Measure Theory Cheat Sheet

Topology

A collection T os subsets of a set X is said to be a **topology** in X if T satisfies the following properties,

- $\emptyset \in T$ and $X \in T$
- Closed under finite intersections
- Closed under arbitrary unions

Members of T are called open sets.
If X, Y are topological spaces then $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for all open sets $V \in Y$.

σ-algebra

A collection F of subsets of X is called a $\sigma-$algebra if the following properties hold,

- $X \in F$
- If $A \in F$ then $A^C = A - X \in F$
- Closed under unions

Measureability

- If F is a $\sigma-$algebra of X then X is a **measurable space** and members of F are **measurable sets** in X.
- If X is a measurable space and Y is a topological space, then $f : X \to Y$ is said to be **measurable** if $f^(-1)(V)$ is a measurable set in X for all open sets V in Y.

Characteristic function: It is a measurable function defined as follows .If E is a measurable set in X define $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$

Borel $\sigma-$algebra

Generated $\sigma-$algebra: For any collection of subsets F of X there exists a smallest $\sigma-$algebra which contains F. It is the intersection of all $\sigma-$algebras containing F.
Borel $\sigma-$ algebra: For a topological space X the $\sigma-$algebra generated by the family of open sets of X. Elements of a Borel $\sigma-$algebra are called Borel sets.
Borel mapping: A map between two topological spaces $f : X \to Y$if the inverse image of an open set in Y is an element of the Borel $\sigma-$algebra of X.

- If $f : X \to [-\infty, \infty]$ and F is a $\sigma-$algebra of X, then f is measurable if $f^{(-1)}((a, \infty)) \in F$ for all a.

Pointwise convergence and measurability

- If $f_n : X \to [-\infty, \infty]$ is measurable for all $n \in \mathbb{N}$ then $\sup, \inf, \limsup, \liminf$ of f_n are also measurable.
- o the limit of every pointwise convergent sequence of measurable functions is measurable.
- If f is measurable then so is $f^+ = \max\{f, 0\}, f^{-1} = -\min\{f, 0\}$

Simple functions

A complex function whose range consists of only finitely many points. If $\alpha_1 \ldots, \alpha_n$ are the distinct values of the simple function s and $A_i = x : s(x) = \alpha_i$ then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

- Every measurable function $f : X \to [0, \infty]$ can be written as a pointwise limit of a sequence of simple functions.

Positive measure

A **positive measure** μ is a measure along with the following additional properties,

- Its range is in $[0, \infty]$
- Countable additivity: $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

A *measure* space refers to a measurable space with a positive measure.

Lebesgue integral

Monotone convergence theorem

Fatou's lemma

Dominated convergence theorem

Zero measure

Caratheodory's extension theorem

Riesz representation theorem

Borel measures

Lebesgue measure

Convexity

Jensen's inequality

Hölder's inequality

Minkowski's inequality

L^p norms

Total variation

Complex measure

Absolutely continuity and mutually singular

Lebesgue-Radon-Nikodym theorem

Product measures