Bhoris Dhanjal

Calculus IV

Lecture Notes
for SMAT401

Contents

Chapter 1

Functions of several variables

1.1 Examples of functions of several variables

$f(x,y) = x + y \log x$	$f : \mathbb{R}^2 \to \mathbb{R}$	Scalar valued function
$f(x,y) = (x^2y, \cos x, e^x - 9)$	$f : \mathbb{R}^2 \to \mathbb{R}^3$	Vector valued function

Clearly, $f : \mathbb{R} \to \mathbb{R}$ is a particular case of scalar valued function.

1.2 Non-existence of limit by 2 path test

For a function $f : \mathbb{R} \to \mathbb{R}$ the limit exists if limit value is the same along all possible paths, i.e. left hand and right hand limit are equivalent.

For a multivariate function the 2 path test can be used to show non existence of a limit.

Example 1.1. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$ *doesn't exist.*

Proof. Consider $x = my^2$ and let $y \to 0$, then

$$\lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^2} = \frac{2m}{1+m^2}$$

.

Therefore, the limit value varies for different values of m. □

Example 1.2. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ *doesn't exist.*

Proof. Consider first along x axis (i.e. $y = 0$)

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Consider now along y axis (i.e. $x = 0$)

$$\lim_{y \to 0} \frac{y}{-y} = -1$$

Since the limit is not path independent we can say the limit does not exist. $\square$

Example 1.3. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ *doesn't exist.*

Proof. Along x and y axis the limits are both zero. Consider instead the path $y = x^2$

$$\lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limit is not path independent it does not exist. $\square$

Example 1.4. *Show that the* $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2 - 2x}$ *doesn't exist.*

Proof. Along x, y axis the limit is 0. Consider the path $y = \sqrt{2x}$

$$\lim_{x \to 0} \frac{x^2}{x^2} = 1$$

Since the limit is not path independent it does not exist. $\square$

1.3 Existence of limit with ε, δ definition

Recall the single variable definition of a limit,

Definition 1.5 (Limit of a single valued function)**.** *For a function* $f : \mathbb{R} \to \mathbb{R}$, $\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ *such that* $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$

Definition 1.6 (Limit of a multivariate function)**.** *For a function* $f : \mathbb{R}^2 \to \mathbb{R}$, $\lim_{(x,y)\to(a,b)} f(x) = L \iff \forall \varepsilon > 0, \exists \delta$ *such that*

$$0 < ||(x, y) - (a, b)||_2 < \delta \implies |f(x, y) - L| < \varepsilon$$

, alternatively

$$\sqrt{(x - a)^2 + (x - b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Example 1.7. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x-y}{1+x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$, consider

$$\begin{aligned}|f(x,y)-L| = |f(x,y)| &= \left|\frac{x-y}{1+x^2+y^2}\right| \\ &= \frac{|x-y|}{1+x^2+y^2}\end{aligned}$$

since $1+x^2+y^2 \geq 1$

$$\begin{aligned}&\leq |x-y| \\ &\leq |x|+|y| \\ &\leq \sqrt{x^2+y^2}+\sqrt{x^2+y^2} = 2\sqrt{x^2+y^2}\end{aligned}$$

Therefore, if $2\sqrt{x^2+y^2} < \varepsilon \implies |f(x,y)-L| < \varepsilon$ so take $\delta = \varepsilon/2$. □

Example 1.8 (H.W)**.** *Show that* $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$,consider

$$\begin{aligned}|f(x,y)-L| &= \left|\frac{xy^2}{x^2+y^2} - 0\right| = \frac{|x|y^2}{x^2+y^2} \\ &= \frac{|x|}{\frac{x^2}{y^2}+1} \\ &\leq |x| \\ &\leq \sqrt{x^2+y^2} < \varepsilon \implies |f(x,y)-L| < \varepsilon\end{aligned}$$

So we can just pick $\delta = \varepsilon$. □

1.4 Continuity

Definition 1.9 (Continuity)**.** *A function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is said to be continuous at a point* (a,b) *if* $\forall\varepsilon > 0, \exists\delta > 0$ *such that,*

$$0 < ||(x,y)-(a,b)||_2 < \delta \implies |f(x,y)-f(a,b)| < \varepsilon$$

provided $f(a,b)$ *exists. Alternatively,*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note that, we can show the function is discontinuous if

1. $f(a,b)$ doesn't exist.
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ doesn't exist.
3. Both exist but are not equal to each other.

Example 1.10. *Show that the given function is continous at* $(0,0)$ *where,*

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. Here, $f(0,0) = 0$. Clearly we have that $|x^2 - y^2| \leq |x^2 + y^2|$.
Let $\varepsilon > 0$,

$$\begin{aligned} |f(x,y) - L| &= \left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| \\ &= |x||y| \left| \frac{x^2-y^2}{x^2+y^2} \right| \\ &\leq |x||y| \\ \\ &\leq \sqrt{x^2+y^2}\sqrt{x^2+y^2} = x^2+y^2 \end{aligned}$$

So when $x^2 + y^2 < \varepsilon \implies |f(x,y) = f(0,0)| < \varepsilon$ so we take $\delta = \sqrt{\varepsilon}$. □

Example 1.11. *Show that the given function is discontinuous at* $(0,0)$ *where,*

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. content... □

1.5 Polar Coordinates

The polar coordinates r(the radial coordinate) and θ(the angular coordinate), are defined in terms of cartesian coordinates as below.

$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}, \theta = \arctan\left(\frac{y}{x}\right)$$

1.5.1 Limits in Polar coordinates

Use polar coordinates when you are over (0,0)

Example 1.12. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

Proof. Put $x = r\cos\theta$ and $y = r\sin\theta$

$$f(x,y) = \frac{2xy}{x^2+y^2} \iff f(r,\theta) = \frac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta$$
$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} 2\cos\theta\sin\theta = 2\cos\theta\sin\theta$$

Which depends on θ. □

1.5.2 Epsilon-delta with polar coordinates

Definition 1.13. $\forall\varepsilon > 0 \exists\delta > 0 s.t.$

Example 1.14. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

Proof.

$$f(r,\theta) = \frac{r^3\cos^3\theta}{r^2} = r(\cos\theta)^3 \tag{1.1}$$

Let $\varepsilon > 0$, consider $|f(r,\theta) - L| = |r||\cos\theta|^3 \leq |r|$. So we can set $\delta = \varepsilon$ □

Example 1.15. *Find the domain and range of* $g(x,y) = \sqrt{9 - x^2 - y^2}$

Proof. The sqrt interior must be positive so take $x^2 + y^2 \leq 9$, so its a circle of radius 3 centered at 0. So the domain is the circle. The range is $\{z \mid 0 \leq z \leq 3\} = [0,3]$ □

1.6 Algebra of limits

1.7 General multivariate limit

Theorem 1.16 (Limit of a function $f : \mathbb{R}^n \to \mathbb{R}$)**.** *For a function* $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ *if and only if* $\forall\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$0 < ||x - a||_n < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.17 (ε- neighbourhood)**.** $B(a,\varepsilon)$ *open ball of radius* ε *around* α.

$$0 \leq ||x - a||_n < \varepsilon$$

Definition 1.18 (Deleted ε neighbourhood)**.** $B(a, \varepsilon) - \{a\}$

Definition 1.19 (Alternate definition of a limit)**.** *For a function* $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ *if and only if* $\forall\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in B * (a, \delta) \implies |f(x) - L| < \varepsilon$$

Definition 1.20 (Bounded function)**.** *Let* E *be a non-empty subset of* $\mathbb{R}^n$*. The function* $f : E \to \mathbb{R}$ *is said to be bounded in some* δ*-neighbourhood of point* $p \in \mathbb{R}^n$ *if there exists* $M > 0$ *in* $\mathbb{R}$ *such that ....*

Definition 1.21 (Relation between bounded function and limit of a function in $\mathbb{R}^n$)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $p \in \mathbb{R}^n$*. Let* $f(p)$ *be defined. If* f $\lim_{x\to p} f(x)$ *exists then* f *is bounded in some neighbourhood of point* p*.*

The converse of 1.21 isn't true.

Theorem 1.22 (Uniqueness of limit in $\mathbb{R}^n$)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $p \in \mathbb{R}^n$*. If* $\lim_{x\to p} f(x)$ *exists then it is unique.*

Chapter 2

Differentiation

Chapter 3

Applications