

Introductory Complex Analysis Cheat Sheet

Field of Complex Numbers

We construct the field of complex numbers as the following quotient ring, $\mathbb{C} = \mathbb{R}[x]/\langle x^2 + 1 \rangle$

Algebra of Complex Numbers

- Addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$
- Multiplication: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$
- Division: $\frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$
- Square root: $\sqrt{a + ib} = \pm \left(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i \frac{b}{|b|} \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \right)$
- $\Re(a + ib) = a, \Im(a + ib) = b$

Conjugation, Absolute Value

- **Complex conjugation:** $\overline{a + ib} = a - ib$
 - $\overline{a + b} = \overline{a} + \overline{b}$
 - $\overline{ab} = \overline{a} \cdot \overline{b}$

Geometrically, conjugation is reflection over the real axis.

- **Absolute value:** $|a| = +\sqrt{a\bar{a}}$
 - $|ab| = |a| \cdot |b|$
 - $|a + b|^2 = |a|^2 + |b|^2 + 2\Re(a\bar{b})$
 - $|a - b|^2 = |a|^2 + |b|^2 - 2\Re(a\bar{b})$
 - $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$

The absolute value function forms the metric on \mathbb{C} . \mathbb{C} is complete under this metric.

Basic Topological definitions in \mathbb{C}

Some basic results:

- For $z_0 \in \mathbb{C}, r > 0$ we denote the ball (i.e. disk) of radius r around z_0 to be $D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- A point $z \in \mathbb{C}$ is a **limit point** of $E \subseteq \mathbb{C}$ if $\forall \varepsilon > 0, D(z, \varepsilon) \cap E$ contains a point other than z .
- A subset $E \subseteq \mathbb{C}$ is said to be **open** if $\forall z \in E, \exists r > 0$, s.t. $D(z, r) \subset E$.
- A subset $E \subseteq \mathbb{C}$ is said to be **closed**, if $\mathbb{C} \setminus E$ is open in \mathbb{C} . Or equivalently a set which contains all its limit points.

Some properties of open sets:

- \mathbb{C} and \emptyset are open subsets of \mathbb{C} .
- All finite intersections of open sets are open sets.
- The collection of all open sets on \mathbb{C} form a topology on \mathbb{C} .

Interior, closure, density

- **Interior:** Let $E \subseteq \mathbb{C}$. The interior of E is defined as, E° = set of all interior points of E , or equivalently, $\cup \{\Omega \mid \Omega \subseteq E \wedge \Omega \text{ is open in } \mathbb{C}\}$
- **Closure:** Let $E \subseteq \mathbb{C}$. The closure of E is defined as $\{F \mid E \subseteq F \wedge F \text{ is closed in } \mathbb{C}\}$
- **Density:** Let $E \subseteq D$, the closure of E in D is D . Then E is called dense in D .

Path: A path in a metric space from a point $x \in X$ to $y \in Y$ is a continuous mapping $\gamma : [0, 1] \rightarrow X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

Separated and Connected

For a metric space (X, d) .

- **Separated:** X is separated if \exists disjoint non-empty open subsets A, B of X s.t. $X = A \cup B$.
- **Connected:**
 - X is connected if it has no separation.
 - X is connected $\iff X$ does not contain a proper subset of X which is both open and closed in X .
 - Continuous functions preserve connectedness.
 - An open subset $\Omega \subseteq \mathbb{C}$ is connected \iff for $z, w \in \Omega$, there exists a path from z to w .

Basic Topological definitions in \mathbb{C} contd.

Open cover: Let (X, d) be a metric space and E be a collection of open sets in X . We say that \mathcal{U} is an open cover of a subset $K \subseteq X$, if $K \subset \bigcup \{\mathcal{U} \mid \mathcal{U} \in E\}$

Compactness: For some $K \subseteq X$ is compact if for every open cover E of K , there exists $E_1, \dots, E_n \in E$ s.t. $K \subset \bigcup_{i=1}^n E_i$, i.e. it is compact if it has a finite open cover.

- In a metric space, a compact set is closed.
- A closed subset of a compact set is closed.

Limit point compact: We say a metric space X is limit point compact if every infinite subset of X has a limit point.

- If X is a compact metric space, then it is also limit point compact.

Sequentially compact: We say a metric space X is sequentially compact if every sequence has a convergent sub-sequence.

- If X is limit point compact then X is sequentially compact.
- Let X be sequentially compact, then X is a compact metric space.

Lebesgue number lemma: Let X be sequentially compact, and let \mathcal{U} be an open cover of X . Then $\exists \delta > 0$ s.t. for $x \in X, \exists u \in \mathcal{U}$ s.t. $B(x, \delta) \subseteq u$.

Isometries on the Complex Plane

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called an **isometry** if $|f(z) - f(w)| = |z - w|, \forall z, w \in \mathbb{C}$.

- Let f be an isometry s.t. $f(0) = 0$, then the inner product $\langle f(z), f(w) \rangle = \langle z, w \rangle, \forall z, w \in \mathbb{C}$.
- If f is an isometry s.t. $f(0) = 0$ then f is a linear map.
- The standard argument for $a + ib \in \mathbb{C}, \text{Arg}(a + ib) = \tan^{-1} \frac{b}{a}$

Functions on the Complex Plane

Uniform convergence: Let $\Omega \subseteq \mathbb{C}$ and $f_1, \dots, f_n : \Omega \rightarrow \mathbb{C}$ be a set of functions on Ω . We say, $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f if given $\varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon, \forall x \in \Omega$ and $n \geq N$.

Complex exponential: For $z \in \mathbb{C}, \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Trigonometric functions: For $z \in \mathbb{C}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$

Hyperbolic trigonometric functions: For $z \in \mathbb{C}, \cosh(z) = \frac{e^z + e^{-z}}{2}$ and $\sinh(z) = \frac{e^z - e^{-z}}{2}$

Complex differentiability

Complex derivative: Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \rightarrow \mathbb{C}$, we say that f is complex differentiable at a point $z_0 \in \Omega$ if z_0 is an interior point and the following limit exists $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$. The limit is denoted as $f'(z_0)$ or $\frac{df(z)}{dz}$.

Holomorphic functions: If $f : \Omega \rightarrow \mathbb{C}$ is complex differentiable at every point $z \in \Omega$, then f is said to be a holomorphic on Ω . **Entire function:** Functions which are complex differentiable on \mathbb{C} are called entire functions.

- Complex differentiability implies continuity.
- Complex derivatives of a function are linear transformations.
- **Product rule:** If $f, g : \Omega \rightarrow \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$. Then fg is complex differentiable at z_0 with derivative $f'(z_0)g(z_0) + g'(z_0)f(z_0)$.
- **Quotient rule:** If $f, g : \Omega \rightarrow \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and g doesn't vanish at z_0 . Then $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$
- **Chain rule:** If $f : \Omega \rightarrow \mathbb{C}$ and $g : D \rightarrow \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and $f(\Omega) \subseteq D$. Then $g(f(x))'(z_0) = g'(f(z_0))f'(z_0)$

Power Series

Formal Power Series: A formal power series around $z_0 \in \mathbb{C}$ is a formal expansion $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $a_n \in \mathbb{C}$ and z is indeterminate.

Radius of convergence: For a formal power series $\sum a_n(z - z_0)^n$ the radius of convergence $R \in [0, \infty]$ given by $R = \liminf_{n \rightarrow \infty} |a_n|^{-1/n}$. Using the ratio test is identical i.e. $R = \liminf_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|}$.

- The series converges absolutely when $z \in B(z_0, R)$, and for $r < R$, the series converges uniformly, else if $|z - z_0| > R$ the series diverges.
- Let $z \in \mathbb{C}$ s.t. $|z - z_0| > R$, then \exists infinitely many $n \in \mathbb{N}$ s.t. $|a_n|^{-1/n} < |z - z_0|$.

Abels Theorem: Let $F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with a positive radius of convergence R , suppose $z_1 = z_0 + Re^{i\theta}$ be a point s.t. $F(z_1)$ converges. Then $\lim_{r \rightarrow R^-} F(z_0 + re^{i\theta}) = F(z_1)$

Differentiation of Power Series

Let $F(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a power series around z_0 with a radius of convergence R . Then F is **holomorphic** in $B(z_0, R)$.

- $F'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$ with same radius of convergence R .
- $a_n = \frac{F^{(n)}(z_0)}{n!}$

Cauchy product of two power series: For power series $F(z) = \sum a_n(z - z_0)^n$ and $G(z) = \sum a_n(z - z_0)^n$ with degree of convergence at least R . Then the Cauchy product $F(z)G(z) = \sum c_n(z - z_0)^n$ where $c_n = \sum_{k=0}^n a_k b_{n-k}$ also has degree of convergence at least R .

Cauchy-Riemann Differential Equations

For a complex function $f(z) = u(z) + iv(z)$,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \text{ or } f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Therefore, we get the two **Cauchy-Riemann Differential equations**,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \bullet \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

A function is holomorphic iff it satisfies the Cauchy-Riemann equations.

Wirtinger derivatives:

$$\bullet \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad \bullet \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

If f is holomorphic at z_0 then, $\frac{\partial f}{\partial \bar{z}} = 0$ and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2 \frac{\partial u}{\partial z}(z_0)$

Harmonic Functions

Laplacian: Define $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Harmonic function: Let $u : \Omega \rightarrow \mathbb{R}$ be a twice differentiable function. We say that u is a harmonic function if $\Delta u = 0$

For any holomorphic function $f, \Re(f), \Im(f)$ are examples of harmonic functions, but there are harmonic functions which are not holomorphic.

Boundary of a set: For a metric space $X, \Omega \in X$,

the boundary of $\Omega = \partial\Omega = \overline{\Omega} \cap \overline{\Omega^c}$

Maximum principle for harmonic functions: Let $u : \Omega \rightarrow \mathbb{R}$ be a twice differentiable harmonic function. Let $K \subset \Omega$ be a compact sub set of Ω . Then, $\sup_{z \in K} u(z) = \sup_{z \in \partial K} u(z)$ and $\inf_{z \in K} u(z) = \inf_{z \in \partial K} u(z)$

Maximum principle for holomorphic functions: Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. Then, for compact $K \subseteq \Omega$, we have, $\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$

Harmonic conjugate: Let $u : \Omega \rightarrow \mathbb{R}$ be a twice differentiable harmonic function. We say that $v : \Omega \rightarrow \mathbb{R}$ is a harmonic conjugate of u if $f = u + iv$ is holomorphic.

- For a harmonic function from \mathbb{C} to \mathbb{R} there exists a uniquely determined harmonic conjugate from \mathbb{C} to \mathbb{R} (up to constants).

Riemann Sphere

Extended complex plane: $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Consider S^2 , associate every point $z = x + iy$ with a line L that connects to the point $P = (0, 0, 1)$. $L = (1 - t)z + tP$, where $t \in \mathbb{R}$.

The point at which L for some z touches S^2 is given as $\left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)$, associate P with ∞ . This gives a stereographic projection of the complex plane unto S^2 . This sphere is known as the Riemann sphere.

Möbius transformations

A map $S(z) = \frac{az + b}{cz + d}$ for $a, b, c, d \in \mathbb{C}$ is called a Möbius transformation if $ad - bc \neq 0$.

Every mobius transformation is holomorphic at $\mathbb{C} \setminus \{-d/c\}$, i.e. every point other than is zero.

- The set of all mobius transformations is a group under transposition.
- S forms a bijection with $\widehat{\mathbb{C}}$

Every mobius transformation can be written as composition of,

- Translation: $S(z) = z + b, b \in \mathbb{C}$
- Dilation: $S(z) = az, a \neq 0, a = e^{i\theta}$
- Inversion: $S(z) = 1/z$

Curves in \mathbb{C}

A continuous parametrized curve is a continuous map $\gamma : [a, b] \rightarrow \mathbb{C}$ for $a, b \in \mathbb{R}$.

- If $a = b$ the curve is trivial.
- $\gamma(a)$ is initial point and $\gamma(b)$ is terminal point.
- γ is said to be closed if $\gamma(a) = \gamma(b)$.
- γ is said to be simple if it is injective, i.e. doesn't "cross" itself.
- A curve $-\gamma$ is a reversal of γ if $\gamma : [-a, -b] \rightarrow \mathbb{C}$ and if $-\gamma(t) = \gamma(-t)$
- γ is said to be continuously differentiable if $\gamma'(t_0)$ (defined usually) exists and is continuous.

Reparametrization: We say a curve $\gamma_2 : [a_2, b_2] \rightarrow \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \rightarrow \mathbb{C}$, if there exists a homeomorphism $\varphi : [a_1, b_1] \rightarrow [a_2, b_2]$ s.t. $\varphi(a_1) = a_2, \varphi(b_1) = b_2$ and $\gamma_2(\varphi(t)) = \gamma_1(t) \forall t \in [a_1, b_1]$.

- Reparametrization is an equivalence relation.

Arc length: Arc length of curve $\gamma = |\gamma| = \sup \sum_{i=0}^n |\gamma(x_{i+1}) - \gamma(x_i)|$ for all partitions of $[a, b]$.

- A curve that has a finite arc length is called **rectifiable**.

$$|\gamma| = \int_a^b |\gamma'(t)| dt$$

First Fundamental Theorem of Calculus

Let $f : \Omega \rightarrow \mathbb{C}$ be a continuous function. Let $F : \Omega \rightarrow \mathbb{C}$ be called the anti-derivative of f , i.e. F is holomorphic in Ω and $F'(z) = f(z), \forall z \in \Omega$. For a rectifiable curve $\gamma, \int_{\gamma} f(z) dz = F(z_1) - F(z_0)$, where z_0 is the initial point and z_1 is the terminal point.

Second Fundamental Theorem of Calculus

Let $f : \Omega \rightarrow \mathbb{C}$ be a continuous function such that $\int_{\gamma} f = 0$. Whenever γ is a closed polygonal path contained in Ω . For fixed $z_0 \in \Omega$, define a path γ_1 from z_0 to z_1 such that $F(z_1) = \int_{\gamma_1} f(z) dz$. Then F is a well defined holomorphic function s.t. $F'(z_1) = f(z_1) \forall z_1 \in \Omega$

Properties of complex integration

For continuously differentiable curves $\gamma : [a, b] \rightarrow \mathbb{C}$, and $\sigma : [b, c] \rightarrow \mathbb{C}$

- For a reparametrization $\hat{\gamma}$ of γ we can say that $\int_{\gamma} f(z) dz = \int_{\hat{\gamma}} f(z) dz$
- $\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$
- $\int_{\gamma+\sigma} f(z) dz = \int_{\gamma} f(z) dz + \int_{\sigma} f(z) dz$
- $\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$
- If f is bounded by M then $\int_{\gamma} f(z) dz \leq M|\gamma|$
- For $c \in \mathbb{C}$, we have, $\int_{\gamma} (cf + g)(z) dz = c \int_{\gamma} f(z) dz + \int_{\gamma} g(z) dz$

Homotopy of curves

Consider two curves $\gamma_0, \gamma_1 \rightarrow \Omega$ with the same initial and end point $[a, b]$.

We say that γ_0 is homotopic to γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $H : [0, 1] \times [a, b] \rightarrow \Omega$ s.t. $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t), \forall t \in [a, b]$.

$H(s, a) = z_0, H(s, b) = z_1 \forall s \in [0, 1]$

For **closed curves** γ_0 at z_0 and γ_1 at z_1 , we say that γ_0 is homotopic to γ_1 as closed curves if there exists a continuous map $H : [0, 1] \times [a, b] \rightarrow \Omega$, s.t. $H(0, t) = \gamma_0(t), H(1, t) = \gamma_1(t), \forall t \in [a, b]$. And $H(s, a) = H(s, b), \forall s \in [0, 1]$.

- Homotopy is an equivalence relation.

Cauchy-Goursat Theorem

Cauchy-Goursat theorem: If a curve γ_0 is homotopic to a reparametrization of γ_1 then, the integral of some function $f : \Omega \rightarrow \mathbb{C}$ is homotopy invariant, i.e.,

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

Alternative statement: Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω , and $\gamma_0 : [a, b] \rightarrow \Omega$ is a rectifiable curve which is null-homotopic (i.e. homotopic to a constant map). Then, $\int_{\gamma_0} f(z) dz = 0$

Cauchy's theorem for convex domains

Let $\Omega \subseteq \mathbb{C}$ be a convex and open set and $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω . Then f has an anti derivative F on Ω , and if γ is a closed rectifiable curve on Ω then $\int_{\gamma} f = 0$.

Cauchy's integral formula

Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. Fix $z_0 \in \Omega$ and let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$. Suppose γ is a closed curve in $\Omega \setminus \{z_0\}$ s.t. γ is homotopic to a reparametrization to γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz$$

Complex analytic function

An alternative statement, we say $f : \Omega \rightarrow \mathbb{C}$ is complex analytical if given $z_0 \in \Omega, \exists B(z_0, r) \subseteq \Omega$ s.t. the formal power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to f .

Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω . Suppose for $z_0 \in \Omega, \overline{B(z_0, r)} \subset \Omega$, then for every $n \in \mathbb{N}$, let $a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$ where $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to $f(z)$.

Corollary: If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic then f' is also holomorphic. Therefore f is infinitely differentiable.

Factor theorem for analytic function

For a analytic function $f : \Omega \rightarrow \mathbb{C}$ s.t. $f(z_0) = 0$ at $z_0 \in \Omega, \exists$ a unique analytic function $g : \Omega \rightarrow \mathbb{C}$ s.t. $f(z) = (z - z_0)g(z)$

Principle of analytical continuation

- Let Ω be open and connected subset of \mathbb{C} . and $f, g : \Omega \rightarrow \mathbb{C}$ be analytic functions on Ω . Suppose f, g agree on a non-empty subset of Ω . Then $f \equiv g$ on Ω .
- A consequence to this is that, non-trivial holomorphic functions have isolated zeros.

Higher-order Cauchy integral formula

Let $f : \Omega \rightarrow \mathbb{C}$ be analytic on Ω and $z_0 \in \Omega$ with $\overline{B(z_0, r)} \subseteq \Omega$. Let γ be a closed curve in $\Omega \setminus \{z_0\}$ that is homotopic to a reparametrization of γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Cauchy estimates: If $|f(z)| \leq M \forall z \in \gamma([0, 2\pi])$ then, $\forall n \in \mathbb{N}$, then we have $|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

Liouville's Theorem

Let f be a entire function which is bounded. Then f is a constant function.

Fundamental Theorem of Algebra

Let $p(z) = a_0 + a_1z + \dots + a_nz^n$ be a non constant polynomial s.t. $a_i \in \mathbb{C}, a_n \neq 0$. Then $\exists z_1, z_2, \dots, z_n$ s.t. $p(z) = a_n(z - z_1) \dots (z - z_n)$.

Morera's Theorem

Let $f : \Omega \rightarrow \mathbb{C}$ be a continuous function such that, $\int_{\gamma} f(z) dz = 0, \forall$ closed polygonal paths $\gamma \in \Omega$. Then f is holomorphic on Ω .

Uniform limit of holomorphic functions

Let $f_n : \Omega \rightarrow \mathbb{C}$ be a holomorphic on $\Omega, \forall n \in \mathbb{N}$ s.t. f_n converges uniformly on compact sets to f . Then f is holomorphic.

Winding number

Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a closed curve and let z_0 be a point not in the image of γ . Then the winding number of γ around z_0 is

$$W_{\gamma}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

- Winding number is invariant over homotopy.
- Let z_0 be a point not in the image of γ then $\exists r > 0$ s.t. for $z \in B(z_0, r), W_{\gamma}(z_0) = W_{\gamma}(z)$
- The winding number is always an integer.
- The winding number is locally constant.

Generalized Cauchy Integral formula: Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω and $\gamma : [a, b] \rightarrow \Omega$ be a closed curve which is null homotopic. Then for z_0 not in the image of γ ,

$$f(z_0)W_{\gamma}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)} dz$$

Open Mapping Theorem

- $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on Ω . Then $G : \Omega \times \Omega \rightarrow \mathbb{C}$ given by

$$G(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{when } z \neq w \\ f'(z) & \text{when } z = w \end{cases}$$

then G is continuous.

- Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic on some open set. Suppose $z_0 \in \Omega$ s.t. $f'(z_0) \neq 0$. Then \exists a neighbourhood U of $z_0 \in \Omega$ s.t. f restricted to U is injective. And $V = f(U)$ is an open set and the inverse $g : V \rightarrow U$ of f is holomorphic.
- Let $f : \Omega \rightarrow \mathbb{C}$ be a non-constant holomorphic function on open, connected set Ω . Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Then \exists a neighbourhood U of z_0 and bijective holomorphic function φ on U s.t. $f(z) = w_0 + (\varphi(z))^m$ for $z \in U$ and some integer $m > 0$. And φ maps U onto $D(0, r)$ for some $r > 0$.

Open Mapping Theorem: Let $f : \Omega \rightarrow \mathbb{C}$ be a non-constant holomorphic function on open connected set Ω , then $f(\Omega)$ is an open set.

Schwarz reflection principle

Let Ω be a open connected set which is symmetric w.r.t \mathbb{R} . Then define the following,

- $\Omega_+ = \{z \in \Omega \mid \Im(z) > 0\}$
- $\Omega_- = \{z \in \Omega \mid \Im(z) < 0\}$
- $I = \{z \in \Omega \mid \Im(z) = 0\}$

Schwarz reflection principle: Let Ω be defined as above. Then if $f : \Omega_+ \cup I \rightarrow \mathbb{C}$ which is continuous on $\Omega_+ \cup I$ and holomorphic on Ω_+ . Suppose for $f(x) \in \mathbb{R}, \forall x \in I$ then there exists $g : \Omega \rightarrow \mathbb{C}$ holomorphic on Ω s.t. $g(z) = f(z)$ for $z \in \Omega_+ \cup I$

Singularity of a holomorphic function

- Isolated singularity:** If f is holomorphic on $B(z_0, R) \setminus \{z_0\}$ for some $R > 0$ then z_0 is called an isolated singularity.
- Removable singularity:** Let z_0 be an isolated singularity of a holomorphic function f as defined above. It is called removable if there exists holomorphic function g on $B(z_0, R)$ s.t. $g(z) = f(z)$ on $B(z_0, R) \setminus \{z_0\}$.
- Riemann removable singularity theorem:** Let z_0 be an isolated singularity of a function f , then z_0 is a removable singularity if and only if f is locally bounded around z_0 .
- Pole:** If z_0 is an isolated singularity as defined above and if $\lim_{z \rightarrow z_0} |f(z)| = \infty$ then z_0 is called a pole of f .
- Essential singularity:** A singularity that is neither removable nor a pole.

Doubly infinite series

Let z_n be a function defined for $n = 0, \pm 1, \pm 2, \dots$, then it is doubly infinite.

- A doubly infinite series converges if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ both converge.
- Splitting up the series in similar manners you can define absolute and uniform convergence.

Annulus

An annulus $A(z_0, R_1, R_2)$ around a point z_0 for $0 \leq R_1 \leq R_2$ is the set of all $z \in \mathbb{C}$ s.t. $R_1 \leq |z - z_0| \leq R_2$.

Laurent series expansion

Let f be a function holomorphic on $A(z_0, R_1, R_2)$, then there exists $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ s.t.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the doubly infinite series converges absolutely and uniformly in some $A(z_0, r_1, r_2)$ when $R_1 < r_1 < r_2 < R_2$.

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where $\gamma(z) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r < R_2$.

Important results

- f has a removable singularity at $z_0 \iff a_n = 0$ for $n < 0$ in the Laurent series expansion of f
- f has a pole at z_0 of order $m \iff a_n = 0$ for $n < -m$ in the Laurent series expansion of f .
- f has a essential singularity at $z_0 \iff a_n \neq 0$ for infinitely many negative integers n .

Casorati-Weierstrass theorem

Let z_0 be an essential singularity of f then given $\alpha \in \mathbb{C}$, there exists a sequence $z_n \in B(z_0, R) \setminus \{z_0\}$ s.t. $z_n \rightarrow z_0$ and $f(z_n) \rightarrow \alpha$.

- Alternatively, f approaches any given value arbitrarily closely in any neighborhood of an essential singularity.

Meromorphic functions

Let Ω be a open connected subset of \mathbb{C} and let $S \subset \Omega$. Let $f : \Omega \setminus S \rightarrow \mathbb{C}$ be holomorphic on Ω . We say that f is a meromorphic function on Ω if,

- S is a discrete set.
- f either has removable singularities or poles at point of S .

Operations on meromorphic functions

Let $\mathcal{M}(\Omega)$ denote the equivalence classes of meromorphic functions over Ω .

- We say that two meromorphic functions $f : \Omega \setminus S_1$ and $g : \Omega \setminus S_2$ are equivalent if $f(z) = g(z)$ on $\Omega \setminus (S_1 \cup S_2)$.
- For $f, g \in \mathcal{M}(\Omega)$, define $f + g$ to be the equivalence class of $(f + g) : \Omega \setminus (S_1 \cup S_2)$
- Similarly, fg is the equivalence class of $fg : \Omega \setminus (S_1 \cup S_2)$.

The space of all meromorphic functions is a field.

Order of meromorphic functions

The order of a meromorphic function is defined as follows,

- If $z_0 \in S$ is a removable singularity then the order of f at z_0 is the order of the zero at z_0 of f , i.e., $f(z) = (z - z_0)^m g(z)$ then m is the order.
- If $z_0 \in S$ is a pole and the pole is of order m then order of f at z_0 is $-m$.
- If $f \equiv 0$ then $\text{Ord}_{z_0} = \infty$.
- $\text{Ord}_{z_0}(f + g) \geq \min(\text{Ord}_{z_0}(f), \text{Ord}_{z_0}(g))$
- $\text{Ord}_{z_0}(fg) = \text{Ord}_{z_0}(f) + \text{Ord}_{z_0}(g)$

Residue of a function

Residue of a function: Let $f : \Omega \setminus S \rightarrow \mathbb{C}$ be a holomorphic function, where Ω is an open set and S is a discrete subset of Ω . Then for $z_0 \in S$, let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$ and $B(z_0, r) = \{z_0\}$. Then in $B(z_0, r) \setminus \{z_0\}$, consider the Laurent series expansion of f given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. We define the residue of f at z_0 to be $\text{Res}(f, z_0) = a_{-1}$.

- If z_0 is a removable singularity then $\text{Res}(z_0) = 0$.
- If z_0 is a pole of order m then $(z - z_0)^m f(z) = g(z)$, where $g(z) \neq 0$ on $B(z_0, r) \setminus \{z_0\}$ then, $\text{Res}(z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$.

Residue theorem

Let Ω be an open connected subset of \mathbb{C} and S be a finite subset of Ω and let $f : \Omega \setminus S \rightarrow \mathbb{C}$ be a holomorphic function. Let γ be a null homotopic closed curve on Ω . Then,

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{j=1}^k W_{\gamma}(z_j) \text{Res}(f, z_j)$$

where $S = \{z_1, \dots, z_k\}$ and W_{γ} is the winding number.

Log derivative

For a holomorphic function $f : \Omega \rightarrow \mathbb{C}$. Define the log derivative of f to be the meromorphic function $\frac{f'(z)}{f(z)}$.

- $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$
- $\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}$
- When f has a pole of order m at z_0 then for $f(z) = \frac{g(z)}{(z - z_0)^m}$ the log derivative of f is $\frac{g'(z)}{g(z)} - \frac{m}{(z - z_0)}$

Argument principle

Let $f : \Omega \setminus S \rightarrow \mathbb{C}$ be a meromorphic function s.t. f has zeros of order d_1, \dots, d_n at z_1, \dots, z_n after removing the removable singularities. And f has poles of order e_1, \dots, e_m at points w_1, \dots, w_m . Let γ be a closed curve which is null homotopic in Ω s.t. the zeros and poles don't lie in the image of γ . Then,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{i=0}^n d_i W_{\gamma}(z_i) - \sum_{j=1}^m e_j W_{\gamma}(w_j)$$

Rouche's theorem

Let γ be a closed curve which is null homotopic in Ω . Let f, g be functions holomorphic in Ω and $|g(z)| < |f(z)|$ on γ then f and $f + g$ have the same number of zeros counting multiplicities on the interior of $H([0, 1] \times [a, b])$ where H is the null homotopy from γ to a constant path.

Branch of the complex logarithm

Let Ω be an open connected subset of $\mathbb{C} \setminus \{0\}$. Define a branch of the logarithm on Ω as a function $f : \Omega \rightarrow \mathbb{C}$ s.t. $\exp(f(z)) = z, \forall z \in \Omega$. For $\Omega = \mathbb{C} \setminus \{\Re(x) \leq 0\}$ define the standard branch to be

$$\text{Log}(z) = \ln |z| + i \text{Arg}(z)$$

As defined above $\text{Log}(z)$ is holomorphic on Ω .

Schawrz lemma

Let \mathbb{D} denote the open unit disc. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a holomoprhic function s.t. $f(0) = 0$. Then,

$$|f(z)| \leq |z|, \forall z \in \mathbb{D}, \text{ and } |f'(z)| \leq 1$$

Also, if $|f(z)| = |z|$ for some $z \in \mathbb{D}$ or if $|f'(0)| = 1$ then $\exists \lambda \in \mathbb{C}, |\lambda| = 1$ s.t. $f(z) = \lambda z$.

Automorphism

A function $f : \Omega \rightarrow \Omega$ is an automorphism if f is holomorphic and has a holomorphic inverse.

Automorphisms of the unit disc

Define a function $\varphi_\alpha : \mathbb{D} \rightarrow \mathbb{C}$ defined as $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$.

Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be an automorphism. Then there exists $\alpha \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ s.t.

$$f(z) = \lambda \varphi_\alpha(z)$$

Phragmén–Lindelöf method

Let $\Omega = \{z \in \Omega : a < \Re(z) < b\}$. Let $f : \overline{\Omega} \rightarrow \mathbb{C}$, s.t. f is continous on $\overline{\Omega}$ and holomorphic on Ω . Suppose for some $z = x + iy$, we have $|f(z)| < B$ and let $M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\}$. Then,

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}$$

And further

$$|f(z)| \leq M(x) \leq \max\{M(a), M(b)\} = \sup_{z \in \partial\Omega} |f(z)|$$