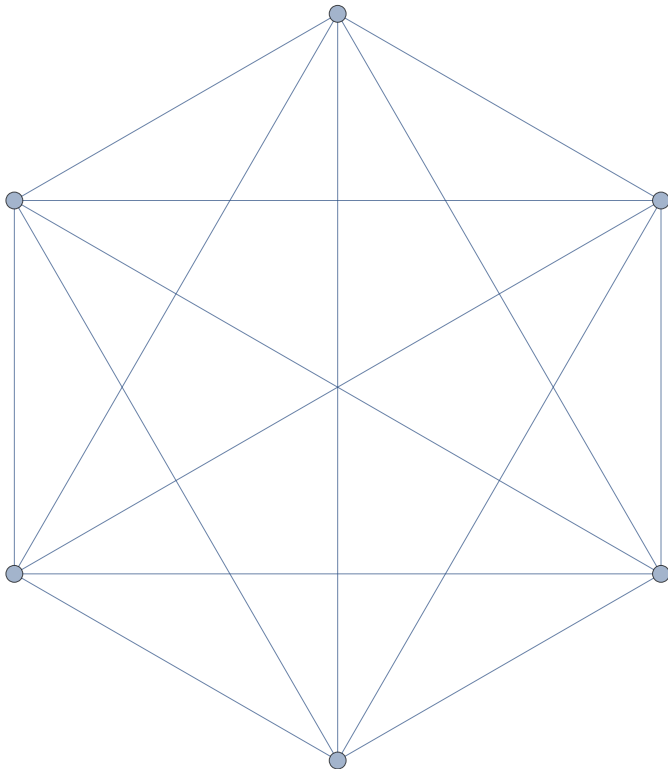


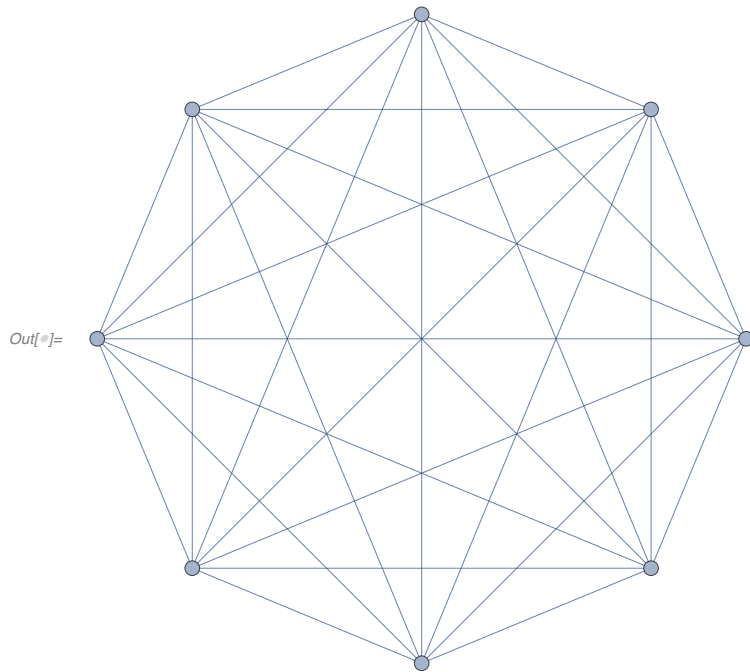
Reciprocal Multifactorial Constants

Perfect matchings for K_6 and K_8

```
In[ ]:= k6 = CompleteGraph[6]  
k8 = CompleteGraph[8]
```

Out[]:=





```
In[ ]:= 16 = Length[FindIndependentEdgeSet[k6]]
        18 = Length[FindIndependentEdgeSet[k8]]
```

```
Out[ ]:= 3
```

```
Out[ ]:= 4
```

```

In[*]:= es16 = Select[Subsets[EdgeList[k6], {16}], IndependentEdgeSetQ[k6, #] &]
es18 = Select[Subsets[EdgeList[k8], {18}], IndependentEdgeSetQ[k8, #] &]

```

```

Out[*]:= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 5}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 6},
{1 ↔ 3, 2 ↔ 5, 4 ↔ 6}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 5}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 6}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 6},
{1 ↔ 4, 2 ↔ 6, 3 ↔ 5}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 6}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 6}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 4},
{1 ↔ 6, 2 ↔ 3, 4 ↔ 5}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 5}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 4}}

```

```

Out[*]:= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 8, 6 ↔ 7},
{1 ↔ 2, 3 ↔ 5, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 8, 6 ↔ 7},
{1 ↔ 2, 3 ↔ 6, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 8, 5 ↔ 7},
{1 ↔ 2, 3 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 8, 5 ↔ 6},
{1 ↔ 2, 3 ↔ 8, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 2, 3 ↔ 8, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 2, 3 ↔ 8, 4 ↔ 7, 5 ↔ 6},
{1 ↔ 3, 2 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 8, 6 ↔ 7},
{1 ↔ 3, 2 ↔ 5, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 5, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 5, 4 ↔ 8, 6 ↔ 7},
{1 ↔ 3, 2 ↔ 6, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 8, 5 ↔ 7},
{1 ↔ 3, 2 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 8, 5 ↔ 6},
{1 ↔ 3, 2 ↔ 8, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 7, 5 ↔ 6},
{1 ↔ 4, 2 ↔ 3, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 8, 6 ↔ 7},
{1 ↔ 4, 2 ↔ 5, 3 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 8, 6 ↔ 7},
{1 ↔ 4, 2 ↔ 6, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 8, 5 ↔ 7},
{1 ↔ 4, 2 ↔ 7, 3 ↔ 5, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 6, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 8, 5 ↔ 6},
{1 ↔ 4, 2 ↔ 8, 3 ↔ 5, 6 ↔ 7}, {1 ↔ 4, 2 ↔ 8, 3 ↔ 6, 5 ↔ 7}, {1 ↔ 4, 2 ↔ 8, 3 ↔ 7, 5 ↔ 6},
{1 ↔ 5, 2 ↔ 3, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 8, 6 ↔ 7},
{1 ↔ 5, 2 ↔ 4, 3 ↔ 6, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 7, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 8, 6 ↔ 7},
{1 ↔ 5, 2 ↔ 6, 3 ↔ 4, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 7, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 8, 4 ↔ 7},
{1 ↔ 5, 2 ↔ 7, 3 ↔ 4, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 6, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 8, 4 ↔ 6},
{1 ↔ 5, 2 ↔ 8, 3 ↔ 4, 6 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 6, 4 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 7, 4 ↔ 6},
{1 ↔ 6, 2 ↔ 3, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 8, 5 ↔ 7},
{1 ↔ 6, 2 ↔ 4, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 8, 5 ↔ 7},
{1 ↔ 6, 2 ↔ 5, 3 ↔ 4, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 7, 4 ↔ 8}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 8, 4 ↔ 7},
{1 ↔ 6, 2 ↔ 7, 3 ↔ 4, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 7, 3 ↔ 5, 4 ↔ 8}, {1 ↔ 6, 2 ↔ 7, 3 ↔ 8, 4 ↔ 5},
{1 ↔ 6, 2 ↔ 8, 3 ↔ 4, 5 ↔ 7}, {1 ↔ 6, 2 ↔ 8, 3 ↔ 5, 4 ↔ 7}, {1 ↔ 6, 2 ↔ 8, 3 ↔ 7, 4 ↔ 5},
{1 ↔ 7, 2 ↔ 3, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 3, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 3, 4 ↔ 8, 5 ↔ 6},
{1 ↔ 7, 2 ↔ 4, 3 ↔ 5, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 4, 3 ↔ 6, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 4, 3 ↔ 8, 5 ↔ 6},
{1 ↔ 7, 2 ↔ 5, 3 ↔ 4, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 5, 3 ↔ 6, 4 ↔ 8}, {1 ↔ 7, 2 ↔ 5, 3 ↔ 8, 4 ↔ 6},
{1 ↔ 7, 2 ↔ 6, 3 ↔ 4, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 6, 3 ↔ 5, 4 ↔ 8}, {1 ↔ 7, 2 ↔ 6, 3 ↔ 8, 4 ↔ 5},
{1 ↔ 7, 2 ↔ 8, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 7, 2 ↔ 8, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 7, 2 ↔ 8, 3 ↔ 6, 4 ↔ 5},
{1 ↔ 8, 2 ↔ 3, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 3, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 3, 4 ↔ 7, 5 ↔ 6},
{1 ↔ 8, 2 ↔ 4, 3 ↔ 5, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 4, 3 ↔ 6, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 4, 3 ↔ 7, 5 ↔ 6},
{1 ↔ 8, 2 ↔ 5, 3 ↔ 4, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 5, 3 ↔ 6, 4 ↔ 7}, {1 ↔ 8, 2 ↔ 5, 3 ↔ 7, 4 ↔ 6},
{1 ↔ 8, 2 ↔ 6, 3 ↔ 4, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 6, 3 ↔ 5, 4 ↔ 7}, {1 ↔ 8, 2 ↔ 6, 3 ↔ 7, 4 ↔ 5},
{1 ↔ 8, 2 ↔ 7, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 6, 4 ↔ 5}}

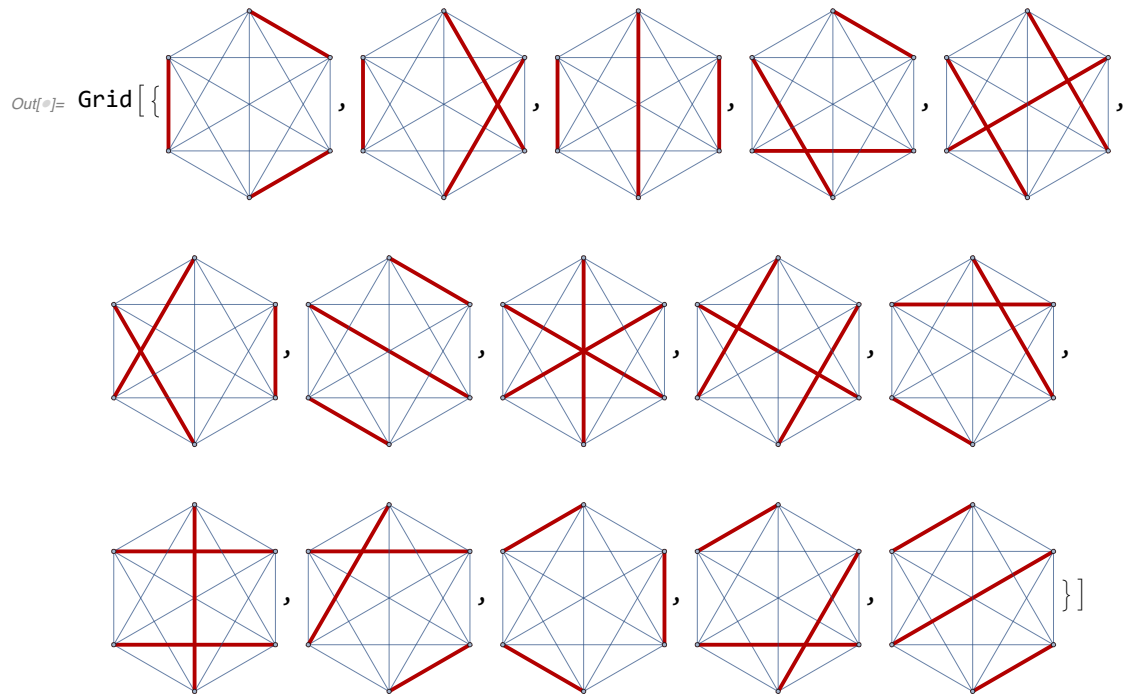
```

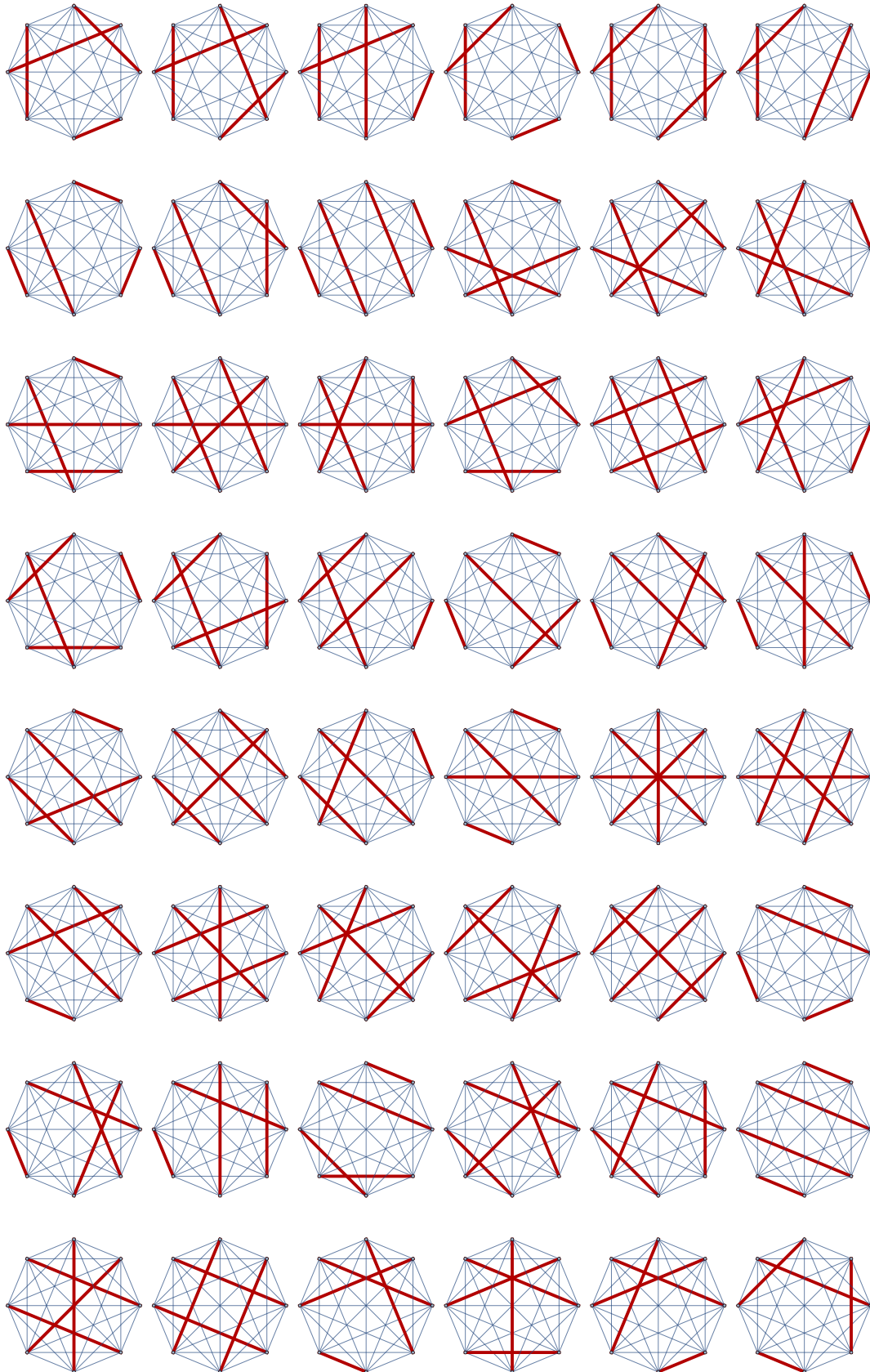
```

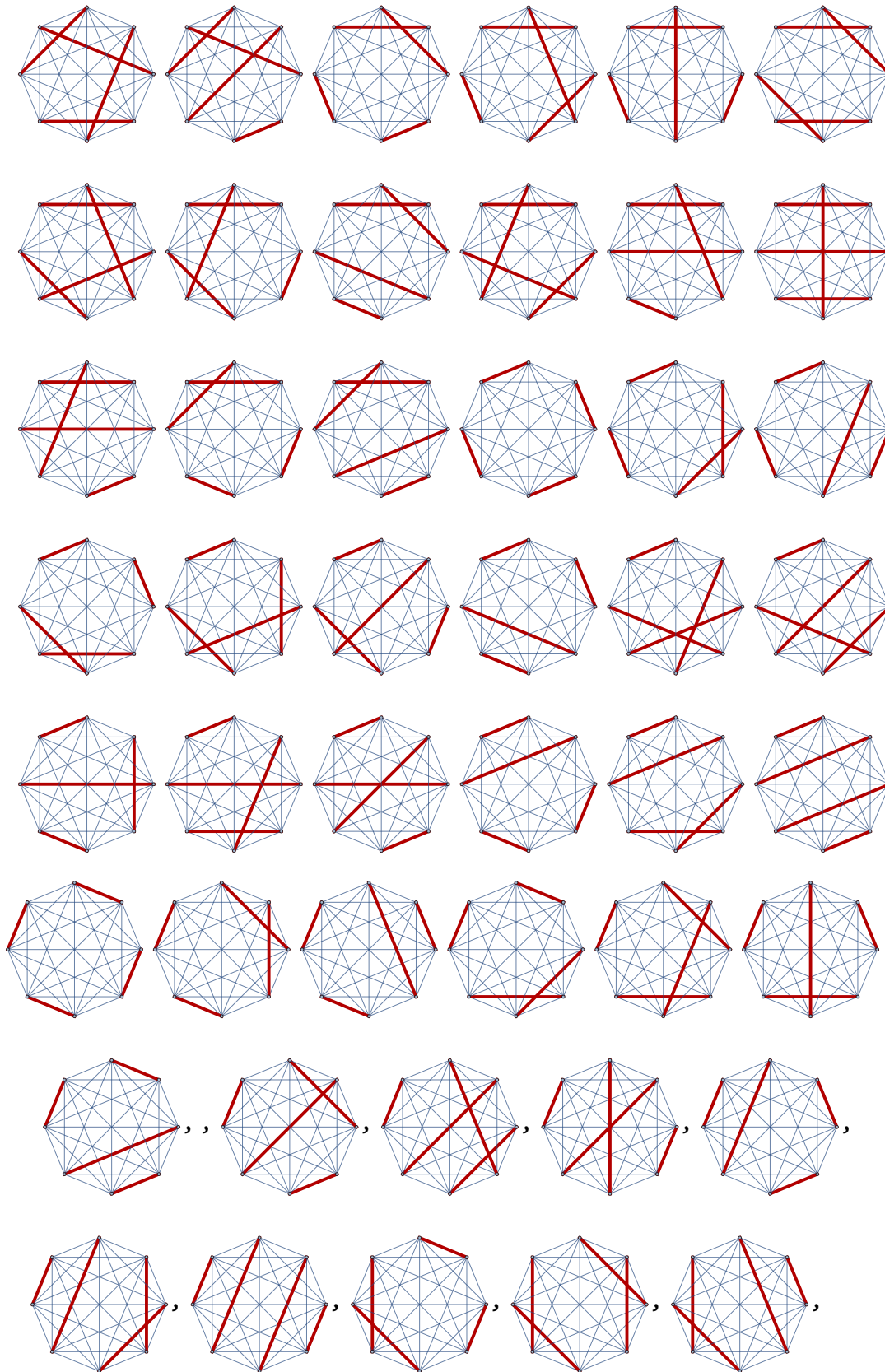
In[*]:=

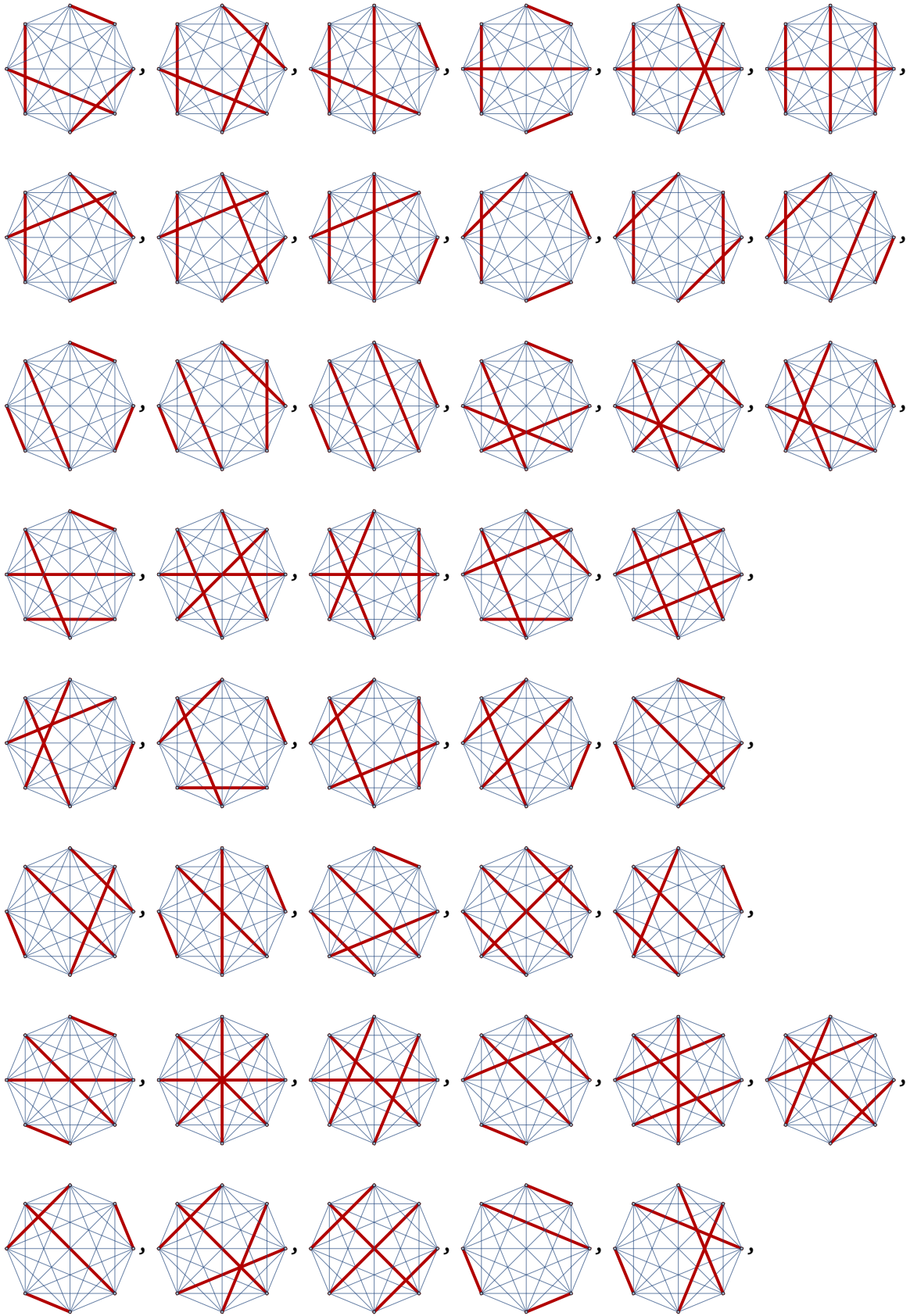
```

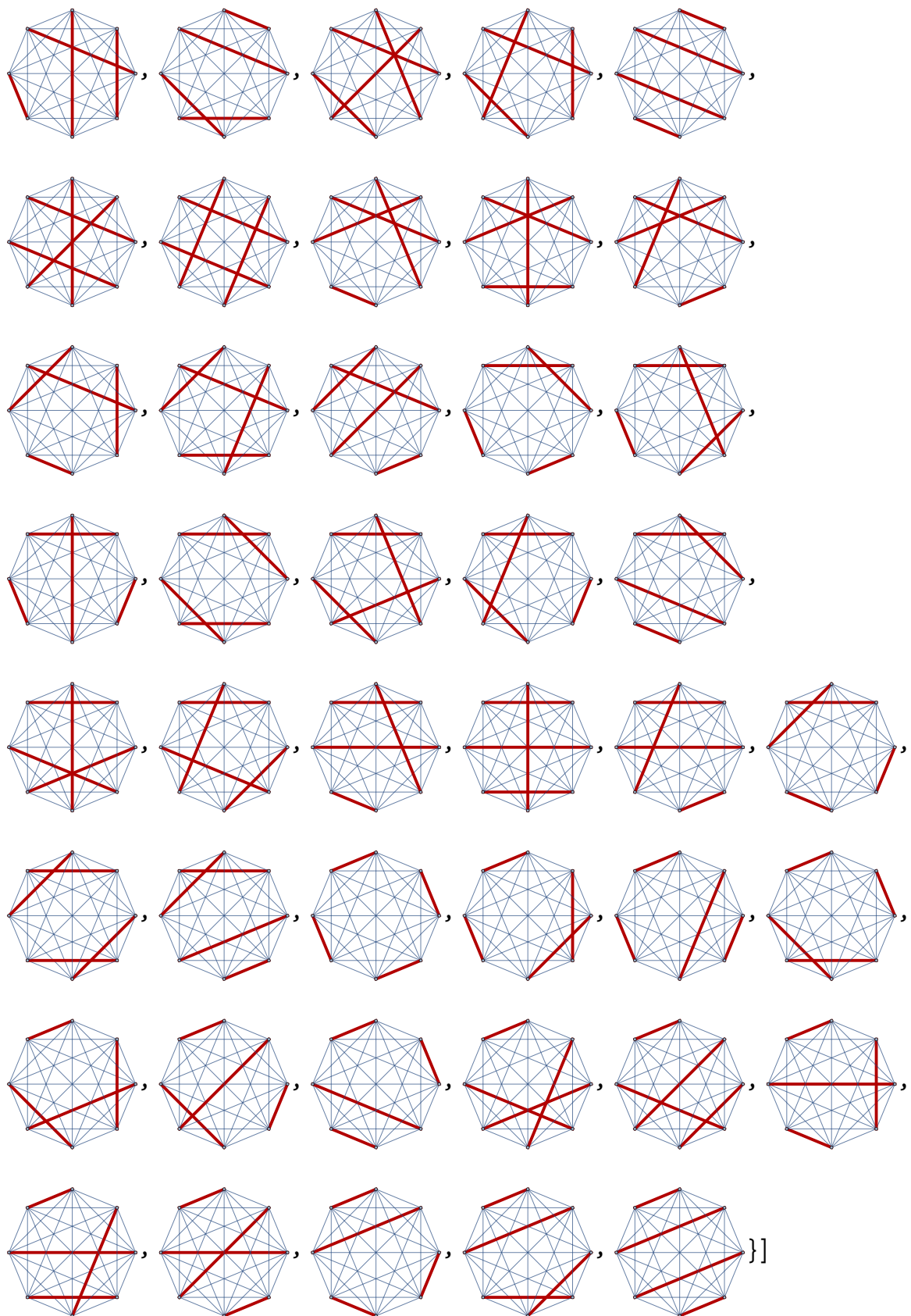
```
In[ ]:= Grid[Table[HighlightGraph[k6, h, GraphHighlightStyle -> "Thick"], {h, es16}]]
```











Stirling Permutations

```
In[ ]:= ClearAll[stringPermutations]
stringPermutations[1] = {{1, 1}};
stringPermutations[k_] := Join @@
  (Function[x, Flatten[Insert[x, {k, k}, #]] & /@ Range[2 k - 1]] /@ stringPermutations[k - 1])
```

```
In[ ]:= Multicolumn[Sort@stringPermutations@4, 5, Appearance -> "Horizontal"]
```

```
Out[ ]:=
```

{1, 1, 2, 2, 3, 3, 4, 4}	{1, 1, 2, 2, 3, 4, 4, 3}	{1, 1, 2, 2, 4, 4, 3, 3}	{1, 1, 2, 3, 3, 2, 4, 4}	{1, 1, 2, 3, 3, 4, 4, 2}
{1, 1, 2, 3, 4, 4, 3, 2}	{1, 1, 2, 4, 4, 2, 3, 3}	{1, 1, 2, 4, 4, 3, 3, 2}	{1, 1, 3, 3, 2, 2, 4, 4}	{1, 1, 3, 3, 2, 4, 4, 2}
{1, 1, 3, 3, 4, 4, 2, 2}	{1, 1, 3, 4, 4, 3, 2, 2}	{1, 1, 4, 4, 2, 2, 3, 3}	{1, 1, 4, 4, 2, 3, 3, 2}	{1, 1, 4, 4, 3, 3, 2, 2}
{1, 2, 2, 1, 3, 3, 4, 4}	{1, 2, 2, 1, 3, 4, 4, 3}	{1, 2, 2, 1, 4, 4, 3, 3}	{1, 2, 2, 3, 3, 1, 4, 4}	{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 4, 4, 3, 1}	{1, 2, 2, 4, 4, 1, 3, 3}	{1, 2, 2, 4, 4, 3, 3, 1}	{1, 2, 3, 3, 2, 1, 4, 4}	{1, 2, 3, 3, 2, 4, 4, 1}
{1, 2, 3, 3, 4, 4, 2, 1}	{1, 2, 3, 4, 4, 3, 2, 1}	{1, 2, 4, 4, 2, 1, 3, 3}	{1, 2, 4, 4, 2, 3, 3, 1}	{1, 2, 4, 4, 3, 3, 2, 1}
{1, 3, 3, 1, 2, 2, 4, 4}	{1, 3, 3, 1, 2, 4, 4, 2}	{1, 3, 3, 1, 4, 4, 2, 2}	{1, 3, 3, 2, 2, 1, 4, 4}	{1, 3, 3, 2, 2, 4, 4, 1}
{1, 3, 3, 2, 4, 4, 2, 1}	{1, 3, 3, 4, 4, 1, 2, 2}	{1, 3, 3, 4, 4, 2, 2, 1}	{1, 3, 4, 4, 3, 1, 2, 2}	{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 1, 2, 2, 3, 3}	{1, 4, 4, 1, 2, 3, 3, 2}	{1, 4, 4, 1, 3, 3, 2, 2}	{1, 4, 4, 2, 2, 1, 3, 3}	{1, 4, 4, 2, 2, 3, 3, 1}
{1, 4, 4, 2, 3, 3, 2, 1}	{1, 4, 4, 3, 3, 1, 2, 2}	{1, 4, 4, 3, 3, 2, 2, 1}	{2, 2, 1, 1, 3, 3, 4, 4}	{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 4, 4, 3, 3}	{2, 2, 1, 3, 3, 1, 4, 4}	{2, 2, 1, 3, 3, 4, 4, 1}	{2, 2, 1, 3, 4, 4, 3, 1}	{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 4, 4, 3, 3, 1}	{2, 2, 3, 3, 1, 1, 4, 4}	{2, 2, 3, 3, 1, 4, 4, 1}	{2, 2, 3, 3, 4, 4, 1, 1}	{2, 2, 3, 4, 4, 3, 1, 1}
{2, 2, 4, 4, 1, 1, 3, 3}	{2, 2, 4, 4, 1, 3, 3, 1}	{2, 2, 4, 4, 3, 3, 1, 1}	{2, 3, 3, 2, 1, 1, 4, 4}	{2, 3, 3, 2, 1, 4, 4, 1}
{2, 3, 3, 2, 4, 4, 1, 1}	{2, 3, 3, 4, 4, 2, 1, 1}	{2, 3, 4, 4, 3, 2, 1, 1}	{2, 4, 4, 2, 1, 1, 3, 3}	{2, 4, 4, 2, 1, 3, 3, 1}
{2, 4, 4, 2, 3, 3, 1, 1}	{2, 4, 4, 3, 3, 2, 1, 1}	{3, 3, 1, 1, 2, 2, 4, 4}	{3, 3, 1, 1, 2, 4, 4, 2}	{3, 3, 1, 1, 4, 4, 2, 2}
{3, 3, 1, 2, 2, 1, 4, 4}	{3, 3, 1, 2, 2, 4, 4, 1}	{3, 3, 1, 2, 4, 4, 2, 1}	{3, 3, 1, 4, 4, 1, 2, 2}	{3, 3, 1, 4, 4, 2, 2, 1}
{3, 3, 2, 2, 1, 1, 4, 4}	{3, 3, 2, 2, 1, 4, 4, 1}	{3, 3, 2, 2, 4, 4, 1, 1}	{3, 3, 2, 4, 4, 2, 1, 1}	{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 2, 2, 1}	{3, 3, 4, 4, 2, 2, 1, 1}	{3, 4, 4, 3, 1, 1, 2, 2}	{3, 4, 4, 3, 1, 2, 2, 1}	{3, 4, 4, 3, 2, 2, 1, 1}
{4, 4, 1, 1, 2, 2, 3, 3}	{4, 4, 1, 1, 2, 3, 3, 2}	{4, 4, 1, 1, 3, 3, 2, 2}	{4, 4, 1, 2, 2, 1, 3, 3}	{4, 4, 1, 2, 2, 3, 3, 1}
{4, 4, 1, 2, 3, 3, 2, 1}	{4, 4, 1, 3, 3, 1, 2, 2}	{4, 4, 1, 3, 3, 2, 2, 1}	{4, 4, 2, 2, 1, 1, 3, 3}	{4, 4, 2, 2, 1, 3, 3, 1}
{4, 4, 2, 2, 3, 3, 1, 1}	{4, 4, 2, 3, 3, 2, 1, 1}	{4, 4, 3, 3, 1, 1, 2, 2}	{4, 4, 3, 3, 1, 2, 2, 1}	{4, 4, 3, 3, 2, 2, 1, 1}

```

In[ ]:= ClearAll[stringPermGraph]
stringPermGraph[sp_, opts : OptionsPattern[]] :=
Module[{v1 = DeleteDuplicates@sp, pos = PositionIndex@sp,
  eL = EdgeList@*TransitiveReductionGraph@*GraphUnion},
  Graph[Prepend[v1, 0], eL[Graph@Thread[0 → v1],
    SimpleGraph@RelationGraph[And @@ Between[pos@#] /@ pos[#2] &, v1]],
  GraphLayout → {"LayeredEmbedding", "RootVertex" → 0},
  EdgeLabels → {e_ → Placed[Last@e, {Left, "Middle"}]}, opts]]

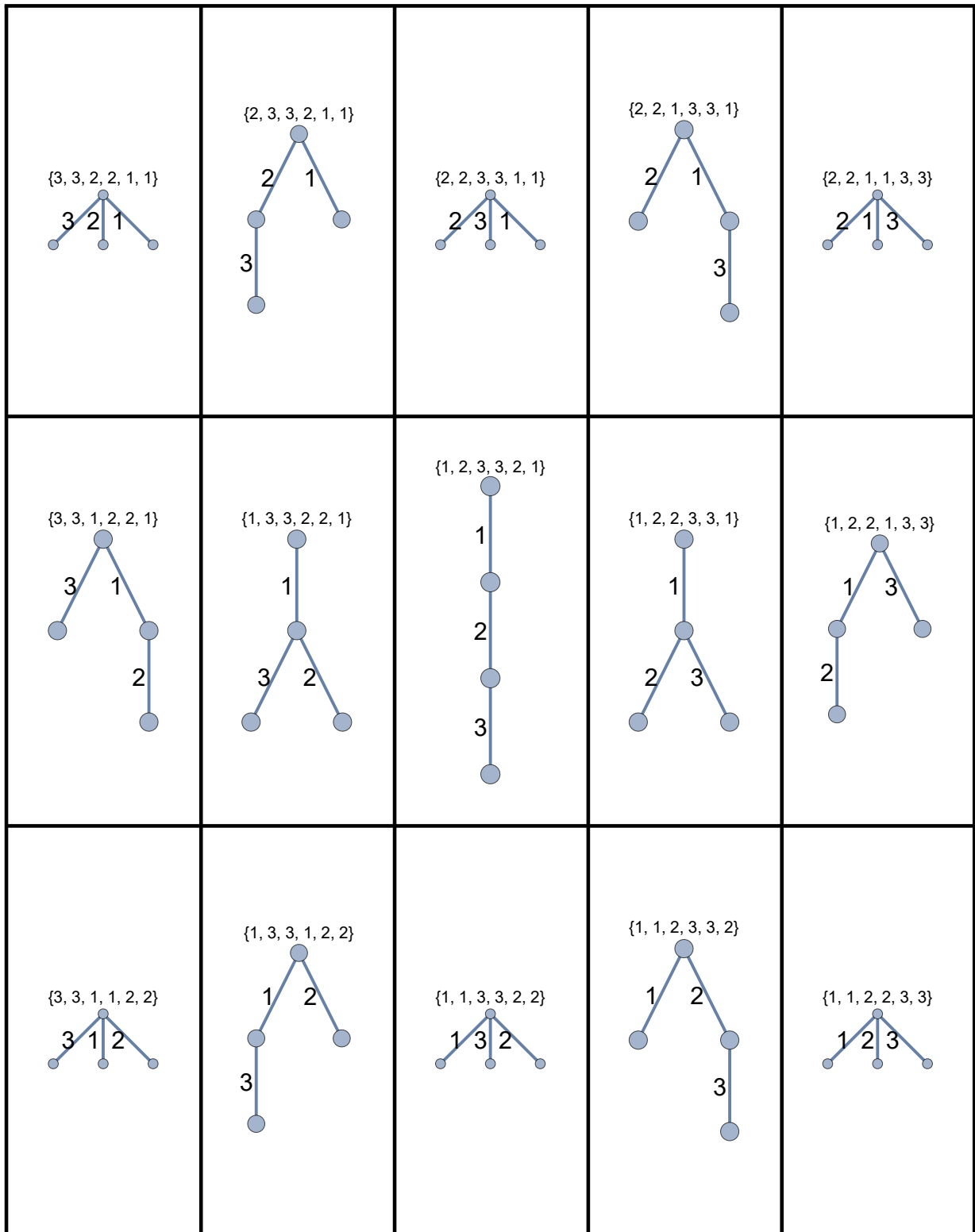
```

```

In[ ]:= Grid[Partition[stringPermGraph[#, PlotLabel -> #, EdgeShapeFunction -> "Line",
  EdgeStyle -> Thick, EdgeLabelStyle -> 16, VertexSize -> Medium] & /@
  stringPermutations[3], 5], Dividers -> All, Spacings -> {4, 4}]

```

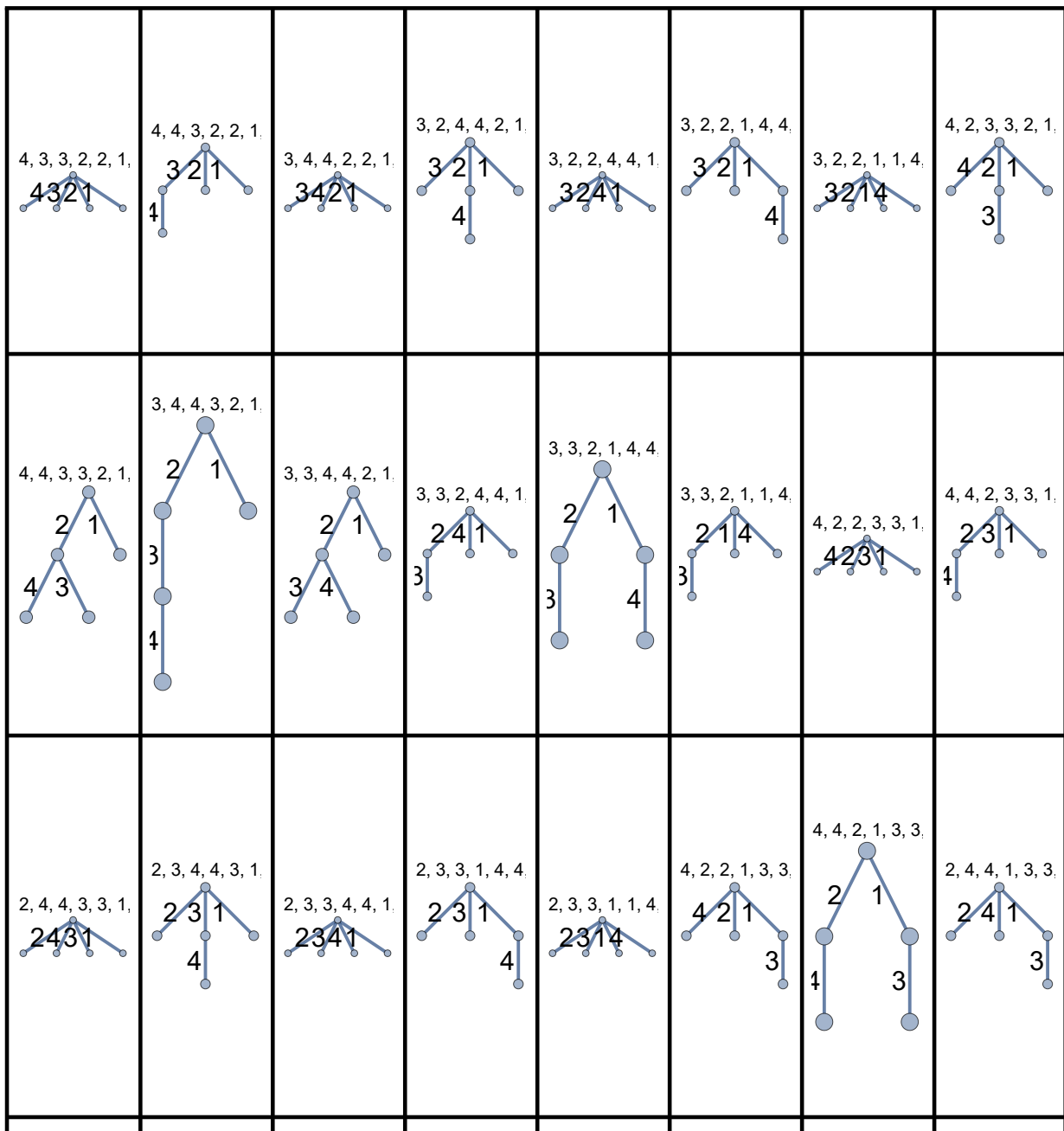
Out[]:=

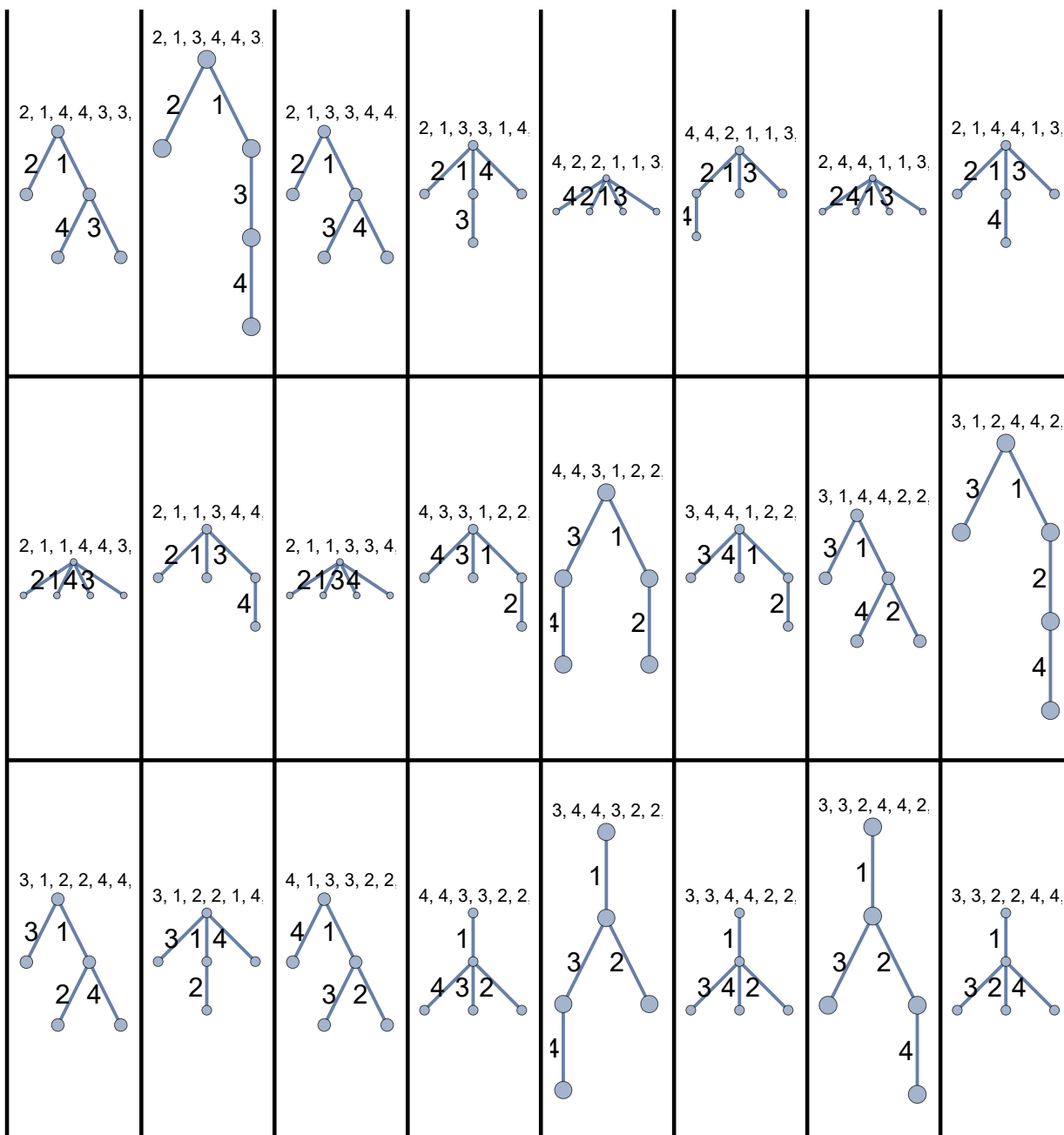


```

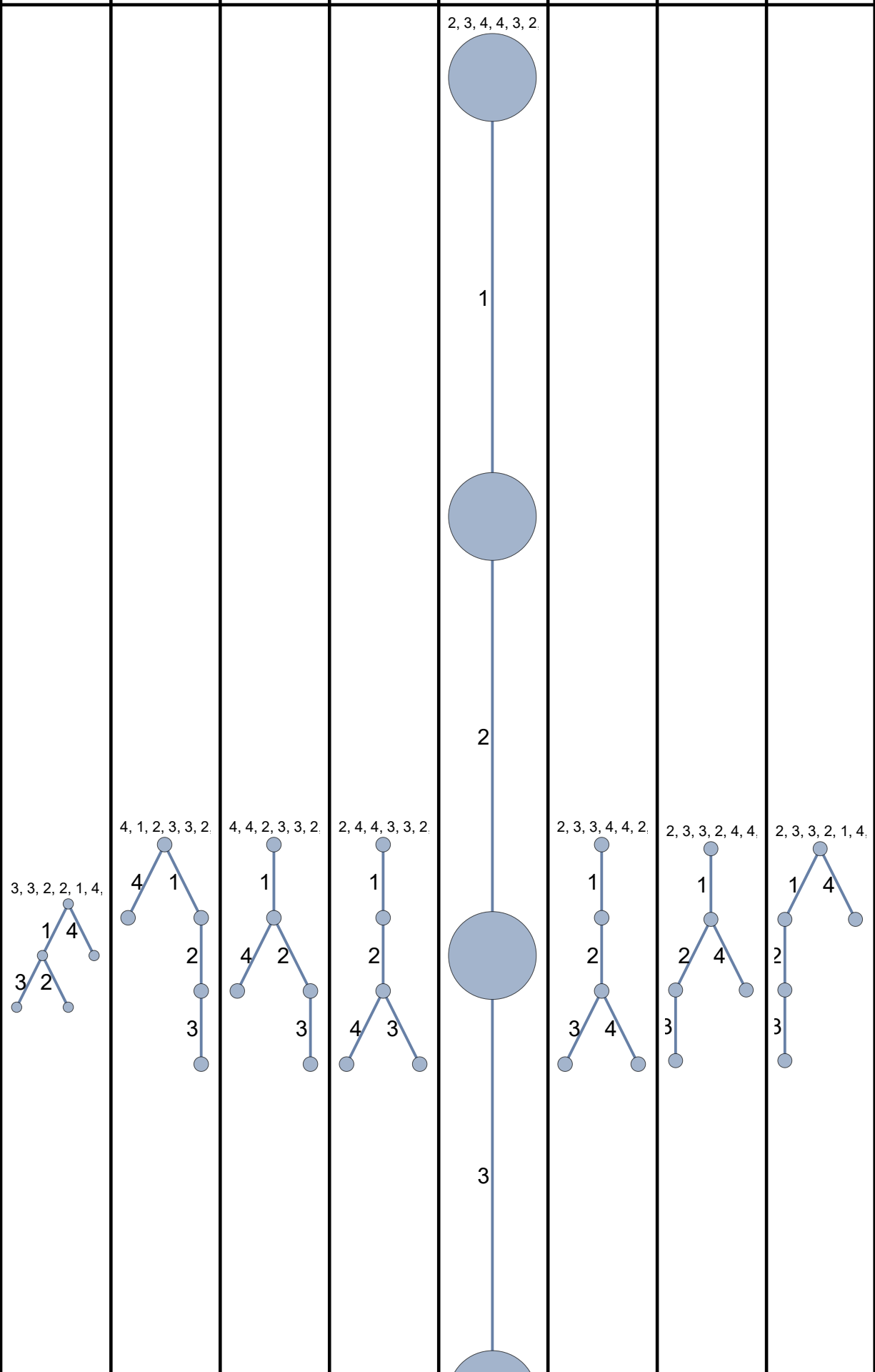
In[ ]:= Grid[Partition[stringPermGraph[#, PlotLabel -> #, EdgeShapeFunction -> "Line",
  EdgeStyle -> Thick, EdgeLabelStyle -> 16, VertexSize -> Medium] & /@
  stringPermutations[4], 8], Dividers -> All, Spacings -> {1, 1}]

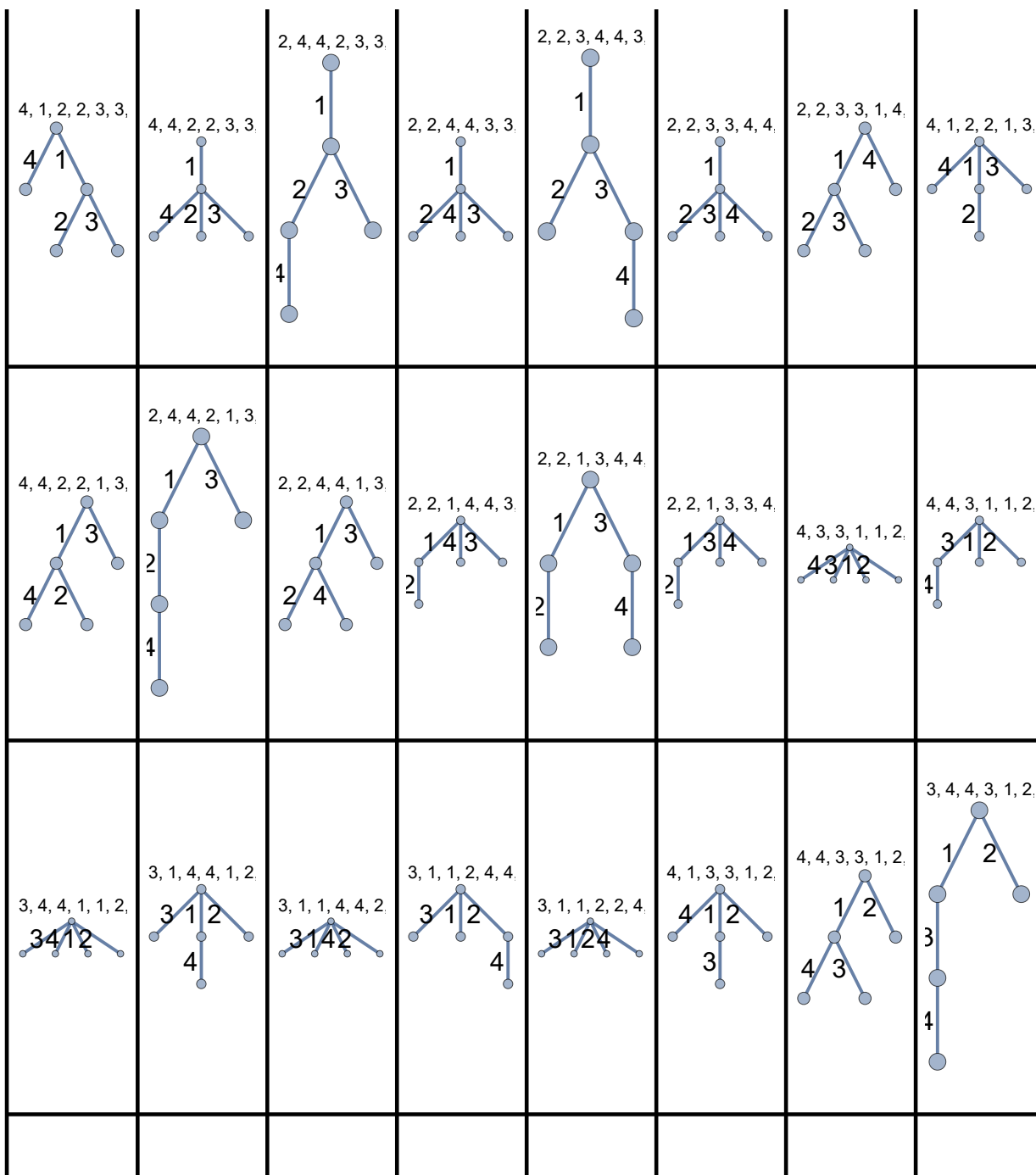
```

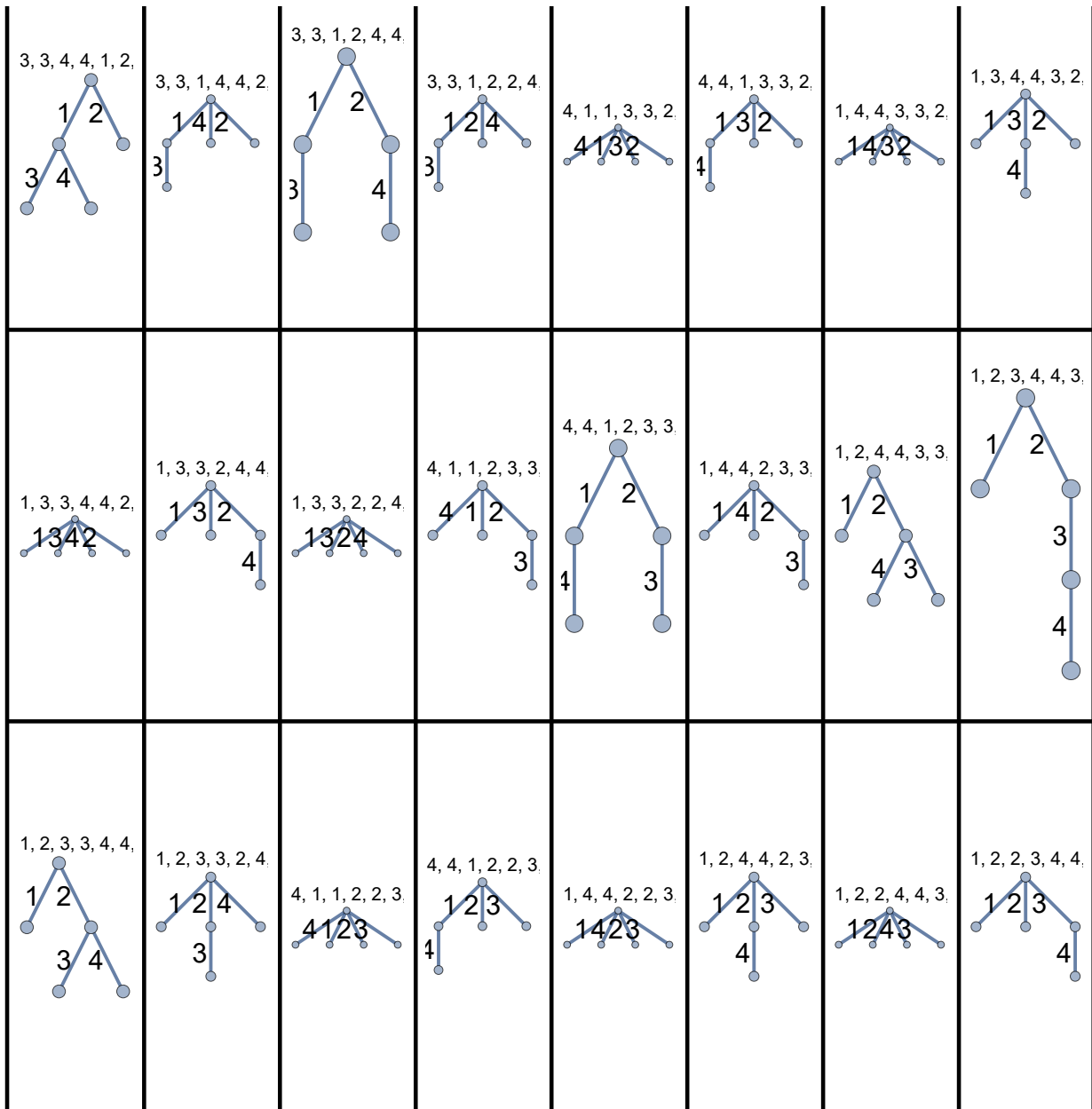




Out[8]=







Limits for Ratio test in $m(2)$

$$\ln[\#] := \text{case1} = \left(2^{1+2n} n! (1+n)! \right) / \left(2 (1+n)! \right) !$$

$$\text{case2} = \left(2^{-1-2n} (2n)! \right) / \left(n! (1+n)! \right)$$

$$\text{Out}[\#] = \frac{2^{1+2n} n! (1+n)!}{(2 (1+n))!}$$

$$\text{Out}[\#] = \frac{2^{-1-2n} (2n)!}{n! (1+n)!}$$

```
In[ ]:= limitm2case1 = Limit[case1, n → Infinity]
      limitm2case2 = Limit[case2, n → Infinity]
```

```
Out[ ]:= 0
```

```
Out[ ]:= 0
```

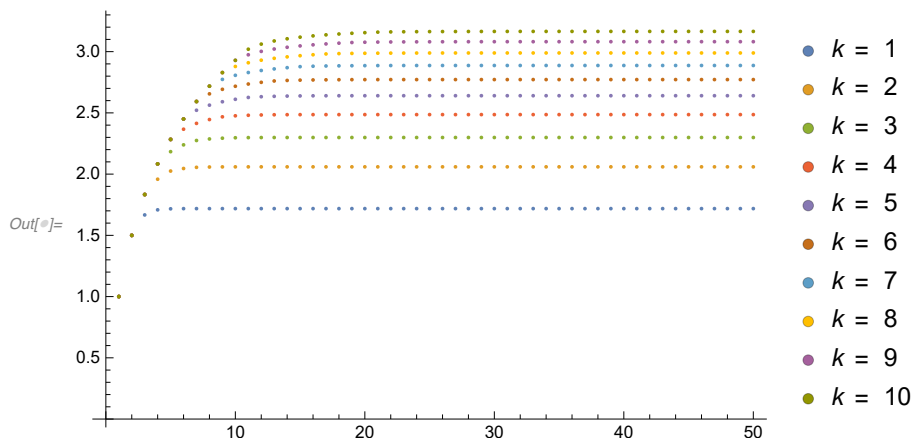
Computing $m(1)$ to $m(10)$ from $n=0$ to $n=2000$

```
In[ ]:= Multifactorial[n_, k_] := Abs[Apply[Times, Range[-n, -1, k]]]
```

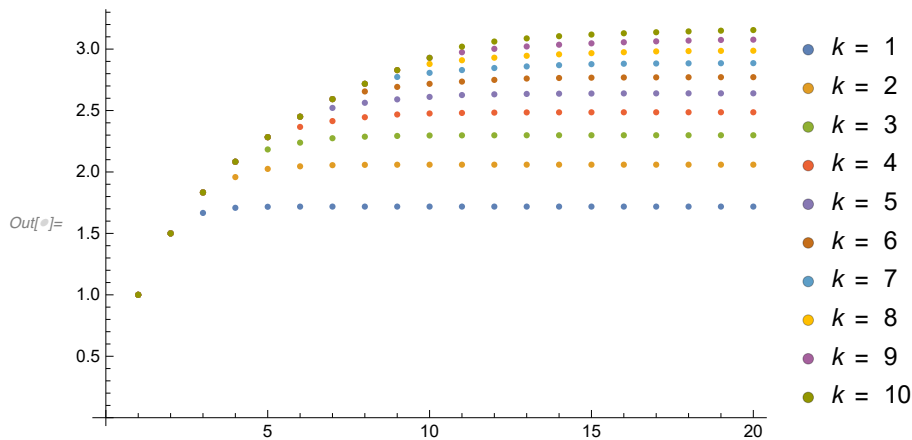
```
In[ ]:= For[i = 1, i < 11, i++, Print[N[Sum[1/Multifactorial[n, i], {n, 0, 150}], 20]]]
2.7182818284590452354
3.0594074053425761445
3.2989135380884190034
3.4859449774535577452
3.6402244677338097342
3.7719023962117584357
3.8869596537408434954
3.9892412126901365441
4.0813755201688985441
4.1652437655583845908
```

Plot of $m(1)$ to $m(10)$ superimposed on each other from $n = 0$ to $n = 2000$

```
ListPlot[Table[Sum[1/Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 20}],
PlotLegends → PointLegend[Automatic,
PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```



```
ListPlot[Table[Sum[1/Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 50}],
PlotLegends -> PointLegend[Automatic,
PromptForm[k, #] & /@ Range[10], LegendMarkers -> {Graphics[Disk[]], 6}]]
```



Computation of RMFCs using the closed form formula

```
In[ ]:= ClosedFormRMFC[n_] := 1 + 1/n Exp[1/n] Sum[n^k/n Gamma[k/n, 0, 1/n], {k, n}]
```

```
In[ ]:= For[i = 1, i < 11, i++, Print[N[ClosedFormRMFC[i], 20]]]
```

```
2.7182818284590452354
```

```
3.0594074053425761445
```

```
3.2989135380884190034
```

```
3.4859449774535577452
```

```
3.6402244677338097342
```

```
3.7719023962117584357
```

```
3.8869596537408434954
```

```
3.9892412126901365441
```

```
4.0813755201688985441
```

```
4.1652437655583845908
```

```
In[ ]:= N[ClosedFormRMFC[100], 100]
```

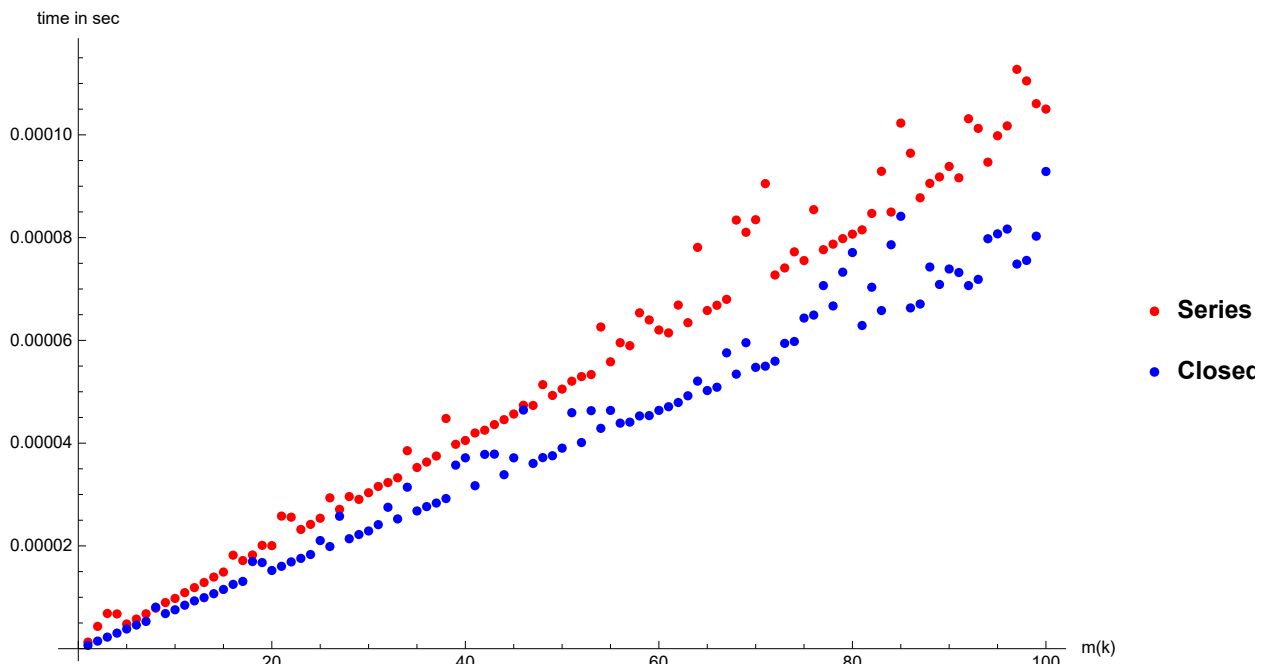
```
Out[ ]:= 6.23255596900487817559483331447484719147772171073261303698458651259212659259992188311253\
0488409940747
```

Analysing efficiency of the two RMFC calculation methods

```
In[ ]:= test1[xx_] :=
  (For[i = 1, i < xx, i++, Print[N[Sum[1/Multifactorial[n, i], {n, 0, 250}], 50]] //
    Inactive] // RepeatedTiming)[[1]]
test2[xx_] := (For[i = 1, i < xx, i++, Print[N[ClosedFormRMFC[i], 50]] // Inactive] //
  RepeatedTiming)[[1]]
```

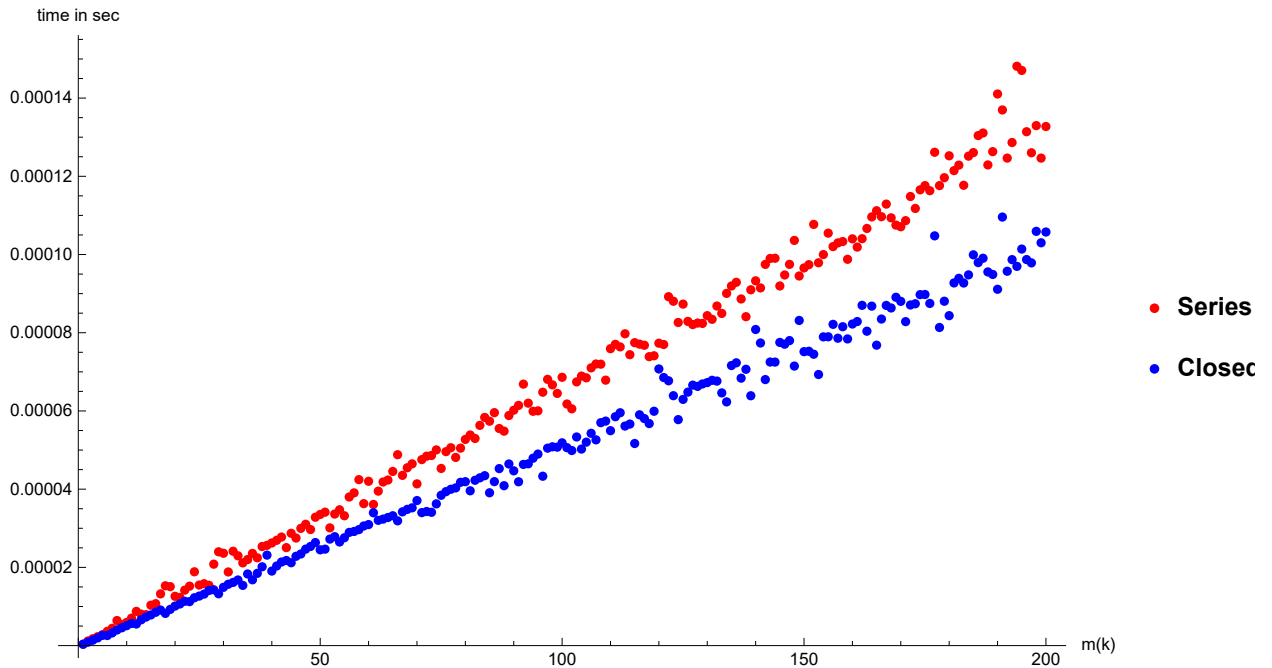
```
In[ ]:= list1 =
  ListPlot[Table[{xx, test1[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
list2 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends →
  {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

```
In[ ]:= Show[list1, list2, ImageSize → Large]
```



```
In[ ]:= list3 =
  ListPlot[Table[{xx, test1[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
list4 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends →
  {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

```
In[ ]:= Show[list3, list4, ImageSize → Large]
```

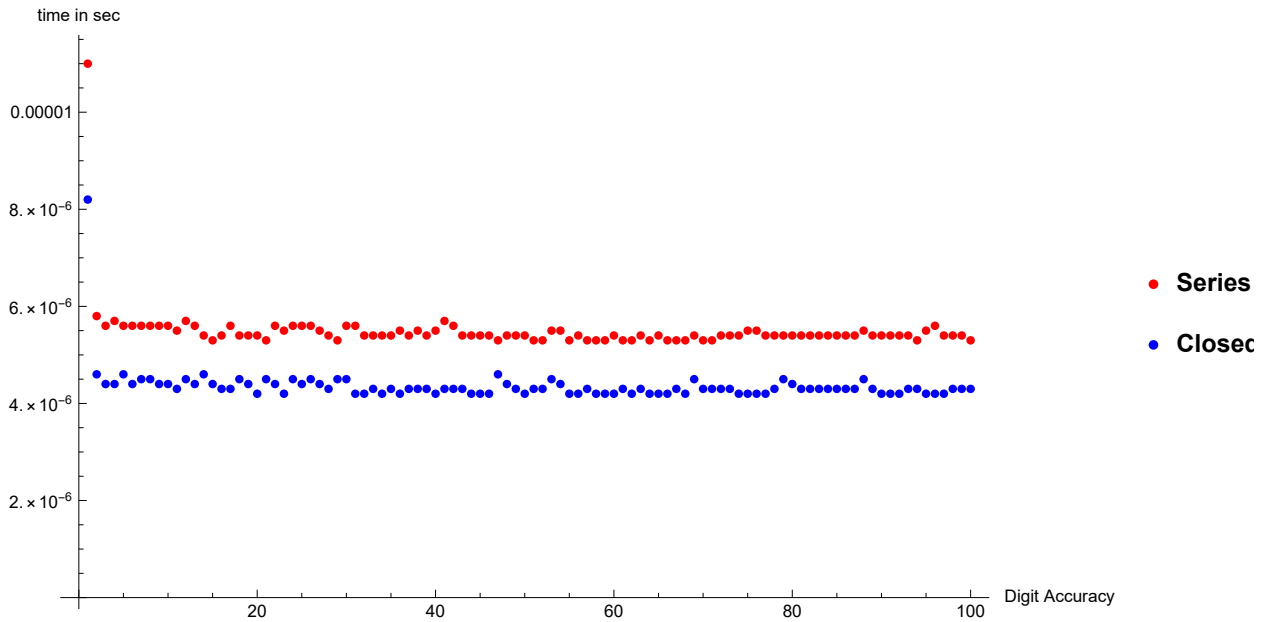


```
test3[xx_] :=
  (For[i = 1, i < 10, i++, Print[N[Sum[1/Multifactorial[n, i], {n, 0, 500}], xx]] //
    Inactive] // AbsoluteTiming)[[1]]
test4[xx_] := (For[i = 1, i < 10, i++, Print[N[ClosedFormRMFC[i], xx]] // Inactive] //
  AbsoluteTiming)[[1]]
```

```
In[ ]:= list5 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

```
In[ ]:= list6 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

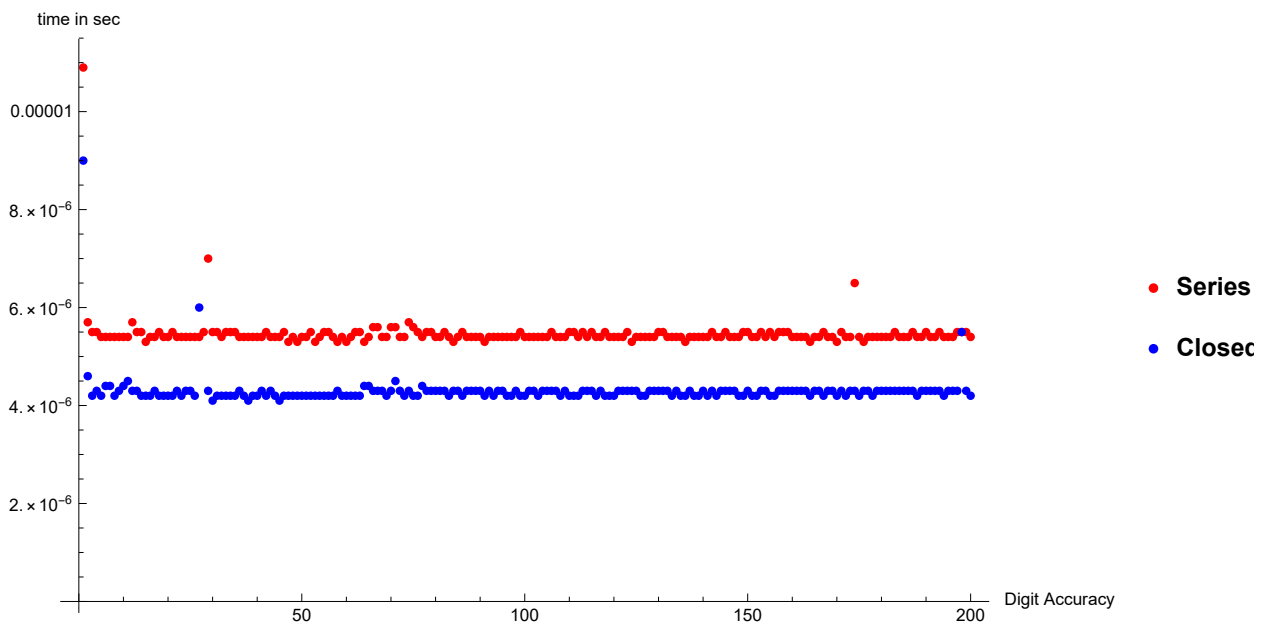
```
In[ ]:= Show[list5, list6, ImageSize → Large]
```



```
In[ ]:= list7 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

```
In[ ]:= list8 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
    AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

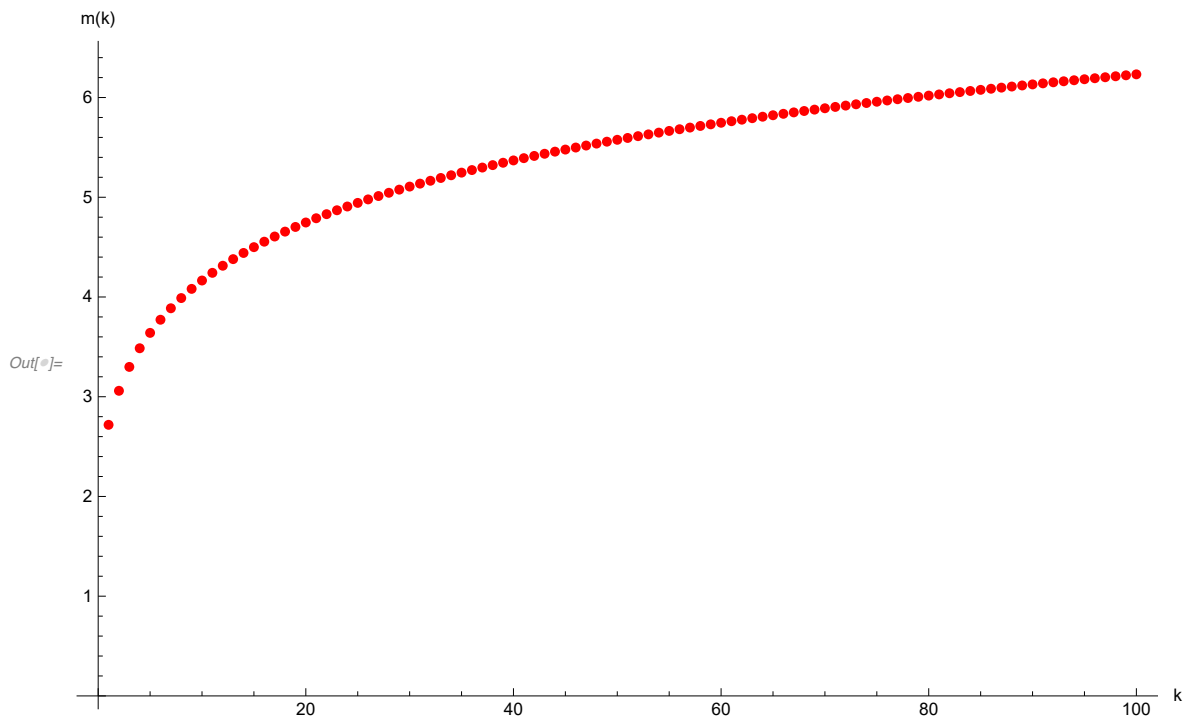
```
In[ ]:= Show[list7, list8, ImageSize → Large]
```



Asymptotics of Reciprocal Multifactorial Series

Simple Graph

```
In[8]:= ListPlot[Table[{x, ClosedFormRMFC[x]}, {x, 1, 100}],
  ImageSize -> Large, AxesLabel -> {"k", "m(k)"}, PlotStyle -> {Red, Thick}]
```



Detailed asymptotics

```
Series[1/k Integrate[(1 - t) (1 - E^(-t/k))]/(t (E^(-t/k Log[t]) - 1)), {t, 0, 1}],
  {k, Infinity, 5}]
```

$$\text{Out}[8]= \frac{\text{Log}[2]}{k} + \frac{-1 - \text{Log}\left[\frac{9}{4}\right]}{4 k^2} + \frac{5 + 8 \text{Log}\left[\frac{4}{3}\right]}{48 k^3} + \frac{-11 - 36 \text{Log}\left[\frac{5}{4}\right]}{864 k^4} + \frac{-79 + 108 \text{Log}\left[\frac{6}{5}\right]}{12960 k^5} + O\left[\frac{1}{k}\right]^6$$

```
In[9]:= Series[HarmonicNumber[k], {k, Infinity, 5}]
```

$$\text{Out}[9]= \left(\text{EulerGamma} + \text{Log}[k]\right) + \frac{1}{2 k} - \frac{1}{12 k^2} + \frac{1}{120 k^4} + O\left[\frac{1}{k}\right]^6$$

```
In[8]:= Series[E^(1/k), {k, Infinity, 5}]
```

$$\text{Out}[8]= 1 + \frac{1}{k} + \frac{1}{2 k^2} + \frac{1}{6 k^3} + \frac{1}{24 k^4} + \frac{1}{120 k^5} + O\left[\frac{1}{k}\right]^6$$

```
In[6]:= FullSimplify[1 + (1 +  $\frac{1}{2k^2} + \frac{1}{k}$ ) (EulerGamma +  $\frac{1}{6k^2} + \frac{1}{2k} + \frac{\text{Log}[3]}{2k^2} - \frac{\text{Log}[4]}{2k} - \frac{\text{Log}[64]}{12k^2} + \text{Log}[k]$ )]
```

$$\text{Out[6]} = 1 + \frac{(1 + 2k(1+k)) \left(1 + \text{Log}\left[\frac{27}{8}\right] + k(3 + 6\text{EulerGamma}k - \text{Log}[64]) + 6k^2 \text{Log}[k]\right)}{12k^4}$$

```
In[7]:= FullSimplify[1 + (1 +  $\frac{1}{k}$ ) * ((EulerGamma + Log[k] +  $\frac{1}{2k}$ ) - ( $\frac{1}{k} \text{Log}[2]$ ))]
```

$$\text{Out[7]} = 1 + \frac{(1+k)(1 + 2\text{EulerGamma}k - \text{Log}[4] + 2k \text{Log}[k])}{2k^2}$$

```
In[8]:= ClosedFormRMFC[n_] := 1 +  $\frac{1}{n} \text{Exp}[1/n] \text{Sum}[n^{k/n} \text{Gamma}[\frac{k}{n}, 0, \frac{1}{n}], \{k, n\}]$ 
```

```
In[2]:= RMFCApproximation[k_] :=
```

$$1 + \frac{(1 + 2k(1+k)) \left(1 + \text{Log}\left[\frac{27}{8}\right] + k(3 + 6\text{EulerGamma}k - \text{Log}[64]) + 6k^2 \text{Log}[k]\right)}{12k^4}$$

```
RMFCApproximation1[k_] := 1 +  $\frac{(1+k)(1 + 2\text{EulerGamma}k - 2\text{Log}[2] + 2k \text{Log}[k])}{2k^2}$ 
```

```
In[4]:= N[RMFCApproximation[15]]
N[RMFCApproximation1[15]]
```

```
Out[4]= 4.49958
```

```
Out[5]= 4.49055
```

```
N[ClosedFormRMFC[15]]
```

```
Out[6]= 4.49969
```

```
In[9]:= Table[{x, Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation[10^x]])] /
```

$$\text{N[ClosedFormRMFC[10^x], 50]}], \{x, 0, 5\}]$$

```
Out[9]= {{0, 0.0608426}, {1, 0.0000779859}, {2, 1.14278 × 10-7},
```

$$\{3, 1.29004 \times 10^{-10}\}, \{4, 1.37156 \times 10^{-13}\}, \{5, 0.\}}$$

```
In[10]:= Table[{x, Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation1[10^x]])] /
```

$$\text{N[ClosedFormRMFC[10^x], 50]}], \{x, 0, 5\}]$$

```
Out[10]= {{0, 0.349539}, {1, 0.00449172}, {2, 0.0000476604},
```

$$\{3, 4.84353 \times 10^{-7}\}, \{4, 4.87866 \times 10^{-9}\}, \{5, 4.90019 \times 10^{-11}\}}$$