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Probability and Sampling Distributions (B)

Lecture Notes
for SSTA401

Contents

Chapter 1

Transformation of random variables & standard univariate continuous probability distributions

1.1 Uniform/Rectangular distributions

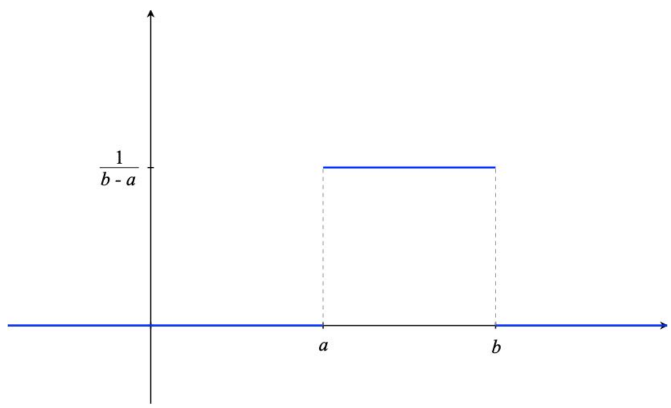
Definition 1.1. *A r.v. X is said to follow uniform distribution over an interval (a, b) if its pdf is constant over the entire range.*

1.1.1 PDF of uniform distribution

Theorem 1.2. *PDF of uniform distribution*

$$\begin{aligned} P(x) &= k & a < x < b \\ &= 0 & \textit{otherwise} \end{aligned}$$

- $\int_a^b f(x)\,dx = \int_a^b k\,dx = k[x]_a^b = k(b-a) = 1$, therefore $k = \frac{1}{b-a}$
- We denote it as, $X \sim U(a, b)$
- $f(x) = \frac{1}{b-a}$



1.1.2 CDF of uniform distribution

Theorem 1.3. *CDF of uniform distribution*

$$\begin{aligned} F(x) &= 0 && x \leq a \\ &= P(X \leq x) = \int_a^x f(x)\,dx = \frac{x-a}{b-a} && a < x < b \\ &= 1 && x \geq b \end{aligned}$$

1.1.3 Expectation and variance of uniform distribution

Theorem 1.4. *Expected value of $X \sim U(a,b)$ is equal to $\frac{(a+b)}{2}$*

Proof. Consider the expectation of the uniform distribution as,

$$\begin{aligned} E[x] &= \int_a^b xP(x)\,dx \\ &= \int_a^b x\frac{1}{b-a}\,dx \\ &= \frac{1}{b-a}\int_a^b x\,dx \\ &= \frac{a+b}{2} \end{aligned}$$

□

Theorem 1.5. *Variance of uniform distribution is equal to $\frac{1}{12}(b-a)^2$*

Proof. We begin by finding out $E[X^2]$

$$
\begin{aligned}
E[X^2] &= \int_a^b x^2 P(x)\,dx \\
&= \int_a^b x^2 \frac{1}{b-a}\,dx \\
&= \frac{1}{b-a}\int_a^b x^2\,dx \\
&= \frac{1}{3}\left(a^2+ab+b^2\right)
\end{aligned}
$$

Now we can find the variance as $V[X] = E[X^2] - E[X]^2$ as follows,

$$
\begin{aligned}
V[X] &= E[X^2] - E[X]^2 \\
&= \frac{1}{3}\left(a^2+ab+b^2\right) - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

□

1.1.4 Raw moments of uniform distribution

The r^{th} raw moment of the uniform distribution is given as

$$
\begin{aligned}
\mu'_r = E[X^r] &= \int_a^b x^r f(x)\,dx \\
&= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b = \frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)}
\end{aligned}
$$

Example 1.6. *Suppose in a quiz there are 30 participants. A question is given to all 30 participants and the time allowed is 25 seconds.*

Proof. Let X denote the time to respond.
$X \sim U(0,25)$, the pdf is given by $f(x) = \frac{1}{25}; 0 < x < 25$ and 0 otherwise.

$$P(x \leq 6) = \int_0^6 f(x)\,dx = \int_0^6 \frac{1}{25}\,dx = \frac{151}{25}$$

$$P(6 \leq x \leq 10) = \int_6^1 0f(x)\,dx = \int_6^{10} \frac{1}{25}\,dx = \frac{101}{25}$$

□

Example 1.7. *A r.v. x is said to follow uniform dist with $\mu = 1$ and $V(x) = 4/3$. Obtain $P(x < 0)$.*

Proof. First begin by finding out the parameters for the unfirom distribution. First consider the mean,

$$\begin{aligned} \mu &= 1 \\ \frac{a+b}{2} &= 1 \\ a+b &= 2 \end{aligned}$$

Then consider the variance,

$$\begin{aligned} V(x) &= \frac{4}{3} \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \\ (b-a)^2 &= 16 \end{aligned}$$

Solving two simultaneous equations we get $a = -1, b = 3$.
Therefore, we have $X \sim U(-1, 3)$

$$P(x \leq 0) = F(0) = \frac{0+1}{4} = \frac{1}{4}$$

□

Example 1.8. *If $X \sim U(-3, 3)$, find $P(x < 2)$, $P(|x| < 2)$, $P(|x-2| < 2)$, also obtain k if $P(x > k) = 1/3$*

Proof.

$$\begin{aligned} P(x < 2) = F(2) &= \frac{2+3}{6} = \frac{5}{6} \\ P(|x| < 2) &= \int_{-} 2^2 \frac{1}{6}\, dx = \frac{2}{3} \\ P(|x-2| < 2) &= \int_0^3 \frac{1}{6} = \frac{1}{2} \\ P(x > k) = 1/3 &\implies \ldots \end{aligned}$$

Complete this l8r alig8r

□

1.1.5 MGF of Uniform distribution

Theorem 1.9. *MGF of Uniform distribution* $= \frac{e^{bt}-e^{at}}{t(b-a)}, t \neq 0$ *and* $= 1, t = 0$

Proof.

$$M_x(t) = E[e^{tx}] = \int_a^b \frac{e^{tx}}{b-a}\,dt = \frac{e^{bt}-e^{at}}{(b-a)t}$$

The Taylor series for this can be expressed as the following,

$$M_x(t) = \frac{b-a}{b-a} + \frac{b^2-a^2}{2(b-a)}t + \frac{b^3-a^3}{3(b-a)}\frac{t^2}{2!} + \cdots$$

Therefore we can say,

$$\mu_1' = \text{coeff of } t = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$
$$\mu_2' = \text{coeff of } \frac{t^2}{2!} = \frac{b^3-a^3}{3(b-a)}$$

And we can say $\mu_2 = \dots$ $\square$

1.1.6 Applications of uniform distribution

1. Assumption of uniform death for insurance

   $\vdots$

Write sumthin here

1.2 Gamma distribution

Definition 1.10 (Gamma distribution)**.** *A r.v. 'X' is said to follow gamma distribution $X \sim G(\lambda, \theta)$. Where $\lambda =$ shape parameter and $\theta =$ scale parameter.*

1.2.1 PDF of Gamma distribution

Definition 1.11 (PDF of Gamma distribution)**.**

$$\begin{aligned} f(x,\lambda,\theta) &= \frac{\theta^\lambda}{\Gamma(\lambda)}e^{-\theta x}x^{\lambda-1} && x>0, \lambda>0, \theta>0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

Where $\Gamma(\lambda) = (\lambda-1)! = (\lambda-1)\Gamma(\lambda-1)$.

Corollary 1.12. *If $\theta = 1$ we will have gamma distribution with a single parameter λ which is called the standard gamma distribution.*

$$\begin{aligned} X \sim G(\lambda) &= \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)} && x > 0, \lambda > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

Corollary 1.13. *If $\lambda = 1, X \sim G(1, \theta) = Exp(\theta)$.*

Corollary 1.14. *If $\lambda = 1, \theta = 1$, $X \sim$ Standard exponential distribution, i.e.*

$$\begin{aligned} f(x) &= e^{-x} && x > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

Definition 1.15 (Gamma function)**.**

$$\Gamma(\lambda) = \int_0^\infty e^{-x}x^{\lambda-1}\,dx$$

Definition 1.16 (Gamma integral)**.**

$$\int_0^\infty e^{-\theta x}x^{\lambda-1}\,dx = \frac{\Gamma(\lambda)}{\theta^\lambda}$$

1.2.2 CDF of Gamma distribution

Theorem 1.17. *CDF of Gamma distribution is given as*

$$F(x) =$$

Proof.

$$\begin{aligned} F(x) = P(X < x) &= \int_0^x \frac{\theta^\lambda e^{-\theta x}x^{\lambda-1}}{\Gamma(\lambda)}\,dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)}\int_0^x x^{\lambda-1}e^{-\theta x}\,dx \end{aligned}$$

□

1.2.3 Raw moments of Gamma distribution

Theorem 1.18. *The r^{th} raw moment of the Gamma distribution is given by*

$$\mu'_r = \frac{\Gamma(\lambda + r)}{\Gamma(\lambda)\theta^r}$$

Proof.

$$\begin{aligned}
\mu'_r = E[x^r] &= \int_0^\infty \frac{x^r \theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\, dx \\
&= \int_0^\infty \frac{\theta^\lambda e^{-\theta x} x^{\lambda+r-1}}{\Gamma(\lambda)}\, dx \\
&= \frac{\Gamma(\lambda + r)}{\Gamma(\lambda)\theta^r}
\end{aligned}$$

□

1.2.4 Mean and Variance of Gamma distribution

Now we can find μ'_1, μ'_2

$$\begin{aligned}
E[x] = \mu'_1 &= \frac{\lambda}{\theta} \\
\mu'_2 &= \frac{\lambda(\lambda+1)}{\theta^2} \\
V[x] = \mu_2 = \mu'_2 - \mu'^2_1 &= \frac{\lambda(\lambda+1)}{\theta^2} - \frac{\lambda^2}{\theta^2} = \frac{\lambda}{\theta^2}
\end{aligned}$$

1.2.5 MGF of Gamma distribution

$$\begin{aligned}
E[e^{tx}] &= \int_0^\infty e^{tx} \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\, dx \\
&= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(\theta-t)x} x^{\lambda-1}\, dx \\
&= \frac{\theta^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda)}{(\theta-t)^\lambda} = \left(\frac{\theta}{\theta-t}\right)^\lambda \\
&= \left(1 - \frac{t}{\theta}\right)^{-\lambda}
\end{aligned}$$

1.2.6 CGF of Gamma distribution

$$\begin{aligned} K_x(t) &= \log\left(1-\frac{t}{\theta}\right)^{-\lambda} \\ &= -\lambda\log\left(1-\frac{t}{\theta}\right) \\ &= \frac{\lambda t}{\theta}+\frac{\lambda t^2}{2\theta^2}+\frac{\lambda t^3}{3\theta^3}+\cdots \end{aligned}$$

Using this we can get the mean and variance easily.

$$\begin{aligned} \text{Mean } &= k_1 = \frac{\lambda}{\theta} \\ \text{Variance } &= k_2 = \frac{\lambda}{\theta^2} \end{aligned}$$

1.2.7 Additive property of Gamma distribution

If $X_i (i = 1, \ldots, k)$ are k independent Gamma distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ and θ respectively, then,

$$\begin{aligned} \sum_{i=1}^{k} X_i &\sim G\left(\sum_{i=1}^{k}\lambda_i, \theta\right) \\ M_{X_i}(t) &= \left(1-\frac{t}{\theta}\right)^{-\lambda_i} \end{aligned}$$

Let $Z = \sum X_i$

$$\begin{aligned} M_Z(t) &= \prod_{i=1}^{k}\left(1-\frac{t}{\theta}\right)^{-\lambda_i} \\ &= \left(1-\frac{t}{\theta}\right)^{-\sum\lambda_i} \end{aligned}$$

By uniqueness property of mgf

$$\sum_i X_i \sim G\left(\sum_i \lambda_i, \theta\right)$$

1.2.8 Limiting form of Gamma distribution

$$\beta_1 = \frac{4}{\lambda}, \text{ as } \lambda \to \infty, \beta_1 \to 0 \implies \text{Normal dist}$$
$$\beta_2 = 3 + \frac{6}{\lambda} \text{ as } \lambda \to \infty, \beta_2 \to 3 \implies \text{Normal dist}$$

Note that they are both independent of θ.
Therefore, as $\lambda \to \infty$ we have $G(\lambda, \infty) \to N\left(\frac{\lambda}{\theta}, \frac{\lambda}{\theta^2}\right)$.

1.2.9 Applications of Gamma distribution

Idk write something bruh

1.3 Exponential distribution

1.3.1 PDF of Exponential Distribution

Definition 1.19 (PDF of Exponential distribution)**.** *A r.v. x is said to follow the exponential distribution with parameter θ if its pdf is given by*

$$\begin{aligned} f(x) &= \theta e^{-\theta x} & & x \geq 0, \theta > 0 \\ &= 0 & & \textit{otherwise} \end{aligned}$$

1.3.2 INCOMPLETE CDF of exponential distribution

$$F[x] = 1 - e^{-\theta x}$$

FILL THIS UP

1.3.3 Raw moment of exponential distribution

Theorem 1.20. *The r^{th} raw moment for exponential distribution is given by*

$$\mu'_r = \frac{r!}{\theta^r}$$

Proof.

$$\begin{aligned} \mu'_r = E[x^r] &= \int_0^\infty x^r \theta e^{-\theta x}\, dx \\ &= \frac{\Gamma(r+1)}{\theta^r} \\ &= \frac{r!}{\theta^r} \end{aligned}$$

□

1.3.4 Mean and variance of exponential distribution

Theorem 1.21. *The mean of exponential distribution is given be*

$$\mu = \frac{1}{\theta}$$

Proof. Consider $r = 1$,

$$\mu_1' = \frac{1}{\theta}$$

□

Theorem 1.22. *The variance of the exponential distribution is given by*

$$\mu_2 = \frac{1}{\theta^2}$$

Proof. First find μ_2'

$$\mu_2' = \frac{2}{\theta^2}$$

So now we can compute the variance as $\frac{1}{\theta^2}$ □

1.3.5 MGF of exponential distribution

Theorem 1.23. *MGF of exponential distribution is given by*

$$M_x(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$$

Proof.

$$\begin{aligned}
M_x(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx}\theta e^{-\theta x}\,dx \\
&= \theta \int_0^\infty e^{x(t-\theta)} x^{1-1}\,dx \\
&= \frac{\theta\Gamma(1)}{\theta - t} \\
&= \frac{\theta}{\theta - t} \\
&= \left(1 - \frac{t}{\theta}\right)^{-1}
\end{aligned}$$

□

1.3.6 CGF of exponential distribution

Theorem 1.24. *CGF of exponential distribution is given by*

$$K_x(t) = -\log\left(1 - \frac{t}{\theta}\right)$$

Proof.

$$\begin{aligned} K_x(t) &= \log\left(1 - \frac{t}{\theta}\right)^{-1} \\ &= -\log\left(1 - \frac{t}{\theta}\right) \\ &= \frac{t}{\theta} + \frac{t^2}{2\theta^2} + \frac{t^3}{3\theta^3} \end{aligned}$$

We can say the general r^{th} cumulant is given by $K_r = \frac{(r-1)!}{\theta^r}$ □

1.3.7 Additive property of exponential variates

Theorem 1.25. *If $x_1, x_2, \ldots, x_k$ are k independent exponential variates each with parameter θthen*

$$\sum_{i=1}^{k} x_i \sim G(k, \theta)$$

Proof. We will do this with the MGF. Consider taht $Z = \sum i = 1^k x_i$.

$$\begin{aligned} M_z(t) &= \prod_{i=1}^{k} M_x(t) \\ &= \prod_{i=1}^{k} \left(1 - \frac{t}{\theta}\right)^{-1} \\ &= \left(1 - \frac{t}{\theta}\right)^{-k} \end{aligned}$$

Therefore, (by uniqueness property of MGF) comparing this MGF to that of the gamma distribution we can say that,

$$\sum_{i=1}^{k} x_i = Z \sim G(k, \theta)$$

□

1.3.8 Lack of memory of exponential distribution

Theorem 1.26. *For a exponentially distributed random variate,* $P[x > a+b \mid x > a] = P[x > b]$

Proof. Let $X \sim E(\theta)$. Consider first case

$$\begin{aligned} P[x > a + b \mid x > a] &= \frac{P[x > a + b]}{P[x > a]} \\ &= \frac{\int_{a+b}^{\infty} \theta e^{-\theta x} \, dx}{\int_{a}^{\infty} \theta e^{-\theta x} \, dx} \\ &= \frac{e^{-\theta a + b}}{e^{-\theta a}} \\ &= e^{-\theta b} \end{aligned}$$

Consider second case now,

$$P[x > b] = \int_{b}^{\infty} \theta e^{-\theta x} \, dx = e^{-\theta b}$$

Equality holds. □

1.4 INCOMPLETE Laplace distribution (Double exponential)

1.4.1 PDF

Definition 1.27 (PDF of Laplace distribution)**.** $X \sim L(\lambda, \mu)$

$$f(x) = \begin{cases} \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|} & -\infty < x < \infty \\ 0 & otherwise \end{cases}$$

1.4.2 CDF

Definition 1.28 (CDF of Laplace distribution)**.**

$$F[x] = \begin{cases} content... \end{cases}$$

1.4.3 Raw moment

Theorem 1.29. *The r^{th} raw moment for the Laplace distribution is given by*

$$\mu'_r =$$

Proof.

$$\mu'_r = E[x^r] = \int_{-\infty}^{\infty} \frac{x^r}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\,dx\right|}$$

Transform $(x - \mu)/\lambda = z$

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\infty} (z\lambda + \mu)^r e^{-|z|}\,\lambda\,dz\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (z-\lambda)^k \mu^{r-k} e^{-|z|\,dz}\right) \\
&= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k e^{-|z|}\,dz\right]
\end{aligned}$$

Complete this up

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} k!(1 + (-1)^k)\right]$$

□

1.4.4 Mean and variance

We can do this with the raw moments above but instead we will do it with the PDF.

Theorem 1.30. *Expectation of laplace distribution is given as*

$$E[x] =$$

Proof.

$$\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_{-\infty}^{\infty} \frac{x}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}\,dx
\end{aligned}$$

Split it around μ

$$
\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} xe^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} xe^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} xe^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} xe^{-x/\lambda}\,dz\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\lambda(x-\lambda)e^{x/\lambda} - e^{\mu/\lambda}(\lambda(x+\lambda)e^{-x/\lambda})\right] \\
&= \mu
\end{aligned}
$$

□

Theorem 1.31. *Expectation of x^2 in Laplace distribution is given be*

$$E[x^2] = bruh$$

Proof.

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}$$

Split it around μ

$$
\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} x^2e^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} x^2e^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} x^2e^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} x^2e^{-x/\lambda}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}(\lambda(x^2-2\lambda x+2\lambda^2)e^{x/\lambda}) - e^{\mu/\lambda}(\lambda(x^2+2\lambda x+2\lambda^2)e^{-x/\lambda})\right] \\
&= 2\lambda^2
\end{aligned}
$$

□

Theorem 1.32. *Variance of Laplace distribution is given as*

$$V[x] =$$

1.4.5 MGF

Theorem 1.33. *MGF of the Laplace distribution is given by*

$$M_x(t) = bruh$$

Proof.

$$
\begin{aligned}
M_x(t) = E[e^{tx}] &= \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{tx - \left|\frac{x-\mu}{\lambda}\right|} \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} e^{x\left(t+\frac{1}{\lambda}\right)}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} e^{-x\left(\frac{1}{\lambda}-t\right)}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\left(\frac{e^{\mu\left(\frac{1}{\lambda}+t\right)}}{\frac{1}{\lambda}+t}\right) + e^{\mu/\lambda}\left(\frac{-e^{\mu\left(\frac{1}{\lambda}-t\right)}}{-\frac{1}{\lambda}+t}\right)\right] \\
&= \frac{1}{2\lambda}\left[\frac{e^{\mu t}}{t+\frac{1}{\lambda}} - \frac{e^{\mu t}}{t-\frac{1}{\lambda}}\right]
\end{aligned}
$$

□

Plot a graph for the beta-1 dsitribution when alpha=5, beta=2

1.4.6 CGF

1.5 Beta distribution of Type-I

1.5.1 PDF

Definition 1.34 (PDF of Beta I)**.**

$$
f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 < x < 1; m,n > 0 \\ 0 & \textit{otherwise} \end{cases}
$$

Where $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Note the following,

1. We can say, $X \sim \beta_1(m,n)$ where m,n are the parameters of the distribution.

2. Since $f(x)$ is a pdf we have the following,

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx \\
&= \int_0^1
\end{aligned}
$$

1.5.2 Raw moments

Theorem 1.35. *The r^{th} raw moment of the Beta I distribution is given by*

$$\mu'_r = \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}$$

Proof.

$$\begin{aligned}\mu'_r = E[x^r] &= \int_0^1 \frac{1}{\beta(m,n)} x^{r+m-1}(1-x)^{n-1}\,dx \\ &= \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}\end{aligned}$$

□

1.5.3 Mean and Variance

Theorem 1.36. *Mean of Beta I distribution is given by*

$$E[x] = \frac{m}{m+n}$$

Proof.

$$E[x] = \mu'_1 = \frac{\Gamma(m+n)\Gamma(m+1)}{\Gamma(m)+\Gamma(m+n+1)} = \frac{m}{m+n}$$

□

Theorem 1.37. *Variance of Beta I distribution is given by*

$$V[x] = \frac{mn}{(m+n)^2(m+n+1)}$$

Proof.

$$\mu'_2 = \frac{(m+1)(m)}{(m+n)(m+n+1)}$$

So now we have the variance given as,

$$\begin{aligned}\mu_2 &= \mu'_2 - \mu'^2_1 \\ &= \frac{mn}{(m+n)^2(m+n+1)}\end{aligned}$$

□

1.6 Beta distribution of Type-II

1.6.1 PDF

Definition 1.38 (PDF of Beta-II distribution)**.**

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & otherwise \end{cases}$$

Note the following,

1. X is said to follow $\beta_2(m,n)$ as $X \sim \beta_2(m,n)$

2. $$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^{m+1}}{(1+x)^{m+n}} = \beta(m,n)$$

1.6.2 Raw moments

Theorem 1.39 (Raw moments of Beta-2 distribution)**.** *The raw moments of the Beta-2 distribution is given by*

$$\mu_r' = \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)}$$

Proof.

$$\begin{aligned} \mu_r' = E[x^r] &= \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx \\ &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)} \end{aligned}$$

□

1.6.3 Mean and variance

Theorem 1.40 (Mean of Beta-2 distribution)**.** *The mean of Beta-2 distribution is given by*

$$E[x] = \frac{m}{n-1}$$

Proof.

$$E[x] = \mu_1' = \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma(m)\Gamma(n)}$$
$$= \frac{m}{n-1}$$

□

Theorem 1.41 (Variance of Beta-2 distribution)**.** *The variance of Beta-2 distribution is given by*

$$V[x] = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

Proof. First consider the 2nd raw moment,

$$\mu_2' = \frac{m(m+1)}{(n-2)(n-2)}$$

Now we can compute the variance as follows

$$V[x] = \mu_2 = \mu_2' - \mu_1'^2 = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

□

1.7 Transformation of variables

1.7.1 One dimensional random variable

Let X be a continuous random variable with pdf $f(x)$ and let $Y = g(x)$ be a strictly monotonic function of X with unique inverse.
Assume that $g(x)$ is differentiable and is continuous for all x, then the pdf of r.v. Y is given by

$$h(y) = f(x) \cdot \det \left| \frac{dx}{dy} \right| = \left| \frac{dx}{dy} \right|$$

where r.v. x is expressed in terms of y. Steps to solve,

1. Write pdf of r.v. X.
2. Express old variable X in terms of new variable Y.

3. Write the range of the new variable.
4. Obtain J where $J = \left|\frac{dx}{dy}\right|$ and $|J|$.
5. Obtain $h(y) = f(x) \cdot |J|$, where X is expressed in terms of Y.

Remark 1.42. *For $2-1$ correspondence, i.e. for ever 2 values of X is there is only one value of Y, then multiple $|J|$ with 2.*

Remark 1.43.

For $1-2$ correspondence i.e., for every 1 value of x if there are 2 values of Y then multiply $|J|$ with $\frac{1}{2}$.

Example 1.44. *If a r.v. $X \sim B_1(m,n)$ obtain the distribution of $Y = 1-X$.*

Proof. First begin by stating the pdf of X.

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 < x < 1; m,n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $X = 1-Y$ this ranges from $1-Y = 0$ to $1-Y = 1$. So $0 < Y < 1$ again.
Now compute J

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{dy}(1-y) \\ J &= -1 \\ |J| &= 1 \end{aligned}$$

We multiply this with $f(x)$ to get $h(y)$.

$$\begin{aligned} h(y) &= f(x) \cdot |J| \\ h(y) &= f(x) \end{aligned}$$

So $h(y) \sim B(n,m)$. The order changes. □

Example 1.45. *A r.v. $X \sim B_2(m,n)$. Obtain the distribution of Y where $Y = \frac{1}{1+X}$.*

Proof. First state the pdf,

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m,n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now state X in terms of Y, we have $X = \frac{1}{Y} - 1$.
Compute the new ranges now we have $\frac{1}{Y} - 1 = 0$ so $Y = 1$ as one side then $\frac{1}{Y} - 1 = \lim_{m\to\infty} m$ so to $Y = 0$.
The new ranges are $0 < Y < 1$. Now compute $|J|$,

$$\begin{aligned} J &= \frac{dx}{dy} = \frac{1}{dy}\left(\frac{1}{y} - 1\right) \\ &= -\frac{1}{y^2} \\ |J| &= \frac{1}{y^2} \end{aligned}$$

So now we can compute $h(y)$ as follows,

$$\begin{aligned} h(y) &= f(x)|J| \\ &= \frac{1}{\beta(m,n)} \frac{\left(\frac{1}{y} - 1\right)^{m-1}}{(1/y)^{m+n}} \frac{1}{y^2} \\ &= \frac{1}{\beta(m,n)} y^{n-1}(1-y)^{m-1} \end{aligned}$$

This is for the range we have and 0 otherwise. But I'm too lazy to typeset that out as a cases.
So we now have $Y \sim B_1(n, m)$. □

1.8 Two dimensional r.v.

Let X and Y be two continuous independent r.v. with joint pdf $f(x, y)$. Say $U = g(x, y)$ and $V = h(x, y)$ are two other r.v. then the joint pdf of U and V is given by,

$$k(u, v) = f(x, y) \cdot |J|$$

where X, Y are expressed in terms of U, V. Here we have the Jacobian as follows,

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} \end{bmatrix}$$

1.8.1 Steps to solve

1. Write the pdf of X and Y, i.e. $f(x, y)$.

2. Express old variable in terms of new variable.
3. Obtain range of the new variable.
4. Obtain J and $|J|$.
5. Obtain $k(u,v) = f(x,y)|J|$.

Example 1.46. *X and Y are two independent gamma variates with parameters a and b respectively.*

1. *Obtain the joint distribution of u and v where $u = x + y, v = \frac{x}{x+y}$.*
2. *Show that u, v are independent and identify their distributions.*

Proof. Consider the pdf of gamma function as follows,

$$\begin{aligned} X \sim G(\lambda) &= \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)} && x > 0, \lambda > 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

Where $\Gamma(\lambda) = (\lambda - 1)! = (\lambda - 1)\Gamma(\lambda - 1)$.

$$\begin{aligned} f_1(x) &= \frac{1}{\Gamma(a)} e^{-x} x^{a-1} \\ f_2(x) &= \frac{1}{\Gamma(b)} e^{-x} x^{b-1} \end{aligned}$$

Find $f(x,y) = f_1(x)f_2(y)$

$$\begin{aligned} f(x,y) &= \frac{1}{\Gamma(a)\Gamma(b)} e^{-x-y} x^{a-1} x^{b-1} && x, y, a, b, > 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

We now have the new variables U, V $U = X + Y, V = \frac{X}{X+Y}$. This implies that $X = UV, Y = U(1 - V)$.
We need to find the new ranges now. Since $X, Y > 0$ we have $U > 0$ and $X < X + Y \implies \frac{x}{x+y} < 1 \implies v < 1$. And $0 < V < 1$.

Find the Jacobian,

$$\begin{aligned} J &= \begin{bmatrix} v & u \\ 1 - v & -u \end{bmatrix} = -u \\ |J| &= u \end{aligned}$$

The joint distribution is then given as,

$$\begin{aligned}
k(u,v) &= \frac{1}{\Gamma(a)\Gamma(b)}e^{-(uv+u-uv)}(uv)^{a-1}[u(1-v)]^{b-1}\cdot u \\
&= \frac{1}{\Gamma(a)\Gamma(b)}e^{-u}u^{a-1+b-1+1}v^{a-1}(1-v)^{b-1}\times\frac{\Gamma(a+b)}{\Gamma(a+b)} \\
&= \frac{1}{\Gamma(a+b)}e^{-u}u^{a+b-1}\times\frac{1}{\beta(a,b)}v^{a-1}(1-v)^{b-1} \\
&= k_1(u)k_2(v)
\end{aligned}$$

So u and v are independent r.v. and $U \sim G(a+b), V \sim \beta_1(a,b)$ □

Example 1.47. *X and Y are two independent r.v. $X \sim G(a)$ and $Y \sim G(b)$. We have $U = X+Y$ and $W = \frac{X}{Y}$. Show that U, W are independent and identify the distribution.*

Proof. We know the following,

$$\begin{aligned}
f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} && x>0, a>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

and,

$$\begin{aligned}
f_2(y) &= \frac{e^{-y}y^{b-1}}{\Gamma(b)} && x>0, b>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

Now the joint distribution $f(x,y)$ is given by its product since they are independent,

$$\begin{aligned}
f(x,y) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)}\times\frac{e^{-y}y^{b-1}}{\Gamma(b)} && x>0, y>0; a,b>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

Now we compute the new ranges $X = \frac{UW}{W+1}$ and $Y = \frac{U}{W+1}$. Now when $X = 0$ we have $U = 0, W = 0$ when $X \to \infty$ $U \to \infty, V \to \infty$. So we have $U > 0$ and $W > 0$.

Now compute the Jacobian as follows,

$$\begin{aligned}
J &= \begin{bmatrix} \frac{w}{1+w} & \frac{-uw}{(1+w)^2}+\frac{u}{1+w} \\ \frac{1}{1+w} & \frac{-u}{(1+w)^2} \end{bmatrix} \\
|J| &= \frac{u}{(1+w)^2}
\end{aligned}$$

Since for 2 values of Y we get one value of X we will multiply the jacobian by 2. Now we compute $k(u, w)$ as follows,

$$\begin{aligned} k(u,w) &= f(x,y)|J| \\ &= \frac{e^{-\frac{uw}{w+1}} \frac{uw}{w+1}^{a-1}}{\Gamma(a)} \times \frac{e^{-\frac{u}{w+1}} \frac{u}{w+1}^{b-1}}{\Gamma(b)} \times \frac{u}{(1+w)^2} \\ &= \frac{1}{\Gamma(a+b)} e^{-u} u^{a+b-1} \times \frac{1}{\beta(a,b)} \end{aligned}$$

Complete this □

Example 1.48. $X \sim N(\mu, \sigma^2)$*. Obtain the distribution of* $Y = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

Proof. Begin by stating the pdf of r.v. X,

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} && -\infty < x < \infty, \sigma > 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

We now state X in terms of Y as follows, $X = \mu \pm \sqrt{2}\sigma\sqrt{y}$. Range of y is $0 < y < \infty$. And since it is 2-1 correspondence we will multiply the Jacobian by 2.
Compute the value of Jacobian first,

$$|J| = \frac{\sigma}{\sqrt{2}\sqrt{y}}$$

Now compute the new function,

$$\begin{aligned} h(y) &= f(x)|J|2 \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \times \frac{\sigma}{\sqrt{2}\sqrt{y}} \times 2 \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-y} \frac{2\sigma}{\sqrt{2}\sqrt{y}} \\ &= \frac{2}{\sqrt{2}\sqrt{y}\sqrt{2\pi}} e^{-y} \\ &= \frac{e^{-y}}{\sqrt{\pi}\sqrt{y}} \\ &= \frac{1}{\Gamma(\frac{1}{2})} e^{-y} y^{1-\frac{1}{2}} \end{aligned}$$

So we have $Y \sim G\left(\frac{1}{2}\right)$. □

Example 1.49.

$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & x,y>0 \\ 0 & otherwise \end{cases}$$

Prove that $h(u) = 2u^3e^{-u^2}, u>0$ *where* $u=\sqrt{x^2+y^2}$ *and* $v=x$.

Proof. The variables we are dealing with are,

$$x = v, y = \sqrt{u^2-v^2}$$

The range for v,u is $(0,\infty)$ but $0<v<u<\infty$.
Begin by computing the Jacobian,

$$|J| = \frac{u}{\sqrt{u^2-v^2}}$$

Consider now the joint distribution with the change of variables,

$$\begin{aligned} g(u,v) &= f(x,y)|J| \\ &= 4xye^{-(x^2+y^2)}|J| \\ &= 4(v)(\sqrt{u^2-v^2})e^{-(v^2+u^2-v^2)}\frac{u}{\sqrt{u^2-v^2}} \\ &= 4v\sqrt{u^2-v^2}e^{-u^2}\frac{u}{\sqrt{u^2-v^2}} \\ &= 4vue^{-u^2} \end{aligned}$$

Integrate out v

$$\begin{aligned} h(u) &= 4ue^{-u^2}\int_0^u v\,dv \\ &= 4ue^{-u^2}\frac{u^2}{2} \\ &= 2u^3e^{-u^2} \end{aligned}$$

□

Example 1.50.

$$f(x,y) = \begin{cases} \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)} & x,y>0 \\ 0 & otherwise \end{cases}$$

Obtain pdf of u *where* $u=\frac{x}{x+y}$ *take* $v=x+y$ *also obtain* $E[u], V[u]$.

Proof. Consider the new variables, $x = uv, y = v - uv$. The range for v is $(0, \infty)$ and for u is $(0, 1)$
Compute the Jacobian,

$$|J| = v$$

Compute the joint pdf,

$$\begin{aligned} g(u, v) &= f(x, y)|J| \\ &= \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)}|J| \\ &= \frac{e^{-(uv+v-uv)}(uv)^3(v-uv)^4}{\Gamma(4)\Gamma(5)}v \\ &= \frac{e^{-v}u^3(1-u)^4v^8}{\Gamma(4)\Gamma(5)} \end{aligned}$$

Integrate out v

$$\begin{aligned} h(u) &= \frac{u^3(1-u)^4}{\Gamma(4)\Gamma(5)}\int_0^\infty e^{-v}v^8\,dv \\ &= \frac{\Gamma(9)}{\Gamma(4)\Gamma(5)}u^3(1-u)^4 \end{aligned}$$

So $U \sim \beta_1(m = 4, n = 5)$. Compute the mean and variance as follows,

$$\begin{aligned} E[U] &= \frac{m}{m+n} = \frac{4}{9} \\ V[U] &= \frac{mn}{(m+n)^2(m+n+1)} = \frac{20}{810} = \frac{2}{81} \end{aligned}$$

□

Example 1.51. *X, Y are two independent gamma variates with parameters a, b respectively. Show that $U + X + Y, V = \frac{X-Y}{X+Y}$ are independent.*

Proof. Consider the original pdfs,

$$\begin{aligned} f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} \\ f_2(y) &= \frac{e^{-y}x^{b-1}}{\Gamma(b)} \end{aligned}$$

Since they are independent

$$f(x,y) = \frac{e^{-(x+y)}x^{a-1}y^{b-1}}{\Gamma(a)\Gamma(b)}$$

Consider the new variables, $x = \frac{1}{2}(uv+u), y = \frac{1}{2}(u-uv)$, the ranges for u is $(0,\infty)$ but for v is $(-1,1)$
Compute the Jacobian,

$$|J| = \frac{u}{2}$$

Compute the joint pdf,

$$\begin{aligned} g(u,v) &= f(x,y)|J| \\ &= \frac{e^{-(x+y)}x^{a-1}y^{b-1}}{\Gamma(a)\Gamma(b)}|J| \\ &= -\frac{e^{-u}(v+1)2^{-a-b+2}(u(v+1))^{a-2}(u-uv)^{b}}{(v-1)\Gamma(a)\Gamma(b)} \end{aligned}$$

Split this up I'm too lazy to type it. $\square$

Chapter 2

Chi-square distribution

Chapter 3

F-distribution