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Probability and Sampling Distributions (B)

Lecture Notes
for SSTA401

Contents

Chapter 1

Transformation of random variables & standard univariate continuous probability distributions

1.1 Uniform/Rectangular distributions

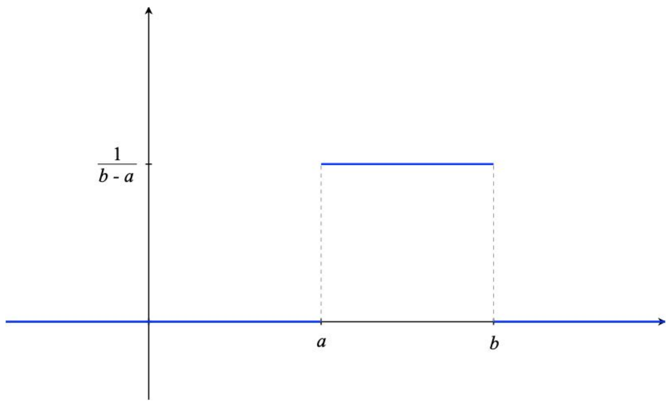
Definition 1.1. *A r.v. X is said to follow uniform distribution over an interval (a, b) if its pdf is constant over the entire range.*

1.1.1 PDF of uniform distribution

Theorem 1.2. *PDF of uniform distribution*

$$\begin{aligned} P(x) &= k & a < x < b \\ &= 0 & \textit{otherwise} \end{aligned}$$

- $\int_a^b f(x)\,dx = \int_a^b k\,dx = k[x]_a^b = k(b-a) = 1$, therefore $k = \frac{1}{b-a}$
- We denote it as, $X \sim U(a, b)$
- $f(x) = \frac{1}{b-a}$



1.1.2 CDF of uniform distribution

Theorem 1.3. *CDF of uniform distribution*

$$\begin{aligned} F(x) &= 0 && x \leq a \\ &= P(X \leq x) = \int_a^x f(x)\,dx = \frac{x-a}{b-a} && a < x < b \\ &= 1 && x \geq b \end{aligned}$$

1.1.3 Expectation and variance of uniform distribution

Theorem 1.4. *Expected value of $X \sim U(a,b)$ is equal to $\frac{(a+b)}{2}$*

Proof. Consider the expectation of the uniform distribution as,

$$\begin{aligned} E[x] &= \int_a^b xP(x)\,dx \\ &= \int_a^b x\frac{1}{b-a}\,dx \\ &= \frac{1}{b-a}\int_a^b x\,dx \\ &= \frac{a+b}{2} \end{aligned}$$

□

Theorem 1.5. *Variance of uniform distribution is equal to $\frac{1}{12}(b-a)^2$*

Proof. We begin by finding out $E[X^2]$

$$\begin{aligned} E[X^2] &= \int_a^b x^2 P(x)\, dx \\ &= \int_a^b x^2 \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \int_a^b x^2\, dx \\ &= \frac{1}{3}\left(a^2 + ab + b^2\right) \end{aligned}$$

Now we can find the variance as $V[X] = E[X^2] - E[X]^2$ as follows,

$$\begin{aligned} V[X] &= E[X^2] - E[X]^2 \\ &= \frac{1}{3}\left(a^2 + ab + b^2\right) - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

□

1.1.4 Raw moments of uniform distribution

The r^{th} raw moment of the uniform distribution is given as

$$\begin{aligned} \mu_r' = E[X^r] &= \int_a^b x^r f(x)\, dx \\ &= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \end{aligned}$$

Example 1.6. *Suppose in a quiz there are 30 participants. A question is given to all 30 participants and the time allowed is 25 seconds.*

Proof. Let X denote the time to respond.
$X \sim U(0, 25)$, the pdf is given by $f(x) = \frac{1}{25}; 0 < x < 25$ and 0 otherwise.

$$P(x \leq 6) = \int_0^6 f(x)\, dx = \int_0^6 \frac{1}{25}\, dx = \frac{151}{25}$$

$$P(6 \leq x \leq 10) = \int_6^1 0 f(x)\, dx = \int_6^{10} \frac{1}{25}\, dx = \frac{101}{25}$$

□

Example 1.7. *A r.v. x is said to follow uniform dist with $\mu = 1$ and $V(x) = 4/3$. Obtain $P(x < 0)$.*

Proof. First begin by finding out the parameters for the unfirom distribution. First consider the mean,

$$\begin{aligned} \mu &= 1 \\ \frac{a+b}{2} &= 1 \\ a+b &= 2 \end{aligned}$$

Then consider the variance,

$$\begin{aligned} V(x) &= \frac{4}{3} \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \\ (b-a)^2 &= 16 \end{aligned}$$

Solving two simultaneous equations we get $a = -1, b = 3$.
Therefore, we have $X \sim U(-1, 3)$

$$P(x \leq 0) = F(0) = \frac{0+1}{4} = \frac{1}{4}$$

□

Example 1.8. *If $X \sim U(-3, 3)$, find $P(x < 2)$, $P(|x| < 2)$, $P(|x-2| < 2)$, also obtain k if $P(x > k) = 1/3$*

Proof.

$$\begin{aligned} P(x < 2) = F(2) &= \frac{2+3}{6} = \frac{5}{6} \\ P(|x| < 2) &= \int_- 2^2 \frac{1}{6}\, dx = \frac{2}{3} \\ P(|x-2| < 2) &= \int_0^3 \frac{1}{6} = \frac{1}{2} \\ P(x > k) = 1/3 \implies \ldots \end{aligned}$$

Complete this l8r alig8r

□

1.1.5 MGF of Uniform distribution

Theorem 1.9. *MGF of Uniform distribution* $= \frac{e^{bt}-e^{at}}{t(b-a)}, t \neq 0$ *and* $= 1, t = 0$

Proof.

$$M_x(t) = E[e^{tx}] = \int_a^b \frac{e^{tx}}{b-a}\, dt = \frac{e^{bt}-e^{at}}{(b-a)t}$$

The Taylor series for this can be expressed as the following,

$$M_x(t) = \frac{b-a}{b-a} + \frac{b^2-a^2}{2(b-a)}t + \frac{b^3-a^3}{3(b-a)}\frac{t^2}{2!} + \cdots$$

Therefore we can say,

$$\mu_1' = \text{coeff of } t = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

$$\mu_2' = \text{coeff of } \frac{t^2}{2!} = \frac{b^3-a^3}{3(b-a)}$$

And we can say $\mu_2 = \ldots$ □

1.1.6 Applications of uniform distribution

1. Assumption of uniform death for insurance

   ⋮

Write sumthin here

1.2 Gamma distribution

Definition 1.10 (Gamma distribution)**.** *A r.v. 'X' is said to follow gamma distribution $X \sim G(\lambda, \theta)$. Where λ = shape parameter and θ = scale parameter.*

1.2.1 PDF of Gamma distribution

Definition 1.11 (PDF of Gamma distribution)**.**

$$\begin{aligned} f(x,\lambda,\theta) &= \frac{\theta^\lambda}{\Gamma(\lambda)} e^{-\theta x} x^{\lambda-1} && x > 0, \lambda > 0, \theta > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

Where $\Gamma(\lambda) = (\lambda-1)! = (\lambda-1)\Gamma(\lambda-1)$.

Corollary 1.12. *If $\theta = 1$ we will have gamma distribution with a single parameter λ which is called the standard gamma distribution.*

$$
\begin{aligned}
X \sim G(\lambda) &= \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)} && x > 0, \lambda > 0 \\
&= 0 && \textit{otherwise}
\end{aligned}
$$

Corollary 1.13. *If $\lambda = 1, X \sim G(1, \theta) = Exp(\theta)$.*

Corollary 1.14. *If $\lambda = 1, \theta = 1$, $X \sim$ Standard exponential distribution, i.e.*

$$
\begin{aligned}
f(x) &= e^{-x} && x > 0 \\
&= 0 && \textit{otherwise}
\end{aligned}
$$

Definition 1.15 (Gamma function)**.**

$$\Gamma(\lambda) = \int_0^\infty e^{-x} x^{\lambda-1} \, dx$$

Definition 1.16 (Gamma integral)**.**

$$\int_0^\infty e^{-\theta x} x^{\lambda-1} \, dx = \frac{\Gamma(\lambda)}{\theta^\lambda}$$

1.2.2 CDF of Gamma distribution

Theorem 1.17. *CDF of Gamma distribution is given as*

$$F(x) =$$

Proof.

$$
\begin{aligned}
F(x) = P(X < x) &= \int_0^x \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)} \, dx \\
&= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^x x^{\lambda-1} e^{-\theta x} \, dx
\end{aligned}
$$

□

1.2.3 Raw moments of Gamma distribution

Theorem 1.18. *The r^{th} aw moment of the Gamma distribution is given by*

$$\mu_r' = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)\theta^r}$$

Proof.

$$\begin{aligned}
\mu_r' = E[x^r] &= \int_0^\infty \frac{x^r e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\,dx \\
&= \int_0^\infty \frac{\theta^\lambda e^{-\theta x} x^{\lambda+r-1}}{\Gamma(\lambda)}\,dx \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)\theta^r}
\end{aligned}$$

□

1.2.4 Mean and Variance of Gamma distribution

Now we can find μ_1', μ_2'

$$\begin{aligned}
E[x] = \mu_1' &= \frac{\lambda}{\theta} \\
\mu_2' &= \frac{\lambda(\lambda+1)}{\theta^2} \\
V[x] = \mu_2 = \mu_2' - \mu_1'^2 &= \frac{\lambda(\lambda+1)}{\theta^2} - \frac{\lambda^2}{\theta^2} = \frac{\lambda}{\theta^2}
\end{aligned}$$

1.2.5 MGF of Gamma distribution

$$\begin{aligned}
E[e^{tx}] &= \int_0^\infty e^{tx} \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\,dx \\
&= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(\theta-t)x} x^{\lambda-1}\,dx \\
&= \frac{\theta^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda)}{(\theta-t)^\lambda} = \left(\frac{\theta}{\theta-t}\right)^\lambda \\
&= \left(1 - \frac{t}{\theta}\right)^{-\lambda}
\end{aligned}$$

1.2.6 CGF of Gamma distribution

$$\begin{aligned} K_x(t) &= \log\left(1-\frac{t}{\theta}\right)^{-\lambda} \\ &= -\lambda \log\left(1-\frac{t}{\theta}\right) \\ &= \frac{\lambda t}{\theta} + \frac{\lambda t^2}{2\theta^2} + \frac{\lambda t^3}{3\theta^3} + \cdots \end{aligned}$$

Using this we can get the mean and variance easily.

$$\begin{aligned} \text{Mean } &= k_1 = \frac{\lambda}{\theta} \\ \text{Variance } &= k_2 = \frac{\lambda}{\theta^2} \end{aligned}$$

1.2.7 Additive property of Gamma distribution

If $X_i (i = 1, \ldots, k)$ are k independent Gamma distributions with parametrs $\lambda_1, \lambda_2, \ldots, \lambda_k$ and θ respectively, then,

$$\begin{aligned} \sum_{i=1}^{k} X_i &\sim G(\sum_{i=1}^{k} \lambda_i, \theta) \\ M_{X_i}(t) &= \left(1-\frac{t}{\theta}\right)^{-\lambda_i} \end{aligned}$$

Let $Z = \sum X_i$

$$\begin{aligned} M_Z(t) &= \prod_{i=1}^{k} \left(1-\frac{t}{\theta}\right)^{-\lambda_i} \\ &= \left(1-\frac{t}{\theta}\right)^{-\sum \lambda_i} \end{aligned}$$

By uniqueness property of mgf

$$\sum_i X_i \sim G\left(\sum_i \lambda_i, \theta\right)$$

1.2.8 Limiting form of Gamma distribution

$$\beta_1 = \frac{4}{\lambda}, \text{ as } \lambda \to \infty, \beta_1 \to 0 \implies \text{Normal dist}$$
$$\beta_2 = 3 + \frac{6}{\lambda} \text{ as } \lambda \to \infty, \beta_2 \to 3 \implies \text{Normal dist}$$

Note that they are both independent of θ.
Therefore, as $\lambda \to \infty$ we have $G(\lambda, \infty) \to N\left(\frac{\lambda}{\theta}, \frac{\lambda}{\theta^2}\right)$.

1.2.9 Applications of Gamma distribution

Idk write something bruh

1.3 Exponential distribution

1.3.1 PDF of Exponential Distribution

Definition 1.19 (PDF of Exponential distribution)**.** *A r.v. x os said to follow the exponential distribution with parameter θ if its pdf is given by*

$$\begin{aligned} f(x) &= \theta e^{-\theta x} && x \geq 0, \theta > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

1.3.2 INCOMPLETE CDF of exponential distribution

$$F[x] = 1 - e^{-\theta x}$$

FILL THIS UP

1.3.3 Raw moment of exponential distribution

Theorem 1.20. *The r^{th} raw moment for exponential distribution is given by*

$$\mu_r' = \frac{r!}{\theta^r}$$

Proof.

$$\begin{aligned} \mu_r' = E[x^r] &= \int_0^\infty x^r \theta e^{-\theta x} \, dx \\ &= \frac{\Gamma(r+1)}{\theta^r} \\ &= \frac{r!}{\theta^r} \end{aligned}$$

□

1.3.4 Mean and variance of exponential distribution

Theorem 1.21. *The mean of exponential distribution is given be*

$$\mu = \frac{1}{\theta}$$

Proof. Consider $r = 1$,

$$\mu_1' = \frac{1}{\theta}$$

□

Theorem 1.22. *The variance of the exponential distribution is given by*

$$\mu_2 = \frac{1}{\theta^2}$$

Proof. First find μ_2'

$$\mu_2' = \frac{2}{\theta^2}$$

So now we can compute the variance as $\frac{1}{\theta^2}$ □

1.3.5 MGF of exponential distribution

Theorem 1.23. *MGF of exponential distribution is given by*

$$M_x(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$$

Proof.

$$\begin{aligned}
M_x(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx}\theta e^{-\theta x}\,dx \\
&= \theta\int_0^\infty e^{x(t-\theta)}x^{1-1}\,dx \\
&= \frac{\theta\Gamma(1)}{\theta - t} \\
&= \frac{\theta}{\theta - t} \\
&= \left(1 - \frac{t}{\theta}\right)^{-1}
\end{aligned}$$

□

1.3.6 CGF of exponential distribution

Theorem 1.24. *CGF of exponential distribution is given by*

$$K_x(t) = -\log\left(1 - \frac{t}{\theta}\right)$$

Proof.

$$\begin{aligned} K_x(t) &= \log\left(1 - \frac{t}{\theta}\right)^{-1} \\ &= -\log\left(1 - \frac{t}{\theta}\right) \\ &= \frac{t}{\theta} + \frac{t^2}{2\theta^2} + \frac{t^3}{3\theta^3} \end{aligned}$$

We can say the general r^{th} cumulant is given by $K_r = \frac{(r-1)!}{\theta^r}$ □

1.3.7 Additive property of exponential variates

Theorem 1.25. *If $x_1, x_2, \ldots, x_k$ are k independent exponential variates each with parameter θthen*

$$\sum_{i=1}^{k} x_i \sim G(k, \theta)$$

Proof. We will do this with the MGF. Consider taht $Z = \sum i = 1^k x_i$.

$$\begin{aligned} M_z(t) &= \prod_{i=1}^{k} M_x(t) \\ &= \prod_{i=1}^{k} \left(1 - \frac{t}{\theta}\right)^{-1} \\ &= \left(1 - \frac{t}{\theta}\right)^{-k} \end{aligned}$$

Therefore, (by uniqueness property of MGF) comparing this MGF to that of the gamma distribution we can say that,

$$\sum_{i=1}^{k} x_i = Z \sim G(k, \theta)$$

□

1.3.8 Lack of memory of exponential distribution

Theorem 1.26. *For a exponentially distributed random variate, $P[x > a+b \mid x > a] = P[x > b]$*

Proof. Let $X \sim E(\theta)$. Consider first case

$$\begin{aligned} P[x > a+b \mid x > a] &= \frac{P[x > a+b]}{P[x > a]} \\ &= \frac{\int_{a+b}^{\infty} \theta e^{-\theta x}\, dx}{\int_{a}^{\infty} \theta e^{-\theta x}\, dx} \\ &= \frac{e^{-\theta a + b}}{e^{-\theta a}} \\ &= e^{-\theta b} \end{aligned}$$

Consider second case now,

$$P[x > b] = \int_{b}^{\infty} \theta e^{-\theta x}\, dx = e^{-\theta b}$$

Equality holds. $\square$

1.4 INCOMPLETE Laplace distribution (Double exponential)

1.4.1 PDF

Definition 1.27 (PDF of Laplace distribution)**.** $X \sim L(\lambda, \mu)$

$$f(x) = \begin{cases} \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|} & -\infty < x < \infty \\ 0 & otherwise \end{cases}$$

1.4.2 CDF

Definition 1.28 (CDF of Laplace distribution)**.**

$$F[x] = \begin{cases} content... \end{cases}$$

1.4.3 Raw moment

Theorem 1.29. *The r^{th} raw moment for the Laplace distribution is given by*

$$\mu'_r =$$

Proof.

$$\mu'_r = E[x^r] = \int_{-\infty}^{\infty} \frac{x^r}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\, dx\right|}$$

Transform $(x-\mu)/\lambda = z$

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\infty} (z\lambda+\mu)^r e^{-|z|}\, \lambda\, dz\right)\\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (z-\lambda)^k \mu^{r-k} e^{-|z|\, dz}\right)\\
&= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k e^{-|z|}\, dz\right]
\end{aligned}$$

Complete this up

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} k!(1+(-1)^k)\right]$$

□

1.4.4 Mean and variance

We can do this with the raw moments above but instead we will do it with the PDF.

Theorem 1.30. *Expectation of laplace distribution is given as*

$$E[x] =$$

Proof.

$$\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x f(x)\, dx\\
&= \int_{-\infty}^{\infty} \frac{x}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}\, dx
\end{aligned}$$

Split it around μ

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} xe^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} xe^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} xe^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} xe^{-x/\lambda}\,dz\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\lambda(x-\lambda)e^{x/\lambda} - e^{\mu/\lambda}(\lambda(x+\lambda)e^{-x/\lambda})\right] \\
&= \mu
\end{aligned}$$

□

Theorem 1.31. *Expectation of x^2 in Laplace distribution is given be*

$$E[x^2] = bruh$$

Proof.

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}$$

Split it around μ

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} x^2e^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} x^2e^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} x^2e^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} x^2e^{-x/\lambda}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}(\lambda(x^2-2\lambda x+2\lambda^2)e^{x/\lambda}) - e^{\mu/\lambda}(\lambda(x^2+2\lambda x+2\lambda^2)e^{-x/\lambda})\right] \\
&= 2\lambda^2
\end{aligned}$$

□

Theorem 1.32. *Variance of Laplace distribution is given as*

$$V[x] =$$

1.4.5 MGF

Theorem 1.33. *MGF of the Laplace distribution is given by*

$$M_x(t) = bruh$$

Proof.

$$\begin{aligned}
M_x(t) = E[e^{tx}] &= \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{tx - \left|\frac{x-\mu}{\lambda}\right|} \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} e^{x\left(t+\frac{1}{\lambda}\right)}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} e^{-x\left(\frac{1}{\lambda}-t\right)}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\left(\frac{e^{\mu\left(\frac{1}{\lambda}+t\right)}}{\frac{1}{\lambda}+t}\right) + e^{\mu/\lambda}\left(\frac{-e^{\mu\left(\frac{1}{\lambda}-t\right)}}{-\frac{1}{\lambda}+t}\right)\right] \\
&= \frac{1}{2\lambda}\left[\frac{e^{\mu t}}{t+\frac{1}{\lambda}} - \frac{e^{\mu t}}{t-\frac{1}{\lambda}}\right]
\end{aligned}$$

□

Plot a graph for the beta-1 dsitribution when alpha=5, beta=2

1.4.6 CGF

1.5 Beta distribution of Type-I

1.5.1 PDF

Definition 1.34 (PDF of Beta I)**.**

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 < x < 1; m, n > 0 \\ 0 & otherwise \end{cases}$$

Where $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Note the following,

1. We can say, $X \sim \beta_1(m,n)$ where m,n are the parameters of the distribution.

2. Since $f(x)$ is a pdf we have the following,

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx \\
&= \int_0^1
\end{aligned}$$

1.5.2 Raw moments

Theorem 1.35. *The r^{th} raw moment of the Beta I distribution is given by*

$$\mu'_r = \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}$$

Proof.

$$\begin{aligned}\mu'_r = E[x^r] &= \int_0^1 \frac{1}{\beta(m,n)} x^{r+m-1}(1-x)^{n-1}\,dx \\ &= \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}\end{aligned}$$

□

1.5.3 Mean and Variance

Theorem 1.36. *Mean of Beta I distribution is given by*

$$E[x] = \frac{m}{m+n}$$

Proof.

$$E[x] = \mu'_1 = \frac{\Gamma(m+n)\Gamma(m+1)}{\Gamma(m)+\Gamma(m+n+1)} = \frac{m}{m+n}$$

□

Theorem 1.37. *Variance of Beta I distribution is given by*

$$V[x] = \frac{mn}{(m+n)^2(m+n+1)}$$

Proof.

$$\mu'_2 = \frac{(m+1)(m)}{(m+n)(m+n+1)}$$

So now we have the variance given as,

$$\begin{aligned}\mu_2 &= \mu'_2 - \mu'^2_1 \\ &= \frac{mn}{(m+n)^2(m+n+1)}\end{aligned}$$

□

1.6 Beta distribution of Type-II

1.6.1 PDF

Definition 1.38 (PDF of Beta-II distribution)**.**

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & otherwise \end{cases}$$

Note the following,

1. X is said to follow $\beta_2(m,n)$ as $X \sim \beta_2(m,n)$

2. $$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^{m+1}}{(1+x)^{m+n}} = \beta(m,n)$$

1.6.2 Raw moments

Theorem 1.39 (Raw moments of Beta-2 distribution)**.** *The raw moments of the Beta-2 distribution is given by*

$$\mu_r' = \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)}$$

Proof.

$$\begin{aligned} \mu_r' = E[x^r] &= \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx \\ &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)} \end{aligned}$$

□

1.6.3 Mean and variance

Theorem 1.40 (Mean of Beta-2 distribution)**.** *The mean of Beta-2 distribution is given by*

$$E[x] = \frac{m}{n-1}$$

Proof.

$$\begin{aligned} E[x] = \mu_1' &= \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma(m)\Gamma(n)} \\ &= \frac{m}{n-1} \end{aligned}$$

□

Theorem 1.41 (Variance of Beta-2 distribution)**.** *The variance of Beta-2 distribution is given by*

$$V[x] = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

Proof. First consider the 2nd raw moment,

$$\mu_2' = \frac{m(m+1)}{(n-2)(n-2)}$$

Now we can compute the variance as follows

$$V[x] = \mu_2 = \mu_2' - \mu_1'^2 = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

□

Chapter 2

Chi-square distribution

Chapter 3

F-distribution