Commutative Algebra Cheat Sheet

Rings

A **ring** A is a set with two binary operations addition and multiplication such that

- A is an abelian group with addition.
- Multiplication is associative and distributive over addition.

Additionally we consider rings with commutativity and existence of multiplicative identity 1.
A function $\varphi : A \to B$ between rings is a **homomorphism** if it preserves addition multiplication and sends 1 to 1.
A **subring** is a subset of a ring that is also a ring with the induced relations.

Ideals

An **ideal** $\mathfrak{a}$ of a ring A is a subset of A which is a additive subgroup group and for $x \in \mathfrak{a}, xA \subseteq \mathfrak{a}$.
The cosets of $\mathfrak{a} \in A$ form a quotient ring $A/\mathfrak{a}$.
Correspondence theorem for rings: There is a bijection between ideals of A containing $\mathfrak{a}$ and the ideals of $A/\mathfrak{a}$.

Zero divisors, units

An element is called a **zero divisor** if its product with a non zero element gives 0.
A commutative ring with the only zero divisor being zero is called an **integral domain**.
An element is called a **unit** if its product with some element gives 1.

- $x \in A$ is a unit $\iff \langle x \rangle = \{ax \mid a \in A\} = A = \langle 1 \rangle$

A ring in which every non zero element is a unit is called a **field**.

- All fields are integral domains.
- All finite integral domains are fields.
- The only ideals in a field F are 0 and $\langle 1 \rangle = F$

Prime and Maximal ideals

A proper ideal $\mathfrak{p} \in A$ is called **prime** if for $xy \in \mathfrak{a} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ alternatively if $A/\mathfrak{p}$ is an integral domain.
A proper ideal $\mathfrak{m} \in A$ is called **maximal** if it is maximal with respect to inclusion alternatively if $A/\mathfrak{m}$ is a field.
A ring with exactly one maximal ideal is called a **local** ring. And its subsequent quotient is called the **residue field** of the ring. If number of maximal ideals are finite then it is called **semi local**.

Ideal operations

Radical ideals, ideal quotients

Nilradical and Jacobson ideal

Extension and Contraction of ideals