Algebraic Topology Cheat Sheet

Topology

A topology on a set X is the tuple (X, T) where T is a collection of (open) subsets of X such that

- $\emptyset \in T, X \in T$.
- Closed under unions.
- Closed under finite intersections.

Constructions on topologies

For a topological space (X, T).
For a subset Y of X there exists a **subspace topology** naturally endowed upon Y characterized by the following UMP, continuous mappings from another topology Z to X factors through Y with the inclusion mapping, i.e. g is continuous iff f is,

$$\begin{tikzcd} & X \\ Z \arrow[ur, "g=f\circ i"] \arrow[r, dashed, "f"] & Y \arrow[u, "i"'] \end{tikzcd}$$

The **quotient topology** of X on some set S given a surjection $\pi : X \to X$ is characterized by the following UMP, continuous mappings from $X \to Z$ factor through S,

$$\begin{tikzcd} & X \arrow[dl, "g=\pi\circ f"'] \arrow[d, "\pi"] \\ Z & S \arrow[l, dashed, "f"'] \end{tikzcd}$$

For a family of topologies on X_a the product topology $X = \prod_a X_a$ is characterized by the following UMP, continuous mappings from a topology $Z \to X_a$ factor through X for all a,

$$\begin{tikzcd} & X_a \\ Z \arrow[ur, "g=f\circ\pi_a"] \arrow[r, dotted, "f"'] & X \arrow[u, "\pi_a"] \end{tikzcd}$$

The coproduct topology (i.e. the disjoint unions) is characterized similarly simply as the dual of the above.

Deformation retraction

A **deformation retract** of a space onto a subspace is a family of continuous maps from identity to the subspace and every map restricted to the subspace is identity. In particular for a topology X and subspace A it is a family of maps $f_t(x), t \in [0, 1]$ continuous in x such that $f_0 = \text{Id}$ and $f_1(X) = A, f_t|_A = \text{Id}$

Homotopy

A **homotopy** is a generalization of deformation retractions. It is defined to be a family of maps between arbitrary topologies X, Y, $f_t : X \to Y$ for $t \in [0, 1]$ such that $f_0 = X, f_1 = Y$ and the map $H : X \times [0, 1] \to Y, H(x, t) = f_t(x)$ is continuous.
Topologies are said to be **homotopically equivalent** if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.
Homeomorphism $\implies$ homotopy equivalent but the converse need not be true.
A topology is **contractible** if its identity map is homotopic to a constant function (i.e. **null homotopic**)

CW Complex

A CW complex is a space $X = \cup_n X^n$ where each X^n is inductively built with the following process,

- X_0 is an discrete set.
- X_n is constructed from X_{n-1} by 'attaching' $n-$cells e^n_α (spaces homeomorphic to $n-$dimensional discs D^n) this attachment is via continuous maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$ (note that S^{n-1} is just ∂D^k), in particular this is just a quotient map where the points on the boundaries are being identified with points in X_{n-1}

To labour the point, X_n is given as the pushout of the corner consisting of the inclusion mapping from the disjoint union of spheres into disks and φ_α

Constructions on CW complexes