Category Theory Cheat Sheet

Category

A **category** consists of the following,

- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each f there exists, $\text{dom}(f), \text{cod}(f)$ called domain and codomain of f. We write $f : A \to B$ to indicate $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of f and g.
- For each A, there exists $1_A : A \to A$ called the *identity arrow* of A.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

$$
\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\
 & C \arrow[loop above, "1_C"] \arrow[r, "h"'] & D \arrow[loop above, "1_D"]
\end{tikzcd}
$$

Functor

For categories $\mathbf{C}, \mathbf{D}$ we define a **functor** $F : C \to D$ to be a a mapping of objects and arrows to objects and arrows, such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an **isomorphism** if there exists an arrow $g : B \to C$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., A is isomorphic to B.

Monoid

A set M with binary operation $\cdot$ is called a **monoid** if it is associative and has an identity

- A monoid can be understood as a single element category.
- $\text{Hom}_{\mathbf{C}}(C, C)$ forms a monoid under composition.
- A monoid with existence of inverses is a group.
- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form (C, D) for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.

Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^{\to}$ has the arrows of $\mathbf{C}$ as objects and an arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^{\to}$ is the following commutative square

$$
\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "g_2"'] & B'
\end{tikzcd}
$$

  where g_1, g_2 are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.
- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ denoted as $\mathbf{C}/C$. it contains objects as all arrows in $\mathbf{C}$ who map to C. And arrows in $\mathbf{C}/C$ are arrows between the dom of the object arrows, i.e., a as seen below.

$$
\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
 & C &
\end{tikzcd}
$$

  - The prototypical example is that of a slice of an element in a poset category being the principal ideal.
- **Co-slice category:** Denoted as $C/\mathbf{C}$ is the dual of a slice category with objects as arrows mapping from C.

Free monoid

For a set A a *word* over A is any finite sequence of its elements.
The **Kleene closure** of A is defined to be the set of all words over A denoted as A^*. With the binary operation of concatenation A^* forms a monoid and is called the **free monoid** on A.
Universal mapping property (UMP) of free monoid: Let $M(A)$ be the free monoid on a set A. There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$
\begin{tikzcd}
M(A) \arrow[d, dashed, "\overline{f}"] \\
N
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\
A \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

$A*$ has the UMP of the free monoid on A.

Free category

A directed graph G "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.
UMP of $\mathbf{C}(G)$ There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$
\begin{tikzcd}
\mathbf{C}(G) \arrow[d, dashed, "\overline{h}"] \\
\mathbf{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{|\mathbf{C}(G)|} \arrow[r, "|\overline{h}|"] & {|\mathbf{D}|} \\
G \arrow[u, "i"] \arrow[ur, "h"'] &
\end{tikzcd}
$$

Small categories

A category is called **small** if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.
A category $\mathbf{C}$ is **locally small** if for all objects $X, Y \in \mathbf{C}$, the collection $\text{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

Types of morphisms

Monomorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$
\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B
\end{tikzcd}
$$

Epimorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "j"] \arrow[r, shift right, "i"'] & D
\end{tikzcd}
$$

- We say, $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse r. The inverse arrow r is called the **retraction**, m is called a *section* of r and A is called a **retract** of B.

Initial and terminal objects

An object $0 \in \mathbf{C}$ is **initial** if for any object $C \in \mathbf{C} \exists!$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is **terminal** if for any object $C \in \mathbf{C} \exists!$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the **generalized elements** of A with stage of definition given by X.

Product of objects

In any category $\mathbf{C}$, a product diagram for the objects A, B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$
\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

The product P is unique up to isomorphism.

Categories with products

A category which has a product for every pair of objects is said to have **binary products**.
A category is said to have **all finite products**, if it has a terminal object and all binary products.
A category has **all small products** if every set of objects has a product.

Covariant representable functor

The functor $\mathrm{Hom}(A,-) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).
For a category with products a covariant representable functor preserves products.

Duality

If any statement about categories holds for all categories then so does the dual statement.

Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$\begin{tikzcd}
& Z & \\
A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"]
\end{tikzcd}$$

Equalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say an **equalizer** of f, g consists of an object E and an arrow $e : E \to A$ universal such that

$$f \circ e = g \circ e$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, dashed, "u"] \arrow[ur, "z"] & &
\end{tikzcd}$$

- Equalizers are monic.
- It is analogous to the notion of a kernel.

Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

We say a **coequalizer** of f, g consists of an object Q and an arrow $q : B \to Q$ universal such that

$$q \circ f = q \circ g$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\
& & Z
\end{tikzcd}$$

- Coequalizers are epic.
- It is analogous to the notion of a quotient.

Groups in a category

A group ($\mathrm{Group}(\mathbf{C})$) can be defined over a category $\mathbf{C}$.

$$\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& 1 \arrow[u, "u"] &
\end{tikzcd}$$

Where the arrows obey the following, m is associative, u is a unit, and i is an inverse for m, i.e. the following diagrams commute

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\cong"] \arrow[d, "m \times 1"'] & & G \times (G \times G) \arrow[d, "1 \times m"] \\
G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle u, 1_G \rangle"] \arrow[d, "\langle 1_G, u \rangle"'] \arrow[dr, "1_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[d, "1_G \times i"'] & G \arrow[l, "\langle 1_G, 1_G \rangle"'] \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[d, "u"'] & G \times G \arrow[d, "i \times 1_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

- A homomorphism $h : G \to H$ of groups in a category $\mathbf{C}$ is an arrow such that, h preserves m, u, i, i.e. the following diagrams commmute.

$$\begin{tikzcd}
G \times G \arrow[r, "h \times h"] \arrow[d, "m"'] & H \times H \arrow[d, "m"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] & H \\
1 \arrow[u, "u"] \arrow[ur, "u"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, "i"'] & H \arrow[d, "i"] \\
G \arrow[r, "h"'] & H
\end{tikzcd}$$

- The objects in the category of groups (i.e. $\mathrm{Group}(\mathbf{Grp})$) are abelian groups.

Congruence

A **congruence** on a category is a equivalance relation on arrows ($f \sim g$) s.t.
- $f \sim g \implies \mathrm{dom}(f) = \mathrm{dom}(f)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.
- $f \sim g \implies bfa \sim bga$

Let C_0, C_1 denote the class of objects and arrows for a category $\mathbf{C}$. Then a **congruence category $\mathbf{C}^\sim$** is defined as follows,
- $(\mathbf{C}^\sim)_0 = \mathbf{C}_0$
- $(\mathbf{C}^\sim)_1 = \{\langle f, g \rangle | f \sim g\}$
- $\tilde{1}_C = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$\begin{tikzcd}
\mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C}
\end{tikzcd}$$

We define the **quotient category** of the congruence as the coequalizer, i.e,

$$\begin{tikzcd}
\mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C} \arrow[r, "\pi"] & \mathbf{C}/\sim
\end{tikzcd}$$

Finitely presented category

Consider the free category $\mathbf{C}(G)$ on a finite graph G. And the finite set of relations $\sum$ to be relations of the form $(g_1 \circ \cdots \circ g_n) = (g_1' \circ \cdots \circ g_m')$ for $g_i \in G$ and $\mathrm{dom}(g_n) = \mathrm{dom}(g_m')$ and $\mathrm{cod}(g_1) = \mathrm{cod}(g_1')$. Let $\sim_\Sigma$ be the smallest congruence $g \sim g'$ if $g = g' \in \sum$. We call the quotient by this congruence to be a **fintely presented category**.

Subobjects

A **subobject** for some $X \in \mathbf{C}$ is a monomorphism into X.
- Arrows between subobjects of the same X are arrows in the slice category of X. So collection of subobjects form a category with a preorder (with inclusion) we call $\mathrm{Sub}_{\mathbf{C}}(X)$

Pullback

In a category $\mathbf{C}$ a **pullback** of arrows f, g with the same image

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

is the pair of universal arrows p_1, p_2 such that $fp_1 = gp_2$ (i.e. u unique below)

$$\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, "z_2"] \arrow[ddr, "z_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

- P is often denoted as $A \times_C B$. Rephrased in terms of products the pullback can be considered as a subobject of $A \times B$ determined as the equalizer of projection maps composed with f, g. Every category with products and equalizers has pullbacks defined like this and vice versa.
- For two pullback squares side by side sharing a morphism the larger rectangle forms a pullback square too.
- The pullback of a commutative triangle is also a commutative triangle by the above point.
- Pullbacks define a functor between slice categories, for fixed $f : A \to B$ $f^* : \mathbf{C}/B \to \mathbf{C}/A$ defined as $(D \xrightarrow{\alpha} B) \mapsto (A \times_B D \xrightarrow{\alpha^*} A)$ is functorial.
- This pullback functor makes the following diagram commute,

$$\begin{tikzcd}
\mathrm{Sub}(A) \arrow[d] & \mathrm{Sub}(B) \arrow[l, "f^{-1}"'] \arrow[d] \\
\mathbf{C}/A & \mathbf{C}/B \arrow[l, "f^*"]
\end{tikzcd}$$

  where f^{-1} is the restriction of f^*.
- A category with pullbacks and terminal objects $\iff$ it has finite products and equalizers

Diagram

For categories $\mathbf{J}, \mathbf{C}$ a **diagram** of type $\mathbf{J}$ in $\mathbf{C}$ is a functor $D : \mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ admits an indexing. This is a formalization of the notion of 'diagram' we use intuitively. It can be thought of as the image of $\mathbf{J}$ in $\mathbf{C}$, the actual stucture of $\mathbf{J}$ is largely irrelevant.
For example,

$\mathbf{J}$	Diagram
$\bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet$	$D_1 \underset{D_g}{\overset{D_f}{\rightrightarrows}} D_2$

Cone

Given $\mathbf{J}, \mathbf{C}$ and a diagram of type $\mathbf{J}$ in $\mathbf{C}$, $D : \mathbf{J} \to \mathbf{C}$ we define a **cone** to the diagram D for an object (vertex) C of $\mathbf{C}$ and family of arrows $c_j : C \to D_j$ for all $j \in \mathbf{J}$ such for $\alpha : i \to j$ the following commute,

$$\begin{tikzcd} & C \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

Furthermore we can have a morphism between cones in the natural way $\vartheta : (C, c_j) \to (C', c'_j)$ making every such triangle commute,

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

This lets us define a category of cones into D denoted as $\mathbf{Cone}(D)$. Its dual is called a cocone.

Comma category

We define the **comma category** $(S \downarrow T)$ categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ which are related as $\mathbf{A} \xrightarrow{S} \mathbf{C} \xleftarrow{T} \mathbf{B}$. With objects as $3-$tuples $(A, B, h), A \in \mathbf{A}, B \in \mathbf{B}, (h : S(A) \to T(B)) \in \mathbf{C}$ and arrows between them defined naturally as follows, all (f, g) for $f : A \to A', g : B \to B'$ such that the following commutes,

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"'] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"'] & T(B') \end{tikzcd}$$

A cone can alternatively be understood as a comma category $(\Delta \downarrow D)$, for the diagram D as a functor from $\Delta : C \to \mathrm{Fun}(\mathbf{J}, \mathbf{C})$ sometimes denoted as C^J. $\mathrm{Fun}(\mathbf{J}, \mathbf{C})$ is the functor category which is defined later.
Defined as sending $\Delta(C) : \mathbf{J} \to \mathbf{C}$ which just maps C to C. This functor is usually called the **diagonal functor**.

Limit

Given a diagram $D : \mathbf{J} \to \mathbf{C}$ its **limit** is a terminal object in $\mathbf{Cone}(D)$, denoted as $p_i : \lim_{\leftarrow_j} D_j \to D_i$.
If $\mathbf{J}$ is finite the limit is called a finite limit.

- A category has finite limits $\iff$ it has finite products and equalizers (and so pullbacks and terminal objects.)

A functor F is said to **preserve limits** of type J if $F(\lim_{\leftarrow} D_j) \cong \lim_{\leftarrow} F(D_j)$. Such a functor is called continuous.

- Representable functors in locally small categories are continuous.
- Colimts are the dual notion of limits, e.g. direct limit of groups.

Exponentials

For a category $\mathbf{C}$ with binary products there exists an exponential of objects B, C which consists of an object B^C and an arrow $\epsilon : C^B \times B \to C$ universal as seen below,

$$\begin{tikzcd} C^B & & C^B \times B \arrow[r, "\epsilon"] & C \\ Z \arrow[u, dashed, "\tilde{f}"] & & Z \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

Cartesian closed categories

A category is **cartesian closed** if it has finite products and exponentials.

- Exponentiation is functorial in a cartesian closed category.

Heyting algebra

A Heyting algebra is a poset with finite meets, joins, least and greatest element (0 and 1) and exponentials defined as implications, $a \wedge b \leq c$ iff $a \leq b \implies c$.

- A Heyting algebra is a distributive lattice. But only complete distributive lattices form Heyting algebras.
- Every boolean algebra is a Heyting algebra with implication defined classically $p \implies q$ iff $\neg p \vee q$.
- Heyting algebras form an algebraic analogue for intuitionistic propositional calculi as every IPC gives rise to an associated Heyting algebra where formulae are identified by syntactic equivalence. In particular this gives a correspondence between Heyting algebras and IPC.

$\lambda-$calculus

$\lambda-$calculus is a formal system relying on two symbols λ and a dot ".".
There exists a correspondence between typed $\lambda-$ calculus and cartesian closed categories. With the objects in the associated category being types and arrows defined as equivalence classes of terms of a type identified when equal.