Category Theory Cheat Sheet

Category

A **category** consists of the following,

- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each f there exists, $\mathrm{dom}(f), \mathrm{cod}(f)$ called domain and codomain of f. We write $f : A \to B$ to indicate $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of f and g.
- For each A, there exists $1_A : A \to A$ called the *identity arrow* of A.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

$$\begin{tikzcd} A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & C \arrow[loop above, "1_C"] \arrow[r, "h"'] & D \arrow[loop above, "1_D"] \end{tikzcd}$$

Functor

For categories $\mathbf{C}, \mathbf{D}$ we define a **functor** $F : \mathbf{C} \to \mathbf{D}$ to be a a mapping of objects and arrows to objects and arrows, such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_A) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there exists an arrow $g : B \to A$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., A is isomorphic to B.

Monoid

A set M with binary operation $\circ$ is called a **monoid** if it is associative and has an identity

- A monoid can be understood as a single element category.
- $\mathrm{Hom}_{\mathbf{C}}(C, C)$ forms a monoid under composition.
- A monoid with existence of inverses is a group.
- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form (C, D) for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.

Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^{\to}$ has the arrows of $\mathbf{C}$ as objects and an arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^{\to}$ is the following commutative square

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g_2"'] & B' \end{tikzcd}$$

  where g_1, g_2 are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.
- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ denoted as $\mathbf{C}/C$. it contains objects as all arrows in $\mathbf{C}$ who map to C. And arrows in $\mathbf{C}/C$ are arrows between the dom of the object arrows, i.e., a as seen below.

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

  - The prototypical example is that of a slice of an element in a poset category being the principal ideal.
- **Co-slice category:** Denoted as $C/\mathbf{C}$ is the dual of a slice category with objects as arrows mapping from C.

Free monoid

For a set A a *word* over A is any finite sequence of its elements.
The **Kleene closure** of A is defined to be the set of all words over A denoted as A^*. With the binary operation of concatenation A^* forms a monoid and is called the **free monoid** on A.
Universal mapping property (UMP) of free monoid: Let $M(A)$ be the free monoid on a set A. There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$\begin{tikzcd} M(A) \arrow[d, dashed, "\overline{f}"] \\ N \end{tikzcd} \qquad \begin{tikzcd} {|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

$A*$ has the UMP of the free monoid on A.

Free category

A directed graph G "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.
UMP of $\mathbf{C}(G)$ There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$\begin{tikzcd} \mathbf{C}(G) \arrow[d, dashed, "\overline{h}"] \\ \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} {|\mathbf{C}(G)|} \arrow[r, "|\overline{h}|"] & {|\mathbf{D}|} \\ G \arrow[u, "i"] \arrow[ur, "h"'] & \end{tikzcd}$$

Small categories

A category is called **small** if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.
A category $\mathbf{C}$ is **locally small** if for all objects $X, Y \in \mathbf{C}$, the collection $\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

Types of morphisms

Monomorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Epimorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "j"] \arrow[r, shift right, "i"'] & D \end{tikzcd}$$

- We say, $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse r. The inverse arrow r is called the **retraction**, m is called a *section* of r and A is called a **retract** of B.

Initial and terminal objects

An object $0 \in \mathbf{C}$ is **initial** if for any object $C \in \mathbf{C} \exists!$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is **terminal** if for any object $C \in \mathbf{C} \exists!$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the **generalized elements** of A with stage of definition given by X.

Product of objects

In any category $\mathbf{C}$, a product diagram for the objects A, B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

The product P is unique up to isomorphism.

Categories with products

A category which has a product for every pair of objects is said to have **binary products**.
A category is said to have **all finite products**, if it has a terminal object and all binary products.
A category has **all small products** if every set of objects has a product.

Covariant representable functor

The functor $\text{Hom}(A,-) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).
For a category with products a covariant representable functor preserves products.

Duality

If any statement about categories holds for all categories then so does the dual statement.

Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$\begin{tikzcd}
& Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

Equalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say an **equalizer** of f, g consists of an object E and an arrow $e : E \to A$ universal such that

$$f \circ e = g \circ e$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"] & &
\end{tikzcd}$$

- Equalizers are monic.
- It is analogous to the notion of a kernel.

Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say a **coequalizer** of f, g consists of an object Q and an arrow $q : B \to Q$ universal such that

$$q \circ f = q \circ g$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\
& & Z
\end{tikzcd}$$

- Coequalizers are epic.
- It is analogous to the notion of a quotient.

Groups in a category

A group ($\text{Group}(\mathbf{C})$) can be defined over a category $\mathbf{C}$.

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& 1 \arrow[u, "u"] &
\end{tikzcd}$$

Where the arrows obey the following, m is associative, u is a unit, and i is an inverse for m, i.e. the following diagrams commute

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\cong"] \arrow[d, "m \times 1"'] & & G \times (G \times G) \arrow[d, "1 \times m"] \\
G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle u, 1_G \rangle"] \arrow[d, "\langle 1_G, u \rangle"'] \arrow[dr, "1_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[d, "1_G \times i"'] & G \arrow[l, "\langle 1_G, 1_G \rangle"'] \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[d, "u"'] & G \times G \arrow[d, "i \times 1_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

- A homomorphism $h : G \to H$ of groups in a category $\mathbf{C}$ is an arrow such that, h preserves m, u, i, i.e. the following diagrams commmute.

$$\begin{tikzcd}
G \times G \arrow[r, "h \times h"] \arrow[d, "m"'] & H \times H \arrow[d, "m"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] & H \\
1 \arrow[u, "u"] \arrow[ur, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "i"'] & H \arrow[d, "i"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}$$

- The objects in the category of groups (i.e. $\text{Group}(\mathbf{Grp})$) are abelian groups.

Congruence

A **congruence** on a category is a equivalance relation on arrows ($f \sim g$) s.t.
- $f \sim g \implies \text{dom}(f) = \text{dom}(f)$ and $\text{cod}(f) = \text{cod}(g)$.
- $f \sim g \implies bfa \sim bga$

Let C_0, C_1 denote the class of objects and arrows for a category $\mathbf{C}$. Then a **congruence category $\mathbf{C}^\sim$** is defined as follows,
- $(\mathbf{C}^\sim)_0 = \mathbf{C}_0$
- $(\mathbf{C}^\sim)_1 = \{\langle f, g \rangle | f \sim g\}$
- $\tilde{1}_C = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$\begin{tikzcd}
\mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C}
\end{tikzcd}$$

We define the **quotient category** of the congruence as the coequalizer, i.e,

$$\begin{tikzcd}
\mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C} \arrow[r, "\pi"] & \mathbf{C}/\sim
\end{tikzcd}$$

Finitely presented category

Consider the free category $\mathbf{C}(G)$ on a finite graph G. And the finite set of relations $\sum$ to be relations of the form $(g_1 \circ \cdots \circ g_n) = (g'_1 \circ \cdots \circ g'_m)$ for $g_i \in G$ and $\text{dom}(g_n) = \text{dom}(g'_m)$ and $\text{cod}(g_1) = \text{cod}(g'_1)$. Let $\sim_\Sigma$ be the smallest congruence $g \sim g'$ if $g = g' \in \sum$. We call the quotient by this congruence to be a **fintely presented category**.

Subobjects

A **subobject** for some $X \in \mathbf{C}$ is a monomorphism into X.
- Arrows between subobjects of the same X are arrows in the slice category of X. So collection of subobjects form a category with a preorder (with inclusion) we call $\text{Sub}_{\mathbf{C}}(X)$

Pullback

In a category $\mathbf{C}$ a **pullback** of arrows f, g with the same image

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

is the pair of universal arrows p_1, p_2 such that $fp_1 = gp_2$ (i.e. u unique below)

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[ddr, bend right, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

- P is often denoted as $A \times_C B$. Rephrased in terms of products the pullback can be considered as a subobject of $A \times B$ determined as the equalizer of projection maps composed with f, g. Every category with products and equalizers has pullbacks defined like this and vice versa.
- For two pullback squares side by side sharing a morphism the larger rectangle forms a pullback square too.
- The pullback of a commutative triangle is also a commutative triangle by the above point.
- Pullbacks define a functor between slice categories, for fixed $f : A \to B$ $f^* : \mathbf{C}/B \to \mathbf{C}/A$ defined as $(D \xrightarrow{\alpha} B) \mapsto (A \times_B D \xrightarrow{\alpha^*} A)$ is functorial.
- This pullback functor makes the following diagram commute,

$$\begin{tikzcd}
\text{Sub}(A) \arrow[d] & \text{Sub}(B) \arrow[l, "f^{-1}"'] \arrow[d] \\
\mathbf{C}/A & \mathbf{C}/B \arrow[l, "f^*"]
\end{tikzcd}$$

  where f^{-1} is the restriction of f^*.
- A category with pullbacks and terminal objects $\iff$ it has finite products and equalizers

Diagram

For categories $\mathbf{J}, \mathbf{C}$ a **diagram** of type $\mathbf{J}$ in $\mathbf{C}$ is a functor $D : \mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ admits an indexing. This is a formalization of the notion of 'diagram' we use intuitively. It can be thought of as the image of $\mathbf{J}$ in $\mathbf{C}$, the actual stucture of $\mathbf{J}$ is largely irrelevant.
For example,

$\mathbf{J}$	Diagram
$\bullet \overset{f}{\underset{g}{\rightrightarrows}} \bullet$	$D_1 \overset{D_f}{\underset{D_g}{\rightrightarrows}} D_2$

Cone

Given $\mathbf{J}, \mathbf{C}$ and a diagram of type $\mathbf{J}$ in $\mathbf{C}$, $D : \mathbf{J} \to \mathbf{C}$ we define a **cone** to the diagram D for an object (vertex) C of $\mathbf{C}$ and family of arrows $c_j : C \to D_j$ for all $j \in \mathbf{J}$ such for $\alpha : i \to j$ the following commute,

$$\begin{tikzcd} & C \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Furthermore we can have a morphism between cones in the natural way $\vartheta : (C, c_j) \to (C', c_j')$ making every such triangle commute,

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c_j'"] \\ & D_j \end{tikzcd}$$

This lets us define a category of cones into D denoted as $\mathbf{Cone}(D)$. Its dual is called a cocone.

Comma category

We define the **comma category** $(S \downarrow T)$ categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ which are related as $\mathbf{A} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{B}$.
With objects as 3−tuples $(A, B, h), A \in \mathbf{A}, B \in \mathbf{B}, (h : S(A) \to T(B)) \in \mathbf{C}$ and arrows between them defined naturally as follows, all (f, g) for $f : A \to A', g : B \to B'$ such that the following commutes,

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"'] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"'] & T(B') \end{tikzcd}$$

A cone can alternatively be understood as a comma category $(\Delta \downarrow D)$, for the diagram D as a functor from $\Delta : C \to \mathrm{Fun}(\mathbf{J}, \mathbf{C})$ sometimes denoted as C^J. $\mathrm{Fun}(\mathbf{J}, \mathbf{C})$ is the functor category which is defined later.
Defined as sending $\Delta(C) : \mathbf{J} \to \mathbf{C}$ which just maps C to C. This functor is usually called the **diagonal functor**.

- (Co)slice categories are a special case of comma categories.

Limit

Given a diagram $D : \mathbf{J} \to \mathbf{C}$ its **limit** is a terminal object in $\mathbf{Cone}(D)$, denoted as $p_i : \lim_{\leftarrow j} D_j \to D_i$.
If $\mathbf{J}$ is finite the limit is called a finite limit.

- A category has finite limits $\iff$ it has finite products and equalizers (and so pullbacks and terminal objects.)

A functor F is said to **preserve limits** of type J if $F(\lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} F(D_j)$. Such a functor is called continuous.

- Representable functors in locally small categories are continuous.
- Colimts are the dual notion of limits, e.g. direct limit of groups.

Exponentials

For a category $\mathbf{C}$ with binary products there exists an exponential of objects B, C which consists of an object B^C and an arrow $\epsilon : C^B \times B \to C$ universal as seen below,

$$\begin{tikzcd} C^B & C^B \times B \arrow[r, "\epsilon"] & C \\ Z \arrow[u, dashed, "\tilde{f}"] & Z \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

Cartesian closed categories

A category is **cartesian closed** if it has finite products and exponentials.

- Exponentiation is functorial in a cartesian closed category.

Heyting algebra

A Heyting algebra is a poset with finite meets, joins, least and greatest element (0 and 1) and exponentials defined as implications, $a \wedge b \leq c$ iff $a \leq b \implies c$.

- A Heyting algebra is a distributive lattice. But only complete distributive lattices form Heyting algebras.
- Every boolean algebra is a Heyting algebra with implication defined classically $p \implies q$ iff $\neg p \vee q$.
- Heyting algebras form an algebraic analogue for intuitionistic propositional calculi as every IPC gives rise to an associated Heyting algebra where formulae are identified by syntactic equivalence. In particular this gives a correspondence between Heyting algebras and IPC.

$\lambda-$calculus

$\lambda-$calculus is a formal system relying on two symbols λ and a dot ".".
There exists a correspondence between typed $\lambda-$ calculus and Cartesian closed categories. With the objects in the associated category being types and arrows defined as equivalence classes of terms of a type identified when equal.

Full and faithful functors

A functor $F : \mathbf{C} \to \mathbf{D}$ induces a map in locally small categories $F_{A,B} = \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$.
The functor F is said to be **faithful** if $F_{A,B}$ is injective for all pairs of objects and **full** if it is surjective.
Note that this is not the same as the functor being injective/surjective on objects/arrows.

- Covariant and contravariant representable functors are faithful.

Natural transformations

A **natural transformation** is a map between functors.
For functors $F, G : \mathbf{C} \to \mathbf{D}$ a natural transformation $\eta : F \to G$ is a family of morphisms (in $\mathbf{D}$) which consist of **components** η_X which associates for every object $C \in \mathbf{C}$ a morphism between objects in $\mathbf{D}$, $\eta_C : F(C) \to G(C)$.
Also components must commute naturally, in particular for $f : C \to C'$ we have $\eta_{C'} \circ F(f) = G(f) \circ \eta_C$, i.e. below diagram commutes,

$$\begin{tikzcd} C \arrow[d, "f"'] & F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ C' & F(C') \arrow[r, "\eta_{C'}"] & G(C') \end{tikzcd}$$

Functor categories and exponentials

For two categories $\mathbf{C}, \mathbf{D}$ we can define the **functor category** as the category whose objects are functors between $\mathbf{C}$ and $\mathbf{D}$ and morphisms are natural transformations. It is denoted as $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$
In **Cat** (Category of small categories) the functor category between two categories defines its exponential. Therefore, **Cat** is Cartesian closed.

- A **natural isomorphism** is a natural transformation which is an isomorphism in the functor category

Bifunctors and profunctors

A **bifunctor** is any functor of two variables i.e. domain in terms of a product category.
A mapping $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ is a bifunctor if its functorial in each component and the following square commutes,

$$\begin{tikzcd} A' & B' & F(A,B) \arrow[r, "F(A,g)"] \arrow[d, "F(f,B)"'] & F(A,B') \arrow[d, "F(f,B')"] \\ A \arrow[u, "f"] & B \arrow[u, "g"] & F(A',B) \arrow[r, "F(A',g)"'] & F(A',B') \end{tikzcd}$$

A bifunctor is called a **profunctor** if its of the form $\mathbf{B}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ like Hom/Internal hom.

Equivalence of categories

For categories $\mathbf{C}, \mathbf{D}$ they is said to be an equivalence between them if there exist functors $E : \mathbf{C} \rightleftarrows \mathbf{D} : F$ and a pair of natural *isomorphisms* $\alpha : 1_{\mathbf{C}} \to F \circ E$ and $\beta : 1_{\mathbf{D}} \to E \circ F$.
This is a weaker condition than isomorphism.
$F : \mathbf{C} \to \mathbf{D}$ is a part of an equivalence of categories if it is full and faithful and for all $D \in \mathbf{D}$ there exists $C \in \mathbf{C}, F(C) \cong D$

Yoneda embedding

The **Yoneda embedding** is a functor $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ mapping objects to their contravariant representable functor (i.e. presheaves). In particular $y(C) = \mathrm{Hom}_{\mathbf{C}}(-, C)$ it takes arrows to the natural transformation $yf = \mathrm{Hom}_{\mathbf{C}}(-, f) : \mathrm{Hom}_{\mathbf{C}}(-, C) \to \mathrm{Hom}_{\mathbf{C}}(-, D)$

Yoneda lemma

For a locally small category $\mathbf{C}$ we have $\mathrm{Hom}(yC, F) \cong FC$ for $C \in \mathbf{C}$ and a functor $F \in \mathbf{Sets}^{\mathbf{C}^{op}}$. The isomorphism is natural in both C and F.

Applications of Yoneda lemma

For locally small categories

- The Yoneda embedding $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.
- $yC \cong yC' \implies C \cong C'$ for objects C, C'
- All objects in presheaves are colimits of some representable functors (in particular the end).
- So Yoneda embedding is the free cocompletion of any category. In particular there is a UMP for maps from $\mathbf{C}$ to any cocomplete categories factoring through presheaves wrt Yoneda embedding

Category of presheaves

For locally small category $\mathbf{C}$ its category of presheaves denoted as $\mathbf{Sets}^{\mathbf{C}^{op}}$ or $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$ or $\hat{\mathbf{C}}$ is complete, cocomplete and is cartesian closed. Due to these properties its often useful to examine the presheaves of a category.
We also have that the Yoneda embedding preserves products, exponentials and is continuous (preserves limits).

Monoidal categories

A monoidal category is a category $\mathbf{C}$ with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a 'unit' element I, and natural isomorphisms that make the functor associative and unital with I as expected. It is a generalization of the notion of a 'tensor product'. Its used to define enriched categories.
This can be formalized as follows, there exists $I \in \mathbf{C}$ and natural isomorphisms,

- $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ (read as associator)
- $\lambda_A : I \otimes A \to A$ (read as left unitor)
- $\rho_A : A \otimes I \to A$ (read as right unitor)

And the following diagrams commute,

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A \otimes B, C, D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C \otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B \otimes C,D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes 1_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
 & A \otimes B &
\end{tikzcd}$$

Enriched categories (monoidal)

For a monoidal category as defined above $(\mathbf{V}, \otimes, I, \alpha, \lambda, \rho)$ a category $\mathbf{C}$ enriched over $\mathbf{V}$ is a category along with the following data

- Associated to all ordered pair (A, B) of objects in $\mathbf{C}$, a hom object $\mathbf{C}(A, B) \in \mathrm{Ob}(\mathbf{V})$.
- Associated to all objects A in $\mathbf{C}$, an identity morphism $\mathrm{id}_A : I \to \mathbf{C}(A, A)$
- Associated to each ordered triple (A, B, C) of objects in $\mathbf{C}$ say (A, B, C), a morphism in $\mathrm{Ob}(\mathbf{V})$, $\circ_{A,B,C} : \mathbf{C}(B, C) \otimes \mathbf{C}(A, B) \to \mathbf{C}(A, C)$
- Also the following diagrams commute,

$$\begin{tikzcd}
 & \mathbf{C}(A,D) & \\
\mathbf{C}(B,D) \otimes \mathbf{C}(A,B) \arrow[ur, "\circ_{A,B,D}"] & & \mathbf{C}(C,D) \otimes \mathbf{C}(A,C) \arrow[ul, "\circ_{A,C,D}"'] \\
(\mathbf{C}(C,D) \otimes \mathbf{C}(B,C)) \otimes \mathbf{C}(A,B) \arrow[u, "\circ_{B,C,D} \otimes 1_{\mathbf{C}(A,B)}"] \arrow[rr, "\alpha"'] & & \mathbf{C}(C,D) \otimes (\mathbf{C}(B,C) \otimes \mathbf{C}(A,B)) \arrow[u, "1_{\mathbf{C}(C,D)} \otimes \circ_{A,B,C}"']
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes \mathbf{C}(A,B) \arrow[dr, "\lambda"] \arrow[dd, "\mathrm{id}_B \otimes 1_{\mathbf{C}(A,B)}"'] & & \mathbf{C}(A,B) \otimes I \arrow[dl, "\rho"'] \arrow[dd, "1_{\mathbf{C}(A,B)} \otimes \mathrm{id}_A"] \\
 & \mathbf{C}(A,B) & \\
\mathbf{C}(B,B) \otimes \mathbf{C}(A,B) \arrow[ur, "\circ_{A,B,B}"] & & \mathbf{C}(A,B) \otimes \mathbf{C}(A,A) \arrow[ul, "\circ_{A,A,B}"']
\end{tikzcd}$$

2-categories

The classical definition for 2-categories also called strict 2-categories are categories enriched over **Cat**, it gives us the following data

- Collection of objects
- For each pair of objects we have associated to it a hom-category.
- 1-morphisms between objects, realized as objects of hom-categories.
- 2-morphisms between 1-morphisms, realized as morphisms of hom-categories.

Topoi

A **topoi** is a category which is has small limits, has exponentials and has a subobject classifier.
A **subobject classifer** is an object Ω along with an arrow $1 \to \Omega$ universal such that it induces a subobject S as the pullback to the following diagram for some arbitrary object C,

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail] & 1 \arrow[d, "t"] \\
C \arrow[r, dashed, "u"'] & \Omega
\end{tikzcd}$$

The notion of a subobject classifier is intuitively understood as a 'truth value' function which assigns a true value for elements of $U \in C$. In particular topoi are used in logic due to a similar notion as just described.
Presheaves form a topoi.

Adjoint functors

The functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$ form an **adjunction** between categories if there exists a natural transformation $\eta : 1_{\mathbf{C}} \to U \circ F$ with this unit having the following UMP,

$$\begin{tikzcd}
D & & U(F(C)) \arrow[r, "U(g)"] & U(D) \\
F(C) \arrow[u, dashed] & & C \arrow[u, "\eta_C"] \arrow[ur, "f"'] &
\end{tikzcd}$$

F is called the left adjoint of U and vice versa, and is denoted as $F \dashv U$.
Alternatively we also have the following formulation, $F \dashv U$ if for $C \in \mathbf{C}, D \in \mathbf{D}$ there exist natural isomorphisms $\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD) : \psi$
A third more general formulation is given later.

Properties and examples of adjunctions

- Adjunctions are unique up to isomorphisms.
- Adjunctions between prosets are called 'Galois connections', they involve order preserving maps.
- Operations in Heyting algebras can interpreted as biconditions and now its clear to see how they are built on adjoints.
- In first order logic, quantifiers form adjoints, in particular if you have a functor between propositions of some $\mathcal{L}-$structures then $\forall$ is its right adjoint and $\exists$ is its left adjoint
- For arbitrary diagonal functors over some small index category say J, $\lim_{\to_J} \dashv \Delta_J \dashv \lim_{\leftarrow_J}$, in particular when diagonal functor is over 2 we get $+ \dashv \Delta_2 \dashv \times$
- Free functors are right adjoint to forgetful functors
- Right adjoints preserve limits and left adjoints preserve colimits.

Wedges

Ends and coends

Nerves and realizations

Kan extensions

A functor $f : \mathbf{C} \to \mathbf{D}$ induces a functor between its presheaves, $f^* : [\mathbf{D}^{\mathrm{op}}, \mathbf{Sets}] \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$, given by precomposition $f^*(Q)(C) = Q(fC)$ and this induced functor has both left and right adjoints which we call left and right **Kan extensions**, i.e. $f_! \dashv f^* \dashv f_*$ and left Kan extensions are naturally isomorphic with the Yoneda embedding as such, $f_! y_{\mathbf{C}} \cong y_{\mathbf{D}} f$

Locally cartesian closed categories

A category is said to be locally cartesian closed if its slice categories are cartesian closed.

Adjoint functor theorem

For a locally small, complete category $\mathbf{C}$, some other category $\mathbf{X}$. A limit preserving functor $U : \mathbf{C} \to \mathbf{X}$ has a left adjoint iff

Adjoints in terms of units and counits

For a locally small, complete category $\mathbf{C}$, some other category $\mathbf{X}$. A limit preserving functor $U : \mathbf{C} \to \mathbf{X}$ has a left adjoint iff

Monads

Abelian/Derived categories

Simplical sets

∞ groupoids

Higher categories

Model categories