# Category Theory Cheat Sheet

## Category

A *category* consists of the following,

- Objects: A,B,C,...
- Arrows/Morphisms: f,g,h,...
- For each $f$ there exists, $\mathrm{dom}(f), \mathrm{cod}(f)$ called domain and codomain of $f$. We write $f : A \to B$ to indicate $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given $f : A \to B$ and $g : B \to C$ there exists, $g \circ f : A \to C$ called the *composite* of $f$ and $g$.
- For each $A$, there exists $1_A : A \to A$ called the *identity arrow* of $A$.
- Arrows should also satisfy the following,
  - Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f : A \to B, g : B \to C, h : C \to D$.
  - Unit: $f \circ 1_A = f = 1_B \circ f$, for all $f : A \to B$.

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[loop above, "1_B"] \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[loop above, "1_C"] \arrow[r, "h"'] & D \arrow[loop above, "1_D"]
\end{tikzcd}$$

## Functor

A *functor* $F : C \to D$ between $C$ and $D$ is a mapping of objects and arrows to arrows, such that

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(1_a) = 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

## Isomorphism

In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an *isomorphism* if there exists an arrow $g : B \to C$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. We say, $g = f^{-1}$. And that $A \cong B$, i.e., $A$ is isomorphic to $B$.

## Monoid

A set $M$ with binary operation $\cdot$ is called a *monoid* if it is associative and has an identity $u \in M$, i.e., for $x, y, z \in M$

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $u \cdot x = x = x \cdot u$.

A monoid with an inverse for each element is called a *group*.

- *Cayley's theorem:* Every group G is isomorphic to a group of permutations.

## Constructions on categories

- **Product category:** The product of two categories $\mathbf{C}$ and $\mathbf{D}$ written as $\mathbf{C} \times \mathbf{D}$ has objects of the form $(C, D)$ for $C \in \mathbf{C}$ and $D \in \mathbf{D}$, and arrows of the form $(f, g) : (C, D) \to (C', D')$ for $f : C \to C' \in \mathbf{C}$ and $g : D \to D' \in \mathbf{D}$.
  Composition and units are defined componentwise, i.e. $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ and $1_{(C,D)} = (1_C, 1_D)$.
- **Opposite/Dual category:** For category $\mathbf{C}$ its opposite category $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$ but an arrow $f : C \to D$ in $\mathbf{C}^{op}$ is an arrow $f : D \to C$ in $\mathbf{C}$.
  For notational simplicity we say $f^* : D^* \to C^*$ in $\mathbf{C}^{op}$ for $f : C \to D$ in $\mathbf{C}$. Composition and units are therefore defined as follows, $f^* \circ g* = (g \circ f)^*$ and $(1_{C^*} = 1_C)^*$

## Constructions on categories contd.

- **Arrow category:** For category $\mathbf{C}$ its arrow category $\mathbf{C}^{\to}$ has the arrows of $\mathbf{C}$ as objects and an arrow $g$ from $f : A \to B$ to $f' : A' \to B'$ in $\mathbf{C}^{\to}$ is the following commutative square

$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "g_2"'] & B'
\end{tikzcd}$$

  where $g_1, g_2$ are arrows in $\mathbf{C}$, i.e. an arrow is a pair of arrows $g = (g_1, g_2)$ s.t. $g_2 \circ f = f' \circ g_1$. The identity of an object $f : A \to B$ is the pair $(1_A, 1_B)$ and composition is componentwise.
- **Slice category:** For category $\mathbf{C}$ its slice category over $C \in \mathbf{C}$ is $\mathbf{C}/C$ where its objects are all arrows $f \in \mathbf{C}$ s.t. $\mathrm{cod}(f) = C$. An arrow $a$ from $f : X \to C$ to $f' : X' \to C$ is an arrow $a : X \to X'$ in $\mathbf{C}$ s.t. $f' \circ a = f$, i.e.

$$\begin{tikzcd}
X \arrow[rr, "a"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& C &
\end{tikzcd}$$

  If $g : C \to D$ is any arrow then there exists a composition functor $g_* : \mathbf{C}/C \to \mathbf{C}/D$ defined as $g_*(f) = g \circ f$. Therefore the slice category of $\mathbf{C}$ with any of its objects is a functor from $\mathbf{C} \to \mathbf{Cat}$. This is called the *forgetful* functor as the base object is "forgetten".

## Free monoid

For a set $A$ a *word* over $A$ is any finite sequence of its elements. The *Kleene closure* of $A$ is defined to be the set of all words over $A$. Define the binary operation of concatenation on $A*$. Since it is associative, $A$ along with $*$ with the empty word "$-$" is a monoid called the **free monoid** on the set $A$.

**Universal mapping property (UMP) of free monoid:** Let $M(A)$ be the free monoid on a set $A$. There is a function $i : A \to |M(A)|$, and given any monoid $N$ and any function $f : A \to |N|$, there is a unique monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $|\overline{f}| \circ i = f$.

$$\begin{tikzcd}
M(A) \arrow[d, dashed, "\overline{f}"] \\
N
\end{tikzcd}
\qquad
\begin{tikzcd}
{|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\
A \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$A*$ has the UMP of the free monoid on $A$.

## Free category

A directed graph $G$ "generates" a free category $\mathbf{C}(G)$ whose objects are the vertices of the graph and its arrows are paths. Composition of arrows is defined as concatenation of paths.

**UMP of $\mathbf{C}(G)$** There is a graphic homomorphism $i : G \to |\mathbf{C}(G)|$, and given any category $\mathbf{D}$ and any graph homomorphism $h : G \to |\mathbf{D}|$, there is a unique functor $\overline{h} : \mathbf{C}(G) \to \mathbf{D}$ with $\overline{h} \circ i = h$.

$$\begin{tikzcd}
\mathbf{C}(G) \arrow[d, dashed, "\overline{h}"] \\
\mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
{|\mathbf{C}(G)|} \arrow[r, "|\overline{h}|"] & {|\mathbf{D}|} \\
G \arrow[u, "i"] \arrow[ur, "h"'] &
\end{tikzcd}$$

## Small categories

A category is called *small* if it has a small set of objects and arrows. (i.e., not classes). It is called large otherwise.

A category $\mathbf{C}$ is *locally small* if for all objects $X, Y \in \mathbf{C}$, the collection $\mathrm{Hom}_{\mathbf{C}}(X, Y) = \{f \in \mathbf{C}_1 \mid f : X \to Y\}$ is a small set.

## Types of morphisms

**Monomoprhism:** In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism (monic), if for any $g, h, : C \to A, fg = fh \implies g = h$.

$$\begin{tikzcd}
C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

**Epimorphism:** In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism (epic), if for any $i, j : B \to D$ $if = jf \implies i = j$.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, shift left, "j"] \arrow[r, shift right, "i"'] & D
\end{tikzcd}$$

- We say, $f : A \rightarrowtail B$ if $f$ is a monomorphism and $f : A \twoheadrightarrow B$ if $f$ is an epimorphism.
- Every isomorphism is both a monomorphism and an epimorphism. The converse need not be true.
- A **split** mono (epi) is an arrow $m : A \to B$ with a left (right) inverse $r$. The inverse arrow $r$ is called the *retraction*, $m$ is called a *section* of $r$ and $A$ is called a *retract* of $B$.

## Initial and terminal objects

An object $0 \in \mathbf{C}$ is *initial* if for any object $C \in \mathbf{C} \exists !$ morphism $0 \to C$.
An object $1 \in \mathbf{C}$ is *terminal* if for any object $C \in \mathbf{C} \exists !$ morphism $C \to 1$.
Initial and terminal objects are unique up to isomorphism.

## Generalized elements

For an object $A \in \mathbf{C}$ arbitrary arrows $x : X \to A$ are called the *generalized elements* of $A$ with stage of definition given by $X$.

## Product of objects

In any category $\mathbf{C}$, a product diagram for the objects $A, B$ consists of an object $P$ and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP. Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique arrow $u : X \to P$, making the following diagram commute

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

The product $P$ is unique up to isomorphism.

## Categories with products

A category which has a product for every pair of objects is said to have *binary products*.

A category is said to have *all finite products*, if it has a terminal object and all binary products.

A category has *all small products* if every set of objects has a product.

## Covariant representable functor

The functor $\mathrm{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$ is called a covariant representable functor (for some object $A \in \mathbf{C}$).
For a category with products a covariant representable functor preserves products.

## Duality

If any statement about categories holds for all categories then so does the dual statement.

## Coproducts

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of $A$ and $B$ if for any $Z$ and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ making the diagram commute.

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dashed, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

## Equalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We say an *equalizer* of $f, g$ consists of an object $E$ and an arrow $e : E \to A$ universal such that

$$f \circ e = g \circ e$$

i.e., for any $z : Z \to A$ with $f \circ z = g \circ z$, there exists a unique $u : Z \to E$ with $e \circ u = z$

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dashed, "u"] \arrow[ur, "z"] & & \end{tikzcd}$$

- Equalizers are monic.

## Coequalizers

In some category $\mathbf{C}$ given the following diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

We say a *coequalizer* of $f, g$ consists of an object $Q$ and an arrow $q : B \to Q$ universal such that

$$q \circ f = q \circ g$$

i.e., for any $z : B \to Z$ with $z \circ f = z \circ g$, there exists a unique $u : Q \to Z$ with $u \circ q = z$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

- Coequalizers are epic.

## Groups in a category

A group ($\mathrm{Group}(\mathbf{C})$) can be defined over a category $\mathbf{C}$.

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$$

Where the arrows obey the following, $m$ is associative, $u$ is a unit, and $i$ is an inverse for $m$, i.e. the following diagrams commute

$$\begin{tikzcd} (G \times G) \times G \arrow[rr, "\cong"] \arrow[d, "m \times 1"'] & & G \times (G \times G) \arrow[d, "1 \times m"] \\ G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\ & G & \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r, "\langle u, 1_G \rangle"] \arrow[d, "\langle 1_G, u \rangle"'] \arrow[dr, "1_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

$$\begin{tikzcd} G \times G \arrow[d, "1_G \times i"'] & G \arrow[l, "\langle 1_G, 1_G \rangle"'] \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[d, "u"'] & G \times G \arrow[d, "i \times 1_G"] \\ G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"] \end{tikzcd}$$

- A homomorphism $h : G \to H$ of groups in a category $\mathbf{C}$ is an arrow such that, $h$ preserves $m, u, i$, i.e. the following diagrams commmute.

$$\begin{tikzcd} G \times G \arrow[r, "h \times h"] \arrow[d, "m"'] & H \times H \arrow[d, "m"] \\ G \arrow[r, "h"'] & H \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r, "h"] & H \\ 1 \arrow[u, "u"] \arrow[ur, "u"'] & \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "i"'] & H \arrow[d, "i"] \\ G \arrow[r, "h"'] & H \end{tikzcd}$$

- The objects in the category of groups (i.e. $\mathrm{Group}(\mathbf{Grp})$) are abelian groups.

## Congruence

A *congruence* on a category is a equivalance relation on arrows ($f \sim g$) s.t.

- $f \sim g \implies \mathrm{dom}(f) = \mathrm{dom}(f)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$.
- $f \sim g \implies bfa \sim bga$

Let $C_0, C_1$ denote the class of objects and arrows for a category $\mathbf{C}$. Then a *congruence category* $\mathbf{C}^\sim$ is defined as follows,

- $(\mathbf{C}^\sim)_0 = \mathbf{C}_0$
- $(\mathbf{C}^\sim)_1 = \{\langle f, g \rangle | f \sim g\}$
- $\tilde{1}_C = \langle 1_C, 1_C \rangle$
- $\langle f', g' \rangle \circ \langle f, g \rangle = \langle f'f, g'g \rangle$

$$\begin{tikzcd} \mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C} \end{tikzcd}$$

We define the *quotient category* of the congruence as the coequalizer, i.e,

$$\begin{tikzcd} \mathbf{C}^\sim \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbf{C} \arrow[r, "\pi"] & \mathbf{C}/\sim \end{tikzcd}$$

## Finitely presented category

Consider the free category $\mathbf{C}(G)$ on a finite graph $G$. And the finite set of relations $\sum$ to be relations of the form $(g_1 \circ \cdots \circ g_n) = (g'_1 \circ \cdots \circ g'_m)$ for $g_i \in G$ and $\mathrm{dom}(g_n) = \mathrm{dom}(g'_m)$ and $\mathrm{cod}(g_1) = \mathrm{cod}(g'_1)$. Let $\sim_\Sigma$ be the smallest congruence $g \sim g'$ if $g = g' \in \sum$. We call the quotient by this congruence to be a *fintely presented category.*