Introductory Complex Analysis Cheat Sheet

Field of Complex Numbers

We construct the field of complex numbers as the following quotient ring, $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$

Algebra of Complex Numbers

- Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$
- Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$
- Division: $\dfrac{a+ib}{c+id} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$
- Square root: $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\right)$
- $\Re(a+ib) = a, \Im(a+ib) = b$

Conjugation, Absolute Value

- **Complex conjugation:** $\overline{a+ib} = a-ib$
  - $\overline{a+b} = \bar{a}+\bar{b}$
  - $\overline{ab} = \bar{a}\cdot\bar{b}$

  Geometrically, conjugation is reflection over the real axis.
- **Absolute value:** $|a| = +\sqrt{a\bar{a}}$
  - $|ab| = |a|\cdot|b|$
  - $|a+b|^2 = |a|^2+|b|^2+2\Re(a\bar{b})$
  - $|a-b|^2 = |a|^2+|b|^2-2\Re(a\bar{b})$
  - $|a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$

  The absolute value function forms the metric on $\mathbb{C}$. $\mathbb{C}$ is complete under this metric.

Basic Topological definitions in $\mathbb{C}$

Some basic results:

- For $z_0 \in \mathbb{C}, r>0$ we denote the ball (i.e. disk) of radius r around z_0 to be $B(z_0,r) = \{z\in\mathbb{C} \mid |z-z_0|<r\}$
- A point $z\in\mathbb{C}$ is a **limit point** of $E\subseteq\mathbb{C}$ if $\forall\varepsilon>0$, $B(z,\varepsilon)\cap E$ contains a point other than z.
- A subset $E\subseteq\mathbb{C}$ is said to be **open** if $\forall z\in E, \exists\, r>0$, s.t. $B(z,r)\subset E$.
- A subset $E\subseteq\mathbb{C}$ is said to be **closed**, if $\mathbb{C}\backslash E$ is open in C. Or equivalently a set which contains all its limit points.

Some properties of open sets:

- $\mathbb{C}$ and $\emptyset$ are open subsets of $\mathbb{C}$.
- All finite intersections of open sets are open sets.
- The collection of all open sets on $\mathbb{C}$ form a topology on $\mathbb{C}$.

Interior, closure, density

- **Interior:** Let $E\subseteq\mathbb{C}$. The interior of E is defined as, E°=set of all interior points of E, or equivalently, $\cup\{\Omega \mid \Omega\subseteq E\wedge\Omega$ is open in $\mathbb{C}\}$
- **Closure:** Let $E\subseteq\mathbb{C}$. The closure of E is defined as $\hat{\{}F \mid E\subseteq F\wedge F$ is closed in $\mathbb{C}\}$
- **Density:** Let $E\subseteq D$, the closure of E in D is D. Then E is called dense in D.

Path : A path in a metric space from a point $x\in X$ to $y\in Y$ is a continuous mapping $\gamma:[0,1]\to X$ s.t. $\gamma(0)=x$ and $\gamma(1)=y$.

Separated and Connected

For a metric space (X,d).

- **Separated:** X is separated if $\exists$ disjoint non-empty open subsets A,B of X s.t. $X=A\cup B$.
- **Connected:**
  - X is connected if it has no separation.
  - X is connected $\iff$ X does not contain a proper subset of X which is both open and closed in X.
  - Continuous functions preserve connectedness.
  - An open subset $\Omega\in\mathbb{C}$ is connected $\iff$ for $z,w\in\Omega$, there exists a path from z to w.

Basic Topological definitions in C contd.

Open cover: Let (X,d) be a metric space and E be a collection of open sets in X. We say that $\mathscr{U}$ is an open cover of a subset $K\subseteq X$, if $K\subset\bigcup\{\mathscr{U}\mid\mathscr{U}\in E\}$

Compactness: For some $K\subseteq X$ is compact if for every open cover E of K, there exists $E_1,\cdots.E_n\in E$ s.t. $K\subset U_{i=1}^n E_n$, i.e. it is compact if it has a finite open cover.

- In a metric space, a compact set is closed.
- A closed subset of a compact set is closed.

Limit point compact: We say a metric space X is limit point compact if every infinite subset of X has a limit point.

- If X is a compact metric space, then it is also limit point compact.

Sequentially compact: We say a metric space X is sequentially compact if every sequence has a convergent sub-sequence.

- If X is limit point compact then X is sequentially compact.
- Let X be sequentially compact, then X is a compact metric space.

Lebesgue number lemma: Let X be sequentially compact, and let $\mathscr{U}$ be an open cover of X. Then $\exists\,\delta>0$ s.t. for $x\in X$, $\exists\, u\in\mathscr{U}$ s.t. $B(x,\delta)\subseteq u$.

Isometries on the Complex Plane

A function $f:\mathbb{C}\to\mathbb{C}$ is called an **isometry** if $|f(z)-f(w)| = |z-w|\ ,\forall z,w\in\mathbb{C}$.

- Let f be an isometry s.t. $f(0)=0$, then the inner product $\langle f(z),f(w)\rangle = \langle z,w\rangle\ ,\forall z,w\in\mathbb{C}$.
- If f is an isometry s.t. $f(0)=0$ then f is a linear map.
- The standard argument for $a+ib\in\mathbb{C}$, $\mathrm{Arg}(a+ib) = \tan^{-1}\frac{b}{a}$

Functions on the Complex Plane

Uniform convergence: Let $\Omega\subseteq\mathbb{C}$ and $f_1,\cdots,f_n:\Omega\to\mathbb{C}$ be a set of functions on Ω. We say, $\{f\}_{n\in\mathbb{N}}$ converges uniformly to f if given $\varepsilon>0,\exists n\in\mathbb{N}$ s.t. $|f_n(x)-f(x)|<\varepsilon,\forall x\in\Omega$ and $n\geq N$.

Complex exponential: For $z\in C$, $\exp(z) = \sum_{n=0}^\infty \frac{z^n}{n!}$

Trigonometric functions: For $z\in\mathbb{C}$, $\cos(x) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(x) = \frac{e^{iz}-e^{iz}}{2}$

Hyperbolic trigonometric functions: For $z\in\mathbb{C}$, $\cosh(x) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$

Complex differentiability

Complex derivative: Let $\Omega\subseteq\mathbb{C}$ and $f:\Omega\to\mathbb{C}$, we say that f is complex differentiable at a point $z_0\in\Omega$ if z_0 is an interior point and the following limit exists $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$. The limit is denoted as $f'(z_0)$ or $\frac{\mathrm{d}f(z)}{\mathrm{d}z}$.

Holomorphic functions: If $f:\Omega\to\mathbb{C}$ is complex differentiable at every point $z\in\Omega$, then f is said to be a holomorphic on Ω. **Entire function:** Functions which are complex differentiable on $\mathbb{C}$ are called entire functions.

- Complex differentiability implies continuity.
- Complex derivatives of a function are linear transformations.
- **Product rule:** If $f,g:\Omega\to\mathbb{C}$ are complex differentiable at $z_0\in\Omega$. Then fg is complex differentiable at z_0 with derivative $f'(z_0)g(z_0)+g'(z_0)f(z_0)$.
- **Quotient rule:** If $f,g:\Omega\to\mathbb{C}$ are complex differentiable at $z_0\in\Omega$, and g doesn't vanish at z_0. Then $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0)-g'(z_0)f(z_0)}{g(z_0)^2}$
- **Chain rule:** If $f:\Omega\to\mathbb{C}$ and $g:D\to\mathbb{C}$ are complex differentiable at $z_0\in\Omega$, and $f(\Omega)\subseteq D$. Then $g(f(x))'(z_0) = g'(f(z_0))f'(z_0)$

Power Series

Formal Power Series: A formal power series around $z_0\in\mathbb{C}$ is a formal expansion $\sum_{n=0}^\infty a_n(z-z_0)^n$, where $a_n\in\mathbb{C}$ and z is indeterminate.

Radius of convergence: For a formal power series $\sum a_n(z-z_0)^n$ the radius of convergence $R\in[0,\infty]$ given by $R = \liminf_{n\to\infty}|a_n|^{-1/n}$. Using the ratio test is identical i.e. $R = \liminf_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}$.

- The series converges absolutely when $z\in B(z_0,R)$, and for $r<R$, the series converges uniformly, else if $|z-z_0|>R$ the series diverges.
- Let $z\in\mathbb{C}$ s.t. $|z-z_0|>R$, then $\exists$ infinitely many $n\in N$ s.t. $|a_n|^{-1/n}<|z-z_0|$.

Abel's Theorem: Let $F(z) = \sum_{n=0}^\infty a_n(z-z_0)^n$ be a power series with a positive radius of convergence R, suppose $z_1 = z_0+Re^{i\theta}$ be a point s.t. $F(z_1)$ converges. Then $\lim_{r\to R^-}F(z_0+re^{i\theta}) = F(z_1)$

Differentiation of Power Series

Let $F(z) = \sum_{n=1}^\infty a_n(z-z_0)^n$ be a power series around z_0 with a radius of convergence R. Then F is **holomorphic** in $B(z_0,R)$.

- $F(x)' = \sum_{n=1}^\infty na_n(z-z_0)^{n-1}$ with same radius of convergence R.
- $a_n = \frac{F^n(z_0)}{n!}$

Cauchy product of two power series: For power series $F(z) = \sum a_n(z-z_0)^n$ and $G(z) = \sum a_n(z-z_0)^n$ with degree of convergence at least R. Then the Cauchy product $F(z)G(z) = \sum c_n(z-z_0)^n$ where $c_n = \sum_{k=0}^n a_kb_{n-k}$ also has degree of convergence at least R.

Cauchy-Riemann Differential Equations

For a complex function $f(z) = u(z)+iv(z)$,

$f'(x) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ or $f'(x) = -i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$

Therefore, we get the two **Cauchy-Riemann Differential equations**,

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$
- $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

A function is holomorphic **iff** it satisfies the Cauchy-Riemann equations.

Wirtinger derivatives:

- $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x}+\dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$
- $\dfrac{\partial}{\partial\bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x}-\dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$

If f is holomorphic at z_0 then, $\frac{\partial f}{\partial\bar{z}} = 0$ and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$

Harmonic Functions

Laplacian: Define $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Harmonic function: Let $u:\Omega\to\mathbb{R}$ be a twice differentiable function. We say that u is a harmonic function if $\Delta u = 0$

For any holomorphic function f, $\Re(f),\Im(f)$ are examples of harmonic functions, but there are harmonic functions which are not holomorphic.

Boundary of a set: For a metric space X, $\Omega\in X$, the boundary of $\Omega = \partial\Omega = \overline{\Omega}\cap\overline{\Omega^C}$

Maximum principle for harmonic functions: Let $u:\Omega\to\mathbb{R}$ be a twice differentiable harmonic function. Let $k\subset\Omega$ be a compact sub set of Ω. Then, $\sup_{z\in k}u(z) = \sup_{z\in\partial k}u(z)$ and $\inf_{z\in k}u(z) = \inf_{z\in\partial k}u(z)$

Maximum principle for holomorphic functions: Let $\Omega\subseteq\mathbb{C}$ be open and connected and let $f:\Omega\to\mathbb{C}$ be a holomorphic function. Then, for compact $k\subseteq\Omega$, we have, $\sup_{z\in k}|f(z)| = \sup_{\partial k}|f(z)|$

Harmonic conjugate: Let $u:\Omega\to\mathbb{R}$ be a twice differentiable harmonic function. We say that $v:\Omega\to\mathbb{R}$ is a harmonic conjugate of u if $f = u+iv$ is holomorphic.

- For a harmonic function from $\mathbb{C}$ to $\mathbb{R}$ there exists a uniquely determined harmonic conjugate from $\mathbb{C}$ to $\mathbb{R}$ (up to constants).

Riemann Sphere

Extended complex plane: $\widehat{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$
Consider S^2, associate every point $z = x + iy$ with a line L that connects to the point $P = (0, 0, 1)$. $L = (1 - t)z + tP$, where $t \in \mathbb{R}$.
The point at which L for some z touches S^2 is given as $\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$, associate P with ∞. This gives a stereographic projection of the complex plane unto S^2. This sphere is known as the Riemann sphere.

Möbius transformations

A map $S(z) = \dfrac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ is called a Möbius transformation if $ad - bc \neq 0$.
Every mobius transformation is holomorphic at $\mathbb{C} \setminus \{-d/c\}$, i.e. every point other than is zero.

- The set of all mobius transformations is a group under transposition.
- S forms a bijection with $\widehat{\mathbb{C}}$

Every mobius transformation can be written as composition of,

1. Translation: $S(z) = z + b, b \in \mathbb{C}$
2. Dilation: $S(z) = az, a \neq 0, a = e^{i\theta}$
3. Inversion: $S(z) = 1/z$

Curves in $\mathbb{C}$

A continuous parametrized curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$ for $a, b \in \mathbb{R}$.

- If $a = b$ the curve is trivial.
- $\gamma(a)$ is initial point and $\gamma(b)$ is terminal point.
- γ is said to be closed if $\gamma(a) = \gamma(b)$.
- γ is said to be simple if it is injective, i.e. doesn't "cross" itself.
- A curve $-\gamma$ is a reversal of γ if $\gamma : [-a, -b] \to \mathbb{C}$ and if $-\gamma(t) = \gamma(-t)$
- γ is said to be continuously differentiable if $\gamma'(t_0)$ (defined usually) exists and is continuous.

Reparametrization: We say a curve $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, if there exists a homeomorphism $\varphi : [a_1, b_1] \to [a_2, b_2]$ s.t.$\varphi(a_1) = a_2, \varphi(b_1) = b_2$ and $\gamma_2(\varphi(t)) = \gamma_1(t) \forall t \in [a_1, b_1]$.

- Reparametrization is an equivalence relation.

Arc length: Arc length of curve $\gamma = |\gamma| = \sup \sum_{i=0}^{n} |\gamma(x_{i+1} - \gamma(x_i))|$ for all partitions of $[a, b]$.

- A curve that has a finite arc length is called **rectifiable**.
- $|\gamma| = \displaystyle\int_a^b |\gamma'(t)|\, dt$

First Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $F : \Omega \to \mathbb{C}$ be called the anti-derivative of f, i.e. F is holomorphic in Ω and $F'(z) = f(z), \forall z \in \Omega$. For a rectifiable curve γ, $\int_\gamma f(z)dz = F(z_1) - F(z_0)$, where z_0 is the initial point and z_1 is the terminal point.

Second Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that $\int_\gamma f = 0$. Whenever γ is a closed polygonal path contained in Ω. For fixed $z_0 \in \Omega$, define a path γ_1 from z_0 to z_1 such that $F(z_1) = \int_{\gamma_1} f(z)\, dz$. Then F is a well defined holomorphic function s.t. $F'(z_1) = f(z_1)\ \forall z_1 \in \Omega$

Properties of complex integration

For continuously differentiable curves $\gamma : [a, b] \to \mathbb{C}$, and $\sigma : [b, c] \to \mathbb{C}$

- For a reparametrization $\widehat{\gamma}$ of γ we can say that $\int_\gamma f(z)\, dz = \int_{\widehat{\gamma}} f(z)\, dz$
- $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$
- $\int_{\gamma+\sigma} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\sigma f(z)\, dz$
- $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$
- If f is bounded by M then $\int_\gamma f(z)\, dz \leq M|\gamma|$
- For $c \in \mathbb{C}$, we have, $\int_\gamma (cf + g)(z)\, dz = c\int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$

Homotopy of curves

Consider two curves $\gamma_0, \gamma_1 \to \Omega$ with the same initial and end point $[a, b]$.
We say that γ_0 is homotopic to γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $H : [0, 1] \times [a, b] \to \Omega$ s.t. $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t),\ \forall t \in [a, b]$.
$H(s, a) = z_0, H(s, b) = z_1\ \forall s \in [0, 1]$
For **closed curves** γ_0 at z_0 and γ_1 at z_1, we say that γ_0 is homotopic to γ_1 as closed curves if there exists a continuous map $H : [0, 1] \times [a, b] \to \Omega$, s.t. $H(0, t) = \gamma_0(t), H(1, t) = \gamma_1(t),\ \forall t \in [a, b]$. And $H(s, a) = H(s, b),\ \forall s \in [0, 1]$.

- Homotopy is an equivalence relation.

Cauchy-Goursat Theorem

Cauchy-Goursat theorem: If a curve γ_0 is homotopic to a reparametrization of γ_1 then, the integral of some function $f : \Omega \to \mathbb{C}$ is homotopy invariant, i.e., $\displaystyle\int_{\gamma_0} f = \int_{\gamma_1} f$
Alternative statement: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, and $\gamma_0 : [a, b] \to \Omega$ is a rectifiable curve which is null-homotopic (i.e. homotopic to a constant map). Then, $\displaystyle\int_{\gamma_0} f(z)\, dz = 0$

Cauchy's theorem for convex domains

Let $\Omega \subseteq \mathbb{C}$ be a convex and open set and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then f has an anti derivative F on Ω, and if γ is a closed rectifiable curve on Ω then $\int_\gamma f = 0$.

Cauchy's integral formula

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Fix $z_0 \in \Omega$ and let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$. Suppose γ is a closed curve in $\Omega \backslash \{z_0\}$ s.t. γ is homotopic to a reparametrization to γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz$$

Complex analytic function

An alternative statement, we say $f : \Omega \to \mathbb{C}$ is complex analytical if given $z_0 \in \Omega, \exists\, B(z_0, r) \subseteq \Omega$ s.t. the formal power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to f.
Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Suppose for $z_0 \in \Omega, \overline{B(z_0, r)} \subset \Omega$, then for every $n \in \mathbb{N}$, let $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ where $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$.
Then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to $f(z)$.
Corollary: If $f : \Omega \to \mathbb{C}$ is holomorphic then f' is also holomorphic. Therefore f is infinitely differentiable.

Factor theorem for analytic function

For a analytic function $f : \Omega \to \mathbb{C}$ s.t. $f(z_0) = 0$ at $z_0 \in \Omega, \exists$ a unique analytic function $g : \Omega \to \mathbb{C}$ s.t. $f(z) = (z - z_0)g(z)$

Principle of analytical continuation

- Let Ω be open and connected subset of $\mathbb{C}$. and $f, g : \Omega \to \mathbb{C}$ be analytic functions on Ω. Suppose f, g agree on a non-empty subset of Ω, and this subset has an accumulation point. Then $f \equiv g$ on Ω.
- A consequence to this is that, non-trivial holomorphic functions have isolated zeros.

Higher-order Cauchy integral formula

Let $f : \Omega \to \mathbb{C}$ be analytic on Ω and $z_0 \in \Omega$ with $\overline{B(z_0, r)} \subseteq \Omega$. Let γ be a closed curve in $\Omega \setminus \{z_0\}$ that is homotopic to a reparametrization of γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Cauchy estimates: If $|f(z)| \leq M\ \forall z \in \gamma([0, 2\pi])$ then, $\forall n \in \mathbb{N}$, then we have $|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

Liouville's Theorem

Let f be a entire function which is bounded. Then f is a constant function.

Fundamental Theorem of Algebra

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a non constant polynomial s.t. $a_i \in \mathbb{C}, a_n \neq 0$. Then $\exists z_1, z_2, \ldots, z_n$ s.t. $p(z) = a_n(z - z_1)\ldots(z - z_n)$.

Morera's Theorem

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that, $\int_\gamma f(z)\, dz = 0, \forall$ closed polygonal paths $\gamma \in \Omega$. Then f is holomorphic on Ω.

Uniform limit of holomorphic functions

Let $f_n : \Omega \to \mathbb{C}$ be a holomorphic on $\Omega, \forall n \in \mathbb{N}$ s.t. f_n converges uniformly on compact sets to f. Then f is holomorphic.

Winding number

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve and let z_0 be a point not in the image of γ. Then the winding number of γ around z_0 is

$$W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$$

- Winding number is invariant over homotopy.
- Let z_0 be a point not in the image of γ then $\exists r > 0$ s.t. for $z \in B(z_0, r), W_\gamma(z_0) = W_\gamma(z)$
- The winding number is always an integer.
- The winding number is locally constant.

Generalized Cauchy Integral formula: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and $\gamma : [a, b] \to \Omega$ be a closed curve which is null homotopic. Then for z_0 not in the image of γ,

$$f(z_0)W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)}\, dz$$

Open Mapping Theorem

- $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then $G : \Omega \times \Omega \to \mathbb{C}$ given by

$$G(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{when } z \neq w \\ f'(z) & \text{when } z = w \end{cases}$$

  then G is continuous.
- Let $f : \Omega \to \mathbb{C}$ be holomorphic on some open set. Suppose $z_0 \in \Omega$ s.t. $f'(z_0) \neq 0$. Then $\exists$ a neighbourhood U of $z_0 \in \Omega$ s.t. f restricted to U is injective. And $V = f(U)$ is an open set and the inverse $g : V \to U$ of f is holomorphic.
- Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open, connected set Ω. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Then $\exists$ a neighbourhood U of z_0 and bijective holomorphic function φ on U s.t. $f(z) = w_0 + (\varphi(z))^m$ for $z \in U$ and some integer $m > 0$. And φ maps U unto $B(0,r)$ for some $r > 0$.

Open Mapping Theorem: Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open connected set Ω, then $f(\Omega)$ is an open set.

Schwarz reflection principle

Let Ω be a open connected set which is symmetric w.r.t $\mathbb{R}$. Then define the following,

- $\Omega_+ = \{z \in \Omega \mid \Im(z) > 0\}$
- $\Omega_- = \{z \in \Omega \mid \Im(z) < 0\}$
- $I = \{z \in \Omega \mid \Im(z) = 0\}$

Schwarz reflection principle: Let Ω be defined as above. Then if $f : \Omega_+ \bigcup I \to \mathbb{C}$ which is continuous on $\Omega_+ \bigcup I$ and holomorphic on Ω_+. Suppose for $f(x) \in \mathbb{R}$, $\forall x \in I$ then there exists $g : \Omega \to \mathbb{C}$ holomorphic on Ω s.t. $g(z) = f(z)$ for $z \in \Omega_+ \bigcup I$

Singularity of a holomorphic function

- **Isolated singularity:** If f is holomorphic on $B(z_0, R) \setminus \{z_0\}$ for some $R > 0$ then z_0 is called an isolated singularity.
- **Removable singularity:** Let z_0 be an isolated singularity of a holomorphic function f as defined above. It is called removable if there exists holomorphic function g on $B(z_0, R)$ s.t. $g(z) = f(z)$ on $B(z_0, R) \setminus \{z_0\}$.
- **Riemann removable singularity theorem:** Let z_0 be an isolated singularity of a function f, then z_0 is a removable singularity if and only if f is locally bounded around z_0.
- **Pole:** If z_0 is an isolated singularity as defined above and if $\lim_{z \to z_0} |f(z)| = \infty$ then z_0 is called a pole of f.
- **Essential singularity:** A singularity that is neither removable nor a pole.

Doubly infinite series

Let z_n be a function defined for $n =, 0, \pm 1, \pm 2, \cdots$, then it is doubly infinite.

- A doubly infinite series converges if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{=n}$ both converge.
- Splitting up the series in similar manners you can define absolute and uniform convergence.

Annulus

An annulus $A(z_0, R_1, R_2)$ around a point z_0 for $0 \leq R_1 \leq R_2$ is the set of all $z \in \mathbb{C}$ s.t. $R_1 \leq |z - z_0| \leq R_2$.

Laurent series expansion

Let f be a function holomorphic on $A(z_0, R_1, R_2)$, then there exists $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ s.t.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the doubly infinite series converges absolutely and uniformly in some $A(z_0, r_1, r_2)$ when $R_1 < r_1 < r_2 < R_2$.

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

where $\gamma(z) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r < R_2$.

Important results

- f has a removable singularity at $z_0 \iff a_n = 0$ for $n < 0$ in the Laurent series expansion of f
- f has a pole at z_0 of order $m \iff a_n = 0$ for $n < -m$ in the Laurent series expansion of f.
- f has a essential singularity at $z_0 \iff a_n \neq 0$ for infinitely many negative integers n.

Casorati–Weierstrass theorem

Let z_0 be an essential singularity of f then given $\alpha \in \mathbb{C}$, there exists a sequence $z_n \in B(z_0, R) \setminus \{z_0\}$ s.t. $z_n \to z_0$ and $f(z_n) \to \alpha$.

- Alternatively, f approaches any given value arbitrarily closely in any neighborhood of an essential singularity.

Meromorphic functions

Let Ω be a open connected subset of $\mathbb{C}$ and let $S \subset \Omega$. Let $f : \Omega \setminus S \to \mathbb{C}$ be holomorphic on Ω. We say that f is a meromorphic function on Ω if,

- S is a discrete set.
- f either has removable singularities or poles at point of S.

Operations on meromorphic functions

Let $\mathcal{M}(\Omega)$ denote the equivalence classes of meromorphic functions over Ω.

- We say that two meromorphic functions $f : \Omega \setminus S_1$ and $g : \Omega \setminus S_2$ are equivalent if $f(z) = g(z)$ on $\Omega \setminus (S_1 \bigcup S_2)$.
- For $f, g \in \mathcal{M}(\Omega)$, define $f + g$ to be the equivalence class of $(f + g) : \Omega \setminus (S_1 \bigcup S_2)$
- Similarly, fg is the equivalence class of $fg : \Omega \setminus (S_1 \bigcup S_2)$.

The space of all meromorphic functions is a field.

Order of meromorphic functions

The order of a meromorphic function is defined as follows,

- If $z_0 \in S$ is a removable singularity then the order of f at z_0 is the order of the zero at z_0 of f, i.e., $f(z) = (z - z_0)^m g(z)$ then m is the order.
- If $z_0 \in S$ is a pole and the pole is of order m then order of f at z_0 is $-m$.
- If $f \equiv 0$ then $\text{Ord}_{z_0} = \infty$.
- $\text{Ord}_{z_0}(f + g) \geq \min(\text{Ord}_{z_0}(f), \text{Ord}_{z_0}(g))$
- $\text{Ord}_{z_0}(fg) = \text{Ord}_{z_0}(f) + \text{Ord}_{z_0}(g)$

Residue of a function

Residue of a function: Let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function, where Ω is an open set and S is a discrete subset of Ω. Then for $z_0 \in S$, let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$ and $B(z_0, r) = \{z_0\}$. Then in $B(z_0, r) \setminus \{z_0\}$, consider the Laurent series expansion of f given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. We define the residue of f at z_0 to be $\text{Res}(f, z_0) = a_{-1}$.

1. If z_0 is a removable singularity then $\text{Res}(z_0) = 0$.
2. If z_0 is a pole of order m then $(z - z_0)^m f(z) = g(z)$, where $g(z) \neq 0$ on $B(z_0, r) \setminus \{z_0\}$then, $\text{Res}(z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$.

Residue theorem

Let Ω be an open connected subset of $\mathbb{C}$ and S be a finite subset of Ω and let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function. Let γ be a null homotopic closed curve on Ω. Then,

$$\frac{1}{2\pi i} \int_\gamma f(z) \, dz = \sum_{j=1}^{k} W_\gamma(z_j) \text{Res}(f, z_j)$$

where $S = \{z_1, \cdots, z_k\}$ and W_γ is the winding number.

Log derivative

For a holomorphic function $f : \Omega \to \mathbb{C}$. Define the log derivative of f to be the meromorphic function $\frac{f'(z)}{f(z)}$.

1. $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$
2. $\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}$
3. When f has a pole of order m at z_0 then for $f(z) = \frac{g(z)}{(z - z_0)^m}$ the log derivative of f is $\frac{g'(z)}{g(z)} - \frac{m}{(z - z_0)}$

Argument principle

Let $f : \Omega \setminus S \to \mathbb{C}$ be a meromorphic function s.t. f has zeros of order $d_1, \ldots, d_n$ at $z_1, \ldots z_n$ after removing the removable singularities. And f has poles of order $e_1, \ldots, e_m$ at points $w_1, \ldots, w_m$. Let γ be a closed curve which is null homotopic in Ω s.t. the zeros and poles don't lie in the image of γ. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{i=0}^{n} d_i W_\gamma(z_i) - \sum_{j=1}^{m} e_j W_\gamma(w_j)$$

Rouche's theorem

Let γ be a closed curve which is null homotopic in Ω. Let f, g be functions holomorphic in Ω and $|g(z)| < |f(z)|$ on γ then f and $f + g$ have the same number of zeros counting multiplicities on the interior of $H([0,1] \times [a,b])$ where H is the null homotopy from γ to a constant path.

Branch of the complex logarithm

Let Ω be an open connected subset of $\mathbb{C} \setminus \{0\}$. Define a branch of the logarithm on Ω as a function $f : \Omega \to \mathbb{C}$ s.t. $\exp(f(z)) = z, \forall z \in \Omega$.
For $\Omega = \mathbb{C} \setminus \{\Re(x) \leq 0\}$ define the standard branch to be

$$\text{Log}(z) = \ln |z| + i\text{Arg}(z)$$

As defined above $\text{Log}(z)$ is holomorphic on Ω.

Schwarz lemma

Let $\mathbb{D}$ denote the open unit disc. Let $f:\mathbb{D}\to\mathbb{D}$ be a holomoprhic function s.t. $f(0)=0$. Then,

$$|f(z)|\leq |z|, \forall z\in\mathbb{D}, \text{ and } |f'(z)|\leq 1$$

Also, if $|f(z)|=|z|$ for some $z\in\mathbb{D}$ or if $|f'(0)|=1$ then $\exists\lambda\in\mathbb{C}, |\lambda|=1$ s.t. $f(z)=\lambda z$.

Automorphism

A function $f:\Omega\to\Omega$ is an automorphism if f is holomorphic and has a holomorphic inverse.

Automorphisms of the unit disc

Define a function $\varphi_\alpha:\mathbb{D}\to\mathbb{C}$ defined as $\varphi_\alpha(z)=\frac{z-\alpha}{1-\overline{\alpha}z}$.
Let $f:\mathbb{D}\to\mathbb{D}$ be an automorphism. Then there exists $\alpha\in\mathbb{D}$ and $\lambda\in\partial\mathbb{D}$ s.t.

$$f(z)=\lambda\varphi_\alpha(z)$$

Phragmén–Lindelöf method

Let $\Omega=\{z\in\Omega : a<\Re(z)<b\}$. Let $f:\overline{\Omega}\to\mathbb{C}$, s.t. f is continuous on $\overline{\Omega}$ and holomorphic on Ω. Suppose for some $z=x+iy$, we have $|f(z)|<B$ and let $M(x)=\sup\{|f(x+iy)| : -\infty<y<\infty\}$. Then,

$$M(x)^{b-a}\leq M(a)^{b-x}M(b)^{x-a}$$

And further

$$|f(z)|\leq M(x)\leq \max\{M(a),M(b)\}=\sup_{z\in\partial\Omega}|f(z)|$$

Schwarz-Pick theorem

First define $\rho(z,w)=\left|\frac{z-w}{1-\overline{w}z}\right|$ for $z,w\in\mathbb{D}$. Let $f:\mathbb{D}\to\mathbb{D}$ be holomorphic. Then,

$$\rho(f(z),f(w))\leq\rho(z,w)\ \forall z,w\in\mathbb{D}$$

and,

$$\frac{|f'(z)|}{1-|f(z)|^2}\leq\frac{1}{1-|z|^2}\ \forall z\in\mathbb{D}$$

Lifting of maps

Let X,Y,Z be open subsets of $\mathbb{C}$ and let $f:Y\to X$ and $g:Z\to X$ be continuous maps. Then we say, a map $\widetilde{g}:Z\to Y$ is a lift of g w.r.t. f if $f\circ\widetilde{g}=g$.

Uniqueness of lifts: Let X,Y,Z be open connected subsets of $\mathbb{C}$ and let $f:Y\to X$ be a *local* homeomorphism. Let $g:Z\to X$ be a continuous map. Let $\widetilde{g_1}$ and $\widetilde{g_2}$ be lifts of g w.r.t. f and suppose they are equal at some point in Z. Then $\widetilde{g_1}\equiv\widetilde{g_2}$.

- Let $f:Y\to X$ be a holomorphic map s.t. $f'(y)\neq 0$ on Y. Let $g:Z\to X$ be a holomorphic map s.t. $\widetilde{g}:Z\to Y$ is a lift of g w.r.t. f. Then $\widetilde{g}$ is holomorphic.
- Let X,Y be open subsets of $\mathbb{C}$ let, $f:Y\to X$ be a local homeomorphism. Let γ_0,γ_1 be curves in X from z_1 to z_2 which are homotopic. Suppose that for every $s\in[0,1]$, we can lift $\gamma_s(t)=H(s,t)$ to a path $\widetilde{\gamma_s}:[a,b]\to Y$ w.r.t. f s.t. $\widetilde{\gamma_s}(a)=\widetilde{z_1}$, $\forall s\in[0,1]$. Then $\widetilde{\gamma_0},\widetilde{\gamma_1}$ are homotopic in Y.

Covering spaces

Let X,Y be open subsets of $\mathbb{C}$. We say that a continuous map $f:Y\to X$ is a covering map if given $x\in X$ there exists a neighbourhood U of X and open sets $\{V_\alpha\}_{\alpha\in A}$ in Y s.t. $f^{-1}(U)=\coprod_{\alpha\in A}V_\alpha$ (disjoint union of V_α) and $f|_{V_\alpha}:V_\alpha\to U$ is a homeomorphism. Then Y is called a cover of X.

- Let $f:Y\to X$ be a covering map and $\gamma[a,b]\to X$ be a curve from x_0 to x_1 in X. Suppose $y_0\in f^{-1}(\{x_0\})$. Then there exists a unique lift $\widetilde{\gamma}[a,b]\to Y$ of γ w.r.t. f s.t. $\widetilde{\gamma}(a)=y_0$.
- For connected X let $f:Y\to X$ be a covering map. Suppose $x_0,x_1\in X$. Then the cardinality of $f^{-1}(x_0)$ is the same as the cardinality of $f^{-1}(x)$.
- For open subsets X,Y of $\mathbb{C}$ let, $f:Y\to X$ be a covering map from Y to X. Let Z be an open connected subset of $\mathbb{C}$, which is simply connected and locally connected. Suppose $g:Z\to X$ is a continuous map. Then given $z_0\in C$ and $y_0\in Y$ s.t. $g(z_0)=f(y_0)$, then there exists a unique lift $\widetilde{g}:Z\to Y$ of g w.r.t f.
- Let Ω be a simply connected, locally connected, open connected subset of $\mathbb{C}$ and $g:\Omega\to\mathbb{C}^*$ be a holomorphic map. Then there exists a lift $\widetilde{g}:\Omega\to\mathbb{C}$ s.t. $\exp(\widetilde{g})=g$.

Bloch's theorem

- For $f:\mathbb{D}\to\mathbb{C}$ s.t. $f(0)=0, f'(0)=1$ and $|f(z)|\leq M\ \forall z\in\mathbb{D}$. Then $B(0,\frac{1}{6M})\subseteq f(\mathbb{D})$.
- Let $f:B(0,R)\to\mathbb{C}$ be holomorphic s.t. $f(0)=0, f'(0)=\mu$ for some $\mu>0$ and $f|(z)|\leq M\ \forall z\in B(0,R)$. Then, $B(0,\frac{R^2\mu^2}{6M})\subseteq f(B(0,R))$.

Bloch's theorem: Let Ω be an open connected subset of $\mathbb{C}$ s.t. $\overline{\mathbb{D}}\subset\Omega$. Let$f:\Omega\to\mathbb{C}$ s.t. $f(0)=0, f'(0)=1$. Then there exists a ball B' contained in $\mathbb{D}$ s.t. $f|_{B'}$ is injective and $B(0,\frac{1}{72})\subseteq f(B')\subseteq f(\mathbb{D})$.

Little Picard's theorem

- Let Ω be an open connected subset of $\mathbb{C}$ which is simply connected. Let $f:\Omega\to\mathbb{C}$ which omits 0 and 1. Then there exists a holomorphic function $g:\Omega\to\mathbb{C}$ s.t. $f(z)=-\exp(\pi i\cosh(2g(z)))$
- The function g as defined above doesn't contain any disk of radius 1.

Little Picard's theorem: If f is an entire function which omits two points, then f is a constant function.