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Calculus IV

Lecture Notes
for SMAT401

Contents

Chapter 1

Tutorial 1

Show that the limits as the function approaches (0,0) dont exist

1.1 Question 1

$$\frac{x^2 - y^2}{x^2 + y^2}$$

use x and y axis

1.2 Question 2

$$\frac{x^3 y}{x^6 + y^2}$$

consider $y = 0$ then we have limit equal 0. Consider now $y = x^3$ so now limit of $\frac{x^6}{2x^6} = \frac{1}{2}$.

1.3 Question 3

$$\frac{\sin(x^2 + y)}{x + y}$$

along the x axis (y=0) we get $\sin(x^2)/x$ and the lim is 0. But for y axis (x=0) we get $\sin(y)/y$ and the lim is 1.

1.4 Question 4

$$\frac{x^3 + y^3}{x - y}$$

take the line $y = 0$ we get $\frac{x^3+m^3x^3}{x-mx} = \frac{(1+m)x^3}{x(1-m)}$ is 0 but with $y = x - x^3$ is equal to 2.

Try with $y = x - x^2$ we get

$$\begin{aligned}\lim \frac{x^3 + (x - x^3)^3}{x - (x - x^3)} &= \lim \frac{x^3}{x^3} + \frac{()x - x^3)^3}{x^3} \\ &= 1 + \lim \frac{(x - x^3)^3}{x^3} = 1 + 1 = 2\end{aligned}$$

1.5 Question 5

$$\frac{x^2y^2}{x^2y^2 + (x - y)^2}$$

consider the line $x = 0$ then the limit is obviously 0. Now consider $y = x$ then the limit of $\frac{x^4}{x^4} = 1$.

1.6 Question 6

$$\frac{2xy^2}{x^3 + y^3}$$

take y=0 and x=y.

Tutorial 1.5

1.7 Question 1

$$\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) = 0$$

Tutorial 2

1.8 Question 1

Using polar coordinates, show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$$

with $f(0, 0) = 0$ is continuous at $(0, 0)$.

Proof.

$$f(r, \theta) = r \cos\theta^3 - \sin\theta^3$$

Let $\varepsilon > 0$ so

$$\begin{aligned} |f(r, \theta) - f(0, 0)| &= |r||\cos\theta^3 - \sin\theta^3 \\ &\leq |r|(|\cos\theta|^3 + |\sin\theta|^3) \\ &\leq r(1 + 1) && = 2r \end{aligned}$$

So pick $\delta = \varepsilon/2$ $\square$

1.9 Question 2

Prove that

$$\lim_{(x,y)\to(0,0)} xy \sin\left(\frac{1}{x^2 + y^2}\right) = 0$$

using without polar form.

Proof. Let $\varepsilon > 0$

$$|f(x, y) - f(0, 0)| = |xy| \left|\sin\left(\frac{1}{x^2 + y^2}\right)\right|$$

$$\begin{aligned}
&\leq |x||y| \\
&\leq \sqrt{x^2+y^2}\sqrt{x^2+y^2} \\
&\leq x^2+y^2
\end{aligned}$$

So just take $\delta = \sqrt{\varepsilon}$ $\square$

1.10 Question 3

Prove that

$$\lim_{(x,y)\to(0,0)} \frac{1}{xy}\sin(x^2y+xy^2) = 0$$

with epsilon delta.

Proof. We will use the fact that for small θ, $|\sin\theta| \leq \theta$.
Let $\varepsilon < 0$ then,

$$\begin{aligned}
\left|\frac{1}{xy}\sin(x^2y+xy^2)\right| &\leq \frac{1}{|x||y|}|x^2y+xy^2| \\
&\leq |x|+|y| \\
&\leq \sqrt{x^2+y^2}+\sqrt{x^2+y^2} \\
&\leq 2\sqrt{x^2+y^2}
\end{aligned}$$

So pick $\delta = \varepsilon/2$ $\square$

1.11 Question 4

Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}$$

for non zero and $f(0,0) = (0,0)$ is continuous at $(0,0)$.

Proof. Consider limit as f approaches $(0,0)$. Let $\varepsilon > 0$

$$|f(0,0)-L| = \left|\frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}\right|$$

$$= \left| \frac{x^2 y^2}{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)} \right|$$

Hint, $\sqrt{x} + 1 \geq 1$, $\frac{1}{\sqrt{x}+1} \leq 1$

$$= \left| \frac{1}{\sqrt{x^2 + y^2 + 1} + 1} \right|$$

$\square$

Tutorial 3: Incomplete

Tutorial 4

1.12 Question 1

If $u = x^2 \arctan\left(\frac{y}{x}\right) - y^2 \arctan\left(\frac{x}{y}\right)$, find u_{yx}

Proof. First begin by finding $\partial u/\partial y$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \lim_{h \to 0}\left(\frac{f(x, y+h) - f(x, y)}{h}\right) \\
&= \frac{x^3}{x^2 + y^2} - y\left(2\arctan(x/y) - \frac{xy}{x^2 + y^2}\right) \\
&= x - 2y\arctan(x/y)
\end{aligned}$$

Now find its partial derivative with erespect to y

$$\begin{aligned}
u_{yx} &= \frac{\partial}{\partial x}(x - 2y\arctan(x/y)) \\
&= 1 - \frac{2y^2}{x^2 + y^2} \\
&= \frac{x^2 - y^2}{x^2 + y^2}
\end{aligned}$$

$\square$

1.13 Question 2

If $u = x^y$ prove that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Proof. First consider LHS we will find partial wrt y first

$$\frac{\partial u}{\partial y} = x^y \log(x)$$

Now double derivative w.r.t. x

$$\frac{\partial u}{\partial x} = x^{y-1}(y\log(x) + 1)$$

Again

$$\frac{\partial^3 u}{\partial x^2 \partial y} = x^{y-2}((y-1)y\log x + 2y - 1)$$

Now consider the RHS, partial with x first

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

Then with y

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1}(y\log x + 1)$$

Finally with x

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = x^{y-2}((y-1)y\log x + 2y - 1)$$

□

1.14 Question 3

If $v = \frac{c}{\sqrt{t}} \exp\left(\frac{-x^2}{4a^2 t}\right)$