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Probability and Sampling Distributions (B)

Lecture Notes
for SSTA401

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Chapter 1

Continuous probability distributions

1.1 Uniform/Rectangular distributions

Definition 1.1. A r.v. X is said to follow uniform distribution over an interval (a, b) if its pdf is constant over the entire range.

1.1.1 PDF of uniform distribution

Theorem 1.2. PDF of uniform distribution

$$\begin{aligned} P(x) &= k & a < x < b \\ &= 0 & \text{otherwise} \end{aligned}$$

- $\int_a^b f(x) dx = \int_a^b k dx = k[x]_a^b = k(b - a) = 1$, therefore $k = \frac{1}{b-a}$
- We denote it as, $X \sim U(a, b)$
- $f(x) = \frac{1}{b-a}$

1.1.2 CDF of uniform distribution

Theorem 1.3. *CDF of uniform distribution*

$$\begin{aligned}
F(x) &= 0 & x &\leq a \\
&= P(X \leq x) = \int_a^x f(x) dx = \frac{x-a}{b-a} & a < x < b \\
&= 1 & x &\geq b
\end{aligned}$$

1.1.3 Expectation and variance of uniform distribution

Theorem 1.4. *Expected value of $X \sim U(a, b)$ is equal to $\frac{(a+b)}{2}$* *Proof.* Consider the expectation of the uniform distribution as,

$$\begin{aligned}
E[x] &= \int_a^b xP(x) dx \\
&= \int_a^b x \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x dx \\
&= \frac{a+b}{2}
\end{aligned}$$

□

Theorem 1.5. *Variance of uniform distribution is equal to $\frac{1}{12}(b-a)^2$* *Proof.* We begin by finding out $E[X^2]$

$$\begin{aligned}
E[X^2] &= \int_a^b x^2 P(x) dx \\
&= \int_a^b x^2 \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x^2 dx \\
&= \frac{1}{3} (a^2 + ab + b^2)
\end{aligned}$$

Now we can find the variance as $V[X] = E[X^2] - E[X]^2$ as follows,

$$\begin{aligned} V[X] &= E[X^2] - E[X]^2 \\ &= \frac{1}{3} (a^2 + ab + b^2) - \left(\frac{a+b}{2} \right)^2 \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

□

1.1.4 Raw moments of uniform distribution

The r^{th} raw moment of the uniform distribution is given as

$$\begin{aligned} \mu'_r &= E[X^r] = \int_a^b x^r f(x) dx \\ &= \frac{1}{b-a} \left[\frac{x^{r+1}}{r+1} \right]_a^b = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \end{aligned}$$

Example 1.6. Suppose in a quiz there are 30 participants. A question is given to all 30 participants and the time allowed is 25 seconds.

Proof. Let X denote the time to respond.

$X \sim U(0, 25)$, the pdf is given by $f(x) = \frac{1}{25}; 0 < x < 25$ and 0 otherwise.

$$\begin{aligned} P(x \leq 6) &= \int_0^6 f(x) dx = \int_0^6 \frac{1}{25} dx = \frac{151}{25} \\ P(6 \leq x \leq 10) &= \int_6^{10} f(x) dx = \int_6^{10} \frac{1}{25} dx = \frac{101}{25} \end{aligned}$$

□

Example 1.7. A r.v. x is said to follow uniform dist with $\mu = 1$ and $V(x) = 4/3$. Obtain $P(x < 0)$.

Proof. First begin by finding out the parameters for the uniform distribution. First consider the mean,

$$\begin{aligned} \mu &= 1 \\ \frac{a+b}{2} &= 1 \\ a+b &= 2 \end{aligned}$$

Then consider the variance,

$$\begin{aligned} V(x) &= \frac{4}{3} \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \\ (b-a)^2 &= 16 \end{aligned}$$

Solving two simultaneous equations we get $a = -1, b = 3$.
Therefore, we have $X \sim U(-1, 3)$

$$P(x \leq 0) = F(0) = \frac{0+1}{4} = \frac{1}{4}$$

□

Example 1.8. If $X \sim U(-3, 3)$, find $P(x < 2)$, $P(|x| < 2)$, $P(|x - 2| < 2)$, also obtain k if $P(x > k) = 1/3$

Proof.

$$\begin{aligned} P(x < 2) &= F(2) = \frac{2+3}{6} = \frac{5}{6} \\ P(|x| < 2) &= \int_{-2}^2 \frac{1}{6} dx = \frac{2}{3} \\ P(|x - 2| < 2) &= \int_0^3 \frac{1}{6} dx = \frac{1}{2} \\ P(x > k) &= 1/3 \implies \dots \end{aligned}$$

Complete this l8r alig8r

□

1.1.5 MGF of Uniform distribution

Theorem 1.9. MGF of Uniform distribution $= \frac{e^{bt}-e^{at}}{t(b-a)}, t \neq 0$ and $= 1, t = 0$

Proof.

$$M_x(t) = E[e^{tx}] = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{e^{bt} - e^{at}}{(b-a)t}$$

The Taylor series for this can be expressed as the following,

$$M_x(t) = \frac{b-a}{b-a} + \frac{b^2-a^2}{2(b-a)}t + \frac{b^3-a^3}{3(b-a)}\frac{t^2}{2!} + \dots$$

Therefore we can say,

$$\begin{aligned}\mu'_1 &= \text{coeff of } t = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2} \\ \mu'_2 &= \text{coeff of } \frac{t^2}{2!} = \frac{b^3 - a^3}{3(b-a)}\end{aligned}$$

And we can say $\mu_2 = \dots$

□

1.1.6 Applications of uniform distribution

1. Assumption of uniform death for insurance
- ⋮

Write sumthin here

1.2 Gamma distribution

Definition 1.10 (Gamma distribution). *A r.v. 'X' is said to follow gamma distribution $X \sim G(\lambda, \theta)$. Where λ = shape parameter and θ = scale parameter.*

1.2.1 PDF of Gamma distribution

Definition 1.11 (PDF of Gamma distribution).

$$\begin{aligned}f(x, \lambda, \theta) &= \frac{\theta^\lambda}{\Gamma(\lambda)} e^{-\theta x} x^{\lambda-1} & x > 0, \lambda > 0, \theta > 0 \\ &= 0 & \text{otherwise}\end{aligned}$$

Where $\Gamma(\lambda) = (\lambda - 1)! = (\lambda - 1)\Gamma(\lambda - 1)$.

Corollary 1.12. *If $\theta = 1$ we will have gamma distribution with a single parameter λ which is called the standard gamma distribution.*

$$\begin{aligned}X \sim G(\lambda) &= \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)} & x > 0, \lambda > 0 \\ &= 0 & \text{otherwise}\end{aligned}$$

Corollary 1.13. *If $\lambda = 1$, $X \sim G(1, \theta) = \text{Exp}(\theta)$.*

Corollary 1.14. *If $\lambda = 1, \theta = 1$, $X \sim$ Standard exponential distribution, i.e.*

$$\begin{aligned} f(x) &= e^{-x} & x > 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

Definition 1.15 (Gamma function).

$$\Gamma(\lambda) = \int_0^\infty e^{-x} x^{\lambda-1} dx$$

Definition 1.16 (Gamma integral).

$$\int_0^\infty e^{-\theta x} x^{\lambda-1} dx = \frac{\Gamma(\lambda)}{\theta^\lambda}$$

1.2.2 CDF of Gamma distribution

Theorem 1.17. *CDF of Gamma distribution is given as*

$$F(x) = \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^x x^{\lambda-1} e^{-\theta x} dx$$

Proof.

$$\begin{aligned} F(x) &= P(X < x) = \int_0^x \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)} dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^x x^{\lambda-1} e^{-\theta x} dx \end{aligned}$$

□

1.2.3 Raw moments of Gamma distribution

Theorem 1.18. *The r^{th} raw moment of the Gamma distribution is given by*

$$\mu'_r = \frac{\Gamma(\lambda + r)}{\Gamma(\lambda) \theta^r}$$

Proof.

$$\begin{aligned} \mu'_r &= E[x^r] = \int_0^\infty \frac{x^r \theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)} dx \\ &= \int_0^\infty \frac{\theta^\lambda e^{-\theta x} x^{\lambda+r-1}}{\Gamma(\lambda)} dx \\ &= \frac{\Gamma(\lambda + r)}{\Gamma(\lambda) \theta^r} \end{aligned}$$

□

1.2.4 Mean and Variance of Gamma distribution

Now we can find μ'_1, μ'_2

$$\begin{aligned} E[x] &= \mu'_1 = \frac{\lambda}{\theta} \\ \mu'_2 &= \frac{\lambda(\lambda+1)}{\theta^2} \\ V[x] &= \mu_2 = \mu'_2 - \mu'^2_1 = \frac{\lambda(\lambda+1)}{\theta^2} - \frac{\lambda^2}{\theta^2} = \frac{\lambda}{\theta^2} \end{aligned}$$

1.2.5 MGF of Gamma distribution

$$\begin{aligned} E[e^{tx}] &= \int_0^\infty e^{tx} \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)} dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(\theta-t)x} x^{\lambda-1} dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda)}{(\theta-t)^\lambda} = \left(\frac{\theta}{\theta-t} \right)^\lambda \\ &= \left(1 - \frac{t}{\theta} \right)^{-\lambda} \end{aligned}$$

1.2.6 CGF of Gamma distribution

$$\begin{aligned} K_x(t) &= \log \left(1 - \frac{t}{\theta} \right)^{-\lambda} \\ &= -\lambda \log \left(1 - \frac{t}{\theta} \right) \\ &= \frac{\lambda t}{\theta} + \frac{\lambda t^2}{2\theta^2} + \frac{\lambda t^3}{3\theta^3} + \dots \end{aligned}$$

Using this we can get the mean and variance easily.

$$\begin{aligned} \text{Mean} &= k_1 = \frac{\lambda}{\theta} \\ \text{Variance} &= k_2 = \frac{\lambda}{\theta^2} \end{aligned}$$

1.2.7 Additive property of Gamma distribution

If $X_i (i = 1, \dots, k)$ are k independent Gamma distributions with parameters $\lambda_1, \lambda_2, \dots, \lambda_k$ and θ respectively, then,

$$\sum_{i=1}^k X_i \sim G\left(\sum_{i=1}^k \lambda_i, \theta\right)$$

$$M_{X_i}(t) = \left(1 - \frac{t}{\theta}\right)^{-\lambda_i}$$

Let $Z = \sum X_i$

$$M_Z(t) = \prod_{i=1}^k \left(1 - \frac{t}{\theta}\right)^{-\lambda_i}$$

$$= \left(1 - \frac{t}{\theta}\right)^{-\sum \lambda_i}$$

By uniqueness property of mgf

$$\sum_i X_i \sim G\left(\sum_i \lambda_i, \theta\right)$$

1.2.8 Limiting form of Gamma distribution

$$\beta_1 = \frac{4}{\lambda}, \text{ as } \lambda \rightarrow \infty, \beta_1 \rightarrow 0 \implies \text{Normal dist}$$

$$\beta_2 = 3 + \frac{6}{\lambda} \text{ as } \lambda \rightarrow \infty, \beta_2 \rightarrow 3 \implies \text{Normal dist}$$

Note that they are both independent of θ .

Therefore, as $\lambda \rightarrow \infty$ we have $G(\lambda, \infty) \rightarrow N\left(\frac{\lambda}{\theta}, \frac{\lambda}{\theta^2}\right)$.

1.2.9 Applications of Gamma distribution

Idk write something bruh

1.3 Exponential distribution

1.3.1 PDF of Exponential Distribution

Definition 1.19 (PDF of Exponential distribution). *A r.v. x is said to follow the exponential distribution with parameter θ if its pdf is given by*

$$\begin{aligned} f(x) &= \theta e^{-\theta x} & x \geq 0, \theta > 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

1.3.2 INCOMPLETE CDF of exponential distribution

Theorem 1.20 (CDF of Exponential distribution).

$$F[x] = 1 - e^{-\theta x}$$

Proof.

$$\begin{aligned} F[x] &= \int_0^x f(x) dx \\ &= \int_0^x \theta e^{-\theta x} dx \\ &= \theta \left[-\frac{e^{-\theta x}}{\theta} \right]_0^x \\ &= 1 - e^{-\theta x} \end{aligned}$$

□

1.3.3 Raw moment of exponential distribution

Theorem 1.21. *The r^{th} raw moment for exponential distribution is given by*

$$\mu'_r = \frac{r!}{\theta^r}$$

Proof.

$$\begin{aligned} \mu'_r = E[x^r] &= \int_0^\infty x^r \theta e^{-\theta x} dx \\ &= \frac{\Gamma(r+1)}{\theta^r} \\ &= \frac{r!}{\theta^r} \end{aligned}$$

□

1.3.4 Mean and variance of exponential distribution

Theorem 1.22. *The mean of exponential distribution is given by*

$$\mu = \frac{1}{\theta}$$

Proof. Consider $r = 1$,

$$\mu'_1 = \frac{1}{\theta}$$

□

Theorem 1.23. *The variance of the exponential distribution is given by*

$$\mu_2 = \frac{1}{\theta^2}$$

Proof. First find μ'_2

$$\mu'_2 = \frac{2}{\theta^2}$$

So now we can compute the variance as $\frac{1}{\theta^2}$

□

1.3.5 MGF of exponential distribution

Theorem 1.24. *MGF of exponential distribution is given by*

$$M_x(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$$

Proof.

$$\begin{aligned} M_x(t) &= E[e^{tx}] \\ &= \int_0^\infty e^{tx} \theta e^{-\theta x} dx \\ &= \theta \int_0^\infty e^{x(t-\theta)} x^{1-1} dx \\ &= \frac{\theta \Gamma(1)}{\theta - t} \\ &= \frac{\theta}{\theta - t} \\ &= \left(1 - \frac{t}{\theta}\right)^{-1} \end{aligned}$$

□

1.3.6 CGF of exponential distribution

Theorem 1.25. *CGF of exponential distribution is given by*

$$K_x(t) = -\log \left(1 - \frac{t}{\theta} \right)$$

Proof.

$$\begin{aligned} K_x(t) &= \log \left(1 - \frac{t}{\theta} \right)^{-1} \\ &= -\log \left(1 - \frac{t}{\theta} \right) \\ &= \frac{t}{\theta} + \frac{t^2}{2\theta^2} + \frac{t^3}{3\theta^3} \end{aligned}$$

We can say the general r^{th} cumulant is given by $K_r = \frac{(r-1)!}{\theta^r}$ □

1.3.7 Additive property of exponential variates

Theorem 1.26. *If x_1, x_2, \dots, x_k are k independent exponential variates each with parameter θ then*

$$\sum_{i=1}^k x_i \sim G(k, \theta)$$

Proof. We will do this with the MGF. Consider $Z = \sum_{i=1}^k x_i$.

$$\begin{aligned} M_z(t) &= \prod_{i=1}^k M_{x_i}(t) \\ &= \prod_{i=1}^k \left(1 - \frac{t}{\theta} \right)^{-1} \\ &= \left(1 - \frac{t}{\theta} \right)^{-k} \end{aligned}$$

Therefore, (by uniqueness property of MGF) comparing this MGF to that of the gamma distribution we can say that,

$$\sum_{i=1}^k x_i = Z \sim G(k, \theta)$$

□

1.3.8 Lack of memory of exponential distribution

Theorem 1.27. For a exponentially distributed random variate, $P[x > a+b \mid x > a] = P[x > b]$

Proof. Let $X \sim E(\theta)$. Consider first case

$$\begin{aligned} P[x > a+b \mid x > a] &= \frac{P[x > a+b]}{P[x > a]} \\ &= \frac{\int_{a+b}^{\infty} \theta e^{-\theta x} dx}{\int_a^{\infty} \theta e^{-\theta x} dx} \\ &= \frac{e^{-\theta(a+b)}}{e^{-\theta a}} \\ &= e^{-\theta b} \end{aligned}$$

Consider second case now,

$$P[x > b] = \int_b^{\infty} \theta e^{-\theta x} dx = e^{-\theta b}$$

Equality holds. □

1.4 INCOMPLETE Laplace distribution (Double exponential)

1.4.1 PDF

Definition 1.28 (PDF of Laplace distribution). $X \sim L(\lambda, \mu)$

$$f(x) = \begin{cases} \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|} & -\infty < x < \infty \\ 0 & otherwise \end{cases}$$

1.4.2 CDF

Definition 1.29 (CDF of Laplace distribution).

$$F[x] = \left\{ \begin{array}{l} content... \end{array} \right.$$

1.4.3 Raw moment

Theorem 1.30. *The r^{th} raw moment for the Laplace distribution is given by*

$$\mu'_r = \frac{1}{2} \sum_{k=0}^r \left[\binom{r}{k} \lambda^k \mu^{r-k} k! \left(1 + (-1)^k \right) \right]$$

Proof.

$$\mu'_r = E[x^r] = \int_{-\infty}^{\infty} \frac{x^r}{2\lambda} e^{-|\frac{x-\mu}{\lambda}|} dx$$

Transform $(x - \mu)/\lambda = z$

$$\begin{aligned} &= \frac{1}{2\lambda} \left(\int_{-\infty}^{\infty} (z\lambda + \mu)^r e^{-|z|} \lambda dz \right) \\ &= \frac{1}{2} \left(\int_{-\infty}^{\infty} \sum_{k=0}^r \binom{r}{k} (z\lambda)^k \mu^{r-k} e^{-|z|} dz \right) \\ &= \frac{1}{2} \sum_{k=0}^r \left[\binom{r}{k} \lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k e^{-|z|} dz \right] \end{aligned}$$

Complete this up

$$= \frac{1}{2} \sum_{k=0}^r \left[\binom{r}{k} \lambda^k \mu^{r-k} k! \left(1 + (-1)^k \right) \right]$$

□

1.4.4 Mean and variance

We can do this with the raw moments above but instead we will do it with the PDF.

Theorem 1.31. *Expectation of laplace distribution is given as*

$$E[x] =$$

Proof.

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{\infty} \frac{x}{2\lambda} e^{-|\frac{x-\mu}{\lambda}|} dx \end{aligned}$$

Split it around μ

$$\begin{aligned}
&= \frac{1}{2\lambda} \left(\int_{-\infty}^{\mu} x e^{\frac{x-\mu}{\lambda}} dx + \int_{\mu}^{\infty} x e^{-\frac{x-\mu}{\lambda}} dx \right) \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} \int_{-\infty}^{\mu} x e^{x/\lambda} dx + e^{\mu/\lambda} \int_{\mu}^{\infty} x e^{-x/\lambda} dz \right] \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} \lambda (x - \lambda) e^{x/\lambda} - e^{\mu/\lambda} (\lambda (x + \lambda) e^{-x/\lambda}) \right] \\
&= \mu
\end{aligned}$$

□

Theorem 1.32. *Expectation of x^2 in Laplace distribution is given be*

$$E[x^2] = 2\lambda^2 + \mu^2$$

Proof.

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-|\frac{x-\mu}{\lambda}|} dx$$

Split it around μ

$$\begin{aligned}
&= \frac{1}{2\lambda} \left(\int_{-\infty}^{\mu} x^2 e^{\frac{x-\mu}{\lambda}} dx + \int_{\mu}^{\infty} x^2 e^{-\frac{x-\mu}{\lambda}} dx \right) \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} \int_{-\infty}^{\mu} x^2 e^{x/\lambda} dx + e^{\mu/\lambda} \int_{\mu}^{\infty} x^2 e^{-x/\lambda} dx \right] \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} (\lambda(x^2 - 2\lambda x + 2\lambda^2) e^{x/\lambda}) - e^{\mu/\lambda} (\lambda(x^2 + 2\lambda x + 2\lambda^2) e^{-x/\lambda}) \right] \\
&= 2\lambda^2 + \mu^2
\end{aligned}$$

□

Theorem 1.33. *Variance of Laplace distribution is given as*

$$V[x] = 2\lambda^2$$

1.4.5 MGF

Theorem 1.34. *MGF of the Laplace distribution is given by*

$$M_x(t) = \frac{e^{\mu t}}{1 - \lambda^2 t^2}$$

Proof.

$$\begin{aligned}
M_x(t) &= E[e^{tx}] = \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{tx - |\frac{x-\mu}{\lambda}|} \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} \int_{-\infty}^{\mu} e^{x(t+\frac{1}{\lambda})} dx + e^{\mu/\lambda} \int_{\mu}^{\infty} e^{-x(\frac{1}{\lambda}-t)} dx \right] \\
&= \frac{1}{2\lambda} \left[e^{-\mu/\lambda} \left(\frac{e^{\mu(\frac{1}{\lambda}+t)}}{\frac{1}{\lambda}+t} \right) + e^{\mu/\lambda} \left(\frac{-e^{\mu(\frac{1}{\lambda}-t)}}{-\frac{1}{\lambda}+t} \right) \right] \\
&= \frac{1}{2\lambda} \left[\frac{e^{\mu t}}{t + \frac{1}{\lambda}} - \frac{e^{\mu t}}{t - \frac{1}{\lambda}} \right] \\
&= \frac{e^{\mu t}}{1 - \lambda^2 t^2}
\end{aligned}$$

□

Plot a graph for the beta-1 distribution when alpha=5, beta=2

1.4.6 CGF

1.5 Beta distribution of Type-I

1.5.1 PDF

Definition 1.35 (PDF of Beta I).

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} & 0 < x < 1; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Note the following,

1. We can say, $X \sim \beta_1(m, n)$ where m, n are the parameters of the distribution.
2. Since $f(x)$ is a pdf we have the following,

$$\begin{aligned}
\int_0^1 f(x) dx &= \int_0^1 \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1} dx \\
&= \frac{1}{\beta(m, n)} \int_0^1 x^{m-1} (1-x)^{n-1} dx \\
&= \frac{1}{\beta(m, n)} \beta(m, n)
\end{aligned}$$

1.5.2 Raw moments

Theorem 1.36. *The r^{th} raw moment of the Beta I distribution is given by*

$$\mu'_r = \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}$$

Proof.

$$\begin{aligned} \mu'_r = E[x^r] &= \int_0^1 \frac{1}{\beta(m,n)} x^{r+m-1} (1-x)^{n-1} dx \\ &= \frac{\beta(r+m,n)}{\beta(m,n)} \\ &= \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)} \end{aligned}$$

□

1.5.3 Mean and Variance

Theorem 1.37. *Mean of Beta I distribution is given by*

$$E[x] = \frac{m}{m+n}$$

Proof.

$$E[x] = \mu'_1 = \frac{\Gamma(m+n)\Gamma(m+1)}{\Gamma(m) + \Gamma(m+n+1)} = \frac{m}{m+n}$$

□

Theorem 1.38. *Variance of Beta I distribution is given by*

$$V[x] = \frac{mn}{(m+n)^2(m+n+1)}$$

Proof.

$$\mu'_2 = \frac{(m+1)(m)}{(m+n)(m+n+1)}$$

So now we have the variance given as,

$$\begin{aligned} \mu_2 &= \mu'_2 - \mu'^2_1 \\ &= \frac{mn}{(m+n)^2(m+n+1)} \end{aligned}$$

□

1.6 Beta distribution of Type-II

1.6.1 PDF

Definition 1.39 (PDF of Beta-II distribution).

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note the following,

1. X is said to follow $\beta_2(m, n)$ as $X \sim \beta_2(m, n)$
- 2.

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m, n)$$

1.6.2 Raw moments

Theorem 1.40 (Raw moments of Beta-2 distribution). *The raw moments of the Beta-2 distribution is given by*

$$\mu'_r = \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)}$$

Proof.

$$\begin{aligned} \mu'_r &= E[x^r] = \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m+r-1}}{(1+x)^{m+n}} dx \\ &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)} \end{aligned}$$

□

1.6.3 Mean and variance

Theorem 1.41 (Mean of Beta-2 distribution). *The mean of Beta-2 distribution is given by*

$$E[x] = \frac{m}{n-1}$$

Proof.

$$\begin{aligned} E[x] = \mu'_1 &= \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma(m)\Gamma(n)} \\ &= \frac{m}{n-1} \end{aligned}$$

□

Theorem 1.42 (Variance of Beta-2 distribution). *The variance of Beta-2 distribution is given by*

$$V[x] = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

Proof. First consider the 2nd raw moment,

$$\mu'_2 = \frac{m(m+1)}{(n-2)(n-2)}$$

Now we can compute the variance as follows

$$V[x] = \mu_2 = \mu'_2 - \mu_1'^2 = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

□

1.7 Transformation of variables

1.7.1 One dimensional random variable

Let X be a continuous random variable with pdf $f(x)$ and let $Y = g(x)$ be a strictly monotonic function of X with unique inverse.

Assume that $g(x)$ is differentiable and is continuous for all x , then the pdf of r.v. Y is given by

$$h(y) = f(x) \cdot \det \left| \frac{dx}{dy} \right| = \left| \frac{dx}{dy} \right|$$

where r.v. x is expressed in terms of y . Steps to solve,

1. Write pdf of r.v. X .
2. Express old variable X in terms of new variable Y .

3. Write the range of the new variable.
4. Obtain J where $J = \left| \frac{dx}{dy} \right|$ and $|J|$.
5. Obtain $h(y) = f(x) \cdot |J|$, where X is expressed in terms of Y .

Remark 1.43. For 2 – 1 correspondence, i.e. for every 2 values of X is there is only one value of Y , then multiply $|J|$ with 2.

Remark 1.44.

For 1 – 2 correspondence i.e., for every 1 value of x if there are 2 values of Y then multiply $|J|$ with $\frac{1}{2}$.

Example 1.45. If a r.v. $X \sim B_1(m, n)$ obtain the distribution of $Y = 1 - X$.

Proof. First begin by stating the pdf of X .

$$f(x) = \begin{cases} \frac{1}{\beta(m, n)} x^{m-1} (1-x)^{n-1} & 0 < x < 1; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $X = 1 - Y$ this ranges from $1 - Y = 0$ to $1 - Y = 1$. So $0 < Y < 1$ again.

Now compute J

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{dy} (1 - y) \\ J &= -1 \\ |J| &= 1 \end{aligned}$$

We multiply this with $f(x)$ to get $h(y)$.

$$\begin{aligned} h(y) &= f(x) \cdot |J| \\ h(y) &= f(x) \end{aligned}$$

So $h(y) \sim B(n, m)$. The order changes. □

Example 1.46. A r.v. $X \sim B_2(m, n)$. Obtain the distribution of Y where $Y = \frac{1}{1+X}$.

Proof. First state the pdf,

$$f(x) = \begin{cases} \frac{1}{\beta(m, n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now state X in terms of Y , we have $X = \frac{1}{Y} - 1$.

Compute the new ranges now we have $\frac{1}{Y} - 1 = 0$ so $Y = 1$ as one side then $\frac{1}{Y} - 1 = \lim_{m \rightarrow \infty} m$ so to $Y = 0$.

The new ranges are $0 < Y < 1$. Now compute $|J|$,

$$\begin{aligned} J &= \frac{dx}{dy} = \frac{1}{dy} \left(\frac{1}{y} - 1 \right) \\ &= -\frac{1}{y^2} \\ |J| &= \frac{1}{y^2} \end{aligned}$$

So now we can compute $h(y)$ as follows,

$$\begin{aligned} h(y) &= f(x)|J| \\ &= \frac{1}{\beta(m, n)} \frac{\left(\frac{1}{y} - 1\right)^{m-1}}{(1/y)^{m+n}} \frac{1}{y^2} \\ &= \frac{1}{\beta(m, n)} y^{n-1} (1-y)^{m-1} \end{aligned}$$

This is for the range we have and 0 otherwise. But I'm too lazy to typeset that out as a cases.

So we now have $Y \sim B_1(n, m)$. □

1.8 Two dimensional r.v.

Let X and Y be two continuous independent r.v. with joint pdf $f(x, y)$. Say $U = g(x, y)$ and $V = h(x, y)$ are two other r.v. then the joint pdf of U and V is given by,

$$k(u, v) = f(x, y) \cdot |J|$$

where X, Y are expressed in terms of U, V . Here we have the Jacobian as follows,

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

1.8.1 Steps to solve

1. Write the pdf of X and Y , i.e. $f(x, y)$.

2. Express old variable in terms of new variable.
3. Obtain range of the new variable.
4. Obtain J and $|J|$.
5. Obtain $k(u, v) = f(x, y)|J|$.

Example 1.47. X and Y are two independent gamma variates with parameters a and b respectively.

1. Obtain the joint distribution of u and v where $u = x + y, v = \frac{x}{x+y}$.
2. Show that u, v are independent and identify their distributions.

Proof. Consider the pdf of gamma function as follows,

$$\begin{aligned} X \sim G(\lambda) &= \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)} & x > 0, \lambda > 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

Where $\Gamma(\lambda) = (\lambda - 1)! = (\lambda - 1)\Gamma(\lambda - 1)$.

$$\begin{aligned} f_1(x) &= \frac{1}{\Gamma(a)}e^{-x}x^{a-1} \\ f_2(x) &= \frac{1}{\Gamma(b)}e^{-x}x^{b-1} \end{aligned}$$

Find $f(x, y) = f_1(x)f_2(y)$

$$\begin{aligned} f(x, y) &= \frac{1}{\Gamma(a)\Gamma(b)}e^{-x-y}x^{a-1}y^{b-1} & x, y, a, b, > 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

We now have the new variables U, V $U = X + Y, V = \frac{X}{X+Y}$. This implies that $X = UV, Y = U(1 - V)$.

We need to find the new ranges now. Since $X, Y > 0$ we have $U > 0$ and $X < X + Y \implies \frac{x}{x+y} < 1 \implies v < 1$. And $0 < V < 1$.

Find the Jacobian,

$$\begin{aligned} J &= \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -u \\ |J| &= u \end{aligned}$$

The joint distribution is then given as,

$$\begin{aligned}
 k(u, v) &= \frac{1}{\Gamma(a)\Gamma(b)} e^{-(uv+u-uv)} (uv)^{a-1} [u(1-v)]^{b-1} \cdot u \\
 &= \frac{1}{\Gamma(a)\Gamma(b)} e^{-u} u^{a-1+b-1+1} v^{a-1} (1-v)^{b-1} \times \frac{\Gamma(a+b)}{\Gamma(a+b)} \\
 &= \frac{1}{\Gamma(a+b)} e^{-u} u^{a+b-1} \times \frac{1}{\beta(a, b)} v^{a-1} (1-v)^{b-1} \\
 &= k_1(u)k_2(v)
 \end{aligned}$$

So u and v are independent r.v. and $U \sim G(a+b), V \sim \beta_1(a, b)$ \square

Example 1.48. X and Y are two independent r.v. $X \sim G(a)$ and $Y \sim G(b)$. We have $U = X + Y$ and $W = \frac{X}{Y}$. Show that U, W are independent and identify the distribution.

Proof. We know the following,

$$\begin{aligned}
 f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} & x > 0, a > 0 \\
 &= 0 & \text{otherwise}
 \end{aligned}$$

and,

$$\begin{aligned}
 f_2(y) &= \frac{e^{-y}y^{b-1}}{\Gamma(b)} & x > 0, b > 0 \\
 &= 0 & \text{otherwise}
 \end{aligned}$$

Now the joint distribution $f(x, y)$ is given by its product since they are independent,

$$\begin{aligned}
 f(x, y) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} \times \frac{e^{-y}y^{b-1}}{\Gamma(b)} & x > 0, y > 0; a, b > 0 \\
 &= 0 & \text{otherwise}
 \end{aligned}$$

Now we compute the new ranges $X = \frac{UW}{W+1}$ and $Y = \frac{U}{W+1}$. Now when $X = 0$ we have $U = 0, W = 0$ when $X \rightarrow \infty, U \rightarrow \infty, W \rightarrow \infty$. So we have $U > 0$ and $W > 0$.

Now compute the Jacobian as follows,

$$\begin{aligned}
 J &= \begin{bmatrix} \frac{w}{1+w} & \frac{-uw}{(1+w)^2} + \frac{u}{1+w} \\ \frac{1}{1+w} & \frac{-u}{(1+w)^2} \end{bmatrix} \\
 |J| &= \frac{u}{(1+w)^2}
 \end{aligned}$$

Since for 2 values of Y we get one value of X we will multiply the jacobian by 2. Now we compute $k(u, w)$ as follows,

$$\begin{aligned} k(u, w) &= f(x, y)|J| \\ &= \frac{e^{-\frac{uw}{w+1}} \frac{uw}{w+1} a^{-1}}{\Gamma(a)} \times \frac{e^{-\frac{u}{w+1}} \frac{u}{w+1} b^{-1}}{\Gamma(b)} \times \frac{u}{(1+w)^2} \\ &= \frac{1}{\Gamma(a+b)} e^{-u} u^{a+b-1} \times \frac{1}{\beta(a, b)} \end{aligned}$$

Complete this □

Example 1.49. $X \sim N(\mu, \sigma^2)$. Obtain the distribution of $Y = \frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2$

Proof. Begin by stating the pdf of r.v. X ,

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2} & -\infty < x < \infty, \sigma > 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

We now state X in terms of Y as follows, $X = \mu \pm \sqrt{2}\sigma\sqrt{y}$. Range of y is $0 < y < \infty$. And since it is 2-1 correspondence we will multiply the Jacobian by 2.

Compute the value of Jacobian first,

$$|J| = \frac{\sigma}{\sqrt{2}\sqrt{y}}$$

Now compute the new function,

$$\begin{aligned} h(y) &= f(x)|J|2 \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2} \times \frac{\sigma}{\sqrt{2}\sqrt{y}} \times 2 \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-y} \frac{2\sigma}{\sqrt{2}\sqrt{y}} \\ &= \frac{2}{\sqrt{2}\sqrt{y}\sqrt{2\pi}} e^{-y} \\ &= \frac{e^{-y}}{\sqrt{\pi}\sqrt{y}} \\ &= \frac{1}{\Gamma(\frac{1}{2})} e^{-y} y^{1-\frac{1}{2}} \end{aligned}$$

So we have $Y \sim G\left(\frac{1}{2}\right)$. □

Example 1.50.

$$f(x, y) = \begin{cases} 4xye^{-(x^2+y^2)} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove that $h(u) = 2u^3e^{-u^2}$, $u > 0$ where $u = \sqrt{x^2 + y^2}$ and $v = x$.

Proof. The variables we are dealing with are,

$$x = v, y = \sqrt{u^2 - v^2}$$

The range for v, u is $(0, \infty)$ but $0 < v < u < \infty$.

Begin by computing the Jacobian,

$$|J| = \frac{u}{\sqrt{u^2 - v^2}}$$

Consider now the joint distribution with the change of variables,

$$\begin{aligned} g(u, v) &= f(x, y)|J| \\ &= 4xye^{-(x^2+y^2)}|J| \\ &= 4(v)(\sqrt{u^2 - v^2})e^{-(v^2+u^2-v^2)}\frac{u}{\sqrt{u^2 - v^2}} \\ &= 4v\sqrt{u^2 - v^2}e^{-u^2}\frac{u}{\sqrt{u^2 - v^2}} \\ &= 4vue^{-u^2} \end{aligned}$$

Integrate out v

$$\begin{aligned} h(u) &= 4ue^{-u^2} \int_0^u v \, dv \\ &= 4ue^{-u^2} \frac{u^2}{2} \\ &= 2u^3e^{-u^2} \end{aligned}$$

□

Example 1.51.

$$f(x, y) = \begin{cases} \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Obtain pdf of u where $u = \frac{x}{x+y}$ take $v = x + y$ also obtain $E[u]$, $V[u]$.

Proof. Consider the new variables, $x = uv, y = v - uv$. The range for v is $(0, \infty)$ and for u is $(0, 1)$
 Compute the Jacobian,

$$|J| = v$$

Compute the joint pdf,

$$\begin{aligned} g(u, v) &= f(x, y)|J| \\ &= \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)}|J| \\ &= \frac{e^{-(uv+v-uv)}(uv)^3(v-uv)^4}{\Gamma(4)\Gamma(5)}v \\ &= \frac{e^{-v}u^3(1-u)^4v^8}{\Gamma(4)\Gamma(5)} \end{aligned}$$

Integrate out v

$$\begin{aligned} h(u) &= \frac{u^3(1-u)^4}{\Gamma(4)\Gamma(5)} \int_0^\infty e^{-v}v^8 dv \\ &= \frac{\Gamma(9)}{\Gamma(4)\Gamma(5)}u^3(1-u)^4 \end{aligned}$$

So $U \sim \beta_1(m=4, n=5)$. Compute the mean and variance as follows,

$$\begin{aligned} E[U] &= \frac{m}{m+n} = \frac{4}{9} \\ V[U] &= \frac{mn}{(m+n)^2(m+n+1)} = \frac{20}{810} = \frac{2}{81} \end{aligned}$$

□

Example 1.52. X, Y are two independent gamma variates with parameters a, b respectively. Show that $U = X/(X+Y), V = Y/(X+Y)$ are independent.

Proof. Consider the original pdfs,

$$\begin{aligned} f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} \\ f_2(y) &= \frac{e^{-y}y^{b-1}}{\Gamma(b)} \end{aligned}$$

Since they are independent

$$f(x, y) = \frac{e^{-(x+y)} x^{a-1} y^{b-1}}{\Gamma(a)\Gamma(b)}$$

Consider the new variables, $x = \frac{1}{2}(uv + u)$, $y = \frac{1}{2}(u - uv)$, the ranges for u is $(0, \infty)$ but for v is $(-1, 1)$

Compute the Jacobian,

$$|J| = \frac{u}{2}$$

Compute the joint pdf,

$$\begin{aligned} g(u, v) &= f(x, y)|J| \\ &= \frac{e^{-(x+y)} x^{a-1} y^{b-1}}{\Gamma(a)\Gamma(b)} |J| \\ &= \frac{e^{-u}(v+1)2^{-a-b+2}(u(v+1))^{a-2}(u-uv)^b}{(v-1)\Gamma(a)\Gamma(b)} \end{aligned}$$

Split this up I'm too lazy to type it. □

Chapter 2

Chi-square distribution

The chi squared distribution with n degrees of freedom is the distribution of a random variable that is the sum of squares of n independent standard normal variables.

2.1 PDF

Definition 2.1 (PDF of Chi-square distribution).

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}, 0 < x < \infty, n > 0$$

Proof. The PDF of $Z \sim SNV$ is given as

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$$

Let $X = Z^2 \implies Z = \pm\sqrt{X}$ and $X > 0$

The Jacobian is $J = \frac{dz}{dx} = \pm 1 \frac{1}{2\sqrt{x}}$ we multiply by 2 too.

$$\begin{aligned} f(x) &= f(z) \cdot |J| \cdot 2 \\ &= \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2\sqrt{x}} \cdot 2 \\ &= \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2} \end{aligned}$$

This is pdf of Gamma variate with $\lambda = 1/2, \theta = 1/2$. Since we want the sum of Z^2, n times by the additive property of the Gamma distribution we get that chi square with n d.f. follows $\chi_n^2 \sim G(\lambda = n/2, \theta = 1/2)$ \square

2.2 MGF

Theorem 2.2 (MGF of Chi square distribution).

$$M_x(t) = (1 - 2t)^{-n/2} \text{ if } |2t| < 1$$

Proof.

$$\begin{aligned} M_x(t) &= E[e^{tx}] = \int_0^\infty e^{tx} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} dx \\ &= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{n/2-1} e^{-(1/2-t)x} dx \\ &= \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \frac{\Gamma(n/2)}{(1/2-t)^{n/2}} \\ &= \frac{1}{2^{n/2}(1/2-t)^{n/2}} \\ &= \frac{1}{(1-2t)^{n/2}} \\ &= (1-2t)^{-n/2} \end{aligned}$$

□

2.3 CGF

Theorem 2.3. *The CGF of Chi squared distributed is given by*

$$K_x(t) = -\frac{n}{2} \log(1 - 2t)$$

Proof. MGF is given by $(1 - 2t)^{-n/2}$ and since CGF is just \log MGF. We have the result as required. □

The values of the cumulants are given as follows,

$$K_1 = n$$

$$K_2 = 2n$$

$$K_3 = 8n$$

$$K_4 = 48n \text{ so } \mu_4 = 48n + 12n^2$$

2.4 Skewness and kurtosis

Theorem 2.4. *Skewness is*

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{8}{n}$$

so it is positively skewed.

Theorem 2.5. *Kurtosis is*

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{12}{n}$$

so it is leptokurtic and approaches normal as $n \rightarrow \infty$.

Plot a graph of chi square with 6 degrees of freedom.

2.5 Additive property of chi squared distribution

Theorem 2.6. *If X_1, X_2, \dots, X_k are k independent χ^2 variates with degrees of freedom n_i for $i = 1, 2, \dots, k$ then*

$$\sum_i^n X_i \sim \chi_{\sum n_i}^2$$

Proof. Take the MGF multiply it that's it. □

2.6 Mode

We will obtain the mode as follows,

1. $f'(x) = 0$
2. Check if $f''(x) < 0$

Begin with the pdf,

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{n/2-1} & x > 0, n > 2 \\ 0 & o.w. \end{cases}$$

Now take its logarithm,

$$\log f(x) = \frac{f'(x)}{f(x)}$$

$$\begin{aligned}\frac{d \log f(x)}{dx} &= 0 \implies x = n - 2 \\ \frac{d^2 \log f(x)}{dx^2} &= 0 - \frac{n - 2}{2x^2} < 0\end{aligned}$$

You can also do it by just taking the derivative and doing it the long way.

You can do it in a third method by getting a recurrence relation after derivative it once.

Theorem 2.7 (Mode of Chi-square distribution). *Mode of Chi-square distribution with n degrees of freedom is $n - 2$*

2.7 Applications of Chi-square distribution

2.7.1 Goodness of fit

This is a very powerful test for testing significance of difference between theoretical and experimental values. It was given by Prof. Karl Pearson in 1900.

If $O_i, i = 1, 2, \dots, k$ is a set of observed or experimental frequencies and $E_i, i = 1, 2, \dots, k$ is a set of corresponding expected (hypothetical) frequencies, then Karl Pearson's χ^2 statistic is given by,

H_0 : There is no significant difference between the observed and expected frequencies (the dist. is a good fit).

H_1 : There is significant difference.

The test statistic is given by,

$$\chi^2 = \frac{\sum_{i=1}^k (O_i - E_i)^2}{E_i} \sim \chi_{k-1, \alpha}^2$$

The decision criteria is,

Reject H_0 if $\chi_{cal}^2 > \chi_{tab}^2 = \chi_{\alpha, k-1}^2$

For computation

$$\chi^2 = \frac{\sum O_i^2}{E_i} - N$$

The test is one sided (right sided).

Conditions for validity of the χ^2 test,

1. Sample observations should be independent.
2. Constraints on the cell frequencies should be similar, i.e. $\sum O_i = \sum E_i$.

3. The total frequencies (N) should be reasonably large. Say $N > 50$.
4. No theoretical frequencies should be less than 5. If a frequency is less than 5 then it is pooled with the preceding or succeeding frequency such that the pooled frequency is greater than 5. The degrees of freedom will decrease by the number of observations that are pooled.

Note: Sometimes while fitting a distribution the given data, some parameters have to be estimated. If p parameters are estimated, the degrees of freedom will be $k - p - 1$ where k is the number of classes. and p is the number of parameters estimated. Therefore, $d.f. = (k - 1) - (\text{no of pooled values}) - (\text{no. of parameters estimated})$

Example 2.8. Four identical coins are tossed 160 times, and the number of heads is recorded as follows, X : No. of heads

X	0	1	2	3	4
O_i	14	30	70	35	11

Test the hypothesis that the coins are perfect.

Proof. H_0 : Coins are perfect or Binomial is a good fit.

H_1 : Not H_0

We say X : No. of heads in 4 coin tosses and $X \sim B(n = 4, p = 0.5)$ and $N = 160$.

We get the following,

X	0	1	2	3	4
O_i	14	30	70	35	11
E_i	10	40	60	40	10

So $\chi_{cal}^2 = \frac{\sum O_i^2}{E_i} - N = 6.4917$ and $\chi_{tab}^2 = 9.488$. So $\chi_{cal}^2 < \chi_{tab}^2$ so it doesn't fall in critical reason and we fail to reject H_0 . \square

Example 2.9. A study of printing mistakes in a book of 550 pages gives the following data.

X : No of printing mistakes per page

X	0	1	2	3	4
O_i	485	52	8	4	1

Proof. H_0 : Poisson distribution is a good fit

H_1 : Not H_0

$X \sim P(0.1527 = \frac{42}{275})$

We get the following,

X	0	1	2	3	4
O_i	485	52	8	4	1
E_i	472	72	6	0	0

Pool the last 3 classes.

So $\chi_{cal}^2 = \frac{\sum O_i^2}{E_i} - N = 14.08027$ and $\chi_{tab}^2 = \chi_{0.05,1}^2 = 3.841$. So $\chi_{cal}^2 > \chi_{tab}^2$ so it falls in critical reason and we accept H_1 . \square

2.7.2 Test for variance

To test, $H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

Let α be l.o.s.

The test statistic is

$$T = \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma_0^2} \sim \chi_n^2$$

Note: If μ is unknown then $T = \frac{\sum (x_i - \bar{x})^2}{\sigma_0^2} \sim \chi_{n-1}^2$

Decision criteria:

Reject H_0 if $\chi_{cal}^2 > \chi_2^2$ or $< \chi_1^2$ Where $\chi_1^2 = \chi_{1-\alpha/2,n}^2$ and $\chi_2^2 = \chi_{\alpha/2,n}^2$

If the alternate hypothesis is one-sided, then it becomes a one-sided test.

If $H_1 : \sigma^2 > \sigma_0^2$, then D.C. is reject H_0 if $\chi_{cal}^2 > \chi_{\alpha,n}^2$

Similarly, if $H_1 : \sigma^2 < \sigma_0^2$ then, reject H_0 if $\chi_{cal}^2 < \chi_{1-\alpha,n}^2$

Confidence interval for σ^2 [(100(1 - σ))% Ci.i for population variance].

Let $\phi = \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2}$ be the pivotal quantity.

Then 100(1 - α)% C.I. for σ^2 is given as follows,

$$\begin{aligned} P[\chi_1^2 < \phi < \chi_2^2] &= (1 - \alpha) \\ \Rightarrow P\left[\chi_1^2 < \frac{\sum (x_i - \mu)^2}{\sigma^2} < \chi_2^2\right] &= 1 - \alpha \\ \Rightarrow P\left[\frac{\chi_1^2}{\sum (x_i - \mu)^2} < \frac{1}{\sigma^2} < \frac{\chi_2^2}{\sum (x_i - \mu)^2}\right] &= 1 - \alpha \\ P\left[\frac{\sum (x_i - \mu)^2}{\chi_2^2} < \sigma^2 < \frac{\sum (x_i - \mu)^2}{\chi_1^2}\right] &= 1 - \alpha \end{aligned}$$

So the $100(1 - \alpha)\%$ C.I. for σ^2 is

$$\left[\frac{\sum (x_i - \mu)^2}{\chi_{n, \alpha/2}^2}, \frac{\sum (x_i - \mu)^2}{\chi_{n, 1-\alpha/2}^2} \right] = \left[\frac{(n-1)s_{n-1}^2}{\chi_{n, \alpha/2}^2}, \frac{(n-1)s_{n-1}^2}{\chi_{n, 1-\alpha/2}^2} \right]$$

Where $\chi_1^2 = \dots$

2.7.3 Limiting form of χ^2 as $n \rightarrow \infty$

Let $X \sim \chi_n^2$ if $n \rightarrow \infty$ then $X \sim N(n, 2n)$

Proof. $M_x(t) = (1 - 2t)^{-n/2}$ let $z = \frac{x-\mu}{\sigma}$ then we have

$$\begin{aligned} M_{\frac{x-\mu}{\sigma}}(t) &= E \left[e^{t(\frac{x-\mu}{\sigma})} \right] \\ &= E \left[e^{t \frac{x-n}{\sqrt{2n}}} \right] \\ &= e^{-nt/\sqrt{2n}} E \left[e^{\frac{tx}{\sqrt{2n}}} \right] \\ M_z(t) &= e^{-t\sqrt{\frac{n}{2}}} \left(1 - \frac{2t}{\sqrt{2n}} \right)^{\frac{-n}{2}} \end{aligned}$$

Take log

$$\log M_z(t) = K_z(t) = -t\sqrt{n/2} - n/2 \log \left(1 - t\sqrt{2/n} \right)$$

Take the series expansion of log

$$\begin{aligned} &= -t\sqrt{n/2} + n/2 \left[t\sqrt{2/n} + \frac{t^2}{2} \frac{2}{n} + \frac{t^3}{3} \left(\frac{2}{n} \right)^{3/2} + \dots \right] \\ &= \frac{t^2}{2} + \mathcal{O}(n^{-1/2}) \end{aligned}$$

Ignore $\mathcal{O}(n^{-1/2})$.

Therefore, $\lim_{n \rightarrow \infty} K_z(t) = \frac{t^2}{2} \implies M_z(t) = e^{t^2/2}$ which is m.g.f. of a standard normal variance.

Hence by uniqueness theorem of mgf of Z is asymptotically standard normal. \square

2.8 Independence of attributes

Let us consider two attributes A and B . A divided into r classes and B divided into s classes.

Such a classification in which attributes are divided into more than two classes is known as manifold classification. The various cell frequencies can be expressed in the following table known as a $r \times s$ manifold contingency table.

Where (A_i) is the no. of persons possessing the attribute A_i and similarly for (B_j) . So $(A_i B_j)$ denotes the no. of persons possessing both A_i and B_j

$B \setminus A$	A_1	A_2	\dots	A_i	\dots	A_r	Total
B_1	$(A_1 B_1)$	$(A_2 B_1)$	\dots	$(A_i B_1)$	\dots	$(A_r B_1)$	(B_1)
B_2	$(A_1 B_2)$	$(A_2 B_2)$	\dots	$(A_i B_2)$	\dots	$(A_r B_2)$	(B_2)
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
B_j	$(A_1 B_j)$	$(A_2 B_j)$	\dots	$(A_i B_j)$	\dots	$(A_r B_j)$	(B_j)
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots
B_s	$(A_1 B_s)$	$(A_2 B_s)$	\dots	$(A_i B_s)$	\dots	$(A_r B_s)$	(B_s)
Total	(A_1)	(A_2)	\dots	(A_i)	\dots	(A_r)	N

The problem is to test if the two attributes A and B under consideration are independent or not.

H_0 : The two attributes are independent

H_1 : The two attributes are dependent.

Under the null hypothesis that the attributes are independent, the theoretical cell frequencies are calculated as follows:

$$P(A_i) = \text{Probability that a person possesses the attribute } A_i = \frac{(A_i)}{N}$$

$$P(B_j) = \text{Probability that a person possesses the attribute } B_j = \frac{(B_j)}{N}$$

$$P(A_i B_j) = \text{Probability that a person possesses the attributes } A_i, B_j = \frac{(A_i)}{N} \frac{(B_j)}{N} = P(A_i)P(B_j)^1$$

$$\text{Expected number of persons possessing both the attributes } A_i, B_j = (A_i B_j)_0 = NP[A_i B_j] = \frac{(A_i)(B_j)}{N}$$

Under the null hypothesis of independence of attributes the test statistic is,

$$\chi^2 = \sum_{i=1}^r \sum_{j=1}^s \left[\frac{((A_i B_j) - (A_i B_j)_0)^2}{(A_i B_j)_0} \right]$$

¹Since we are assuming they are independent under the null hypothesis.

which is distributed as χ^2 variate with $(r-1)(s-1)$ d.f. for large N .

Note on degrees of freedom: The number of independent variates which make up the statistic is known as the degrees of freedom and is usually denoted by ν .

The number of degrees of freedom, in general, is the total number of observations less than the number of independent constraints imposed on the observations. For example, if k is the number of independent constraints in a set of data of n observations then $\nu = n - k$

Thus in a set of n observations usually, the degrees of freedom for χ^2 are $n - 1$, one d.f. being lost because of the linear constraint $\sum O_i = \sum E_i = N$ on the frequencies.

If r independent linear constraints are imposed on the cell frequencies, then the d.f. are reduced by r .

In addition, if any of the population parameters are calculated from the given data and used for computing the expected frequencies then in applying χ^2 - test of goodness of fit, we have to subtract one d.f. from each parameter calculated. Thus if s is the number of population parameters estimated from the sample observations (n in number) then the required number of degrees of freedom for χ^2 test is $(n - s - 1)$.

If any one or more of the theoretical frequencies is less than 5 then in applying χ^2 test we have also to subtract the degrees of freedom lost in pooling these frequencies with the proceeding or succeeding frequency(ies).

In a $r \times s$ contingency table, in calculating the expected frequencies, the row totals, the column totals and the grand totals remain fixed. The fixation of r column totals and s row totals imposes $r + s$ constraints on the cell frequencies. But since

$$\sum_{i=1}^r (A_i) = \sum_{j=1}^s (B_j) = N$$

the total number of independent constraints is only $r + s - 1$. Further, since the total number of cell frequencies is $r \times s$ the required number of degrees of freedom is

$$\nu = rs - (r + s - 1) = (r - 1)(s - 1)$$

Example 2.10. *Two sample polls of votes for two candidates A and B for a public office are taken, one from among the residents of rural areas. The results are given the table. Examine whether the nature of the area is related*

to voting preference in this election.

Area \ Votes for	A	B	Total
Rural	620	380	1000
Urban	550	450	1000
Total	1170	830	2000

Proof. Note that $(r - 1)(s - 1) = 1 \cdot 1 = 1$.

$$(A_1B_1) = 620$$

$$(A_2B_1) = 380$$

$$(A_1B_2) = 550$$

$$(A_2B_2) = 450$$

Now check the expected values

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{1170 \cdot 1000}{2000} = 585$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{830 \cdot 1000}{2000} = 415$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{1170 \cdot 1000}{2000} = 585$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{830 \cdot 1000}{2000} = 415$$

So the test statistic is as follows,

$$\chi_{cal}^2 = 10.0916$$

Now the tabulated value is as follows,

$$\chi_{tab}^2 = \chi_{1,0.05}^2 = 3.841$$

Since $\chi_{cal}^2 > \chi_{tab}^2$ we reject H_0 . That is to say, that the nature of area is related to voting preference in the election. \square

Example 2.11.

Class \ Autonomy preference	In favour	In opposition	Total
FY	120	80	200
SY	130	70	200
TY	70	30	100
MA	80	20	100
Total	400	200	600

Proof. Write the hypothesis here I'm lazy.

Note that there are $(r - 1)(s - 1) = 3 \cdot 1 = 3$ degrees of freedom.

$$(A_1B_1) = 120$$

$$(A_2B_1) = 80$$

$$(A_1B_2) = 130$$

$$(A_2B_2) = 70$$

$$(A_3B_1) = 70$$

$$(A_3B_2) = 30$$

$$(A_4B_1) = 80$$

$$(A_4B_2) = 20$$

Now compute the expectations,

$$(A_1B_1)_0 = 133.333$$

$$(A_2B_1)_0 = 66.666$$

$$(A_1B_2)_0 = 133.333$$

$$(A_2B_2)_0 = 66.666$$

$$(A_3B_1)_0 = 66.666$$

$$(A_3B_2)_0 = 33.333$$

$$(A_4B_1)_0 = 66.666$$

$$(A_4B_2)_0 = 33.333$$

The test statistic is as follows,

$$\chi_{cal}^2 = 12.750$$

Check the χ_{tab}^2 and conclude. □

For a $r \times s$ contingency table in case of any expected value less than 5 it will be pooled with the nearby class and the degrees of freedom will be $(r - 1)(s - 1)$ —no. of pooling.

Example 2.12. *Observed matrix*

$$\left[\begin{array}{ccc|c} 50 & 40 & 10 & T = 100 \\ 30 & 20 & 1 & = 51 \\ T = 80 & = 60 & = 11 & = 151 \end{array} \right]$$

Expected values,

$$\begin{bmatrix} 53 & 40 & 7 & T = 100 \\ 27 & 20 & 3 & = 51 \\ T = 80 & = 60 & = 11 & = 151 \end{bmatrix}$$

Proof. We pool it as follows,

O_i	E
50	53
40	40
10	7
30	27
20	20
1	3

Just pool the last two rows, so $\chi_{tab}^2 = \chi_{(2-1) \times (3-1)-1}^2 = \chi_1^2 = 3.841$.

And the test statistic is as,

$$\chi_{cal}^2 = 1.9628$$

□

Theorem 2.13. *For a 2×2 contingency table as follows,*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the chi square test statistic is given as follows,

$$\chi^2 = \frac{N(ad - bc)^2}{(a + c)(b + d)(a + b)(c + d)}$$

Proof.

$$\begin{aligned} (A_1 B_1) &= a \\ (A_2 B_1) &= b \\ (A_1 B_2) &= c \\ (A_2 B_2) &= d \end{aligned}$$

and the expected values are as follows,

$$\begin{aligned}(A_1B_1)_0 &= \frac{(a+b)(a+c)}{N} \\ (A_2B_1)_0 &= \frac{(a+b)(b+d)}{N} \\ (A_1B_2)_0 &= \frac{(a+c)(c+d)}{N} \\ (A_2B_2)_0 &= \frac{(b+d)(c+d)}{N}\end{aligned}$$

The chi squared test statistic is as follows. This is why we have computers I don't wanna type this out bruh wtf I know how to solve it.

$$\chi_{cal}^2 =$$

□

Example 2.14. If X and Y are independent χ^2 variates with n_1, n_2 d.f. respectively, obtain the dist of $U = \frac{X}{Y}$

Proof. Let $U = \frac{X}{Y}$ and let $V = Y$ so we have $U > 0, V > 0$ and $Y = V, X = UV$.

$$\begin{aligned}f(x, y) &= f_1(x)f_2(y) \\ &= \frac{1}{2^{n_1/2}\Gamma(n_1/2)}e^{-x/2}x^{n_1/2-1} \frac{1}{2^{n_2/2}\Gamma(n_2/2)}e^{-y/2}y^{n_2/2-1} \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)}e^{-\frac{1}{2}(x+y)}x^{n_1/2-1}y^{n_2/2-1}\end{aligned}$$

Compute the Jacobian as follows,

$$|J| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1 \end{bmatrix} = \frac{1}{y} = v$$

So now our required $h(u, v)$ is as follows,

$$\begin{aligned}h(u, v) &= f(x, y)|J| \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)}e^{-\frac{1}{2}(v(1+u))}u^{n_1/2-1}v^{\frac{n_1+n_2}{2}-1} \times v\end{aligned}$$

Integrate out v

$$\begin{aligned}
&= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)} u^{n_1/2-1} \int_0^\infty e^{-\frac{v}{2}(1+u)} v^{\frac{n_1+n_2}{2}-1} dv \\
&= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)} u^{n_1/2-1} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}} \\
&= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}
\end{aligned}$$

So we get $U \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ □

2.8.1 Pre-requisite for Application 2 (Test for variance) using MGF

Theorem 2.15. Let X_1, X_2, \dots, X_n be a random sample from $N(\mu, \sigma^2)$, then

1. $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2. \bar{X} and $\sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n-1}$ are independently distributed
3. $\sum_{i=1}^n \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi_{n-1}^2$

$$s^2 = s_{n-1}^2 = \sum \frac{(X_i - \bar{X})^2}{n-1} = \text{sample variance.}$$

Proof. To prove A), B) we shall first prove that $\bar{X} = \sum \frac{X_i}{n}$ and $(X_i - \bar{X})$ are independently distributed.

Joint mgf of \bar{X} and $(X_i - \bar{X})$ is,

$$\begin{aligned}
M(t_1, t_2) &= E[\exp(t_1 \bar{X} - t_2(X_i - \bar{X}))] = E[\exp((t_1 - t_2)\bar{X} + t_2 X_i)] \\
&= E\left[\exp\left(\frac{t_1 - t_2}{n} \sum_{i=1}^n X_i + t_2 X_i\right)\right] \\
&= E\left[\exp\left(\left(\frac{t_1 - t_2}{n} + t_2\right) X_i\right)\right] E\left[\exp\left(\left(\frac{t_1 - t_2}{n}\right) \sum_{j=1, j \neq i}^n X_j\right)\right] \tag{2.1}
\end{aligned}$$

$U = \sum_{j=1, j \neq i}^n X_j$ being sum of $n-1$ iid $N(\mu, \sigma^2)$ is a $N((n-1)\mu, (n-1)\sigma^2)$.

$$M_u(t) = \exp \left[t(n-1)\mu + \frac{t^2}{2}(n-1)\sigma^2 \right]$$

$$\begin{aligned} \therefore E \left[\exp \left(\frac{t_1 - t_2}{n} \sum_{j=1, j \neq i}^n X_j \right) \right] &= E \left[\exp \left(\frac{t_1 - t_2}{n} U \right) \right] = M_u \left[\frac{t_1 - t_2}{n} \right] \\ &= \exp \left[\frac{t_1 - t_2}{n} (n-1)\mu + \left(\frac{t_1 - t_2}{n} \right)^2 (n-1) \frac{\sigma^2}{2} \right] \end{aligned} \quad (2.2)$$

$$\begin{aligned} E \left[\exp \left(\frac{t_1 - t_2}{n} + t_2 \right) X_i \right] &= M_{X_i} \left[\frac{t_1 - t_2}{n} + t_2 \right] \\ &= \exp \left[\left(\frac{t_1 - t_2}{n} + t_2 \right) \mu + \left(\frac{t_1 - t_2}{n} + t_2 \right)^2 \frac{\sigma^2}{2} \right] \end{aligned} \quad (2.3)$$

Substitute *ii*, *iii* to *i* we get,

$$\begin{aligned} M(t_1, t_2) &= \exp \left[\left(\frac{t_1 - t_2}{n} (n-1) + \frac{t_1 - t_2}{n} + t_2 \right) U \right] \cdot \exp \left[\left(\left(\frac{t_1 - t_2}{n} \right)^2 (n-1) + \left(\frac{t_1 - t_2}{n} + t_2 \right)^2 \right) \frac{\sigma^2}{2} \right] \\ &= \exp \left[t_1 \mu + \frac{1}{2} t_1^2 \frac{\sigma^2}{n} \right] \cdot \exp \left[\frac{1}{2} t_2^2 \frac{n-1}{n} \sigma^2 \right] \\ &= M(t_1) \cdot M(t_2) \end{aligned}$$

Now consider,

A) \bar{X} and $X_i \bar{X}$ are independently distributed -iv

B) $\bar{X} \sim N \left(\mu, \frac{\sigma^2}{n} \right)$ -v and $(X_i - \bar{X}) \sim N \left(0, \frac{n-1}{n} \sigma^2 \right)$ -vi

So \bar{X} and $(X_i - \bar{X})$ are independently distributed, \bar{X} and $s^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$ are independently distributed -vii

To derive distribution of s^2 we note that,

$$\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X} + \bar{X} - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

Product term vanishes since $\sum (X_i - \bar{X}) = 0$

$$\therefore \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{\sigma^2} + \left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right]^2 - viii$$

$$v = w + z$$

Here v being sum of squares of n independent SNV is $\chi_{(n)}^2$, therefore $M_v(t) = (1 - 2t)^{-n/2}$ -ix

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \implies \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \implies Z = \left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right]^2 \sim \chi_{(1)}^2$$

So $M_z(t) = (1 - 2t)^{-1/2}$ -x

So \bar{X} and s^2 are independent $\implies w, z$ are independent.

$$\begin{aligned} M_u(t) &= M_{w+z}(t) = M_w(t) + M_z(t) \\ (1 - 2t)^{-n/2} &= M_w(t)(1 - 2t)^{-1/2} \\ \therefore M_w(t) &= (1 - 2t)^{-\frac{(n-1)}{2}} \end{aligned}$$

Which is mgf of $\chi_{(n-1)}^2$

$$\therefore w = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} \sim \chi_{(n-1)}^2 - xi$$

c part proved

Now consider,

$$E \left[\sum_{i=1}^n \frac{(X_i - \bar{X})^2}{\sigma^2} \right] = n - 1 \implies E \left[\sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n - 1} \right] = \sigma^2$$

So s^2 is an unbiased estimator of σ^2 .

Note that,

1. If μ is known, $\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi_{(n)}^2$
2. When n is large, $\hat{\sigma}^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n}$

Conclusion is

1. $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2. $\sum_{i=1}^n \frac{(X_i - \bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$
3. \bar{X} and $\sum_{i=1}^n \dots$

□

2.9 Students t-distribution

Definition 2.16 (Definition of t -distribution). *If U, V are two independent random variables which $U \sim SN(0, 1)$ and $V \sim \chi_n^2$ then,*

$$X = \frac{U}{\sqrt{V/n}}$$

is said to follows t - distribution where n is small $n < 30$.

Definition 2.17 (PDF of t -distribution).

$$f(x) = \begin{cases} = \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} & -\infty < x < \infty \\ = 0 & \text{otherwise} \end{cases}$$

2.9.1 Derivation of PDF of t -distribution

Proof. Let U and V be tow independent random variables where $U \sim SN(0, 1)$ and $V \sim \chi_n^2$. We have $-\infty < U < \infty$ and $0 < v < \infty$.

Let $Y = V, X = \frac{U}{\sqrt{V/n}}$, range of $-\infty < X < \infty$ and $0 < Y < \infty$.

Consider the Jacobian, Compute the Jacobian as follows,

$$|J| = \sqrt{\frac{Y}{n}}$$

Now compute $f(x, y) = h(u, v)|J|$

$$\begin{aligned} f(x, y) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2 y}{2n}} \frac{1}{2^{n/2} \Gamma(n/2)} e^{-y/2} y^{n/2-1} \sqrt{\frac{y}{n}} \\ &= \frac{1}{\Gamma(1/2) \Gamma(n/2) 2^{\frac{n+1}{2}} n^{1/2}} e^{-y/2 \left(\frac{x^2}{n} + 1\right)} y^{\frac{n-1}{2}} \end{aligned}$$

$\therefore X, Y$ are independent,

$$\begin{aligned}
 f_1(x) &= \int_0^\infty f(x, y) dy \\
 &= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \int_0^\infty e^{\frac{-y}{2}\left(\frac{x^2}{n}+1\right)} y^{\frac{n-1}{2}} dy \\
 &= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \int_0^\infty \underbrace{y^{\frac{n}{2}+\frac{1}{2}-1} e^{-\left(\frac{1}{2}\left(\frac{x^2}{n}+1\right)\right)y}}_{\text{Gamma function}} dy \\
 f_1(x) &= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\left(\frac{x^2}{2n} + \frac{1}{2}\right)^{\frac{n+1}{2}}} \\
 &= \frac{1}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)n^{1/2}2^{\frac{n+1}{2}}} \frac{2^{\frac{n+1}{2}}}{\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}}} \\
 &= \frac{1}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}
 \end{aligned}$$

□

2.9.2 Moments of t distribution

Theorem 2.18 (Odd moments are zero).

$$\mu'_{2r+1} = E[x^{2r+1}] = \int_{-\infty}^\infty x^{2r+1} f(x) dx = 0$$

So mean=0 that implies raw moments are equal to central moments.

Theorem 2.19 (Even ordered moments).

$$\mu'_{2r} = \frac{n^r \Gamma\left(r + \frac{1}{2}\right) \Gamma\left(\frac{n}{2} - r\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}$$

Proof.

$$\begin{aligned}
 \mu'_{2r} &= \mu_{2r} = E[X^{2r}] \\
 &= \int_{-\infty}^{\infty} x^{2r} f(x) dx \\
 &= 2 \int_0^{\infty} x^{2r} f(x) dx \\
 &= \frac{2}{\beta\left(\frac{1}{2}, \frac{n}{2}\right) \sqrt{n}} \int_0^{\infty} x^{2r} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} dx
 \end{aligned}$$

Let $\frac{x^2}{n} = y \implies x^2 = ny$ so we have $\frac{dy}{dx} = \frac{2x}{n} \implies dx = \frac{ndy}{2x}$ Let beta term be c .

$$\begin{aligned}
 \mu_{2r} &= c \int_0^{\infty} (ny)^r (1+y)^{-\frac{n+1}{2}} \frac{n}{2\sqrt{ny}} dy \\
 &= \frac{cn}{2} \int_0^{\infty} (ny)^{r-1/2} (1+y)^{-\frac{n+1}{2}} dy \\
 &= \frac{nc}{2} n^{r-1/2} \int_0^{\infty} y^{r-1/2} (1+y)^{-\frac{n+1}{2}} dy \\
 &= \frac{nn^r}{2\sqrt{n} \frac{2}{\beta\left(\frac{1}{2}, \frac{n}{2}\right) \sqrt{n}}} \int_0^{\infty} \frac{y^{r-1/2}}{(1+y)^{\frac{n+1}{2}}} dy \\
 &= \frac{n^r}{\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \beta\left(r + \frac{1}{2}, \frac{n}{2} - r\right) \\
 &= \frac{n^r \Gamma\left(r + \frac{1}{2}\right) \Gamma\left(\frac{n}{2} - r\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}
 \end{aligned}$$

□

Theorem 2.20 (Variance).

$$V[x] = \frac{n}{n-2}$$

Proof.

$$\begin{aligned}
 V[x] &= \left. \frac{n^r \Gamma\left(r + \frac{1}{2}\right) \Gamma\left(\frac{n}{2} - r\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)} \right|_{r=2} \\
 &= \frac{n^2 \Gamma(3/2) \Gamma(n/2 - 1)}{\Gamma(1/2) \Gamma(n/2)} \\
 &= \frac{n}{n-2} \quad \text{for } n > 2
 \end{aligned}$$

□

Theorem 2.21 (Skewness).

$$\beta_1 = 0$$

Proof. Since $\mu_3 = 0$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

□

Theorem 2.22 (Kurtosis).

$$\beta_2 = 3 + \frac{6}{n-4}$$

Proof. We know $\mu_4 = \frac{3n^2}{(n-2)(n-4)}$ so then,

$$\begin{aligned}
 \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3n^2}{(n-2)(n-4)} \frac{(n-2)^2}{n^2} \\
 &= 3 + \frac{6}{n-4}
 \end{aligned}$$

□

2.10 Applications of t -distributions

2.10.1 Test for single mean

Assumptions

1. The parent population from which the sample is drawn is normal.

2. The sample drawn is a random sample, i.e. the observations are chosen independently.
3. Population variance is unknown.

To test if the random sample of size n is drawn from a normal population with mean $\mu = \mu_0$, or,

To test if the sample mean \bar{x} differs significantly from the hypothetical values μ_0 ,

$$H_0 : \mu = \mu_0 \text{ and } H_1 : \mu \neq \mu_0 \text{ with } LOS = \alpha$$

Test Statistic

For test statistic consider,

$$T = \frac{\bar{x} - \mu}{\hat{\sigma}/\sqrt{n}} = \frac{(\bar{x} - \mu)\sqrt{n}}{\hat{\sigma}} = \frac{(\bar{x} - \mu)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}}$$

Divide numerator and denominator by σ ,

$$\begin{aligned} T &= \frac{(x - \mu)\sqrt{n}/\sigma}{\sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}} \frac{1}{\sigma^2}} \\ &= \frac{\frac{(\bar{x} - \mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum (x_i - \bar{x})^2}{(n-1)\sigma^2}}} = \frac{SNV}{\sqrt{\frac{\chi^2}{n-1}}} \sim t_{n-1} \end{aligned}$$

Where,

$$\begin{aligned} \bar{x} &= \frac{\sum x}{n}, \hat{\sigma} = s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} \text{ for raw data} \\ \bar{x} &= \frac{\sum fx}{\sum f}, \hat{\sigma} = s = \sqrt{\frac{\sum f(x_i - \bar{x})^2}{\sum f - 1}} \text{ for frequency distribution} \end{aligned}$$

Decision criteria

Reject H_0 if T_{cal} lies in the critical region, i.e. $|T_{cal}| < -t_{\alpha/2, n-1}$ or $|T_{cal}| > t_{\alpha/2, n-1}$.

100(1 - α)% **confidence interval for μ**

We know that

$$\frac{\bar{x} - \mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1}$$

So 100(1 - α)% confidence intervals are given as,

$$P \left[-t_{n-1, \alpha/2} < \frac{\bar{x} - \mu}{\hat{\sigma}/\sqrt{n}} < t_{n-1, \alpha/2} \right] = (1 - \alpha)$$

$$P \left[\frac{-t\hat{\sigma}}{\sqrt{n}} < \bar{x} - \mu < \frac{t\hat{\sigma}}{\sqrt{n}} \right] = (1 - \alpha)$$

$$P \left[\frac{-t\hat{\sigma}}{\sqrt{n}} - \bar{x} < -\mu < \frac{t\hat{\sigma}}{\sqrt{n}} - \bar{x} \right] = (1 - \alpha)$$

$$P \left[\bar{x} - \frac{t\hat{\sigma}}{\sqrt{n}} < \mu < \bar{x} + \frac{t\hat{\sigma}}{\sqrt{n}} \right] = (1 - \alpha)$$

So the confidence interval is,

$$\bar{x} \pm t_{\alpha/2, n-1} \frac{\hat{\sigma}}{\sqrt{n}}$$

Example 2.23. *It is claimed that the depth of the sea has increased from the previous record for 55 meters. Test this claim given a sample of size 6 where the depths are 60, 62, 53, 57, 60, 59*

Proof. $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$ with $LOS = \alpha$

The test statistic is $t_{6-1} = t_{5, 0.05}$.

The table values are $[-2.57058, 2.57058]$

$$T_{cal} = \frac{58.5 - 55}{3.14642/\sqrt{6}} = 2.72475$$

Since $T_{cal} > T_{tab}$ we reject H_0 . □

2.10.2 Test for two means for two independent samples

Let x_1, x_2, \dots, x_{n_1} and y_1, y_2, \dots, y_{n_2} be two random samples from two normal populations with mean μ_x and variance σ^2 and μ_y and variance σ^2 .

Assumptions

1. The parent population follows normal distribution.
2. The two samples are independent.
3. The variances are unknown but equal.

To test if the two independent samples come from 2 populations with equal means, we proceed as follows.

Hypothesis

To test,

$$H_0 : \mu_x - \mu_y = d,$$

$$H_1 : \mu_x - \mu_y \neq d,$$

$$\text{LOS} = \alpha$$

T statistic

$$T = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\hat{\sigma} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\hat{\sigma}^2 = s_p^2 = \frac{\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2}{n_1 + n_2 - 2} \quad \text{Divide by } \sigma$$

$$\begin{aligned} T &= \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\hat{\sigma} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{SNV}{\sqrt{\frac{\chi^2}{n_1 + n_2 - 2}}} \sim t_{n_1 + n_2 - 2} \end{aligned}$$

Decision criteria

Reject H_0 if $T_{cal} < -t_{n_1 + n_2 - 2, \alpha/2}$ or $T_{cal} > t_{n_1 + n_2 - 2, \alpha/2}$

For $H_1 : \mu_x - \mu_y > d$ reject H_0 if $T_{cal} > t_{n_1 + n_2 - 2, \alpha}$

For $H_1 : \mu_x - \mu_y < d$ reject H_0 if $T_{cal} < -t_{n_1 + n_2 - 2, \alpha}$

Confidence intervals

$(1 - \alpha)100\%$ confidence intervals for $\mu_x - \mu_y$,

$$P \left[-t < \frac{\bar{x} - \bar{y} - (\mu_x - \mu_y)}{s_p \sqrt{1/n_1 + 1/n_2}} < t \right] = (1 - \alpha)$$

$$P \left[-ts_p \sqrt{1/n_1 + 1/n_2} - (\bar{x} - \bar{y}) < -(\mu_x - \mu_y) < ts_p \sqrt{1/n_1 + 1/n_2} - (\bar{x} - \bar{y}) \right] = (1 - \alpha)$$

$$P \left[(\bar{x} - \bar{y}) - ts_p \sqrt{1/n_1 + 1/n_2} < \mu_x - \mu_y < (\bar{x} - \bar{y}) + ts_p \sqrt{1/n_1 + 1/n_2} \right] = (1 - \alpha)$$

So the $100(1 - \alpha)\%$ confidence interval for $\mu_x - \mu_y$ is,

$$(\bar{x} - \bar{y}) \pm t_{(\alpha/2, n_1 + n_2 - 2)} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Example 2.24. Test for equality of mean for the following sets of values, 54, 60, 62, 55, 63 and 40, 60, 70, 50, 60, 50, 52. Also find confidence interval for difference in the means.

Proof. To test,

$$H_0 : \mu_x - \mu_y = d = 0,$$

$$H_1 : \mu_x - \mu_y \neq d = 0,$$

$$\text{LOS} = \alpha = 0.05$$

$n_1 = 5, n_2 = 7$ so the test statistic is given as t_{10} .

$$T = \frac{(\bar{x} - \bar{y})}{\hat{\sigma} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{58.8 - 54.5714}{7.9026 \sqrt{\frac{1}{5} + \frac{1}{7}}}$$

□

2.10.3 Paired t-test

If we want to test for a significant difference in the readings before and after a training program, we consider the difference in the readings as the variable d .

Consider n readings $d_i = (x_i - y_i)$

Hypothesis

To test,

$$H_0 : \mu = \alpha \text{ for some constant } \alpha$$

$$H_1 : \mu_d \neq \alpha$$

Test statistic

The test statistic is,

$$T = \frac{\bar{d} - \mu_d}{\hat{\sigma}_d / \sqrt{n}} \sim t_{n-1, \alpha/2}$$

where $\hat{\sigma}_d = \sqrt{\sum_{i=1}^n \frac{(d_i - \bar{d})^2}{n-1}}$, $\bar{d} = \sum_{i=1}^n \frac{d_i}{n}$

To prove that $T \sim t_{n-1}$ multiply and divide T by $\frac{1}{\hat{\sigma}_d}$

$$T = \frac{\frac{\bar{d} - \mu_d}{\hat{\sigma}_d}}{\frac{1}{\hat{\sigma}_d \sqrt{n}}} = \frac{SNV}{\sqrt{\frac{\chi^2}{n-1}}}$$

Decision criteria

Reject H_0 if $T_{cal} < -t_{n-1,\alpha/2}$ or $T_{cal} > t_{n-1,\alpha/2}$

Confidence intervals

$$\mu_d = \bar{d} \pm \frac{\hat{\sigma}_d}{\sqrt{n}} t_{n-1,\alpha/2}$$

Example 2.25. *content...*

Chapter 3

F-distribution

3.1 Applications of F-distributions

Example 3.1. $F_{0.01(9,4)} = 14.66$

Example 3.2. $F_{0.05(3,12)} = 3.49$

Example 3.3. $F_{0.99(4,10)} = \frac{1}{F_{0.01(10,4)}} = \frac{1}{14.5} = 0.0689$

Example 3.4. $F_{0.025(3,12)} = 4.47$

Example 3.5. $F_{0.005(4,9)} = 7.96$

Example 3.6. $F_{0.95(7,8)} = \frac{1}{F_{0.05(8,7)}} = \frac{1}{3.75} = 0.27$

Example 3.7. *content...*

3.1.1 One way ANOVA

Layout of one way ANOVA

Factor level	Observations	Total	Mean
1	$y_{11} y_{12} \dots y_{1n_1}$	$\sum_{j=1}^{n_1} y_{1j} = y_{1\cdot}$	$\frac{y_{1\cdot}}{n_1} = \overline{y_{1\cdot}}$
2	$y_{21} y_{22} \dots y_{2n_2}$	$\sum_{j=1}^{n_2} y_{2j} = y_{2\cdot}$	$\frac{y_{2\cdot}}{n_2} = \overline{y_{2\cdot}}$
\vdots	$\vdots \quad \vdots \quad \ddots \quad \vdots$	\vdots	\vdots
k	$y_{k1} y_{k2} \dots y_{kn_k}$	$\sum_j y_{kj} = y_{k\cdot}$	$\frac{y_{k\cdot}}{n_k} = \overline{y_{k\cdot}}$

y_{ij} is j^{th} observation due to i^{th} level of factor, $i = 1, 2, \dots, k; j = 1, 2, \dots, n_i$

$$\sum_{i=1}^k n_i = n_1 + n_2 + \cdots + n_k = N$$

$$y_{i\cdot} = \sum_{j=1}^{n_i} y_{ij}, \text{ when, } i = 1, 2, \dots, k \text{ total of } i^{\text{th}} \text{ row}$$

$$y_{\cdot\cdot} = \sum_{i=1}^k y_{i\cdot} = G = \text{Grand total} = \sum_{i=1}^k \left[\sum_{j=1}^{n_i} y_{ij} \right]$$

$$x\overline{y_{i\cdot}} = \frac{y_{i\cdot}}{n_i} = \sum_{j=1}^{n_i} \frac{y_{ij}}{n_i} = \text{Mean of obs of } i^{\text{th}} \text{ level of factor, } i = 1, 2, \dots, k$$

$$\overline{y_{\cdot\cdot}} = \frac{1}{N} \sum_i \sum_j y_{ij} = \frac{1}{N} \sum_i n_i \overline{y_{i\cdot}} = \text{Overall mean}$$

Null hypothesis

We want to test the equality of the population means, i.e. the homogeneity of different treatment. Hence, the null hypothesis is given by

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k = \mu$$

which reduces to

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Alternate hypothesis

At least two of the means $(\mu_1, \mu_2, \dots, \mu_k)$ are different.

One way ANOVA table

One way ANOVA table, Replace R with T below here,

Source of variance	d.f.	s.s.	m.s.s	F
Between factor (treatment)	$k - 1$	$\sum_i \frac{T_i^2}{n_i} - \frac{G^2}{N} = S_T^2$	$s_T^2 = \frac{S_T^2}{k-1}$	$F = \frac{s_T^2}{s_E^2}$
Within factor (error)	$N - k$	$+ = S_E^2$	$s_E^2 = \frac{s_E^2}{N-k}$	—
Total	$N - 1$	$\sum_i \sum_j y_{ij}^2 - \frac{G^2}{N}$	—	—

Test statistics

Using Cochran's theorem.

We have $\frac{s.s.due\ to\ factor}{\sigma^2}$, $\frac{s.s.due\ to\ error}{sigma^2}$ follows chi squared with $(k-1)$ and $(N-k)$ degrees of freedom respectively.

Therefore, the test statistics are defined as follows

$$f = \frac{\chi_1^2/d.f.}{\chi_2^2/d.f.} = \frac{\text{s.s. due to factors}/\sigma^2/(k-1)}{\text{s.s. due to error}/\sigma^2/(N-k)}$$

$$\frac{\text{M.s.s for factor}}{\text{m.s.s. for error}}$$

We will reject $H_0 : \alpha_i = 0 \forall i$ if $F_{cal} > F_{tab}$ as two independent χ^2 variates with $(k-1), (N-k)$ d.f. respectively.

3.1.2 Two way ANOVA**Layout of two-way classification**

Levels	1	2	...	j	...	h	Total	Mean
1	y_{11}	y_{12}	...	y_{1j}	...	y_{1h}	$y_{1\cdot}$	$\overline{y_{1\cdot}}$
2	y_{21}	y_{22}	...	y_{2j}	...	y_{2h}	$y_{2\cdot}$	$\overline{y_{2\cdot}}$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\vdots
i	y_{i1}	y_{i2}	...	y_{ij}	...	y_{ih}	$y_{i\cdot}$	$\overline{y_{i\cdot}}$
\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\vdots
k	y_{k1}	y_{k2}	...	y_{kj}	...	y_{kh}	$y_{k\cdot}$	$\overline{y_{k\cdot}}$
Total	$y_{\cdot 1}$	$y_{\cdot 2}$...	$y_{\cdot j}$...	$y_{\cdot h}$		
Mean	$\overline{y_{\cdot 1}}$	$\overline{y_{\cdot 2}}$...	$\overline{y_{\cdot j}}$...	$\overline{y_{\cdot h}}$		

Notation

$$N = kh$$

$$y_{..} = \sum_i \sum_j y_{ij} = G$$

$$\overline{y_{..}} = \sum_i \sum_j \frac{y_{ij}}{N} = \frac{y_{..}}{kh}$$

y_{ij} = The observation receiving i^{th} level of factor A and j^{th} level of factor B

$$R_i = y_{i\cdot} = \sum_{j=1}^h y_{ij}$$

$$\overline{y_{i\cdot}} = \frac{y_{i\cdot}}{h}$$

$$C_j = y_{\cdot j} = \sum_{i=1}^k y_{ij}$$

$$\overline{y_{\cdot j}} = \frac{y_{\cdot j}}{k}$$

Null hypothesis

H_{0_α} : There is no significant difference of factor A,

$$\mu_{1\cdot} = \mu_{2\cdot} = \cdots = \mu_{k\cdot} = \mu$$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

H_{0_β} : There is no significant difference of factor B,

$$\mu_{\cdot 1} = \mu_{\cdot 2} = \cdots = \mu_{\cdot h} = \mu$$

$$\beta_1 = \beta_2 = \cdots = \beta_h = 0$$

Alternative hypothesis

H_{1_α} : At least two of the $\mu_{i\cdot}$'s are different.

i.e. At least one of the α_i 's is not zero.

H_{1_β} : At least two of the $\mu_{\cdot j}$'s are different.

i.e. At least one of the β_j 's is not zero.

Degrees of freedom

- Total d.f. $N - 1$.
- Factor A d.f. $k - 1$.
- Factor B d.f. $h - 1$
- Error d.f. $N - k - h + 1 = (k - 1)(h - 1)$

Test statistics

Total s.s. = s.s. due to factor A + s.s. due to factor B + SSE $N - 1 = (k - 1) + (h - 1) + (k - 1)(h - 1)$

Therefore by Cochran's theorem,

$$\begin{aligned}\chi_1^2 &= \frac{SS_A}{\sigma^2} \sim \chi_{k-1}^2 \\ \chi_2^2 &= \frac{SS_B}{\sigma^2} \sim \chi_{h-1}^2 \\ \chi_3^2 &= \frac{SS_{err}}{\sigma^2} \sim \chi_{(k-1)(h-1)}^2\end{aligned}$$

And we have the test statistics as follows,

$$\begin{aligned}F_1 &= \frac{\chi_1^2/(k-1)}{\chi_3^2/((h-1)(k-1))} \\ F_2 &= \frac{\chi_2^2/(h-1)}{\chi_3^2/((h-1)(k-1))}\end{aligned}$$

Also $E[MSS_A] > E[MSSE]$ then H_{0_1} is rejected.
 $E[MSS_B] > E[MSSE]$ then H_{0_2} is rejected.

Two way ANOVA Table

Replace R with T below here,

Source of variance	d.f.	S.S.	S.S.S	F_{cal}
Between factor A	$k - 1$	$RSS = \sum_i^k \frac{R_i^2}{n_i} - \frac{G^2}{N}$	$MRSS = \frac{RSS}{k-1}$	$F_1 = \frac{MRSS}{MESS}$
Between factor B	$h - 1$	$CSS = \sum_j^h \frac{R_j^2}{n_j} - \frac{G^2}{N}$	$MCSS = \frac{CSS}{h-1}$	$F_2 = \frac{MLSS}{MESS}$
Error	$(k - 1)(h - 1)$	$ESS = +$	$MESS = \frac{CSS}{(h-1)(k-1)}$	—
Total	$N - 1$	$T_{ss} = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{N}$	—	—

3.2 Test for equality of variance

Let x_1, x_2, \dots, x_{n_1} and y_1, y_2, \dots, y_{n_2} be two independent random samples drawn from 2 normal populations with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively where μ_1, μ_2 are unknown.

To test,

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ vs } H_1 : \sigma_1^2 \neq \sigma_2^2$$

Now $\sum_{i=1}^{n_1} \frac{(x_i - \bar{x})^2}{\sigma_1^2} \sim \chi_{n_1-1}^2$ and $\sum_{i=1}^{n_2} \frac{(y_i - \bar{y})^2}{\sigma_2^2} \sim \chi_{n_2-1}^2$

$$F = \frac{\sum_{i=1}^{n_1} \frac{(x_i - \bar{x})^2}{\sigma_1^2}}{\sum_{i=1}^{n_2} \frac{(y_i - \bar{y})^2}{\sigma_2^2}} \sim F(n_1 - 1, n_2 - 1)$$

Under H_0 , $F = \frac{s_1^2}{s_2^2}$

3.3 PDF

F distribution is the ratio of 2 mean chi square variates, i.e. if U and V are tow independent χ^2 variates with n_1, n_2 degrees of freedom respectively then,

$$X = \frac{U/n_1}{V/n_2} \sim F_{n_1, n_2}$$

and its pdf is given as follows,

Definition 3.8 (PDF of F distribution).

$$f(x) = \begin{cases} \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{x n_1}{n_2}\right)^{\frac{n_1+n_2}{2}}} & x > 0, n_1, n_2 > 0 \\ 0 & otherwise \end{cases}$$

3.3.1 Derivation of PDF

Proof. Let U, V be 2 independent random variates where $U \sim \chi_{n_1}^2, V \sim \chi_{n_2}^2$

Let $X = \frac{U/n_1}{V/n_2} = \frac{n_2}{n_1} - \frac{U}{V}$

Let $Y = V$

$$h(u, v) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma(n_1/2) \Gamma(n_2/2)} e^{-\left(\frac{u+v}{2}\right)} u^{n_1/2-1} v^{n_2/2-1}$$

Compute the Jacobian,

$$|J| = y \frac{n_1}{n_2}$$

So we have $f(x, y) = h(u, v)|J|$

$$\begin{aligned}
 f(x, y) &= \frac{e^{-y/2(1+xn_1/n_2)} x^{n_1/2-1} (n_1/n_2)^{n_1/2} y^{n_1/2+n_2/2-1}}{2^{(n_1+n_2)/2} \Gamma(n_1/2) \Gamma(n_2/2)} \\
 f_1(x) &= \int_0^\infty f(x, y) dy \\
 &= \frac{x^{n_1/2-1} (n_1/n_2)^{n_1/2}}{2^{(n_1+n_2)/2} \Gamma(n_1/2) \Gamma(n_2/2)} \int_0^\infty y^{n_1/2+n_2/2-1} e^{-y/2(1+xn_1/n_2)} dy \\
 &= \frac{x^{n_1/2-1} (n_1/n_2)^{n_1/2}}{2^{(n_1+n_2)/2} \Gamma(n_1/2) \Gamma(n_2/2)} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\left[\frac{1}{2} \left(1 + \frac{xn_1}{n_2}\right)\right]^{\frac{n_1+n_2}{2}}} \\
 &= \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} x^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{xn_1}{n_2}\right)^{\frac{n_1+n_2}{2}}}
 \end{aligned}$$

□

3.4 r^{th} order raw moment of F-distribution

Theorem 3.9. *The r^{th} order raw moment of the F-distribution is given as*

$$\mu'_r = \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right)$$

Proof.

$$\begin{aligned}
 \mu'_r &= E[x^r] = \int_0^\infty x^r f(x) dx \\
 &= \int_0^\infty x^r \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{xn_1}{n_2}\right)^{\frac{n_1+n_2}{2}}} dx \\
 &= \int_0^\infty \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}+r-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{xn_1}{n_2}\right)^{\frac{n_1+n_2}{2}}} dx \\
 &= \frac{(n_1/n_2)^{n_1/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{x^{n_1/2+r-1}}{\left(1 + \frac{xn_1}{n_2}\right)^{\frac{n_1+n_2}{2}}} dx
 \end{aligned}$$

Let $xn_1/n_2 = y$ we get $dx = n_2/n_1 dy$

$$\begin{aligned} &= \frac{(n_1/n_2)^{n_1/2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{y^{n_1/2+r-1}}{(1+y)^{\frac{n_1+n_2}{2}+r-r}} \frac{n_2}{n_1} dy \\ &= \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right) \end{aligned}$$

□

3.5 Mean and variance

Theorem 3.10 (Mean). *Mean of F distribution is given as*

$$E[x] = \mu'_1 = ???$$

Proof.

$$\begin{aligned} E[x] &= \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right) \Bigg|_{r=1} \\ &= \frac{n_2/2}{(n_2-2)/2} = \frac{n_2}{n_2-2} \end{aligned}$$

□

Theorem 3.11 (Variance). *Variance of F distribution is given as*

$$V[x] = \mu'_2 - \mu'^2_1 = ???$$

Proof. First find second raw moment

$$\begin{aligned} \mu_2 &= \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + r, \frac{n_2}{2} - r\right) \Bigg|_{r=2} \\ &= \frac{\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2} + 2, \frac{n_2}{2} - 2\right) \\ &= \frac{(n_1+2)n_2^2}{n_1(n_2-4)(n_2-2)} \end{aligned}$$

Now the variance is given as $\mu_2 = \mu'_2 - \mu'^2_1$

$$\mu_2 = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-4)(n_2-2)^2}$$

□

3.6 Mode of F-distribution

Theorem 3.12 (Mode). *Mode of F distribution is given as*

????

Proof. We need to do the following,

1. $f'(x) = 0$
2. Check if $f''(x) < 0$.

Take $\log f(x)$ so we get

$$\begin{aligned}\frac{d \log f(x)}{dx} &= \frac{f'(x)}{f(x)} = \frac{n_1 - 2}{2} \frac{1}{x} - \frac{n_1 + n_2}{2} \frac{n_1/n_2}{1 + xn_1/n_2} \\ f'(x) &= \left[\frac{n_1 - 2}{2} \frac{1}{x} - \frac{n_1 + n_2}{2} \frac{n_1}{n_2 + xn_1} \right] f(x)\end{aligned}$$

$f'(x) = 0$

$$\begin{aligned}\implies \frac{n_1 - 2}{2} \frac{1}{x} - \frac{n_1 + n_2}{2} \frac{n_1}{n_2 + xn_1} &= 0 \\ \implies \frac{n_1 - 2}{2x} &= \frac{n_1^2 + n_1 n_2}{2n_2 + 2xn_1} \\ \implies 2(n_2 + xn_1)(n_1 - 2) &= 2xn_1(n_1 + n_2) \\ &\vdots \\ x &= \frac{n_2(n_1 - 2)}{n_1(n_2 - 2)}\end{aligned}$$

Don't bother to do the second derivative just assume.

□

3.7 Relation between $F(n_1, n_2)$ and $F(n_2, n_1)$

Theorem 3.13. $F(n_1, n_2)$ and $F(n_2, n_1)$

Proof. $X \sim \chi_{n_1}^2$ and $Y \sim \chi_{n_2}^2$

We know $F = \frac{X/n_1}{Y/n_2} \sim F(n_1, n_2)$

and $1/F = \frac{Y/n_2}{X/n_1} \sim F(n_2, n_1)$

Consider $P[F(n_1, n_2) \geq c] = 1 - \alpha \implies c = F_{1-\alpha}(n_1, n_2)$

also,

$$\begin{aligned}
\alpha &= 1 - P(F(n_1, n_2) \geq c) \\
&= 1 - P\left[\frac{1}{F(n_1, n_2)} < \frac{1}{c}\right] \\
&= 1 - P\left[F(n_2, n_1) < \frac{1}{c}\right] \\
&= P\left[F(n_2, n_1) \geq \frac{1}{c}\right] \\
&\implies \frac{1}{c} = F_\alpha(n_2, n_1) \\
c &= \frac{1}{F_\alpha(n_2, n_1)}
\end{aligned}$$

So we get $F_{1-\alpha}(n_1, n_2) = \frac{1}{F_\alpha(n_2, n_1)}$. □

3.8 Relation between t, F, χ^2

3.8.1 Relationship between t and F

Theorem 3.14. *If $X \sim t_n$ then $X^2 \sim F(1, n)$*

Proof. Let $X \sim t_n$ and $U \sim N(0, 1)$ and $V \sim \chi_n^2$

$$X = \frac{SNV}{\sqrt{\frac{\chi_n^2}{n}}} = \frac{U}{\sqrt{V/n}}$$

Let $Y = X^2 = \frac{U^2}{V/n} = \frac{\chi_1^2/1}{\chi_n^2/n} \sim F(1, n)$ this is because square of SNV follows χ^2 with 1 d.f. by definition. □

Theorem 3.15. *If $X \sim F(1, n)$ show that $\pm\sqrt{X} \sim t_n$ using transformation.*

Proof. Let $Y = \pm\sqrt{X}$ and $X = Y^2$ so $dx = 2ydy$. And $x > 0 \implies -\infty < y < \infty$ so there is a one-to-one correspondence so divide Jacobian by 1/2.

So $dx/dy = |J| = 2y$

$$\text{Now } f(x) = \frac{(1/n)^{1/2} x^{-1/2}}{\beta(1/2, n/2)(1+x/n)^{(n+1)/2}}$$

So we have

$$\begin{aligned}
h(y) &= f(x)|J|\frac{1}{2} \\
&= \frac{(1/n)^{1/2} y^{2(-1/2)}}{\beta(1/2, n/2)(1+y^2/n)^{(n+1)/2}} \times y \\
&= \frac{(1+y^2/n)^{-(n+1)/2}}{\beta(1/2, n/2)\sqrt{n}} \sim t_n
\end{aligned}$$

□

3.8.2 Relationship between F and χ^2

Theorem 3.16. Let $X \sim F(n_1, n_2)$ and $n_2 \rightarrow \infty$ obtain the distribution of $Y = n_1 x$

Proof.

$$f(x) = \frac{(n_1/n_2)^{n_1/2} x^{n_1/2-1}}{\beta(n_1/2, n_2/2) (1 + x n_1/n_2)^{\frac{n_1+n_2}{2}}}$$

Exxpand out the beta function

$$= \frac{(n_1/n_2)^{n_1/2} (y/n_1)^{n_1/2-1}}{\Gamma(n_1/2)} \times \frac{1}{(1 + x n_1/n_2)^{\frac{n_1+n_2}{2}}} \times \frac{\Gamma((n_1 + n_2)/2)}{\Gamma(n_2/2)}$$

$$Y = n_1 x \implies x = y/n_1 \implies J = dx/dy = 1/n_1.$$

Now we know that,

$$\lim_{m \rightarrow \infty} \frac{\Gamma[m+k]}{\Gamma[m]} = m^k$$

$$\lim_{m \rightarrow \infty} \left(1 + \frac{x}{m}\right)^m = e^{-x}$$

Now consider $n_2 \rightarrow \infty$

$$\lim_{n_2 \rightarrow \infty} \frac{\Gamma[(n_1 + n_2)/2]}{\Gamma(n_1/2)} = \left(\frac{n_2}{2}\right)^{\frac{n_1}{2}}$$

$$\lim_{n_2 \rightarrow \infty} \frac{???}{n_2^{\frac{n_1}{2}}}$$

$$h(y) = f(x)|J| \text{ wen } n_2 \rightarrow \infty$$

$$h_2(y) = \frac{1}{2^{n_1/2} \Gamma\left(\frac{n_1}{2}\right)} e^{-y/2} y^{\frac{n_1}{2}-1} \frac{n_1^{\frac{n_1}{2}-\frac{n_1}{2}+1-1} n_2^{\frac{n_1}{2}}}{n_2^{\frac{n_1}{2}}}$$

$$= \frac{1}{2^{n_1/2} \Gamma(n_1/2)} e^{-y/2} y^{n_1/2-1}$$

So $Y \sim \chi_{n_1}^2$

□

