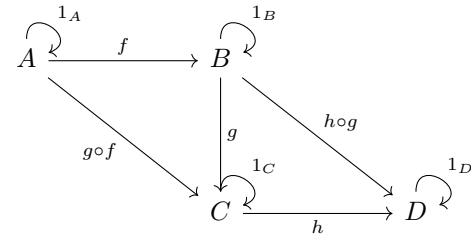


# Category Theory Cheat Sheet

## Category

A **category** consists of the following,

- Objects:  $A, B, C, \dots$
- Arrows/Morphisms:  $f, g, h, \dots$
- For each  $f$  there exists,  $\text{dom}(f), \text{cod}(f)$  called domain and codomain of  $f$ . We write  $f : A \rightarrow B$  to indicate  $A = \text{dom}(f)$  and  $B = \text{cod}(f)$ .
- Given  $f : A \rightarrow B$  and  $g : B \rightarrow C$  there exists,  $g \circ f : A \rightarrow C$  called the *composite* of  $f$  and  $g$ .
- For each  $A$ , there exists  $1_A : A \rightarrow A$  called the *identity arrow* of  $A$ .
- Arrows should also satisfy the following,
  - Associativity:  $h \circ (g \circ f) = (h \circ g) \circ f$ , for all  $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$ .
  - Unit:  $f \circ 1_A = f = 1_B \circ f$ , for all  $f : A \rightarrow B$ .



## Functor

For categories  $\mathbf{C}, \mathbf{D}$  we define a **functor**  $F : \mathbf{C} \rightarrow \mathbf{D}$  to be a mapping of objects and arrows to objects and arrows, such that

- $F(f : A \rightarrow B) = F(f) : F(A) \rightarrow F(B)$
- $F(1_A) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$ .

## Isomorphism

In any category  $\mathbf{C}$ , an arrow  $f : A \rightarrow B$  is called an **isomorphism** if there exists an arrow  $g : B \rightarrow A$  s.t.  $g \circ f = 1_A$  and  $f \circ g = 1_B$ . We say,  $g = f^{-1}$ . And that  $A \cong B$ , i.e.,  $A$  is isomorphic to  $B$ .

## Monoid

A set  $M$  with binary operation  $\cdot$  is called a **monoid** if it is associative and has an identity

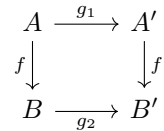
- A monoid can be understood as a single element category.
- $\text{Hom}_{\mathbf{C}}(C, C)$  forms a monoid under composition.
- A monoid with existence of inverses is a group.
- *Cayley's theorem*: Every group  $G$  is isomorphic to a group of permutations.

## Constructions on categories

- **Product category**: The product of two categories  $\mathbf{C}$  and  $\mathbf{D}$  written as  $\mathbf{C} \times \mathbf{D}$  has objects of the form  $(C, D)$  for  $C \in \mathbf{C}$  and  $D \in \mathbf{D}$ , and arrows of the form  $(f, g) : (C, D) \rightarrow (C', D')$  for  $f : C \rightarrow C' \in \mathbf{C}$  and  $g : D \rightarrow D' \in \mathbf{D}$ . Composition and units are defined componentwise.
- **Opposite/Dual category**: For category  $\mathbf{C}$  its opposite category  $\mathbf{C}^{op}$  has the same objects as  $\mathbf{C}$  but an arrow  $f : C \rightarrow D$  in  $\mathbf{C}^{op}$  is an arrow  $f : D \rightarrow C$  in  $\mathbf{C}$ .

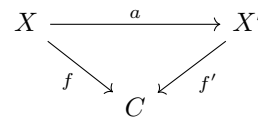
## Constructions on categories contd.

- **Arrow category**: For category  $\mathbf{C}$  its arrow category  $\mathbf{C}^{\rightarrow}$  has the arrows of  $\mathbf{C}$  as objects and an arrow  $g$  from  $f : A \rightarrow B$  to  $f' : A' \rightarrow B'$  in  $\mathbf{C}^{\rightarrow}$  is the following commutative square



where  $g_1, g_2$  are arrows in  $\mathbf{C}$ , i.e. an arrow is a pair of arrows  $g = (g_1, g_2)$  s.t.  $g_2 \circ f = f' \circ g_1$ . The identity of an object  $f : A \rightarrow B$  is the pair  $(1_A, 1_B)$  and composition is componentwise.

- **Slice category**: For category  $\mathbf{C}$  its slice category over  $C \in \mathbf{C}$  denoted as  $\mathbf{C}/C$ . it contains objects as all arrows in  $\mathbf{C}$  who map to  $C$ . And arrows in  $\mathbf{C}/C$  are arrows between the dom of the object arrows, i.e.,  $a$  as seen below.



- The prototypical example is that of a slice of an element in a poset category being the principal ideal.

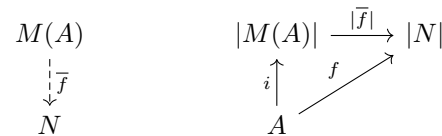
- **Co-slice category**: Denoted as  $C/\mathbf{C}$  is the dual of a slice category with objects as arrows mapping from  $C$ .

## Free monoid

For a set  $A$  a *word* over  $A$  is any finite sequence of its elements.

The **Kleene closure** of  $A$  is defined to be the set of all words over  $A$  denoted as  $A^*$ . With the binary operation of concatenation  $A^*$  forms a monoid and is called the **free monoid** on  $A$ .

**Universal mapping property (UMP) of free monoid**: Let  $M(A)$  be the free monoid on a set  $A$ . There is a function  $i : A \rightarrow |M(A)|$ , and given any monoid  $N$  and any function  $f : A \rightarrow |N|$ , there is a unique monoid homomorphism  $\bar{f} : M(A) \rightarrow N$  s.t.  $|\bar{f}| \circ i = f$ .

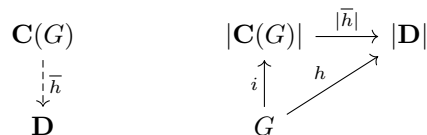


$A^*$  has the UMP of the free monoid on  $A$ .

## Free category

A directed graph  $G$  “generates” a free category  $\mathbf{C}(G)$  whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.

**UMP of  $\mathbf{C}(G)$**  There is a graphic homomorphism  $i : G \rightarrow |\mathbf{C}(G)|$ , and given any category  $\mathbf{D}$  and any graph homomorphism  $h : G \rightarrow |\mathbf{D}|$ , there is a unique functor  $\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$  with  $\bar{h} \circ i = h$ .



## Small categories

A category is called **small** if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.

A category  $\mathbf{C}$  is **locally small** if for all objects  $X, Y \in \mathbf{C}$ , the collection  $\text{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \rightarrow Y\}$  is a small set.

## Types of morphisms

**Monomorphism**: In any category  $\mathbf{C}$ , an arrow  $f : A \rightarrow B$  is called a monomorphism (monic), if for any  $g, h : C \rightarrow A, fg = fh \implies g = h$ .

$$C \xrightarrow[g]{g} A \xrightarrow{f} B$$

**Epimorphism**: In any category  $\mathbf{C}$ , an arrow  $f : A \rightarrow B$  is called an epimorphism (epic), if for any  $i, j : B \rightarrow D$   $if = jf \implies i = j$ .

$$A \xrightarrow{f} B \xrightarrow[i]{j} D$$

- We say,  $f : A \rightarrowtail B$  if  $f$  is a monomorphism and  $f : A \twoheadrightarrow B$  if  $f$  is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow  $m : A \rightarrow B$  with a left (right) inverse  $r$ . The inverse arrow  $r$  is called the **retraction**,  $m$  is called a *section* of  $r$  and  $A$  is called a **retract** of  $B$ .

## Initial and terminal objects

An object  $0 \in \mathbf{C}$  is **initial** if for any object  $C \in \mathbf{C}$ ! morphism  $0 \rightarrow C$ .

An object  $1 \in \mathbf{C}$  is **terminal** if for any object  $C \in \mathbf{C}$ ! morphism  $C \rightarrow 1$ .

Initial and terminal objects are unique up to isomorphism.

## Generalized elements

For an object  $A \in \mathbf{C}$  arbitrary arrows  $x : X \rightarrow A$  are called the **generalized elements** of  $A$  with stage of definition given by  $X$ .

## Product of objects

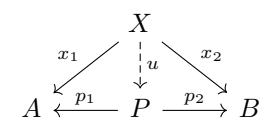
In any category  $\mathbf{C}$ , a product diagram for the objects  $A, B$  consists of an object  $P$  and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow  $u : X \rightarrow P$ , making the following diagram commute



The product  $P$  is unique up to isomorphism.

## Categories with products

A category which has a product for every pair of objects is said to have **binary products**.

A category is said to have **all finite products**, if it has a terminal object and all binary products.

A category has **all small products** if every set of objects has a product.

### Covariant representable functor

The functor  $\text{Hom}(A, -) : \mathbf{C} \rightarrow \mathbf{Sets}$  is called a covariant representable functor (for some object  $A \in \mathbf{C}$ ).  
For a category with products a covariant representable functor preserves products.

### Duality

If any statement about categories holds for all categories then so does the dual statement.

### Coproducts

A diagram  $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$  is a coproduct of  $A$  and  $B$  if for any  $Z$  and  $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$  there is a unique  $u : Q \rightarrow Z$  making the diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow z_1 & \uparrow u & \nwarrow z_2 & \\ A & \xrightarrow{q_1} & Q & \xleftarrow{q_2} & B \end{array}$$

### Equalizers

In some category  $\mathbf{C}$  given the following diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

We say an **equalizer** of  $f, g$  consists of an object  $E$  and an arrow  $e : E \rightarrow A$  universal such that

$$f \circ e = g \circ e$$

i.e., for any  $z : Z \rightarrow A$  with  $f \circ z = g \circ z$ , there exists a unique  $u : Z \rightarrow E$  with  $e \circ u = z$

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\ \uparrow u & \nearrow z & & & \\ Z & & & & \end{array}$$

- Equalizers are monic.
- It is analogous to the notion of a kernel.

### Coequalizers

In some category  $\mathbf{C}$  given the following diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

We say a **coequalizer** of  $f, g$  consists of an object  $Q$  and an arrow  $q : B \rightarrow Q$  universal such that

$$q \circ f = q \circ g$$

i.e., for any  $z : B \rightarrow Z$  with  $z \circ f = z \circ g$ , there exists a unique  $u : Q \rightarrow Z$  with  $u \circ q = z$

$$\begin{array}{ccccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B & \xrightarrow{q} & Q \\ & & \searrow z & \downarrow u & \\ & & & Z & \end{array}$$

- Coequalizers are epic.
- It is analogous to the notion of a quotient.

### Groups in a category

A group ( $\text{Group}(\mathbf{C})$ ) can be defined over a category  $\mathbf{C}$ .

$$\begin{array}{ccccc} G \times G & \xrightarrow{m} & G & \xleftarrow{i} & G \\ & & \uparrow u & & \\ & & 1 & & \end{array}$$

Where the arrows obey the following,  $m$  is associative,  $u$  is a unit, and  $i$  is an inverse for  $m$ , i.e. the following diagrams commute

$$\begin{array}{ccc} (G \times G) \times G & \xrightarrow{\cong} & G \times (G \times G) \\ m \times 1 \downarrow & & \downarrow 1 \times m \\ G \times G & \xrightarrow{m} & G \times G \\ & \searrow m & \swarrow m \\ & G & \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\langle u, 1_G \rangle} & G \times G \\ \langle 1_G, u \rangle \downarrow & \searrow 1_G & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccccc} G \times G & \xleftarrow{\langle 1_G, 1_G \rangle} & G & \xrightarrow{\langle 1_G, 1_G \rangle} & G \times G \\ 1_G \times i \downarrow & & u \downarrow & & \downarrow i \times 1_G \\ G \times G & \xrightarrow{m} & G & \xleftarrow{m} & G \times G \end{array}$$

- A homomorphism  $h : G \rightarrow H$  of groups in a category  $\mathbf{C}$  is an arrow such that,  $h$  preserves  $m, u, i$ , i.e. the following diagrams commute.

$$\begin{array}{ccc} G \times G & \xrightarrow{h \times h} & H \times H \\ m \downarrow & & \downarrow m \\ G & \xrightarrow{h} & H \end{array} \quad \begin{array}{ccc} G & \xrightarrow{h} & H \\ u \uparrow & \nearrow u & \\ 1 & & \end{array} \quad \begin{array}{ccc} G & \xrightarrow{h} & H \\ i \downarrow & & \downarrow i \\ G & \xrightarrow{h} & H \end{array}$$

- The objects in the category of groups (i.e.  $\text{Group}(\mathbf{Grp})$ ) are abelian groups.

### Congruence

A **congruence** on a category is a equivalence relation on arrows ( $f \sim g$ ) s.t.

- $f \sim g \implies \text{dom}(f) = \text{dom}(g)$  and  $\text{cod}(f) = \text{cod}(g)$ .
- $f \sim g \implies bfa \sim bga$

Let  $C_0, C_1$  denote the class of objects and arrows for a category  $\mathbf{C}$ . Then a **congruence category**  $\mathbf{C}^\sim$  is defined as follows,

- $(\mathbf{C}^\sim)_0 = \mathbf{C}_0$
- $(\mathbf{C}^\sim)_1 = \{ \langle f, g \rangle \mid f \sim g \}$
- $1_{\mathbf{C}^\sim} = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$\mathbf{C}^\sim \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathbf{C}$$

We define the **quotient category** of the congruence as the coequalizer, i.e.,

$$\mathbf{C}^\sim \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathbf{C} \xrightarrow{\pi} \mathbf{C} / \sim$$

### Finitely presented category

Consider the free category  $\mathbf{C}(G)$  on a finite graph  $G$ . And the finite set of relations  $\sum$  to be relations of the form  $(g_1 \circ \dots \circ g_n) = (g'_1 \circ \dots \circ g'_m)$  for  $g_i \in G$  and  $\text{dom}(g_n) = \text{dom}(g'_m)$  and  $\text{cod}(g_1) = \text{cod}(g'_1)$ . Let  $\sim_\Sigma$  be the smallest congruence  $g \sim g'$  if  $g = g' \in \sum$ . We call the quotient by this congruence to be a **finitely presented category**.

### Subobjects

A **subobject** for some  $X \in \mathbf{C}$  is a monomorphism into  $X$ .

- Arrows between subobjects of the same  $X$  are arrows in the slice category of  $X$ . So collection of subobjects form a category with a preorder (with inclusion) we call  $\text{Sub}_{\mathbf{C}}(X)$

### Pullback

In a category  $\mathbf{C}$  a **pullback** of arrows  $f, g$  with the same image

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ A & \xrightarrow{f} & C \end{array}$$

is the pair of universal arrows  $p_1, p_2$  such that  $f p_1 = g p_2$  (i.e.  $u$  unique below)

$$\begin{array}{ccccc} Z & & & & \\ & \searrow z_2 & & \nearrow z_1 & \\ & & P & \xrightarrow{p_2} & B \\ & \nearrow z_1 & \downarrow p_1 & & \downarrow g \\ & & A & \xrightarrow{f} & C \end{array}$$

- $P$  is often denoted as  $A \times_C B$ . Rephrased in terms of products the pullback can be considered as a subobject of  $A \times B$  determined as the equalizer of projection maps composed with  $f, g$ . Every category with products and equalizers has pullbacks defined like this and vice versa.
- For two pullback squares side by side sharing a morphism the larger rectangle forms a pullback square too.
- The pullback of a commutative triangle is also a commutative triangle by the above point.
- Pullbacks define a functor between slice categories, for fixed  $f : A \rightarrow B$   $f^* : \mathbf{C}/B \rightarrow \mathbf{C}/A$  defined as  $(D \xrightarrow{\alpha} B) \mapsto (A \times_B D \xrightarrow{\alpha^*} A)$  is functorial.
- This pullback functor makes the following diagram commute,

$$\begin{array}{ccc} \text{Sub}(A) & \xleftarrow{f^{-1}} & \text{Sub}(B) \\ \downarrow & & \downarrow \\ \mathbf{C}/A & \xleftarrow{f^*} & \mathbf{C}/B \end{array}$$

where  $f^{-1}$  is the restriction of  $f^*$ .

- A category with pullbacks and terminal objects  $\iff$  it has finite products and equalizers

### Diagram

For categories  $\mathbf{J}, \mathbf{C}$  a **diagram** of type  $\mathbf{J}$  in  $\mathbf{C}$  is a functor  $D : \mathbf{J} \rightarrow \mathbf{C}$  where  $\mathbf{J}$  admits an indexing. This is a formalization of the notion of 'diagram' we use intuitively. It can be thought of as the image of  $\mathbf{J}$  in  $\mathbf{C}$ , the actual structure of  $\mathbf{J}$  is largely irrelevant.

For example,

$$\begin{array}{ccc} \mathbf{J} & & \text{Diagram} \\ \bullet & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & \bullet \end{array} \quad \begin{array}{ccc} D_1 & \begin{array}{c} \xrightarrow{D_f} \\ \xrightarrow{D_g} \end{array} & D_2 \end{array}$$

### Cone

Given  $\mathbf{J}, \mathbf{C}$  and a diagram of type  $\mathbf{J}$  in  $\mathbf{C}$ ,  $D : \mathbf{J} \rightarrow \mathbf{C}$  we define a **cone** to the diagram  $D$  for an object (vertex)  $C$  of  $\mathbf{C}$  and family of arrows  $c_j : C \rightarrow D_j$  for all  $j \in \mathbf{J}$  such for  $\alpha : i \rightarrow j$  the following commute,

$$\begin{array}{ccc} & C & \\ c_i \swarrow & & \searrow c_j \\ D_i & \xrightarrow{D_\alpha} & D_j \end{array}$$

Furthermore we can have a morphism between cones in the natural way  $\vartheta : (C, c_j) \rightarrow (C', c'_j)$  making every such triangle commute,

$$\begin{array}{ccc} C & \xrightarrow{\vartheta} & C' \\ & \searrow c_j & \downarrow c'_j \\ & & D_j \end{array}$$

This lets us define a category of cones into  $D$  denoted as  $\mathbf{Cone}(D)$ . Its dual is called a cocone.

### Comma category

We define the **comma category**  $(S \downarrow T)$  categories  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  which are related as  $\mathbf{A} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{B}$ . With objects as 3-tuples  $(A, B, h), A \in \mathbf{A}, B \in \mathbf{B}, (h : S(A) \rightarrow T(B)) \in \mathbf{C}$  and arrows between them defined naturally as follows, all  $(f, g)$  for  $f : A \rightarrow A', g : B \rightarrow B'$  such that the following commutes,

$$\begin{array}{ccc} S(A) & \xrightarrow{S(f)} & S(A') \\ \downarrow h & & \downarrow h' \\ T(B) & \xrightarrow{T(g)} & T(B') \end{array}$$

A cone can alternatively be understood as a comma category  $(\Delta \downarrow D)$ , for the diagram  $D$  as a functor from  $\Delta : \mathbf{C} \rightarrow \mathbf{Fun}(\mathbf{J}, \mathbf{C})$  sometimes denoted as  $C^J$ .  $\mathbf{Fun}(\mathbf{J}, \mathbf{C})$  is the functor category which is defined later. Defined as sending  $\Delta(C) : \mathbf{J} \rightarrow \mathbf{C}$  which just maps  $C$  to  $C$ . This functor is usually called the **diagonal functor**.

### Limit

Given a diagram  $D : \mathbf{J} \rightarrow \mathbf{C}$  its **limit** is a terminal object in  $\mathbf{Cone}(D)$ , denoted as  $p_i : \lim_{\leftarrow j} D_j \rightarrow D_i$ .

If  $\mathbf{J}$  is finite the limit is called a finite limit.

- A category has finite limits  $\iff$  it has finite products and equalizers (and so pullbacks and terminal objects.)

A functor  $F$  is said to **preserve limits** of type  $J$  if  $F(\lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} F(D_j)$ . Such a functor is called continuous.

- Representable functors in locally small categories are continuous.