# Category Theory Cheat Sheet

Category

A **category** consists of the following,
- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each f there exists, $\text{dom}(f), \text{cod}(f)$ called domain and codomain of f. We write $f : A \to B$ to indicate $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of f and g.
- For each A, there exists $1_A : A \to A$ called the *identity arrow* of A.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.
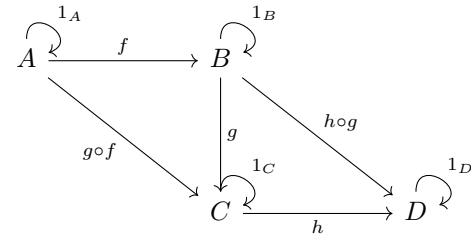
Functor

For categories $\mathbf{C}, \mathbf{D}$ we define a **functor** $F : C \to D$ to be a a mapping of objects and arrows to objects and arrows, such that
- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an *isomorphism* if there exists an arrow $g : B \to C$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., A is isomorphic to B.

Monoid

A set M with binary operation $\cdot$ is called a **monoid** if it is associative and has an identity
- A monoid can be understood as a single element category.
- $\text{Hom}_\mathbf{C}(C, C)$ forms a monoid under composition.
- A monoid with existence of inverses is a group.
- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form (C, D) for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.

Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^\to$ has the arrows of $\mathbf{C}$ as objects and an arrow g from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^\to$ is the following commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{g_1} & A' \\
\downarrow{f} & & \downarrow{f'} \\
B & \xrightarrow{g_2} & B'
\end{array}
$$
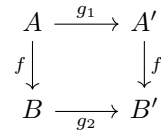
where g_1, g_2 are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.
- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ is $\mathbf{C}/C$ where its objects are all arrows $f \in \mathbf{C}$ s.t. $\text{cod}(f) = C$. An arrow a from $f : X \to C$ to $f' : X' \to C$ is an arrow $a : X \to X'$ in $\mathbf{C}$ s.t. $f' \circ a = f$, i.e.

$$
\begin{array}{ccc}
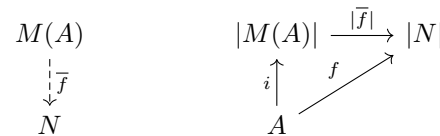X & \xrightarrow{a} & X' \\
& {f}\searrow \quad \swarrow{f'} & \\
& C &
\end{array}
$$
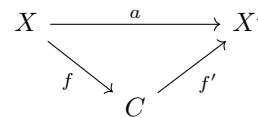
If $g : C \to D$ is any arrow then there exists a composition functor $g_* : \mathbf{C}/C \to \mathbf{C}/D$ defined as $g_*(f) = g \circ f$. Therefore the slice category of $\mathbf{C}$ with any of its objects is a functor from $\mathbf{C} \to \mathbf{Cat}$. This is called the *forgetful* functor as the base object is "forgotten".

Free monoid

For a set A a *word* over A is any finite sequence of its elements.
The **Kleene closure** of A is defined to be the set of all words over A denoted as A^*. With the binary operation of concatenation A^* forms a monoid and is called the **free monoid** on A.
Universal mapping property (UMP) of free monoid: Let $M(A)$ be the free monoid on a set A. There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$
\begin{array}{ccc}
M(A) & & |M(A)| \xrightarrow{|\overline{f}|} |N| \\
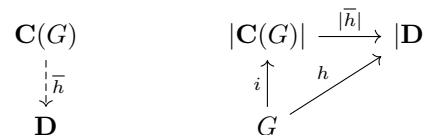\downarrow{\overline{f}} & & {i}\uparrow \quad \nearrow{f} \\
N & & A
\end{array}
$$

$A*$ has the UMP of the free monoid on A.

Free category

A directed graph G "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.
UMP of $\mathbf{C}(G)$ There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$
\begin{array}{ccc}
\mathbf{C}(G) & & |\mathbf{C}(G)| \xrightarrow{|\overline{h}|} |\mathbf{D}| \\
\downarrow{\overline{h}} & & {i}\uparrow \quad \nearrow{h} \\
\mathbf{D} & & G
\end{array}
$$

Small categories

A category is called *small* if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.
A category $\mathbf{C}$ is *locally small* if for all objects $X, Y \in \mathbf{C}$, the collection $\text{Hom}_\mathbf{C}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

Types of morphisms

Monomorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$
C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B
$$

Epimorphism: In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$
A \xrightarrow{f} B \underset{i}{\overset{j}{\rightrightarrows}} D
$$

- We say, $f : A \rightarrowtail B$ if f is a monomorphism and $f : A \twoheadrightarrow B$ if f is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse r. The inverse arrow r is called the *retraction*, m is called a *section* of r and A is called a *retract* of B.

Initial and terminal objects

An object $0 \in \mathbf{C}$ is *initial* if for any object $C \in \mathbf{C} \exists!$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is *terminal* if for any object $C \in \mathbf{C} \exists!$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the *generalized elements* of A with stage of definition given by X.
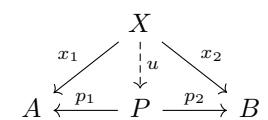
Product of objects

In any category $\mathbf{C}$, a product diagram for the objects A, B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$
\begin{array}{ccc}
 & X & \\
{x_1}\swarrow & \downarrow{u} & \searrow{x_2} \\
A \xleftarrow{p_1} & P & \xrightarrow{p_2} B
\end{array}
$$

The product P is unique up to isomorphism.

Categories with products

A category which has a product for every pair of objects is said to have *binary products*.
A category is said to have *all finite products*, if it has a terminal object and all binary products.
A category has *all small products* if every set of objects has a product.

Covariant representable functor

The functor $\text{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).
For a category with products a covariant representable functor preserves products.

Duality

If any statement about categories holds for all categories then so does the dual statement.

Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$
\begin{array}{ccc}
 & Z & \\
{}^{z_1}\nearrow & \uparrow{\scriptstyle u} & \nwarrow{}^{z_2} \\
A \xrightarrow{q_1} & Q & \xleftarrow{q_2} B
\end{array}
$$

Equalizers

In some category $\mathbf{C}$ given the following diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

We say an *equalizer* of f, g consists of an object E and an arrow $e : E \to A$ universal such that

$$f \circ e = g \circ e$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$
\begin{array}{ccc}
E & \xrightarrow{e} A \xrightarrow{f} B \\
{\scriptstyle u}\uparrow \quad {}^{z}\nearrow & \xrightarrow{\quad g \quad} \\
Z & &
\end{array}
$$

- Equalizers are monic.

Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

We say a *coequalizer* of f, g consists of an object Q and an arrow $q : B \to Q$ universal such that

$$q \circ f = q \circ g$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$
\begin{array}{ccc}
A \underset{g}{\overset{f}{\rightrightarrows}} B & \xrightarrow{q} Q \\
& {}^{z}\searrow \quad \downarrow{\scriptstyle u} \\
& Z
\end{array}
$$

- Coequalizers are epic.

Groups in a category

A group ($\text{Group}(\mathbf{C})$) can be defined over a category $\mathbf{C}$.

$$
G \times G \xrightarrow{m} G \xleftarrow{i} G
$$
$$
{\scriptstyle u}\uparrow
$$
$$
1
$$

Where the arrows obey the following, m is associative, u is a unit, and i is an inverse for m, i.e. the following diagrams commute

$$
\begin{array}{ccc}
(G \times G) \times G & \xrightarrow{\cong} & G \times (G \times G) \\
{\scriptstyle m \times 1}\downarrow & & \downarrow{\scriptstyle 1 \times m} \\
G \times G & & G \times G \\
& {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\
& G &
\end{array}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{\langle u, 1_G \rangle} & G \times G \\
{\scriptstyle \langle 1_G, u \rangle}\downarrow & {\scriptstyle 1_G} & \downarrow{\scriptstyle m} \\
G \times G & \xrightarrow{m} & G
\end{array}
$$

$$
\begin{array}{ccccc}
G \times G & \xleftarrow{\langle 1_G, 1_G \rangle} & G & \xrightarrow{\langle 1_G, 1_G \rangle} & G \times G \\
{\scriptstyle 1_G \times i}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle i \times 1_G} \\
G \times G & \xrightarrow{m} & G & \xleftarrow{m} & G \times G
\end{array}
$$

- A homomorphism $h : G \to H$ of groups in a category $\mathbf{C}$ is an arrow such that, h preserves m, u, i, i.e. the following diagrams commute.

$$
\begin{array}{ccc}
G \times G & \xrightarrow{h \times h} & H \times H \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\
G & \xrightarrow{h} & H
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
{\scriptstyle u}\uparrow & \nearrow{\scriptstyle u} & \\
1 & &
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
G & \xrightarrow{h} & H
\end{array}
$$

- The objects in the category of groups (i.e. $\text{Group}(\mathbf{Grp})$) are abelian groups.

Congruence

A *congruence* on a category is a equivalance relation on arrows ($f \sim g$) s.t.
- $f \sim g \implies \text{dom}(f) = \text{dom}(f)$ and $\text{cod}(f) = \text{cod}(g)$.
- $f \sim g \implies bfa \sim bga$

Let C_0, C_1 denote the class of objects and arrows for a category $\mathbf{C}$. Then a *congruence category* $\mathbf{C}^\sim$ is defined as follows,
- $(\mathbf{C}^\sim)_0 = \mathbf{C}_0$
- $(\mathbf{C}^\sim)_1 = \{\langle f, g \rangle | f \sim g\}$
- $\tilde{1}_C = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$
\mathbf{C}^\sim \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{C}
$$

We define the *quotient category* of the congruence as the coequalizer, i.e,

$$
\mathbf{C}^\sim \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{C} \xrightarrow{\pi} \mathbf{C}/\sim
$$

Finitely presented category

Consider the free category $\mathbf{C}(G)$ on a finite graph G. And the finite set of relations $\sum$ to be relations of the form $(g_1 \circ \cdots \circ g_n) = (g'_1 \circ \cdots \circ g'_m)$ for $g_i \in G$ and $\text{dom}(g_n) = \text{dom}(g'_m)$ and $\text{cod}(g_1) = \text{cod}(g'_1)$. Let $\sim_\Sigma$ be the smallest congruence $g \sim g'$ if $g = g' \in \sum$. We call the quotient by this congruence to be a *finitely presented category*.