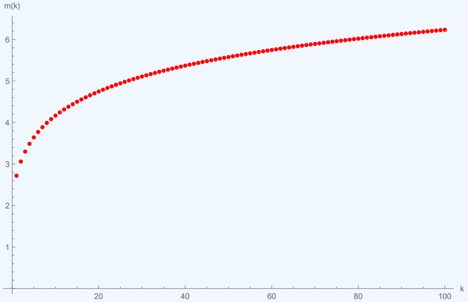
A Look at Reciprocal Multifactorial Constants

Bhoris Dhanjal

St. Xavier's College (Autonomous)

July 9, 2021



Section 1: Multifactorials

Definition of a Multifactorial

We define the multifactorial $n \underbrace{!\ldots!}_{\text{k times}}$ or $n!_{(k)}$ for $n \in \mathbb{N}_0, k \in \mathbb{N}$ by comparing it to the factorial.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \ldots \quad \textit{Terminates with 1}$$

Using steps of larger integer values we get,

$$\begin{aligned} n!! &= n \cdot (n-2) \cdot (n-4) \cdot (n-6) \ldots \quad \textit{Terminates with 2 or 1} \\ n!!! &= n \cdot (n-3) \cdot (n-6) \cdot (n-9) \ldots \quad \textit{Terminates with 3, 2 or 1} \\ &\vdots \end{aligned}$$

Using this we can alternatively define the multifactorial of any $n > 0$ of order $k > 0$ as the follows:

$$\begin{aligned} n!_{(k)} &= \prod_{j=0}^{q} kj + r \qquad \text{where } n = kq + r, q \geq 0, \text{and } 1 \leq r \leq k \\ &= 1 \qquad\qquad\qquad n = 0 \end{aligned} \tag{1}$$

The above definition is identical to the recursive relation:

$$n!_{(k)} = \begin{cases} 1 & \text{if } n = 0 \\ n & \text{if } 0 < n \leq k \\ n\left((n-k)!_{(k)}\right) & \text{if } n > k \end{cases} \tag{2}$$

The Reciprocal Multifactorial Series for multifactorial of order k is defined as the follows:

$$m(k) = \sum_{n=0}^{\infty} \frac{1}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}} \tag{3}$$

m(k)	$\sum_{n=0}^{2000} \frac{1}{n!_{(k)}}$ Rounded to 10 significant digits
m(1)	2.718281828
m(2)	3.059407405
m(3)	3.298913538
m(4)	3.485944977
m(5)	3.640224468
m(6)	3.771902396
m(7)	3.886959654
m(8)	3.989241213
m(9)	4.081375520
m(10)	4.165243766

Table: Computed values of first 10 RMFCs [1]

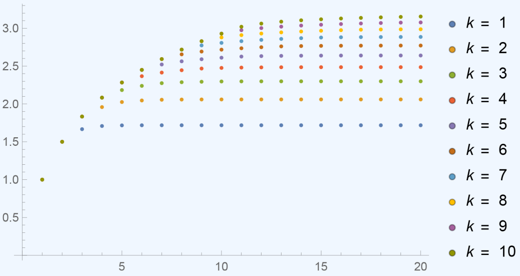


Figure: First 20 partial sums of first 10 Reciprocal Multifactorial Series

Section 2: Closed Form Formula for RMFCs

A few prerequisite definitions [2] [3] [4]

$$\text{Gamma Function } \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \text{ for } \mathrm{Re}(z) > 0 \tag{4}$$

$$\text{Beta Function } \mathrm{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \text{ for } \mathrm{Re}(x), \mathrm{Re}(y) > 0 \tag{5}$$

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{6}$$

$$\text{Lower incomplete gamma function } \gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt \tag{7}$$

Closed Form Formula for RMFCs

Lemma

Relation between k^{th} multifactorial and Beta function is given by

$$n!_{(k)} = \frac{k^{q+1} q!}{\mathrm{B}\left(\frac{r}{k}, q+1\right)}$$

Proof: Recall the definition of multifactorial as described in eq. 1.

$$n \underbrace{!\ldots!}_{\text{k times}} = n!_{(k)} = \prod_{j=0}^{q}(kj+r) = k^{q+1}\prod_{j=0}^{q}\left(j+\frac{r}{k}\right)$$

$$= k^{q+1}\frac{\Gamma(q+1+\frac{r}{k})}{\Gamma(r/k)} = \frac{k^{q+1}\Gamma(q+1)}{\mathrm{B}\left(\frac{r}{k},q+1\right)} \text{ (from eq. 6)}$$

$$= \frac{k^{q+1}q!}{\mathrm{B}\left(\frac{r}{k},q+1\right)} \text{ (since q is a positive integer)}$$

$\square$

Closed Form Formula for RMFCs contd.

Theorem

A closed form formula for Reciprocal Multifactorial Constants is given by the following expression [5]

$$m(k) = 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right)$$

Proof of Closed Form Formula for RMFCs

Proof: Recall definition 3.1 for reciprocal multifactorial series.

$$m(k) = \sum_{n=0}^{\infty} \frac{1}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}}$$

Now, using the lemma we get,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{k^{q+1} q!} \mathrm{B}\left(\frac{r}{k}, q+1\right)$$

Using the integral definition for the Beta function (see 4.1.2) we can say,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{k^{q+1} q!} \int_0^1 t^{r/k-1}(1-t)^q \, dt$$

Proof of Closed Form Formula for RMFCs contd.

$$= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}\sum_{q=0}^{\infty}\frac{1}{q!}\left(\frac{1-t}{k}\right)^q dt$$

$$= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}e^{(1-t)/k}\,dt$$

Taking $t = kx$ we simplify the function as follows,

$$= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^{1/k}(kx)^{r/k-1}e^{(1-kx)/k}\,kdx$$

$$= 1 + \frac{e^{1/k}}{k}\sum_{r=1}^{k}k^{r/k}\int_0^{1/k}x^{r/k-1}e^{-x}\,dx$$

Finally using the definition for incomplete gamma function

$$= 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right)$$

$\square$

Reciprocal Multifactorial Series contd.

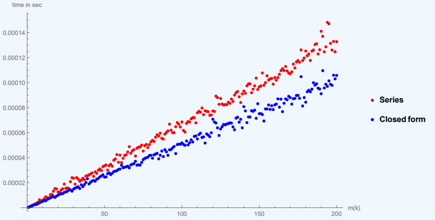


Figure: Time taken to compute a list of first k RMFCs (k ranging from 1 to 200) up to 50 digit accuracy.

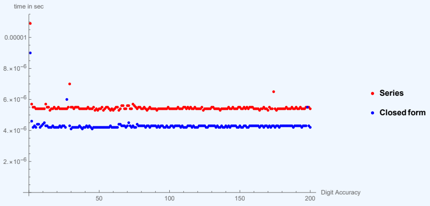


Figure: Time taken to compute first 10 RMFCs with increasing accuracy up to 200 digit accuracy.

Section 3: Asymptotics of Reciprocal Multifactorial Series

Theorem

$\lim_{k\to\infty} m(k) = 1 + H_k$

Proof: Continuing from the previous proof, [6]

$$m(k) = 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} \int_0^1 t^{r/k-1} e^{-t/k}\, dt$$

Observe that $e^{-t/k} = 1 - (1 - e^{-t/k})$

$$= 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} \int_0^1 t^{r/k-1} (1 - (1 - e^{-t/k}))\, dt$$

Since, $\int_0^1 t^{r/k-1}\, dt = \frac{k}{r}$, we can further simplify the above expression as

Divergent nature of RMFCs for large k contd.

$$= 1 + e^{1/k}\left(H_k - \frac{1}{k}\int_0^1 \sum_{r=1}^{k} t^{r/k}\frac{(1-e^{-t/k})}{t}\,dt\right)$$

$$= 1 + e^{1/k}\left(H_k - \frac{1}{k}\int_0^1 \frac{(1-t)(1-e^{-t/k})}{t(t^{-1/k}-1)}\,dt.\right)$$

Replacing $\frac{1}{k}\int_0^1 \frac{(1-t)(1-e^{-t/k})}{t(t^{-1/k}-1)}\,dt.$ with Δk we get,

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$

Taking the limit as k approaches infinity we get,

$$\lim_{k\to\infty} m(k) = \lim_{k\to\infty} 1 + e^{1/k}(H_k - \Delta k)$$

$$\lim_{k\to\infty} m(k) = 1 + H_k \qquad \square$$

We will generate asymptotic approximations for RMFCs using the following series, [7][8][9]

$$e^{1/k} = 1 + \frac{1}{k} + \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right) \tag{8}$$

$$H_k = \log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + O\left(\frac{1}{k^4}\right) \tag{9}$$

$$\Delta k = \frac{\log(2)}{k} + \frac{1}{4k^2}\left(-1 - \log\frac{9}{4}\right) + O\left(\frac{1}{k^3}\right) \tag{10}$$

Approximation 1: Using asymptotic series truncated at 1^{st} order terms

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$
$$m(k) \sim 1 + \left(1 + \frac{1}{k}\right)\left(\left(\log(k) + \gamma + \frac{1}{2k}\right) - \left(\frac{\log(2)}{k}\right)\right)$$

Which can be simplified as the following,

$$m(k) \sim 1 + \frac{(1+k)(1+2\gamma k - \log(4) + 2k\log(k))}{2k^2} \tag{11}$$

Approximation 2: Using asymptotic series truncated at 2^{nd} order terms

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$

$$m(k) \sim 1 + \left(1 + \frac{1}{k} + \frac{1}{2k^2}\right)\left[\left(\log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\right) - \left(\frac{\log(2)}{k} + \frac{1}{4k^2}\left(-1 - \log\frac{9}{4}\right)\right)\right]$$

Which can be simplified as the following,

$$m(k) \sim 1 + \frac{(1 + 2k(1+k) \cdot (1 + \log\frac{27}{8} + 6k^2 \log k + k(3 + 6\gamma k - \log 64))}{12k^4} \tag{12}$$

Accuracy of asymptotic approximations for RMFCs

m(k)	Absolute error between asymptotic approximation and exact solution
1	$3.49539 \cdot 10^{-1}$
10	$4.49172 \cdot 10^{-3}$
10^2	$4.76604 \cdot 10^{-5}$
10^3	$4.84353 \cdot 10^{-7}$
10^4	$4.87866 \cdot 10^{-9}$

Table: Comparison between first asymptotic equation and exact solution for a few RMFCs

m(k)	Absolute error between asymptotic approximation and exact solution
1	$6.08426 \cdot 10^{-2}$
10	$7.79859 \cdot 10^{-5}$
10^2	$1.14278 \cdot 10^{-7}$
10^3	$1.29004 \cdot 10^{-10}$
10^4	$1.37156 \cdot 10^{-13}$

Table: Comparison between second asymptotic equation and exact solution for a few RMFCs

Section 4: Generalized Reciprocal Multifactorial Constants

Definition of Generalized Reciprocal Multifactorial Constants

In this section we will discuss the power series reciprocal multifactorial series. For simplicity sake we will refer to these power series as *Generalized Reciprocal Multifactorial Series*. Which we define as follows,

$$m_x(k) = \sum_{n=0}^{\infty} \frac{x^n}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}} \tag{13}$$

The radius for the above power series can be seen to be infinity for all finite k.

Closed form formula for GRMFCs

Theorem

A closed form formula for Generalized Reciprocal Multifactorial Constants is given by the following expression

$$m_x(k) = 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{x^k}{k}\right)$$

Proof: We continue similarly to the previous theorem,

$$m_x(k) = \sum_{n=0}^{\infty} \frac{x^n}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{(kq+r) \underbrace{! \ldots !}_{\text{k times}}}$$

Using the lemma we get,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{k^{q+1} q!} \mathrm{B}\left(\frac{r}{k}, q+1\right)$$

Using the integral definition for the Beta function,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{k^{q+1} q!} \int_0^1 t^{r/k-1} (1-t)^q \, dt$$

Proof of Closed Form Formula for GRMFCs contd.

$$= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^1 x^r t^{r/k-1} \sum_{q=0}^{\infty} \frac{1}{q!} \left(x^k \frac{1-t}{k} \right)^q dt$$

$$= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^1 x^r t^{r/k-1} e^{(x^k - x^k t)/k} \, dt$$

$$= 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} x^r \int_0^1 t^{r/k-1} e^{(-x^k t)/k} \, dt$$

Let, $u = x^k t/k, dt = kx^{-k} du, t = uk/x^k$

$$= 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} x^r \int_0^{x^k/k} \left(\frac{uk}{x^k} \right)^{r/k-1} e^{-u} k x^{-k} \, du$$

Proof of Closed Form Formula for GRMFCs contd.

$$= 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} x^r (kx^{-k})(k^{r/k-1} x^{k-r}) \int_0^{x^k/k} u^{r/k-1} e^{-u}\, du$$

Simplifying we get,

$$= 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} k^{r/k} \int_0^{x^k/k} u^{r/k-1} e^{-u}\, du$$

$$= 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{x^k}{k}\right) \qquad \square$$

x\k	1	2	3	4	5
0.5	1.64872	1.67697	1.68663	1.69039	1.69195
1.0	2.71828	3.05941	3.29891	3.48594	3.64022
1.5	4.48169	6.42488	9.06146	13.1533	20.2725
2.0	7.38906	16.2285	45.7755	219.471	$2.85031 \cdot 10^3$
2.5	12.1825	50.9309	589.300	$7.02993 \cdot 10^4$	$1.43833 \cdot 10^9$

Table: Few computed values of GRMFCs using the closed form formula.

References I

OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences.
A001113, A143280, A288055, A288091, A288092, A288093, A288094, A288095, A288096. A342033,.

Gamma function. From MathWorld—A Wolfram Web Resource.
https://mathworld.wolfram.com/GammaFunction.html.

Beta function. From MathWorld—A Wolfram Web Resource.
https://mathworld.wolfram.com/BetaFunction.html.

Incomplete gamma function. From MathWorld—A Wolfram Web Resource.
https://mathworld.wolfram.com/IncompleteGammaFunction.html.

Mathematics Stack Exchange.
https://math.stackexchange.com/q/3993774 (version: 2021-01-21).

Mathematics Stack Exchange.
https://math.stackexchange.com/q/4021528 (version: 2021-02-11).

References II

- Harmonic number. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/HarmonicNumber.html.
- Exponential function. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/ExponentialFunction.html.
- Euler-mascheroni constant. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/Euler-MascheroniConstant.html.

Thanks for listening