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Differential equations

Lecture Notes
for SMAT403

Contents

Introduction

> *"This course is justly viewed as the most unpleasant undergraduate course in mathematics, by both teachers and students. Some of my colleagues have publicly announced that they would rather resign from MIT than lecture in sophomore differential equations"*
>
> – Gian-Carlo Rota

These are lecture notes for the SMAT403 Differential Equations course. If you spot mistakes just message me and let me know. There probably might be some typos.

Differential equations might well be the most boring topic in maths I think I will ever study.

Hopefully these notes make it more tolerable. If I remember to do it, there might be an appendix with common derivates and integrals at the end.

1

First order ordinary linear differential equations

"Just use Mathematica bro."

– Isaac Newton

1.1 Homogenous equations

The differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be homogenous if M and N are of the same degree.

Substitute $y = vx$ to solve a homogenous ODE.

Example 1.1. *Solve* $(x+y)\ dx - (x-y)\ dy = 0$

Proof.

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Let $y = vx$

$$\begin{aligned} \frac{dy}{dx} &= v + x\frac{dv}{dx} \\ v + x\frac{dv}{dx} &= \frac{x+vx}{x-vx} \\ \frac{(1-v)}{1+v^2}\,dv &= \frac{1}{x}\,dx \end{aligned}$$

Integrate both sides

$$\arctan v - \frac{1}{2}\log(1+v^2) = \log(x) + c$$
$$\arctan\left(\frac{y}{x}\right) = \log(\sqrt{x^2+y^2}) + c$$

□

1.2 Exact differential equation

IF you have $M\,dx + N\,dy = 0$ and if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then the differential equation is exact.

The solution is $f(x,y) = c$ where $\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$

Example 1.2. $e^y\,dx + (xe^y + 2y)\,dy = 0$

Proof. Here we have, $M = e^y, N = xe^y + 2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(e^y) = e^y$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xe^y + 2y) = e^y$$

Therefore, it is exact.

$$\frac{\partial f}{\partial x} = e^y$$
$$f(x,y) = \int e^y\,dx = xe^y + g(y)$$
$$\frac{\partial f}{\partial y} = xe^y + \frac{dg}{dy} = N = xe^y + 2y$$
$$\frac{dg}{dy} = 2y$$

□

2

Higher order ordinary linear differential equations

> *"Damn this is boring."*
> – Euclid, *When a time traveller showed him differential equations*

2.1 Second order linear differential equations

2.1.1 Definition

One dependent variable y and one independent variable x.
The general second order linear differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

or

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$

2.1.2 Existence and uniqueness theorem

Theorem 2.1. *Let $P(x), Q(x), R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if y_0, y_0' are any numbers. Then eq. 2.1 has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$*

Proof. Not covered in class. Just trust me bro. $\square$

Example 2.2. *Find the largest interval where* $(x^2-1)y''+3xy'+(\cos x)y = e^x, y(0)=4, y'(0)=5$ *is guaranteed to have a unique solution.*

Proof. We need to divide by (x^2-1) but this automatically implies that from $\mathbb{R}$ we cannot include $-1, 1$.
$P(x) = \frac{3x}{x^2-1}$, it is continuous over $\mathbb{R} \setminus \{-1,1\}$, $Q(x) = \frac{\cos(x)}{x^2-1}$ same case, $R(x) = \frac{e^x}{x^2-1}$ same.
Therefore the interval is simply, $(-\infty,-1)\bigcap(-1,1)\bigcap(1,\infty)$ we will choose just the middle interval since we need 0.
Therefore the largest interval in which the DE is guaranteed to have a unique solution is (-1,1). □

Example 2.3. $(x+2)y''+xy'+(\cot x)y = x^2+1, y(2)=11, y'(2)=-2$

Proof. Divide by $(x+2)$ so $x=-2$ cannot be included. Also $\cot x$ isnt defined for $x=n\pi$.
So the required largest interval would be $(0,\pi)$ □

Example 2.4. *Find the largest interval where* $(x^2-4x)y''+3xy'+4y = 2, y(3)=0, y'(3)=-1$

Proof. We need to divide by (x^2-4x) so we cannot include the points $0,4$ so the interval is $(0,4)$. □

Example 2.5. $(x-3)y''+xy'+\log|x|y = 0, y(1)=0, y'(1)=1$

Proof. We need to divide by $(x-3)$ so $x \neq 3$. Also $\log x$ is not defined for $x=0$. So the required interval is just $(0,3)$ □

2.1.3 Homogenous equation

Definition 2.6. *The equation*

$$y''+P(x)y'+Q(x)=0$$

is called a homogenous equation.

Theorem 2.7. *If* $y_p(x)$ *is a fixed particular solution of eq 2.1 and* $y(x)$ *is any general solution of eq. 2.1, then* $y(x)-y_p(x)$ *is a solution of 2.6*

Proof. Let y_1 be some solution to 2.1 and y_2 be some other solution to 2.1. Then y_1-y_2 will be a solution to 2.6.??? (*Check this later it makes no sense*) □

Theorem 2.8. *If $y_1(x)$ and $y_2(x)$ are any two solutions of eq. 2.6, then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution for any constants c_1 and c_2.*

Proof. The proof is trivial and is left as an exercise to the next person who reads this.

□

Example 2.9. *Verify that $y = c_1 \cos x + c_2 \sin x$ is a solution of $y'' + y = 0$ Find the solution which satisfies (A)(???) and*

- $y(0) = 0, y'(0) = 1$
- $y(0) = 1, y'(0) = 0$

Proof. Consider case i,

$$\begin{aligned} y(0) &= c_1 \cos(0) + c_2 \sin(0) \\ &= c_1 \end{aligned}$$

Therefore, $c_1 = 0$ but c_2 is undecided.... *finish this later* □

Example 2.10. *Solve $y'' + y' = 0$*

Proof.

$$\begin{aligned} y'' + y' &= 0 \\ y' = t(x) \ \rightarrow y'' &= t'(x) \\ t' + tz &= 0 \\ \frac{dt}{t} + dx &= 0 \\ \ln|t| + x &= C \\ \ln|t| + \ln e^x &= \ln e^C \\ t \cdot e^x &= C_1 \\ y' \cdot e^x &= C_1 \\ dy =& C_1 \frac{dx}{e^x} \\ \int dy =& C_1 \int \frac{dx}{e^x} \\ & \dots \end{aligned}$$

General solution is $y = c_1 e^{-x} + c_2$ □

Example 2.11. *Solve* $x^2y'' + 2xy' - 2y = 0$

Proof. Let $y = x^m$ so $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$

$$x^2m(m-1)x^{m-2} + 2x(mx^{m-1}) - 2x^m = 0$$
$$x^m(m^2 - m + 2m - 2) = 0$$

$m = 1$ or $m = -2$

$$\therefore y_1(x) = x' = x \text{ and } y_2(x) = x^{-2}$$
$$\text{General solution } y(x) = c_1x + c_2x^{-2}$$

□

Example 2.12. *Verify that* $y_1 = 1, y_2 = x^2$ *are solutions of* $xy'' - y' = 0$ *and write down the general solution.*

Proof. We will not insult the intelligence of the reader by verifying the solution.
For the general solution consider that $1, x^2$ are Linearly independent, so the general solution is $y(x) = c_1(1) + c_2(x^2)$ □

2.2 General solution of the homogenous system

Definition 2.13. *If two functions* $f(x), g(x)$ *are defined on an interval* I *and have the property that one is a constant multiple of the other, then they are said to be linearly dependent on* I*. Otherwise they are called linearly independent.*

Note that, if $f(x) \equiv 0$ and $g(x)$ are linearly dependent for every function $g(x)$.

Definition 2.14. *Let* $y_1(x), y_2(x)$ *be linearly independent solutions of the homogenous equations* $y'' + P(x)y' + Q(x)y = 0$*. Then the function* $W(y_1, y_2) = y_1y_2' - y_1'y_2$ *is called the Wronskian of* y_1, y_2*.*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If two functions are dependent then their Wronskian is identically zero.

Lemma 2.15. *If* $y_1(x), y_2(x)$ *are any two solutions to eq. 2.6 on interval* I *then their Wronskian is either identically zero or never zero on* I*.*

Proof. I trust my professor. □

Example 2.16. *By eliminating c_1 and c_2, find the differential equation for the family of the curves $y = c_1e^x + c_2e^{-3x}$. Then use Abel's formula to find the Wronskian.*

Proof.

$$y' = c_1e^x - 3c_2e^{-3x} \tag{2.1}$$
$$y'' = c_1e^x + 9c_2e^{-3x} \tag{2.2}$$

Now do $y - y'$

$$y - y'' = 4c_2e^{3x} \implies c_2 = \frac{y - y'}{4e^{-3x}} \tag{2.3}$$

Now do $y'' - y'$

$$y'' - y' = 12c_2e^{-3x} \implies c_2 = \frac{y'' - y'}{12e^{-3x}} \tag{2.4}$$

But now we have the following equality

$$y'' - y' = 3(y - y') \tag{2.5}$$

Expanding this gives us the differential equation required as,

$$y'' + 2y' - 3y = 0 \tag{2.6}$$

□

Lemma 2.17. *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 2.6 on I, then they are linearly dependent on this interval $\iff$ their Wronskian is identically 0.*

Proof. **Case 1:** If the function is linearly dependent then its Wronskian is equal to 0.

$$\begin{aligned} W(y_1, y_2) &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ (cy_2)y_2' - (cy_2')y_2 &= 0 \end{aligned}$$

Case 2: If the Wronskian is identically equal to 0 then we have to prove the function is linearly dependent.

Case 2a: If $y_1 \equiv 0 \to y_1$ is the zero function then y_1, y_2 are L.D.
Case 2b: If $y_1 \not\equiv 0 \implies y_1(x_0) \neq 0$, for some $x_0 \in I$. This implies $\exists [c,d] \subseteq I$ s.t. $y_1(x_0) \neq 0 \forall x_0 \in [c,d]$.
Also $W = 0$ on $[c,d] \implies y_1 y_2' - y_1' y_2 = 0 \implies \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \implies \left(\frac{y_2}{y_1}\right)'$.
And we get $y_2(x) = ky_1(x)$ for some $[c,d] \in I$. We need to extend this to all I. $y_2(x_0) = ky_1(x_0) = y_0 \forall x_0 \in [c,d]$ $y_2'(x_0) = ky_1'(x_0) = y_0' \forall x_0 \in [c,d]$, then use existence and uniqueness theorem. □

Lemma 2.18. *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 2.6 on I, then they are linearly independent on this interval iff their Wronskian is never zero on I.*

Proof. This proof is very boring and is left as an exercise to the reader. Use lemma 2.17.
Assume that y_1, y_2 are L.I. then we have to show $W \neq 0$. Assume $W = 0$ then use lemma 2.17.
If W is never zero then show that y_1, y_2 are L.I. □

Theorem 2.19. *Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogenous equation 2.6 on I. Then $c_1 y_1(x) + c_2 y_2(x)$ is the general soltuion of equation 2.6 on I.*

Proof. Show $c_1 y_1 + c_2 y_2$ be a solution of eq 2.6. Next, let $y(x)$ be any other solution of 2.6 then show that there exists $c_1, c_2 \in \mathbb{R}$ such taht $y(x) = c_1 y_1 + c_2 y_2$. That is, to show that for some $x_0 \in I$ we can find c_1, c_2 s.t. $c_1 y_1(x_0) + c_2' y_2(x_0) = y(x_0)$ and $c_1 y_1'(x_0) + c_2 y_2'(x_0)$. That is we have to show,

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \end{bmatrix}$$

Complete this later its boring □

Example 2.20. *Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval, and find the particular solution for which $y(0) = 2$ and $y'(0) = 3$.*

Proof.

$$y_1 = \sin x, y_2 = \cos x$$

It is easy to see that it a solution. Now we just have to show that it is linearly independent and them by the previous theorem we can say its linear combination is a general solution. Using the initial conditions we then solve and find the values for c_1 and c_2 □

Example 2.21 (H.W). *Show that the space of solutions for the homogenous equation $y'' + P(x)y' + Q(x)y = 0$ is a vector space over $\mathbb{R}$.*

Example 2.22 (Review). *Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ in any interval.*

Proof. Verify that it is a solution. Consider then its Wronskian. □

2.2.1 Using a known solution to find another

Theorem 2.23 (Use of a known solution to find another). [1] *Consider $y'' + P(x)y' + Q(x)y = 0 \to (1)$. Assume $y_1(x)$ is a known non-zero solution of (1). Assume $y_2 = vy_1$ is another solution to (1), where $v = v(x)$. Then $y_2'' + P(x)y_2' + Q(x)y_2 = 0 \implies [vy_1]'' + P(x)[vy_1]' + Q(x)[vy_1] = 0$.*

$$v = \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$$

Example 2.24. *Show that $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution.*

Proof. $y_1' = 1, y_1'' = 0$, therefore we have

$$\begin{aligned} x^2y'' + xy' - y &= 0 \\ x^2(0) + x(1) - x &= 0 \end{aligned}$$

Therefore, it is a solution. Now we need to find y_2 another solution. Here in the standard form $P(x) = \frac{1}{x}, x \neq= 0$.
Now compute v

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{x^2}\frac{1}{x}\,dx \\ &= \int \frac{1}{x^3}\,dx \\ &= -\frac{1}{2x^2} \end{aligned}$$

So now we can say $y_2 = vy_1 = \frac{-1}{2x^2}x = \frac{-1}{2x}$.
Therefore, the general solution for the given differential solution is $y(x) = c_1y_1 + c_2y_2 = c_1x + c_2\frac{-1}{2x}$ in any interval not containing 0. □

[1] Just memorize this garbage.

Example 2.25. *Use the method of this section to find y_2 and the general solution of each of the following equations from the given solution y_1*

1. $y'' + y = 0, y_1 = \sin x$
2. $y'' - y = 0, y_1 = e^x$
3. $x^2y'' + xy' - 4y = 0, y_1 = x^2$

Proof. Consider 1) first, we have $P(x) = 0$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{\sin x^2} 1\,dx \\ &= -\cot x \end{aligned}$$

So now we can say $y_2 = vy_1 = -\cos x$. And the general solution is given by $y(x) = c_1 \sin x + c_2 \cos x$
Consider 2) now, we have again $P(x) = 0$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{e^{2x}}\,dx \\ &= \frac{-e^{-2x}}{2} \end{aligned}$$

Complete this up $\cdots$
Now consider 3), we have $P(x) = \frac{1}{x}$. We will now find v,

$$\begin{aligned} v &= \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ &= \int \frac{1}{x^4}\frac{1}{x}\,dx \\ &= \int \frac{1}{x^5}\,dx \\ &= \frac{-1}{4x^4} \end{aligned}$$

Now we can get $y_2 = vy_1 = \frac{-1}{4x^2}$. And get the general solution. $\square$

Example 2.26. *Show that $y_1 = x$ is a solution to $(1-x^2)y'' + 2xy' + 2y = 0$. Find the general solution.*

Proof. First make the DE into standard form by dividing by $1 - x^2$. So we have $P = \frac{-2x}{1-x^2}$. We will now compute v,

$$\begin{aligned} v &= \int \frac{1}{x^2} e^{-\int \frac{-2x}{1-x^2}\,dx}\,dx \\ &= \int \frac{1}{x^2}\frac{1}{x^2-1}\,dx \\ &= \int \frac{1}{x^4-x^2}\,dx \\ &= \frac{1}{x} + \frac{1}{2}\log(1-x) - \frac{1}{2}\log(x+1) \end{aligned}$$

Now we compute $y_2 = vy_1 = 1 + \frac{1}{2}\log(1-x) - \frac{1}{2}x\log(1+x)$ □

2.2.2 Homogenous equation with constants

Theorem 2.27. *If we have $y'' + P(x)y' + Q(x)y = 0$ and $P(x), Q(x)$ are constants. Consider the Auxiliary equation $m^2 + pm + q = 0$ and let the roots of the auxiliary equation be m_1, m_2.*

1. *If the roots are real and distinct ($m_1 \neq m_2$), $y = c_1e^{m_1x} + c_2e^{m_2x}$*
2. *If the roots are real and repeated ($m = m_1 = m_2$), $y = c_1e^{mx} + c_2xe^{mx}$*
3. *If the roots are complex ($m = \alpha \pm \beta i$), $y = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$*

Example 2.28. $y'' + 3y' - y = 0$

Proof. Assume e^{mx} is a solution to the following differential equation, then we have the following,

$$\begin{aligned} y'' + 3y' - y &= 0 \\ m^2e^{mx} + 3me^{mx} - e^{mx} &= 0 \end{aligned}$$

We can know solve for m

$$m = \frac{-3}{2} \pm \frac{\sqrt{13}}{2}$$

Complete this ... □

Example 2.29. *Find the general solution for each of the following equations:*

1. $y'' + y' - 6y = 0$

2. $y'' + 3y' + y = 0$

Proof. Consider 1) first, use the auxiliary equation that appeared out of a hat then,

$$\begin{aligned}
k^2 + pk + q &= 0 \\
k^2 + k - 6 &= 0 \\
k &= -3, k = 2
\end{aligned}$$

So now the general solution is $y = c_1 e^{-3x} + c_2 e^{2x}$

□

Example 2.30. *Solve* $2y'' + 5y' - 12y = 0$

Proof. Use the auxiliary equation,

$$2m^2 + 5m - 12 = 0$$

We get the values of $m = 3/4, -4$. These are real and distinct roots. So the general solution is

$$y = c_1 e^{4x} + c_2 e^{3/2x}$$

. □

2.3 Method of undetermined coefficients (UDC)

This is a method to solve non-homogenous linear DE of order 2.

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.2}$$

The second equation is just the associated homogenous equation of the first.
Let y_g be the general solution of eq. 2.2 be known as a complementary function (CF).
Let y_p be a particular solution of eq. 2.1 then it is known as particular integral.
You begin by computing y_g then compute y_p depending on one of the three stupid cases and their subcases [2]. Once you have the y_p you just add that (**without** a constant) to the y_g and you have the general solution for 2.1

[2]Just have to memorize it unfortunately. You want a intuitive reason for doing it? Cry more lmao.

2.3.1 Assumptions of UDC

1. $P(x), Q(x)$ are constants
2. $R(x)$ is either exponential, sine or cosine or polynomial.

2.3.2 Case 1: Exponential

Theorem 2.31.

$$y'' + py' + qy = e^{ax} \tag{2.3}$$

1. *If a is not a root of AE, then $y_p = Ae^{ax}$*
2. *If a is a simple root (i.e. multiplicity 1) of AE, then $y_p = Axe^{ax}$*
3. *If a is a double root of AE, then $y_p = Ax^2e^{ax}$*

Example 2.32. $y'' - 4y' + 4y = e^{2x}$

Proof. Here $a = 2$ and computing m we get a double root with $m = 2, 2$. □

2.3.2.1 Subcase 1: a is not a root of AE

If a is not a root of the AE then you take $y_p = Ae^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Ae^{ax} \\ y_p' &= Aae^{ax} \\ y_p'' &= Aa^2e^{ax} \end{aligned}$$

Plug this into the original DE then just append the y_p to y_g without constants for your general solution.

For subcase i, you can compute A directly as follows,

$$A = \frac{1}{a^2 + pa + q}$$

Example 2.33. $y'' - 5y' + 6y = e^{4x}$

Proof. We get $a = 4$ and this is not a root to the auxiliary equation ($m = 2, 3$).
So now,

$$\begin{aligned} y_p &= Ae^{4x} \\ y_p' &= 4Ae^{4x} \\ y_p'' &= 16Ae^{4x} \end{aligned}$$

Now plug this into the DE

$$\begin{aligned} y'' - 5y' + 6y &= e^{4x} \\ 16Ae^{4x} - 5(4Ae^{4x}) + 6(Ae^{4x}) &= e^{4x} \end{aligned}$$

So we get $A = 1/2$. □

2.3.2.2 Subcase 2: a is a simple root of AE

If a is a simple root of the AE then you take $y_p = Axe^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Axe^{ax} = Ae^{ax}(x) \\ y_p' &= Ae^{ax}(ax + 1) \\ y_p'' &= Ae^{ax}(a^2x + 2a) \end{aligned}$$

Now just plug these values into the original differential equation. You will get y_p just append that to y_g and you have your solution.

For subcase ii, you can compute A directly as follows,

$$A = \frac{1}{2a + p}$$

Example 2.34. *Solve* $y'' - 5y' + 6y = 3e^{2x}$

Proof. The AHE is $y'' - 5y' + 6y = 0$, we get the auxiliary equation as $m^2 - 5m + 6 = 0$ we get $m = 2, 3$. This is a simple root.

$$\begin{aligned} y_p &= Axe^{2x} \\ y_p' &= Ae^{2x} + 2Axe^{2x} \\ y_p'' &= 4Ae^{2x} + 4Axe^{2x} \end{aligned}$$

Now plug this into the DE,

$$
\begin{aligned}
y'' - 5y' + 6y &= 3e^{2x} \\
4Ae^{2x} + 4Axe^{2x} - 5(Ae^{2x} + 2Axe^{2x}) + 6(Axe^{2x}) &= 3e^{2x}
\end{aligned}
$$

We get $A = -3$. □

2.3.2.3 Subcase 3: a is a double root of AE

If a is a double root of the AE then you take $y_p = Ax^2e^{ax}$. So you have the following,

$$
\begin{aligned}
y_p &= Ax^2e^{ax} = Ae^{ax}(x^2) \\
y_p' &= Ae^{ax}(ax^2 + 2x) \\
y_p'' &= Ae^{ax}(a^2x^2 + 4ax + 2)
\end{aligned}
$$

Example 2.35 (Review)**.** *Find the general solution of* $y'' + 3y' - 10y = 6e^{4x}$

Proof. Consider the auxiliary equation,

$$m^2 + 3m - 10 = 0$$

We have $m = -5, 2$ real and distinct roots but a is not a root of AE. The solution to the AE is $y = c_1e^{2x} + c_2e^{-5x}$ So assume the general solution is given as,

$$
\begin{aligned}
y_p &= Ae^{4x} \\
y_p' &= 4Ae^{4x} \\
y_p'' &= 16Ae^{4x}
\end{aligned}
$$

Substitute this into the original equation to get $A = \frac{1}{3}$. So the general solution is given as $y = c_1e^{2x} + c_2e^{-5x} + \frac{1}{3}e^{4x}$

□

2.3.3 Case 2: Trigonometric

If $y'' + py' + qy = \sin kx$ or $\cos kx$

2.3.3.1 Subcase 1: If it is not a solution of AHE

If we have $\sin kx$ or $\cos kx$ not being a solution to the AHE we take $y_p = A\sin kx + B\cos kx$. Using this we get the following,

$$\begin{aligned}
y_p &= A\sin kx + B\cos kx \\
y_p' &= k(A\cos kx - B\cos kx) \\
y_p'' &= -k^2(A\sin(kx) + B\cos kx)
\end{aligned}$$

Plug this ugly mess into the original differential equation and you will get y_p append that without a constant to y_g for the general solution.

2.3.3.2 Subcase 2: If it is a solution of AHE

If we have $\sin kx$ or $\cos kx$ is a solution of the AHE then we take $y_p = x(A\sin kx + B\cos kx)$. Using this we get the following,

$$\begin{aligned}
y_p &= x(A\sin kx + B\cos kx) \\
y_p' &= A(\sin kx + kx\cos kx) + B(\cos kx - kx\sin kx) \\
y_p'' &= A(2k\cos kx - k^2x\sin kx) + B(-xk^2\cos kx - 2k\sin kx)
\end{aligned}$$

Example 2.36. *Solve* $y'' + y = \sin x$

Proof. AHE is given $y'' + y = 0$ and AE $m^2 + 1 = 0$ so $m = \pm i$.

$$y_g = c_1\cos x + c_2\sin x$$

Since $\sin x$ is a solution of y_p let

$$\begin{aligned}
y_p &= x(A\sin x + B\cos x) \\
y_p' &= A\sin x + B\cos x + x(A\cos x - B\sin x) \\
y_p'' &= A\cos x - B\sin x + (A\cos x - B\sin x) + x(-A\sin x - B\cos x)
\end{aligned}$$

Plug this into the diff eq.

$$y'' + y = \sin x$$

$$A\cos x - B\sin x + (A\cos x - B\sin x) + x(-A\sin x - B\cos x) + x(A\sin x + B\cos x) = \sin x$$

$2A = 0, B = \frac{-1}{2}$ so we have $y_p = -1/2x\cos x$ and then $y(x) = y_g + y_p = c_1\cos x + c_2\sin x - 1/2x\cos x$ $\square$

2.3.4 Case 3: Polynomial

For $y'' + py' + qy = a_0 + a_1x + \cdots + a_nx^n$

2.3.4.1 Subcase 1: If $\mathbf{q \neq 0}$

Take y_p as follows

$$\begin{aligned} y_p &= A_0 + A_1x + \cdots A_nx^n \\ y_p' &= A_1 + \cdots + nA_nx^{n-1} \\ y_p'' &= 2A_2 + \cdots + n(n-1)A_n^{n-2} \end{aligned}$$

2.3.4.2 Subcase 2: If $\mathbf{q = 0, p \neq 0}$

Take y_p as follows, the derivates are obvious,

$$y_p = x(A_0 + A_1x + \cdots A_nx^n)$$

2.3.4.3 Subcase 3: If $\mathbf{q = 0, p = 0}$

Take y_p as follows,

$$y_p = x^2(A_0 + A_x + \cdots + A_nx^n)$$

Example 2.37. *Find the general solution of* $y'' - y' - 2y = 4x^2$

Proof. AHE $y'' - y' - 2y = 0$ and AE $m^2 - m - 2 = 0$ so $m = 2, -1$ so $y_g = c_1e^{2x} + c_2e^{-x}$.
Let

$$\begin{aligned} y_p &= A_0 + A_1x + A_2x^2 \\ y_p' &= A_1 + 2A_2x \\ y_p'' &= 2A_2 \end{aligned}$$

Substitute this into the original DE

$$2A_2 - A_1 - 2A_2x - 2(A_0 + A_1x + A_2x^2) = 4x^2$$

We get $A_0 = -3, A_1 = 2, A_2 = -2$. So the general solution is $y(x) = c_1e^{2x} + c_2e^{-x} - 3 + 2x - 2x^2$ □

Example 2.38. *Find general solution of* $y'' - 2y' + 5y = 25x^2 + 12$

Proof. AHE $y'' - 2y' + 5y = 0$ and AE $m^2 - 2m + 5 = 0$ we get $m = 1 \pm 2i$.So we have $\alpha = 1, \beta = 2$.
So we get $y_g = e^x(c_1 \cos 2x + c_2 \sin 2x)$.
Let

$$\begin{aligned}
y_p &= A_0 + A_1 x + A_2 x^2 \\
y_p' &= A_1 + 2A_2 x \\
y_p'' &= 2A_2
\end{aligned}$$

Substitute this into the original DE

$$2A_2 - 2(A_1 + 2A_2 x) + 5(A_0 + A_1 x + A_2 x^2) = 25x^2 + 12$$

Upon solving we get $A_0 = 2, A_1 = 4, A_2 = 5$. So now we have $y(x) = y_g + y_p = e^x(c_1 \cos 2x + c_2 \sin 2x) + 2 + 4x + 5x^2$ □

Example 2.39 (Review)**.** *Solve* $y'' - 2y' = 12x - 10$

Proof. Begin with AHE $y'' - 2y' = 0$ then see HE $m^2 - 2m = 0$ so we got $m = 0, 2$ so me and my homies say $y_g = 1 + e^{2x}$.
Now since $q = 0$ we say that

$$\begin{aligned}
y_p &= A_1 x^2 + A_0 x \\
y_p' &= 2A_1 x + A_0 \\
y_p'' &= 2A_1
\end{aligned}$$

Now plug this back into the original differential equation.

$$2A_1 - 2(2A_1 x + A_0) = 12x - 10$$

Upon solving, a smart dog or a slow student would see that $A_0 = 2, A_1 = -3$. This gives us $y_p = -3x^2 + 2x$. And as such our final solution is indubitably,

$$\begin{aligned}
y(x) &= y_g + y_p \\
&= c_1 + c_2(e^{2x}) + (-3x^2 + 2x)
\end{aligned}$$

□

2.4 Variation of Parameters (VOP) method

Consider the non-homogenous equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

Consider its AHE. Say it has two solutions y_1, y_2. So that $y_g = c_1y_1 + c_2y_2$. We will say that the required particular solution y_p is a combination of y_g as follows,

$$y_p = v_1y_1 + v_2y_2$$

Now our job is just to get v_1, v_2 such that

$$y_p(x) = v_1y_1 + v_2y_2$$

Now begin differentiating this by repeated use of the product rule [3].

$$\begin{aligned} y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \\ y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2 \end{aligned}$$

We also now choose $v_1'y_1 + v_2'y_2 = 0$ to simplify the derivatives [4], Now with those terms we can restate the original differential equation as follows,

$$\begin{aligned} y_p' &= v_1y_1' + v_2y_2' \\ y_p'' &= v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2' \end{aligned}$$

Substitute these into the original differential equation and you get,

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1'y_1' + v_2'y_2' = R(x)$$

Since y_1, y_2 are solutions to the AHE the terms in the parentheses vanish and we are left with

$$v_1'y_1' + v_2'y_2' = R(x)$$

Recall from footnote 4 that we now have a system of equations as follows,

$$\begin{aligned} v_1' + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= R(x) \end{aligned}$$

Now upon solving we get

$$v_1' = \frac{-y_2R(x)}{W(y_1, y_2)} \qquad \text{and} \qquad v_2' = \frac{y_1R(x)}{W(y_1, y_2)}$$

$$v_1 = -\int \frac{y_2R(x)}{W(y_1, y_2)}\,dx \qquad \text{and} \qquad v_2 = \int \frac{R(x)y_1}{W(y_1, y_2)}\,dx$$

[3] See why this is so boring yet?

[4] lmao

Example 2.40. *Find a particular solution of $y'' + y = \csc x$*

Proof. Find the AHE $y'' + y = 0$ the solution for this is

$$y_p = c_1 \cos x + c_2 \sin x$$

. We have $R(x) = \csc x$ and we also know that $W(y_1, y_2) = 1$
So we got

$$v_1 = \int \frac{-\sin x(\csc x)}{1}\, dx == x$$
$$v_2 = \int \frac{\cos x \sec x}{1}\, dx = \log(|\sin x|)$$

□

Example 2.41. *Find a particular solution to $y'' - 2y' + y = 2x$*

Proof. Take the AHE $y'' - 2y' + y = 0$ and the HE $m^2 - 2m + 1 = 0$ so we get $y_p = c_1 e^x + c_2 x e^x$.
The Wronskian is equal to $W(y_1, y_2) = e^{2x}$.
Now do some magic and get

$$v_2 = \int \frac{2xe^x}{e^{2x}}\, dx = -2e^{-x}(x+1)$$

and

$$v_1 = \int \frac{-2x^2 e^x}{e^{2x}}\, dx = 2e^{-x}(x^2 + 2x + 2)$$

So the final solution is given as follows

$$\begin{aligned} y &= v_1 y_1 + v_2 y_2 \\ &= 2e^{-x}(x^2 + 2x + 2)(e^x) + (-2e^{-x}(x+1))(xe^x) \\ &= 2x + 4 \end{aligned}$$

□

Example 2.42. *Find a particular solution of $y'' - y' - 6y = e^x$ with UDC then with VOP*

Proof. First do with UDC find the AHE and HE $m^2 - m - 6 = 0$ so we get $m = -2, m = 3$ and the $y_g = c_1 e^{-2x} + c_2 e^{3x}$. A is not a root so we go

$$\begin{aligned} y_p &= Ae^x \\ y_p' &= Ae^x \\ y_p'' &= Ae^x \end{aligned}$$

Plug thing into the original differential equation

$$Ae^x - Ae^x - 6Ae^x = e^x$$

we got $A = -1/6$. So the general solution is $y = e^{-2x} + e^{3x} - 1/6e^x$.
Now do VOP, from AHE and HE we get $y_g = c_1e^{-2x} + c_2e^{3x}$. We also now got the Wronskian as $5e^x$. **Complete this later...**

content...

□

Example 2.43. *Find a particular solution for each of the following equations*

1. $y'' + 4y = \tan 2x$
2. $y'' + 2y' + y = e^{-x} \log x$

Example 2.44 (Review)**.** *Solve $y'' - 2y' - 3y = 64xe^{-x}$ using VOP and UDC.*

Proof. Solving with VOP first. Consider its AHE and its HE $m^2 - 2m - 3 = 0$ we get $m = -1, 3$ so $y_g = c_1e^{-1} + c_2e^3$.
The Wronskian is $-4e^{2x}$. We now find v_1, v_2 as follows,

$$\begin{aligned} v_1 &= -\int \frac{y_2R(x)}{W(y_1, y_2)}\, dx \\ &= -\int \frac{-e^{-x}64xe^{-x}}{-4e^{2x}}\, dx \\ &= e^{-4x}(4x + 1) \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{R(x)y_1}{W(y_1, y_2)}\, dx \\ &= \int \frac{e^{3x}64xe^{-x}}{-4e^{2x}}\, dx \\ &= -8x^2 \end{aligned}$$

So the particular solution is given by,

$$y_p = v_1y_1 + v_2y_2 = -e^{-4x}(4x + 1)e^{3x} - 8x^2(e^{-x})$$

Then use this for the general solution.
Now do it with UDC.

If we just had x we would use $y_{p_1} = \alpha_0 + \alpha_1 x$. If we just had e^{-x} we would use $y_{p_2} = \beta x e^{-x}$.
We will use $y_{p_1} y_{p_2} = (\alpha_0+\alpha_1 x)(\beta x e^{-x}) = \alpha_0 \beta x e^{-x} + \alpha_1 \beta x^2 e^{-x}$ this simplifies to the following trial solution,

$$y_p = A_0 x e^{-x} + A_1 x^2 e^{-x}$$

Differentiate it and get the following,

$$\begin{aligned} y_p &= A_0 x e^{-x} + A_1 x^2 e^{-x} \\ y_p' &= e^{-x}[A_0(1-x) - A_1(x-2)x] \\ y_p'' &= e^{-x}[A_0(x-2) + A_1(x^2-4x+2)] \end{aligned}$$

Plug this behemoth into the original differential equation,

$$e^{-x}[A_0(x-2) + A_1(x^2-4x+2)] - 2(e^{-x}[A_0(1-x) - A_1(x-2)x]) - 3(A_0 x e^{-x} + A_1 x^2 e^{-x}) = 64xe^{-x}$$

Simplify this $A_0 = -4, A_1 = -8$
so $y_p = -4xe^{-x} - 8x^2 e^{-x}$ □

2.5 Higher order linear equations

Definition 2.45. *content...*

Example 2.46. *Solve* $y^{(4)} - 5y'' + 4y = 0$.

Proof. The auxiliary equation is $m^4 - 5m^2 + 4 = 0$. Substitute $m^2 = p$ so we got $p^2 - 5p + 4 = 0$ so $p = 4, 1$ and then $m = \pm 2, \pm 1$ General solution is then $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^x + c_4 e^{-x}$ □

Example 2.47. *Solve* $y^{(4)} - 8y'' + 16y = 0$

Proof. Auxiliary equation is $m^2 - 8m + 16 = 0$ substitute $p^2 = m$ and solve we get $m = \pm 2, \pm 2$ so general solution is $y = (c_1 e^{2x} + c_2 x e^{2x}) + (c_2 e^{-2x} + c_4 x e^{-2x})$ □

Proof. Solving $y^{(4)} + 2y''' + 2y'' - 2y' + y = 0$ □

Proof. Take the auxiliary equation $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$. We got the thing as $m = 1, 1, \pm i$. So the general solution is given by

$$y = (c_1 e^x + c_2 x e^x) + (c_3 \cos x + c_4 \sin x)$$

□

Example 2.48. *Solve* $y''' + 2y'' - y' = 3x^2 - 2x + 1$

Proof. AHE and HE so auxiliary equation is $m^3 + 2m^2 - m = 0$ we get $m = 0, -1 \pm \sqrt{2}$ □

Example 2.49 (Review-9)**.** *Find the general solution of* $y'' - 3y'' + 2y' = 0$.

Proof. Consider the AHE and HE, $m^3 - 3m^2 + 2m = 0$ we get the solutions as $m = 0, 1, 2$. So the general solution is just given by

$$y = c_1 + c_2 e^x + c_3 e^{2x}$$

□

2.6 Operator method

- Consider the differential equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = R(x)$
- Using the differential operator D, we can re write it as $D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = R(x)$
- $\implies$ $p(D)y = R(x)$ where $p(m)$ is called the axuliary polynomial and $p(m) = (m - m_1)(m - m_2) \cdot (m - m_n)$ where m_i are roots of the auxiliary equation.

Example 2.50. $p(D)y = R(x) \implies y = \frac{1}{p(D)} R(x)$

Proof.

$$\begin{aligned} y &= \frac{1}{D} R(x) \\ &= \int R(x)\, dx \end{aligned}$$

□

Example 2.51. $(D - m)y = R(x) \implies y = \frac{1}{(D-m)} R(x)$

Proof.

$$y = e^{mx} \int e^{-mx} R(x)\, dx \tag{2.1}$$

$$\frac{1}{D - m} R(x) = e^{mx} \int e^{-mx} R(x)\, dx \tag{2.2}$$

□

2.6.1 Successive integration

$$y = \frac{1}{p(D)} R(x) = \frac{1}{[(D - m_1)(D - m_2) \cdots (D - m_n)]} R(x)$$

Using 2.2 successively we get the particular solution.

Example 2.52. *Find a particular solution of* $y'' - 3y' + 2y = xe^x$

Proof.

$$\begin{aligned} (D^2 - 3D + 2)y &= xe^x \\ (D - 1)(D - 2)y &= xe^x \\ y &= \frac{1}{D-1}\left[\frac{1}{D-2} xe^x\right] \end{aligned}$$

Here $R(x) = xe^x, m_1 = 2$. So we do

$$\frac{1}{D-2} xe^x = e^{2x} \int e^{-2x} xe^x \, dx = -(1+x)e^x$$

Step 2,

$$\begin{aligned} y &= \frac{1}{D-1}[-(1+x)e^x] \\ \frac{1}{D-1}[(1+x)e^x] &= e^x \int e^{-x}(1+x)e^x \, dx \\ &= e^x \, (....) \end{aligned}$$

□

2.6.2 Partial fraction decomposition

$$p(D)y = R(x) \implies y = \frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)}$$

Use partial fractions... **complete this...**

Example 2.53. *Find a particular solution of* $y'' - 3y' + 2y = xe^x$ *using partial fraction decomposition.*

Proof.

$$
\begin{aligned}
(D^2 - 3D + 2)y &= xe^x \\
\implies y &= \frac{1}{(D-1)(D-2)} xe^x \\
y &= \left(\frac{-1}{D-1} + \frac{1}{D-2} \right) xe^x
\end{aligned}
$$

i am not typing this out lmao □

Example 2.54.

$$
\begin{aligned}
\frac{1}{1-r} &= 1 + r + r^2 + r^3 + \cdots, \ \textit{if} |r| < 1 \\
\frac{1}{1+r} &= 1 - r + r^2 - r^3 + \cdots, \ \textit{if} |r| < 1
\end{aligned}
$$

Example 2.55 (Review-10)**.** *Find a particular solution of $y'' - 4y = e^{2x}$ by using methods 1 and 2.*

Proof. Consider the eq as follows,

$$D^2 y - 4Dy = e^{2x}$$

□

2.6.3 Series expansion

Used when $R(x)$ is a polynomial.

$$y = \frac{1}{p(D)} R(x) = [1 + b_1 D + b_2 D^2 + \cdots] R(x)$$

higher order derivates vanish

Example 2.56. *Find particular solution of $y''' - 2y'' + y' = x^4 + 2x + 5$*

Proof.

$$
\begin{aligned}
&D^3 - 2D^2 + Dy = x^4 + 2x + 5 \\
&\implies y = \frac{1}{1 - (2D^2 - D^3)} R(x) \quad = \left[1 + (2D^2 - D^3) + (2D^2 - D^3)^2 + (2D^2 - D^3)^3 + \cdots \right] R(x \\
&= \left[1 + 2D^2 - D^{+} 4D^4\right]
\end{aligned}
$$

complete later... □

2.6.4 Exponential shift rule

If $R(x) = e^{kx}g(x)$

$$y = \frac{1}{p(D)}e^{kx}g(x)$$
$$y = e^{kx}\left[\frac{1}{p(D+k)}g(x)\right]$$

Example 2.57. *Find a particular solution of $y'' - 3y' + 2y = xe^x$*

Proof.

$$\begin{aligned}(D^2 - 3D + 2)y &= e^x x \\ y &= \frac{1}{p(D)}e^x x \\ &= e^x \frac{1}{p(D)} x \\ &= e^x \frac{1}{p(D+1)} x\end{aligned}$$

magick

$$\begin{aligned}&= -e^x \frac{1}{1-D}\frac{x^2}{2} \\ &= \frac{-e^x}{2}[1 + D + D^2 + \cdots]\end{aligned}$$

□

Example 2.58. *Find a particular solution of $y'' - y = x^2e^{2x}$ using methods 1,2,4 then find general solution.*

Proof.

$$\begin{aligned}(D^2 - 1)y &= x^2e^{2x} \\ y &= \frac{1}{D^2-1}x^2e^{2x} \\ &= \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)x^2e^{2x} \\ &= \frac{1}{2}(-2 - 2D^2 - 2D^4 + \cdots)x^2e^{2x}\end{aligned}$$

□

Example 2.59. *Find a particular solution of $y'' - y' + y = x^3 - 3x^2 + 1$*

Proof.

$$
\begin{aligned}
(D^2 - D + 1)y &= x^3 - 3x^2 + 1 \\
y &= \frac{1}{D^2 - D + 1} x^3 - 3x^2 + 1 \\
&= (1 + D - D^3 - D^4 + \cdots)x^3 - 3x^2 + 1 \\
&= (x^3 - 3x^2 + 1) + (3x^2 - 6x) - 6 \\
&= x^3 - 6x - 5
\end{aligned}
$$

□

3

Linear systems of ordinary differential equations

"I love differential equations. It is so fun. It has so many real life applications."

– Someone I don't like

3.1 Linear system of DE

We are mainly concerned with first order linear system of ODEs.

The general system of linear differential equations is given by the following,

content...

Observe that the single n^{th} order equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \tag{3.1}$$

Is in fact equivalent to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, y_2, \dots, y_n) \end{aligned}$$

Example 3.1. *$y'' = x^2y' + xy$ make system.*

Proof. $n = 2$ so take $f = x^2y' + xy$. So the system is given as,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= x^2y_2 + xy_1 \end{aligned}$$

□

Theorem 3.2 (Existence and uniqueness theorem for general system of linear differential equations)**.** *Let the functions $f_1, f_2, \dots, f_n$ and the partial derivatives $\partial f_1/\partial y_1, \dots, \partial f_1/\partial y_n, \dots \partial f_n/\partial y_1, \dots, \partial f_n/\partial y_n$ be continuous in a region R of $(x, y_1, y_2, \dots, y_n)$ space. If $(x_0, a_1, a_2, \dots, a_n)$ is an internet point of R, then the system has an unique solution $y_1(x), y_2(x), \dots, y_n(x)$ that satisfies the initial conditions (5).*

Theorem 3.3 (Existence and uniqueness theorem for 3.1)**.** *Let the function f and the partial derivatives $\partial f/\partial y, \partial f/\partial y', \dots, \partial f/\partial y^{(n-1)}$ be continuous in a region R of $(x, y, y', \dots, y^{(n-1)})$ space. If $(x_0, a_1, a_2, \dots, a_n)$ is an interior point of R, then equation 3.1 has a unique solution $y(x)$ that satisfies the initial conditions $y(x_0) = a, y'(x_0) = a_2, \dots, y^{(n-1)}(x_0) = a_n$.*

Example 3.4. *Replace each of the following differential equations by an equivalent system of first order equations,*

- $2y^{(4)} + xy'' + e^x y' - y\sin(x) = 0$
- *bleh*

Proof. Consider the first one,
The equation is as follows,

$$y^{(4)} = \frac{1}{2}\left(y\sin x - y'(e^x) - xy''\right)$$

System is then,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= \frac{1}{2}\left(y_1\sin x - y_2e^x - xy_3\right) \end{aligned}$$

□

Example 3.5 (Review-1)**.** *Replace the following differential equation by an equivalent system of first order equations*

$$2y^{(4)} - xy''' + x^2(y'')^3 - yy' = 7$$

Proof. Consider that $n = 4$ and $f = \frac{1}{2}\left(7 + xy''' - x^2(y'')^3 + yy'\right)$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= \frac{1}{2}\left(7 + xy_4 - x^2y_3^2 - y_1y_2\right) \end{aligned}$$

□

We will only see systems of two first order equations in two unknown functions of the following form,

$$\begin{aligned} \frac{dx}{dt} &= F(t, x, y) \\ \frac{dy}{dt} &= G(t, x, y) \end{aligned}$$

More specifically we have **linear** systems of the form,

Definition 3.6 (Linear system of two ODE)**.**

$$\begin{aligned} \frac{dx}{dt} &= a_1(t) + b_1(t) + f_1(t) \\ \frac{dy}{dt} &= a_2(t) + b_2(t) + f_2(t) \end{aligned}$$

Definition 3.7 (Homogenous linear system of two ODE)**.**

$$\begin{aligned} \frac{dx}{dt} &= a_1(t) + b_1(t) \\ \frac{dy}{dt} &= a_2(t) + b_2(t) \end{aligned}$$

We assume that $a_i(t), b_i(t), f_i(t)$ for $i = 1, 2$ are continuous on some closed interval $[a, b]$ on the t-axis.
If $f_i(t)$ are both identically zero, then the system is called homogenous else.
A solution of 3.6 is of the following form,

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

Example 3.8 (Review-2). *Solution for the system*

$$\begin{cases} \frac{dx}{dt} = 4x - t^2 y \\ \frac{dy}{dt} = 2x + y \end{cases}$$

1. $x = e^{3t}, y = -e^{-3t}$
2.

3.2 Existence and uniqueness theorems

Theorem 3.9. *If t_0 is any point of the interval $[a, b]$ and if x_0 and y_0 are any numbers then def. 3.6 has one and only one solution*

$$x = x(t)$$
$$y = y(t)$$

valid throughout $[a, b]$, such that $x(t_0) = x_0, y(t_0) = y_0$.

3.3 Homogenous linear system of ODE in two variables

Now consider the system of linear homogenous equations (def 3.7).

Theorem 3.10. *If the homogenous system (def. 3.7) has two solutions on the interval $[a, b]$*

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

then we also have another solution of the form

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$

for any constants c_1, c_2.

3.4 Wronskian of homogenous linear system of ODE

Definition 3.11 (Wronskian).

$$W(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix}$$

3.5 General solution of Homogenous linear system of ODE in two variables

Theorem 3.12. *If the two solutions for the homogenous system 3.7 have a Wronskian that does not vanish on* $[a, b]$ *then its linear combination of the solutions as described in theorem 3.10 is the general solution of the homogenous system 3.7 on that interval.*

Theorem 3.13. *If* $W(t)$ *is the Wronskian of two solutions of the homogenous system then* $W(t)$ *is either identically zero or nowhere zero on* $[a, b]$.

Theorem 3.14. *If the two solutions of the homogeneous system are linearly independent then the linear combination 3.10 is its general solution.*

Lemma 3.15. W *is never zero* $\iff$ *the solutions are L.I.*

Example 3.16 (Review-2 again?)**.** *The Wronskian of the two solutions* $x_1 = e^{2t}, y_1 = 2e^{3t}$ *and* $x_2 = e^{3t}, y_2 = 3e^{2t}$ *is zero.*

Example 3.17. *Find wronskian*

$$\begin{cases} x_1 = 2e^{4t} \\ y_1 = 3e^{4t} \end{cases} \quad \begin{cases} x_2 = e^{-t} \\ y_2 = -e^{-t} \end{cases}$$

and check if L.I.

Proof. The Wronskian is $-2e^{3t} - 3e^{3t} = -5e^{3t}$ so the solutions are L.I. □

3.6 Non-homogenous linear system in two variables

Theorem 3.18. *If the two solutions for the homogenous system (Th. 3.10) are linearly independent on* $[a, b]$ *and if*

$$\begin{cases} x &= x_p(t) \\ y &= y_p(t) \end{cases}$$

is any particular solution of the non-homogenous linear system of ODEs (def: 3.6) then

$$\begin{cases} x &= c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y &= c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$$

is the general solution of the non homogenous system 3.6.

Example 3.19. *Show that*

$$\begin{cases} x_1 = 2e^{4t} \\ y_1 = 3e^{4t} \end{cases} \quad \begin{cases} x_2 = e^{-t} \\ y_2 = -e^{-t} \end{cases}$$

satisfies the following homogeneous system,

$$\begin{cases} \dot{x} & = x + 2y \\ \dot{y} & = 3x + 2y \end{cases}$$

If we then have the non-homogenous system

$$\begin{cases} \dot{x} & = x + 2y + t - 1 \\ \dot{y} & = 3x + 2y - 5t - 2 \end{cases}$$

show that the following is the particular solution,

$$\begin{cases} x_p = 3t - 2 \\ y_p = -2t + 3 \end{cases}$$

Proof. Consider x_1, y_1 first verify that it satisfies it,

$$\begin{aligned} \dot{x} &= x + 2y \\ 8e^{4t} &= 2e^{4t} + 6e^{4t} \\ \dot{y} &= 3x + 2y \\ 12e^{4t} &= 6e^{4t} + 6e^{4t} \end{aligned}$$

now check x_2, y_2

$$\begin{aligned} \dot{x} &= x + 2y \\ -e^{-t} &= e^{-t} - 2e^{-t} \\ \dot{y} &= 3x + 2y \\ e^{-t} &= 3e^{-t} - 2e^{-t} \end{aligned}$$

So we have verified that they are both solutions. And since they are linearly independent we have their general solution as follows,

$$y_g = \begin{cases} x_g = c_1 2e^{4t} + c_2 e^{-t} \\ y_g = c_1 3e^{4t} + c_2(-e^{-t}) \end{cases}$$

Consider the non-homogenous system now and lets verify it is a solution,

$$\begin{aligned} \dot{x} &= x + 2y + t - 1 \\ 3 &= 3t - 2 - 4t + 6 + t - 1 = 3 \\ \dot{y} &= 3x + 2y - 5t - 2 \\ -2 &= 9t - 6 - 4t + 6 - 5t - 2 = -2 \end{aligned}$$

So our particular solution is verified. And now we have the general solution to be as follows,

$$\begin{cases} x = c_1 2e^{4t} + c_2 e^{-t} + (3t - 2) \\ y = c_1 3e^{4t} + c_2(-e^{-t}) + (-2t + 3) \end{cases}$$

□

Example 3.20 (Problem)**.** *a) Show that,*

$$\begin{cases} x_1 = e^{4t} \\ y_1 = e^{4t} \end{cases} \quad \begin{cases} x_2 = e^{-2t} \\ y_2 = -e^{-2t} \end{cases}$$

are solutions to the homogenous system

$$\begin{cases} \dot{x} = x + 3y \\ \dot{y} = 3x + y \end{cases}$$

b) Show in two ways that the given solutions are linearly independent on every closed interval and write the general solution of this system. c) Find the particular solution $x(0) = 5, y(0) = 1$

Proof. Consider the first set of solutions first,

$$\begin{aligned} \dot{x} &= x + 3y \\ 4e^{4t} &= e^{4t} + 3e^{4t} \end{aligned}$$

... complete part a) its easy
For part b) now begin by computing its wronskian,

$$W = -e^{4t}e^{-2t} - e^{-2t}e^{4t} = -e^{-2t}$$

Also not that there does not exist any constant k such that $x_1 = kx_2, y_1 = ky_2$ so its also L.I. So we have the general solution to be

$$\begin{cases} x_g = c_1 e^{4t} + c_2 e^{-2t} \\ y_g = c_1 e^{4t} + c_2(-e^{-2t}) \end{cases}$$

For part c) consider the following,

$$c_1 + c_2 = 5$$
$$c_1 - c_2 = 1$$

So $c_1 = 3, c_2 = 2$ this forms the particular solution. $\square$

3.7 Homogenous linear systems with constant coefficients

Technically this section doesn't seem to be in her syllabus but I'm pretty sure its gonna be covered so I've done it anyway.
In this section we will examine the following system of linear ODEs,

$$\begin{cases} \frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y \end{cases} \tag{3.1}$$

where a_i, b_i are constants.
Assume $x = Ae^{mt}, y = Be^{mt}$ substitute into the system and we get

$$\begin{cases} Ame^{mt} = a_1 Ae^{mt} + b_1 Be^{mt} \\ Bme^{mt} = a_2 Ae^{mt} + b_2 Be^{mt} \end{cases}$$

Dividing by e^{mt} we get

$$(a_1 - m)A + b_1 B = 0$$
$$a_2 A + (b_2 - m)B = 0$$

We want non trivial so we require zero determinant,

$$\begin{bmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{bmatrix} = 0 \implies (a_1 - m)(b_2 - m) - a_2 b_1 = 0$$

We are concerned with the following auxiliary equation that is related to this system,

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \tag{3.2}$$

and the solutions are

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases} \quad \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$$

Example 3.21. *Find the auxiliary equation for the following homogenous system with constant coefficients,*

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = 4x - 2y \end{cases}$$

Proof. The auxiliary equation is given by,

$$\begin{aligned} m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) &= 0 \\ m^2 + m - 6 &= 0 \end{aligned}$$

□

Example 3.22. *Find Auxiliary equation of*

$$\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 8x - 6y \end{cases}$$

Proof. The auxiliary equation is given by,

$$\begin{aligned} m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) &= 0 \\ m^2 + 2m &= 0 \end{aligned}$$

So we have $m_1 = -2, m_2 = 0$. □

Example 3.23 (Review=4)**.** *Find auxiliary equation for the following system*

$$\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$$

Proof. Auxiliary equation is $m^2 - 6m + 18 = 0$. □

3.7.1 Distinct real roots

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases} \quad \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$$

Consider its Wronskian as $(A_1 B_2 - A_2 B_1) e^{(m_1 + m_2)t}$.

If eq. 3.2 has distinct real roots m_1, m_2 then the general solution of eq. 3.1 is given as,

$$\begin{cases} x = c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y = c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{cases}$$

Example 3.24.

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$$

Proof. The auxiliary polynomial is given $m^2 + m - 6 = 0$ with roots $m_1 = -3, m_2 = 2$. So the general solution is given as

$$\begin{cases} x = c_1 A_1 e^{-3t} + c_2 A_2 e^{2t} \\ y = c_1 B_1 e^{-3t} + c_2 B_2 e^{2t} \end{cases}$$

Also consider that

$$\begin{cases} (a_1 - m)A + b_1 B = 0 \\ a_2 A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (1 - m)A + B = 0 \\ 4A + (-2 - m)B = 0 \end{cases}$$

Substitute $m = -3$ first

$$\begin{cases} 4A + B = 0 \\ 4A + B = 0 \end{cases} \implies 4A + B = 0$$

So let $A_1 = 1, B_1 = -4$

Substitute $m = 2$ and we will get $A = B$ $A_2 = 1, B_2 = 1$ Compute the Wronskian as a sanity check $W = e^{-t} + 4e^{-t} = 5e^{-t}$

$$\begin{cases} x = c_1 e^{-3t} + c_2 e^{2t} \\ y = c_1(-4e^{-3t}) + c_2 e^{2t} \end{cases}$$

□

Example 3.25.

$$\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = -2x + 3y \end{cases}$$

Proof. The auxiliary equation is ... roots are $-1, 1$. So the general solution is given as

$$\begin{cases} x = c_1 A_1 e^{-t} + c_2 A_2 e^{t} \\ y = c_1 B_1 e^{-t} + c_2 B_2 e^{t} \end{cases}$$

Also consider that

$$\begin{cases} (a_1 - m)A + b_1 B = 0 \\ a_2 A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (-3 - m)A + 4B = 0 \\ -2A + (3 - m)B = 0 \end{cases}$$

Substitute $m = -1$

$$\begin{cases} (-3-m)A + 4B = 0 \\ -2A + (3-m)B = 0 \end{cases}$$

We get $A = 2B$ so pick $A_1 = 2, B_2 = 1$

Substitute $m = 1$, we get $A_2 = 1, B_2 = 1$ □

Example 3.26.

$$\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 8x - 6y \end{cases}$$

Proof. The roots of aux eq are $m_1 = 0, m_2 = -2$,

$$\begin{cases} x = c_1A_1 + c_2A_2e^{-2t} \\ y = c_1B_1 + c_2B_2e^{-2t} \end{cases}$$

Also consider that

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (4-m)A - 3B = 0 \\ 8A + (-6-m)B = 0 \end{cases}$$

Substitute $m = 0$ we get $A_1 = 3, B_1 = 4$ Substitute $m = -2$ we get $A_2 = 1, B_2 = 1$ □

3.7.2 Equal real root

$$\begin{cases} x_1 = Ae^{mt} \\ y_1 = Be^{mt} \end{cases} \quad \begin{cases} x_2 = (A_1 + A_2t)e^{mt} \\ y_2 = (B_1 + B_2t)e^{mt} \end{cases}$$

If eq. 3.2 has equal real roots $m = m_1 = m_2$ then the general solution of eq. 3.1 is given as,

$$\begin{cases} x &= c_1Ae^{mt} + c_2(A_1 + A_2t)e^{mt} \\ y &= c_1Be^{mt} + c_2(B_1 + B_2t)e^{mt} \end{cases}$$

Example 3.27.

$$\begin{cases} \frac{dx}{dt} = 3x - 4y \\ \frac{dy}{dt} = x - y \end{cases}$$

Proof. AE is $m^2 - 2m + 1 = 0$ and we get $m = 1, 1$.

$$\begin{cases}(3-m)A - 4B = 0\\ A + (-1-m)B = 0\end{cases}$$

Solving with $m = 1$ we get

$$\begin{cases}2A - 4B = 0\\ A - 2B = 0\end{cases}$$

We get $B = 1, A = 2$ this is the first solution.

$$\begin{cases}x_1 = 2e^t\\ y_2 = e^t\end{cases}$$

We know the second solution should be of the form

$$\begin{cases}x_2 = (A_1 + A_2 t)e^t\\ y_2 = (B_1 + B_2 t)e^t\end{cases}$$

Assume this is the solution and substitute it in the original differential equation

$$\begin{cases}(2A_2 - 4B_2)t + (2A_1 - A_2 - 4B_1) = 0\\ (A_2 - 2B_2)t + (A_1 - 2B_1 - B_2) = 0\end{cases}$$

We get

$$\begin{cases}2A_2 - 4B_2 = 0 \text{ and } 2A_1 - A_2 - 4B_1 = 0\\ A_2 - 2B_2 = 0 \text{ and } A_1 - 2B_1 - B_2 = 0\end{cases}$$

We get $A_2 = 2B_2$ and $A_1 - 2B_1 - B_2 = 0$ So pick $A_2 = 2, B_2 = 1$ and $A_1 - 2B_1 = 2$ and $A_1 = 1, B_1 = 0$

So finally,

$$\begin{cases}x_2 = (1 + 2t)e^t\\ y_2 = (0 + t)e^t\end{cases}$$

consider the Wronskian as $2e^t(te^t) - (1 - 2t)e^t e^2 = -e^{2t}$. So they are L.I. So we have the general solution as its linear combination, □

Example 3.28.

$$\begin{cases}\frac{dx}{dt} = 5x + 4y\\ \frac{dy}{dt} = -x + y\end{cases}$$

Proof. The auxiliary equation is given as $m^2 - (a_1+b_2)m+(a_1b_2-a_2b_1) = 0$ so $m^2 - 6m + 9 = 0$ which has repeated roots $m = 3, 3$.

Also consider that

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} 2A + 4B = 0 \\ -A - 2B = 0 \end{cases}$$

Which on solving gives $A = -1, B = 2$ So we have,

$$\begin{cases} x_1 = 2e^{3t} \\ y_2 = -e^{3t} \end{cases}$$

We know the second solution should be of the form

$$\begin{cases} x_2 = (A_1 + A_2t)e^t \\ y_2 = (B_1 + B_2t)e^t \end{cases}$$

Assume this is the solution and substitute it in the original differential equation

$$\begin{cases} e^t(A_2t + A_1 + A_2) = 5((A_1 + A_2t)e^t) + 4((B_1 + B_2t)e^t) \\ e^t(B_2t + B_1 + B_1) = -(A_1 + A_2t)e^t + (B_1 + B_2t)e^t \end{cases}$$

We get ??????

$$\begin{cases} x_2 = (-3 - 2t)e^{3t} \\ y_2 = (1 + t)e^{3t} \end{cases}$$

□

3.7.3 Distinct complex roots

If $m_1 = a + ib, m_2 = a - ib$ we will have

$$\begin{cases} x_1 = e^{at}(A_1 \cos bt - A_2 \sin bt) \\ y_1 = e^{at}(B_1 \cos bt - B_2 \sin bt) \end{cases} \quad \begin{cases} x_2 = e^{at}(A_1 \cos bt + A_2 \sin bt) \\ y_2 = e^{at}(B_1 \cos bt + B_2 \sin bt) \end{cases}$$

If eq. 3.2 has distinct complex roots $a \pm ib$ then the general solution of eq. 3.1 is given as,

$$\begin{cases} x &= e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y &= e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)] \end{cases}$$

Example 3.29.
$$\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$$

Proof. Consider the auxiliary equation associated to the above system,

$$\begin{aligned} m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0 \\ m^2 - (4 + 2)m + (4 \cdot 2 + 2 \cdot 5) = 0 \\ m^2 - 6m + 18 = 0 \end{aligned}$$

This has roots $m_1 = 3 - 3i, m_2 = 3 + 3i$. These are complex roots with $a = 3, b = 3$, so we know the general solution will be of the form,

$$\begin{cases} x &= e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y &= e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)] \end{cases}$$

Also consider that,

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (4 - m)A - 2B = 0 \\ 5A + (2 - m)B = 0 \end{cases}$$

Begin by substituting $m = 3 - 3i$

$$\begin{cases} (4 - (3 + 3i))A - 2B = 0 \\ 5A + (2 - (3 + 3i))B = 0 \end{cases}$$

Which upon solving gives $(1 + 3i)B = 5A$ using this we can say $B = 1, A = \frac{1+3i}{5}$ and as such $A_1 = \frac{1}{5}, A_2 = \frac{3}{5}$ and $B_1 = 1, B_2 = 0$ So our general solution is,

$$\begin{cases} x &= e^{3t}[c_1\left(\frac{1}{5}\cos 3t - \frac{3}{5}\sin 3t\right) + c_2\left(\frac{1}{5}\sin 3t + \frac{3}{5}\cos 3t\right)] \\ y &= e^{3t}[c_1 \cos 3t + c_2 \sin 3t] \end{cases}$$

□

Example 3.30. $\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 4x + 5y \end{cases}$

Proof. AE $m^2 - 6m + 13 = 0$ so $m = 3 \pm 2i$

$$\begin{cases} (-2 - 2i)A - 2B = 0 \\ 4A + (2 - 2i)B = 0 \end{cases}$$

Which upon solving gives us $A_1 = 1, A_2 = 0, B_1 = -1, B_2 = -1$.

So the general solution is,

$$\begin{cases} x = e^{3t}[c_1 \sin 2t + c_2 \cos 2t] \\ y = e^{3t}[c_1(-\cos 2t + \sin 2t) + c_2(-\sin 2t - \cos 2t)] \end{cases}$$

□

Example 3.31. *Solve* $\begin{cases} \frac{dx}{dt} = -3x + 4y \\ \frac{dy}{dt} = -2x + 3y \end{cases}$

Proof. Consider the auxiliary equation associated to the above system,

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$$
$$m^2 - (-3 + 3)m + (-3 \cdot 3 - (-2)4) = 0$$
$$m^2 - 1 = 0$$

This has roots $m_1 = -1, m_2 = 1$. These are distinct and real roots, so we know the general solution will be of the form,

$$\begin{cases} x = c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y = c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{cases}$$

Also consider that,

$$\begin{cases} (a_1 - m)A + b_1 B = 0 \\ a_2 A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (-3 - m)A + 4B = 0 \\ -2A + (3 - m)B = 0 \end{cases}$$

Begin by substituting $m = -1$

$$\begin{cases} (-3 + 1)A + 4B = 0 \\ -2A + (3 + 1)B = 0 \end{cases}$$

Which upon solving gives $A = 2B$ using this we can say $A_1 = 2, B_1 = 1$.

Now by substituting $m = 1$

$$\begin{cases} (-3 - 1)A + 4B = 0 \\ -2A + (3 - 1)B = 0 \end{cases}$$

Which upon solving gives $A = B$ using this we can say $A_2 = 1, B_2 = 1$

So our general solution is,

$$\begin{cases} x = c_1 2e^{-t} + c_2 e^t \\ y = c_1 e^{-t} + c_2 e^t \end{cases}$$

□

Example 3.32. $\begin{cases} \dot{x} = 7x + 6y \\ \dot{y} = 2x + 6y \end{cases}$

Proof. Consider the auxiliary equation associated to the above system,

$$\begin{aligned} m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) &= 0 \\ m^2 - (7+6)m + (42 - 12) &= 0 \\ m^2 - 13m + 30 &= 0 \end{aligned}$$

So we have $m_1 = 3, m_2 = 10$. Also consider that,

$$\begin{cases} (a_1 - m)A + b_1 B = 0 \\ a_2 A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (7 - m)A + 6B = 0 \\ 2A + (6 - m)B = 0 \end{cases}$$

Solve with $m = 3$

$$\begin{cases} (7 - 3)A + 6B = 0 \\ 2A + (6 - 3)B = 0 \end{cases}$$

So we have $A_1 = 3, B_1 = -2$. Solve with $m = 10$

$$\begin{cases} (7 - 10)A + 6B = 0 \\ 2A + (6 - 10)B = 0 \end{cases}$$

So we have $A_2 = 2, B_2 = 1$

So our general solution is

$$\begin{cases} x = c_1 3e^{3t} + c_2 2e^{10t} \\ y = c_1 - 2e^{3t} + c_2 e^{10t} \end{cases}$$

□

3.8 Non-homogenous linear system

Consider the nonhomogenous linear system

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases} \tag{3.1}$$

and the corresponding homogenous system,

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases} \tag{3.2}$$

Let

$$\begin{cases} x = c_1x_1 + c_2x_2 \\ y = c_2y_1 + c_2y_2 \end{cases}$$

be solutions to 3.2

$$\begin{cases} x_p = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y_p = v_1(t)y_1(t) + v_2(t)y_2(t) \end{cases}$$

will be a particular solution of 3.1 if the functions $v_1(t)$ and $v_2(t)$ satisfy the system,

$$\begin{cases} v_1'x_1 + v_2'x_2 = f_1 \\ v_1'y_1 + v_2'y_2 = f_2 \end{cases}$$

This technique for finding particular solutions of nonhomogenous linear systems is called method of variation of parameters.

Example 3.33. *Apply the method before to find a particular solution to the nonhomogenous system*

$$\begin{cases} \frac{dx}{dt} = x + y - 5t + 2 \\ \frac{dy}{dt} = 4x - 2y - 8t - 8 \end{cases}$$

Proof. The corresponding homogenous system is,

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$$

And we have $f_1 = -5t + 2, f_2 = -8t - 8$.

The solution to the auxiliary equation of the homogenous system is $m = -2, 3$ Also consider that,

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (1 - m)A + 1B = 0 \\ 4A + (-2 - m)B = 0 \end{cases}$$

Begin by substituting $m = -2$

$$\begin{cases} (1 + 2))A + B = 0 \\ 4A + (-2 + 2))B = 0 \end{cases}$$

$A_1 = 1, B_1 = 1$ Begin by substituting $m = 3$

$$\begin{cases} (1 - 3)A + B = 0 \\ 4A + (-2 - 3)B = 0 \end{cases}$$

$A_2 = 4, B_2 = -2$.

Now $v_1'x_1 + v_2'x_1 = f_1, v_1'y_1 + v_2'y_2 = f_2$

$$\begin{aligned} v_1'e^{2t} + v_2'e^{-3t} &= -5t + 2 \\ v_1'e^{2t} + v_2'(-4e^{-3t}) &= -8t - 8 \end{aligned}$$

Subtract second from first

$$\begin{aligned} v_2'5e^{-3t} &= 3t + 10 \\ v_2' &= \frac{3t+10}{5}e^{3t} \end{aligned}$$

Integrate

$$\begin{aligned} v_2 &= \frac{1}{5}e^{3t}(t+3) \\ v_1'e^{2t} &= .... \end{aligned}$$

Integrate

$$v_1 = \frac{14}{5}te^{-2t} + \frac{7}{5}e^{-2t}$$

Now we get the particular solutions as follows,

$$\begin{cases} x_p = \frac{14}{5}te^{-2t} + \frac{7}{5}e^{-2t}x_1(t) + (\frac{1}{5}e^{3t}(t+3))x_2(t) \\ y_p = (\frac{14}{5}te^{-2t} + \frac{7}{5}e^{-2t})y_1(t) + (\frac{1}{5}e^{3t}(t+3))y_2(t) \end{cases}$$

□

Example 3.34. *Solve the IVP,*

$$\begin{cases} \frac{dx}{dt} = x + 2y + 12e^{3t} \\ \frac{dy}{dt} = 4x + 3y + 18e^{2t} \end{cases} , x(0) = 3, y(0) = 0$$

Proof. The associated auxiliary equation is $m^2 - 4m - 5 = 0$ with roots $m_1 = -1, m_2 = 5$.

Also consider that,

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (1-m)A + 2B = 0 \\ 4A + (3-m)B = 0 \end{cases}$$

Solve with $m = -1$

$$\begin{cases} (1-(-1))A + 2B = 0 \\ 4A + (3-(-1))B = 0 \end{cases}$$

$A = -B$ so take $A_1 = 1, B_1 = -1$ Solve with $m = 5$

$$\begin{cases} (1-(5))A + 2B = 0 \\ 4A + (3-(5))B = 0 \end{cases}$$

$B = 2A$ so take $A_2 = 1, B_2 = 2$.

$$\begin{cases} x_g = c_1 e^{-t} + c_2 e^{5t} \\ y_g = c_2 - e^{-t} + c_2 2e^{5t} \end{cases}$$

Now $v_1' x_1 + v_2' x_2 = f_1, v_1' y_1 + v_2' y_2 = f_2$

$$\begin{aligned} v_1' e^{-t} + v_2' e^{5t} &= 12e^{3t} \\ v_1'(-e^{-t}) + v_2'(2e^{5t}) &= 18e^{2t} \end{aligned}$$

Add them

$$\begin{aligned} v_2' &= \frac{12e^{3t} + 18e^{2t}}{3e^{5t}} \\ v_2 &= -2e^{-3t} - 2e^{-2t} \end{aligned}$$

second minus twice first

$$\begin{aligned} v_1'(-3e^{-t}) &= 18e^{2t} - 24e^{3t} \\ v_1' &= \frac{18e^{2t} - 24e^{3t}}{-3e^{-t}} \\ v_1 &= 2e^{4t} - 2e^{3t} \end{aligned}$$

So the particular solution is given as,

$$\begin{cases} x_p = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y_p = v_1(t)y_1(t) + v_2(t)y_2(t) \end{cases}$$

$$\begin{cases} x_p = -4e^{2t} \\ y_p = -2e^{2t} - 6e^{3t} \end{cases}$$

The general solution is given as,

$$\begin{cases} x = c_1e^{-t} + c_2e^{5t} - 4e^{2t} \\ y = c_1 - e^{-t} + c_2 2e^{5t} - 2(e^{2t} - 3e^{3t}) \end{cases}$$

Solve for initial conditions now,
We get $c_1 = 5, c_2 = -2$ □

Example 3.35.

$$\begin{cases} \frac{dx}{dt} = x + 2y + 2t \\ \frac{dy}{dt} = 3x + 2y - 4t \end{cases}$$

Proof. The associated auxiliary equation is $m^2 - 3m - 4 = 0$ with roots $m_1 = -1, m_2 = 4$.

Also consider that,

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} (1 - m)A + 2B = 0 \\ 3A + (2 - m)B = 0 \end{cases}$$

Solve with $m = -1$

$$\begin{cases} (1 - (-1))A + 2B = 0 \\ 3A + (2 - (-1))B = 0 \end{cases}$$

$A = -B$ so take $A_1 = 1, B_1 = -1$.

Solve with $m = 4$

$$\begin{cases} (1 - (4))A + 2B = 0 \\ 3A + (2 - (4))B = 0 \end{cases}$$

$B = 3/2A$ so pick $A_2 = 2, B_2 = 3$.

The solution to the homogenous system is given as,

$$\begin{cases} x_g = c_1e^{-t} + c_2(2e^{4t}) \\ y_g = c_1(-e^{-t}) + c_2(3e^{4t}) \end{cases}$$

Now $v_1'x_1 + v_2'x_2 = f_1, v_1'y_1 + v_2'y_2 = f_2$

$$v_1'e^{-t} + v_2'(2e^{4t}) = 2t$$
$$v_1'(-e^{-t}) + v_2'(3e^{4t}) = -4t$$

Add them

$$v_2' = \frac{-2t}{5e^{4t}}$$
$$v_2 = \frac{1}{40}e^{-4t}(4t+1)$$

2 second minus 3 first

$$v_1'(-5e^{-t}) = -14t$$
$$v_1' = \frac{14t}{5e^{-t}}$$
$$v_1 = \frac{14}{5}e^{t}(t-1)$$

The particular solution is given by,

$$\begin{cases} x_p = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y_p = v_1(t)y_1(t) + v_2(t)y_2(t) \end{cases}$$
$$\begin{cases} x_p = 3t - \frac{11}{4} \\ y_p = \frac{23}{8} - \frac{5t}{2} \end{cases}$$

So the general solution is given as,

$$\begin{cases} x = c_1e^{-t} + c_2(2e^{4t}) + (3t - \frac{11}{4}) \\ y = c_1(-e^{-t}) + c_2(3e^{4t}) + (\frac{23}{8} - \frac{5t}{2}) \end{cases}$$

□

3.9 Nonlinear systems

If x is the number of rabbits at time t then,

$$\frac{dx}{dt} = ax(a > 0)$$

as a consequence of unlimited supply of clover, if the number y of foxes is zero. Assume that the number of encounters per unit time between rabbits and foxes is jointly proportional to x and y. If we further assume that a certain proportion fo these encounters result in a rabbit being eaten then we have.

Assume that the number of encounters per unit time between rabbits and foxes is jointly proportional to x and y. Further assume that a certain proportion of these encounters result in a rabbit being eaten, then we have,

$$\frac{dx}{dt} = ax - bxy$$

Similarly, in the absence of rabbits the foxes dies out and their increase depends on the number of their encounters with rabbits,

$$\frac{dy}{dt} = -cy + dxy$$

We get the following system,

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = -y(c - dx) \end{cases} \tag{3.1}$$

Equation 3.1 is called Volterra's prey-predator equation. Let the unknown solutions be thought as constituting

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

the parametric equations of a curve in the $xy-$plane, then we can find the rectangular equation of this curve.

Eliminating t in 3.1 we get

$$\frac{(a - by)dy}{y} = \frac{(c - dx)dx}{x}$$

?????

Where the constant K in terms of initial values is given by,

$$K = x_0^c y_0^a e^{-dx_0 - by_0} \tag{3.2}$$

and

$$z = y?? \tag{3.3}$$

if the rabbit and fox population is

$$x = \frac{c}{d}, y = \frac{a}{b} \tag{3.4}$$

then system 3.1 os satisfied and we have $dx/dt = 0$ and $dy/dt = 0$ so there is no increase or decrease in x or y. The population above is called equilibrium population, for x, y can maintain themselves indefinitely at these constant levels.

Let $x = \frac{c}{d} + X$ and $y = \frac{a}{b} + Y$ then X, Y can be thought of as the deviations of x, y from their equilibrium values.

If x, y in 3.1 are replaced with X, Y then it becomes

$$\begin{cases} \frac{dX}{dt} = -\frac{bc}{d}Y - bXY \\ \frac{dY}{dt} = \frac{ad}{b}X + dXY \end{cases} \tag{3.5}$$

To "linearize" the system, assume that if X, Y are small then XY can be discarded without serious error. Thus simplifying it,

$$\begin{cases} \frac{dX}{dt} = -\frac{bc}{d}Y \\ \frac{dY}{dt} = \frac{ad}{b}X \end{cases} \tag{3.6}$$

Eliminating t we get,

$$\frac{dY}{dX} = -\frac{ad^2}{b^2c}\frac{X}{Y}$$

Whose solution is

$$ad^2X^2 + b^2cY^2 = C^2$$

This is a family of ellipses surrounding the origin in the XY plane.

Example 3.36. *Find solution to*

$$\begin{cases} \frac{dx}{dy} = x \\ \frac{dy}{dt} = y \end{cases}$$

Proof. Consider the auxiliary equation associated to the above system,

$$m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$$
$$m^2 - 2m + 1 = 0$$

So we have $m_1 = m_2 = 1$. Also consider that

$$\begin{cases} (a_1 - m)A + b_1B = 0 \\ a_2A + (b_2 - m)B = 0 \end{cases} \implies \begin{cases} 0 + 0 = 0 \\ 0 + 0 = 0 \end{cases}$$

Which on solving gives $A = 0, B = 0$ So we have,

$$\begin{cases} x_1 = 2e^{3t} \\ y_2 = -e^{3t} \end{cases}$$

We know the second solution should be of the form

$$\begin{cases} x_2 = (A_1 + A_2 t)e^t \\ y_2 = (B_1 + B_2 t)e^t \end{cases}$$

Assume this is the solution and substitute it in the original differential equation

$$\begin{cases} e^t(A_2 t + A_1 + A_2) = 5((A_1 + A_2 t)e^t) + 4((B_1 + B_2 t)e^t) \\ e^t(B_2 t + B_1 + B_1) = -(A_1 + A_2 t)e^t + (B_1 + B_2 t)e^t \end{cases}$$

We get the general solution as

$$\begin{cases} x = c_1 e^t \\ y = c_2 e^t \end{cases}$$

Show that any second order equation obtained from the system in a) is not equivalent to this system, in the sense that it has solutions that are not part of any solution to the system ???????? higher order equations are equivalent to systems, the reverse is not true ????? □

Example 3.37. *Show that*

$$\begin{cases} x = 2e^{4t} \\ y = 3e^{4t} \end{cases}, \begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$$

are solutions to the homogenous system

$$\begin{cases} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = 3x + 2y \end{cases}$$

Proof. Check first,

$$\begin{aligned} 8e^{4t} &= 2e^{4t} + 6e^{4t} \\ 12e^{4t} &= 6e^{4t} + 6e^{4t} \end{aligned}$$

Check second,

$$\begin{aligned} -e^{-t} &= e^{-t} - 2e^{-t} \\ e^{-t} &= 3e^{-t} - 2e^{-t} \end{aligned}$$

They are also both linearly independent since their ratios aren't constant so their linear combination forms the general solution.

Also check linear independence with Wronskian

Then show that

$$\begin{cases} x = 3t - 2 \\ y = -2t + 3 \end{cases}$$

is a particular solution of the non-homogenous system

$$\frac{dx}{dt} = x + 2y + t - 1 \frac{dy}{dt} = 3x + 2y - 5t - 2$$

and write its general solution. This was done in practical 3.

Plug in x_p, y_p into the given differential equation,

$$\begin{aligned} \frac{dx}{dt} &= x + 2y + t - 1 \\ 3 &= 3t - 2 + 2(-2t + 3) + t - 1 = 3 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2 \\ -2 &= 3(3t - 2) + 2(-2t + 3) - 5t - 2 = -2 \end{aligned}$$

So x_p, y_p form a particular solution of the given non-homogenous system.

It is evident that x_1, y_1 and x_2, y_2 are linearly independent (since their ratio is not a constant). So we have the general solution of the given non-homogenous system as,

$$\begin{cases} x_g &= c_1 2e^{4t} + c_2 e^{-t} + 3t - 2 \\ y_g &= c_1 3e^{4t} + c_2(-e^{-t}) - 2t + 3 \end{cases}$$

$\square$

4

Partial differential equations

"Stop studying differential equations."

– Gauss

Definition 4.1 (Partial derivatives)**.** *Partial derivates are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.*
For a function f in n variables $x_1, x_2, \ldots, x_n$ we can define the m^{th} partial derivative as,

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}$$

Partial derivatives can be taken with respect to multiple variables and are denoted as follows,

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

Differential equations that use partial derivates of a function of two or more variables are called PDEs.

Example 4.2. *Reduce to PDE $z = (x-a)^2 + (y-b)^2$*

Proof.

$$\frac{\partial z}{\partial x} = 2(x-a)$$
$$\frac{\partial z}{\partial y} = 2(y-b)$$

□

Example 4.3. *Reduce to PDE* $z = a(x+y) + b$

Proof.

$$z_x = a, z_y = a$$

So z_x, z_y □

Proof. $z = ax + by + ab$ □

Proof. $z_x = a, z_y = b \implies z = xz_x + yz_y + z_x z_y$ □

Example 4.4. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

Proof. content... □

Example 4.5. $z = (x^2+a)(y^2+b)$

Proof.

$$z_x = 2x(y^2+b)$$
$$z_y = 2y(x^2+a)$$

So

$$z = \frac{z_y}{2y}\frac{z_x}{2x}$$

□

Example 4.6. $z = ae^{by}\sin(bx)$

Proof. $z_{yy} = ab^2e^{by}\sin(bx), z_{xx} = -ab^2e^{by}\sin(bx)$ So $z_{yy} = -z_{xx}$ □

Example 4.7. *fnd the differential equation of all spheres of radius* r *around having center in the* x, y *plane.*

Proof. $(x-a)^2 + (y-b)^2 + z^2 = r^2$ So the equation is

$$z^2 = r^2 - (x-a)^2 - (y-b)^2$$

$$\begin{aligned} z_x = 2(x-a) + 2zz_x = 0 \implies (x-a) &= -zz_x \\ (y-b) &= -zz_y \end{aligned}$$

□

Example 4.8. $z = x^2 - y^2$

Proof. $x = uv, y = u + v$ so, $\frac{\partial z}{\partial u} = 2uv^2 - 2u - 2v$ and $\frac{\partial z}{\partial u} = 2vu^2 - 2v - 2u$
$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} + \frac{\partial y}{\partial u}$
$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = -2y, \frac{\partial x}{\partial u} = v, \frac{\partial y}{\partial u} = 1$ □

4.1 Classification of Second order PDE

Second order PDE are usually divided into three types.

Definition 4.9 (General form of a second order PDE)**.**

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

Linear second order PDEs are classified according to the properties of the following 2×2 matrix,

$$Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.1}$$

4.1.1 Elliptic PDE

If Z (eq. 4.1) has determinant strictly greater than 0, it is called an elliptic PDE, i.e. $\det Z > 0$.

4.1.2 Hyperbolic PDE

If Z (eq. 4.1) has determinant strictly lesser than 0, it is called an hyperbolic PDE, i.e. $\det Z < 0$.

4.1.3 Parabolic PDE

If Z (eq. 4.1) has determinant equal to 0, it is called an parabolic PDE, i.e. $\det Z = f0$.

4.2 One dimensional wave equation

Definition 4.10 (One dimension wave equation). *The one dimensional wave equation is given by,*

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

where a is a positive constant.

4.2.1 Vibration of an infinite string

4.2.2 Vibration of an semi-infinite string

4.2.3 Vibrating of a finite string

4.3 Laplace equation

Definition 4.11 (Laplacian). *The Laplacian of a three dimensional function ϕ is given as follows,*

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is generalized for higher dimensions in the expected way.

Definition 4.12 (Laplace's equation). *Laplace's equation is the following PDE*

$$\Delta f = 0$$

4.3.1 Green's function

Definition 4.13 (Green's function). *content...*

4.4 Heat conduction principle

Definition 4.14 (General heat equation). *The temperature function w satisfies the following heat equation*

$$a^2 \Delta w = \frac{\partial w}{\partial t}$$

4.4.1 Infinite rod case

4.4.2 Finite rod case

Appendix

If you are seeing this, I forgot to do it.