*

$$v = \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$$

Example 2.24. *Show that $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution.*

Proof. $y_1' = 1, y_1'' = 0$, therefore we have

$$\begin{aligned} x^2y'' + xy' - y &= 0 \\ x^2(0) + x(1) - x &= 0 \end{aligned}$$

Therefore, it is a solution. Now we need to find y_2 another solution. Here in the standard form $P(x) = \frac{1}{x}, x \neq= 0$.
Now compute v

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{x^2}\frac{1}{x}\,dx \\ &= \int \frac{1}{x^3}\,dx \\ &= -\frac{1}{2x^2} \end{aligned}$$

So now we can say $y_2 = vy_1 = \frac{-1}{2x^2}x = \frac{-1}{2x}$.
Therefore, the general solution for the given differential solution is $y(x) = c_1y_1 + c_2y_2 = c_1x + c_2\frac{-1}{2x}$ in any interval not containing 0. □

[1] Just memorize this garbage.

Example 2.25. *Use the method of this section to find y_2 and the general solution of each of the following equations from the given solution y_1*

1. $y'' + y = 0, y_1 = \sin x$
2. $y'' - y = 0, y_1 = e^x$
3. $x^2y'' + xy' - 4y = 0, y_1 = x^2$

Proof. Consider 1) first, we have $P(x) = 0$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{\sin x^2} 1\,dx \\ &= -\cot x \end{aligned}$$

So now we can say $y_2 = vy_1 = -\cos x$. And the general solution is given by $y(x) = c_1 \sin x + c_2 \cos x$
Consider 2) now, we have again $P(x) = 0$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{e^{2x}}\,dx \\ &= \frac{-e^{-2x}}{2} \end{aligned}$$

Complete this up $\cdots$
Now consider 3), we have $P(x) = \frac{1}{x}$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{x^4}\frac{1}{x}\,dx \\ &= \int \frac{1}{x^5}\,dx \\ &= \frac{-1}{4x^4} \end{aligned}$$

Now we can get $y_2 = vy_1 = \frac{-1}{4x^2}$. And get the general solution. □

Example 2.26. *Show that $y_1 = x$ is a solution to $(1-x^2)y'' + 2xy' + 2y = 0$. Find the general solution.*

Proof. First make the DE into standard form by dividing by $1-x^2$. So we have $P = \frac{-2x}{1-x^2}$. We will now compute v,

$$\begin{aligned} v &= \int \frac{1}{x^2} e^{-\int \frac{-2x}{1-x^2}\, dx}\, dx \\ &= \int \frac{1}{x^2}\frac{1}{x^2-1}\, dx \\ &= \int \frac{1}{x^4-x^2}\, dx \\ &= \frac{1}{x} + \frac{1}{2}\log(1-x) - \frac{1}{2}\log(x+1) \end{aligned}$$

Now we compute $y_2 = vy_1 = 1 + \frac{1}{2}\log(1-x) - \frac{1}{2}x\log(1+x)$ □

2.2.2 Homogenous equation with constants

Theorem 2.27. *If we have $y'' + P(x)y' + Q(x)y = 0$ and $P(x), Q(x)$ are constants. Consider the Auxiliary equation $m^2 + pm + q = 0$ and let the roots of the auxiliary equation be m_1, m_2.*

1. *If the roots are real and distinct ($m_1 \neq m_2$), $y = c_1e^{m_1x} + c_2e^{m_2x}$*
2. *If the roots are real and repeated ($m = m_1 = m_2$), $y = c_1e^{mx} + c_2xe^{mx}$*
3. *If the roots are complex ($m = \alpha \pm \beta i$), $y = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$*

Example 2.28. $y'' + 3y' - y = 0$

Proof. Assume e^{mx} is a solution to the following differential equation, then we have the following,

$$\begin{aligned} y'' + 3y' - y &= 0 \\ m^2e^{mx} + 3me^{mx} - e^{mx} &= 0 \end{aligned}$$

We can know solve for m

$$m = \frac{-3}{2} \pm \frac{\sqrt{13}}{2}$$

Complete this ... □

Example 2.29. *Find the general solution for each of the following equations:*

1. $y'' + y' - 6y = 0$

2. $y'' + 3y' + y = 0$

Proof. Consider 1) first, use the auxiliary equation that appeared out of a hat then,

$$k^2 + pk + q = 0$$
$$k^2 + k - 6 = 0$$
$$k = -3, k = 2$$

So now the general solution is $y = c_1 e^{-3x} + c_2 e^{2x}$

□

Example 2.30. *Solve* $2y'' + 5y' - 12y = 0$

Proof. Use the auxiliary equation,

$$2m^2 + 5m - 12 = 0$$

We get the values of $m = 3/4, -4$. These are real and distinct roots. So the general solution is

$$y = c_1 e^{4x} + c_2 e^{3/2x}$$

. □

2.3 Method of undetermined coefficients (UDC)

This is a method to solve non-homogenous linear DE of order 2.

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$
$$y'' + P(x)y' + Q(x)y = 0 \tag{2.2}$$

The second equation is just the associated homogenous equation of the first. Let y_g be the general solution of eq. 2.2 be known as a complementary function (CF).
Let y_p be a particular solution of eq. 2.1 then it is known as something fuck knows she scrolled down.
You begin by computing y_g then compute y_p depending on one of the three stupid cases and their subcases [2]. Once you have the y_p you just add that (**without** a constant) to the y_g and you have the general solution for 2.1

[2]Just have to memorize it unfortunately. You want a intuitive reason for doing it? Cry more lmao.

2.3.1 Assumptions of UDC

1. $P(x), Q(x)$ are constants
2. $R(x)$ is either exponential, sine or cosine or polynomial.

2.3.2 Case 1: Exponential

Theorem 2.31.

$$y'' + py' + qy = e^{ax} \tag{2.3}$$

1. *If a is not a root of AE, then $y_p = Ae^{ax}$*
2. *If a is a simple root (i.e. multiplicity 1) of AE, then $y_p = Axe^{ax}$*
3. *If a is a double root of AE, then $y_p = Ax^2e^{ax}$*

Example 2.32. $y'' - 4y' + 4y = e^{2x}$

Proof. Here $a = 2$ and computing m we get a double root with $m = 2, 2$. □

2.3.2.1 Subcase i) a is not a root of AE

If a is not a root of the AE then you take $y_p = Ae^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Ae^{ax} \\ y_p' &= Aae^{ax} \\ y_p'' &= Aa^2e^{ax} \end{aligned}$$

Plug this into the original DE then just append the y_p to y_g without constants for your general solution.

Also there's some random thing like this in the notes idk it doesn't work,

$$A = \frac{1}{a^2 + pa + q}$$

Example 2.33. $y'' - 5y' + 6y = e^{4x}$

Proof. We get $a = 4$ and this is not a root to the auxiliary equation ($m = 2, 3$).
So now,

$$\begin{aligned} y_p &= Ae^{4x} \\ y_p' &= 4Ae^{4x} \\ y_p'' &= 16Ae^{4x} \end{aligned}$$

Now plug this into the DE

$$\begin{aligned} y'' - 5y' + 6y &= e^{4x} \\ 16Ae^{4x} - 5(4Ae^{4x}) + 6(Ae^{4x}) &= e^{4x} \end{aligned}$$

So we get $A = 1/2$. □

2.3.2.2 Subcase ii) a is a simple root of AE

If a is a simple root of the AE then you take $y_p = Axe^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Axe^{ax} = Ae^{ax}(x) \\ y_p' &= Ae^{ax}(ax + 1) \\ y_p'' &= Ae^{ax}(a^2x + 2a) \end{aligned}$$

Now just plug these values into the original differential equation. You will get y_p just append that to y_g and you have your solution.

Obligatory thing in the notes which I have no idea what it means,

$$A = \frac{1}{2a + p}$$

Example 2.34. *Solve* $y'' - 5y' + 6y = 3e^{2x}$

Proof. The AHE is $y'' - 5y' + 6y = 0$, we get the auxiliary equation as $m^2 - 5m + 6 = 0$ we get $m = 2, 3$. This is a simple root.

$$\begin{aligned} y_p &= Axe^{2x} \\ y_p' &= Ae^{2x} + 2Axe^{2x} \\ y_p'' &= 4Ae^{2x} + 4Axe^{2x} \end{aligned}$$

Now plug this into the DE,

$$\begin{aligned} y'' - 5y' + 6y &= 3e^{2x} \\ 4Ae^{2x} + 4Axe^{2x} - 5(Ae^{2x} + 2Axe^{2x}) + 6(Axe^{2x}) &= 3e^{2x} \end{aligned}$$

We get $A = -3$. □

Example 2.35 (Review)**.** *Find the general solution of* $y'' + 3y' - 10y = 6e^{4x}$

Proof. Consider the auxiliary equation,

$$m^2 + 3m - 10 = 0$$

We have $m = -5, 2$ real and distinct roots but a is not a root of AE. The solution to the AE is $y = c_1e^{2x} + c_2e^{-5x}$ So assume the general solution is given as,

$$\begin{aligned} y_p &= Ae^{4x} \\ y_p' &= 4Ae^{4x} \\ y_p'' &= 16Ae^{4x} \end{aligned}$$

Substitute this into the original equation to get $A = \frac{1}{3}$. So the general solution is given as $y = c_1e^{2x} + c_2e^{-5x} + \frac{1}{3}e^{4x}$

□

2.3.3 Case 2: Trigonometric

If $y'' + py' + qy = \sin kx$ or $\cos kx$

2.3.3.1 Subcase 1: If it is not a solution of AHE

If we have $\sin kx$ or $\cos kx$ not being a solution to the AHE we take $y_p = A\sin kx + B\cos kx$. Using this we get the following,

$$\begin{aligned} y_p &= A\sin kx + B\cos kx \\ y_p' &= k(A\cos kx - B\cos kx) \\ y_p'' &= -k^2(A\sin(kx) + B\cos kx) \end{aligned}$$

Plug this ugly mess into the original differential equation and you will get y_p append that without a constant to y_g for the general solution.

2.3.3.2 Subcase 2: If it is a solution of AHE

If we have $\sin kx$ or $\cos kx$ is a solution of the AHE then we take $y_p = x(A\sin kx + B\cos kx)$. Using this we get the following,

$$\begin{aligned}
y_p &= x(A\sin kx + B\cos kx)\\
y_p' &= A(\sin kx + kx\cos kx) + B(\cos kx - kx\sin kx)\\
y_p'' &= A(2k\cos kx - k^2 x\sin kx) + B(-xk^2\cos kx - 2k\sin kx)
\end{aligned}$$

Example 2.36. *Solve* $y'' + y = \sin x$

Proof. AHE is given $y'' + y = 0$ and AE $m^2 + 1 = 0$ so $m = \pm i$.

$$y_g = c_1\cos x + c_2\sin x$$

Since $\sin x$ is a solution of y_p let

$$\begin{aligned}
y_p &= x(A\sin x + B\cos x)\\
y_p' &= A\sin x + B\cos x + x(A\cos x - B\sin x)\\
y_p'' &= A\cos x - B\sin x + (A\cos x - B\sin x) + x(-A\sin x - B\cos x)
\end{aligned}$$

Plug this into the diff eq.

$$y'' + y = \sin x$$

$$A\cos x - B\sin x + (A\cos x - B\sin x) + x(-A\sin x - B\cos x) + x(A\sin x + B\cos x) = \sin x$$

$2A = 0, B = \frac{-1}{2}$ so we have $y_p = -1/2x\cos x$ and then $y(x) = y_g + y_p = c_1\cos x + c_2\sin x - 1/2x\cos x$ □

2.3.4 Case 3: Polynomial

For $y'' + py' + qy = a_0 + a_1x + \cdots + a_nx^n$

2.3.4.1 Subcase 1: If q $\neq$ 0

Take y_p as follows

$$\begin{aligned}
y_p &= A_0 + A_1x + \cdots A_nx^n\\
y_p' &= A_1 + \cdots + nA_nx^{n-1}\\
y_p'' &= 2A_2 + \cdots + n(n-1)A_n^{n-2}
\end{aligned}$$

2.3.4.2 Subcase 2: If $\mathbf{q = 0, p \neq 0}$

Take y_p as follows, the derivates are obvious,

$$y_p = x(A_0 + A_1 x + \cdots A_n x^n)$$

2.3.4.3 Subcase 3: If $\mathbf{q = 0, p = 0}$

Take y_p as follows,

$$y_p = x^2(A_0 + A_x + \cdots + A_n x^n)$$

Example 2.37. *Find the general solution of* $y'' - y' - 2y = 4x^2$

Proof. AHE $y'' - y' - 2y = 0$ and AE $m^2 - m - 2 = 0$ so $m = 2, -1$ so $y_g = c_1 e^{2x} + c_2 e^{-x}$.
Let

$$\begin{aligned} y_p &= A_0 + A_1 x + A_2 x^2 \\ y_p' &= A_1 + 2A_2 x \\ y_p'' &= 2A_2 \end{aligned}$$

Substitute this into the original DE

$$2A_2 - A_1 - 2A_2 x - 2(A_0 + A_1 x + A_2 x^2) = 4x^2$$

We get $A_0 = -3, A_1 = 2, A_2 = -2$. So the general solution is $y(x) = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$ □

Example 2.38. *Find general solution of* $y'' - 2y' + 5y = 25x^2 + 12$

Proof. AHE $y'' - 2y' + 5y = 0$ and AE $m^2 - 2m + 5 = 0$ we get $m = 1 \pm 2i$.So we have $\alpha = 1, \beta = 2$.
So we get $y_g = e^x(c_1 \cos 2x + c_2 \sin 2x)$.
Let

$$\begin{aligned} y_p &= A_0 + A_1 x + A_2 x^2 \\ y_p' &= A_1 + 2A_2 x \\ y_p'' &= 2A_2 \end{aligned}$$

Substitute this into the original DE

$$2A_2 - 2(A_1 + 2A_2 x) + 5(A_0 + A_1 x + A_2 x^2) = 25x^2 + 12$$

Upon solving we get $A_0 = 2, A_1 = 4, A_2 = 5$. So now we have $y(x) = y_g + y_p = e^x(c_1 \cos 2x + c_2 \sin 2x) + 2 + 4x + 5x^2$ □

Example 2.39 (Review). *Solve* $y'' - 2y' = 12x - 10$

Proof. Begin with AHE $y'' - 2y' = 0$ then see HE $m^2 - 2m = 0$ so we got $m = 0, 2$ so me and my homies say $y_g = 1 + e^{2x}$.
Now since $q = 0$ we say that

$$\begin{aligned} y_p &= A_1x^2 + A_0x \\ y_p' &= 2A_1x + A_0 \\ y_p'' &= 2A_1 \end{aligned}$$

Now plug this back into the original differential equation.

$$2A_1 - 2(2A_1x + A_0) = 12x - 10$$

Upon solving, a smart dog or a slow student would see that $A_0 = 2, A_1 = -3$. This gives us $y_p = -3x^2 + 2x$. And as such our final solution is indubitably,

$$\begin{aligned} y(x) &= y_g + y_p \\ &= c_1 + c_2(e^{2x}) + (-3x^2 + 2x) \end{aligned}$$

□

2.4 Variation of Parameters (VOP) method

Consider the non-homogenous equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

Consider its AHE. Say it has two solutions y_1, y_2. So that $y_g = c_1y_1 + c_2y_2$. We will say that the required particular solution y_p is a combination of y_g as follows,

$$y_p = v_1y_1 + v_2y_2$$

Now our job is just to get v_1, v_2 such that

$$y_p(x) = v_1y_1 + v_2y_2$$

Now begin differentiating this by repeated use of the product rule [3].

$$\begin{aligned} y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \\ y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2 \end{aligned}$$

[3] See why this is so boring yet?

We also now choose $v_1'y_1 + v_2'y_2 = 0$ to simplify the derivatives [4], Now with those terms we can restate the original differential equation as follows,

$$
\begin{aligned}
y_p' &= v_1y_1' + v_2y_2' \\
y_p'' &= v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2'
\end{aligned}
$$

Substitute these into the original differential equation and you get,

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1'y_1' + v_2'y_2' = R(x)$$

Since y_1, y_2 are solutions to the AHE the terms in the parentheses vanish and we are left with

$$v_1'y_1' + v_2'y_2' = R(x)$$

Recall from footnote 4 that we now have a system of equations as follows,

$$
\begin{aligned}
v_1' + v_2'y_2 &= 0 \\
v_1'y_1' + v_2'y_2' &= R(x)
\end{aligned}
$$

Now upon solving we get

$$
\begin{aligned}
v_1' &= \frac{-y_2R(x)}{W(y_1, y_2)} && \text{and} && v_2' = \frac{y_1R(x)}{W(y_1, y_2)} \\
v_1 &= -\int \frac{-y_2R(x)}{W(y_1, y_2)}\,dx && \text{and} && v_2 = \int \frac{R(x)y_1}{W(y_1, y_2)}\,dx
\end{aligned}
$$

Example 2.40. *Find a particular solution of $y'' + y = \csc x$*

Proof. Find the AHE $y'' + y = 0$ the solution for this is

$$y_p = c_1 \cos x + c_2 \sin x$$

. We have $R(x) = \csc x$ and we also know that $W(y_1, y_2) = 1$
So we got

$$
\begin{aligned}
v_1 &= \int \frac{-\sin x(\csc x)}{1}\,dx == x \\
v_2 &= \int \frac{\cos x \sec x}{1}\,dx = \log(|\sin x|)
\end{aligned}
$$

□

[4] lmao

Example 2.41. *Find a particular solution to $y'' - 2y' + y = 2x$*

Proof. Take the AHE $y'' - 2y' + y = 0$ and the HE $m^2 - 2m + 1 = 0$ so we get $y_p = c_1 e^x + c_2 x e^x$.
The Wronskian is equal to $W(y_1, y_2) = e^{2x}$.
Now do some magic and get

$$v_2 = \int \frac{2xe^x}{e^{2x}}\, dx = -2e^{-x}(x+1)$$

and

$$v_1 = \int \frac{-2x^2 e^x}{e^{2x}}\, dx = 2e^{-x}(x^2 + 2x + 2)$$

So the final solution is given as follows

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= 2e^{-x}(x^2 + 2x + 2)(e^x) + (-2e^{-x}(x+1))(xe^x) \\ &= 2x + 4 \end{aligned}$$

□

Example 2.42. *Find a particular solution of $y'' - y' - 6y = e^x$ with UDC then with VOP*

Proof. First do with UDC find the AHE and HE $m^2 - m - 6 = 0$ so we get $m = -2, m = 3$ and the $y_g = c_1 e^{-2x} + c_2 e^{3x}$. A is not a root so we go

$$\begin{aligned} y_p &= Ae^x \\ y_p' &= Ae^x \\ y_p'' &= Ae^x \end{aligned}$$

Plug thing into the original differential equation

$$Ae^x - Ae^x - 6Ae^x = e^x$$

we got $A = -1/6$. So the general solution is $y = e^{-2x} + e^{3x} - 1/6e^x$.
Now do VOP, from AHE and HE we get $y_g = c_1 e^{-2x} + c_2 e^{3x}$. We also now got the Wronskian as $5e^x$. **Complete this later...**

content...

□

Example 2.43. *Find a particular solution for each of the following equations*

1. $y'' + 4y = \tan 2x$
2. $y'' + 2y' + y = e^{-x} \log x$

3

Linear systems of ordinary differential equations

> *"I love differential equations. It is so fun. It has so many real life applications."*
>
> – Someone I don't like

3.1 Linear system of DE

We are mainly concerned with first order linear system of ODEs.

Observe that the single n^{th} order equation

$$y^{(n)} = f(x, y, y')$$

Is in fact equivalent to the system

$$\begin{aligned} y_1 &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

We will only see systems of two first order equations in two unknown functions of the following form,

$$\frac{dx}{dt} = F(t, x, y)$$
$$\frac{dy}{dt} = G(t, x, y)$$

More specifically we have **linear** systems of the form,

Definition 3.1 (Linear system of two ODE)**.**

$$\frac{dx}{dt} = a_1(t) + b_1(t) + f_1(t)$$
$$\frac{dy}{dt} = a_2(t) + b_2(t) + f_2(t)$$

Definition 3.2 (Homogenous linear system of two ODE)**.**

$$\frac{dx}{dt} = a_1(t) + b_1(t)$$
$$\frac{dy}{dt} = a_2(t) + b_2(t)$$

We assume that $a_i(t), b_i(t), f_i(t)$ for $i = 1, 2$ are continuous on some closed interval $[a, b]$ on the t-axis.
If $f_i(t)$ are both identically zero, then the system is called homogenous else. A solution of 3.1 is of the following form,

$$x = x(t)$$
$$y = y(t)$$

3.2 Existence and uniqueness theorems

Theorem 3.3. *If t_0 is any point of the interval $[a, b]$ and if x_0 and y_0 are any numbers then def. 3.1 has one and only one solution*

$$x = x(t)$$
$$y = y(t)$$

valid throughout $[a, b]$, such that $x(t_0) = x_0, y(t_0) = y_0$.

3.3 Homogenous linear system of ODE in two variables

Now consider the system of linear homogenous equations (def 3.2).

Theorem 3.4. *If the homogenous system (def. 3.2) has two solutions on the interval* $[a, b]$

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \textit{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

then we also have another solution of the form

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$

for any constants c_1, c_2.

3.4 Wronskian of homogenous linear system of ODE

Theorem 3.5. *If* $W(t)$ *is the Wronskian of two solutions of the homogenous system then* $W(t)$ *is either identically zero or nowhere zero on* $[a, b]$.

3.5 Linearly independent solutions

god nose what fits here

3.6 General solution of Homogenous linear system of ODE in two variables

Theorem 3.6. *If the two solutions for the homogenous system 3.2 have a Wronskian that does not vanish on* $[a, b]$ *then its linear combination of the solutions as described in theorem 3.4 is the general solution of the homogenous system 3.2 on that interval.*

Theorem 3.7. *If the two solutions of the homogeneous system are linearly independent then the linear combination 3.4 is its general solution.*

3.7 Non-homogenous linear system in two variables

Theorem 3.8. *If the two solutions for the homogenous system (Th. 3.4) are linearly independent on* $[a, b]$ *and if*

$$\begin{cases} x &= x_p(t) \\ y &= y_p(t) \end{cases}$$

is any particular solution of the non-homogenous linear system of ODEs (def: 3.1) then

$$\begin{cases} x &= c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y &= c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$$

is the general solution of the non homogenous system 3.1.

3.8 Homogenous linear systems with constant coefficients

Technically this section doesn't seem to be in her syllabus but I'm pretty sure its gonna be covered so I've done it anyway.
In this section we will examine the following system of linear ODEs,

$$\begin{cases} \frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y \end{cases} \tag{3.1}$$

where a_i, b_i are constants.
We are concerned with the following auxiliary equation that is related to this system,

$$m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0 \tag{3.2}$$

3.8.1 Distinct real roots

If eq. 3.2 has distinct real roots m_1, m_2 then the general solution of eq. 3.1 is given as,

$$\begin{cases} x = c_1A_1e^{m_1t} + c_2A_2e^{m_2t} \\ y = c_1B_1e^{m_1t} + c_2B_2e^{m_2t} \end{cases}$$

3.8.2 Equal real root

If eq. 3.2 has equal real roots $m = m_1 = m_2$ then the general solution of eq. 3.1 is given as,

$$\begin{cases} x &= c_1 A e^{mt} + c_2(A_1 + A_2 t)e^{mt} \\ y &= c_1 B e^{mt} + c_2(B_1 + B_2 t)e^{mt} \end{cases}$$

3.8.3 Distinct complex roots

If eq. 3.2 has distinct complex roots $a \pm ib$ then the general solution of eq. 3.1 is given as,

$$\begin{cases} x &= e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y &= e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)] \end{cases}$$

4

Partial differential equations

"Stop studying differential equations."

– Gauss

Definition 4.1 (Partial derivatives). *Partial derivates are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.*
For a function f in n variables $x_1, x_2, \ldots, x_n$ we can define the m^{th} partial derivative as,

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}$$

Partial derivatives can be taken with respect to multiple variables and are denoted as follows,

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

Differential equations that use partial derivates are called PDEs.

4.1 Classification of Second order PDE

Second order PDE are usually divided into three types.

Definition 4.2 (General form of a second order PDE)**.**

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

Linear second order PDEs are classified according to the properties of the following 2×2 matrix,

$$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.1}$$

4.1.1 Elliptic PDE

If Z (eq. 4.1) has determinant strictly greater than 0, it is called an elliptic PDE, i.e. $\det Z > 0$.

4.1.2 Hyperbolic PDE

If Z (eq. 4.1) has determinant strictly lesser than 0, it is called an hyperbolic PDE, i.e. $\det Z < 0$.

4.1.3 Parabolic PDE

If Z (eq. 4.1) has determinant equal to 0, it is called an parabolic PDE, i.e. $\det Z = f0$.

4.2 One dimensional wave equation

Definition 4.3 (One dimension wave equation)**.** *The one dimensional wave equation is given by,*

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

where a is a positive constant.

4.2.1 Vibration of an infinite string

4.2.2 Vibration of an semi-infinite string

4.2.3 Vibrating of a finite string

4.3 Laplace equation

Definition 4.4 (Laplacian). *The Laplacian of a three dimensional function ϕ is given as follows,*

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is generalized for higher dimensions in the expected way.

Definition 4.5 (Laplace's equation). *Laplace's equation is the following PDE*

$$\Delta f = 0$$

4.3.1 Green's function

Definition 4.6 (Green's function). *content...*

4.4 Heat conduction principle

Definition 4.7 (General heat equation). *The temperature function w satisfies the following heat equation*

$$a^2 \Delta w = \frac{\partial w}{\partial t}$$

4.4.1 Infinite rod case

4.4.2 Finite rod case

Appendix

If you are seeing this, I forgot to do it.