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# Probability and Sampling Distributions (B)

**Lecture Notes**
**for SSTA401**

# Contents

# Chapter 1

# Transformation of random variables & standard univariate continuous probability distributions

## 1.1 Uniform/Rectangular distributions

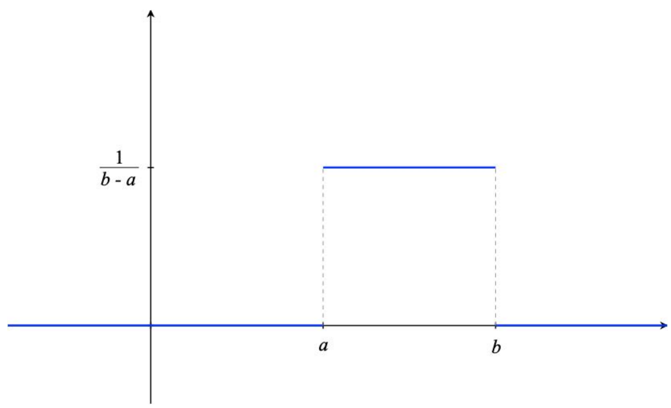
**Definition 1.1.** *A r.v. $X$ is said to follow uniform distribution over an interval $(a, b)$ if its pdf is constant over the entire range.*

### 1.1.1 PDF of uniform distribution

**Theorem 1.2.** *PDF of uniform distribution*

$$\begin{aligned} P(x) &= k && a < x < b \\ &= 0 && \textit{otherwise} \end{aligned}$$

- $\int_a^b f(x)\,dx = \int_a^b k\,dx = k[x]_a^b = k(b-a) = 1$, therefore $k = \frac{1}{b-a}$
- We denote it as, $X \sim U(a, b)$
- $f(x) = \frac{1}{b-a}$




### 1.1.2 CDF of uniform distribution

**Theorem 1.3.** *CDF of uniform distribution*

$$\begin{aligned} F(x) &= 0 && x \leq a \\ &= P(X \leq x) = \int_a^x f(x)\,dx = \frac{x-a}{b-a} && a < x < b \\ &= 1 && x \geq b \end{aligned}$$

### 1.1.3 Expectation and variance of uniform distribution

**Theorem 1.4.** *Expected value of $X \sim U(a,b)$ is equal to $\frac{(a+b)}{2}$*

*Proof.* Consider the expectation of the uniform distribution as,

$$\begin{aligned} E[x] &= \int_a^b xP(x)\,dx \\ &= \int_a^b x\frac{1}{b-a}\,dx \\ &= \frac{1}{b-a}\int_a^b x\,dx \\ &= \frac{a+b}{2} \end{aligned}$$

□

**Theorem 1.5.** *Variance of uniform distribution is equal to $\frac{1}{12}(b-a)^2$*

*Proof.* We begin by finding out $E[X^2]$

$$\begin{aligned}
E[X^2] &= \int_a^b x^2 P(x)\,dx \\
&= \int_a^b x^2 \frac{1}{b-a}\,dx \\
&= \frac{1}{b-a}\int_a^b x^2\,dx \\
&= \frac{1}{3}\left(a^2+ab+b^2\right)
\end{aligned}$$

Now we can find the variance as $V[X] = E[X^2] - E[X]^2$ as follows,

$$\begin{aligned}
V[X] &= E[X^2] - E[X]^2 \\
&= \frac{1}{3}\left(a^2+ab+b^2\right) - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

□

### 1.1.4 Raw moments of uniform distribution

The $r^{th}$ raw moment of the uniform distribution is given as

$$\begin{aligned}
\mu_r' = E[X^r] &= \int_a^b x^r f(x)\,dx \\
&= \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b = \frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)}
\end{aligned}$$

**Example 1.6.** *Suppose in a quiz there are 30 participants. A question is given to all 30 participants and the time allowed is 25 seconds.*

*Proof.* Let $X$ denote the time to respond.
$X \sim U(0,25)$, the pdf is given by $f(x) = \frac{1}{25}; 0 < x < 25$ and 0 otherwise.

$$P(x \le 6) = \int_0^6 f(x)\,dx = \int_0^6 \frac{1}{25}\,dx = \frac{151}{25}$$

$$P(6 \le x \le 10) = \int_6^1 0f(x)\,dx = \int_6^{10} \frac{1}{25}\,dx = \frac{101}{25}$$

□

**Example 1.7.** *A r.v. $x$ is said to follow uniform dist with $\mu = 1$ and $V(x) = 4/3$. Obtain $P(x < 0)$.*

*Proof.* First begin by finding out the parameters for the unfirom distribution. First consider the mean,

$$\begin{aligned} \mu &= 1 \\ \frac{a+b}{2} &= 1 \\ a+b &= 2 \end{aligned}$$

Then consider the variance,

$$\begin{aligned} V(x) &= \frac{4}{3} \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \\ (b-a)^2 &= 16 \end{aligned}$$

Solving two simultaneous equations we get $a = -1, b = 3$.
Therefore, we have $X \sim U(-1, 3)$

$$P(x \leq 0) = F(0) = \frac{0+1}{4} = \frac{1}{4}$$

□

**Example 1.8.** *If $X \sim U(-3, 3)$, find $P(x < 2)$, $P(|x| < 2)$, $P(|x-2| < 2)$, also obtain $k$ if $P(x > k) = 1/3$*

*Proof.*

$$\begin{aligned} P(x < 2) = F(2) &= \frac{2+3}{6} = \frac{5}{6} \\ P(|x| < 2) &= \int_{-} 2^2 \frac{1}{6}\, dx = \frac{2}{3} \\ P(|x-2| < 2) &= \int_0^3 \frac{1}{6} = \frac{1}{2} \\ P(x > k) = 1/3 \implies \ldots \end{aligned}$$

Complete this l8r alig8r

□

### 1.1.5 MGF of Uniform distribution

**Theorem 1.9.** *MGF of Uniform distribution* $= \frac{e^{bt}-e^{at}}{t(b-a)}, t \neq 0$ *and* $= 1, t = 0$

*Proof.*

$$M_x(t) = E[e^{tx}] = \int_a^b \frac{e^{tx}}{b-a}\,dt = \frac{e^{bt}-e^{at}}{(b-a)t}$$

The Taylor series for this can be expressed as the following,

$$M_x(t) = \frac{b-a}{b-a} + \frac{b^2-a^2}{2(b-a)}t + \frac{b^3-a^3}{3(b-a)}\frac{t^2}{2!} + \cdots$$

Therefore we can say,

$$\mu_1' = \text{coeff of } t = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$
$$\mu_2' = \text{coeff of } \frac{t^2}{2!} = \frac{b^3-a^3}{3(b-a)}$$

And we can say $\mu_2 = \ldots$ □

### 1.1.6 Applications of uniform distribution

1. Assumption of uniform death for insurance

   ⋮

Write sumthin here

## 1.2 Gamma distribution

**Definition 1.10** (Gamma distribution)**.** *A r.v. '$X'$ is said to follow gamma distribution* $X \sim G(\lambda, \theta)$*. Where* $\lambda =$ *shape parameter and* $\theta =$ *scale parameter.*

### 1.2.1 PDF of Gamma distribution

**Definition 1.11** (PDF of Gamma distribution)**.**

$$\begin{aligned} f(x,\lambda,\theta) &= \frac{\theta^\lambda}{\Gamma(\lambda)} e^{-\theta x} x^{\lambda-1} && x > 0, \lambda > 0, \theta > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

*Where* $\Gamma(\lambda) = (\lambda-1)! = (\lambda-1)\Gamma(\lambda-1)$.

**Corollary 1.12.** *If $\theta = 1$ we will have gamma distribution with a single parameter $\lambda$ which is called the standard gamma distribution.*

$$\begin{aligned} X \sim G(\lambda) &= \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)} && x > 0, \lambda > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

**Corollary 1.13.** *If $\lambda = 1, X \sim G(1, \theta) = Exp(\theta)$.*

**Corollary 1.14.** *If $\lambda = 1, \theta = 1$, $X \sim$ Standard exponential distribution, i.e.*

$$\begin{aligned} f(x) &= e^{-x} && x > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

**Definition 1.15** (Gamma function)**.**

$$\Gamma(\lambda) = \int_0^\infty e^{-x}x^{\lambda-1}\,dx$$

**Definition 1.16** (Gamma integral)**.**

$$\int_0^\infty e^{-\theta x}x^{\lambda-1}\,dx = \frac{\Gamma(\lambda)}{\theta^\lambda}$$

### 1.2.2 CDF of Gamma distribution

**Theorem 1.17.** *CDF of Gamma distribution is given as*

$$F(x) =$$

*Proof.*

$$\begin{aligned} F(x) = P(X < x) &= \int_0^x \frac{\theta^\lambda e^{-\theta x}x^{\lambda-1}}{\Gamma(\lambda)}\,dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)}\int_0^x x^{\lambda-1}e^{-\theta x}\,dx \end{aligned}$$

□

### 1.2.3 Raw moments of Gamma distribution

**Theorem 1.18.** *The $r^{th}$ raw moment of the Gamma distribution is given by*

$$\mu'_r = \frac{\Gamma(\lambda + r)}{\Gamma(\lambda)\theta^r}$$

*Proof.*

$$\begin{aligned} \mu'_r = E[x^r] &= \int_0^\infty \frac{x^r \theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\, dx \\ &= \int_0^\infty \frac{\theta^\lambda e^{-\theta x} x^{\lambda+r-1}}{\Gamma(\lambda)}\, dx \\ &= \frac{\Gamma(\lambda + r)}{\Gamma(\lambda)\theta^r} \end{aligned}$$

□

### 1.2.4 Mean and Variance of Gamma distribution

Now we can find $\mu'_1, \mu'_2$

$$\begin{aligned} E[x] = \mu'_1 &= \frac{\lambda}{\theta} \\ \mu'_2 &= \frac{\lambda(\lambda+1)}{\theta^2} \\ V[x] = \mu_2 = \mu'_2 - \mu'^2_1 &= \frac{\lambda(\lambda+1)}{\theta^2} - \frac{\lambda^2}{\theta^2} = \frac{\lambda}{\theta^2} \end{aligned}$$

### 1.2.5 MGF of Gamma distribution

$$\begin{aligned} E[e^{tx}] &= \int_0^\infty e^{tx} \frac{\theta^\lambda e^{-\theta x} x^{\lambda-1}}{\Gamma(\lambda)}\, dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(\theta-t)x} x^{\lambda-1}\, dx \\ &= \frac{\theta^\lambda}{\Gamma(\lambda)} \frac{\Gamma(\lambda)}{(\theta-t)^\lambda} = \left(\frac{\theta}{\theta - t}\right)^\lambda \\ &= \left(1 - \frac{t}{\theta}\right)^{-\lambda} \end{aligned}$$

### 1.2.6 CGF of Gamma distribution

$$
\begin{aligned}
K_x(t) &= \log\left(1-\frac{t}{\theta}\right)^{-\lambda} \\
&= -\lambda \log\left(1-\frac{t}{\theta}\right) \\
&= \frac{\lambda t}{\theta} + \frac{\lambda t^2}{2\theta^2} + \frac{\lambda t^3}{3\theta^3} + \cdots
\end{aligned}
$$

Using this we can get the mean and variance easily.

$$
\begin{aligned}
\text{Mean } &= k_1 = \frac{\lambda}{\theta} \\
\text{Variance } &= k_2 = \frac{\lambda}{\theta^2}
\end{aligned}
$$

### 1.2.7 Additive property of Gamma distribution

If $X_i (i = 1, \ldots, k)$ are $k$ independent Gamma distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\theta$ respectively, then,

$$
\begin{aligned}
\sum_{i=1}^{k} X_i &\sim G\left(\sum_{i=1}^{k} \lambda_i, \theta\right) \\
M_{X_i}(t) &= \left(1-\frac{t}{\theta}\right)^{-\lambda_i}
\end{aligned}
$$

Let $Z = \sum X_i$

$$
\begin{aligned}
M_Z(t) &= \prod_{i=1}^{k} \left(1-\frac{t}{\theta}\right)^{-\lambda_i} \\
&= \left(1-\frac{t}{\theta}\right)^{-\sum \lambda_i}
\end{aligned}
$$

By uniqueness property of mgf

$$
\sum_i X_i \sim G\left(\sum_i \lambda_i, \theta\right)
$$

### 1.2.8 Limiting form of Gamma distribution

$$\beta_1 = \frac{4}{\lambda}, \text{ as } \lambda \to \infty, \beta_1 \to 0 \implies \text{Normal dist}$$
$$\beta_2 = 3 + \frac{6}{\lambda} \text{ as } \lambda \to \infty, \beta_2 \to 3 \implies \text{Normal dist}$$

Note that they are both independent of $\theta$.
Therefore, as $\lambda \to \infty$ we have $G(\lambda, \infty) \to N\left(\frac{\lambda}{\theta}, \frac{\lambda}{\theta^2}\right)$.

### 1.2.9 Applications of Gamma distribution

Idk write something bruh

## 1.3 Exponential distribution

### 1.3.1 PDF of Exponential Distribution

**Definition 1.19** (PDF of Exponential distribution)**.** *A r.v. $x$ is said to follow the exponential distribution with parameter $\theta$ if its pdf is given by*

$$\begin{aligned} f(x) &= \theta e^{-\theta x} && x \geq 0, \theta > 0 \\ &= 0 && \textit{otherwise} \end{aligned}$$

### 1.3.2 INCOMPLETE CDF of exponential distribution

$$F[x] = 1 - e^{-\theta x}$$

**FILL THIS UP**

### 1.3.3 Raw moment of exponential distribution

**Theorem 1.20.** *The $r^{th}$ raw moment for exponential distribution is given by*

$$\mu'_r = \frac{r!}{\theta^r}$$

*Proof.*

$$\begin{aligned} \mu'_r = E[x^r] &= \int_0^\infty x^r \theta e^{-\theta x}\, dx \\ &= \frac{\Gamma(r+1)}{\theta^r} \\ &= \frac{r!}{\theta^r} \end{aligned}$$

□

### 1.3.4 Mean and variance of exponential distribution

**Theorem 1.21.** *The mean of exponential distribution is given be*

$$\mu = \frac{1}{\theta}$$

*Proof.* Consider $r = 1$,

$$\mu_1' = \frac{1}{\theta}$$

□

**Theorem 1.22.** *The variance of the exponential distribution is given by*

$$\mu_2 = \frac{1}{\theta^2}$$

*Proof.* First find $\mu_2'$

$$\mu_2' = \frac{2}{\theta^2}$$

So now we can compute the variance as $\frac{1}{\theta^2}$ □

### 1.3.5 MGF of exponential distribution

**Theorem 1.23.** *MGF of exponential distribution is given by*

$$M_x(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$$

*Proof.*

$$\begin{aligned}
M_x(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx}\theta e^{-\theta x}\,dx \\
&= \theta \int_0^\infty e^{x(t-\theta)}x^{1-1}\,dx \\
&= \frac{\theta\Gamma(1)}{\theta - t} \\
&= \frac{\theta}{\theta - t} \\
&= \left(1 - \frac{t}{\theta}\right)^{-1}
\end{aligned}$$

□

### 1.3.6 CGF of exponential distribution

**Theorem 1.24.** *CGF of exponential distribution is given by*

$$K_x(t) = -\log\left(1 - \frac{t}{\theta}\right)$$

*Proof.*

$$\begin{aligned} K_x(t) &= \log\left(1 - \frac{t}{\theta}\right)^{-1} \\ &= -\log\left(1 - \frac{t}{\theta}\right) \\ &= \frac{t}{\theta} + \frac{t^2}{2\theta^2} + \frac{t^3}{3\theta^3} \end{aligned}$$

We can say the general $r^{th}$ cumulant is given by $K_r = \frac{(r-1)!}{\theta^r}$ □

### 1.3.7 Additive property of exponential variates

**Theorem 1.25.** *If $x_1, x_2, \ldots, x_k$ are $k$ independent exponential variates each with parameter $\theta$then*

$$\sum_{i=1}^{k} x_i \sim G(k, \theta)$$

*Proof.* We will do this with the MGF. Consider taht $Z = \sum i = 1^k x_i$.

$$\begin{aligned} M_z(t) &= \prod_{i=1}^{k} M_x(t) \\ &= \prod_{i=1}^{k} \left(1 - \frac{t}{\theta}\right)^{-1} \\ &= \left(1 - \frac{t}{\theta}\right)^{-k} \end{aligned}$$

Therefore, (by uniqueness property of MGF) comparing this MGF to that of the gamma distribution we can say that,

$$\sum_{i=1}^{k} x_i = Z \sim G(k, \theta)$$

□

### 1.3.8 Lack of memory of exponential distribution

**Theorem 1.26.** *For a exponentially distributed random variate,* $P[x > a+b \mid x > a] = P[x > b]$

*Proof.* Let $X \sim E(\theta)$. Consider first case

$$
\begin{aligned}
P[x > a + b \mid x > a] &= \frac{P[x > a + b]}{P[x > a]} \\
&= \frac{\int_{a+b}^{\infty} \theta e^{-\theta x}\, dx}{\int_{a}^{\infty} \theta e^{-\theta x}\, dx} \\
&= \frac{e^{-\theta a + b}}{e^{-\theta a}} \\
&= e^{-\theta b}
\end{aligned}
$$

Consider second case now,

$$P[x > b] = \int_{b}^{\infty} \theta e^{-\theta x}\, dx = e^{-\theta b}$$

Equality holds. □

## 1.4 INCOMPLETE Laplace distribution (Double exponential)

### 1.4.1 PDF

**Definition 1.27** (PDF of Laplace distribution)**.** $X \sim L(\lambda, \mu)$

$$f(x) = \begin{cases} \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|} & -\infty < x < \infty \\ 0 & otherwise \end{cases}$$

### 1.4.2 CDF

**Definition 1.28** (CDF of Laplace distribution)**.**

$$F[x] = \begin{cases} content... \end{cases}$$

### 1.4.3 Raw moment

**Theorem 1.29.** *The $r^{th}$ raw moment for the Laplace distribution is given by*

$$\mu'_r =$$

*Proof.*

$$\mu'_r = E[x^r] = \int_{-\infty}^{\infty} \frac{x^r}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\,dx\right|}$$

Transform $(x-\mu)/\lambda = z$

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\infty} (z\lambda+\mu)^r e^{-|z|}\,\lambda\,dz\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (z-\lambda)^k \mu^{r-k} e^{-|z|\,dz}\right) \\
&= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k e^{-|z|}\,dz\right]
\end{aligned}$$

Complete this up

$$= \frac{1}{2}\sum_{k=0}^{r}\left[\binom{r}{k}\lambda^k \mu^{r-k} k!(1+(-1)^k)\right]$$

□

### 1.4.4 Mean and variance

We can do this with the raw moments above but instead we will do it with the PDF.

**Theorem 1.30.** *Expectation of laplace distribution is given as*

$$E[x] =$$

*Proof.*

$$\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_{-\infty}^{\infty} \frac{x}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}\,dx
\end{aligned}$$

Split it around $\mu$

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} xe^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} xe^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} xe^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} xe^{-x/\lambda}\,dz\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\lambda(x-\lambda)e^{x/\lambda} - e^{\mu/\lambda}(\lambda(x+\lambda)e^{-x/\lambda})\right] \\
&= \mu
\end{aligned}$$

□

**Theorem 1.31.** *Expectation of $x^2$ in Laplace distribution is given be*

$$E[x^2] = bruh$$

*Proof.*

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-\left|\frac{x-\mu}{\lambda}\right|}$$

Split it around $\mu$

$$\begin{aligned}
&= \frac{1}{2\lambda}\left(\int_{-\infty}^{\mu} x^2 e^{\frac{x-\mu}{\lambda}}\,dx + \int_{\mu}^{\infty} x^2 e^{-\frac{x-\mu}{\lambda}}\,dx\right) \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} x^2 e^{x/\lambda}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} x^2 e^{-x/\lambda}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}(\lambda(x^2-2\lambda x+2\lambda^2)e^{x/\lambda}) - e^{\mu/\lambda}(\lambda(x^2+2\lambda x+2\lambda^2)e^{-x/\lambda})\right] \\
&= 2\lambda^2
\end{aligned}$$

□

**Theorem 1.32.** *Variance of Laplace distribution is given as*

$$V[x] =$$

### 1.4.5 MGF

**Theorem 1.33.** *MGF of the Laplace distribution is given by*

$$M_x(t) = bruh$$

*Proof.*

$$
\begin{aligned}
M_x(t) = E[e^{tx}] &= \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{tx - \left|\frac{x-\mu}{\lambda}\right|} \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\int_{-\infty}^{\mu} e^{x\left(t+\frac{1}{\lambda}\right)}\,dx + e^{\mu/\lambda}\int_{\mu}^{\infty} e^{-x\left(\frac{1}{\lambda}-t\right)}\,dx\right] \\
&= \frac{1}{2\lambda}\left[e^{-\mu/\lambda}\left(\frac{e^{\mu\left(\frac{1}{\lambda}+t\right)}}{\frac{1}{\lambda}+t}\right) + e^{\mu/\lambda}\left(\frac{-e^{\mu\left(\frac{1}{\lambda}-t\right)}}{-\frac{1}{\lambda}+t}\right)\right] \\
&= \frac{1}{2\lambda}\left[\frac{e^{\mu t}}{t+\frac{1}{\lambda}} - \frac{e^{\mu t}}{t-\frac{1}{\lambda}}\right]
\end{aligned}
$$

□

Plot a graph for the beta-1 dsitribution when alpha=5, beta=2

### 1.4.6 CGF

## 1.5 Beta distribution of Type-I

### 1.5.1 PDF

**Definition 1.34** (PDF of Beta I)**.**

$$
f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 < x < 1; m, n > 0 \\ 0 & otherwise \end{cases}
$$

*Where* $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Note the following,

1. We can say, $X \sim \beta_1(m,n)$ where $m, n$ are the parameters of the distribution.

2. Since $f(x)$ is a pdf we have the following,

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \int_0^1 \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx \\
&= \int_0^1
\end{aligned}
$$

### 1.5.2 Raw moments

**Theorem 1.35.** *The $r^{th}$ raw moment of the Beta I distribution is given by*

$$\mu_r' = \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}$$

*Proof.*

$$\begin{aligned}\mu_r' = E[x^r] &= \int_0^1 \frac{1}{\beta(m,n)} x^{r+m-1}(1-x)^{n-1}\,dx \\ &= \frac{\Gamma(m+n)\Gamma(r+m)}{\Gamma(m)\Gamma(m+n+r)}\end{aligned}$$

□

### 1.5.3 Mean and Variance

**Theorem 1.36.** *Mean of Beta I distribution is given by*

$$E[x] = \frac{m}{m+n}$$

*Proof.*

$$E[x] = \mu_1' = \frac{\Gamma(m+n)\Gamma(m+1)}{\Gamma(m)+\Gamma(m+n+1)} = \frac{m}{m+n}$$

□

**Theorem 1.37.** *Variance of Beta I distribution is given by*

$$V[x] = \frac{mn}{(m+n)^2(m+n+1)}$$

*Proof.*

$$\mu_2' = \frac{(m+1)(m)}{(m+n)(m+n+1)}$$

So now we have the variance given as,

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= \frac{mn}{(m+n)^2(m+n+1)}\end{aligned}$$

□

## 1.6 Beta distribution of Type-II

### 1.6.1 PDF

**Definition 1.38** (PDF of Beta-II distribution)**.**

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Note the following,

1. $X$ is said to follow $\beta_2(m,n)$ as $X \sim \beta_2(m,n)$

2. $$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^{m+1}}{(1+x)^{m+n}} = \beta(m,n)$$

### 1.6.2 Raw moments

**Theorem 1.39** (Raw moments of Beta-2 distribution)**.** *The raw moments of the Beta-2 distribution is given by*

$$\mu_r' = \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)}$$

*Proof.*

$$\begin{aligned} \mu_r' = E[x^r] &= \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx \\ &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma(m)\Gamma(n)} \end{aligned}$$

□

### 1.6.3 Mean and variance

**Theorem 1.40** (Mean of Beta-2 distribution)**.** *The mean of Beta-2 distribution is given by*

$$E[x] = \frac{m}{n-1}$$

*Proof.*

$$\begin{aligned} E[x] = \mu_1' &= \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma(m)\Gamma(n)} \\ &= \frac{m}{n-1} \end{aligned}$$

□

**Theorem 1.41** (Variance of Beta-2 distribution)**.** *The variance of Beta-2 distribution is given by*

$$V[x] = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

*Proof.* First consider the 2nd raw moment,

$$\mu_2' = \frac{m(m+1)}{(n-2)(n-2)}$$

Now we can compute the variance as follows

$$V[x] = \mu_2 = \mu_2' - \mu_1'^2 = \frac{m(m+n-1)}{(n-2)(n-1)^2}$$

□

## 1.7 Transformation of variables

### 1.7.1 One dimensional random variable

Let $X$ be a continuous random variable with pdf $f(x)$ and let $Y = g(x)$ be a strictly monotonic function of $X$ with unique inverse.
Assume that $g(x)$ is differentiable and is continuous for all $x$, then the pdf of r.v. $Y$ is given by

$$h(y) = f(x) \cdot \det\left|\frac{dx}{dy}\right| = \left|\frac{dx}{dy}\right|$$

where r.v. $x$ is expressed in terms of $y$. Steps to solve,

1. Write pdf of r.v. $X$.
2. Express old variable $X$ in terms of new variable $Y$.

3. Write the range of the new variable.
4. Obtain $J$ where $J = \left|\frac{dx}{dy}\right|$ and $|J|$.
5. Obtain $h(y) = f(x) \cdot |J|$, where $X$ is expressed in terms of $Y$.

**Remark 1.42.** *For $2-1$ correspondence, i.e. for ever 2 values of $X$ is there is only one value of $Y$, then multiple $|J|$ with 2.*

**Remark 1.43.**

*For $1-2$ correspondence i.e., for every 1 value of $x$ if there are 2 values of $Y$ then multiply $|J|$ with $\frac{1}{2}$.*

**Example 1.44.** *If a r.v. $X \sim B_1(m, n)$ obtain the distribution of $Y = 1-X$.*

*Proof.* First begin by stating the pdf of $X$.

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1} & 0 < x < 1; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $X = 1 - Y$ this ranges from $1 - Y = 0$ to $1 - Y = 1$. So $0 < Y < 1$ again.
Now compute $J$

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{dy}(1-y) \\ J &= -1 \\ |J| &= 1 \end{aligned}$$

We multiply this with $f(x)$ to get $h(y)$.

$$\begin{aligned} h(y) &= f(x) \cdot |J| \\ h(y) &= f(x) \end{aligned}$$

So $h(y) \sim B(n, m)$. The order changes. □

**Example 1.45.** *A r.v. $X \sim B_2(m, n)$. Obtain the distribution of $Y$ where $Y = \frac{1}{1+X}$.*

*Proof.* First state the pdf,

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} & 0 < x < \infty; m, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now state $X$ in terms of $Y$, we have $X = \frac{1}{Y} - 1$.
Compute the new ranges now we have $\frac{1}{Y} - 1 = 0$ so $Y = 1$ as one side then $\frac{1}{Y} - 1 = \lim_{m\to\infty} m$ so to $Y = 0$.
The new ranges are $0 < Y < 1$. Now compute $|J|$,

$$\begin{aligned} J &= \frac{dx}{dy} = \frac{1}{dy}\left(\frac{1}{y} - 1\right) \\ &= -\frac{1}{y^2} \\ |J| &= \frac{1}{y^2} \end{aligned}$$

So now we can compute $h(y)$ as follows,

$$\begin{aligned} h(y) &= f(x)|J| \\ &= \frac{1}{\beta(m,n)} \frac{\left(\frac{1}{y} - 1\right)^{m-1}}{(1/y)^{m+n}} \frac{1}{y^2} \\ &= \frac{1}{\beta(m,n)} y^{n-1}(1-y)^{m-1} \end{aligned}$$

This is for the range we have and 0 otherwise. But I'm too lazy to typeset that out as a cases.
So we now have $Y \sim B_1(n, m)$. $\square$

## 1.8 Two dimensional r.v.

Let $X$ and $Y$ be two continuous independent r.v. with joint pdf $f(x, y)$. Say $U = g(x, y)$ and $V = h(x, y)$ are two other r.v. then the joint pdf of $U$ and $V$ is given by,

$$k(u, v) = f(x, y) \cdot |J|$$

where $X, Y$ are expressed in terms of $U, V$. Here we have the Jacobian as follows,

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} \end{bmatrix}$$

### 1.8.1 Steps to solve

1. Write the pdf of $X$ and $Y$, i.e. $f(x, y)$.

2. Express old variable in terms of new variable.
3. Obtain range of the new variable.
4. Obtain $J$ and $|J|$.
5. Obtain $k(u,v) = f(x,y)|J|$.

**Example 1.46.** *$X$ and $Y$ are two independent gamma variates with parameters $a$ and $b$ respectively.*

1. *Obtain the joint distribution of $u$ and $v$ where $u = x + y, v = \frac{x}{x+y}$.*
2. *Show that $u, v$ are independent and identify their distributions.*

*Proof.* Consider the pdf of gamma function as follows,

$$\begin{aligned} X \sim G(\lambda) &= \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)} && x > 0, \lambda > 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

Where $\Gamma(\lambda) = (\lambda - 1)! = (\lambda - 1)\Gamma(\lambda - 1)$.

$$\begin{aligned} f_1(x) &= \frac{1}{\Gamma(a)} e^{-x} x^{a-1} \\ f_2(x) &= \frac{1}{\Gamma(b)} e^{-x} x^{b-1} \end{aligned}$$

Find $f(x,y) = f_1(x) f_2(y)$

$$\begin{aligned} f(x,y) &= \frac{1}{\Gamma(a)\Gamma(b)} e^{-x-y} x^{a-1} x^{b-1} && x, y, a, b, > 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

We now have the new variables $U, V$ $U = X + Y, V = \frac{X}{X+Y}$. This implies that $X = UV, Y = U(1 - V)$.
We need to find the new ranges now. Since $X, Y > 0$ we have $U > 0$ and $X < X + Y \implies \frac{x}{x+y} < 1 \implies v < 1$. And $0 < V < 1$.

Find the Jacobian,

$$\begin{aligned} J &= \begin{bmatrix} v & u \\ 1 - v & -u \end{bmatrix} = -u \\ |J| &= u \end{aligned}$$

The joint distribution is then given as,

$$\begin{aligned}
k(u,v) &= \frac{1}{\Gamma(a)\Gamma(b)} e^{-(uv+u-uv)}(uv)^{a-1}[u(1-v)]^{b-1}\cdot u \\
&= \frac{1}{\Gamma(a)\Gamma(b)} e^{-u}u^{a-1+b-1+1}v^{a-1}(1-v)^{b-1}\times\frac{\Gamma(a+b)}{\Gamma(a+b)} \\
&= \frac{1}{\Gamma(a+b)} e^{-u}u^{a+b-1}\times\frac{1}{\beta(a,b)}v^{a-1}(1-v)^{b-1} \\
&= k_1(u)k_2(v)
\end{aligned}$$

So $u$ and $v$ are independent r.v. and $U \sim G(a+b), V \sim \beta_1(a,b)$ □

**Example 1.47.** *$X$ and $Y$ are two independent r.v. $X \sim G(a)$ and $Y \sim G(b)$. We have $U = X+Y$ and $W = \frac{X}{Y}$. Show that $U, W$ are independent and identify the distribution.*

*Proof.* We know the following,

$$\begin{aligned}
f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} && x>0, a>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

and,

$$\begin{aligned}
f_2(y) &= \frac{e^{-y}y^{b-1}}{\Gamma(b)} && x>0, b>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

Now the joint distribution $f(x,y)$ is given by its product since they are independent,

$$\begin{aligned}
f(x,y) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)}\times\frac{e^{-y}y^{b-1}}{\Gamma(b)} && x>0, y>0; a,b>0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

Now we compute the new ranges $X = \frac{UW}{W+1}$ and $Y = \frac{U}{W+1}$. Now when $X = 0$ we have $U = 0, W = 0$ when $X \to \infty$ $U \to \infty, V \to \infty$. So we have $U > 0$ and $W > 0$.
Now compute the Jacobian as follows,

$$\begin{aligned}
J &= \begin{bmatrix} \frac{w}{1+w} & \frac{-uw}{(1+w)^2}+\frac{u}{1+w} \\ \frac{1}{1+w} & \frac{-u}{(1+w)^2} \end{bmatrix} \\
|J| &= \frac{u}{(1+w)^2}
\end{aligned}$$

Since for 2 values of $Y$ we get one value of $X$ we will multiply the jacobian by 2. Now we compute $k(u,w)$ as follows,

$$\begin{aligned}
k(u,w) &= f(x,y)|J| \\
&= \frac{e^{-\frac{uw}{w+1}} \frac{uw}{w+1}^{a-1}}{\Gamma(a)} \times \frac{e^{-\frac{u}{w+1}} \frac{u}{w+1}^{b-1}}{\Gamma(b)} \times \frac{u}{(1+w)^2} \\
&= \frac{1}{\Gamma(a+b)} e^{-u} u^{a+b-1} \times \frac{1}{\beta(a,b)}
\end{aligned}$$

Complete this □

**Example 1.48.** $X \sim N(\mu, \sigma^2)$. *Obtain the distribution of* $Y = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

*Proof.* Begin by stating the pdf of r.v. $X$,

$$\begin{aligned}
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} && -\infty < x < \infty, \sigma > 0 \\
&= 0 && \text{otherwise}
\end{aligned}$$

We now state $X$ in terms of $Y$ as follows, $X = \mu \pm \sqrt{2}\sigma\sqrt{y}$. Range of $y$ is $0 < y < \infty$. And since it is 2-1 correspondence we will multiply the Jacobian by 2.
Compute the value of Jacobian first,

$$|J| = \frac{\sigma}{\sqrt{2}\sqrt{y}}$$

Now compute the new function,

$$\begin{aligned}
h(y) &= f(x)|J|2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \times \frac{\sigma}{\sqrt{2}\sqrt{y}} \times 2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-y} \frac{2\sigma}{\sqrt{2}\sqrt{y}} \\
&= \frac{2}{\sqrt{2}\sqrt{y}\sqrt{2\pi}} e^{-y} \\
&= \frac{e^{-y}}{\sqrt{\pi}\sqrt{y}} \\
&= \frac{1}{\Gamma(\frac{1}{2})} e^{-y} y^{1-\frac{1}{2}}
\end{aligned}$$

So we have $Y \sim G\left(\frac{1}{2}\right)$. □

**Example 1.49.**

$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & x,y>0 \\ 0 & otherwise \end{cases}$$

*Prove that* $h(u) = 2u^3e^{-u^2}, u>0$ *where* $u = \sqrt{x^2+y^2}$ *and* $v = x$.

*Proof.* The variables we are dealing with are,

$$x = v, y = \sqrt{u^2-v^2}$$

The range for $v, u$ is $(0,\infty)$ but $0 < v < u < \infty$.
Begin by computing the Jacobian,

$$|J| = \frac{u}{\sqrt{u^2-v^2}}$$

Consider now the joint distribution with the change of variables,

$$\begin{aligned} g(u,v) &= f(x,y)|J| \\ &= 4xye^{-(x^2+y^2)}|J| \\ &= 4(v)(\sqrt{u^2-v^2})e^{-(v^2+u^2-v^2)}\frac{u}{\sqrt{u^2-v^2}} \\ &= 4v\sqrt{u^2-v^2}e^{-u^2}\frac{u}{\sqrt{u^2-v^2}} \\ &= 4vue^{-u^2} \end{aligned}$$

Integrate out v

$$\begin{aligned} h(u) &= 4ue^{-u^2}\int_0^u v\,dv \\ &= 4ue^{-u^2}\frac{u^2}{2} \\ &= 2u^3e^{-u^2} \end{aligned}$$

□

**Example 1.50.**

$$f(x,y) = \begin{cases} \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)} & x,y>0 \\ 0 & otherwise \end{cases}$$

*Obtain pdf of* $u$ *where* $u = \frac{x}{x+y}$ *take* $v = x+y$ *also obtain* $E[u], V[u]$.

*Proof.* Consider the new variables, $x = uv, y = v - uv$. The range for $v$ is $(0, \infty)$ and for $u$ is $(0, 1)$
Compute the Jacobian,

$$|J| = v$$

Compute the joint pdf,

$$\begin{aligned}
g(u,v) &= f(x,y)|J| \\
&= \frac{e^{-(x+y)}x^3y^4}{\Gamma(4)\Gamma(5)}|J| \\
&= \frac{e^{-(uv+v-uv)}(uv)^3(v-uv)^4}{\Gamma(4)\Gamma(5)}v \\
&= \frac{e^{-v}u^3(1-u)^4v^8}{\Gamma(4)\Gamma(5)}
\end{aligned}$$

Integrate out v

$$\begin{aligned}
h(u) &= \frac{u^3(1-u)^4}{\Gamma(4)\Gamma(5)}\int_0^\infty e^{-v}v^8\,dv \\
&= \frac{\Gamma(9)}{\Gamma(4)\Gamma(5)}u^3(1-u)^4
\end{aligned}$$

So $U \sim \beta_1(m = 4, n = 5)$. Compute the mean and variance as follows,

$$\begin{aligned}
E[U] &= \frac{m}{m+n} = \frac{4}{9} \\
V[U] &= \frac{mn}{(m+n)^2(m+n+1)} = \frac{20}{810} = \frac{2}{81}
\end{aligned}$$

□

**Example 1.51.** *$X, Y$ are two independent gamma variates with parameters $a, b$ respectively. Show that $U + X + Y, V = \frac{X-Y}{X+Y}$ are independent.*

*Proof.* Consider the original pdfs,

$$\begin{aligned}
f_1(x) &= \frac{e^{-x}x^{a-1}}{\Gamma(a)} \\
f_2(y) &= \frac{e^{-y}x^{b-1}}{\Gamma(b)}
\end{aligned}$$

Since they are independent

$$f(x,y) = \frac{e^{-(x+y)}x^{a-1}y^{b-1}}{\Gamma(a)\Gamma(b)}$$

Consider the new variables, $x = \frac{1}{2}(uv+u), y = \frac{1}{2}(u-uv)$, the ranges for $u$ is $(0,\infty)$ but for $v$ is $(-1,1)$
Compute the Jacobian,

$$|J| = \frac{u}{2}$$

Compute the joint pdf,

$$\begin{aligned} g(u,v) &= f(x,y)|J| \\ &= \frac{e^{-(x+y)}x^{a-1}y^{b-1}}{\Gamma(a)\Gamma(b)}|J| \\ &= -\frac{e^{-u}(v+1)2^{-a-b+2}(u(v+1))^{a-2}(u-uv)^{b}}{(v-1)\Gamma(a)\Gamma(b)} \end{aligned}$$

Split this up I'm too lazy to type it. $\square$

# Chapter 2

# Chi-square distribution

## 2.1 PDF

## 2.2 MGF

## 2.3 CGF

**Theorem 2.1.** *The CGF of Chi squared distributed is given by*

$$K_x(t) = -\frac{n}{2}\log(1-2t)$$

*Proof.* MGF is given by $(1-2t)^{-n/2}$ and since CGF is just log MGF. We have the result as required. $\square$

The values of the cumulants are given as follows,

$$\begin{aligned}
K_1 &= n \\
K_2 &= 2n \\
K_3 &= 8n \\
K_4 &= 48n \text{ so } \mu_4 = 48n + 12n^2
\end{aligned}$$

## 2.4 Skewness and kurtosis

**Theorem 2.2.** *Skewness is*

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{8}{n}$$

*so it is positively skewed.*

**Theorem 2.3.** *Kurtosis is*

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{12}{n}$$

*so it is leptokurtic and approaches normal as $n \to \infty$.*

Plot a graph of chi square with 6 degrees of freedom.

## 2.5 Additive property of chi squared distribution

**Theorem 2.4.** *If $X_1, X_2, \ldots, X_k$ are $k$ independent $\chi^2$ variates with degrees of freedom $n_i$ for $i = 1, 2, \ldots, k$ then*

$$\sum_{i}^{n} X_i \sim \chi^2_{\sum n_i}$$

*Proof.* Take the MGF multiply it that's it. □

## 2.6 Mode

We will obtain the mode as follows,

1. $f'(x) = 0$
2. Check if $f''(x) < 0$

Begin with the pdf,

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{n/2-1} & x > 0, n > 2 \\ 0 & o.w. \end{cases}$$

Now take its logarithm,

$$\log f(x) = \frac{f'(x)}{f(x)}$$

$$\frac{d \log f(x)}{dx} = 0 \implies x = n - 2$$
$$\frac{d^2 \log f(x)}{dx^2} = 0 - \frac{n-2}{2x^2} < 0$$

You can also do it by just taking the derivative and doing it the long way.

You can do it in a third method by getting a recurrence relation after derivative it once.

**Theorem 2.5** (Mode of Chi-square distribution). *Mode of Chi-square distribution with $n$ degrees of freedom is $n-2$*

## 2.7 Applications of Chi-square distribution

### 2.7.1 Goodness of fit

This is a very powerful test for testing significance of difference between theoretical and experimental values. It was given by Prof. Karl Pearson in 1900.

If $O_i, i = 1, 2, \dots, k$ is a set of observed or experimental frequencies and $E_I, i = 1, 2, \dots, k$ is a set of corresponding expected (hypothetical) frequencies, then Karl Pearson's $\chi^2$ statistic is given by,
$H_0$ : There is no significant difference between the observed and expected frequencies (the dist. is a good fit).
$H_1$ : There is significant difference.

The test statistic is given by,

$$\chi^2 = \frac{\sum_{i=1}^{k}(O_i - E_i)^2}{E_i} \sim \chi^2_{k-1,\alpha}$$

The decision criteria is,
Reject $H_0$ if $\chi^2_{cal} > \chi^2_{tab} = \chi^2_{\alpha,k-1}$

For computation

$$\chi^2 = \frac{\sum O_i^2}{E_i} - N$$

The test is one sided (right sided).

Conditions for validity of the $\chi^2$ test,

1. Sample observations should be independent.
2. Constraints on the cell frequencies should be similar, i.e. $\sum O_i = \sum E_i$.
3. The total frequencies $(N)$ should be reasonably large. Say $N > 50$.
4. No theoretical frequencies should be less than 5. If a frequency is less than 5 then it is pooled with the preceding or succeeding frequency such that the pooled frequency is greater than 5. The degrees of freedom will decrease by the number of observations that are pooled.

Note: Sometimes while fitting a distribution the given data, some parameters have to be estimated. If $p$ parameters are estimated, the degrees of freedom will be $k-p-1$ where $k$ is the number of classes. and $p$ is the number of parameters estimated. Therefore, $d.f. = (k-1) -$ (no of pooled values) $-$ (no. of parameters estimated)

**Example 2.6.** *Four identical coins are tossed* 160 *times, and the number of heads is recorded as follows,* $X$ : *No. of heads*

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $O_i$ | 14 | 30 | 70 | 35 | 11 |

*Test the hypothesis that the coins are perfect.*

*Proof.* $H_0$ : Coins are perfect or Binomial is a good fit.
$H_1$ : Not $H_0$

We say $X$ : No. of heads in 4 coin tosses and $X \sim B(n = 4, p = 0.5)$ and $N = 160$.

We get the following,

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $O_i$ | 14 | 30 | 70 | 35 | 11 |
| $E_i$ | 10 | 40 | 60 | 40 | 10 |

So $\chi^2_{cal} = \frac{\sum O_i^2}{E_i} - N = 6.4917$ and $\chi^2_{tab} = 9.488$. So $\chi^2_{cal} < \chi^2_{tab}$ so it doesn't fall in critical reason and we fail to reject $H_0$. □

**Example 2.7.** *A study of printing mistakes in a book of* 550 *pages gives the following data.*
$X$ : *No of printing mistakes per page*

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $O_i$ | 485 | 52 | 8 | 4 | 1 |

*Proof.* $H_0$ : Poisson distribution is a good fit
$H_1$ : Not $H_0$
$X \sim P(0.15\overline{27} = \frac{42}{275})$

We get the following,

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $O_i$ | 485 | 52 | 8 | 4 | 1 |
| $E_i$ | 472 | 72 | 6 | 0 | 0 |

Pool the last 3 classes.

So $\chi^2_{cal} = \frac{\sum O_i^2}{E_i} - N = 14.08027$ and $\chi^2_{tab} = \chi^2_{0.05,1} = 3.841$. So $\chi^2_{cal} > \chi^2_{tab}$ so it falls in critical reason and we accept $H_1$. □

### 2.7.2 Test for variance

To test, $H_0 : \sigma^2 = \sigma_0^2$
$H_1 : \sigma^2 \neq \sigma_0^2$

Let $\alpha$ be l.o.s.

The test statistic is

$$T = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma_0^2} \sim \chi_n^2$$

Note: If $\mu$ is unknown then $T = \frac{\sum(x_i - \overline{x})^2}{\sigma_0^2} \sim \chi_{n-1}^2$

Decision criteria:

Reject $H_0$ if $\chi_{cal}^2 > \chi_2^2$ or $< \chi_1^2$ Where $\chi_1^2 = \chi_{1-\alpha/2,n}^2$ and $\chi_2^2 = \chi_{\alpha/2,n}^2$

If the alternate hypothesis is one-sided, then it becomes a one-sided test.

If $H_1 : \sigma^2 > \sigma_0^2$, then D.C. is reject $H_0$ is $\chi_{cal}^2 > \chi_{\alpha,n}^2$

Similarly, if $H_1 : \sigma^2 < \sigma_0^2$ then, reject $H_0$ if $\chi_{cal}^2 < \chi_{1-\alpha,n}^2$

Confidence interval for $\sigma^2$ [$(100(1 - \sigma))\%$ Ci.i for population variance ].

Let $\phi = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^2}$ be the pivotal quantity.

Then $100(1 - \alpha)\%$ C.I. for $\sigma^2$ is given as follows,

$$P[\chi_1^2 < \phi < \chi_2^2] = (1 - \alpha)$$

$$\implies P\left[\chi_1^2 < \frac{\sum(x_i - \mu)^2}{\sigma^2} < \chi_2^2\right] = 1 - \alpha$$

$$\implies P\left[\frac{\chi_1^2}{\sum(x_i - \mu)^2} < \frac{1}{\sigma^2} < \frac{\chi_2^2}{\sum(x_i - \mu)^2}\right] = 1 - \alpha$$

$$P\left[\frac{\sum(x_i - \mu)^2}{\chi_2^2} < \sigma^2 < \frac{\sum(x_i - \mu)^2}{\chi_1^2}\right] = 1 - \alpha$$

So the $100(1 - \alpha)\%$ C.I. for $\sigma^2$ is

$$\left[\frac{\sum(x_i - \mu)^2}{\chi_{n,\alpha/2}^2}, \frac{\sum(x_i - \mu)^2}{\chi_{n,1-\alpha/2}^2}\right] = \left[\frac{(n-1)s_{n-1}^2}{\chi_{n,\alpha/2}^2}, \frac{(n-1)s_{n-1}^2}{\chi_{n,1-\alpha/2}^2}\right]$$

Where $\chi_1^2 = ...$

### 2.7.3 Limiting form of $\chi^2$ as $n \to \infty$

Let $X \sim \chi_n^2$ if $n \to \infty$ then $X \sim N(n, 2n)$

*Proof.* $M_x(t) = (1-2t)^{-n/2}$ let $z = \frac{x-\mu}{\sigma}$ then we have

$$
\begin{aligned}
M_{\frac{x-\mu}{\sigma}}(t) &= E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)}\right] \\
&= E\left[e^{t\frac{x-n}{\sqrt{2n}}}\right] \\
&= e^{-nt/\sqrt{2n}} E\left[e^{\frac{tx}{\sqrt{2n}}}\right] \\
M_z(t) &= e^{-t\sqrt{\frac{n}{2}}}\left(1 - \frac{2t}{\sqrt{2n}}\right)^{\frac{-n}{2}}
\end{aligned}
$$

Take log

$$
\log M_z(t) = K_z(t) = -t\sqrt{n/2} - n/2 \log\left(1 - t\sqrt{2/n}\right)
$$

Take the series expansion of log

$$
\begin{aligned}
&= -t\sqrt{n/2} + n/2\left[t\sqrt{2/n} + \frac{t^2}{2}\frac{2}{n} + \frac{t^3}{3}\left(\frac{2}{n}\right)^{3/2} + \dots\right] \\
&= \frac{t^2}{2} + \mathcal{O}(n^{-1/2})
\end{aligned}
$$

Ignore $\mathcal{O}(n^{-1/2})$.

Therefore, $\lim_{n\to\infty} K_z(t) = \frac{t^2}{2} \implies M_z(t) = e^{t^2/2}$ which is m.g.f. of a standard normal variance.

Hence by uniqueness theorem of mgf of Z is asymptomatically standard normal. $\square$

## 2.8 Independence of attributes

Let us consider two attributes $A$ and $B$. $A$ divided into $r$ classes and $B$ divided into $s$ classes.

Such a classification in which attributes are divided into more than two classes is known as manifold classification. The various cell frequencies can be expressed in the following table known as a $r \times s$ manifold contingency table.

Where $(A_i)$ is the no. of persons possessing the attribute $A_i$ and similarly for $(B_j)$. So $(A_iB_j)$ denotes the no. of persons possessing both $A_i$ and $B_j$

$$\begin{bmatrix} B\setminus A & A_1 & A_2 & \dots & A_i & \dots & A_r & \text{Total} \\ B_1 & (A_1B_1) & (A_2B_1) & \dots & (A_iB1) & \dots & (A_rB_1) & (B_1) \\ B_2 & (A_1B_2) & (A_2B_2) & \dots & (A_iB2) & \dots & (A_rB_2) & (B_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ B_j & (A_1B_j) & (A_2B_j) & \dots & (A_iB_j) & \dots & (A_rB_j) & (B_j) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ B_s & (A_1B_s) & (A_2B_s) & \dots & (A_iB_s) & \dots & (A_rB_s) & (B_s) \\ \text{Total} & (A_1) & (A_2) & \dots & (A_i) & \dots & (A_r) & N \end{bmatrix}$$

The problem is to test if the two attributes $A$ and $B$ under consideration are independent or not.
$H_0$ : The two attributes are independent
$H_1$ : The two attributes are dependent.

Under the null hypothesis that the attributes are independent, the theoretical cell frequencies are calculated as follows:

$P(A_i)$ =Probability that a person posses the attribute $A_i = \frac{(A_i)}{N}$

$P(B_j)$ =Probability that a person posses the attribute $B_j = \frac{(B_j)}{N}$

$P(A_iB_j)$ =Probability that a person posses the attributes $A_i, B_j = \frac{(A_i)}{N}\frac{(B_j)}{N} = P(A_i)P(B_j)$[1]

Expected number of persons possessing both the attributes $A_i, B_j = (A_iB_j)_0 = NP[A_iB_j] = \frac{(A_i)(B_j)}{N}$

Under the null hypothesis of independence of attributes the test statistic is,

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\left[\frac{((A_iB_j) - (A_iB_j)_0)^2}{(A_iB_j)_0}\right]$$

which is distributed as $\chi^2$ variate with $(r-1)(s-1)$ d.f. for large $N$.

**Note on degrees of freedom:** The number of independent variates which make up the statistic is known as the degrees of freedom and is usually denoted by $\nu$.

The number of degrees of freedom, in general, is the total number of observations less than the number of independent constraints imposed on the observations. For example, if $k$ is the number of independent constraints in a set of data of $n$ observations then $\nu = n - k$

[1]Since we are assuming they are independent under the null hypothesis.

Thus in a set of $n$ observations usually, the degrees of freedom for $\chi^2$ are $n-1$, one d.f. being lost because of the linear constraint $\sum O_i = \sum E_i = N$ on the frequencies.

If $r$ independent linear constraints are imposed on the cell frequencies, then the d.f. are reduced by $r$.

In addition, if any of the population parameters are calculated from the given data and used for computing the expected frequencies then in applying $\chi^2$- test of goodness of fit, we have to subtract one d.f. from each parameter calculated. Thus if $s$ is the number of population parameters estimated from the sample observations ($n$ in number) then the required number of degrees of freedom for $\chi^2$ test is $(n-s-1)$.

If any one or more of the theoretical frequencies is less than 5 then in applying $\chi^2$ test we have also to subtract the degrees of freedom lost in pooling these frequencies with the proceeding or succeeding frequency(ies).

In a $r \times s$ contingency table, in calculating the expected frequencies, the row totals, the column totals and the grand totals remain fixed. The fixation of $r$ column totals and $s$ row totals imposes $r+s$ constraints on the cell frequencies. But since

$$\sum_{i=1}^{r}(A_i) = \sum_{j=1}^{s}(B_j) = N$$

the total number of independent constraints is only $r+s-1$. Further, since the total number of cell frequencies is $r \times s$ the required number of degrees of freedom is

$$\nu = rs - (r+s-1) = (r-1)(s-1)$$

**Example 2.8.** *Two sample polls of votes for two candidates A and B for a public office are taken, one from among the residents of rural areas. The results are given the table. Examine whether the nature of the area is related to voting preferncei n this election.*

$$\begin{bmatrix} Area \setminus Votes\ for & A & B & Total \\ Rural & 620 & 380 & 1000 \\ Urban & 550 & 450 & 1000 \\ Total & 1170 & 830 & 2000 \end{bmatrix}$$

*Proof.* Note that $(r-1)(s-1) = 1 \cdot 1 = 1$.

$$\begin{aligned}(A_1B_1) &= 620\\ (A_2B_1) &= 380\\ (A_1B_2) &= 550\\ (A_2B_2) &= 450\end{aligned}$$

Now check the expected values

$$\begin{aligned}(A_1B_1)_0 &= \frac{(A_1)(B_1)}{N} = \frac{1170 \cdot 1000}{2000} = 585\\ (A_2B_1)_0 &= \frac{(A_2)(B_1)}{N} = \frac{830 \cdot 1000}{2000} = 415\\ (A_1B_2)_0 &= \frac{(A_1)(B_2)}{N} = \frac{1170 \cdot 1000}{2000} = 585\\ (A_2B_2)_0 &= \frac{(A_2)(B_2)}{N} = \frac{830 \cdot 1000}{2000} = 415\end{aligned}$$

So the test statistic is as follows,

$$\chi^2_{cal} = 10.0916$$

Now the tabulated value is as follows,

$$\chi^2_{tab} = \chi^2_{1,0.05} = 3.841$$

Since $\chi^2_{cal} > \chi^2_{tab}$ we reject $H_0$. That is to say, that the nature of area is related to voting preference in the election. □

**Example 2.9.**

$$\begin{bmatrix} \textit{Class} \setminus \textit{Autonomy preference} & \textit{In favour} & \textit{In opposition} & \textit{Total}\\ FY & 120 & 80 & 200\\ SY & 130 & 70 & 200\\ TY & 70 & 30 & 100\\ MA & 80 & 20 & 100\\ \textit{Total} & 400 & 200 & 600 \end{bmatrix}$$

*Proof.* Write the hypothesis here I'm lazy.

Note that there are $(r-1)(s-1) = 3 \cdot 1 = 3$ degrees of freedom.

$$\begin{aligned}
(A_1B_1) &= 120\\
(A_2B_1) &= 80\\
(A_1B_2) &= 130\\
(A_2B_2) &= 70\\
(A_3B_1) &= 70\\
(A_3B_2) &= 30\\
(A_4B_1) &= 80\\
(A_4B_2) &= 20
\end{aligned}$$

Now compute the expectations,

$$\begin{aligned}
(A_1B_1)_0 &= 133.333\\
(A_2B_1)_0 &= 66.666\\
(A_1B_2)_0 &= 133.333\\
(A_2B_2)_0 &= 66.666\\
(A_3B_1)_0 &= 66.666\\
(A_3B_2)_0 &= 33.333\\
(A_4B_1)_0 &= 66.666\\
(A_4B_2)_0 &= 33.333
\end{aligned}$$

The test statistic is as follows,

$$\chi^2_{cal} = 12.750$$

Check the $\chi^2_{tab}$ and conclude. □

For a $r \times s$ contingency table in case of any expected value less than 5 it will be pooled with the nearby class and the degrees of freedom will be $(r-1)(s-1)-$no. of pooling.

**Example 2.10.** *Observed matrix*

$$\begin{bmatrix} 50 & 40 & 10 & T = 100\\ 30 & 20 & 1 & = 51\\ T = 80 & = 60 & = 11 & = 151 \end{bmatrix}$$

*Expected values,*

$$\begin{bmatrix} 53 & 40 & 7 & T = 100\\ 27 & 20 & 3 & = 51\\ T = 80 & = 60 & = 11 & = 151 \end{bmatrix}$$

*Proof.* We pool it as follows,

| $O_i$ | $E$ |
|---|---|
| 50 | 53 |
| 40 | 40 |
| 10 | 7 |
| 30 | 27 |
| 20 | 20 |
| 1 | 3 |

Just pool the last two rows, so $\chi^2_{tab} = \chi^2_{(2-1)\times(3-1)-1} = \chi^2_1 = 3.841$.

And the test statistic is as,

$$\chi^2_{cal} = 1.9628$$

□

**Theorem 2.11.** *For a $2 \times 2$ contingency table as follows,*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*the chi square test statistic is given as follows,*

$$\chi^2 = \frac{N(ad - bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

*Proof.*

$$(A_1B_1) = a$$
$$(A_2B_1) = b$$
$$(A_1B_2) = c$$
$$(A_2B_2) = d$$

and the expected values are as follows,

$$(A_1B_1)_0 = \frac{(a+b)(a+c)}{N}$$
$$(A_2B_1)_0 = \frac{(a+b)(b+d)}{N}$$
$$(A_1B_2)_0 = \frac{(a+c)(c+d)}{N}$$
$$(A_2B_2)_0 = \frac{(b+d)(c+d)}{N}$$

The chi squared test statistic is as follows. This is why we have computers I don't wanna type this out bruh wtf I know how to solve it.

$$\chi^2_{cal} =$$

□

**Example 2.12.** *If $X$ and $Y$ are indepedent $\chi^2$ variates with $n_1, n_2$ d.f. respectively, obtain the dist of $U = \frac{X}{Y}$*

*Proof.* Let $U = \frac{X}{Y}$ and let $V = Y$ so we have $U > 0, V > 0$ and $Y = V, X = UV$.

$$\begin{aligned} f(x,y) &= f_1(x)f_2(y) \\ &= \frac{1}{2^{n_1/2}\Gamma(n_1/2)} e^{-x/2} x^{n_1/2-1} \frac{1}{2^{n_2/2}\Gamma(n_2/2)} e^{-y/2} y^{n_2/2-1} \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)} e^{-\frac{1}{2}(x+y)} x^{n_1/2-1} y^{n_2/2-1} \end{aligned}$$

Compute the Jacobian as follows,

$$|J| = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1 \end{bmatrix} = \frac{1}{y} = v$$

So now our required $h(u,v)$ is as follows,

$$\begin{aligned} h(u,v) &= f(x,y)|J| \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)} e^{-\frac{1}{2}(v(1+u))} u^{n_1/2-1} v^{\frac{n_1+n_2}{2}-1} \times v \end{aligned}$$

Integrate out v

$$\begin{aligned} &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)} u^{n_1/2-1} \int_0^\infty e^{-\frac{v}{2}(1+u)} v^{\frac{n_1+n_2}{2}-1}\, dv \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma(n_1/2)\Gamma(n_2/2)} u^{n_1/2-1} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}} \\ &= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}} \end{aligned}$$

So we get $U \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ □

### 2.8.1 Pre-requisite for Application 2 (Test for variance) using MGF

**Theorem 2.13.** *Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, then*

1. $|overline X \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2. $\overline{X}$ *and* $\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$ *are independently distributed*
3. $\sum_{i=1}^{n} \left(\frac{X_i - \overline{X}}{\sigma}\right)^2 \sim \chi^2_{n-1}$

$s^2 = s^2_{n-1} = \sum \frac{(X_i - \overline{X})^2}{n-1}$ =*sample variance.*

*Proof.* To prove $A), B)$ we shall first prove that $\overline{X} = \sum \frac{X_i}{n}$ and $(X_i - \overline{X})$ are independently distributed.

Joint mgf of $\overline{X}$ and $(X_i - \overline{X})$ is,

$$\begin{aligned} M(t_1, t_2) &= E[\exp(t_1 \overline{X} - t_2(X_i - \overline{X}))] = E[\exp((t_1 - t_2)\overline{X} + t_2 X_i)] \\ &= E\left[\exp\left(\frac{t_1 - t_2}{n} \sum_{i=1}^{n} X_i + t_2 X_i\right)\right] \\ &= E\left[\exp\left(\left(\frac{t_1 - t_2}{n} + t_2\right) X_i\right)\right] E\left[\exp\left(\left(\frac{t_1 - t_2}{n}\right) \sum_{j=1, j \neq i}^{n} X_j\right)\right] \end{aligned} \tag{2.1}$$

$U = \sum_{j=1, j\neq i}^{n} X_j$ being sum of $n-1$ iid $N(\mu, \sigma^2)$ is a $N((n-1)\mu, (n-1)\sigma^2)$.

$M_u(t) = \exp\left[t(n-1)\mu + \frac{t^2}{2}(n-1)\sigma^2\right]$

$$\therefore E\left[\exp\left(\frac{t_1-t_2}{n}\sum_{j=1,j\neq i}^{n}X_j\right)\right] = E\left[\exp\left(\frac{t_1-t_2}{n}U\right)\right] = M_u\left[\frac{t_1-t_2}{n}\right]$$
$$= \exp\left[\frac{t_1-t_2}{n}(n-1)\mu + \left(\frac{t_1-t_2}{n}\right)^2(n-1)\frac{\sigma^2}{2}\right] \tag{2.2}$$

$$E\left[\exp\left(\frac{t_1-t_2}{n}+t_2\right)X_i\right] = M_{X_i}\left[\frac{t_1-t_2}{n}+t_2\right]$$
$$= \exp\left[\left(\frac{t_1-t_2}{n}+t_2\right)\mu + \left(\frac{t_1-t_2}{n}+t_2\right)^2\frac{\sigma^2}{2}\right] \tag{2.3}$$

Substitute $ii, iii$ to $i$ we get,

$$M(t_1,t_2) = \exp\left[\left(\frac{t_1-t_2}{n}(n-1)+\frac{t_1-t_2}{n}+t_2\right)U\right]\cdot\exp\left[\left(\left(\frac{t_1-t_2}{n}\right)^2(n-1)+\left(\frac{t_1-t_2}{n}+t_2\right)^2\right)\frac{\sigma^2}{2}\right.$$
$$= \exp\left[t_1\mu+\frac{1}{2}t_1^2\frac{\sigma^2}{n}\right]\cdot\exp\left[\frac{1}{2}t_2^2\frac{n-1}{n}\sigma^2\right]$$
$$= M(t_1)\cdot M(t_2)$$

Now consider,

$A)\overline{X}$ and $X_i\overline{X}$ are independently distributed -iv

$B)\overline{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$ -v and $(X_i-\overline{X})\sim N\left(0,\frac{n-1}{n}\sigma^2\right)$ -vi

So $\overline{X}$ and $(X_i-\overline{X})$ are independently distributed, $\overline{X}$ and $s^2=\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$ are independently distributed -vii

To derive distribution of $s^2$ we note that,

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\overline{X}+\overline{X}-\mu)^2 = \sum_{i=1}^{n}(X_i-\overline{X})^2+n(\overline{X}-\mu)^2$$

Product term vanishes since $\sum(X_i-\overline{X})=0$

$$\therefore\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \sum_{i=1}^{n}\frac{(X_i-\overline{X})^2}{\sigma^2}+\left[\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right]^2 - viii$$
$$v = w + z$$

Here $v$ being sum of squares of $n$ independent SNV is $\chi^2_{(n)}$, therefore $M_v(t) = (1-2t)^{-n/2}$ -ix

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \implies \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \implies Z = \left[\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right]^2 \sim \chi^2_{(1)}$

So $M_z(t) = (1-2t)^{-1/2}$ -x

So $\overline{X}$ and $s^2$ are independent $\implies w, z$ are independent.

$$
\begin{aligned}
M_u(t) &= M_{w+z}(t) = M_w(t) + M_z(t) \\
(1-2t)^{-n/2} &= M_w(t)(1-2t)^{-1/2} \\
\therefore M_w(t) &= (1-2t)^{\frac{-(n-1)}{2}}
\end{aligned}
$$

Which is mgf of $\chi^2_{(n-1)}$

$$\therefore w = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)} - xi$$

c part proved

Now consider,

$$E\left[\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{\sigma^2}\right] = n-1 \implies E\left[\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}\right] = \sigma^2$$

So $s^2$ is an unbiased estimator of $\sigma^2$.

Note that,

1. If $\mu$ is known, $\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2} \sim \chi^2_{(n)}$
2. When $n$ is large, $\hat{\sigma}^2 = \sum_{i=1}^{n} \frac{(X_i-\overline{X})^2}{n}$

Conclusion is

1. $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2. $\sum_{i=1}^{n} \frac{(X_i-\overline{X})^2}{\sigma^2} \sim \chi^2_{n-1}$
3. $\overline{X}$ and $\sum_{i=1}^{n}$ ....

$\square$

## 2.9 Students t-distribution

**Definition 2.14** (Definition of $t-$distribution). *If $U, V$ are two independent random variables which $U \sim SN(0,1)$ and $V \sim \chi_n^2$ then,*

$$X = \frac{U}{\sqrt{V/n}}$$

*is said to follows $t-$ distribution where $n$ is small $n < 30$.*

**Definition 2.15** (PDF of $t-$distribution).

$$f(x) = \begin{cases} = \frac{1}{\beta(\frac{1}{2},\frac{n}{2})\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} & -\infty < x < \infty \\ = 0 & otherwise \end{cases}$$

### 2.9.1 Derivation of PDF of $t-$distribution

*Proof.* Let $U$ and $V$ be tow independent random variables where $U \sim SN(0,1)$ and $V \sim \chi_n^2$. We have $-\infty < U < \infty$ and $0 < v < \infty$.

Let $Y = V, X = \frac{U}{\sqrt{V/n}}$, range of $-\infty < X < \infty$ and $0 < Y < \infty$.

Consider the Jacobian, Compute the Jacobian as follows,

$$|J| = \sqrt{\frac{Y}{n}}$$

Now compute $f(x,y) = h(u,v)|J|$

$$\begin{aligned} f(x,y) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2 y}{2n}} \frac{1}{2^{n/2}\Gamma(n/2)} e^{-y/2} y^{n/2-1} \sqrt{\frac{y}{n}} \\ &= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}} n^{1/2}} e^{-y/2\left(\frac{x^2}{n}+1\right)} y^{\frac{n-1}{2}} \end{aligned}$$

$\therefore X, Y$ are independent,

$$\begin{aligned}
f_1(x) &= \int_0^\infty f(x,y)\,dy \\
&= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \int_0^\infty e^{\frac{-y}{2}\left(\frac{x^2}{n}+1\right)} y^{\frac{n-1}{2}}\,dy \\
&= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \int_0^\infty \underbrace{y^{\frac{n}{2}+\frac{1}{2}-1} e^{-\left(\frac{1}{2}\left(\frac{x^2}{n}+1\right)\right)y}\,dy}_{\text{Gamma function}} \\
f_1(x) &= \frac{1}{\Gamma(1/2)\Gamma(n/2)2^{\frac{n+1}{2}}n^{1/2}} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\left(\frac{x^2}{2n}+\frac{1}{2}\right)^{\frac{n+1}{2}}} \\
&= \frac{1}{\beta\left(\frac{1}{2},\frac{n}{2}\right)n^{1/2}2^{\frac{n+1}{2}}} \frac{2^{\frac{n+1}{2}}}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}} \\
&= \frac{1}{\beta\left(\frac{1}{2},\frac{n}{2}\right)\sqrt{n}} \left(1+\frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}
\end{aligned}$$

□

### 2.9.2 Moments of t distribution

**Theorem 2.16** (Odd moments are zero)**.**

$$\mu'_{2r+1} = E[x^{2r+1}] = \int_{-\infty}^{\infty} x^{2r+1} f(x)\,dx = 0$$

So mean=0 that implies raw moments are equal to central moments.

**Theorem 2.17** (Even ordered moments)**.**

$$\mu'_{2r} = \frac{n^r\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

*Proof.*

$$\begin{aligned}
\mu'_{2r} &= \mu_{2r} = E[X^{2r}] \\
&= \int_{-\infty}^{\infty} x^{2r} f(x)\,dx \\
&= 2\int_{0}^{\infty} x^{2r} f(x)\,dx \\
&= \frac{2}{\beta\left(\frac{1}{2},\frac{n}{2}\right)\sqrt{n}} \int_0^\infty x^{2r}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}\,dx
\end{aligned}$$

Let $\frac{x^2}{n} = y \implies x^2 = ny$ so we have $\frac{dy}{dx} = \frac{2x}{n} \implies dx = \frac{ndy}{2x}$ Let beta term be $c$.

$$\begin{aligned}
\mu_{2r} &= c\int_0^\infty (ny)^r (1+y)^{-\frac{n+1}{2}} \frac{n}{2\sqrt{ny}}\,dy \\
&= \frac{cn}{2}\int_0^\infty (ny)^{r-1/2}(1+y)^{-\frac{n+1}{2}}\,dy \\
&= \frac{nc}{2} n^{r-1/2}\int_0^\infty y^{r-1/2}(1+y)^{-\frac{n+1}{2}}\,dy \\
&= \frac{nn^r}{2\sqrt{n}\frac{2}{\beta\left(\frac{1}{2},\frac{n}{2}\right)\sqrt{n}}}\int_0^\infty \frac{y^{r-1/2}}{(1+y)^{\frac{n+1}{2}}}\,dy \\
&= \frac{n^r}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\beta\left(r+\frac{1}{2},\frac{n}{2}-r\right) \\
&= \frac{n^r\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}
\end{aligned}$$

□

**Theorem 2.18** (Variance)**.**

$$V[x] = \frac{n}{n-2}$$

*Proof.*

$$\begin{aligned} V[x] &= \left.\frac{n^r\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\right|_{r=2} \\ &= \frac{n^2\Gamma(3/2)\Gamma(n/2-1}{\Gamma(1/2)\Gamma(n/2)} \\ &= \frac{n}{n-2} \qquad \text{for } n > 2 \end{aligned}$$

□

**Theorem 2.19** (Skewness)**.**

$$\beta_1 = 0$$

*Proof.* Since $\mu_3 = 0$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

□

**Theorem 2.20** (Kurtosis)**.**

$$\beta_2 = ???$$

*Proof.* We know $\mu_4 = \frac{3n^2}{(n-2)(n-4)}$ so then,

$$\begin{aligned} \beta_2 = \frac{\mu_4}{\mu_2^2} &= \frac{3n^2}{(n-2)(n-4)}\frac{(n-2)^2}{n^2} \\ &= 3+ \end{aligned}$$

□

## 2.10 Applications of $t-$distributions

### 2.10.1 Test for single mean

#### Assumptions

1. The parent population from which the sample is drawn is normal.
2. The sample drawn is a random sample, i.e. the observations are chosen independently.

3. Population variance is unknown.

To test if the random sample of size $n$ is drawn from a normal population with mean $\mu = \mu_0$, or,

To test if the sample mean $\overline{x}$ differs significantly from the hypothetical values $\mu_0$,

$H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$ with $LOS = \alpha$

**Test Statistic**

For test statistic consider,

$$T = \frac{\overline{x} - \mu}{\hat{\sigma}/\sqrt{n}} = \frac{(\overline{x} - \mu)\sqrt{n}}{\hat{\sigma}} = \frac{(\overline{x} - \mu)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}}$$

Divide numerator and denominator by $\sigma$,

$$\begin{aligned} T &= \frac{(x - \mu)\sqrt{n}/\sigma}{\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}\frac{1}{\sigma^2}}} \\ &= \frac{\frac{(\overline{x}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{\sum(x_i - \overline{x})^2}{(n-1)\sigma^2}}} = \frac{SNV}{\sqrt{\frac{\chi^2}{n-1}}} \sim t_{n-1} \end{aligned}$$

Where,

$$\begin{aligned} \overline{x} &= \frac{\sum x}{n}, \hat{\sigma} = s = \sqrt{\frac{\sum(x_i - \overline{x})^2}{n-1}} \text{ for raw data} \\ \overline{x} &= \frac{\sum fx}{\sum f}, \hat{\sigma} = s = \sqrt{\frac{\sum f(x_i - \overline{x})^2}{\sum f - 1}} \text{ for frequency distribution} \end{aligned}$$

**Decision criteria**

Reject $H_0$ if $T_{cal}$ lies in the critical region, i.e. $|T_{cal}| < -t_{\alpha/2,n-1}$ or $|T_{cal}| > t_{\alpha/2,n-1}$.

$100(1 - \alpha)\%$ **confidence interval for** $\mu$

We know that

$$\frac{\overline{x} - \mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1}$$

So $100(1-\alpha)\%$confidence intervals are given as,

$$P\left[-t_{n-1,\alpha/2} < \frac{\overline{x}-\mu}{\hat{\sigma}/\sqrt{n}} < t_{n-1,\alpha/2}\right] = (1-\alpha)$$
$$P\left[\frac{-t\hat{\sigma}}{\sqrt{n}} < \overline{x}-\mu < \frac{t\hat{\sigma}}{\sqrt{n}}\right] = (1-\alpha)$$
$$P\left[\frac{-t\hat{\sigma}}{\sqrt{n}} - \overline{x} < -\mu < \frac{t\hat{\sigma}}{\sqrt{n}} - \overline{x}\right] = (1-\alpha)$$
$$P\left[\overline{x} - \frac{t\hat{\sigma}}{\sqrt{n}} < \mu < \overline{x} + \frac{t\hat{\sigma}}{\sqrt{n}}\right] = (1-\alpha)$$

So the confidence interval is,

$$\overline{x} \pm t_{\alpha/2,n-1}\frac{\hat{\sigma^2}}{\sqrt{n}}$$

**Example 2.21.** *It is claimed that the depth of the sea has increased from the previous record for* 55 *meters. Test this claim given a sample of size* 6 *where the depths are* $60, 62, 53, 57, 60, 59$

*Proof.* $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$ with $LOS = \alpha$
The test statistic is $t_{6-1} = t_{5,0.05}$.
The table values are $[-2.57058, 2.57058]$
$T_{cal} = \frac{58.5-55}{3.14642/\sqrt{6}} = 2.72475$
Since $T_{cal} > T_{tab}$ we reject $H_0$. □

## 2.10.2 Test for two means for two independent samples

Let $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ be two random samples from two normal populations with mean $\mu_x$ and variance $\sigma^2$ and $\mu_y$ and variance $\sigma^2$.

### Assumptions

1. The parent population follows normal distribution.

2. The two samples are independent.

3. The variances are unknown but equal.

To test if he two independent samples come from 2 populations with euqal means, we proceed as follows.

**Hypothesis**

To test,
$H_0 : \mu_x - \mu_y = d$,
$H_1 : \mu_x - \mu_y \neq d$,
LOS=$\alpha$

**T statistic**

$$T = \frac{(\overline{x} - \overline{y}) - (\mu_x - \mu_y)}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$\hat{\sigma}^2 = s_p^2 = \frac{\sum_i (x_i - \overline{x})^2 + \sum_j (y_j - \overline{y})^2}{n_1 + n_2 - 2}$ Divide by $\sigma$

$$\begin{aligned} T &= \frac{(\overline{x} - \overline{y}) - (\mu_x - \mu_y)}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{SNV}{\sqrt{\frac{\chi^2}{n_1+n_2-2}}} \sim t_{n_1+n_2-2} \end{aligned}$$

**Decision criteria**

Reject $H_0$ if $T_{cal} < -t_{n_1+n_2-2,\alpha/2}$ or $T_{cal} > t_{n_1+n_2-2,\alpha/2}$
For $H_1 : \mu_x - \mu_y > d$ reject $H_0$ if $T_{cal} > t_{n_1+n_2-2,\alpha/2}$
For $H_1 : \mu_x - \mu_y < d$ reject $H_0$ if $T_{cal} < t_{n_1+n_2-2,\alpha/2}$

**Confidence intervals**

$(1 - \alpha)100\%$ confidence intervals for $\mu_x - \mu_y$,

$$P\left[-t < \frac{\overline{x} - \overline{y} - (\mu_x - \mu_y)}{s_p\sqrt{1/n_1 + 1/n_2}} < t\right] = (1 - \alpha)$$

$$P\left[-ts_p\sqrt{1/n_1 + 1/n_2} - (\overline{x} - \overline{y}) < -(\mu_x - \mu_y) < ts_p\sqrt{1/n_1 + 1/n_2} - (\overline{x} - \overline{y})\right] = (1 - \alpha)$$

$$P\left[(\overline{x} - \overline{y}) - ts_p\sqrt{1/n_1 + 1/n_2} < \mu_x - \mu_y < (\overline{x} + \overline{y}) + ts_p\sqrt{1/n_1 + 1/n_2}\right] = (1 - \alpha)$$

So the $100(1 - \alpha)\%$ confidence interval for $\mu_x - \mu_y$ is,

$$(\overline{x} - \overline{y}) \pm t_{(\alpha/2, n_1+n_2-2)} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

**Example 2.22.** *Test for equality of mean for the following sets of values,* $54, 60, 62, 55, 63$ *and* $40, 60, 70, 50, 60, 50, 52$. *Also find confidence interval for difference in the means.*

*Proof.* To test,
$H_0 : \mu_x - \mu_y = d = 0$,
$H_1 : \mu_x - \mu_y \neq d = 0$,
LOS$=\alpha = 0.05$

$n_1 = 5, n_2 = 7$ so the test statistic is given as $t_{10}$.

$$T = \frac{(\overline{x} - \overline{y})}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{58.8 - 54.5714}{7.9026\sqrt{\frac{1}{5} + \frac{1}{9}}}$$

□

# Chapter 3

# F-distribution