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# Differential equations

Lecture Notes
for SMAT403

# Contents

# Introduction

> *"This course is justly viewed as the most unpleasant undergraduate course in mathematics, by both teachers and students. Some of my colleagues have publicly announced that they would rather resign from MIT than lecture in sophomore differential equations"*
>
> – Gian-Carlo Rota

A very dull topic indeed. The appendix at the end has some common derivatives and integrals.

# 1

# Second and Higher order ordinary linear differential equations

*"Just use Mathematica bro."*

– Euler

## 1.1 Second order linear differential equations

### 1.1.1 Definition

One dependent variable $y$ and one independent variable $x$.
The general second order linear differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

or

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1.1}$$

### 1.1.2 Existence and uniqueness theorem

**Theorem 1.1.** *Let $P(x), Q(x), R(x)$ be continuous functions on a closed interval $[a, b]$. If $x_0$ is any point in $[a, b]$ and if $y_0, y_0'$ are any numbers. Then eq. 1.1 has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$*

*Proof.* Not covered in class. □

### 1.1.3 Homogenous equation

**Definition 1.2.** *The equation*

$$y'' + P(x)y' + Q(x) = 0$$

*is called a homogenous equation.*

**Theorem 1.3.** *If $y_p(x)$ is a fixed particular solution of eq 1.1 and $y(x)$ is any general solution of eq. 1.1, then $y(x) - y_p(x)$ is a solution of 1.2*

*Proof.*

$$\begin{aligned}&(y - y_p)'' + P(x)(y - y_p)' + Q(x)(y - y_p)\\&= [y'' + P(x)y'' + Q(x)y] - [y_p'' + P(x)y_p' + Q(x)y_p]\\&= R(x) - R(x) = 0\end{aligned}$$

□

**Theorem 1.4.** *If $y_1(x)$ and $y_2(x)$ are any two solutions of eq. 1.2, then $c_1y_1(x) + c_2y_2(x)$ is also a solution for any constants $c_1$ and $c_2$.*

*Proof.* Just plug in $c_1y_1(x) + c_2y_2(x)$ in the original DE then by linearity of the differential everything cancels out to 0.

□

## 1.2 General solution of the homogenous system

**Definition 1.5.** *If two functions $f(x), g(x)$ are defined on an interval $I$ and have the property that one is a constant multiple of the other, then they are said to be linearly dependent on $I$. Otherwise they are called linearly independent.*

Note that, if $f(x) \equiv 0$ and $g(x)$ are linearly dependent for every function $g(x)$.

**Definition 1.6.** *Let $y_1(x), y_2(x)$ be linearly independent solutions of the homogenous equations $y'' + P(x)y' + Q(x)y = 0$. Then the function $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is called the Wronskian of $y_1, y_2$.*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

If two functions are dependent then their Wronskian is identically zero.

**Lemma 1.7.** *If $y_1(x), y_2(x)$ are any two solutions to eq. 1.2 on interval $I$ then their Wronskian is either identically zero or never zero on $I$.*

*Proof.* We know that $W = y_1y_2' - y_2y_1'$. Now consider $W'$ as follows,

$$\begin{aligned} W' &= y_1y_2'' + y_1'y_2' - y_2y_1'' - y_2'y_1' \\ &= y_1y_2'' - y_2y_1'' \end{aligned}$$

Since we know both $y_1, y_2$ are solutions of 1.2 we know that,

$$\begin{aligned} y_1'' + Py_1' + Qy_1 = 0 \\ y_2'' + Py_2' + Qy_2 = 0 \end{aligned}$$

Now do $y_2(eq1) - y_1(eq2)$

$$\begin{aligned} (y_1y_2'' - y_2y_1'') + P(y_1y_2' - y_2y_1') = 0 \\ W' + PW = 0 \end{aligned}$$

The general solution for this first order equation is

$$W = ce^{-\int P\,dx}$$

Since the exponential factor is never zero we see that $W$ is identically zero if the constant is $c = 0$ and never zero if $c \neq 0$. □

**Lemma 1.8.** *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 1.2 on $I$, then they are linearly dependent on this interval $\iff$ their Wronskian is identically zero.*

*Proof.* **Case 1:** If the function is linearly dependent then its Wronskian is equal to 0.

$$\begin{aligned} W(y_1, y_2) &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ y_1y_2' - y_1'y_2 &= 0 \\ (cy_2)y_2' - (cy_2')y_2 &= 0 \end{aligned}$$

**Case 2:** If the Wronskian is identically equal to 0 then we have to prove the function is linearly dependent.
**Case 2a:** If $y_1 \equiv 0 \to y_1$ is the zero function then $y_1, y_2$ are L.D.
**Case 2b:** If $y_1 \not\equiv 0 \implies y_1(x_0) \neq 0$, for some $x_0 \in I$. This implies $\exists [c, d] \subseteq I$ s.t. $y_1(x_0) \neq 0 \forall x_0 \in [c, d]$.

Also $W = 0$ on $[c, d] \implies y_1 y_2' - y_1' y_2 = 0 \implies \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \implies \left(\frac{y_2}{y_1}\right)'$.
And we get $y_2(x) = k y_1(x)$ for some $[c, d] \in I$. We need to extend this to all $I$. $y_2(x_0) = k y_1(x_0) = y_0 \forall x_0 \in [c, d]$ $y_2'(x_0) = k y_1'(x_0) = y_0' \forall x_0 \in [c, d]$, then use existence and uniqueness theorem. □

**Lemma 1.9.** *If $y_1(x)$ and $y_2(x)$ are two solutions of eq. 1.2 on $I$, then they are linearly independent on this interval iff their Wronskian is never zero on $I$.*

*Proof.* Use lemma 1.8.
Assume that $y_1, y_2$ are L.I. then we have to show $W \neq 0$. Assume $W = 0$ then use lemma 1.8.
If $W$ is never zero then show that $y_1, y_2$ are L.I. □

**Theorem 1.10** (Imp)**.** *Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogenous equation 1.2 on $I$. Then $c_1 y_1(x) + c_2 y_2(x)$ is the general solution of equation 1.2 on $I$.*

*Proof.* First show that $c_1 y_1 + c_2 y_2$ be a solution of eq 1.2. This was done in Th. 1.4.

Next, let $y(x)$ be any other solution of 1.2 to show that there exists $c_1, c_2 \in \mathbb{R}$ such that $y(x) = c_1 y_1 + c_2 y_2$. That is, to show that for some $x_0 \in I$ we can find $c_1, c_2$ s.t. $c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$ and $c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$. That is we have to show there exists some solution to,

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \end{bmatrix}$$

It is of the form $Ax = b$. And we know $|A| = W \neq 0$ (from Lemma 1.8). So $|A| \neq 0$.

Since the determinant is non-zero on every interval we can say that $Ax = b$ has a solution.

So any solution $y(x)$ will be a linear combination of $y_1, y_2$. □

### 1.2.1 Using a known solution to find another

**Theorem 1.11.** *Assume $y_1(x)$ is a known non-zero solution of 1.2. We can find the second solution $y_2(x) = v y_1$ where,*

$$v = \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$$

*Proof.* Assume $y_2 = vy_1$ is another solution to 1.2, where $v = v(x)$. Then we have,

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0 \implies [vy_1]'' + P(x)[vy_1]' + Q(x)[vy_1] = 0$$
$$(vy_1'' + 2v'y_1' + v''y_1) + P[vy_1' + v'y_1] + Q[vy_1] = 0$$
$$\underbrace{v(y_1'' + Py_1' + Qy_1)}_{=0, \because y_1 \text{ is a solution}} + v''y_1 + v'(2y_1' + Py_1) = 0$$

$$v''y_1 + v'(2y_1' + Py_1) = 0 \implies \frac{v''}{v'} = -2\frac{y_1'}{y_1} - P$$

Integrate once

$$\log v' = -2\log y_1 - \int P\,dx$$
$$v' = \frac{1}{y_1^2}e^{-\int P\,dx}$$
$$v = \int \frac{1}{y_1^2}e^{-\int P\,dx}\,dx$$

□

### 1.2.2 Homogenous equation with constants

**Theorem 1.12.** *If we have $y'' + P(x)y' + Q(x)y = 0$ and $P(x), Q(x)$ are constants. Consider the Auxiliary equation $m^2 + pm + q = 0$ and let the roots of the auxiliary equation be $m_1, m_2$.*

1. *If the roots are real and distinct ($m_1 \neq m_2$), $y = c_1e^{m_1x} + c_2e^{m_2x}$*
2. *If the roots are real and repeated ($m = m_1 = m_2$), $y = c_1e^{mx} + c_2xe^{mx}$*
3. *If the roots are complex ($m = \alpha \pm \beta i$), $y = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$*

*Proof.* Assume $y = e^{mx}$ is a solution. Substitute it in. For case one it is trivial.

For case 2 you have first solution as $y_1 = e^{mx}$ then use theorem 1.11.

For case 3 we have $m_1 = a + ib, m_2 = a - ib$. Therefore,

$$y_1 = e^{m_1x} = e^{a+ib} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$$
$$y_2 = e^{m_2x} = e^{a-ib} = e^{ax}e^{-ibx} = e^{ax}(\cos bx - i\sin bx)$$

Since we require real solutions only we do the following,

$$u_1 = \frac{y_1 + y_2}{2} = e^{ax}\cos bx$$
$$u_2 = \frac{y_1 - y_2}{2i} = e^{ax}\sin bx$$

$u_1, u_2$ are obviously linearly independent. Show that they are solutions then claim it linear combination is the general solution. □

## 1.3 Method of undetermined coefficients (UDC)

This is a method to solve non-homogenous linear DE of order 2.

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1.1}$$
$$y'' + P(x)y' + Q(x)y = 0 \tag{1.2}$$

The second equation is just the associated homogenous equation of the first. Let $y_g$ be the general solution of eq. 1.2 be known as a complementary function (CF).
Let $y_p$ be a particular solution of eq. 1.1 then it is known as particular integral.
You begin by computing $y_g$ then compute $y_p$ depending on one of the three stupid cases and their subcases [1]. Once you have the $y_p$ you just add that (**without** a constant) to the $y_g$ and you have the general solution for 1.1

### 1.3.1 Assumptions of UDC

1. $P(x), Q(x)$ are constants
2. $R(x)$ is either exponential, sine or cosine or polynomial.

### 1.3.2 Case 1: Exponential

**Theorem 1.13.**

$$y'' + py' + qy = e^{ax} \tag{1.3}$$

1. *If $a$ is not a root of AE, then $y_p = Ae^{ax}$*
2. *If $a$ is a simple root (i.e. multiplicity 1) of AE, then $y_p = Axe^{ax}$*
3. *If $a$ is a double root of AE, then $y_p = Ax^2e^{ax}$*

[1] Just have to memorize it unfortunately. You want a proof? Cry more lmao.

#### 1.3.2.1 Subcase 1: a is not a root of AE

If $a$ is not a root of the $AE$ then you take $y_p = Ae^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Ae^{ax} \\ y_p' &= Aae^{ax} \\ y_p'' &= Aa^2e^{ax} \end{aligned}$$

Plug this into the original DE then just append the $y_p$ to $y_g$ without constants for your general solution.

For subcase i, you can compute $A$ directly as follows,

$$A = \frac{1}{a^2 + pa + q}$$

#### 1.3.2.2 Subcase 2: a is a simple root of AE

If $a$ is a simple root of the $AE$ then you take $y_p = Axe^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Axe^{ax} = Ae^{ax}(x) \\ y_p' &= Ae^{ax}(ax + 1) \\ y_p'' &= Ae^{ax}(a^2x + 2a) \end{aligned}$$

Now just plug these values into the original differential equation. You will get $y_p$ just append that to $y_g$ and you have your solution.

For subcase ii, you can compute $A$ directly as follows,

$$A = \frac{1}{2a + p}$$

#### 1.3.2.3 Subcase 3: a is a double root of AE

If $a$ is a double root of the $AE$ then you take $y_p = Ax^2e^{ax}$. So you have the following,

$$\begin{aligned} y_p &= Ax^2e^{ax} = Ae^{ax}(x^2) \\ y_p' &= Ae^{ax}(ax^2 + 2x) \\ y_p'' &= Ae^{ax}(a^2x^2 + 4ax + 2) \end{aligned}$$

### 1.3.3 Case 2: Trigonometric

If $y'' + py' + qy = \sin kx$ or $\cos kx$

#### 1.3.3.1 Subcase 1: If it is not a solution of AHE

If we have $\sin kx$ or $\cos kx$ not being a solution to the AHE we take $y_p = A\sin kx + B\cos kx$. Using this we get the following,

$$\begin{aligned} y_p &= A\sin kx + B\cos kx \\ y_p' &= k(A\cos kx - B\cos kx) \\ y_p'' &= -k^2(A\sin(kx) + B\cos kx) \end{aligned}$$

Plug this ugly mess into the original differential equation and you will get $y_p$ append that without a constant to $y_g$ for the general solution.

#### 1.3.3.2 Subcase 2: If it is a solution of AHE

If we have $\sin kx$ or $\cos kx$ is a solution of the AHE then we take $y_p = x(A\sin kx + B\cos kx)$. Using this we get the following,

$$\begin{aligned} y_p &= x(A\sin kx + B\cos kx) \\ y_p' &= A(\sin kx + kx\cos kx) + B(\cos kx - kx\sin kx) \\ y_p'' &= A(2k\cos kx - k^2x\sin kx) + B(-xk^2\cos kx - 2k\sin kx) \end{aligned}$$

### 1.3.4 Case 3: Polynomial

For $y'' + py' + qy = a_0 + a_1x + \cdots + a_nx^n$

#### 1.3.4.1 Subcase 1: If $q \neq 0$

Take $y_p$ as follows

$$\begin{aligned} y_p &= A_0 + A_1x + \cdots A_nx^n \\ y_p' &= A_1 + \cdots + nA_nx^{n-1} \\ y_p'' &= 2A_2 + \cdots + n(n-1)A_n^{n-2} \end{aligned}$$

#### 1.3.4.2 Subcase 2: If $q = 0, p \neq 0$

Take $y_p$ as follows, the derivates are obvious,

$$y_p = x(A_0 + A_1x + \cdots A_nx^n)$$

#### 1.3.4.3 Subcase 3: If $q = 0, p = 0$

Take $y_p$ as follows,

$$y_p = x^2(A_0 + A_x + \cdots + A_n x^n)$$

## 1.4 Variation of Parameters (VOP) method

Consider the non-homogenous equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

Consider its AHE. Say it has two solutions $y_1, y_2$. So that $y_g = c_1y_1 + c_2y_2$. We will say that the required particular solution $y_p$ is a combination of $y_g$ as follows,

$$y_p = v_1y_1 + v_2y_2$$

Now our job is just to get $v_1, v_2$ such that

$$y_p(x) = v_1y_1 + v_2y_2$$

Now begin differentiating this by repeated use of the product rule [2].

$$\begin{aligned} y_p' &= v_1y_1' + v_1'y_1 + v_2y_2' + v_2'y_2 \\ y_p'' &= v_1y_1'' + 2v_1'y_1' + v_1''y_1 + v_2y_2'' + 2v_2'y_2' + v_2''y_2 \end{aligned}$$

We also now choose $v_1'y_1 + v_2'y_2 = 0$ to simplify the derivatives [3], Now with those terms we can restate the original differential equation as follows,

$$\begin{aligned} y_p' &= v_1y_1' + v_2y_2' \\ y_p'' &= v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2' \end{aligned}$$

Substitute these into the original differential equation and you get,

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1'y_1' + v_2'y_2' = R(x)$$

Since $y_1, y_2$ are solutions to the AHE the terms in the parentheses vanish and we are left with

$$v_1'y_1' + v_2'y_2' = R(x)$$

[2] See why this is so boring yet?

[3] lmao

Recall from footnote 3 that we now have a system of equations as follows,

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = R(x)$$

Now upon solving we get

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \qquad \text{and} \qquad v_2' = \frac{y_1 R(x)}{W(y_1, y_2)}$$
$$v_1 = -\int \frac{y_2 R(x)}{W(y_1, y_2)}\, dx \qquad \text{and} \qquad v_2 = \int \frac{R(x) y_1}{W(y_1, y_2)}\, dx$$

## 1.5 Higher order linear equations

**Definition 1.14.** $n^{th}$ *order non-homogenous differential equations with constant coefficients is of the form*

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = R(x)$$

$y(x) = y(g) + y(p)$ where $y(g)$ is the general solution to the associated homogenous equation and $y(p)$ is the particular solution. There's no general way to solve this. If the order is $< 5$ then use the auxiliary equation and find the roots and solve as before. Equations with $n \geq 5$ in general won't be solvable by radicals due to Abel-Ruffini, just pray it factors out.

## 1.6 Operator method

- Consider the differential equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = R(x)$
- Using the differential operator D, we can re write it as $D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = R(x)$
- $\implies p(D)y = R(x)$ where $p(m)$ is called the axuliary polynomial and $p(m) = (m - m_1)(m - m_2) \cdot (m - m_n)$ where $m_i$ are roots of the auxiliary equation.

**Lemma 1.15.** $p(D)y = R(x) \implies y = \frac{1}{p(D)} R(x)$

*Proof.*

$$\begin{aligned} y &= \frac{1}{D}R(x) \\ &= \int R(x)\,dx \end{aligned}$$

□

**Lemma 1.16.** $(D-m)y = R(x) \implies y = \frac{1}{(D-m)}R(x)$

*Proof.*

$$y = e^{mx}\int e^{-mx}R(x)\,dx \tag{1.1}$$

$$\frac{1}{D-m}R(x) = e^{mx}\int e^{-mx}R(x)\,dx \tag{1.2}$$

□

### 1.6.1 Successive integration

$$y = \frac{1}{p(D)}R(x) = \frac{1}{[(D-m_1)(D-m_2)\cdots(D-m_n)]}R(x)$$

Using 1.2 successively, i.e. iteratively integrating multiple times till we get the particular solution.

### 1.6.2 Partial fraction decomposition

$$p(D)y = R(x) \implies y = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)}R(x)$$

Use partial fractions, to split it as such

$$\begin{aligned} y &= \left(\frac{A}{D-m_1} + \frac{B}{D-m_2} + \cdots + \frac{N}{D-m_n}\right)R(x) \\ y &= Ae^{m_1x}\int e^{-m_1x}R(x)\,dx + \cdots + Ne^{m_nx}\int e^{-m_nx}R(x)\,dx \end{aligned}$$

### 1.6.3 Series expansion

Used when $R(x)$ is a polynomial.

$$y = \frac{1}{p(D)} R(x) = [1 + b_1 D + b_2 D^2 + \cdots] R(x)$$

higher order derivates vanish.

### 1.6.4 Exponential shift rule

If $R(x) = e^{kx} g(x)$

$$y = \frac{1}{p(D)} e^{kx} g(x)$$
$$y = e^{kx} \left[ \frac{1}{p(D+k)} g(x) \right]$$

# 2

# Linear systems of ordinary differential equations

*"I love differential equations. It is so fun. It has so many real life applications."*

– Someone I don't like

## 2.1 Linear system of DE

Observe that the single $n^{th}$ order equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{2.1}$$

Is in fact equivalent to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

**Theorem 2.1** (Existence and uniqueness theorem for general system of linear differential equations)**.** *Let the functions $f_1, f_2, \ldots, f_n$ and the partial derivatives $\partial f_1/\partial y_1, \ldots, \partial f_1/\partial y_n, \ldots \partial f_n/\partial y_1, \ldots, \partial f_n/\partial y_n$ be continuous in a region $R$ of $(x, y_1, y_2, \ldots, y_n)$ space. If $(x_0, a_1, a_2, \ldots, a_n)$ is an internet point of $R$, then the system has an unique solution $y_1(x), y_2(x), \ldots, y_n(x)$ that satisfies the initial conditions (5).*

*Proof.* Not covered in class. □

**Theorem 2.2** (Existence and uniqueness theorem for 2.1)**.** *Let the function $f$ and the partial derivatives $\partial f/\partial y, \partial f/\partial y', \ldots, \partial f/\partial y^{(n-1)}$ be continuous in a region $R$ of $(x, y, y', \ldots, y^{(n-1)})$ space. If $(x_0, a_1, a_2, \ldots, a_n)$ is an interior point of $R$, then equation 2.1 has a unique solution $y(x)$ that satisfies the initial conditions $y(x_0) = a, y'(x_0) = a_2, \ldots, y^{(n-1)}(x_0) = a_n$.*

*Proof.* Not covered in class. □

## 2.2 Homogenous linear system of ODE in two variables

Now consider the system of linear homogenous equations (def 2.4). We will only see systems of two first order equations in two unknown functions of the following form,

$$\frac{dx}{dt} = F(t, x, y)$$
$$\frac{dy}{dt} = G(t, x, y)$$

More specifically we have **linear** systems of the form,

**Definition 2.3** (Linear system of two ODE)**.**

$$\frac{dx}{dt} = a_1(t) + b_1(t) + f_1(t)$$
$$\frac{dy}{dt} = a_2(t) + b_2(t) + f_2(t)$$

We assume that $a_i(t), b_i(t), f_i(t)$ for $i = 1, 2$ are continuous on some closed interval $[a, b]$ on the $t$-axis.
If $f_i(t)$ are both identically zero, then the system is called homogenous.

**Definition 2.4** (Homogenous linear system of two ODE)**.**

$$\frac{dx}{dt} = a_1(t) + b_1(t)$$
$$\frac{dy}{dt} = a_2(t) + b_2(t)$$

**Theorem 2.5.** *If $t_0$ is any point of the interval $[a, b]$ and if $x_0$ and $y_0$ are any numbers then def. 2.3 has one and only one solution*

$$x = x(t)$$
$$y = y(t)$$

*valid throughout $[a, b]$, such that $x(t_0) = x_0, y(t_0) = y_0$.*

*Proof.* Not covered in class. □

**Theorem 2.6.** *If the homogenous system (def. 2.4) has two solutions on the interval $[a, b]$*

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \textit{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases} \tag{2.1}$$

*then we also have another solution of the form*

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases} \tag{2.2}$$

*for any constants $c_1, c_2$.*

*Proof.* The proof is obvious and is left as an exercise to the next person to read this. □

## 2.3 Wronskian of homogenous linear system of ODE

**Definition 2.7** (Wronskian)**.** *If 2.4 has two solutions*

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

*Then the Wronskian of the two solutions is given as,*

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$

## 2.4 General solution of Homogenous linear system of ODE in two variables

**Theorem 2.8.** *If the two solutions (eq. 2.1) for the homogenous system 2.4 have a Wronskian that does not vanish on $[a, b]$ then its linear combination of the solutions (eq. 2.2) as described in theorem 2.6 is the general solution of the homogenous system 2.4 on that interval.*

*Proof.* Assume there exists a solution $x = x_0(t), y = y_0(t)$ we wish to show that $x_0 = c_1x_1 + c_2x_2, y_0 = c_1y_1 + c_2y_2$ for some $c_1, c_2 \in \mathbb{R}$.

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

It is of the form $Ax = b$ but note that $|A| = W$. So if the Wronskian does not vanish then $|A| \neq 0 \implies Ax = b$ has a solution. □

**Theorem 2.9.** *If $W(t)$ is the Wronskian of two solutions of the homogenous system then $W(t)$ is either identically zero or nowhere zero on $[a, b]$.*

*Proof.* Consider that $W = x_1y_2 - x_2y_1$. Now its derivative is given as,

$$\begin{aligned} W' &= x_1'y_2 + x_1y_2' - x_2y_1' - x_2'y_1 \\ &= (a_1x_1 + b_1y_1)y_2 + x_1(a_2x_2 + b_2y_2) - x_2(a_2x_1 + b_2y_1) - (a_1x_2 + b_1y_2)y_1 \\ &= [a_1 + b_2](x_1y_2 - x_2y_2) = [a_1 + b_2]W \end{aligned}$$

$$\begin{aligned} \frac{dW}{W} &= a_1 + b_2 dt \\ \log W &= \int a_1(t) + b_2(t)\, dt + \log c \\ W &= ce^{a_1 + b_2\, dt} \end{aligned}$$

□

**Theorem 2.10.** *If the two solutions of the homogeneous system are linearly independent then the linear combination 2.6 is its general solution on this interval.*

*Proof.* In view of the previous 2 theorems we need only show that two solutions are linearly dependent $\iff$ Wronskian is identically zero.

$\implies$ is obvious I'm not typing it.

To show $\Longleftarrow$ . Assume the Wronskian of two solutions is identically zero. Let $t_0 \in [a, b]$ be some arbitrary fixed point. Since we know that $W(t_0) = 0$ we have the following system, which has a solution $c_1, c_2$ where not both $c_1, c_2$ are zero.

$$\begin{cases} c_1 x_1(t_0) + c_2 x_2(t_0) = 0 \\ c_1 y_1(t_0) + c_2 y_2(t_0) = 0 \end{cases}$$

But this lead to the solution given below having only the trivial solution

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$

But from the uniqueness theorem we know it must be trivial for the entire interval. □

**Lemma 2.11.** *$W$ is never zero $\iff$ the solutions are L.I.*

## 2.5 Non-homogenous linear system in two variables

**Theorem 2.12.** *If the two solutions (as in eq. 2.1) for the homogenous system (eq. 2.4) are linearly independent on $[a, b]$ and if*

$$\begin{cases} x &= x_p(t) \\ y &= y_p(t) \end{cases}$$

*is any particular solution of the non-homogenous linear system of ODEs (def: 2.3) then*

$$\begin{cases} x &= c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{cases}$$

*is the general solution of the non homogenous system 2.3.*

*Proof.* If we show that for an arbitrary solution of 2.3 given by,

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \implies \begin{cases} x = x(t) - x_p(t) \\ y = y(t) - y_p(t) \end{cases}$$

is a solution to 2.4. We are done.

I will lose my mind if I have to type this just do it, it works. □

## 2.6 Homogenous linear systems with constant coefficients

In this section we will examine the following system of linear ODEs,

$$\begin{cases} \frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y \end{cases} \tag{2.1}$$

where $a_i, b_i$ are constants.
Assume $x = Ae^{mt}, y = Be^{mt}$ substitute into the system and we get

$$\begin{cases} Ame^{mt} = a_1 Ae^{mt} + b_1 Be^{mt} \\ Bme^{mt} = a_2 Ae^{mt} + b_2 Be^{mt} \end{cases}$$

Dividing by $e^{mt}$ we get

$$\begin{aligned} (a_1 - m)A + b_1 B = 0 \\ a_2 A + (b_2 - m)B = 0 \end{aligned}$$

We want non trivial so we require zero determinant,

$$\begin{bmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{bmatrix} = 0 \implies (a_1 - m)(b_2 - m) - a_2 b_1 = 0$$

Expanding that out gives the following auxiliary equation,

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \tag{2.2}$$

and the solutions are

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases} \quad \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$$

### 2.6.1 Distinct real roots

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases} \quad \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$$

If eq. 2.2 has distinct real roots $m_1, m_2$ then the general solution of eq. 2.1 is given as,

$$\begin{cases} x = c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y = c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{cases}$$

### 2.6.2 Equal real root

$$\begin{cases} x_1 = Ae^{mt} \\ y_1 = Be^{mt} \end{cases} \quad \begin{cases} x_2 = (A_1 + A_2 t)e^{mt} \\ y_2 = (B_1 + B_2 t)e^{mt} \end{cases}$$

If eq. 2.2 has equal real roots $m = m_1 = m_2$ then the general solution of eq. 2.1 is given as,

$$\begin{cases} x &= c_1 Ae^{mt} + c_2(A_1 + A_2 t)e^{mt} \\ y &= c_1 Be^{mt} + c_2(B_1 + B_2 t)e^{mt} \end{cases}$$

### 2.6.3 Distinct complex roots

If $m_1 = a + ib, m_2 = a - ib$ we will have

$$\begin{cases} x_1 = e^{at}(A_1 \cos bt - A_2 \sin bt) \\ y_1 = e^{at}(B_1 \cos bt - B_2 \sin bt) \end{cases} \quad \begin{cases} x_2 = e^{at}(A_1 \cos bt + A_2 \sin bt) \\ y_2 = e^{at}(B_1 \cos bt + B_2 \sin bt) \end{cases}$$

If eq. 2.2 has distinct complex roots $a \pm ib$ then the general solution of eq. 2.1 is given as,

$$\begin{cases} x &= e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y &= e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)] \end{cases}$$

## 2.7 Non-homogenous linear system

Consider the non-homogenous linear system

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases} \tag{2.1}$$

and the corresponding homogenous system,

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases} \tag{2.2}$$

Let

$$\begin{cases} x = c_1 x_1 + c_2 x_2 \\ y = c_2 y_1 + c_2 y_2 \end{cases}$$

be solutions to 2.2

$$\begin{cases} x_p = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y_p = v_1(t)y_1(t) + v_2(t)y_2(t) \end{cases}$$

will be a particular solution of 2.1 if the functions $v_1(t)$ and $v_2(t)$ satisfy the system,

$$\begin{cases} v_1'x_1 + v_2'x_2 = f_1 \\ v_1'y_1 + v_2'y_2 = f_2 \end{cases}$$

This technique for finding particular solutions of nonhomogenous linear systems is called method of variation of parameters.

*Proof.* Assume $x_p, y_p$ is as given then differentiate that (use chain rule) and input into the original system. Its a hernia to type but it simplifies out. □

## 2.8 Nonlinear systems

If $x$ is the number of rabbits at time $t$ then,

$$\frac{dx}{dt} = ax(a > 0)$$

as a consequence of unlimited supply of clover, if the number $y$ of foxes is zero.

Assume that the number of encounters per unit time between rabbits and foxes is jointly proportional to $x$ and $y$. Further assume that a certain proportion of these encounters result in a rabbit being eaten, then we have,

$$\frac{dx}{dt} = ax - bxy \quad \text{For } a, b > 0$$

Similarly, in the absence of rabbits the foxes dies out and their increase depends on the number of their encounters with rabbits,

$$\frac{dy}{dt} = -cy + dxy \quad \text{For } c, d > 0$$

We get the following system,

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = -y(c - dx) \end{cases} \tag{2.1}$$

Equation 2.1 is called Volterra's prey-predator equation. Let the unknown solutions be thought as constituting

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

the parametric equations of a curve in the $xy-$plane, then we can find the rectangular equation of this curve.

Eliminating $t$ in 2.1 we get

$$\frac{(a - by)dy}{y} = \frac{(c - dx)dx}{x}$$

Integrating, we get

$$a \log y - by = -c \log x + dx + \log K$$

Where the constant $K$ in terms of initial values is given by,

$$K = x_0^c y_0^a e^{-dx_0 - by_0} \tag{2.2}$$

If the rabbit and fox population is

$$x = \frac{c}{d}, y = \frac{a}{b} \tag{2.3}$$

then system 2.1 os satisfied and we have $dx/dt = 0$ and $dy/dt = 0$ so there is no increase or decrease in $x$ or $y$. The population above is called equilibrium population, for $x, y$ can maintain themselves indefinitely at these constant levels.

Let $x = \frac{c}{d} + X$ and $y = \frac{a}{b} + Y$ then $X, Y$ can be thought of as the deviations of $x, y$ from their equilibrium values.

If $x, y$ in 2.1 are replaced with $X, Y$ then it becomes

$$\begin{cases} \frac{dX}{dt} = -\frac{bc}{d}Y - bXY \\ \frac{dY}{dt} = \frac{ad}{b}X + dXY \end{cases} \tag{2.4}$$

To "linearize" the system, assume that if $X, Y$ are small then $XY$ can be discarded without serious error. Thus simplifying it,

$$\begin{cases} \frac{dX}{dt} = -\frac{bc}{d}Y \\ \frac{dY}{dt} = \frac{ad}{b}X \end{cases} \tag{2.5}$$

Eliminating $t$ we get,

$$\frac{dY}{dX} = -\frac{ad^2}{b^2c}\frac{X}{Y}$$

Whose solution is

$$ad^2X^2 + b^2cY^2 = C^2$$

This is a family of ellipses surrounding the origin in the $XY$ plane.

# 3

# Partial differential equations

*"Differential equations is mad boring yo."*

– Gauss

**Definition 3.1** (Partial derivatives). *Partial derivates are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.*
*For a function $f$ in $n$ variables $x_1, x_2, \ldots, x_n$ we can define the $m^{th}$ partial derivative as,*

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}$$

*Partial derivatives can be taken with respect to multiple variables and are denoted as follows,*

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

Differential equations that use partial derivates of a function of two or more variables are called PDEs.

## 3.1 Classification of Second order PDE

Second order PDE are usually divided into three types.

**Definition 3.2** (General form of a second order PDE)**.**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Linear second order PDEs are classified according to the properties of its discriminant $d = B^2 - 4AC$

### 3.1.1 Elliptic PDE

If $d < 0\,\forall(x, y) \in R \in \mathbb{R}^2$

### 3.1.2 Hyperbolic PDE

If if $d > 0\,\forall(x, y) \in R \in \mathbb{R}^2$

### 3.1.3 Parabolic PDE

If $d = 0\,\forall(x, y) \in R \in \mathbb{R}^2$.

## 3.2 Classification with more than two variables

Let $u = u(x_1, x_2, \cdots, x_n)$Consider the second order PDE,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu + D = 0$$

Assume $A = [A_{ij}]$ is symmetric then $A$ is diagonalisable with real characteristic values $\lambda_1, \lambda_2, \cdots, \lambda_n$.

1. If all $\lambda_i > 0$ or $\lambda_i < 0$ then its elliptic.
2. If one or more $\lambda_i = 0$ then parabolic.
3. If only one $\lambda_i > 0$ or $\lambda_i < 0$ and all remaining are of opposite sign then it is hyperbolic.

Compute the characteristic values as solutions to $\det(A - \lambda I) = 0$

## 3.3 Canonical form

Equation 3.2 obtains a simple form when the variables $(x, y)$ are transformed to $(\xi, \eta)$ under the transformation $\xi = \xi(x, y), \eta = \eta(x, y)$. We assume that $\xi, \eta$ are twice differentiable and their Jacobian is non-zero, i.e.,

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix}$$

The non zero Jacobian is required to satisfy the inverse function theorem [1].

Using the chain rule we get,

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta n_{yy} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}
\end{aligned}$$

We pick $\xi, \eta$ as curves that satisfy,

$$\frac{dy}{dx} + \lambda_i = 0$$

where $\lambda_i$ are roots of the equation $A\lambda^2 + B\lambda + C = 0$.

$$\implies \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

If the two curves obtained are complex conjugates (happens when elliptic) pick $\xi$ as the real part and $\eta$ as the complex part.

### 3.3.1 Hyperbolic

The hyperbolic equation has two possible canonical forms

$$u_{\xi\eta} + lots = 0$$

or

$$u_{\xi\xi} - u_{\eta\eta} + lots = 0$$

Where lots denotes lower order terms.

---

[1] For $f : \mathbb{R}^n \to \mathbb{R}^n$, and $f \in C^1$. If the Jacobian is invertible (i.e. not zero) at some point $p$ then $f$ has a diffeomorphism in some ball centred at $p$ . Note this is only local invertibility, the global case is a famous open problem.

### 3.3.2 Parabolic

The parabolic equation has the following canonical form

$$u_{\xi\xi} + lots = 0$$

### 3.3.3 Elliptic

The elliptic equation has the following canonical form

$$u_{\xi\xi} + u_{\eta\eta} + lots = 0$$

## 3.4 One dimensional wave equation

### 3.4.1 Vibration of an infinite string

**Definition 3.3** (Infinite one dimension wave equation)**.** *The one dimensional wave equation is given by,*

$$PDE : u_{tt} = c^2 u_{xx} \qquad (c \neq 0)(-\infty < x < \infty, t \geq 0) \tag{3.1}$$
$$IC : u(x,0) = f(x) \qquad (-\infty < x < \infty) \tag{3.2}$$
$$u_t(x,0) = g(x) \qquad (-\infty < x < \infty) \tag{3.3}$$

*where c is a positive constant.*

*Proof.* First re-arrange in standard form,

$$c^2 u_{xx} - u_{tt} = 0$$

Note that $A = c^2, B = 0, C = -1$ so $d = 4c^2 > 0$ if $c \neq 0$.

So the equation is of hyperbolic type. We will now reduce it to the canonical form.

Consider $A\lambda^2 + B\lambda + C = 0 \implies \lambda^2 c^2 - 1 = 0 \implies \lambda = \pm\frac{1}{c}$.

Proceed to find the required characteristic curves as follows,

$$\frac{dt}{dx} \pm \frac{1}{c} = 0$$

so the curves are $ct \pm x = k$. Pick $\xi = x + ct, \eta = x - ct$

$$\xi_x = 1, \xi_t = c, \xi_{xx} = \xi_{xt} = \xi_{tt} = 0$$
$$\eta_x = 1, \eta_t = -c, \eta_{xx} = \eta_{xt} = \eta_{tt} = 0$$

Using the chain rule we get,

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{nn} \qquad (3.4) \\ u_{tt} &= u_{\xi\xi}\xi_t^2 + 2u_{\xi\eta}\xi_t\eta_t + u_{\eta\eta}\eta_t^2 + u_\xi\xi_{tt} + u_\eta n_{tt} \\ &= c^2u_{\xi\xi} - 2c^2u_{\xi\eta} + c^2u_{\eta\eta} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \qquad (3.5) \end{aligned}$$

Canonical form is

$$u_{\xi\eta} = 0 \qquad (3.6)$$

Now integrate 3.6 w.r.t. $\xi$ then $\eta$

$$u(\xi, \eta) = \phi(\eta) + \psi(\xi) \qquad (3.7)$$

where $\phi, \psi$ are some arbitrary functions of $\eta, \xi$.

Now re-substitute $\eta = x - ct, \xi = x + ct$,

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \qquad (3.8)$$

No we use the first initial condition,

$$u(x, 0) = f(x) \implies \phi(x) + \psi(x) = f(x) \qquad (3.9)$$

Now differentiate 3.8 w.r.t. $t$,

$$u_t(x, t) = -c\phi'(x - ct) + c\psi'(x + ct)$$

Put $t = 0$ and apply the other initial condition.

$$u_t(x, 0) = g(x) \implies -c\phi'(x - ct) + c\psi'(x + ct) = g(x) \qquad (3.10)$$

Integrate the above from $x_0$ to $x$ ($x_0$ is choosen in such a manner that $\phi'(x_0) - \psi'(x_0) = 0$) and you get

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\omega)\, d\omega \qquad (3.11)$$

Now adding and subtracting 3.9 and 3.11 you get

$$\begin{aligned} \phi(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\omega)\, d\omega \\ \psi(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\omega)\, d\omega \end{aligned}$$

Furthermore

$$\phi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(\omega)\,d\omega$$
$$\psi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\omega)\,d\omega$$

So the solution is

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)\,d\omega$$

□

### 3.4.2 Vibration of an semi-infinite string

**Definition 3.4** (Semi-infinite one dimension wave equation)**.**

$$\begin{aligned} PDE :& u_{tt} = c^2 u_{xx} && (c \neq 0)(0 \le x < \infty, t \ge 0) \\ BC :& u(0,t) = 0 && (t \ge 0) \\ IC :& u(x,0) = f(x) && (0 \le x < \infty) \\ & u_t(x,0) = g(x) && (0 \le x < \infty) \end{aligned}$$

*where c is a positive constant.*

*Proof.* Proceed in a similar manner as the infinite string case until the last stage,

$$\phi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(\omega)\,d\omega$$
$$\psi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\omega)\,d\omega$$

But we require solutions to $u(x,t)$ for $x > 0, t > 0$. We can see that $x + ct$ is always greater than zero while $x - ct$ need not be. So consider the cases as such,

**Case i: $x - ct \ge 0$**

Here we can just use the d'Alembert solution.

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)\,d\omega \qquad (3.12)$$

**Case ii: $x - ct < 0$** From the Boundary condition we have,

$$\begin{aligned} u(0,t) = 0 \implies \phi(-ct) + \psi(ct) &= 0 \\ \phi(-ct) &= -\psi(ct) \end{aligned} \tag{3.13}$$

Replace $ct$ with $ct - x$ in 3.13,

$$\phi(x - ct) = -\psi(ct - x)$$

Then solving we get

$$\begin{aligned} u(x,t) &= \phi(x-ct) + \psi(x+ct) = -\psi(ct-x) + \psi(x+ct) \\ &= -\left(\frac{1}{2}f(ct-x) + \frac{1}{2c}\int_{x_0}^{ct-x} g(\omega)\,d\omega\right) + \left(\frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(\omega)\,d\omega\right) \\ &= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\omega)\,d\omega \end{aligned}$$

Considering both cases finally the solution is given as follows,

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)\,d\omega, x \geq ct \\ \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\omega)\,d\omega, x < ct \end{cases}$$

□

### 3.4.3 Vibration of a finite string

**Definition 3.5** (Finite one dimension wave equation)**.**

$$\begin{aligned} PDE : & u_{tt} = c^2 u_{xx} && (c \neq 0, 0 \leq x \leq L, t \geq 0) \\ BC : & u(0,t) = 0, u(L,t) = 0 && (t \geq 0) \\ IC : & u(x,0) = f(x), u_t(x,0) = g(x) && (0 \leq x \leq L) \end{aligned}$$

*Proof.* We solve this problem by separation of variables. Assume the following,

$$u(x,t) = X(x)T(t)$$

The PDE then simplifies as such,

$$XT'' = c^2 X'' T$$

Dividing both sides by $c^2 X'' T$ yields,

$$\frac{T''}{c^2 T} = \frac{X''}{X} = k$$

where $k$ is a constant. Now our PDE has simplified to the following two ODEs,

$$\begin{aligned} T'' - kc^2T &= 0 \\ X'' - kX &= 0 \end{aligned}$$

If $k = 0$ then it will lead to a trivial solution as $X(x)$ becomes a linear function thats zero everywhere. Also if $k > 0$ the problem grows without bound so we must choose $k < 0$. Set $k = -\lambda^2$. So we have,

$$\begin{aligned} T'' + c^2\lambda^2T &= 0 \\ X'' + \lambda^2X &= 0 \end{aligned}$$

Solve these ODEs normally (using the auxiliary equation methods in unit 1). We get the following,

$$\begin{aligned} T(t) &= A\sin(c\lambda t) + B\cos(c\lambda t) \\ X(x) &= C\sin(\lambda t) + D\cos(\lambda t) \end{aligned}$$

Apply the first BC to get,

$$X(0) = 0 \implies C\sin(0) + D\cos(0) = 0 \implies D = 0$$

Apply the second BC to get,

$$X(L) = 0 \implies c\sin\lambda L = 0 \implies c = 0 \text{ or } \sin\lambda L = 0$$

Since $c = 0$ yields a trivial solution we consider $\sin\lambda L = 0 \implies \lambda_n = \frac{n\pi}{L}$ for $n \in \mathbb{N}$. So now we have,

$$\begin{cases} T_n(t) = A\sin(c\lambda_n t) + B\cos(c\lambda_n t) \\ X_n(x) = C\sin(\lambda_n x) \end{cases}$$

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n\cos\left(\frac{n\pi ct}{L}\right)\right] \end{aligned}$$

Now use the initial conditions to get the formulas for the constants $a_n, b_n$ as such,

$$u(x,0) = f(x) = \sum_{i=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad \implies b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$
$$u_t(x,0) = g(x) = \sum_{i=1}^{\infty} a_n \left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \quad \implies a_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

□

## 3.5 Fourier transform

**Definition 3.6** (Fourier transform). *Let $f : (-\infty,\infty) \to \mathbb{R}$ or $\mathbb{C}$ the Fourier transform of $f(x)$ is given by*

$$\mathcal{F}\{f(\omega)\} = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

*$\forall \omega \in \mathbb{R}$, provided the integral exists. Where $\omega$ denotes angular frequency.*

**Definition 3.7** (Inverse Fourier transform).

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega$$

*$\forall x \in \mathbb{R}$, provided the integral exists.*

### 3.5.1 Properties

1. Linearity $\mathcal{F}[c_1 f_1 + c_2 f_2] = c_1\mathcal{F}(f_1) + c_2\mathcal{F}(f_2)$
2. Conjugation $\mathcal{F}[\overline{f}] = \overline{\hat{f}(-\omega)}$
3. Continuity $f \to \hat{f} \implies \hat{f}(\omega)$ and it is absolutely integrable ($\int_{-\infty}^{\infty} |f|\,dx$ exists) then $\hat{f}(\omega)$is continuous
4. Convolution $(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$. And $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$
5. Transformation of partial derivatives, $\mathcal{F}[u_x](\omega,t) = i\omega\mathcal{F}[u] = i\omega\hat{u}(\omega,t)$ and $\mathcal{F}[u_t](\omega,t) = \frac{\partial}{\partial t}[\mathcal{F}[u]]$
6. Parseval's identity $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} f(\hat{\omega})\hat{g}(\omega)\,dw$

## 3.6 Heat conduction principle

### 3.6.1 Finite rod case

**Definition 3.8** (Finite rod one dimension heat equation)**.**

$$\begin{aligned} PDE: u_t &= \alpha^2 u_{xx} && (0 \le x \le L, 0 \le t < \infty) \\ BC: u(0,t) &= u(L,t) = 0 && (0 \le t < \infty) \\ IC: u(x,0) &= f(x) && (0 \le x \le L) \end{aligned}$$

*Proof.* We solve this using separation of variables. Assume we can do the following,

$$u(x,t) = X(x)T(t)$$

Substituting this into the PDE we get,

$$\begin{aligned} X(x)T'(t) &= \alpha^2 X''(x)T(t) \\ \frac{T'(t)}{\alpha^2 T(t)} &= \frac{X''(x)}{X(x)} = c \end{aligned}$$

This leads to two ODEs,

$$T'(t) - \alpha^2 c T(t) = 0 \tag{3.1}$$
$$X''(x) - cX(x) = 0 \tag{3.2}$$

**Case 1: $c = \lambda^2 > 0$**
$\implies X''(x) - \lambda^2 X(x) = 0, AE = m^2 - \lambda^2 = 0, m = \pm\lambda$
$X(x) = C_1 e^{\lambda x} + c_2 e^{-\lambda x}$ and $X(0), X(L) = 0$
$X(0) = C_1 + C_2 = 0, \implies -C_1 = C_2$ $X(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0 \implies C_1(e^{\lambda L} - e^{-\lambda L}) = 0 \implies C_1 = 0 \implies C_2 = 0 \implies X(x) = 0 \implies u(x,t) = 0$ Since this is a trivial solution we ignore this case.
**Case 2: $c = 0$** Also leads to a trivial solution.
$X''(x) = 0 \implies X(x) = C_3 + C_4 x$ but with the initial conditions we get its identically zero.
**Case 3 $c = -\lambda^2$**
$X''(x) + \lambda^2 X(x) = 0 \implies AE: m^2 + \lambda^2 = 0 \implies m = \pm\lambda i$
$X(x) = C_5 \cos(\lambda x) + C_6 \sin(\lambda x)$
Apply the initial conditions,

$$X(0) = 0 \implies C_5(1) + C_6(0) = 0 \implies C_5 = 0$$

$$X(L) = 0 \implies C_6 \sin \lambda L = 0$$

So either $C_6 = 0$ or $\sin \lambda L = 0$. The first leads to a trivial solution so we assume $\sin \lambda L = 0 \implies \lambda_n = \frac{n\pi}{L}$ for $n \in \mathbb{N}$.

$$X_n(x) = a_n \sin\left(\frac{n\pi x}{L}\right)$$

Consider now the time equations,

$$\begin{aligned} T'(t) - \alpha^2 c T(t) = 0 &\implies T'(t) + \alpha^2\lambda^2 T(t) = 0 \\ AE : m + \alpha^2\lambda^2 = 0 &\implies m = -\alpha^2\lambda^2 \\ T(t) = b_n e^{-\alpha^2\lambda^2 t} &\implies T_n = b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

Now we must apply the initial conditions to find a formula for $c_n$.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \implies c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

□

### 3.6.2 Infinite rod case

**Definition 3.9** (Infinite rod one dimension heat equation)**.**

$$\begin{aligned} PDE : &u_t(x,t) = \alpha^2 u_{xx}(x,t) && (-\infty < x < \infty, t > 0) \\ IC : &u(x,0) = f(x) && (-\infty < x < \infty) \end{aligned}$$

*Under the assumptions,*

1. *$f(x)$ is continuous*
2. *Either $f(x)$ is absolutely integrable OR it is bounded.*

*Proof.* Begin by applying a Fourier transform

$$\begin{aligned} \mathcal{F}[u_t] &= \frac{d}{dt}\hat{u}(\omega,t) = -\alpha^2\omega^2\hat{u}(\omega,t) \\ \hat{u}(\omega,0) &= \hat{f}(\omega) \end{aligned}$$

So the PDE has simplified to an ODE with the solution as follows,

$$\hat{u}(\omega, t) = a(\omega)e^{-\alpha^2\omega^2 t}$$

But since we knot $\hat{u}(\omega, 0) = \hat{f}(\omega) = a(\omega)$. The solution simplifies to the following,

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-\alpha^2\omega^2 t}$$

Take an inverse Fourier transform,

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}[\hat{u}(\omega, t)] \\
&= \mathcal{F}^{-1}[\hat{f}(\omega)e^{-\alpha^2\omega^2 t}] \\
&= \mathcal{F}^{-1}[\hat{f}(\omega)] * \mathcal{F}^{-1}[e^{-\alpha^2\omega^2 t}] \\
&= f(x) * \left[\frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{x^2}{4\alpha^2 t}}\right] \\
&= \frac{1}{2\sqrt{\alpha^2 \pi t}} \int_{-\infty}^{\infty} f(\omega) e^{-\frac{(x-\omega)^2}{4\alpha^2 t}}\, d\omega
\end{aligned}$$

□

**Lemma 3.10.**

$$\mathcal{F}^{-1}[e^{-\alpha^2\omega^2 t}] = \frac{1}{\sqrt{2\alpha^2 t}} e^{-\frac{x^2}{4\alpha^2 t}}$$

*Proof.*

$$\begin{aligned}
\mathcal{F}^{-1}\{\hat{f}(\omega)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2\omega^2 t} e^{-\omega x}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\omega(ix - t\alpha^2\omega)}\, d\omega
\end{aligned}$$

For simplicity's sake[2]consider $x = a, t\alpha^2 = b$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\omega(ia - b\omega)}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(\sqrt{-b}\omega + \frac{ia}{2\sqrt{-b}}\right)^2 - \frac{a^2}{4b}}\, d\omega
\end{aligned}$$

Substitute $u = \frac{ia-2b\omega}{2\sqrt{b}} \implies d\omega = -\frac{1}{\sqrt{b}}du$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-u^2-\frac{a^2}{4b}}}{\sqrt{b}}\, du \\
&= \frac{e^{\frac{-a^2}{4b}}}{\sqrt{2\pi b}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\text{This is equal to } \sqrt{\pi}\ ^3} \\
&= \frac{e^{\frac{-a^2}{4b}}}{\sqrt{2\pi b}} \sqrt{\pi} \\
&= \frac{e^{\frac{-a^2}{4b}}}{\sqrt{2b}}
\end{aligned}
$$

Recall from footnote 2

$$
= \frac{e^{\frac{-x^2}{4t\alpha^2}}}{\sqrt{2t\alpha^2}}
$$

□

[3] i.e., My sanity's sake.

[3] This is the simplest Gaussian integral you can prove it using another change of variable and the Gamma function but I cba.

# Appendix

## A.1 Common derivatives

| Function | Derivative | Function | Derivative |
|---|---|---|---|
| $x^n$ | $nx^{n-1}$ | $\sec x$ | $\sec x \tan x$ |
| $a^x$ | $a^x \log a$ | $\csc x$ | $-\csc x \cot x$ |
| $e^x$ | $e^x$ | $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\log x$ | $\frac{1}{x}$ | $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\sin x$ | $\cos x$ | $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\cos x$ | $-\sin x$ | $\cot^{-1} x$ | $\frac{-1}{1+x^2}$ |
| $\tan x$ | $\sec^2 x$ | $\sec^{-1} x$ | $\frac{1}{x\sqrt{x^2-1}}$ |
| $\cot x$ | $-\csc^2 x$ | $\csc^{-1} x$ | $\frac{-1}{x\sqrt{x^2-1}}$ |

## B.2 Basic derivative rules

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$
$$(fg)' = f'g + fg'$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \text{ when } g \neq 0$$
$$\text{If } f(x) = h(g(x)), \text{ then, } f'(x) = h'(g(x)) \cdot g'(x)$$

## C.3 Common integrals

| Function | Integral | Function | Integral |
|---|---|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1}$ | $\frac{\sin x}{\cos^2 x}$ | $\sec x$ |
| $a^x$ | $\frac{a^x}{\log a}$ | $\frac{\cos x}{\sin^2 x}$ | $-\csc x$ |
| $e^x$ | $e^x$ | $\tan x$ | $\log \sec x$ |
| $\frac{1}{x}$ | $\log x$ | $\cot x$ | $\log \sin x$ |
| $\cos x$ | $\sin x$ | $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1} \frac{x}{a}$ |
| $\sin x$ | $-\cos x$ | $\frac{1}{a^2+x^2}$ | $\frac{1}{a} \tan^{-1} \frac{x}{a}$ |
| $\sec^2 x$ | $\tan x$ | $\frac{1}{x\sqrt{x^2-a^2}}$ | $\frac{1}{a} \sec^{-1} \frac{x}{a}$ |
| $\csc^2 x$ | $-\cot x$ | | |

## D.4 Tabular integration

Quicker way to implement repeated integration by parts.

Given two functions $f, g$. Let $f = f^{(0)}, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ denote the first $n$ derivatives of $f$ and let $g = g^{(0)}, g^{(-1)}, g^{(-2)}, \ldots, g^{(-n)}$ denote the first $n$ antiderivatives of $g$.

$$\int f(x)g(x)\,dx = \sum_{j=0}^{n-1} (-1)^j f^{(j)} g^{(-(j+1))}(x) + (-1)^n \int f^{(n)}(x) g^{(-n)}(x)\,dx$$

if $f^{(n)} \equiv 0$ the $2^{nd}$ integral term above can be replaced with a constant $C$.

It is easy to implement this using the following table. In one column list $f$ and its first $n$ derivatives and in the next list $g$ and its first $n$ antiderivatives. Then multiply diagonally down and right alternating the sign at each stage.

For most cases pick $f$ as the term whose integrals vanish, or repeat.

$$\begin{tikzcd}
u & dv \\
f \arrow[dr, "+"] & g \\
f^{(1)} \arrow[dr, "-"] & g^{(-1)} \\
f^{(2)} \arrow[dr, "+"] & g^{(-2)} \\
f^{(3)} \arrow[dr, "-"] & g^{(-3)} \\
\vdots & \vdots
\end{tikzcd}$$