Measure Theory Cheat Sheet

Topology

A collection T os subsets of a set X is said to be a **topology** in X if T satisfies the following properties,

- $\emptyset \in T$ and $X \in T$
- Closed under finite intersections
- Closed under arbitrary unions

Members of T are called open sets.
If X, Y are topological spaces then $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for all open sets $V \in Y$.

σ-algebra

A collection F of subsets of X is called a $\sigma-$algebra if the following properties hold

- $X \in F$
- If $A \in F$ then $A^C = X - A \in F$
- Closed under unions

Measureability

- If F is a $\sigma-$algebra of X then X is a **measurable space** and members of F are **measurable sets** in X.
- If X is a measurable space and Y is a topological space, then $f : X \to Y$ is said to be **measurable** if $f^{-1}(V)$ is a measurable set in X for all open sets V in Y.

Characteristic function: It is a measurable function defined as follows .If E is a measurable set in X define $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$

Borel $\sigma-$algebra

Generated $\sigma-$algebra: For any collection of subsets F of X there exists a smallest $\sigma-$algebra which contains F. It is the intersection of all $\sigma-$algebras containing F. Denote it as $\sigma(F)$.
Borel $\sigma-$ algebra: For a topological space X the $\sigma-$algebra generated by the family of open sets of X. Elements of a Borel $\sigma-$algebra are called Borel sets.
Borel mapping: A map between two topological spaces $f : X \to Y$if the inverse image of an open set in Y is an element of the Borel $\sigma-$algebra of X.

- If $f : X \to [-\infty, \infty]$ and F is a $\sigma-$algebra of X, then f is measurable if $f^{-1}((a, \infty)) \in F$ for all a.

Pointwise convergence and measurability

- If $f_n : X \to [-\infty, \infty]$ is measurable for all $n \in \mathbb{N}$ then $\sup$, $\inf$, $\limsup$, $\liminf$ of f_n are also measurable.
- The limit of every pointwise convergent sequence of measurable functions is measurable.
- If f is measurable then so is $f^+ = \max\{f, 0\}$ and , $f^{-1} = -\min\{f, 0\}$

Simple functions

A complex function whose range consists of only finitely many points. If $\alpha_1 \ldots, \alpha_n$ are the distinct values of the simple function s and $A_i = \{x : s(x) = \alpha_i\}$ then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

- Every measurable function $f : X \to [0, \infty]$ can be written as a pointwise limit of a sequence of simple functions.

Positive measure

A **positive measure** μ is a measure along with the following additional properties,

- Its range is in $[0, \infty]$
- Countable additivity: $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

A *measure* space refers to a measurable space with a positive measure.

Arithmetic in $[0, \infty]$

We understand $a + \infty = \infty$ for $0 \le a \le \infty$ and $a \cdot \infty = \infty$ if $0 < a \le \infty$ else 0.

Lebesgue integral

If X is a set with $\sigma-$algebra F and positive measure μ. Then for a measurable simple function $s : X \to [0, \infty]$ as defined previously, its Lebesgue integral over $E \in F$ is defined as follows

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu\left(A_i \bigcap E\right)$$

Lebesgue integrable functions

Define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which $\int_X |f| \, d\mu < \infty$ known as the Lebesgue integrable functions.
For functions f with range in $[-\infty, \infty]$ define $\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$

Zero measure

We say a property holds "almost everywhere (a.e.)" if it holds everywhere except on a set of measure zero.
If any two function $f = g$ a.e. then their Lebesgue integrals are the same. Set of measure zero don't impact the value of the Lebesgue integral

Monotone convergence theorem

Let $\{f_n\}$ be a sequence of measurable functions on X if $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ a.e. and $f_n(x) \to f(x)$ as $n \to \infty$ a.e., then f is measurable and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu$$

Consequences of MCT

- Applying MCT to sequence of partial sums of a convergent series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ where $f_n : X \to [0, \infty]$ measurable for all n we get,

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

- If $f : X \to [0, \infty]$ is measurable with $\sigma-$algebra F of X and $\phi(E) = \int_E f \, d\mu$ for $E \subset F$ then, ϕ is a measure on F and

$$\int_X g \, d\phi = \int_X gf \, d\mu$$

for every measurable $g : X \to [0, \infty]$.

Fatou's lemma

If $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$ then,

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Dominated convergence theorem

If $\{f_n\}$ us a sequence of complex measurable functions on X with $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise. If there exists a function $g \in L^1(\mu)$ such that $|f_n(x)| \le g(x)$ for all $n \in N$ and $x \in X$ then $f \in L(1)(\mu)$ and

$$\lim_{n\to\infty} \int_X f \, d\mu = \int_X \lim_{n\to\infty} f_n(x) \, d\mu$$

Complete measure

A measure is called complete if all subsets of sets of measure 0 are measurable. Every measure can be completed.

Compact support

A function has compact support if it is zero outside of a compact set, i.e. $f \in C_c(X)$ if f has compact support on X.

Locally compact Hausdorff spaces

A topological space X is called locally compact if every point $x \in X$ has a compact neighbourhood.
Uryshon's lemma: If X is a locally compact Hausdorff space. Let $K \subseteq X$ is compact and U open s.t. $K \subseteq U \subset X$, there exists $f \in C_c(X)$ with $0 \le f \le 1$ such that $f_K \equiv 1$ and $f \equiv 0$ otherwise.

Riesz representation theorem

Let X is a locally compact Hausdorff space and T is a positive linear functional on $C_c(X)$. Then there exists a $\sigma-$algebra F of X which contains all Borel sets in X and there exists a unique positive measure μ on F such that for every $f \in C_c(X)$

$$Tf = \int_X f \, d\mu$$

additionally the following properties hold,

1. $\mu(K) < \infty$ for every compact $K \subset X$
2. $\mu(E) = \inf\{\mu(V) : E \subset V, V\text{open}\}$ for all $E \in F$.
3. $\mu(E) = \sup\{\mu(K) : K \subset E, K\text{compact}\}$ holds for every open set $E \subset F$ with $\mu(E) < \infty$
4. If $E \in F$, $A \subset E$ and $\mu(E) = 0$ then $A \in F$.

$\sigma-$finite measure

A set in a measure space is said to have a $\sigma-$finite measure if it is the countable union of sets of finite measure.
A set in a topological space is said to be $\sigma-$compact if it is the countable union of compact sets.

Regular Borel measures

A measure μ defined on the $\sigma-$algebra of all Borel sets in a locally compact Hausdorff space X.
If μ is positive we also say a Borel set is,

- **Outer regular** if satisfies property 3 of the above theorem.
- **Inner regular** if it satisfies property 4 of the above theorem.
- **Regular** if it is both inner and outer regular.

In a locally compact $\sigma-$compact Hausdorff space X. If there exists a positive Borel measure μ defined on X such that for $K \subseteq X$ compact $\mu(K) < \infty \implies \mu$ is regular.

Semiring of sets

A family of subsets S of a set X is called a semiring of sets if it satisfies the following properties,

- $\emptyset \in S$
- $A, B \in S \implies A \bigcap B \in S$
- There exists finite $K_i \in S$ pairwise disjoint s.t. $A \setminus B = \bigcup_{i=1}^{n} K_i$

Premeasure

A function μ from a semiring of sets S to $[0, \infty]$ is called a premeasure if

- $\mu(\emptyset) = 0$
- $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ and $\bigcup_{i=1}^{\infty} A_i \in S$

Carathéodory extension theorem

For a set X, semiring S of X and a pre-measure $\mu : S \to [0, \infty]$ there exists a unique extension to a measure $\tilde{\mu} : \sigma(A) \to [0, \infty]$.

Existence and Uniqueness of Lebesgue measure

For the semiring $S = \{[a, b) : a, b \in \mathbb{R}, a \leq b\}$ say $[a, b) = I$ with $\mu([a, b)) = b - a = \ell(I)$ we have (by the previous theorem) a unique extension $\tilde{\mu}$ on the Borel sets of $\mathbb{R}$ this measure is the Lebesgue measure.

Lebesgue outer measure

The Lebesgue outer measure for an arbitrary subset $E \subseteq \mathbb{R}$ is defined as $\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \ell(I_k)\}$ over a countable family of open intervals $\{I_k\}$ that cover A.

Carathéodory's criterion

$A \subseteq \mathbb{R}$ is Lebesgue measurable iff $\mu^*(B) = \mu^*(B \bigcap A) + \mu^*(B \setminus A)$ for all $B \in \mathbb{R}$.

Lebesgue measure

The Lebesgue measure μ for $\mathbb{R}$ is defined on the sigma algebra of the sets satisfying Carathéodory's criterion as $\mu(A) = \mu^*(A)$.

- Every Borel set is Lebesgue measurable, but the converse need not be true.
- μ is translation invariant, $\mu(A + x) = \mu(A)$ for every Lebesgue measurable A and every $x \in \mathbb{R}$

Product measures

Consider μ_1 a measure defined on $\sigma-$algebra F_1 of X_1 and μ_2 measure defined on σ-algebra F_2 of X_2. Then the product measure $\tilde{\mu} = \mu_1 \times \mu_2$ is the Carathéodory extension of the premeasure $\mu(A \times B) = \mu(A) \cdot \mu(B)$.

- Note that the product $\sigma-$algebra is the generated $\sigma-$algebra of $F_1 \times F_2$ as the ordinary Cartesian product is only a semi ring.
- The product measure is unique if μ_1, μ_2 are $\sigma-$finite.

Fubinis theorem

Similarly as defined above for $\sigma-$finite measure spaces

$$\int_{X \times Y} f(x, y)\, d(\mu_1 \times \mu_2) = \int_Y \left(\int_X f(x, y)\, d\mu_1 \right) d\mu_2$$
$$= \int_X \left(\int_Y f(x, y)\, d\mu_2 \right) d\mu_1$$

Convexity

A real function $\varphi : [a, b] \to \mathbb{R}$ is convex if $\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$ for $x, y \in (a, b)$ and $\lambda \in [0, 1]$.
Convexity implies continuity.

Jensen's inequality

Let μ is a positive measure on a $\sigma-$algebra F in a set X such that $\mu(X) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in X$ and if φ convex on (a,b), then

$$\varphi\left(\int_X f\, d\mu \leq \int_X (\varphi \circ f)\, d\mu \right)$$

Hölder's inequality

Let $p, q \in \mathbb{R}$ s.t. $\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq \infty$, and measure μ on X. If $f, g : X \to [0, \infty]$ measurable then,

$$\int_X fg\, d\mu \leq \left(\int_X f^p\, d\mu \right)^{\frac{1}{p}} \left(\int_X g^q\, d\mu \right)^{\frac{1}{q}}$$

Minkowski's inequality

Let $p, q \in \mathbb{R}$ s.t. $\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq \infty$, and measure μ on X. If $f, g : X \to [0, \infty]$ measurable then,

$$\left(\int_X (f+g)^p\, d\mu \right)^{\frac{1}{p}} \leq \left(\int_X f^p\, d\mu \right)^{\frac{1}{p}} \left(\int_X g^q\, d\mu \right)^{\frac{1}{q}}$$

L^p norms

For X with positive measure μ if $0 < p < \infty$ if f is complex measurable on X define the L^p norm as

$$||f||_p = \left(\int_X |f|^p\, d\mu \right)^{\frac{1}{p}}$$

and let $L^p(\mu)$ be the collection of all functions for which L^p norm is finite. If th measure is the counting measure we denote it as ℓ^p.
Define $||f||_\infty = \sup\{|f(x)| : x \in X\}$

- L^p is a complete metric space.
- $C_c(X)$ is dense in $L^p(\mu)$.

Total variation

For any complex measure μ we define its total variation $|\mu|$ defined for some measurable set E as

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| \right\}$$

where supremum is taken over all of partitions of measurable subsets E_i of E.

- Total variation is a positive measure.
- The total variation of a positive measure is the same as the positive measure itself.
- Total variation of any measurable set is always finite.

Absolutely continuity

An arbitrary measure λ is **absolutely continuous** with respect to a positive measure μ, denoted as $\lambda \ll \mu$ if $\lambda(E) = 0$ for every measurable E for which $\mu(E) = E$.
λ is said to be **concentrated** on measurable set A if $\lambda(A) = \lambda(A \bigcap E)$ for every measurable set E.
For two measures λ_1, λ_2 if there exist disjoint measurable sets A, B such that λ_1 concentrated on A and λ_2 concentrated on B ten we say the measures are **mutually singular**, i.e. $\lambda_1 \perp \lambda_2$.

Lebesgue-Radon-Nikodym theorem

Let μ be a positive $\sigma-$finite measure on a $\sigma-$algebra F on set X and let λ be a complex measure on F. Then,

- There exists a unique pair of complex measures λ_a, λ_s on F such that $\lambda = \lambda_a + \lambda_s$ and $\lambda_a \ll \mu, \lambda_s \perp \mu$
- There exists a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h\, d\mu$$

  for every $E \in F$.

The integral defines a absolutely continuous measure wrt μ and h is referred to as the Radon-Nikodym derivative of λ_a wrt μ.

Consequences of Radon-Nikodym Theorem

- Alternate statement for absolute continuity: If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta \implies |\lambda(E)| < \varepsilon$ for every measurable E, then $\lambda \ll \mu$.
- Polar decomposition of λ : For complex measure λ there exists a measurable function h such that $|h(x)| = 1$ for all $x \in X$ such that $d\mu = hd|\mu|$
- Hahn decomposition theorem: For real measure μ on $\sigma-$ algebra in set X then there exists sets A, B in F such that $A \bigcup B = X, A \bigcap B = \emptyset$ and $\mu^+(E) = \mu(A \cap E), \mu^-(E) = -\mu(B \cap E)$ for measurable E.