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Calculus IV

Lecture Notes
for SMAT401

Contents

Chapter 1

Functions of several variables

1.1 Examples of functions of several variables

$f(x,y) = x + y\log x$	$f : \mathbb{R}^2 \to \mathbb{R}$	Scalar valued function
$f(x,y) = (x^2y, \cos x, e^x - 9)$	$f : \mathbb{R}^2 \to \mathbb{R}^3$	Vector valued function

Clearly, $f : \mathbb{R} \to \mathbb{R}$ is a particular case of scalar valued function.

1.2 Non-existence of limit by 2 path test

For a function $f : \mathbb{R} \to \mathbb{R}$ the limit exists if limit value is the same along all possible paths, i.e. left hand and right hand limit are equivalent.

For a multivariate function the 2 path test can be used to show non existence of a limit.

Example 1.1. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$ *doesn't exist.*

Proof. Consider $x = my^2$ and let $y \to 0$, then

$$\lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^2} = \frac{2m}{1+m^2}$$

.

Therefore, the limit value varies for different values of m. □

Example 1.2. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ *doesn't exist.*

Proof. Consider first along x axis (i.e. $y = 0$)

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Consider now along y axis (i.e. $x = 0$)

$$\lim_{y \to 0} \frac{y}{-y} = -1$$

Since the limit is not path independent we can say the limit does not exist. □

Example 1.3. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ *doesn't exist.*

Proof. Along x and y axis the limits are both zero. Consider instead the path $y = x^2$

$$\lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limit is not path independent it does not exist. □

Example 1.4. *Show that the* $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2-2x}$ *doesn't exist.*

Proof. Along x, y axis the limit is 0. Consider the path $y = \sqrt{2x}$

$$\lim_{x \to 0} \frac{x^2}{x^2} = 1$$

Since the limit is not path independent it does not exist. □

1.3 Existence of limit with ε, δ definition

Recall the single variable definition of a limit,

Definition 1.5 (Limit of a single valued function)**.** *For a function* $f : \mathbb{R} \to \mathbb{R}$, $\lim_{x\to a} f(x) = L \iff \forall\varepsilon > 0, \exists\delta$ *such that* $0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon$

Definition 1.6 (Limit of a multivariate function)**.** *For a function* $f : \mathbb{R}^2 \to \mathbb{R}$, $\lim_{(x,y)\to(a,b)} f(x) = L \iff \forall\varepsilon > 0, \exists\delta$ *such that*

$$0 < ||(x, y) - (a, b)||_2 < \delta \implies |f(x, y) - L| < \varepsilon$$

, alternatively

$$\sqrt{(x-a)^2 + (x-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$$

Example 1.7. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x-y}{1+x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$, consider

$$\begin{aligned}|f(x,y) - L| = |f(x,y)| &= \left|\frac{x-y}{1+x^2+y^2}\right| \\ &= \frac{|x-y|}{1+x^2+y^2}\end{aligned}$$

since $1 + x^2 + y^2 \geq 1$

$$\begin{aligned}&\leq |x-y| \\ &\leq |x| + |y| \\ &\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\sqrt{x^2+y^2}\end{aligned}$$

Therefore, if $2\sqrt{x^2+y^2} < \varepsilon \implies |f(x,y) - L| < \varepsilon$ so take $\delta = \varepsilon/2$. $\square$

Example 1.8 (H.W)**.** *Show that* $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$

Proof. Let $\varepsilon > 0$,consider

$$\begin{aligned}|f(x,y) - L| &= \left|\frac{xy^2}{x^2+y^2} - 0\right| = \frac{|x|y^2}{x^2+y^2} \\ &= \frac{|x|}{\frac{x^2}{y^2} + 1} \\ &\leq |x| \\ &\leq \sqrt{x^2+y^2} < \varepsilon \implies |f(x,y) - L| < \varepsilon\end{aligned}$$

So we can just pick $\delta = \varepsilon$. $\square$

1.4 Continuity

Definition 1.9 (Continuity)**.** *A function* $f : \mathbb{R}^2 \to \mathbb{R}$ *is said to be continuous at a point* (a,b) *if* $\forall\varepsilon > 0, \exists\delta > 0$ *such that,*

$$0 < ||(x,y) - (a,b)||_2 < \delta \implies |f(x,y) - f(a,b)| < \varepsilon$$

provided $f(a,b)$ *exists. Alternatively,*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note that, we can show the function is discontinuous if

1. $f(a,b)$ doesn't exist.
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ doesn't exist.
3. Both exist but are not equal to each other.

Example 1.10. *Show that the given function is continous at* $(0,0)$ *where,*

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. Here, $f(0,0) = 0$. Clearly we have that $|x^2 - y^2| \leq |x^2 + y^2|$.
Let $\varepsilon > 0$,

$$\begin{aligned} |f(x,y) - L| &= \left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| \\ &= |x||y| \left| \frac{x^2-y^2}{x^2+y^2} \right| \\ &\leq |x||y| \\ \\ &\leq \sqrt{x^2+y^2}\sqrt{x^2+y^2} = x^2 + y^2 \end{aligned}$$

So when $x^2 + y^2 < \varepsilon \implies |f(x,y) = f(0,0)| < \varepsilon$ so we take $\delta = \sqrt{\varepsilon}$. □

Example 1.11. *Show that the given function is discontinuous at* $(0,0)$ *where,*

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. content... □

1.5 Polar Coordinates

The polar coordinates r(the radial coordinate) and θ(the angular coordinate), are defined in terms of cartesian coordinates as below.

$$x = r\cos\theta, y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}, \theta = \arctan\left(\frac{y}{x}\right)$$

1.5.1 Limits in Polar coordinates

Use polar coordinates when you are over (0,0)

Example 1.12. *Show that* $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ *doesn't exist.*

Proof. Put $x = r\cos\theta$ and $y = r\sin\theta$

$$f(x,y) = \frac{2xy}{x^2+y^2} \iff f(r,\theta) = \frac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta$$
$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} 2\cos\theta\sin\theta = 2\cos\theta\sin\theta$$

Which depends on θ. □

1.5.2 Epsilon-delta with polar coordinates

Definition 1.13. $\lim_{r\to 0} f(r,\theta) = L \iff \forall\varepsilon > 0 \exists\delta > 0 s.t.$

$$0 < |r| < \delta \implies |f(r,\theta) - L| < \varepsilon$$

Example 1.14. *Show that* $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

Proof.

$$f(r,\theta) = \frac{r^3\cos^3\theta}{r^2} = r(\cos\theta)^3 \tag{1.1}$$

Let $\varepsilon > 0$, consider $|f(r,\theta) - L| = |r||\cos\theta|^3 \leq |r|$. So we can set $\delta = \varepsilon$ □

Example 1.15. *Find the domain and range of* $g(x,y) = \sqrt{9 - x^2 - y^2}$

Proof. The sqrt interior must be positive so take $x^2 + y^2 \leq 9$, so its a circle of radius 3 centred at 0. So the domain is the circle. The range is $\{z \mid 0 \leq z \leq 3\} = [0,3]$ □

1.6 Algebra of limits

Let $f, g : \mathbb{R}^n \to \mathbb{R}, p \in \mathbb{R}^n$ and $k_1, k_2 \in \mathbb{R}$.

Theorem 1.16. *If* $\lim_{x\to p} f(x) = L_1, \lim_{x\to p} g(x) = L_2$, *then*

- $\lim\limits_{x\to p}(k_1 f(x) + k_2 g(x)) = k_1 L_1 + k_2 L_2$
- $\lim\limits_{x\to p}(f(x)g(x)) = L_1 L_2$
- *For non-zero* L_2, $\lim\limits_{x\to p}(f(x)/g(x)) = L_1/L_2$

1.7 General multivariate limit

Theorem 1.17 (Limit of a function $f : \mathbb{R}^n \to \mathbb{R}$). *For a function $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$0 < ||x - a||_n < \delta \implies |f(x) - L| < \varepsilon$$

Definition 1.18 (ε- neighbourhood). *$B(a, \varepsilon)$ open ball of radius ε around α.*

$$0 \leq ||x - a||_n < \varepsilon$$

Definition 1.19 (Deleted ε neighbourhood). $B(a, \varepsilon) - \{a\}$

Definition 1.20 (Alternate definition of a limit). *For a function $f : \mathbb{R}^n \to \mathbb{R}$, $\lim_{x\to a} f(x) = L$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$x \in B * (a, \delta) \implies |f(x) - L| < \varepsilon$$

Definition 1.21 (Bounded function). *Let E be a non-empty subset of $\mathbb{R}^n$. The function $f : E \to \mathbb{R}$ is said to be bounded in some δ-neighbourhood of point $p \in \mathbb{R}^n$ if there exists $M > 0$ in $\mathbb{R}$ such that*

$$|f(x)| \leq M \forall x \in B(p, \delta)$$

Definition 1.22 (Relation between bounded function and limit of a function in $\mathbb{R}^n$). *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$. Let $f(p)$ be defined. If $\lim_{x\to p} f(x)$ exists then f is bounded in some neighbourhood of point p.*

The converse of 1.22 isn't true.

Theorem 1.23 (Uniqueness of limit in $\mathbb{R}^n$). *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$. If $\lim_{x\to p} f(x)$ exists then it is unique.*

1.8 Iterated (Repeated) limits

Let $(a, b) \in E$ and $f : E \to \mathbb{R}$ be a function where $E \subseteq \mathbb{R}^2$,

1. Suppose there exists $\delta > 0$ such that $\forall x$ with $0 < |x - a| < \delta$,we have $\lim_{y\to b} f(x, y)$ exists.
   Define a new function $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = \lim_{y\to b} f(x, y)$. If $\lim_{x\to a} g(x)$ exists then this limit is called **iterated limit** which is given by $\lim_{x\to a} g(x) = \lim_{x\to a} \lim_{y\to b} f(x, y)$.

2. Suppose there exists $\delta > 0$ such that $\forall y$ with $0 < |y - b| < \delta$, we have $\lim_{x \to a} f(x, y)$ exists

Theorem 1.24. *Existence of double limit does not imply existence of iterated limit.*

Proof. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as,

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases}$$

We show that $\lim_{(x,y) \to (0,0)} f(x, y) = 0$ i.e. double limit exists.
Let $\varepsilon > 0$. Consider then

$$\begin{aligned} |f(x, y) - L| = |x \sin(1/y) - 0| = |x|\,|\sin(1/y)| &\leq |x| \\ &\leq \sqrt{x^2} \\ &\leq \sqrt{x^2 + y^2} \end{aligned}$$

so $\sqrt{x^2 + y^2} < \varepsilon \implies |f(x, y) - L| < \varepsilon$. So choose $\delta = \varepsilon$.
We will now check its iterated limit.

$$\begin{aligned} \lim_x \lim_y f(x, y) &= \lim_{x \to 0} \left[\lim_{y \to 0} x \sin 1/y \right] \\ &= \lim_{x \to 0} x \left[\lim_{y \to 0} \sin 1/y \right] \end{aligned}$$

The limit inside doesn't exist.
Claim that $\lim_y \phi(y) = \lim_y \sin 1/y$ doesn't exist, Take $a_n = \frac{1}{(4n+1)\pi/2}, b_n = \frac{1}{(4n-1)\pi/2}$. The sequences converge to zero but their sequences $\phi a_n, \phi b_n$ dont converge to the same limit. ☐

Example 1.25 (Both iterated limits exist but double limit doesn't exist)**.** *consider $f : \mathbb{R}^2 \to \mathbb{R}$, defined as*

$$f(x, y) = \begin{cases} \frac{x^2}{x^2 + y^2 - x} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Proof. Begin with two part test to show that the double limit does not exist.
Consider first the path $x = y$ the limit is 0,

$$\lim_{(x,y) \to (0,0)} \frac{y^2}{y^2 + y^2 - y} = \lim_{(x,y) \to (0,0)} \frac{y}{2y - 1} = 0$$

Then consider the path $x = y^2$

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{y^4 + y^2 - y^2} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4} = 1$$

So double limit does not exist.
Now consider the iterated limits,

$$\begin{aligned} \lim_{x\to 0} \lim_{y\to 0} f(x,y) &= \lim_{x\to 0} \left[\lim_{y\to 0} \frac{x^2}{x^2 + y^2 - x} \right] \\ &= \lim_{x\to 0} \frac{x^2}{x^2 - x} = 0 \end{aligned}$$

Now consider

$$\lim_{y\to 0} \lim_{x\to 0} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

□

Example 1.26 (Both iterated limits exist (not equal) but double limit doesn't exist)**.** *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *definde as,*

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Proof. First show that the double limit does not exist.
Consider the bath $x = 0$ the limit is equal to 1. Consider the path $x = y$ we will have the limit equal to 0.
Consider now the iterated limits,

$$\lim_{x\to 0} \lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} 1 = 1$$

And now the other direction,

$$\lim_{y\to 0} \lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} -1 = -1$$

□

Theorem 1.27. *Suppose* $\lim_{(x,y)\to(0,0)} f(x,y)$ *exists and is equal to* L*. If both iterated limits exist then, the iterated limits are both equal to* L*.*

Proof. Given that $\lim_{(x,y)\to(a,b)} f(x,y) = L$. Let $\lim_{y\to b} f(x,y) = g(x)$. Let $\varepsilon > 0$, then there exists $\delta_1 > 0$ s.t.

$$0 < ||(x,y) - (a,b)||_2 < \delta_1 \implies |f(x,y) - L| < \frac{\varepsilon}{2}$$

and there exists $\delta_2 > 0$ such that

$$0 < |y - b| < \delta_2 \implies |f(x,y) - g(x)| < \frac{\varepsilon}{2}$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta \le \delta_1, \delta \le \delta_2$ which gives

$$0 < |y - b| \le \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L|\frac{\varepsilon}{2}$$

$$0 < |y - b| < \delta \implies |f(x,y) - g(x)| < \varepsilon/2$$

With respect to $0 < |x - a| < \delta$ consider

$$\begin{aligned} |g(x) - L| = |g(x) + f(x,y) - f(x,y) - L| &= |f(x,y) - L) - (f(x,y) - g(x))| \\ &\le |f(x,y) - L| + |f(x,y) - g(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Thus $\forall\varepsilon > 0, \exists\delta > 0$ s.t. **Complete this later...**

Do the same thing for $h(y)$. □

1.9 Limits in 3 variables

1.9.1 Two path test for non-existence of limit

Two path can be used for non-existence of a limit in 3 variables. However a single equation is not enough to define a path in $\mathbb{R}^3$ two Cartesian equations are required for a path in $\mathbb{R}^3$.

Example 1.28. *Show that*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2}$$

Proof. Take $y = x, z = x$ then

$$\lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}$$

Take other path $y = x, z = 0$

$$\lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

□

Definition 1.29 (Limit of a function $\mathbb{R}^3 \to \mathbb{R}$)**.** *For a function $f : \mathbb{R}^3 \to \mathbb{R}, \lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$ if and only if $\forall\varepsilon > 0, \exists\delta > 0$ such that*

$$0 < ||(x,y,z) - (a,b,c)||_3 < \delta \implies |f(x,y,z) - L| < \varepsilon$$

i.e.

$$0 < \sqrt{(x-a)^2 + (y-b)^2(z-c)^2} < \delta \implies |f(x,y,z) - L| < \varepsilon$$

Definition 1.30 (Continuity of a function $\mathbb{R}^3 \to \mathbb{R}$)**.** *Replace L with $f(a,b,c)$.*

Example 1.31. *Show that*

$$\lim_{(x,y,z)\to(1,2,3)} 4x + 2y + z = 11$$

using epsilon delta

Proof. Let $\varepsilon > 0$ consider,

$$\begin{aligned} |f(x,y,z) - L| &= |4x + 2y + z - 11| \\ &= |(4x-4) + (2y-4) + (z-3)| \\ &\le 4|x-1| + 2|y-2| + |z-3| \\ &\le 4\sqrt{(x-1)^2} + 2\sqrt{(y-2)^2} + \sqrt{(z-3)^2} \\ &\le 7\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^3} \end{aligned}$$

So take $\delta = \varepsilon/7$ □

Example 1.32. *Evaluate* $\lim_{(x,y)\to(3,3)} \frac{x^2+xy-2y^2}{x^2-y^2}$

Proof. Factorize $(x-y)$ on numerator and denominator then just plug and chug. □

Chapter 2

Differentiation

2.1 Partial derivatives

Partial derivates are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation. For a function f in n variables $x_1, x_2, \ldots, x_n$ we can define the m^{th} partial derivative as,

$$f_{x_m} = \frac{\partial f}{\partial x_m} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}$$

Partial derivatives can be taken with respect to multiple variables and are denoted as follows,

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

Theorem 2.1. *$E \subseteq \mathbb{R}^2$ Let f_x, f_y, f_{xy}, f_{yx} exist. If f_{xy}, f_{yx} are continuous at (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$*

2.2 Gradient

Definition 2.2 (Gradient)**.** *For $f : \mathbb{R}^3 \to \mathbb{R}$*

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Example 2.3. *If* $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ *find* ∇f_p *where* $p = (\sqrt{2}, \sqrt{2}, -3)$

Proof.

$$f_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

□

Example 2.4. *Find* ∇f *at* $p = (0, \pi/2)$ *if* $f(x, y) = \sin(xy)$ *and its norm at* p.

Theorem 2.5 (Chain rule for two variables)**.** *If* $w = f(x, y)$ *has continuous p.d.* f_x, f_y *and if* $x = x(t), y = y(t)$ *are differentiable functions of* t *then the composite function* $w \circ f(x(t), y(t))$ *is a differentiable function of* t *and*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 2.6. *If* $u = x^2 + y^2$ *and* $x = at^2$ *and* $y = 2at$ *find* $\frac{du}{dt}$

Proof.

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Consider the two partial derivatives first,

$$f_x = 2x, f_y = 2y$$

Now $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$ So we have $\frac{du}{dt} = 2x(2at) + 2y(2a) = 4a^2(t^3 + 2t)$ □

2.3 Level curves

Definition 2.7. *The level curves of a function* f *of two variables are curves with equations* $f(x, y) = k$ *where* k *is a constant (in the range of* f*).*

Theorem 2.8. *The vector* $\nabla f(x, y)$ *is normal (perpendicular to tangent) to level curve of* f.

Chapter 3

Applications