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Algebra IV

Lecture Notes
for SMAT402

Contents

Chapter 1

Groups and subgroups

1.1 Binary operation

For a set V a function from $f : V \times V \to V$ is called a binary function if the following properties hold.

1. f is defined for all pairs of elements of V.
2. f is closed.

Example 1.1. $G = \{1, 2, 3\}$, *then* $+$ *is not a binary operation as it is not closed under addition.*

Example 1.2. $G = \{-1, 0, 1\}$, *then* $+$ *is a binary operation.*

Example 1.3. $\mathbb{N}$, *then both* $+, \times$ *are binary operations.*

1.2 Group axioms

A group is an ordered pair (G, *) where G is a non empty set and * is a binary operation on G satisfying the following axioms:

1. **Closure:** ∀ a, b ∈ G, a * b, is also in G
2. **Associativity:** (a * b) * c = a * (b * c), ∀ a, b, c ∈ G
3. **Identity:** ∃ e ∈ G, called an identity of G, s.t. ∀ a ∈ G we have a * e = e * a = a
4. **Inverse:** ∀ a ∈ G ∃ a^{-1} ∈ G, called an inverse of a, s.t. a * a^{-1} = a^{-1} * a = e.

1.3 Examples of Groups

Example 1.4. $(\mathbb{N}, +)$ *is not a group since it lacks additive identity.*

Example 1.5. $(\mathbb{Z}, +)$ *is a group while* $(\mathbb{Z}, \times)$ *is not a a group since it lacks multiplicative inverses.*

Example 1.6. $(\mathbb{Q}, \times)$ *is not a group since* 0 *doesn't have an inverse. However* $(\mathbb{Q} \setminus 0, \times)$ *is a group.*

Example 1.7. $n\mathbb{Z} = \{\dots, -2n, -n, 0, n, 2n \dots\}$ *with addition are subgroups of* $(\mathbb{Z}, +)$.

Example 1.8. $S = \{1, -1, i, -1\}$, *with multiplication is a cyclic group generated by* i. *Exercise make a Cayley table.*

Example 1.9. $M_{n\times n}(\mathbb{R})$ *for* $n \times n$ *matrices over* $\mathbb{R}$ *forms a group under addition but not under matrix multiplication (because of lack of inverses).*

Example 1.10. $GL_n(\mathbb{R})$ *(i.e. General linear group - matrices with positive determinant) forms a group under multiplication.*

Example 1.11. $SL_n(\mathbb{R})$ *(i.e. Special linear group - matrices with det=1) forms a group under multiplication.*

1.3.1 Group of integers modulo n

Definition 1.12 (Congruence class)**.** *For* $n \in \mathbb{Z}$ *define the congruence relation* R *as* $aRb \iff n|(a-b)$. *This is a equivalence relation.*

Definition 1.13 ($\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}_n$)**.** *Let* $\mathbb{Z}/n\mathbb{Z}$ *be defined as the* $\{x \in \mathbb{Z} \mid xRn\}$.

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$$

Addition $\overline{a} + \overline{b} = \overline{a+b}$ *and multiplication* $\overline{a} \cdot \overline{b} = \overline{ab}$.
$(\mathbb{Z}_n, +)$ *forms a group for all* n, *while* $(\mathbb{Z}_n*, \cdot)$ *forms a group only when* n *is prime.*

Theorem 1.14. $\mathbb{Z}_n*$ *forms a group under multiplication iff* n *is prime.*

Proof. The proof is trivial. □

1.3.2 Klein-4 group (Vierergruppe)

Denoted by V_4 the Klein-4 group is the smallest non-cyclic group. It is abelian. It is a group with 4 elements such that the square of all elements is identity. And product of two distinct elements gives a distinct element.
The symmetry group of a rectangle is isomorphic to V_4.

1.3.3 Symmetric group

The symmetric group is the group whose elements are all the bijections from the set to itself. The order of the n^{th} Symmetric group (S_n) is equal to $n!$.

Two-Line to Cycle notation for permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} = (125)(34) = (34)(125) = (34)(512) = (15)(25)(34)$$

Here, the last form is a case of 2-cycle (transposition).
The parity of any permutation σ is given by the parity of the number of its 2-cycles (transpositions). In the above example it is odd.

1.3.4 Alternating group

The group of all even permutations from S_n is called the alternating group A_n.

1.3.5 Dihedral group

This is the group of symmetries of a regular polygon. Denoted by $D_n, n \geq 3$.

- Order of $D_n = 2n$.
- $D_n = \{e, x, x^2, \ldots, x^{n-1}, y, yx, yx^2, \ldots, yx^{n-1}\}$. Here we can interpret x as rotation by $2\pi/n$and y is reflection about vertical axis.

1.4 Common properties of groups

1.4.1 Abelian group

If a group is commutative it is called Abelian.

- If $a^2 = e \forall a \in G$ then it is Abelian.

1.4.2 Order of a group

If there are a finite number of elements in a group then the group is called a finite group and the number of elements is called the group order of the group.

1.4.3 Order of element

The smallest natural number n such that $a^n = e$ is called the order of a group element a.

Chapter 2

Cyclic groups and cyclic subgroups

Chapter 3

Lagrange's theorem and group homomorphisms

Definition 3.1 (Cosets)**.** *For any $H \leq G$ where $(G, \dot{)}$ and any $a \in G$*

- $aH = \{ah | h \in H\} = \{a, ah_1, ah_2, \dots\}$ *and,*
- $Ha = \{ha | h \in H\} = \{a, ah_1, ah_2, \dots\}$

are called a left coset and right coset respectively.

Lemma 3.2. *1.* $a \in aH$

2. $aH = H \iff a \in H$

3. $(ab)H = a(bH)$ *and* $H(ab) = (Ha)b$

4. $aH = bH \iff a \in bH$

5. $aH = bH$ *or* $aH \cap bH = \emptyset$

6. $aH = bH \iff ab^{-1} \in H$

7. $|aH| = |bH|$

8. $aH = Ha \iff H = aHa^{-1}$

9. aH *is a subgroup of* $G \iff a \in H$.

Proof. 1. H is a subgroup so it will have the identity so, $ae = a \in aH$.

2. Unidirectional part: $aH = H$ then $a \in H$. Since $a \in aH$ then from $aH = H$ we know $a \in H$.

Backwards: Since $a \in H$ and it is closed we know $aH \subseteq H$. Now we must prove $H \subseteq aH$.

We know that $a^{-1} \in H$ so for any $h \in H$ we want to prove $h = ak$ for some $k \in H$ say $k = a^{-1}h$ so $H \subseteq aH$, and so $H = aH$.

3. For $h \in H$, Since $(ab)h = a(bh)$ and $h(ab) = (ha)b$

4. If $aH = bH$ then $a = ae \in aH = bH$. Conversely if $a \in bH$ we have $a = bh$ for $h \in H$ so $aH = (bh)H = b(hH) = bH$

5. $aH = bH$ or $aH \cap bH = \emptyset$. Prove by contradiction if $aH \neq bH$ and $aH \cap bH \neq \emptyset$ but then we have $c \in aH \cap bH$. Then from property 4 $aH = cH = bH$.

6. $aH = bH$ iff $H = a^{-1}bH$ now from property 2.

7. $|aH| = |bH|$ prove there is a 1-1 map. $f(ah) = bh$

8.In forward direction$aH = Ha \implies H = aHa^{-1}$, we have $ah_1 = h_2a \implies ah_1a^{-1} = h_2 = H$.

Prove backward direction as h.w.

9. aH is a subgroup $\iff a \in H$ but $a \in H \iff aH = H \implies aH$ is a subgroup $\iff a \in H$. □

Theorem 3.3 (Lagrange)**.** *If G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$. Moreover the number of distinct left cosets of H in G is $|G|/|H|$.*

Proof. content... □

Corollary 3.4. $[G : H] = |G|/|H|$ *If G is a finite group and H is a subgroup of G, then $[G : H] = |G|/|H|$.*

Corollary 3.5. *$|a|$ divides $|G|$ Order of an element is the order of the subgroup generated by that element.*

Corollary 3.6 (Groups of prime order are cyclic)**.** *A group of prime order is cyclic*

Corollary 3.7. *Let $a \in G$ finite then $a^{|G|} = e$*

Corollary 3.8 (Fermat's little theorem)**.** *For every integer a and every prime p, $a^p \mod p = a \mod p$*

Corollary 3.9 (Euler's theorem)**.** *If n and a are coprime positive integers and $\phi(n)$ denotes Euler's phi function then $a^{\phi(n)} \equiv 1 \mod n$*

Corollary 3.10. *If a finite group G has no non-trivial subgroups then $|G|$ is a prime number and G is cyclic.*

Proof. $|G|$ is divisble by only $|G|$ and 1. □