Commutative Algebra Cheat Sheet

Rings

A **ring** A is a set with two binary operations addition and multiplication such that
- A is an abelian group with addition.
- Multiplication is associative and distributive over addition.

Additionally we consider rings with commutativity and existence of multiplicative identity 1.
A function $\varphi : A \to B$ between rings is a **homomorphism** if it preserves addition multiplication and sends 1 to 1.
A **subring** is a subset of a ring that is also a ring with the induced relations.

Universal mapping property

Ideals

An **ideal** $\mathfrak{a}$ of a ring A is a subset of A which is a additive subgroup group and for $x \in \mathfrak{a}, xA \subseteq \mathfrak{a}$.
The cosets of $\mathfrak{a} \in A$ form a quotient ring $A/\mathfrak{a}$.
Correspondence theorem for rings: There is a bijection between ideals of A containing $\mathfrak{a}$ and the ideals of $A/\mathfrak{a}$.

Zero divisors, units

An element is called a **zero divisor** if its product with a non zero element gives 0.
A commutative ring with the only zero divisor being zero is called an **integral domain**.
An element is called a **unit** if its product with some element gives 1.
- $x \in A$ is a unit $\iff \langle x \rangle = \{ax \mid a \in A\} = A = \langle 1 \rangle$

A ring in which every non zero element is a unit is called a **field**.
- All fields are integral domains.
- All finite integral domains are fields.
- The only ideals in a field F are 0 and $\langle 1 \rangle = F$

Prime and Maximal ideals

A proper ideal $\mathfrak{p} \in A$ is called **prime** if for $xy \in \mathfrak{a} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ alternatively if $A/\mathfrak{p}$ is an integral domain.
A proper ideal $\mathfrak{m} \in A$ is called **maximal** if it is maximal with respect to inclusion alternatively if $A/\mathfrak{m}$ is a field.
A ring with exactly one maximal ideal is called a **local** ring. And its subsequent quotient is called the **residue field** of the ring. If number of maximal ideals are finite then it is called **semi local**.
A ring is local iff its set of non units form an ideal.5

Ideal operations

For ideals $\mathfrak{a}, \mathfrak{b} \in A$,
- $\mathfrak{a} + \mathfrak{b}$ forms an ideal and is the smallest ideal containing $\mathfrak{a}$ and $\mathfrak{b}$.
- Intersection of ideals is an ideal.
- Product of ideals is an ideal.
- Unions are in general not ideals.
- $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq a + b$.
- The distributive laws hold.

The **ideal quotient/colon ideal** is defined as $(\mathfrak{a} : \mathfrak{b}) = \{x \in A : x\mathfrak{b} \subseteq \mathfrak{a}\}$ and $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.
For a ring homomorphism $\varphi : A \to B$ and some ideals $\mathfrak{a} \in A, \mathfrak{b} \in B$ we define the extension of $\mathfrak{a}$ as $\mathfrak{a}^e$ as the ideal generated by $\varphi(\mathfrak{a})$.
And the contraction of $\mathfrak{b}$ just its preimage in A which is always an ideal.

Radical ideals

For an ideal $\mathfrak{a}$ its radical ideal is denoted as $\sqrt{\mathfrak{a}}$ or $r(\mathfrak{a}) = \{x \in A \mid x^n \in \mathfrak{a}\}$

Nilradical and Jacobson ideal

The **nilradical** of a ring A is denoted by $N(R)$ and consists of the set of all nilpotent elements of A. Equivalently it is the intersection of all prime ideals. This shows that a radical of an ideal is just the intersection of prime ideals containing it.
The **Jacobson** radical of a ring A denoted by $J(R)$ is the intersection of all its maximal ideals. An element x is in the Jacobson radical $\iff 1 - xy$ is a unit in $A, \forall y \in A$.

Chinese Remainder Theorem

For a ring A, let $I_1, \ldots, I_n$ be ideals of the ring A. Consider the map $\pi : A \to A/I_1 \times \cdots \times A/I_n$ defined as $\pi(a) = (a \mod I_1, \ldots, a \mod I_n)$. Then $\ker \pi = I_1 \bigcap \cdots \bigcap I_n$, i.e. it is surjective iff $I_1, \cdots I_n$ are pairwise comaximal. If π is a surjection we have,

$$A/\bigcap I_k = A/\prod I_k \cong \prod(A/I_k)$$

Nakayama's Lemma

For M a finitely generated A module then for its Jacobson radical $J(A)$ we have $J(A)M = M \implies M = 0$

Prime avoidance theorem

For ring A consider $\mathfrak{a} \subset A$ that is stable under addition and multiplication and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ ideals such that $\mathfrak{p}_3, \ldots, \mathfrak{p}_n$ are prime in A. If $\mathfrak{a}$ is contained in the union of all $\mathfrak{p}_i$ then $\mathfrak{a} \subset \mathfrak{p}_i$ for some i.