Measure Theory Cheat Sheet

Topology

A collection T os subsets of a set X is said to be a **topology** in X if T satisfies the following properties,

- $\emptyset \in T$ and $X \in T$
- Closed under finite intersections
- Closed under arbitrary unions

Members of T are called open sets.
If X, Y are topological spaces then $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for all open sets $V \in Y$.

σ-algebra

A collection F of subsets of X is called a $\sigma-$algebra if the following properties hold

- $X \in F$
- If $A \in F$ then $A^C = A - X \in F$
- Closed under unions

Measureability

- If F is a $\sigma-$algebra of X then X is a **measurable space** and members of F are **measurable sets** in X.
- If X is a measurable space and Y is a topological space, then $f : X \to Y$ is said to be **measurable** if $f^(- 1)(V)$ is a measurable set in X for all open sets V in Y.

Characteristic function: It is a measurable function defined as follows .If E is a measurable set in X define $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$

Borel $\sigma-$algebra

Generated $\sigma-$algebra: For any collection of subsets F of X there exists a smallest $\sigma-$algebra which contains F. It is the intersection of all $\sigma-$algebras containing F.
Borel $\sigma-$ algebra: For a topological space X the $\sigma-$algebra generated by the family of open sets of X. Elements of a Borel $\sigma-$algebra are called Borel sets.
Borel mapping: A map between two topological spaces $f : X \to Y$if the inverse image of an open set in Y is an element of the Borel $\sigma-$algebra of X.

- If $f : X \to [-\infty, \infty]$ and F is a $\sigma-$algebra of X, then f is measurable if $f^{(-1)}((a, \infty)) \in F$ for all a.

Pointwise convergence and measurability

- If $f_n : X \to [-\infty, \infty]$ is measurable for all $n \in \mathbb{N}$ then $\sup, \inf, \limsup, \liminf$ of f_n are also measurable.
- o the limit of every pointwise convergent sequence of measurable functions is measurable.
- If f is measurable then so is $f^+ = \max\{f, 0\}, f^{-1} = -\min\{f, 0\}$

Simple functions

A complex function whose range consists of only finitely many points. If $\alpha_1 \ldots, \alpha_n$ are the distinct values of the simple function s and $A_i = x : s(x) = \alpha_i$ then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

- Every measurable function $f : X \to [0, \infty]$ can be written as a pointwise limit of a sequence of simple functions.

Positive measure

A **positive measure** μ is a measure along with the following additional properties,

- Its range is in $[0, \infty]$
- Countable additivity: $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

A *measure* space refers to a measurable space with a positive measure.

Arithmetic in $[0, \infty]$

We understand $a + \infty = \infty$ for $0 \le a \le \infty$ and $a \cdot \infty = \infty$ if $0 < a \le \infty$ else 0.

Lebesgue integral

If X is a set with $\sigma-$algebra F and positive measure μ. Then for a measurable simple function $s : X \to [0, \infty]$ as defined previously, its Lebesgue integral over $E \in F$ is defined as follows

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu \left(A_i \bigcap E \right)$$

Lebesgue integrable functions

Define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which $\int_X |f| \, d\mu < \infty$ known as the Lebesgue integrable functions.
For functions f with range in $[-\infty, \infty]$ define $\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$

Zero measure

We say a property holds "almost everywhere (a.e.)" if it holds everywhere except on a set of measure zero.
If any two function $f = g$ a.e. then their Lebesgue integrals are the same. Set of measure zero don't impact the value of the Lebesgue integral

Monotone convergence theorem

Let $\{f_n\}$ be a sequence of measurable functions on X if $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ a.e. and $f_n(x) \to f(x)$ as $n \to \infty$ a.e., then f is measurable and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu$$

Consequences of MCT

- Applying MCT to sequence of partial sums of a convergent series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ we get,

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

- If $f : X \to [0, \infty]$ is measurable with $\sigma-$algebra F of X and $\phi(E) = \int_E f \, d\mu$ for $E \subset F$ then, ϕ is a measure on F and

$$\int_X g \, d\phi = \int_X gf \, d\mu$$

for every measurable $g : X \to [0, \infty]$.

Fatou's lemma

If $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$ then,

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Dominated convergence theorem

If $\{f_n\}$ us a sequence of complex measurable functions on X with $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise. If there exists a function $g \in L^1(\mu)$ such that $|f_n(x)| \le g(x)$ for all $n \in N$ and $x \in X$ then $f \in L^(1)(\mu)$ and

$$\lim_{n\to\infty} \int_X f \, d\mu = \int_X \lim_{n\to\infty} f_n(x) \, d\mu$$

Complete measure

A measure is called complete if all subsets of sets of measure 0 are measurable. Every measure can be completed.

Compact support

A function has compact support if it is zero outside of a compact set, i.e. $f \in C_c(X)$ if f has compact support on X.

Locally compact Hausdorff spaces

A topological space X is called locally compact if every point $x \in X$ has a compact neighbourhood.
Uryshon's lemma: If X is a locally compact Hausdorff space. Let $K \subseteq X$ is compact and U open s.t. $K \subseteq U \subset X$, there exists $f \in C_c(X)$ with $0 \le f \le 1$ such that $f_K \equiv 1$ and $f \equiv 0$ otherwise.

Riesz representation theorem

Let X is a locally compact Hausdorff space and T is a positive linear functional on $C_c(X)$. Then there exists a $\sigma-$algebra F of X which contains all Borel sets in X and there exists a unique positive measure μ on F such that for every $f \in C_c(X)$

$$Tf = \int_X f \, d\mu$$

additionally the following properties hold,

1. $\mu(K) < \infty$ for every compact $K \subset X$
2. $\mu(E) = \inf\{\mu(V) : E \subset V, V \text{open}\}$ for all $E \in F$.
3. $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{compact}\}$ holds for every open set $E \subset F$ with $\mu(E) < \infty$
4. If $E \in F$, $A \subset E$ and $\mu(E) = 0$ then $A \in F$.

Regular Borel measures

A measure μ defined on the $\sigma-$algebra of all Borel sets in a locally compact Hausdorff space X.
If μ is positive we also say a Borel set is,

- **Outer regular** if satisfies property 3 of the above theorem.
- **Inner regular** if it satisfies property 4 of the above theorem.
- **Regular** if it is both inner and outer regular.

In a locally compact Hausdorff space X in which every open set can be written as a finite union of compact sets. If there exists a positive Borel measure μ defined on X such that for $K \subseteq X$ compact $\mu(K) < \infty \implies \mu$ is regular.

Lebesgue measure

Lusin's theorem

Convexity

Jensen’s inequality

Hölder’s inequality

Minkowski’s inequality

L^p norms

Total variation

Complex measure

Absolutely continuity and mutually singular

Lebesgue-Radon-Nikodym theorem

Product measures

Product measures

Differentiation