Category Theory Cheat Sheet

Category

A **category** consists of the following,

- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each f there exists, $\text{dom}(f), \text{cod}(f)$ called domain and codomain of f. We write $f : A \to B$ to indicate $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of f and g.
- For each A, there exists $1_A : A \to A$ called the *identity arrow* of A.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[loop above, "1_C"] \arrow[r, "h"'] & D \arrow[loop above, "1_D"]
\end{tikzcd}$$

Functor

For categories $\mathbf{C}, \mathbf{D}$ we define a **functor** $F : C \to D$ to be a a mapping of objects and arrows to objects and arrows, such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there exists an arrow $g : B \to C$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., A is isomorphic to B.

Monoid

A set M with binary operation $\cdot$ is called a **monoid** if it is associative and has an identity

- A monoid can be understood as a single element category.
- $\text{Hom}_{\mathbf{C}}(C, C)$ forms a monoid under composition.
- A monoid with existence of inverses is a group.
- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form (C, D) for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.

Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^{\to}$ has the arrows of $\mathbf{C}$ as objects and an arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^{\to}$ is the following commutative square

$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "g_2"'] & B'
\end{tikzcd}$$

  where g_1, g_2 are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.
- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ denoted as $\mathbf{C}/C$. it contains objects as all arrows in $\mathbf{C}$ who map to C. And arrows in $\mathbf{C}/C$ are arrows between the dom of the object arrows, i.e., a as seen below.

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& C &
\end{tikzcd}$$

  - The prototypical example is that of a slice of an element in a poset category being the principal ideal.
- **Co-slice category:** Denoted as $C/\mathbf{C}$ is the dual of a slice category with objects as arrows mapping from C.

Free monoid

For a set A a *word* over A is any finite sequence of its elements.
The **Kleene closure** of A is defined to be the set of all words over A denoted as A^*. With the binary operation of concatenation A^* forms a monoid and is called the **free monoid** on A.
Universal mapping property (UMP) of free monoid: Let $M(A)$ be the free monoid on a set A. There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$\begin{tikzcd}
M(A) \arrow[d, dashed, "\overline{f}"] \\
N
\end{tikzcd}
\qquad
\begin{tikzcd}
{|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\
A \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$A*$ has the UMP of the free monoid on A.

Free category

A directed graph G "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.
UMP of $\mathbf{C}(G)$ There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$\begin{tikzcd}
\mathbf{C}(G) \arrow[d, dashed, "\overline{h}"] \\
\mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
{|\mathbf{C}(G)|} \arrow[r, "|\overline{h}|"] & {|\mathbf{D}|} \\
G \arrow[u, "i"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Small categories

A category is called **small** if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.
A category $\mathbf{C}$ is **locally small** if for all objects $X, Y \in \mathbf{C}$, the collection $\text{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

Types of morphisms

Monomorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Epimorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "j"] \arrow[r, shift right, "i"'] & D
\end{tikzcd}$$

- We say, $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse r. The inverse arrow r is called the **retraction**, m is called a *section* of r and A is called a **retract** of B.

Initial and terminal objects

An object $0 \in \mathbf{C}$ is **initial** if for any object $C \in \mathbf{C}\exists!$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is **terminal** if for any object $C \in \mathbf{C}\exists!$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the **generalized elements** of A with stage of definition given by X.

Product of objects

In any category $\mathbf{C}$, a product diagram for the objects A, B consists of an object P and arrows

$$\begin{tikzcd}
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}$$

satisfying the following UMP. Given any diagram of the form

$$\begin{tikzcd}
A & X \arrow[l, "x_1"'] \arrow[r, "x_2"] & B
\end{tikzcd}$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

The product P is unique up to isomorphism.

Categories with products

A category which has a product for every pair of objects is said to have **binary products**.
A category is said to have **all finite products**, if it has a terminal object and all binary products.
A category has **all small products** if every set of objects has a product.

Covariant representable functor

The functor $\mathrm{Hom}(A,-) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).
For a category with products a covariant representable functor preserves products.

Duality

If any statement about categories holds for all categories then so does the dual statement.

Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

Equalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say an **equalizer** of f, g consists of an object E and an arrow $e : E \to A$ universal such that

$$f \circ e = g \circ e$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"] & &
\end{tikzcd}$$

- Equalizers are monic.
- It is analogous to the notion of a kernel.

Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say a **coequalizer** of f, g consists of an object Q and an arrow $q : B \to Q$ universal such that

$$q \circ f = q \circ g$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\
 & & Z
\end{tikzcd}$$

- Coequalizers are epic.
- It is analogous to the notion of a quotient.

Groups in a category

A group (Group($\mathbf{C}$)) can be defined over a category $\mathbf{C}$.

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
 & 1 \arrow[u, "u"] &
\end{tikzcd}$$

Where the arrows obey the following, m is associative, u is a unit, and i is an inverse for m, i.e. the following diagrams commute

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\cong"] \arrow[d, "m \times 1"'] & & G \times (G \times G) \arrow[d, "1 \times m"] \\
G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\
 & G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle u, 1_G \rangle"] \arrow[d, "\langle 1_G, u \rangle"'] \arrow[dr, "1_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[d, "1_G \times i"'] & G \arrow[l, "\langle 1_G, 1_G \rangle"'] \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[d, "u"'] & G \times G \arrow[d, "i \times 1_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

- A homomorphism $h : G \to H$ of groups in a category $\mathbf{C}$ is an arrow such that, h preserves m, u, i, i.e. the following diagrams commmute.

$$\begin{tikzcd}
G \times G \arrow[r, "h \times h"] \arrow[d, "m"'] & H \times H \arrow[d, "m"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] & H \\
1 \arrow[u, "u"] \arrow[ur, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "i"'] & H \arrow[d, "i"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}$$

- The objects in the category of groups (i.e. Group($\mathbf{Grp}$)) are abelian groups.

Congruence

A **congruence** on a category is a equivalance relation on arrows ($f \sim g$) s.t.

- $f \sim g \implies \mathrm{dom}(f) = \mathrm{dom}(f)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.
- $f \sim g \implies bfa \sim bga$

Let C_0, C_1 denote the class of objects and arrows for a category $\mathbf{C}$. Then a **congruence category** $\mathbf{C}^{\sim}$ is defined as follows,

- $(\mathbf{C}^{\sim})_0 = \mathbf{C}_0$
- $(\mathbf{C}^{\sim})_1 = \{\langle f, g \rangle | f \sim g\}$
- $\tilde{1}_C = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$\begin{tikzcd}
\mathbf{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C}
\end{tikzcd}$$

We define the **quotient category** of the congruence as the coequalizer, i.e,

$$\begin{tikzcd}
\mathbf{C}^{\sim} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C} \arrow[r, "\pi"] & \mathbf{C}/\sim
\end{tikzcd}$$

Finitely presented category

Consider the free category $\mathbf{C}(G)$ on a finite graph G. And the finite set of relations $\sum$ to be relations of the form $(g_1 \circ \cdots \circ g_n) = (g'_1 \circ \cdots \circ g'_m)$ for $g_i \in G$ and $\mathrm{dom}(g_n) = \mathrm{dom}(g'_m)$ and $\mathrm{cod}(g_1) = \mathrm{cod}(g'_1)$. Let $\sim_\Sigma$ be the smallest congruence $g \sim g'$ if $g = g' \in \sum$. We call the quotient by this congruence to be a **fintely presented category**.

Subobjects

A **subobject** for some $X \in \mathbf{C}$ is a monomorphism into X.

- Arrows between subobjects of the same X are arrows in the slice category of X. So collection of subobjects form a category with a preorder (with inclusion) we call $\mathrm{Sub}_{\mathbf{C}}(X)$

Pullback

In a category $\mathbf{C}$ a **pullback** of arrows f, g with the same image

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

is the pair of universal arrows p_1, p_2 such that $fp_1 = gp_2$ (i.e. u unique below)

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\
 & P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
 & A \arrow[r, "f"'] & C
\end{tikzcd}$$

- P is often denoted as $A \times_C B$. Rephrased in terms of products the pullback can be considered as a subobject of $A \times B$ determined as the equalizer of projection maps composed with f, g. Every category with products and equalizers has pullbacks defined like this and vice versa.
- For two pullback squares side by side sharing a morphism the larger rectangle forms a pullback square too.
- The pullback of a commutative triangle is also a commutative triangle by the above point.
- Pullbacks define a functor between slice categories, for fixed $f : A \to B$ $f^* : \mathbf{C}/B \to \mathbf{C}/A$ defined as $(D \xrightarrow{\alpha} B) \mapsto (A \times_B D \xrightarrow{\alpha^*} A)$ is functorial.
- This pullback functor makes the following diagram commute,

$$\begin{tikzcd}
\mathrm{Sub}(A) \arrow[d] & \mathrm{Sub}(B) \arrow[l, "f^{-1}"'] \arrow[d] \\
\mathbf{C}/A & \mathbf{C}/B \arrow[l, "f^*"]
\end{tikzcd}$$

  where f^{-1} is the restriction of f^*.
- A category with pullbacks and terminal objects $\iff$ it has finite products and equalizers

Diagram

For categories $\mathbf{J}, \mathbf{C}$ a **diagram** of type $\mathbf{J}$ in $\mathbf{C}$ is a functor $D : \mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ admits an indexing. This is a formalization of the notion of 'diagram' we use intuitively. It can be thought of as the image of $\mathbf{J}$ in $\mathbf{C}$, the actual stucture of $\mathbf{J}$ is largely irrelevant.
For example,

$\mathbf{J}$	Diagram
$\begin{tikzcd} \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet \end{tikzcd}$	$\begin{tikzcd} D_1 \arrow[r, shift left, "D_f"] \arrow[r, shift right, "D_g"'] & D_2 \end{tikzcd}$

Cone

Given $\mathbf{J}, \mathbf{C}$ and a diagram of type $\mathbf{J}$ in $\mathbf{C}$, $D : \mathbf{J} \to \mathbf{C}$ we define a **cone** to the diagram D for an object (vertex) C of $\mathbf{C}$ and family of arrows $c_j : C \to D_j$ for all $j \in \mathbf{J}$ such for $\alpha : i \to j$ the following commute,

$$\begin{tikzcd} & C \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Furthermore we can have a morphism between cones in the natural way $\vartheta : (C, c_j) \to (C', c'_j)$ making every such triangle commute,

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

This lets us define a category of cones into D denoted as $\mathbf{Cone}(D)$. Its dual is called a cocone.

Comma category

We define the **comma category** $(S \downarrow T)$ categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ which are related as $\mathbf{A} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{B}$. With objects as 3−tuples $(A, B, h), A \in \mathbf{A}, B \in \mathbf{B}, (h : S(A) \to T(B)) \in \mathbf{C}$ and arrows between them defined naturally as follows, all (f, g) for $f : A \to A', g : B \to B'$ such that the following commutes,

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"'] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"'] & T(B') \end{tikzcd}$$

A cone can alternatively be understood as a comma category $(\Delta \downarrow D)$, for the diagram D as a functor from $\Delta : C \to \mathrm{Fun}(\mathbf{J}, \mathbf{C})$ sometimes denoted as C^J. $\mathrm{Fun}(\mathbf{J}, \mathbf{C})$ is the functor category which is defined later.
Defined as sending $\Delta(C) : \mathbf{J} \to \mathbf{C}$ which just maps C to C. This functor is usually called the **diagonal functor**.

Limit

Given a diagram $D : \mathbf{J} \to \mathbf{C}$ its **limit** is a terminal object in $\mathbf{Cone}(D)$, denoted as $p_i : \lim_{\leftarrow j} D_j \to D_i$.
If $\mathbf{J}$ is finite the limit is called a finite limit.

- A category has finite limits $\iff$ it has finite products and equalizers (and so pullbacks and terminal objects.)

A functor F is said to **preserve limits** of type J if $F(\lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} F(D_j)$. Such a functor is called continuous.

- Representable functors in locally small categories are continuous.