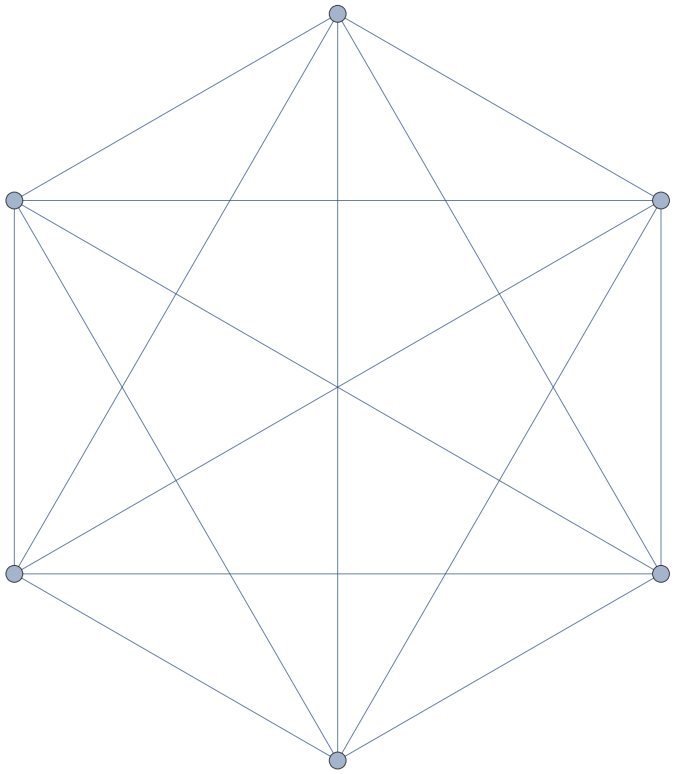
Reciprocal Multifactorial Constants

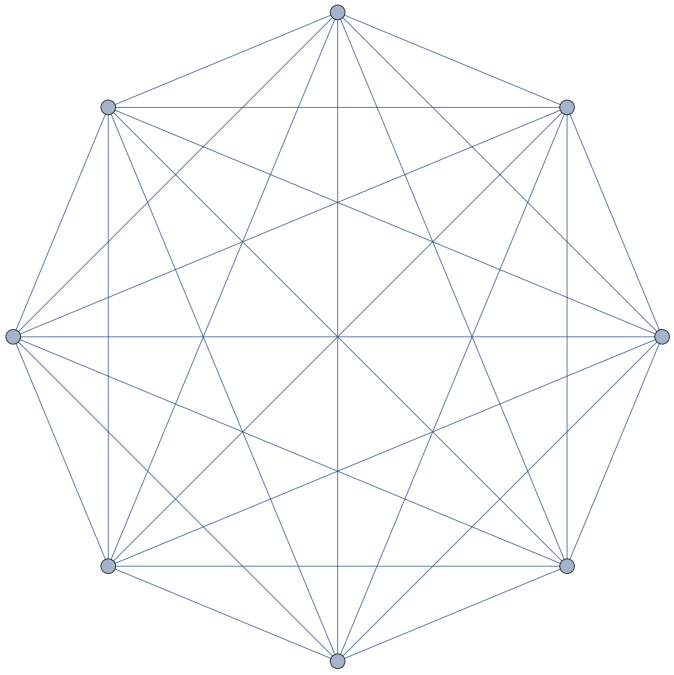
Perfect matchings for K_ 6 and K_8

In[ ]:=
```
k6 = CompleteGraph[6]
k8 = CompleteGraph[8]
```

Out[ ]=



Out[ ]=



In[ ]:=
```
l6 = Length[FindIndependentEdgeSet[k6]]
l8 = Length[FindIndependentEdgeSet[k8]]
```

Out[ ]= 3

Out[ ]= 4

```
In[ ]:= esl6 = Select[Subsets[EdgeList[k6], {l6}], IndependentEdgeSetQ[k6, #] &]
        esl8 = Select[Subsets[EdgeList[k8], {l8}], IndependentEdgeSetQ[k8, #] &]
```

```
Out[ ]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 5}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 6},
  {1 ↔ 3, 2 ↔ 5, 4 ↔ 6}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 5}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 6}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 6},
  {1 ↔ 4, 2 ↔ 6, 3 ↔ 5}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 6}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 6}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 4},
  {1 ↔ 6, 2 ↔ 3, 4 ↔ 5}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 5}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 4}}
```

```
Out[ ]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 8, 6 ↔ 7},
  {1 ↔ 2, 3 ↔ 5, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 8, 6 ↔ 7},
  {1 ↔ 2, 3 ↔ 6, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 8, 5 ↔ 7},
  {1 ↔ 2, 3 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 8, 5 ↔ 6},
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  {1 ↔ 3, 2 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 8, 5 ↔ 6},
  {1 ↔ 3, 2 ↔ 8, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 7, 5 ↔ 6},
  {1 ↔ 4, 2 ↔ 3, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 8, 6 ↔ 7},
  {1 ↔ 4, 2 ↔ 5, 3 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 8, 6 ↔ 7},
  {1 ↔ 4, 2 ↔ 6, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 8, 5 ↔ 7},
  {1 ↔ 4, 2 ↔ 7, 3 ↔ 5, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 6, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 8, 5 ↔ 6},
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  {1 ↔ 5, 2 ↔ 6, 3 ↔ 4, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 7, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 8, 4 ↔ 7},
  {1 ↔ 5, 2 ↔ 7, 3 ↔ 4, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 6, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 8, 4 ↔ 6},
  {1 ↔ 5, 2 ↔ 8, 3 ↔ 4, 6 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 6, 4 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 7, 4 ↔ 6},
  {1 ↔ 6, 2 ↔ 3, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 8, 5 ↔ 7},
  {1 ↔ 6, 2 ↔ 4, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 8, 5 ↔ 7},
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  {1 ↔ 8, 2 ↔ 7, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 6, 4 ↔ 5}}
```
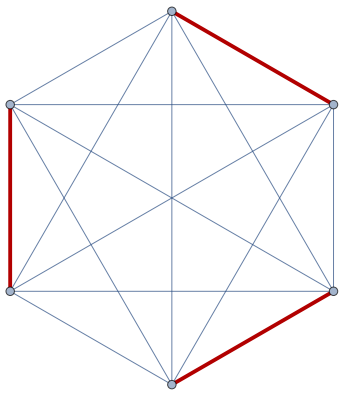

```
In[ ]:=
```

```
In[ ]:= Table[HighlightGraph[k6, h, GraphHighlightStyle → "Thick"], {h, esl6}]
```

Out[•]= {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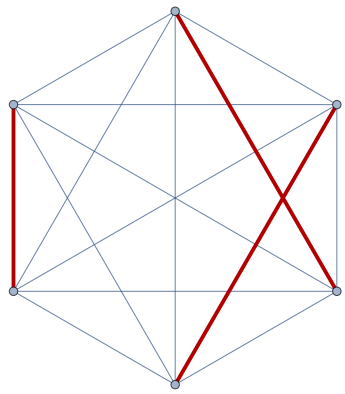
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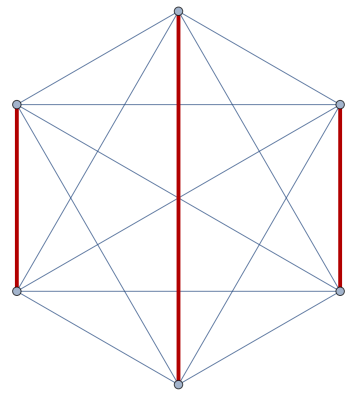
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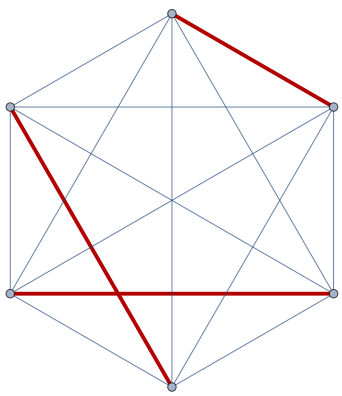
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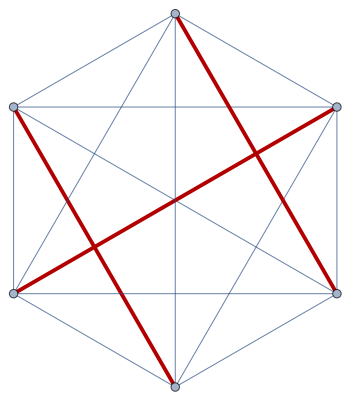
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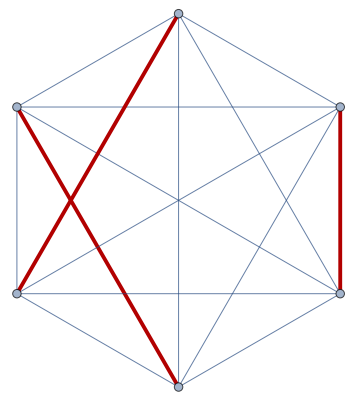
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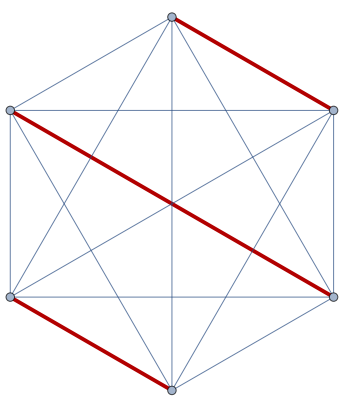
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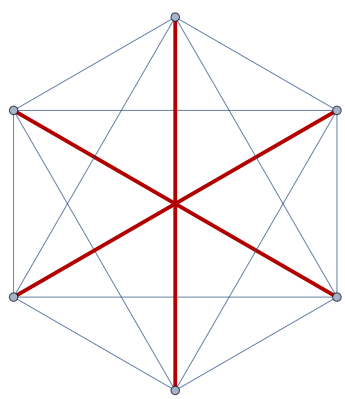
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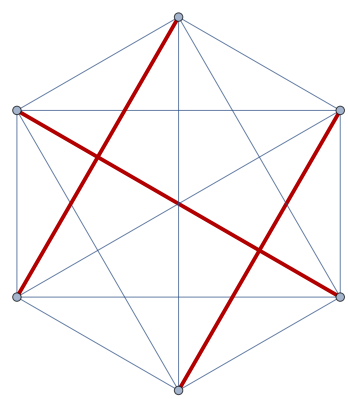
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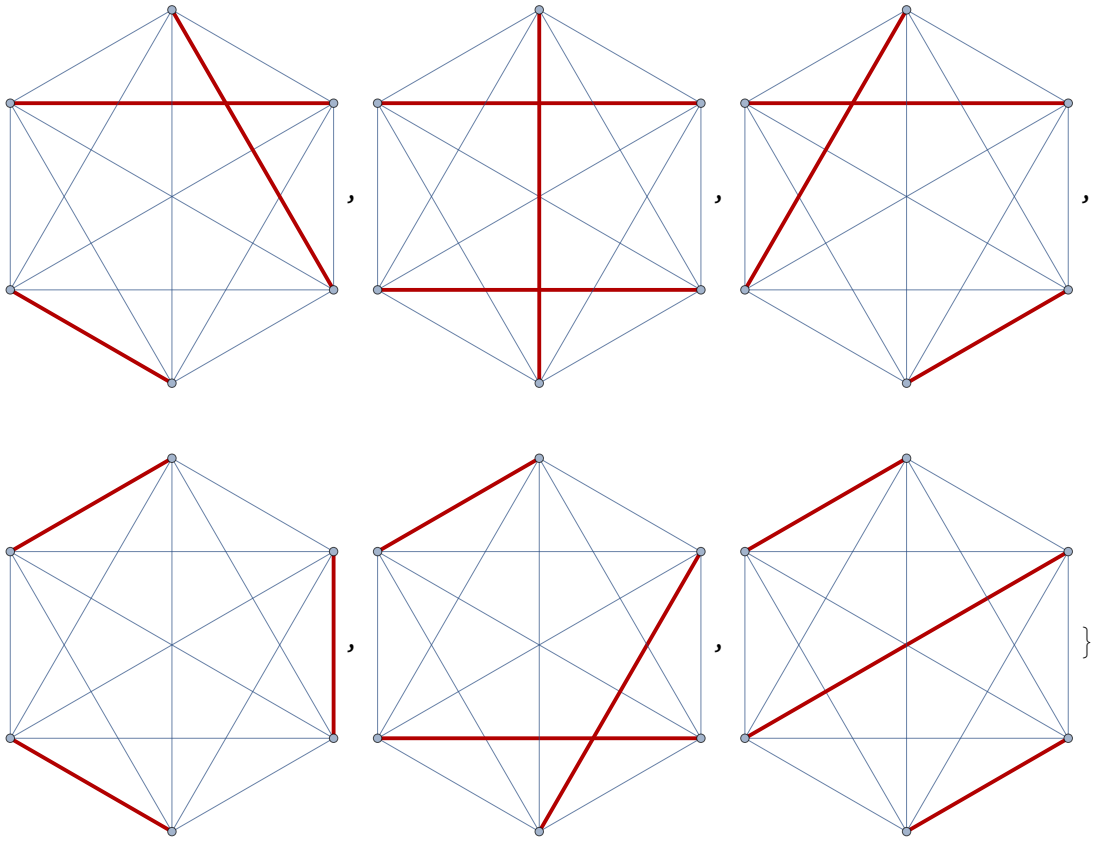
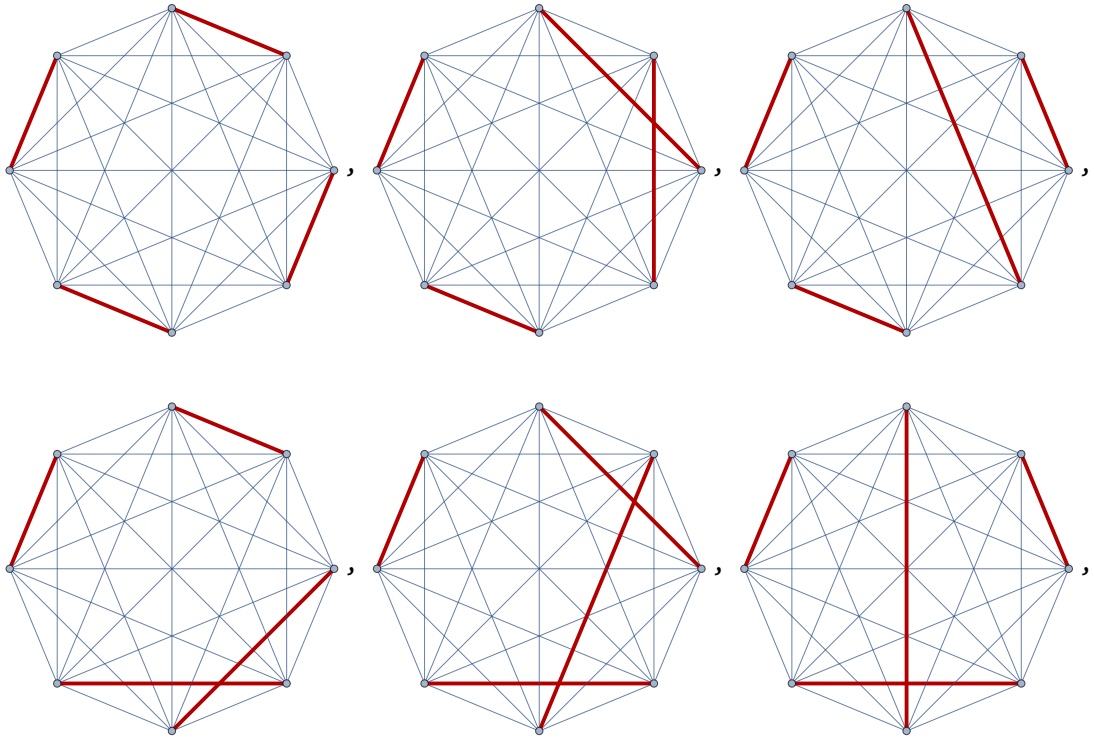
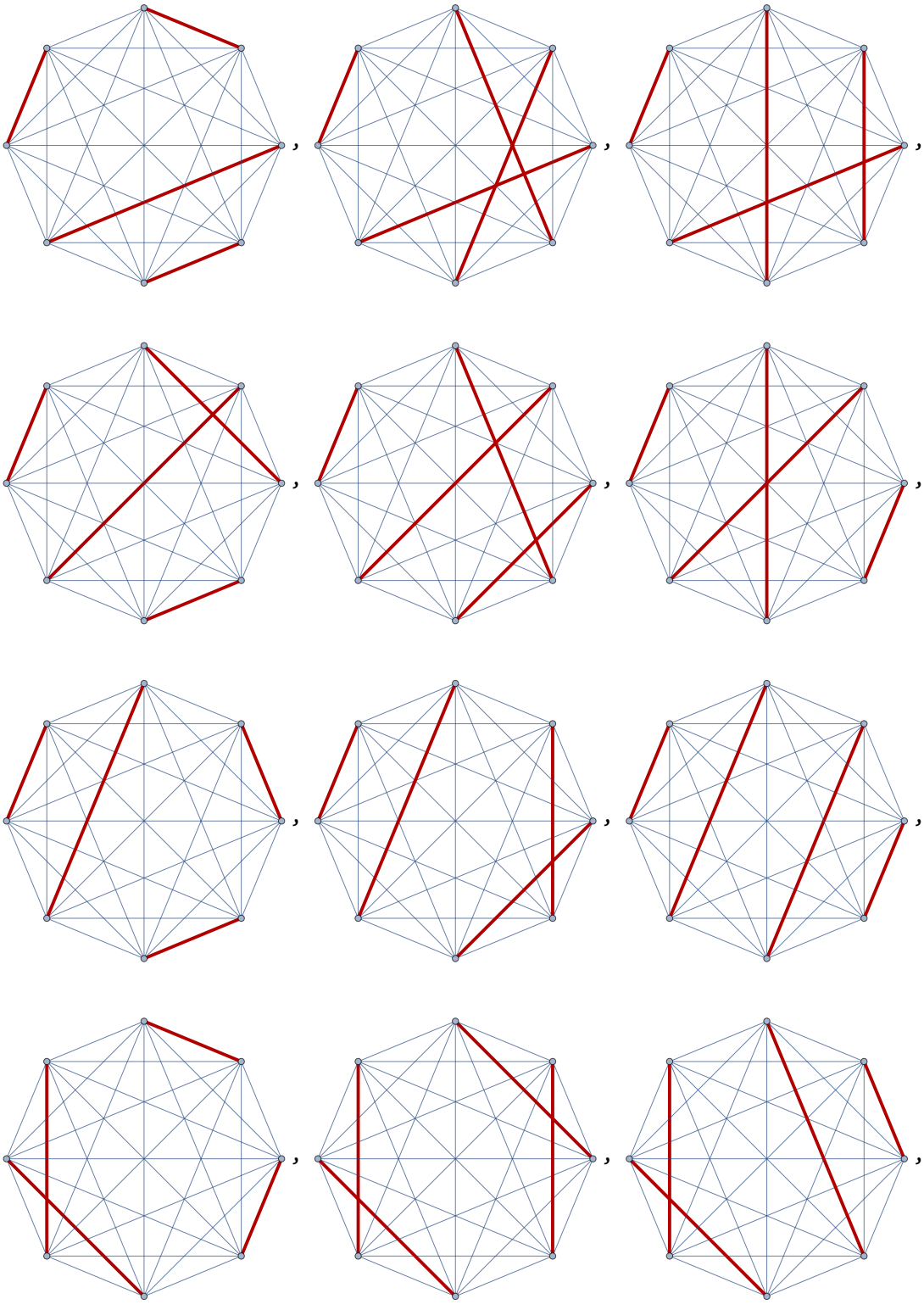
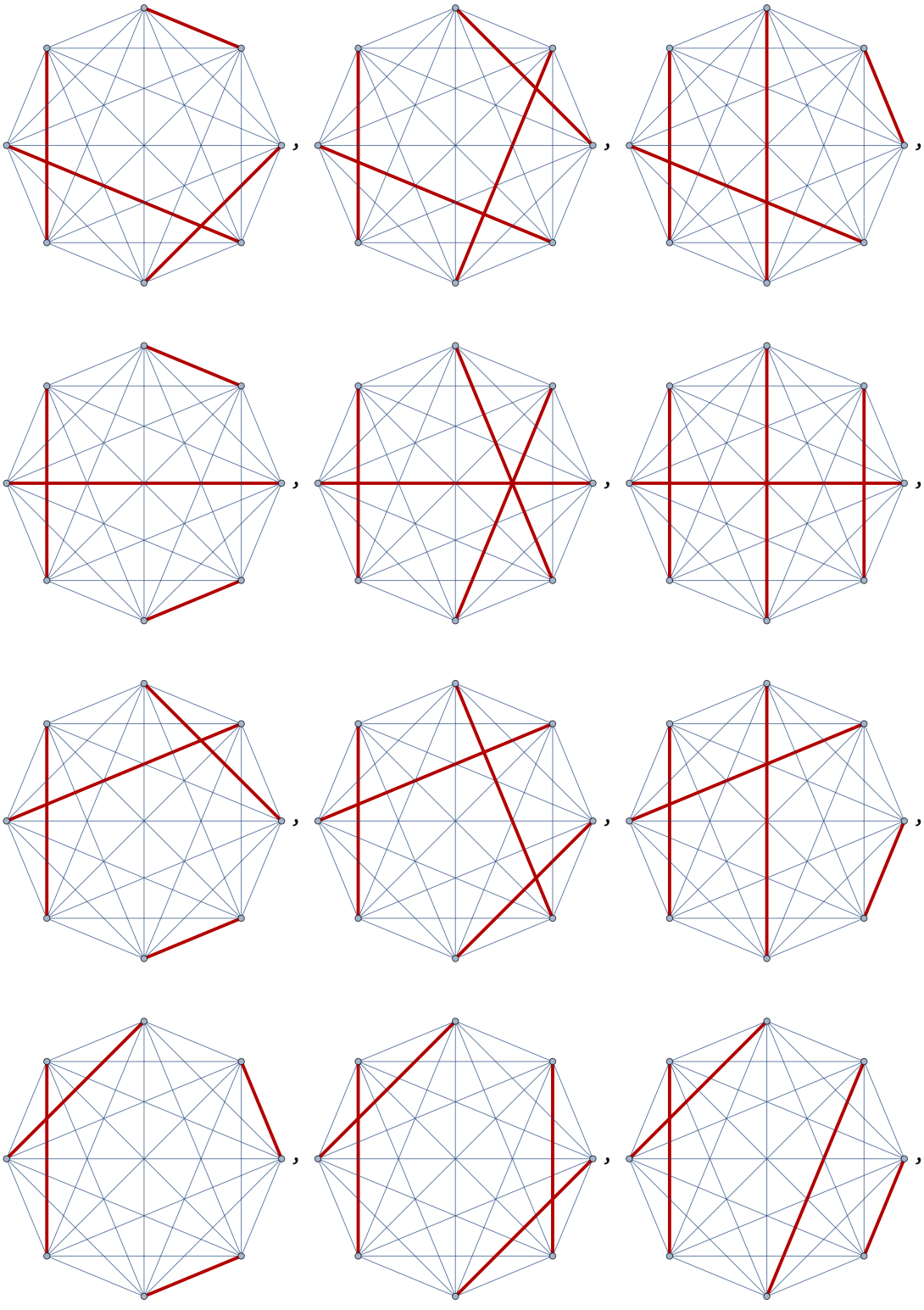


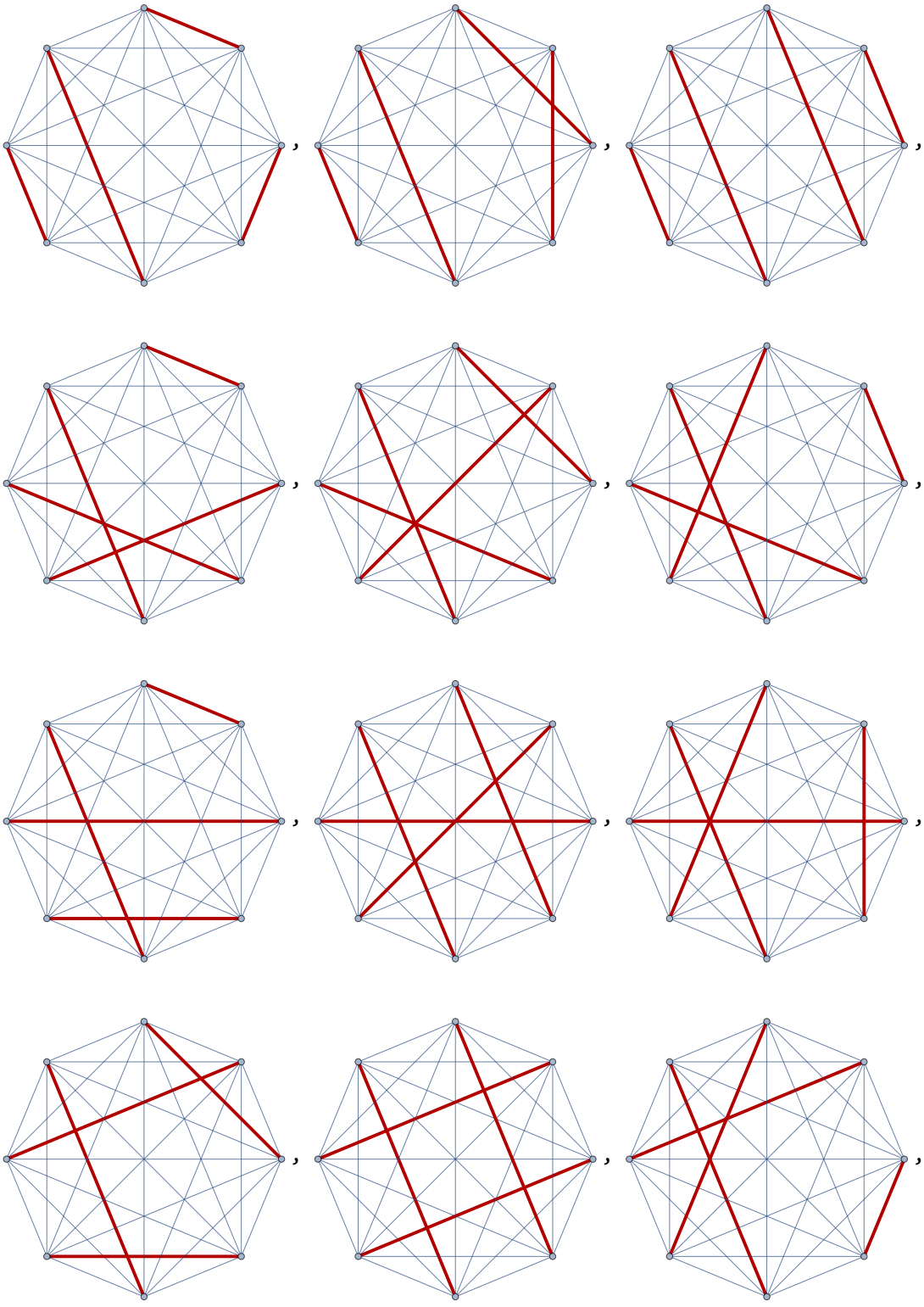
In[ ]:= `Table[HighlightGraph[k8, h, GraphHighlightStyle → "Thick"], {h, esl8}]`

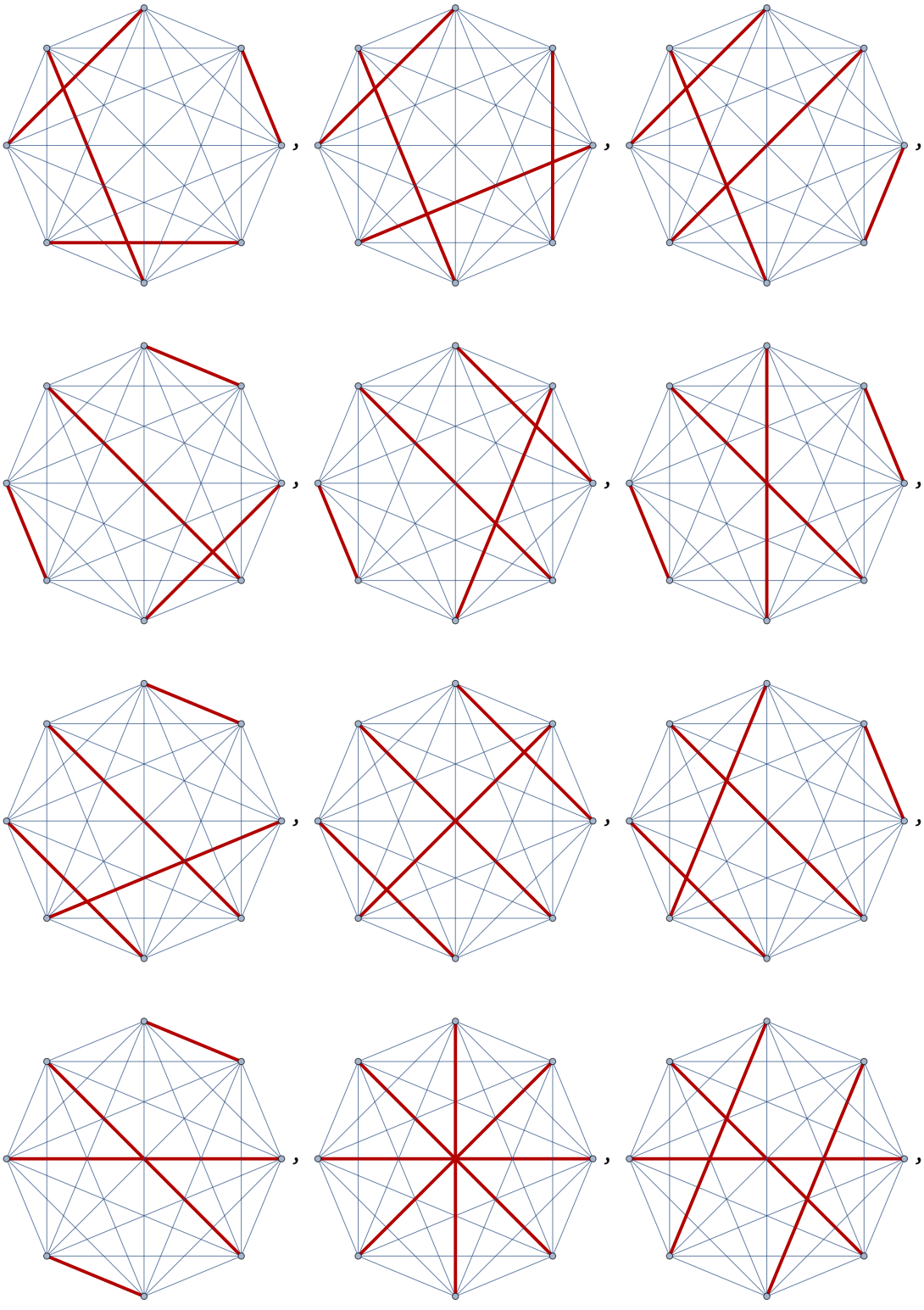
Out[ ]=

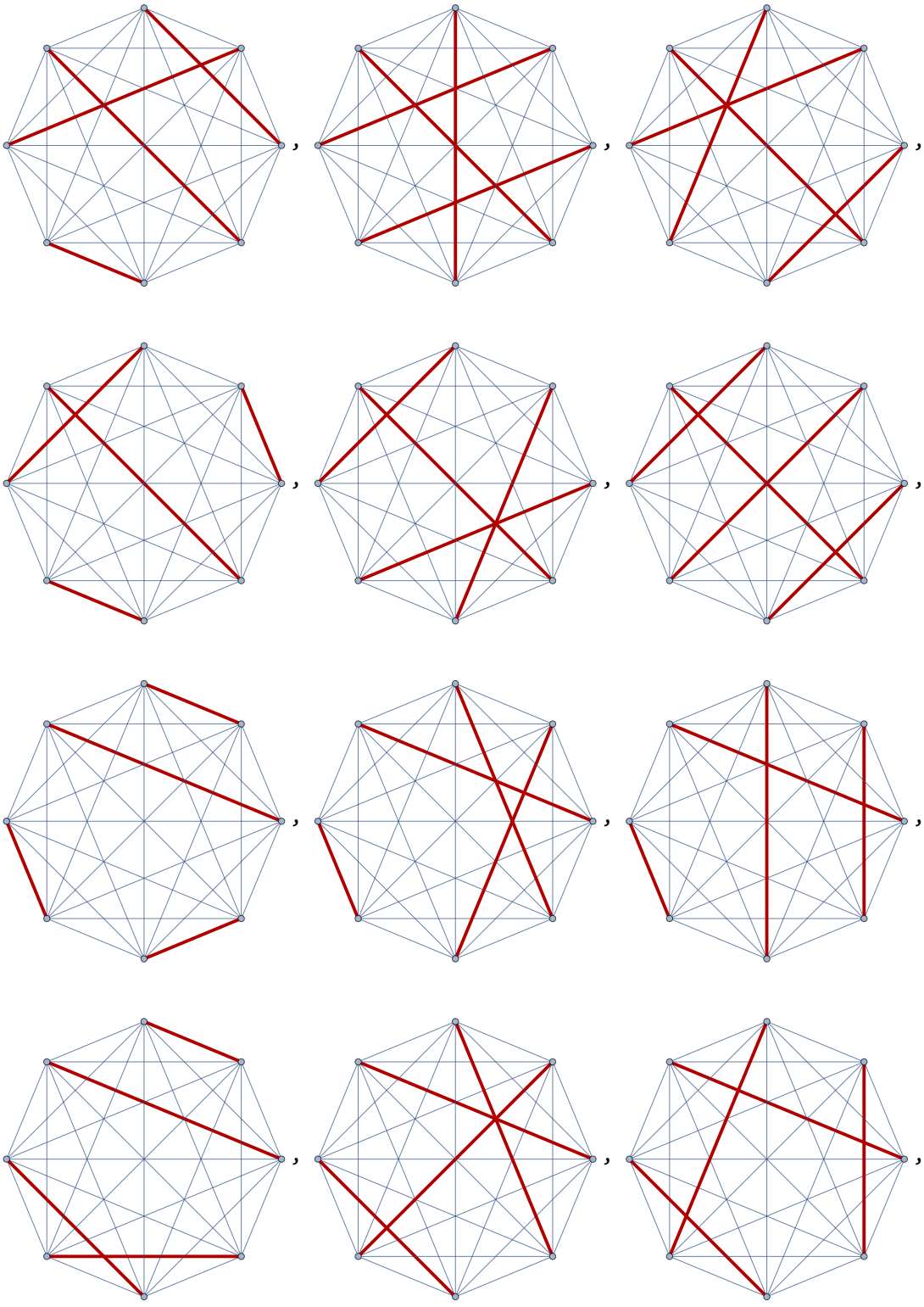


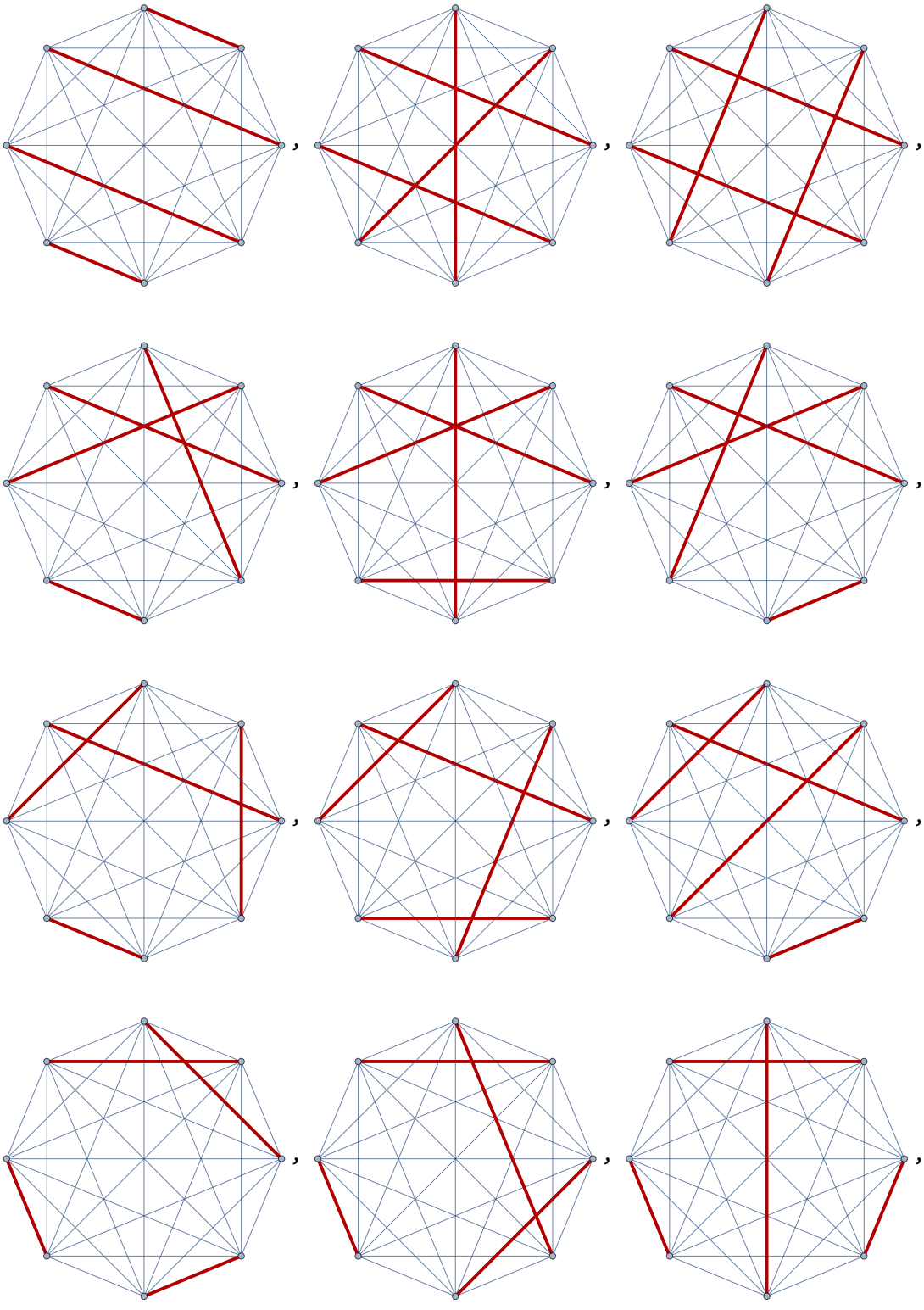


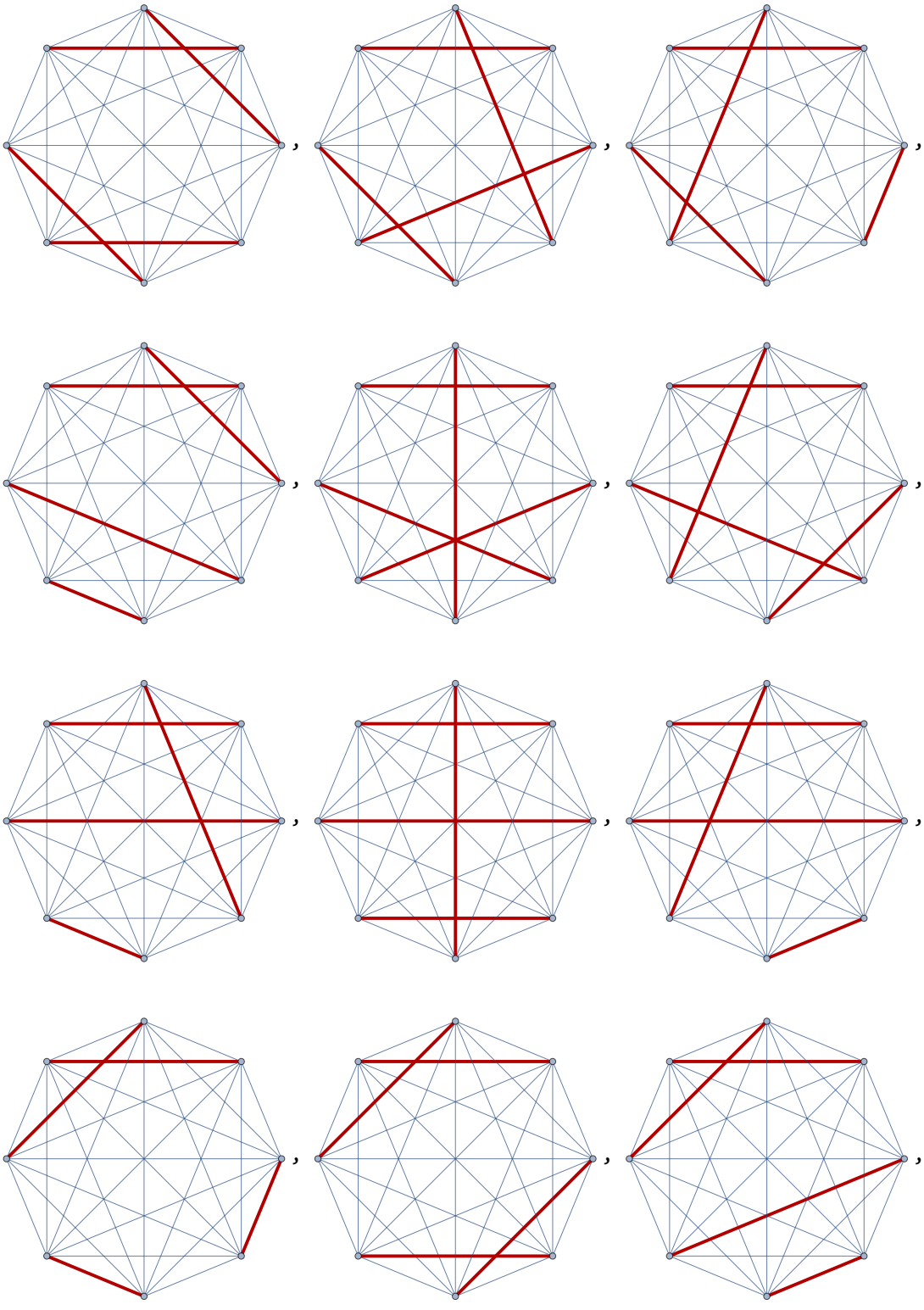


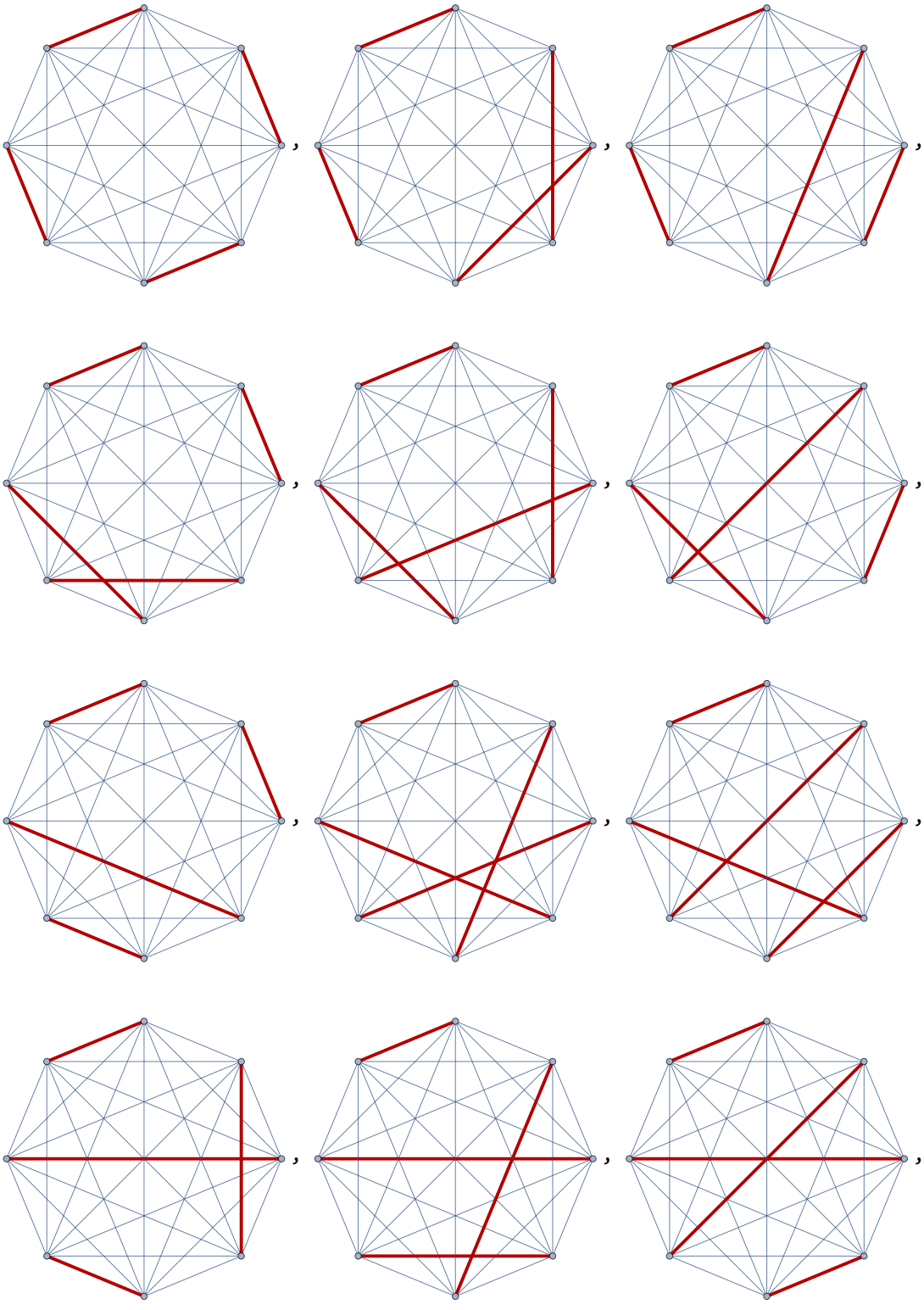


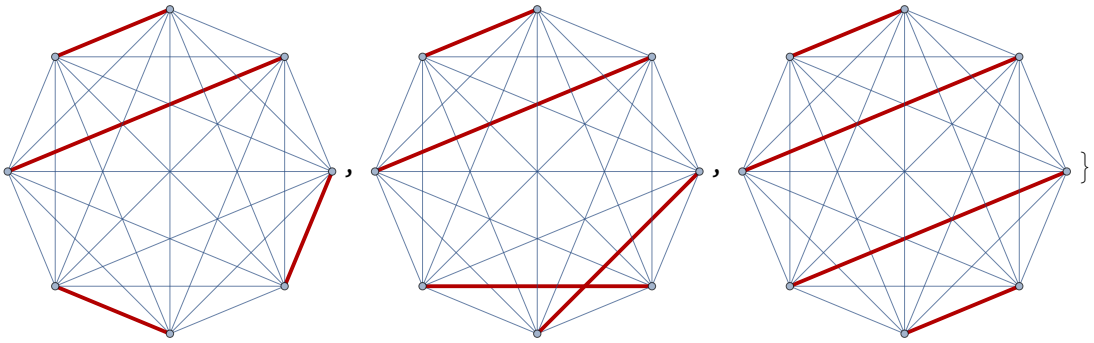












Stirling Permutations

In[ ]:=
```
ClearAll[strlngPermutations]
strlngPermutations[1] = {{1, 1}};
strlngPermutations[k_] := Join @@
  (Function[x, Flatten[Insert[x, {k, k}, #]] & /@ Range[2 k - 1]] /@ strlngPermutations[k - 1])
```

In[•]:=
```
Multicolumn[Sort@strlngPermutations@4, 5, Appearance → "Horizontal"]
```

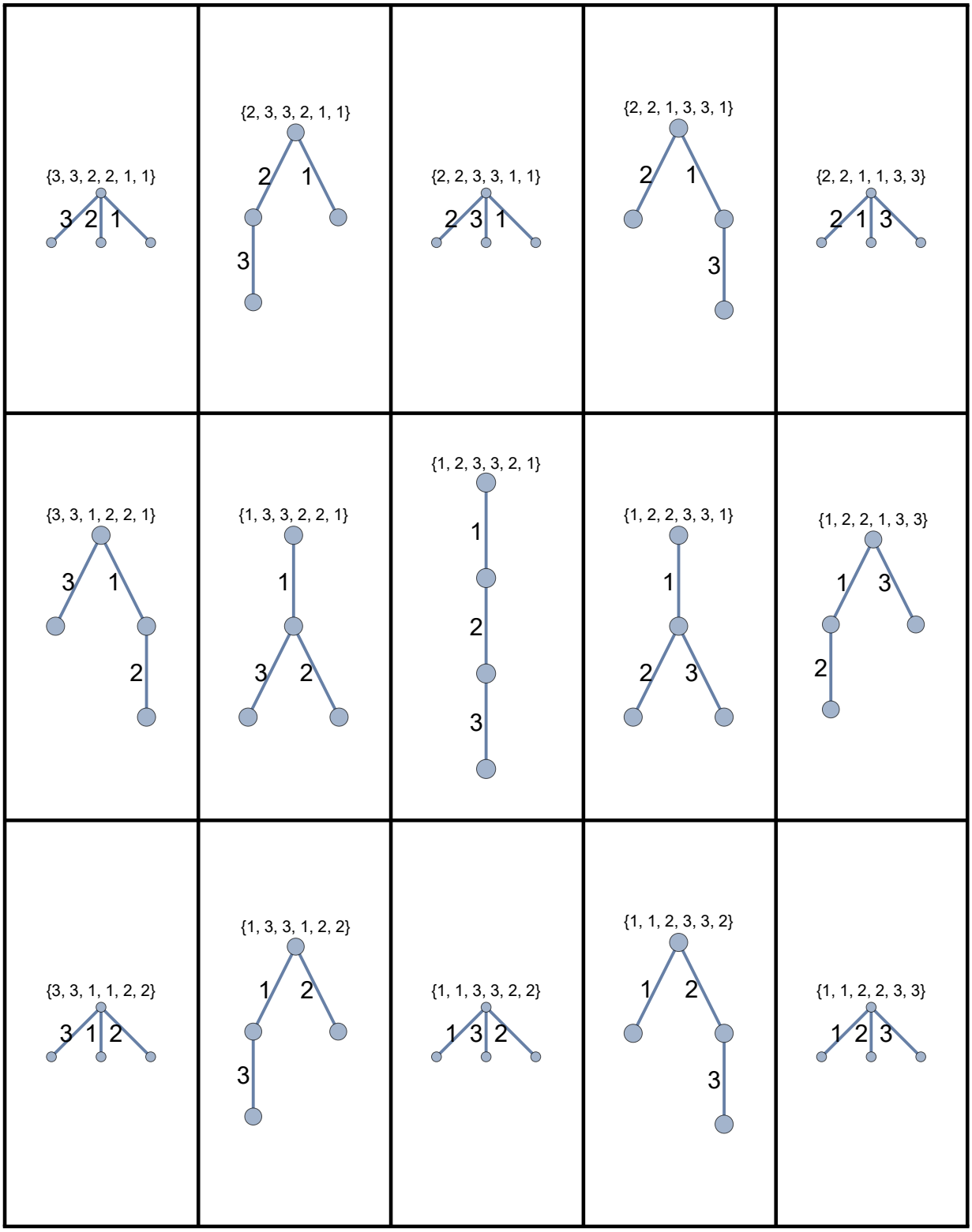
Out[•]=

{1, 1, 2, 2, 3, 3, 4, 4}	{1, 1, 2, 2, 3, 4, 4, 3}	{1, 1, 2, 2, 4, 4, 3, 3}	{1, 1, 2, 3, 3, 2, 4, 4}	{1, 1, 2, 3, 3, 4, 4, 2}
{1, 1, 2, 3, 4, 4, 3, 2}	{1, 1, 2, 4, 4, 2, 3, 3}	{1, 1, 2, 4, 4, 3, 3, 2}	{1, 1, 3, 3, 2, 2, 4, 4}	{1, 1, 3, 3, 2, 4, 4, 2}
{1, 1, 3, 3, 4, 4, 2, 2}	{1, 1, 3, 4, 4, 3, 2, 2}	{1, 1, 4, 4, 2, 2, 3, 3}	{1, 1, 4, 4, 2, 3, 3, 2}	{1, 1, 4, 4, 3, 3, 2, 2}
{1, 2, 2, 1, 3, 3, 4, 4}	{1, 2, 2, 1, 3, 4, 4, 3}	{1, 2, 2, 1, 4, 4, 3, 3}	{1, 2, 2, 3, 3, 1, 4, 4}	{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 4, 4, 3, 1}	{1, 2, 2, 4, 4, 1, 3, 3}	{1, 2, 2, 4, 4, 3, 3, 1}	{1, 2, 3, 3, 2, 1, 4, 4}	{1, 2, 3, 3, 2, 4, 4, 1}
{1, 2, 3, 3, 4, 4, 2, 1}	{1, 2, 3, 4, 4, 3, 2, 1}	{1, 2, 4, 4, 2, 1, 3, 3}	{1, 2, 4, 4, 2, 3, 3, 1}	{1, 2, 4, 4, 3, 3, 2, 1}
{1, 3, 3, 1, 2, 2, 4, 4}	{1, 3, 3, 1, 2, 4, 4, 2}	{1, 3, 3, 1, 4, 4, 2, 2}	{1, 3, 3, 2, 2, 1, 4, 4}	{1, 3, 3, 2, 2, 4, 4, 1}
{1, 3, 3, 2, 4, 4, 2, 1}	{1, 3, 3, 4, 4, 1, 2, 2}	{1, 3, 3, 4, 4, 2, 2, 1}	{1, 3, 4, 4, 3, 1, 2, 2}	{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 1, 2, 2, 3, 3}	{1, 4, 4, 1, 2, 3, 3, 2}	{1, 4, 4, 1, 3, 3, 2, 2}	{1, 4, 4, 2, 2, 1, 3, 3}	{1, 4, 4, 2, 2, 3, 3, 1}
{1, 4, 4, 2, 3, 3, 2, 1}	{1, 4, 4, 3, 3, 1, 2, 2}	{1, 4, 4, 3, 3, 2, 2, 1}	{2, 2, 1, 1, 3, 3, 4, 4}	{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 4, 4, 3, 3}	{2, 2, 1, 3, 3, 1, 4, 4}	{2, 2, 1, 3, 3, 4, 4, 1}	{2, 2, 1, 3, 4, 4, 3, 1}	{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 4, 4, 3, 3, 1}	{2, 2, 3, 3, 1, 1, 4, 4}	{2, 2, 3, 3, 1, 4, 4, 1}	{2, 2, 3, 3, 4, 4, 1, 1}	{2, 2, 3, 4, 4, 3, 1, 1}
{2, 2, 4, 4, 1, 1, 3, 3}	{2, 2, 4, 4, 1, 3, 3, 1}	{2, 2, 4, 4, 3, 3, 1, 1}	{2, 3, 3, 2, 1, 1, 4, 4}	{2, 3, 3, 2, 1, 4, 4, 1}
{2, 3, 3, 2, 4, 4, 1, 1}	{2, 3, 3, 4, 4, 2, 1, 1}	{2, 3, 4, 4, 3, 2, 1, 1}	{2, 4, 4, 2, 1, 1, 3, 3}	{2, 4, 4, 2, 1, 3, 3, 1}
{2, 4, 4, 2, 3, 3, 1, 1}	{2, 4, 4, 3, 3, 2, 1, 1}	{3, 3, 1, 1, 2, 2, 4, 4}	{3, 3, 1, 1, 2, 4, 4, 2}	{3, 3, 1, 1, 4, 4, 2, 2}
{3, 3, 1, 2, 2, 1, 4, 4}	{3, 3, 1, 2, 2, 4, 4, 1}	{3, 3, 1, 2, 4, 4, 2, 1}	{3, 3, 1, 4, 4, 1, 2, 2}	{3, 3, 1, 4, 4, 2, 2, 1}
{3, 3, 2, 2, 1, 1, 4, 4}	{3, 3, 2, 2, 1, 4, 4, 1}	{3, 3, 2, 2, 4, 4, 1, 1}	{3, 3, 2, 4, 4, 2, 1, 1}	{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 2, 2, 1}	{3, 3, 4, 4, 2, 2, 1, 1}	{3, 4, 4, 3, 1, 1, 2, 2}	{3, 4, 4, 3, 1, 2, 2, 1}	{3, 4, 4, 3, 2, 2, 1, 1}
{4, 4, 1, 1, 2, 2, 3, 3}	{4, 4, 1, 1, 2, 3, 3, 2}	{4, 4, 1, 1, 3, 3, 2, 2}	{4, 4, 1, 2, 2, 1, 3, 3}	{4, 4, 1, 2, 2, 3, 3, 1}
{4, 4, 1, 2, 3, 3, 2, 1}	{4, 4, 1, 3, 3, 1, 2, 2}	{4, 4, 1, 3, 3, 2, 2, 1}	{4, 4, 2, 2, 1, 1, 3, 3}	{4, 4, 2, 2, 1, 3, 3, 1}
{4, 4, 2, 2, 3, 3, 1, 1}	{4, 4, 2, 3, 3, 2, 1, 1}	{4, 4, 3, 3, 1, 1, 2, 2}	{4, 4, 3, 3, 1, 2, 2, 1}	{4, 4, 3, 3, 2, 2, 1, 1}

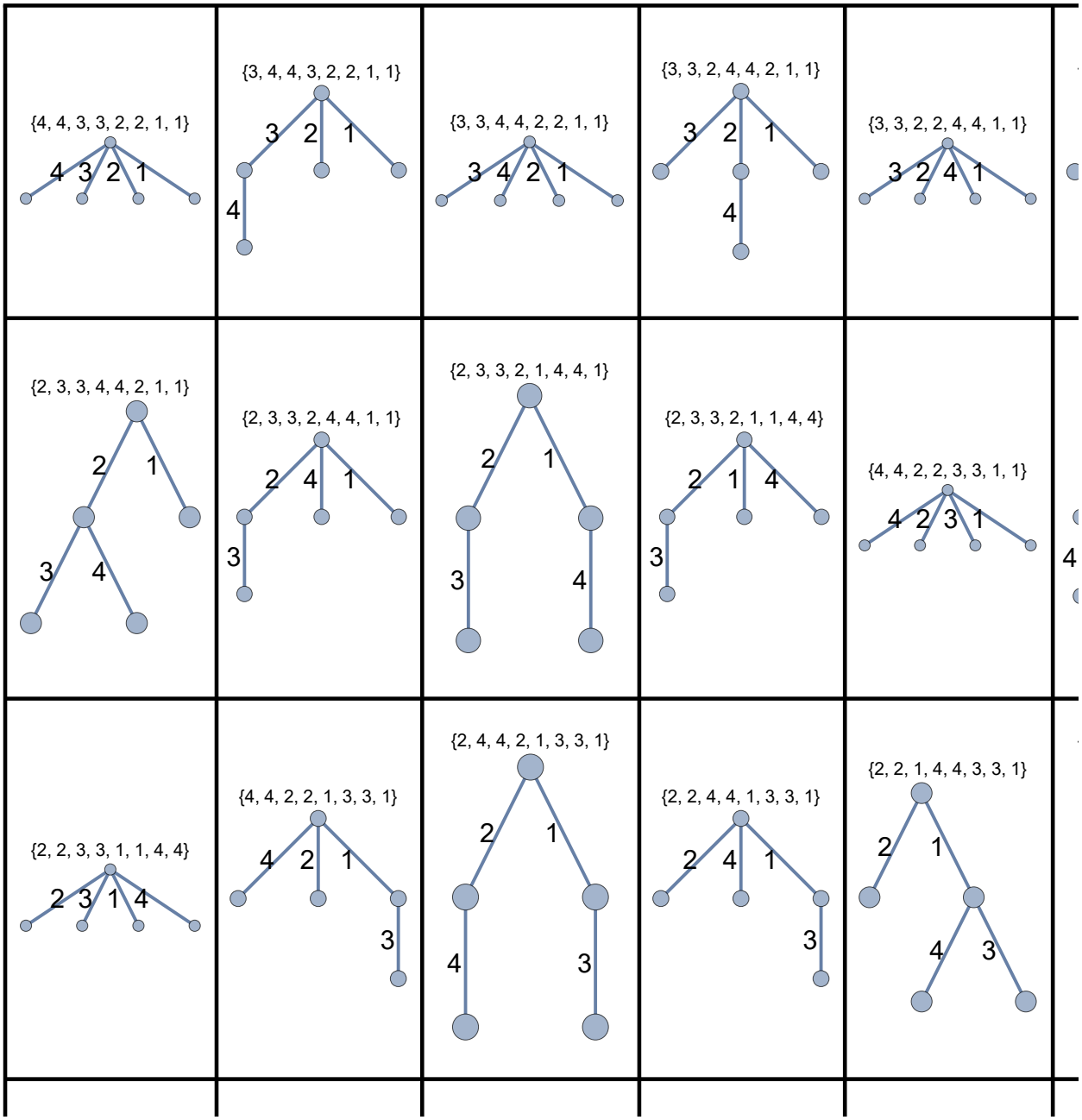
In[ ]:=
```
ClearAll[strlngPermGraph]
strlngPermGraph[sp_, opts : OptionsPattern[]] :=
 Module[{vl = DeleteDuplicates@sp, pos = PositionIndex@sp,
   eL = EdgeList@*TransitiveReductionGraph@*GraphUnion},
  Graph[Prepend[vl, 0], eL[Graph@Thread[0 → vl],
    SimpleGraph@RelationGraph[And @@ Between[pos@#] /@ pos[#2] &, vl]],
   GraphLayout → {"LayeredEmbedding", "RootVertex" → 0},
   EdgeLabels → {e_ :→ Placed[Last@e, {Left, "Middle"}]}, opts]]
```

In[ ]:=
```
Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
      EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
   strlngPermutations[3], 5], Dividers → All, Spacings → {4, 4}]
```

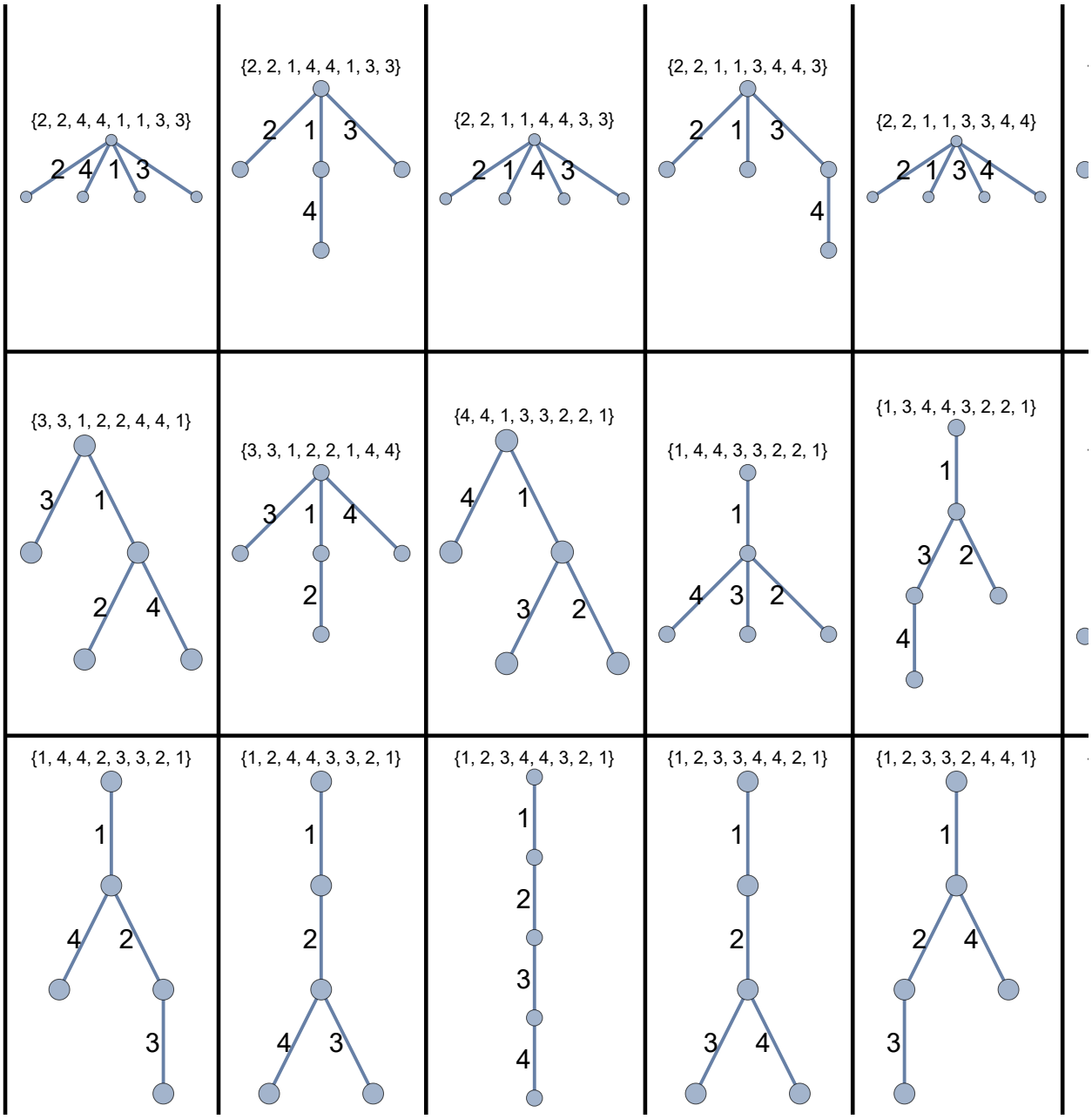
Out[ ]=

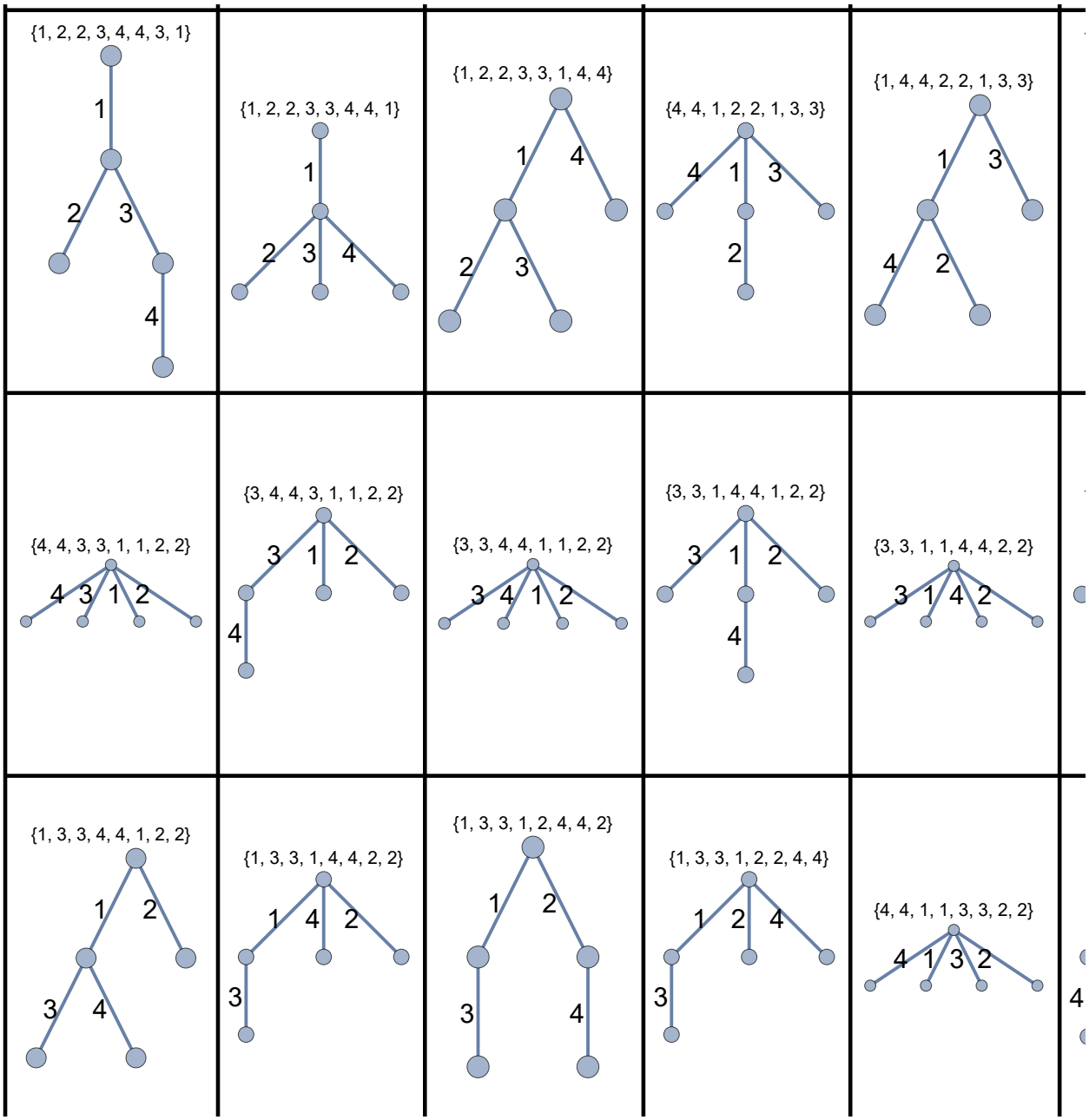


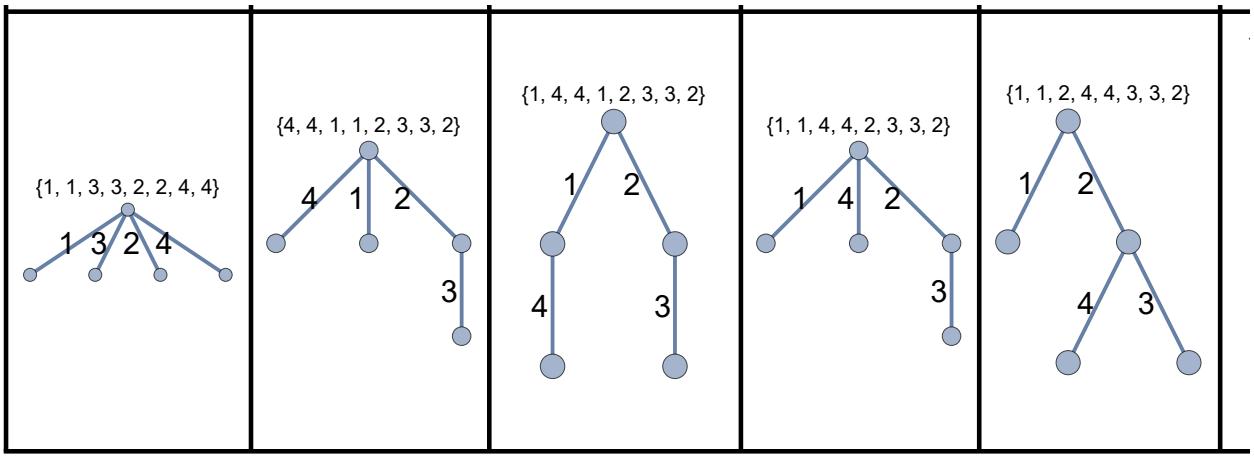
```
In[•]:= Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
          EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
        strlngPermutations[4], 10], Dividers → All, Spacings → {1, 1}]
```



Out[•]=


{2, 2, 4, 4, 1, 1, 3, 3}
{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 1, 4, 4, 3, 3}
{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 3, 3, 4, 4}
{3, 3, 1, 2, 2, 4, 4, 1}
{3, 3, 1, 2, 2, 1, 4, 4}
{4, 4, 1, 3, 3, 2, 2, 1}
{1, 4, 4, 3, 3, 2, 2, 1}
{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 2, 3, 3, 2, 1}
{1, 2, 4, 4, 3, 3, 2, 1}
{1, 2, 3, 4, 4, 3, 2, 1}
{1, 2, 3, 3, 4, 4, 2, 1}
{1, 2, 3, 3, 2, 4, 4, 1}


{1, 2, 2, 3, 4, 4, 3, 1}
{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 3, 1, 4, 4}
{4, 4, 1, 2, 2, 1, 3, 3}
{1, 4, 4, 2, 2, 1, 3, 3}
{4, 4, 3, 3, 1, 1, 2, 2}
{3, 4, 4, 3, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 1, 4, 4, 1, 2, 2}
{3, 3, 1, 1, 4, 4, 2, 2}
{1, 3, 3, 4, 4, 1, 2, 2}
{1, 3, 3, 1, 4, 4, 2, 2}
{1, 3, 3, 1, 2, 4, 4, 2}
{1, 3, 3, 1, 2, 2, 4, 4}
{4, 4, 1, 1, 3, 3, 2, 2}



x

Limits for Ratio test in m(2)

In[]:=
```
case1 = (2^(1 + 2 n) n! (1 + n)!) / (2 (1 + n))!
case2 = (2^(-1 - 2 n) (2 n)!) / (n! (1 + n)!)
```

Out[]=
$$\frac{2^{1+2n}\, n!\, (1+n)!}{(2\,(1+n))!}$$

Out[]=
$$\frac{2^{-1-2n}\, (2n)!}{n!\, (1+n)!}$$

In[]:=
```
limitm2case1 = Limit[case1, n → Infinity]
limitm2case2 = Limit[case2, n → Infinity]
```

Out[]= 0

Out[]= 0

Computing m(1) to m(10) from n=0 to n=2000

In[]:=
```
Multifactorial[n_, k_] := Abs[Apply[Times, Range[-n, -1, k]]]
```

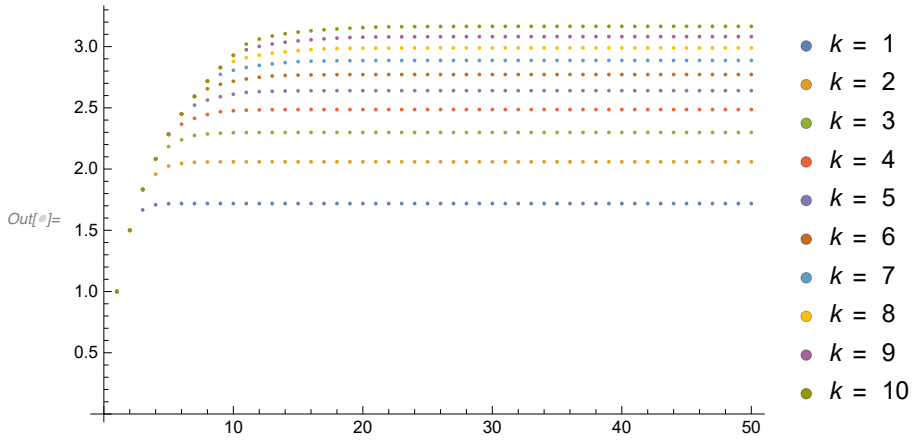
In[]:=
```
For[i = 1, i < 11, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 150}], 20]]]
```

```
2.7182818284590452354

3.0594074053425761445

3.2989135380884190034

3.4859449774535577452

3.6402244677338097342

3.7719023962117584357

3.8869596537408434954

3.9892412126901365441

4.0813755201688985441

4.1652437655583845908
```

Plot of m (1) to m (1) superimposed on each other from n = 0 to n = 2000

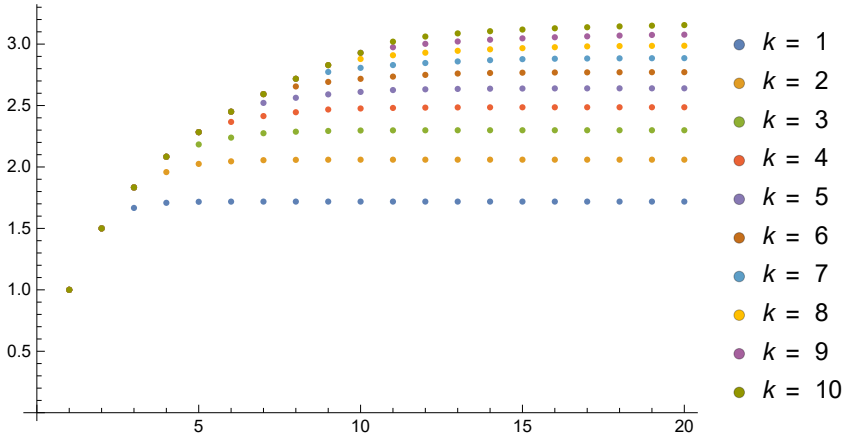
```
ListPlot[Table[Sum[1 / Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 20}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[•]=



```
ListPlot[Table[Sum[1 / Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 50}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[•]=



Computation of RMFCs using the closed form formula

In[•]:=
$$\text{ClosedFormRMFC}[n\_] := 1 + \frac{1}{n}\,\text{Exp}[1/n]\,\text{Sum}\left[n^{k/n}\,\text{Gamma}\left[\frac{k}{n}, 0, \frac{1}{n}\right], \{k, n\}\right]$$

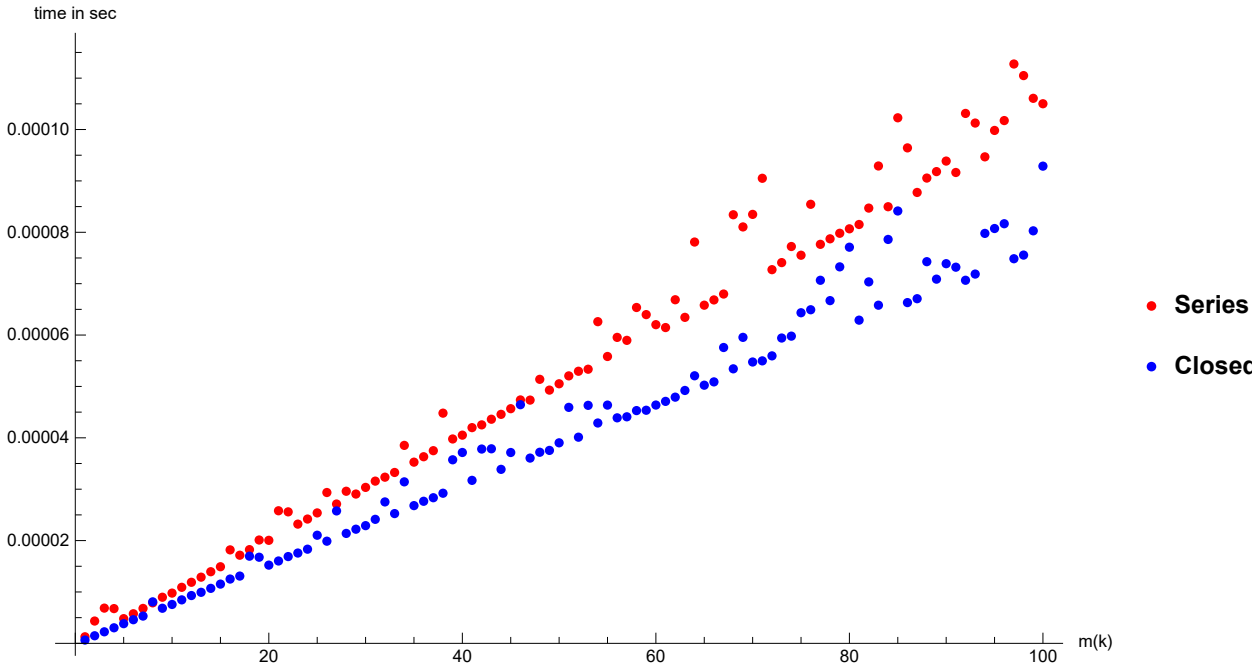
In[•]:=
```
For[i = 1, i < 11, i++, Print[N[ClosedFormRMFC[i], 20]]]
```

```
2.7182818284590452354

3.0594074053425761445

3.2989135380884190034

3.4859449774535577452

3.6402244677338097342

3.7719023962117584357

3.8869596537408434954

3.9892412126901365441

4.0813755201688985441

4.1652437655583845908
```

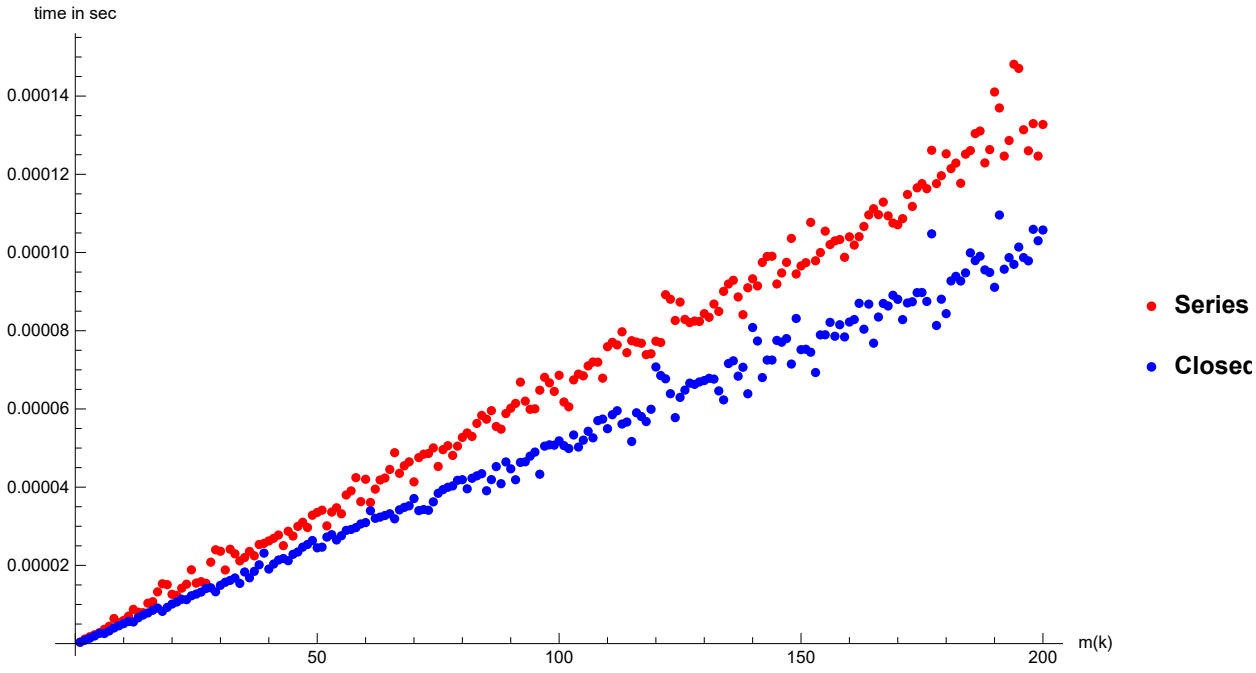
Analysing efficiency of the two RMFC calculation methods

In[•]:=
```
test1[xx_] :=
  (For[i = 1, i < xx, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 250}], 50]] //
      Inactive] // RepeatedTiming)[[1]]
test2[xx_] := (For[i = 1, i < xx, i++, Print[N[ClosedFormRMFC[i], 50]] // Inactive] //
    RepeatedTiming)[[1]]
```

```
In[ ]:= list1 =
          ListPlot[Table[{xx, test1[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
           AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
        list2 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends →
            {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];

In[ ]:= Show[list1, list2, ImageSize → Large]
```




```
In[ ]:= list3 =
          ListPlot[Table[{xx, test1[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
           AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
        list4 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends →
            {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];

In[ ]:= Show[list3, list4, ImageSize → Large]
```

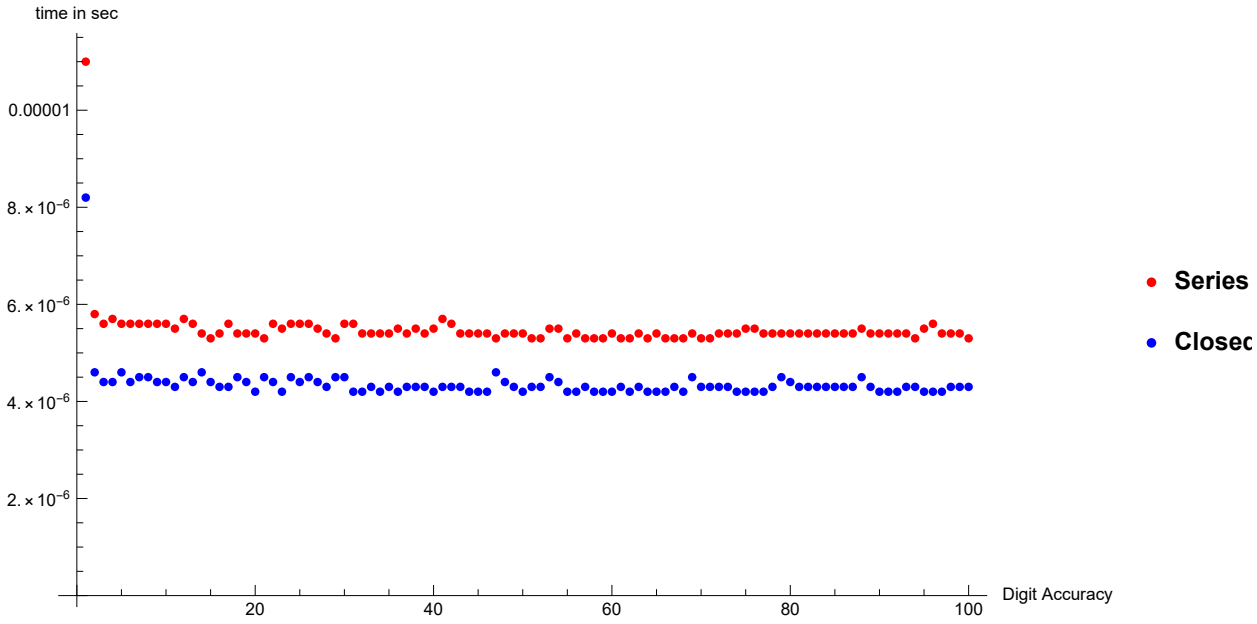



```
test3[xx_] :=
 (For[i = 1, i < 10, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 500}], xx]] //
     Inactive] // AbsoluteTiming)[[1]]
test4[xx_] := (For[i = 1, i < 10, i++, Print[N[ClosedFormRMFC[i], xx]] // Inactive] //
    AbsoluteTiming)[[1]]
```

In[ ]:=
```
list5 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

In[ ]:=
```
list6 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

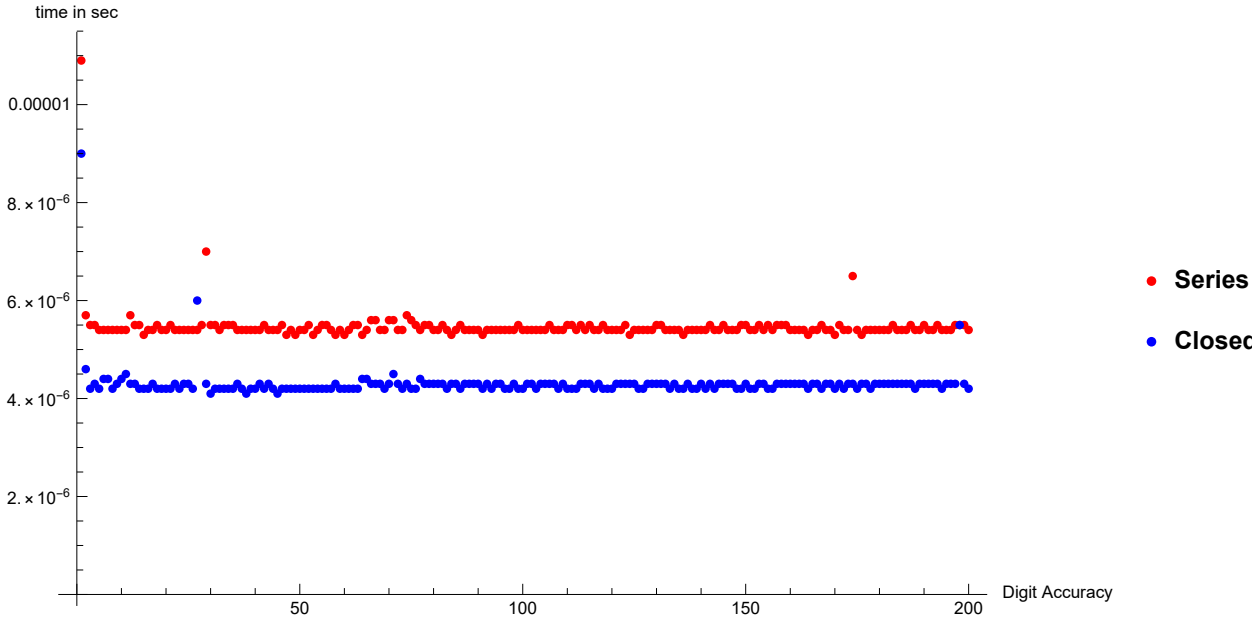
In[ ]:=
```
Show[list5, list6, ImageSize → Large]
```



In[ ]:=
```
list7 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

In[ ]:=
```
list8 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

In[ ]:=
```
Show[list7, list8, ImageSize → Large]
```

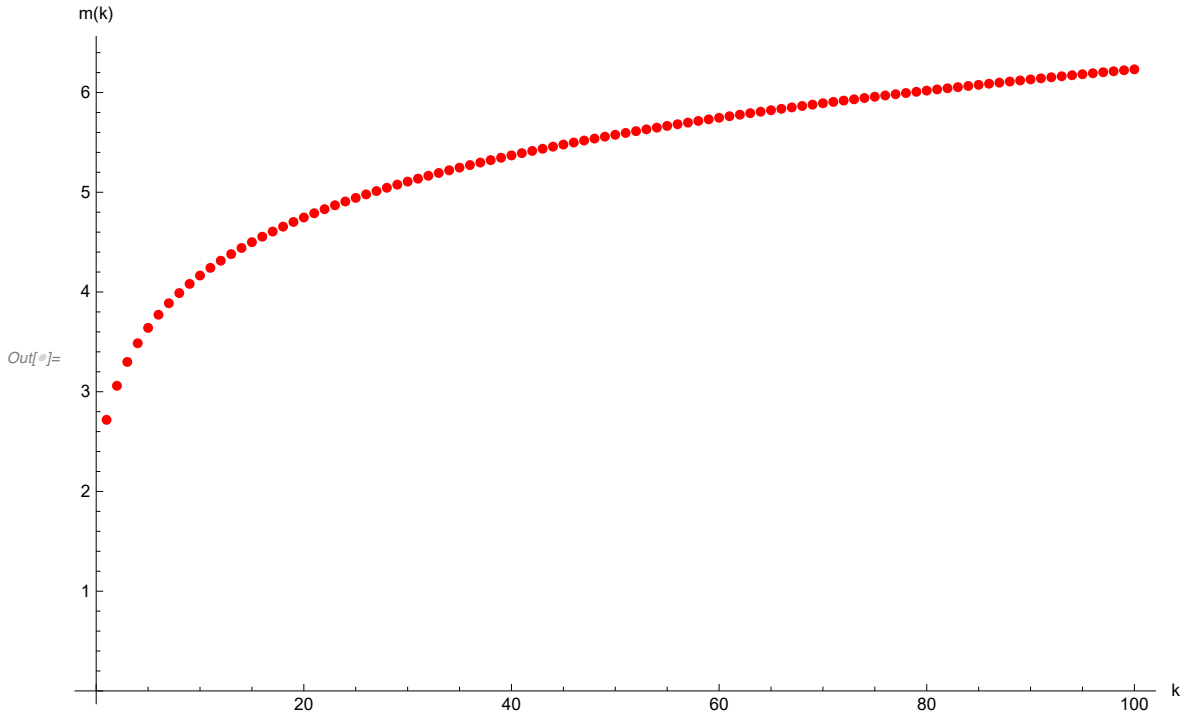


Asymptotics of Reciprocal Multifactorial Series

Simple Graph

In[]:=
```
ListPlot[Table[{x, ClosedFormRMFC[x]}, {x, 1, 100}],
 ImageSize → Large, AxesLabel → {"k", "m(k)"}, PlotStyle → {Red, Thick}]
```

Out[]=



Detailed asymptotics

```
Asymptotic[λ Integrate[((1 - t) (1 - E^(-(λ t)))) / (t (E^(-(λ Log[t])) - 1)), {t, 0, 1}],
 λ → 0, SeriesTermGoal → 2]
```

Out[]=
$$-\frac{\lambda^2}{4}-\frac{1}{2}\lambda^2\,\text{Log}[3]+\frac{1}{2}\lambda\,\text{Log}[4]+\frac{1}{12}\lambda^2\,\text{Log}[64]$$

In[]:=
```
ClosedFormRMFC[n_] := 1 + 1/n Exp[1/n] Sum[n^(k/n) Gamma[k/n, 0, 1/n], {k, n}]
```

In[]:=
```
RMFCApproximation[k_] :=
 1 + ((1 + 2 k (1 + k)) (1 + Log[27/8] + k (3 + 6 EulerGamma k - Log[64]) + 6 k^2 Log[k]))/(12 k^4)
```

In[]:=
$$1+\left(1+\frac{1}{k}+\frac{1}{2\,k\text{^}2}\right)*\left(\left(\text{Log}[k]+\text{EulerGamma}+\frac{1}{2\,k}-\frac{1}{12\,k\text{^}2}\right)-\left(-\frac{\frac{1}{k}^2}{4}-\frac{1}{2}\,\frac{1}{k}^2\,\text{Log}[3]+\frac{1}{2}\,\frac{1}{k}\,\text{Log}[4]+\frac{1}{12}\,\frac{1}{k}^2\,\text{Log}[64]\right)\right)$$

Out[]=
$$1+\left(1+\frac{1}{2\,k^2}+\frac{1}{k}\right)\left(\text{EulerGamma}+\frac{1}{6\,k^2}+\frac{1}{2\,k}+\frac{\text{Log}[3]}{2\,k^2}-\frac{\text{Log}[4]}{2\,k}-\frac{\text{Log}[64]}{12\,k^2}+\text{Log}[k]\right)$$

In[]:=
$$\text{FullSimplify}\left[1+\left(1+\frac{1}{2\,k^2}+\frac{1}{k}\right)\left(\text{EulerGamma}+\frac{1}{6\,k^2}+\frac{1}{2\,k}+\frac{\text{Log}[3]}{2\,k^2}-\frac{\text{Log}[4]}{2\,k}-\frac{\text{Log}[64]}{12\,k^2}+\text{Log}[k]\right)\right]$$

Out[]=
$$1+\frac{\left(1+2\,k\,\left(1+k\right)\right)\,\left(1+\text{Log}\left[\frac{27}{8}\right]+k\,\left(3+6\,\text{EulerGamma}\,k-\text{Log}[64]\right)+6\,k^2\,\text{Log}[k]\right)}{12\,k^4}$$

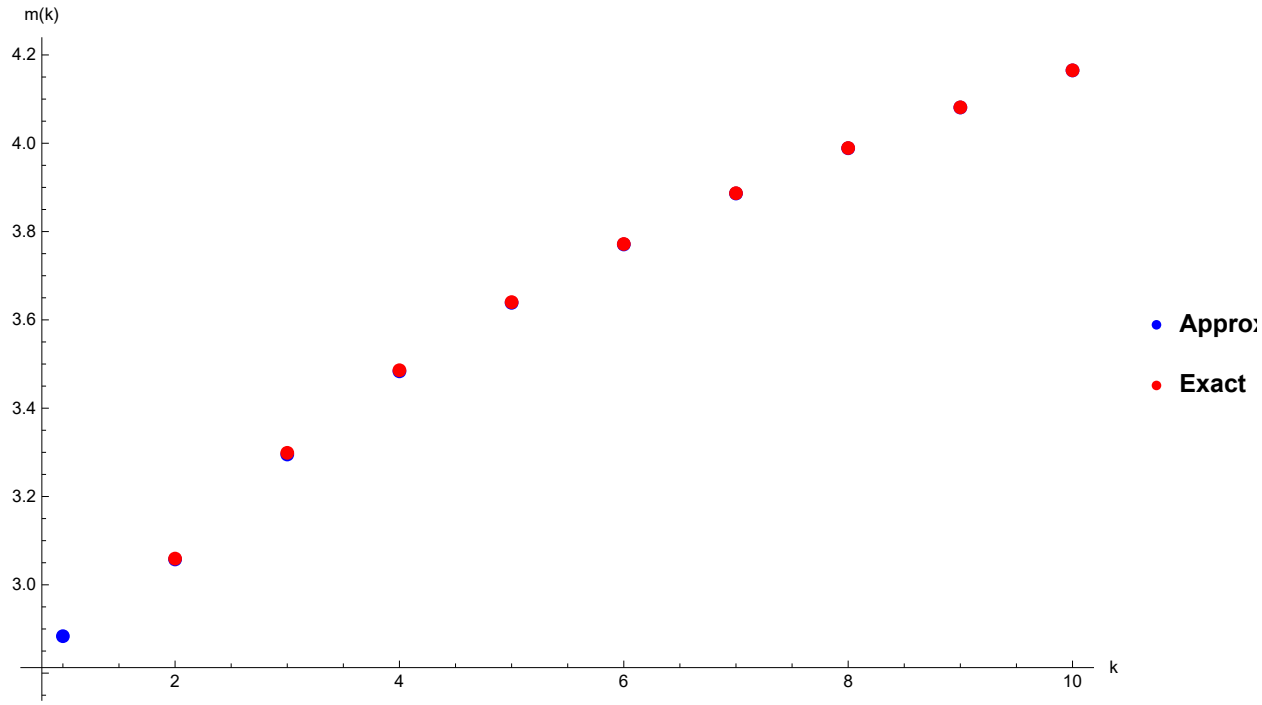
In[]:=
```
N[RMFCApproximation[15]]
```

Out[]= 4.49958

In[]:=
```
N[ClosedFormRMFC[15]]
```

Out[]= 4.49969

In[]:=
```
listaprx = ListPlot[Table[{x, RMFCApproximation[x]}, {x, 1, 10}], PlotRange → All,
   PlotLegends → {Automatic}, AxesLabel → {"k", "m(k)"}, PlotStyle → {Blue}];
listexact = ListPlot[Table[{x, ClosedFormRMFC[x]}, {x, 1, 10}], PlotRange → All,
   PlotLegends → {Automatic}, AxesLabel → {"k", "m(k)"}, PlotStyle → {Red}];
Show[listaprx, listexact, ImageSize → Large]
```



```
In[•]:= Table[{x, N[RMFCApproximation[10^x], 50]}, {x, 0, 5}]
```

```
Out[•]= {{0, 2.8836692626558410221123829825576677447983026999440},
         {1, 4.1649189354117046099965674566757558205746873600579},
         {2, 6.2325552567621081237809508324268568037822079589510},
         {3, 8.4922666866575160699533708545283873810253799369197},
         {4, 10.788515528463920803804219311415785943169176345868},
         {5, 13.090260100433386932193328011615118199331396176765}}
```

```
In[•]:= Table[{x, N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

```
Out[•]= {{0, 2.7182818284590452353602874713526624977572470937000},
         {1, 4.1652437655583845907872624104455607382280307953708},
         {2, 6.2325559690048781755948333144748471914777217107326},
         {3, 8.4922666877530555546922513543645116307527055440479},
         {4, 10.788515528465399974853138837254851897589458808237},
         {5, 13.090260100433388795106669060937341380984702346850}}
```

```
In[•]:= Table[{x, 100 (N[ClosedFormRMFC[10^x]] - N[RMFCApproximation[10^x]]) /
            N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

```
Out[•]= {{0, -6.08426}, {1, 0.00779859}, {2, 0.0000114278},
         {3, 1.29004 × 10^-8}, {4, 1.37156 × 10^-11}, {5, 0.}}
```