Category Theory Cheat Sheet
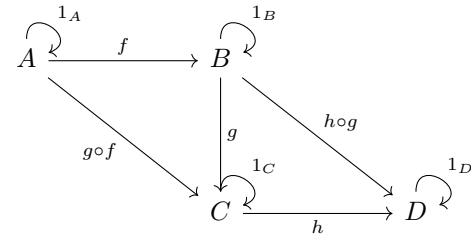
Category

A *category* consists of the following,

- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each f there exists, $\text{dom}(f), \text{cod}(f)$ called domain and codomain of f. We write $f : A \to B$ to indicate $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of f and g.
- For each A, there exists $1_A : A \to A$ called the *identity arrow* of A.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

Functor

A *functor* $F : C \to D$ between C and D is a mapping of objects and arrows to arrows, such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an *isomorphism* if there exists an arrow $g : B \to C$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., A is isomorphic to B.

Monoid

A set M with binary operation $\cdot$ is called a *monoid* if it is associative and has an identity $u \in M$, i.e., for $x, y, z \in M$

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $u \cdot x = x = x \cdot u$.

A monoid with an inverse for each element is called a *group*.

- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form (C, D) for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise, i.e. $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ and $1_{(C,D)} = (1_C, 1_D)$.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.
  For notational simplicity we say $f^* : D^* \to C^*$ in $\mathbf{C}^{op}$ for $f : C \to D$ in $\mathbf{C}$. Composition and units are therefore defined as follows, $f^* \circ g* = (g \circ f)^*$ and $(1_{C^*} = 1_C)^*$

Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^{\to}$ has the arrows of $\mathbf{C}$ as objects and an arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^{\to}$ is the following commutative square

$$A \xrightarrow{g_1} A'$$
$$\downarrow{f} \qquad \downarrow{f'}$$
$$B \xrightarrow{g_2} B'$$
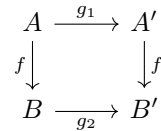
where g_1, g_2 are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.

- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ is $\mathbf{C}/C$ where its objects are all arrows $f \in \mathbf{C}$ s.t. $\text{cod}(f) = C$. An arrow a from $f : X \to C$ to $f' : X' \to C$ is an arrow $a : X \to X'$ in $\mathbf{C}$ s.t. $f' \circ a = f$, i.e.

$$X \xrightarrow{a} X'$$
$$f \searrow \qquad \swarrow f'$$
$$C$$
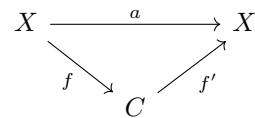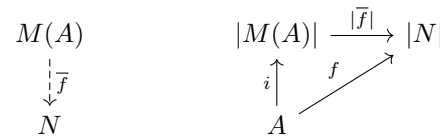
If $g : C \to D$ is any arrow then there exists a composition functor $g_* : \mathbf{C}/C \to \mathbf{C}/D$ defined as $g_*(f) = g \circ f$. Therefore the slice category of $\mathbf{C}$ with any of its objects is a functor from $\mathbf{C} \to \mathbf{C}\mathbf{at}$. This is called the *forgetful* functor as the base object is "forgotten".

Free monoid

For a set A a *word* over A is any finite sequence of its elements. The *Kleene closure* of A is defined to be the set of all words over A. Define the binary operation of concatenation on $A*$. Since it is associative, A along with $*$ with the empty word "$-$" is a monoid called the **free monoid** on the set A.

Universal mapping property (UMP) of free monoid: Let $M(A)$ be the free monoid on a set A. There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$M(A) \qquad\qquad |M(A)| \xrightarrow{|\overline{f}|} |N|$$
$$\downarrow{\overline{f}} \qquad\qquad\qquad i \uparrow \quad \nearrow f$$
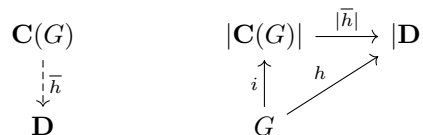$$N \qquad\qquad\qquad\qquad A$$

$A*$ has the UMP of the free monoid on A.

Free category

A directed graph G "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.

UMP of $\mathbf{C}(G)$ There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$\mathbf{C}(G) \qquad\qquad |\mathbf{C}(G)| \xrightarrow{|\overline{h}|} |\mathbf{D}|$$
$$\downarrow{\overline{h}} \qquad\qquad\qquad i \uparrow \quad \nearrow h$$
$$\mathbf{D} \qquad\qquad\qquad\qquad G$$

Small categories

A category is called *small* if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.

A category $\mathbf{C}$ is *locally small* if for all objects $X, Y \in \mathbf{C}$, the collection $\text{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

Types of morphisms

Monomorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$C \xrightarrow[h]{g} A \xrightarrow{f} B$$

Epimorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$A \xrightarrow{f} B \xrightarrow[i]{j} D$$

- We say, $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse r. The inverse arrow r is called the *retraction*, m is called a *section* of r and A is called a *retract* of B.

Initial and terminal objects

An object $0 \in \mathbf{C}$ is *initial* if for any object $C \in \mathbf{C}\exists!$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is *terminal* if for any object $C \in \mathbf{C}\exists!$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the *generalized elements* of A with stage of definition given by X.
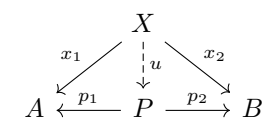
Product of objects

In any category $\mathbf{C}$, a product diagram for the objects A, B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$X$$
$$x_1 \swarrow \quad \downarrow u \quad \searrow x_2$$
$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

The product P is unique up to isomorphism.

Categories with products

A category which has a product for every pair of objects is said to have *binary products*.

A category is said to have *all finite products*, if it has a terminal object and all binary products.

A category has *all small products* if every set of objects has a product.

Covariant representable functor

The functor $\text{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).

For a category with products a covariant representable functor preserves products.

Duality

If any statement about categories holds for all categories then so does the dual statement.

Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$
\begin{array}{ccc}
 & Z & \\
{}^{z_1}\nearrow & \uparrow u \; \nwarrow & {}^{z_2} \\
A \xrightarrow[q_1]{} & Q \xleftarrow[q_2]{} & B
\end{array}
$$

Equalizers

In some category $\mathbf{C}$ given the following diagram

$$
A \underset{g}{\overset{f}{\rightrightarrows}} B
$$

We say an *equalizer* of f, g consists of an object E and an arrow $e : E \to A$ universal such that

$$
f \circ e = g \circ e
$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$
\begin{array}{ccc}
E \xrightarrow{e} & A \underset{g}{\overset{f}{\rightrightarrows}} B \\
u \uparrow \; {}^{z}\nearrow & \\
Z &
\end{array}
$$

- Equalizers are monic.

Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$
A \underset{g}{\overset{f}{\rightrightarrows}} B
$$

We say a *coequalizer* of f, g consists of an object Q and an arrow $q : B \to Q$ universal such that

$$
q \circ f = q \circ g
$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$
\begin{array}{ccc}
A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} & Q \\
 & {}_{z}\searrow & \downarrow u \\
 & & Z
\end{array}
$$

- Coequalizers are epic.