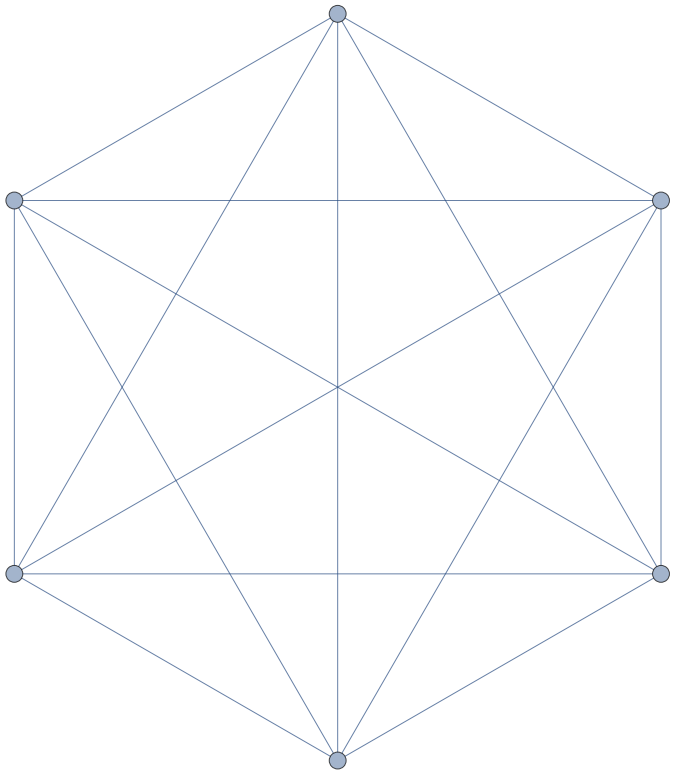
Reciprocal Multifactorial Constants

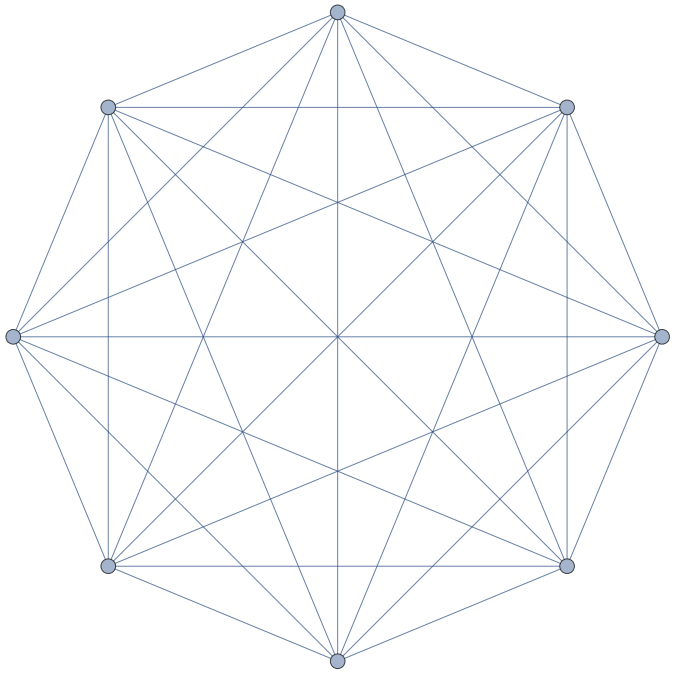
Perfect matchings for K_ 6 and K_8

In[ ]:=
```
k6 = CompleteGraph[6]
k8 = CompleteGraph[8]
```

Out[ ]=



Out[ ]=



In[ ]:=
```
l6 = Length[FindIndependentEdgeSet[k6]]
l8 = Length[FindIndependentEdgeSet[k8]]
```

Out[ ]= 3

Out[ ]= 4

```
In[ ]:= esl6 = Select[Subsets[EdgeList[k6], {l6}], IndependentEdgeSetQ[k6, #] &]
        esl8 = Select[Subsets[EdgeList[k8], {l8}], IndependentEdgeSetQ[k8, #] &]
```

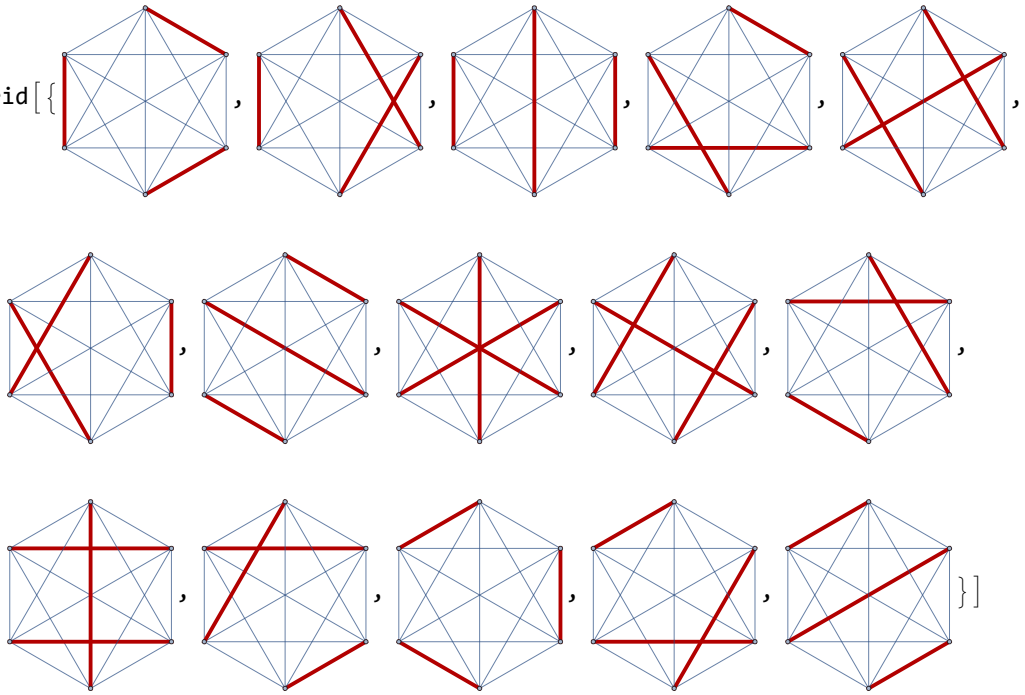
```
Out[ ]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 5}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 6},
  {1 ↔ 3, 2 ↔ 5, 4 ↔ 6}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 5}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 6}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 6},
  {1 ↔ 4, 2 ↔ 6, 3 ↔ 5}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 6}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 6}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 4},
  {1 ↔ 6, 2 ↔ 3, 4 ↔ 5}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 5}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 4}}
```

```
Out[ ]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 8, 6 ↔ 7},
  {1 ↔ 2, 3 ↔ 5, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 8, 6 ↔ 7},
  {1 ↔ 2, 3 ↔ 6, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 8, 5 ↔ 7},
  {1 ↔ 2, 3 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 2, 3 ↔ 7, 4 ↔ 8, 5 ↔ 6},
  {1 ↔ 2, 3 ↔ 8, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 2, 3 ↔ 8, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 2, 3 ↔ 8, 4 ↔ 7, 5 ↔ 6},
  {1 ↔ 3, 2 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 8, 6 ↔ 7},
  {1 ↔ 3, 2 ↔ 5, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 5, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 5, 4 ↔ 8, 6 ↔ 7},
  {1 ↔ 3, 2 ↔ 6, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 3, 2 ↔ 6, 4 ↔ 8, 5 ↔ 7},
  {1 ↔ 3, 2 ↔ 7, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 3, 2 ↔ 7, 4 ↔ 8, 5 ↔ 6},
  {1 ↔ 3, 2 ↔ 8, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 3, 2 ↔ 8, 4 ↔ 7, 5 ↔ 6},
  {1 ↔ 4, 2 ↔ 3, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 3, 5 ↔ 8, 6 ↔ 7},
  {1 ↔ 4, 2 ↔ 5, 3 ↔ 6, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 7, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 5, 3 ↔ 8, 6 ↔ 7},
  {1 ↔ 4, 2 ↔ 6, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 6, 3 ↔ 8, 5 ↔ 7},
  {1 ↔ 4, 2 ↔ 7, 3 ↔ 5, 6 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 6, 5 ↔ 8}, {1 ↔ 4, 2 ↔ 7, 3 ↔ 8, 5 ↔ 6},
  {1 ↔ 4, 2 ↔ 8, 3 ↔ 5, 6 ↔ 7}, {1 ↔ 4, 2 ↔ 8, 3 ↔ 6, 5 ↔ 7}, {1 ↔ 4, 2 ↔ 8, 3 ↔ 7, 5 ↔ 6},
  {1 ↔ 5, 2 ↔ 3, 4 ↔ 6, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 7, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 3, 4 ↔ 8, 6 ↔ 7},
  {1 ↔ 5, 2 ↔ 4, 3 ↔ 6, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 7, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 4, 3 ↔ 8, 6 ↔ 7},
  {1 ↔ 5, 2 ↔ 6, 3 ↔ 4, 7 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 7, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 6, 3 ↔ 8, 4 ↔ 7},
  {1 ↔ 5, 2 ↔ 7, 3 ↔ 4, 6 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 6, 4 ↔ 8}, {1 ↔ 5, 2 ↔ 7, 3 ↔ 8, 4 ↔ 6},
  {1 ↔ 5, 2 ↔ 8, 3 ↔ 4, 6 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 6, 4 ↔ 7}, {1 ↔ 5, 2 ↔ 8, 3 ↔ 7, 4 ↔ 6},
  {1 ↔ 6, 2 ↔ 3, 4 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 3, 4 ↔ 8, 5 ↔ 7},
  {1 ↔ 6, 2 ↔ 4, 3 ↔ 5, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 7, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 4, 3 ↔ 8, 5 ↔ 7},
  {1 ↔ 6, 2 ↔ 5, 3 ↔ 4, 7 ↔ 8}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 7, 4 ↔ 8}, {1 ↔ 6, 2 ↔ 5, 3 ↔ 8, 4 ↔ 7},
  {1 ↔ 6, 2 ↔ 7, 3 ↔ 4, 5 ↔ 8}, {1 ↔ 6, 2 ↔ 7, 3 ↔ 5, 4 ↔ 8}, {1 ↔ 6, 2 ↔ 7, 3 ↔ 8, 4 ↔ 5},
  {1 ↔ 6, 2 ↔ 8, 3 ↔ 4, 5 ↔ 7}, {1 ↔ 6, 2 ↔ 8, 3 ↔ 5, 4 ↔ 7}, {1 ↔ 6, 2 ↔ 8, 3 ↔ 7, 4 ↔ 5},
  {1 ↔ 7, 2 ↔ 3, 4 ↔ 5, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 3, 4 ↔ 6, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 3, 4 ↔ 8, 5 ↔ 6},
  {1 ↔ 7, 2 ↔ 4, 3 ↔ 5, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 4, 3 ↔ 6, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 4, 3 ↔ 8, 5 ↔ 6},
  {1 ↔ 7, 2 ↔ 5, 3 ↔ 4, 6 ↔ 8}, {1 ↔ 7, 2 ↔ 5, 3 ↔ 6, 4 ↔ 8}, {1 ↔ 7, 2 ↔ 5, 3 ↔ 8, 4 ↔ 6},
  {1 ↔ 7, 2 ↔ 6, 3 ↔ 4, 5 ↔ 8}, {1 ↔ 7, 2 ↔ 6, 3 ↔ 5, 4 ↔ 8}, {1 ↔ 7, 2 ↔ 6, 3 ↔ 8, 4 ↔ 5},
  {1 ↔ 7, 2 ↔ 8, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 7, 2 ↔ 8, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 7, 2 ↔ 8, 3 ↔ 6, 4 ↔ 5},
  {1 ↔ 8, 2 ↔ 3, 4 ↔ 5, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 3, 4 ↔ 6, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 3, 4 ↔ 7, 5 ↔ 6},
  {1 ↔ 8, 2 ↔ 4, 3 ↔ 5, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 4, 3 ↔ 6, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 4, 3 ↔ 7, 5 ↔ 6},
  {1 ↔ 8, 2 ↔ 5, 3 ↔ 4, 6 ↔ 7}, {1 ↔ 8, 2 ↔ 5, 3 ↔ 6, 4 ↔ 7}, {1 ↔ 8, 2 ↔ 5, 3 ↔ 7, 4 ↔ 6},
  {1 ↔ 8, 2 ↔ 6, 3 ↔ 4, 5 ↔ 7}, {1 ↔ 8, 2 ↔ 6, 3 ↔ 5, 4 ↔ 7}, {1 ↔ 8, 2 ↔ 6, 3 ↔ 7, 4 ↔ 5},
  {1 ↔ 8, 2 ↔ 7, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 8, 2 ↔ 7, 3 ↔ 6, 4 ↔ 5}}
```

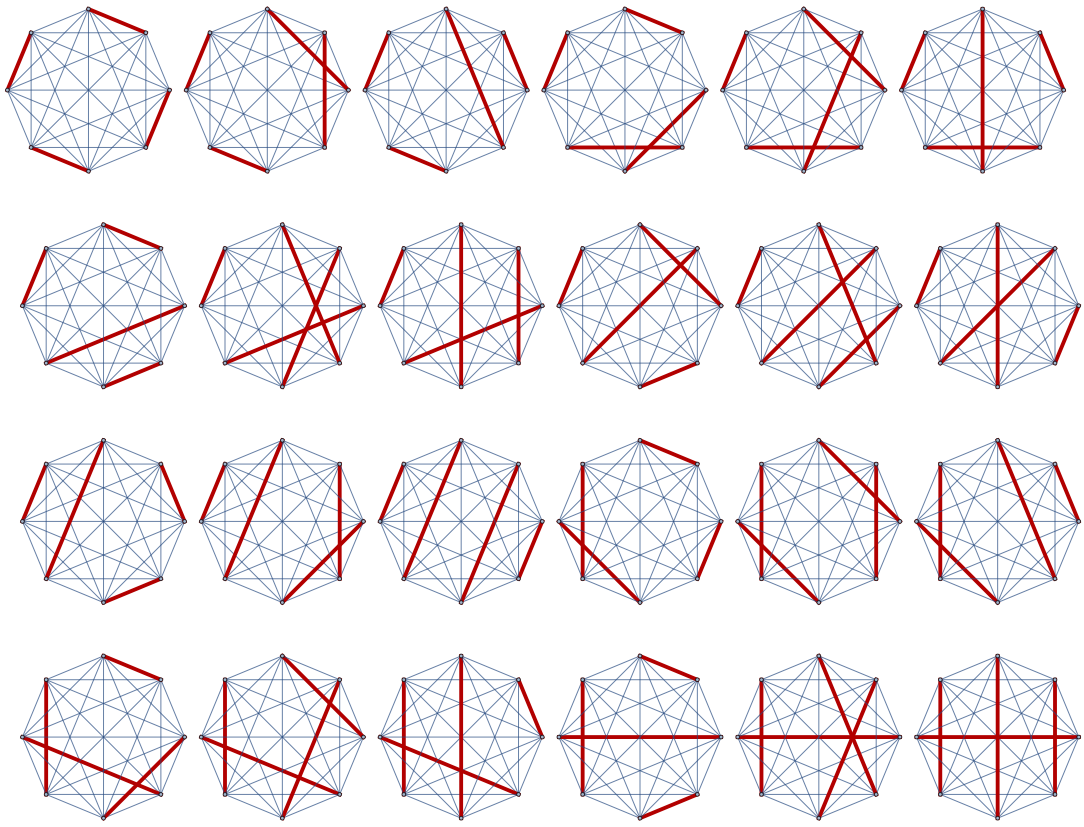
```
In[ ]:=
```

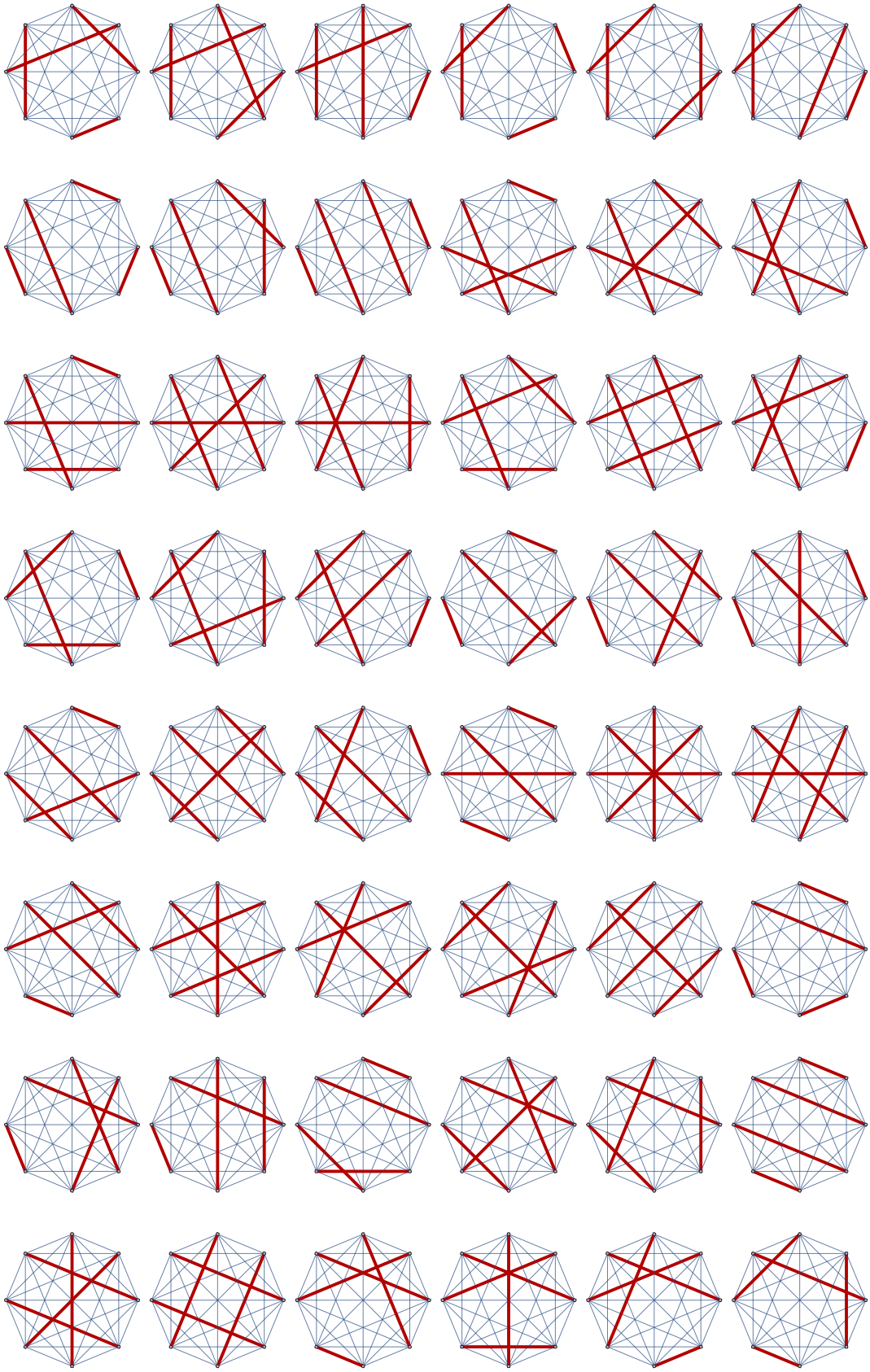
In[ ]:=
```
Grid[Table[HighlightGraph[k6, h, GraphHighlightStyle → "Thick"], {h, esl6}]]
```

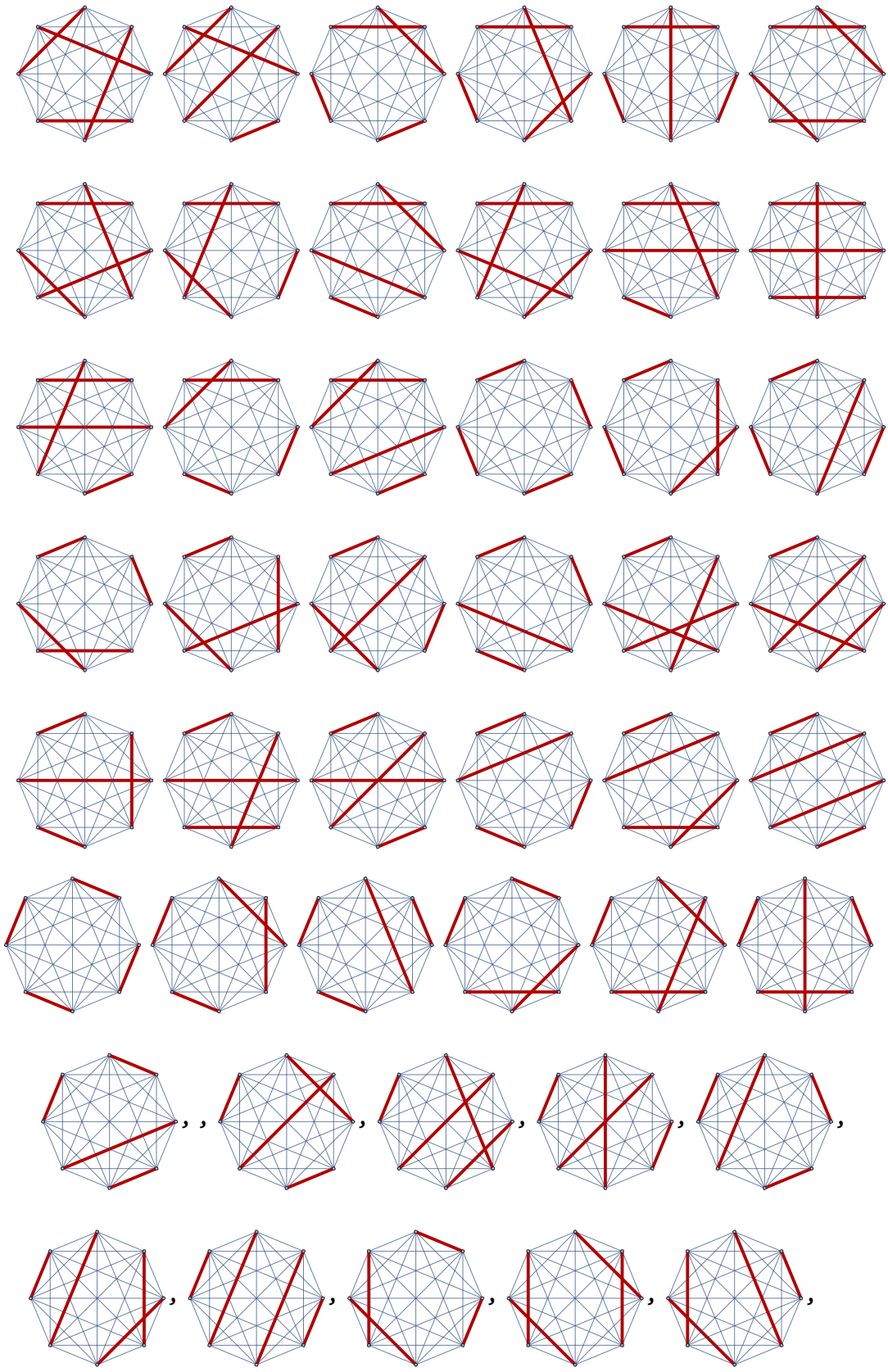
Out[ ]=

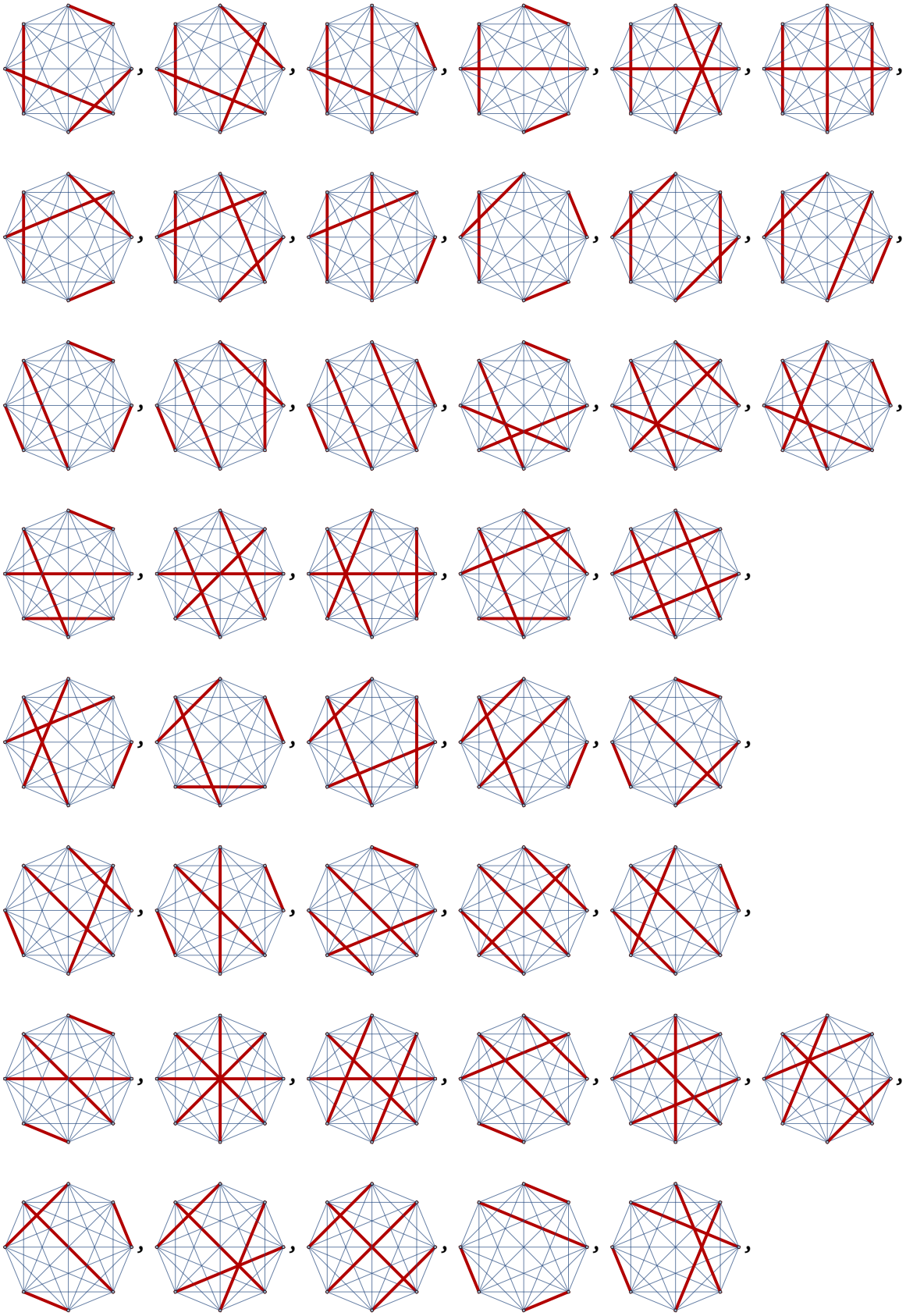


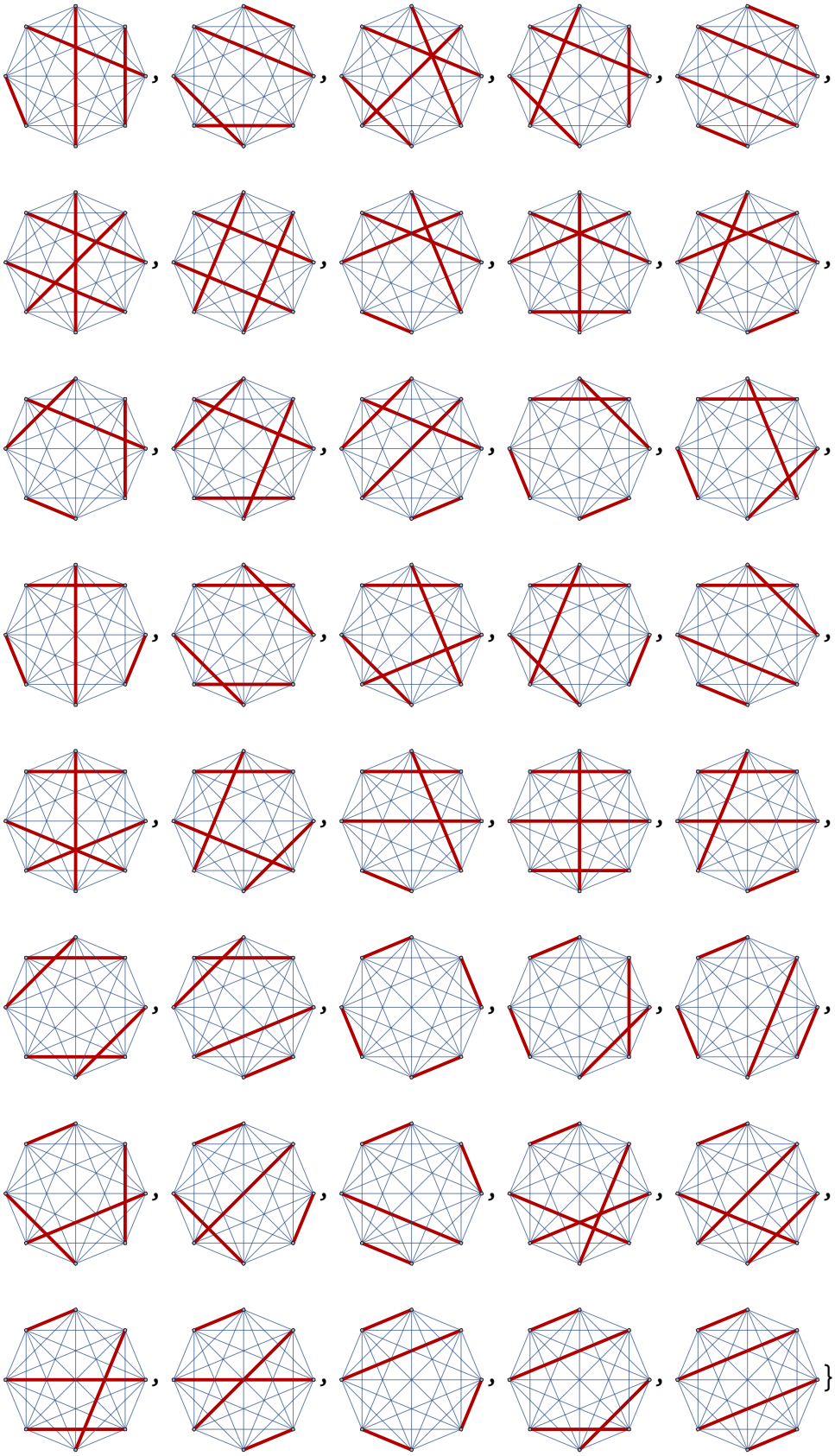
In[ ]:=
```
Grid[Table[HighlightGraph[k8, h, GraphHighlightStyle → "Thick"], {h, esl8}]]
```











Stirling Permutations

In[ ]:=
```
ClearAll[strlngPermutations]
strlngPermutations[1] = {{1, 1}};
strlngPermutations[k_] := Join @@
  (Function[x, Flatten[Insert[x, {k, k}, #]] & /@ Range[2 k - 1]] /@ strlngPermutations[k - 1])
```

In[•]:=
```
Multicolumn[Sort@strlngPermutations@4, 5, Appearance → "Horizontal"]
```

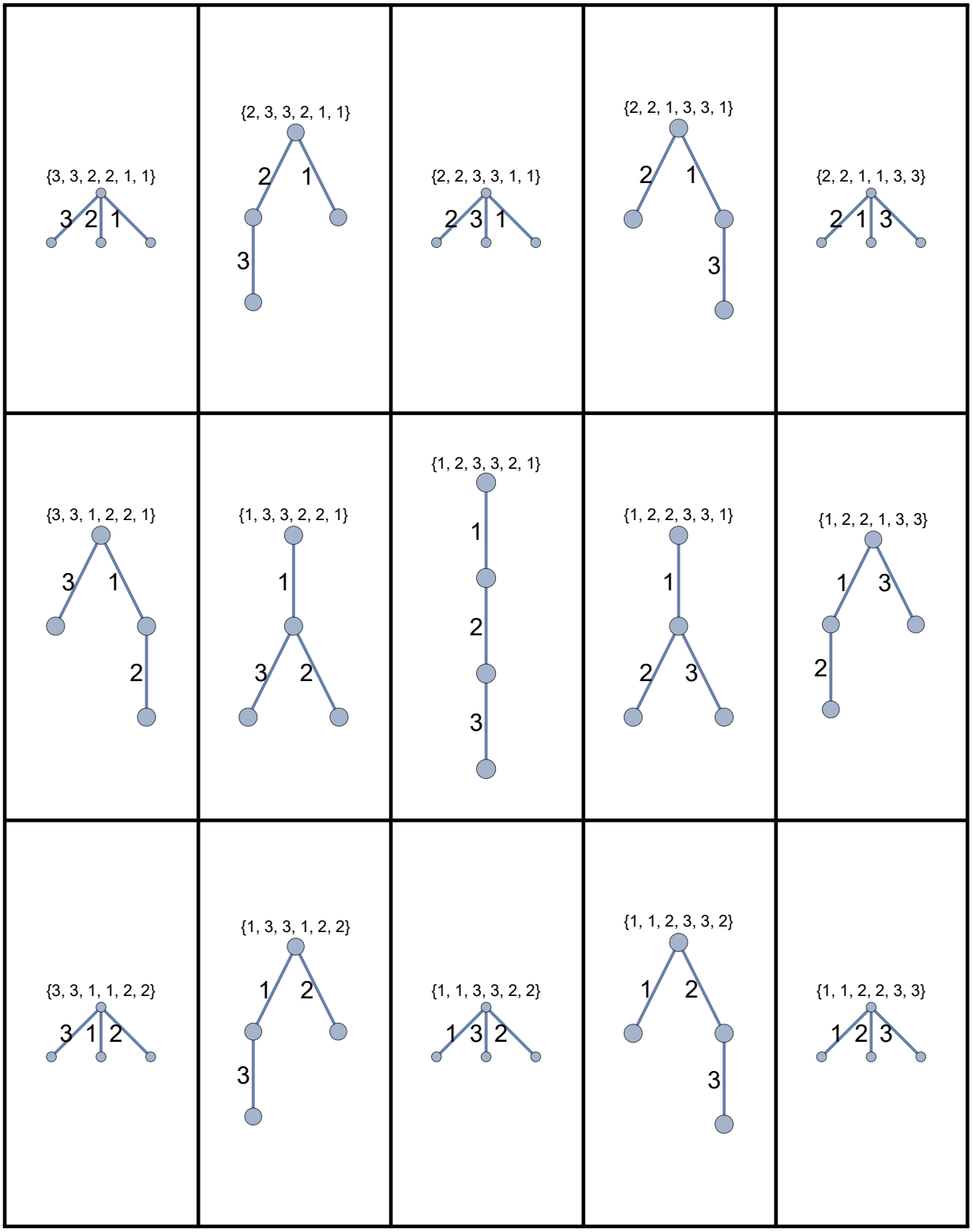
Out[•]=

{1, 1, 2, 2, 3, 3, 4, 4}	{1, 1, 2, 2, 3, 4, 4, 3}	{1, 1, 2, 2, 4, 4, 3, 3}	{1, 1, 2, 3, 3, 2, 4, 4}	{1, 1, 2, 3, 3, 4, 4, 2}
{1, 1, 2, 3, 4, 4, 3, 2}	{1, 1, 2, 4, 4, 2, 3, 3}	{1, 1, 2, 4, 4, 3, 3, 2}	{1, 1, 3, 3, 2, 2, 4, 4}	{1, 1, 3, 3, 2, 4, 4, 2}
{1, 1, 3, 3, 4, 4, 2, 2}	{1, 1, 3, 4, 4, 3, 2, 2}	{1, 1, 4, 4, 2, 2, 3, 3}	{1, 1, 4, 4, 2, 3, 3, 2}	{1, 1, 4, 4, 3, 3, 2, 2}
{1, 2, 2, 1, 3, 3, 4, 4}	{1, 2, 2, 1, 3, 4, 4, 3}	{1, 2, 2, 1, 4, 4, 3, 3}	{1, 2, 2, 3, 3, 1, 4, 4}	{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 4, 4, 3, 1}	{1, 2, 2, 4, 4, 1, 3, 3}	{1, 2, 2, 4, 4, 3, 3, 1}	{1, 2, 3, 3, 2, 1, 4, 4}	{1, 2, 3, 3, 2, 4, 4, 1}
{1, 2, 3, 3, 4, 4, 2, 1}	{1, 2, 3, 4, 4, 3, 2, 1}	{1, 2, 4, 4, 2, 1, 3, 3}	{1, 2, 4, 4, 2, 3, 3, 1}	{1, 2, 4, 4, 3, 3, 2, 1}
{1, 3, 3, 1, 2, 2, 4, 4}	{1, 3, 3, 1, 2, 4, 4, 2}	{1, 3, 3, 1, 4, 4, 2, 2}	{1, 3, 3, 2, 2, 1, 4, 4}	{1, 3, 3, 2, 2, 4, 4, 1}
{1, 3, 3, 2, 4, 4, 2, 1}	{1, 3, 3, 4, 4, 1, 2, 2}	{1, 3, 3, 4, 4, 2, 2, 1}	{1, 3, 4, 4, 3, 1, 2, 2}	{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 1, 2, 2, 3, 3}	{1, 4, 4, 1, 2, 3, 3, 2}	{1, 4, 4, 1, 3, 3, 2, 2}	{1, 4, 4, 2, 2, 1, 3, 3}	{1, 4, 4, 2, 2, 3, 3, 1}
{1, 4, 4, 2, 3, 3, 2, 1}	{1, 4, 4, 3, 3, 1, 2, 2}	{1, 4, 4, 3, 3, 2, 2, 1}	{2, 2, 1, 1, 3, 3, 4, 4}	{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 4, 4, 3, 3}	{2, 2, 1, 3, 3, 1, 4, 4}	{2, 2, 1, 3, 3, 4, 4, 1}	{2, 2, 1, 3, 4, 4, 3, 1}	{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 4, 4, 3, 3, 1}	{2, 2, 3, 3, 1, 1, 4, 4}	{2, 2, 3, 3, 1, 4, 4, 1}	{2, 2, 3, 3, 4, 4, 1, 1}	{2, 2, 3, 4, 4, 3, 1, 1}
{2, 2, 4, 4, 1, 1, 3, 3}	{2, 2, 4, 4, 1, 3, 3, 1}	{2, 2, 4, 4, 3, 3, 1, 1}	{2, 3, 3, 2, 1, 1, 4, 4}	{2, 3, 3, 2, 1, 4, 4, 1}
{2, 3, 3, 2, 4, 4, 1, 1}	{2, 3, 3, 4, 4, 2, 1, 1}	{2, 3, 4, 4, 3, 2, 1, 1}	{2, 4, 4, 2, 1, 1, 3, 3}	{2, 4, 4, 2, 1, 3, 3, 1}
{2, 4, 4, 2, 3, 3, 1, 1}	{2, 4, 4, 3, 3, 2, 1, 1}	{3, 3, 1, 1, 2, 2, 4, 4}	{3, 3, 1, 1, 2, 4, 4, 2}	{3, 3, 1, 1, 4, 4, 2, 2}
{3, 3, 1, 2, 2, 1, 4, 4}	{3, 3, 1, 2, 2, 4, 4, 1}	{3, 3, 1, 2, 4, 4, 2, 1}	{3, 3, 1, 4, 4, 1, 2, 2}	{3, 3, 1, 4, 4, 2, 2, 1}
{3, 3, 2, 2, 1, 1, 4, 4}	{3, 3, 2, 2, 1, 4, 4, 1}	{3, 3, 2, 2, 4, 4, 1, 1}	{3, 3, 2, 4, 4, 2, 1, 1}	{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 2, 2, 1}	{3, 3, 4, 4, 2, 2, 1, 1}	{3, 4, 4, 3, 1, 1, 2, 2}	{3, 4, 4, 3, 1, 2, 2, 1}	{3, 4, 4, 3, 2, 2, 1, 1}
{4, 4, 1, 1, 2, 2, 3, 3}	{4, 4, 1, 1, 2, 3, 3, 2}	{4, 4, 1, 1, 3, 3, 2, 2}	{4, 4, 1, 2, 2, 1, 3, 3}	{4, 4, 1, 2, 2, 3, 3, 1}
{4, 4, 1, 2, 3, 3, 2, 1}	{4, 4, 1, 3, 3, 1, 2, 2}	{4, 4, 1, 3, 3, 2, 2, 1}	{4, 4, 2, 2, 1, 1, 3, 3}	{4, 4, 2, 2, 1, 3, 3, 1}
{4, 4, 2, 2, 3, 3, 1, 1}	{4, 4, 2, 3, 3, 2, 1, 1}	{4, 4, 3, 3, 1, 1, 2, 2}	{4, 4, 3, 3, 1, 2, 2, 1}	{4, 4, 3, 3, 2, 2, 1, 1}

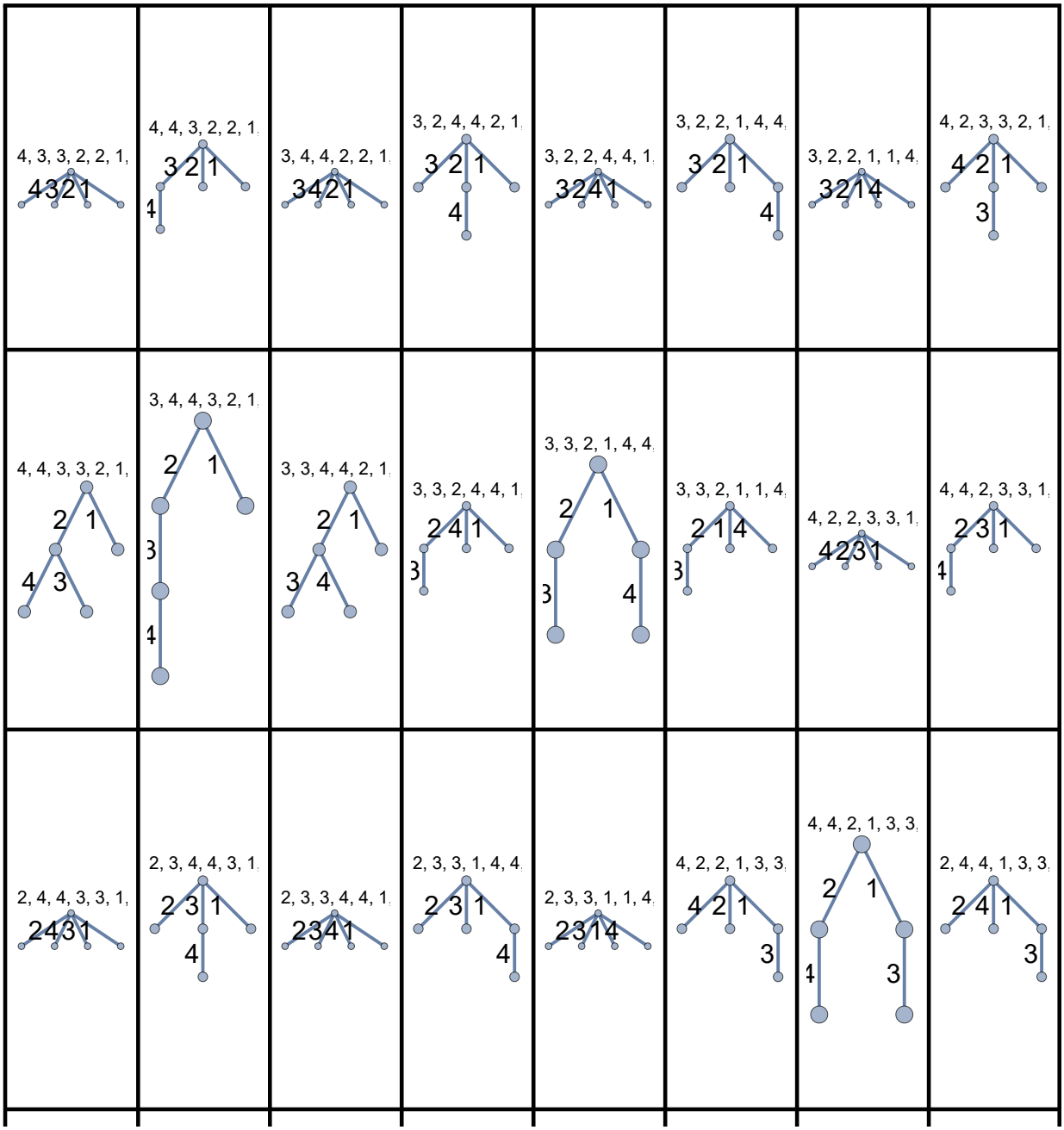
In[ ]:=
```
ClearAll[strlngPermGraph]
strlngPermGraph[sp_, opts : OptionsPattern[]] :=
 Module[{vl = DeleteDuplicates@sp, pos = PositionIndex@sp,
   eL = EdgeList@*TransitiveReductionGraph@*GraphUnion},
  Graph[Prepend[vl, 0], eL[Graph@Thread[0 → vl],
    SimpleGraph@RelationGraph[And @@ Between[pos@#] /@ pos[#2] &, vl]],
   GraphLayout → {"LayeredEmbedding", "RootVertex" → 0},
   EdgeLabels → {e_ :→ Placed[Last@e, {Left, "Middle"}]}, opts]]
```

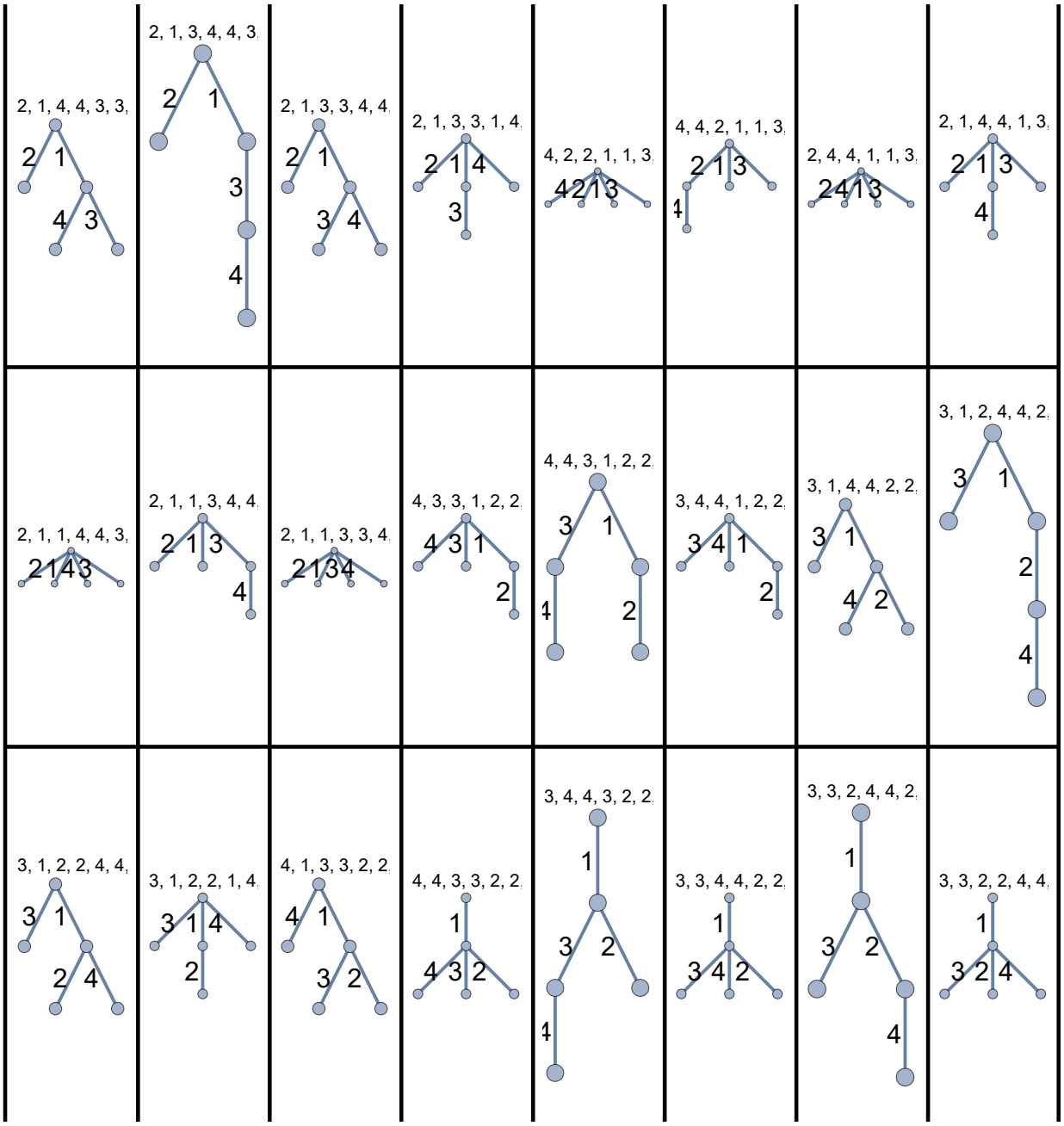
In[]:=
```
Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
      EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
    strlngPermutations[3], 5], Dividers → All, Spacings → {4, 4}]
```

Out[]=



In[ ]:=
```
Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
      EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
    strlngPermutations[4], 8], Dividers → All, Spacings → {1, 1}]
```

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2, 1, 3, 3, 1, 4,
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4, 4, 2, 1, 1, 3,
2, 4, 4, 1, 1, 3,
2, 1, 4, 4, 1, 3,
2, 1, 1, 4, 4, 3,
2, 1, 1, 3, 4, 4,
2, 1, 1, 3, 3, 4,
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4, 4, 3, 1, 2, 2,
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3, 1, 2, 4, 4, 2,
3, 1, 2, 2, 4, 4,
3, 1, 2, 2, 1, 4,
4, 1, 3, 3, 2, 2,
4, 4, 3, 3, 2, 2,
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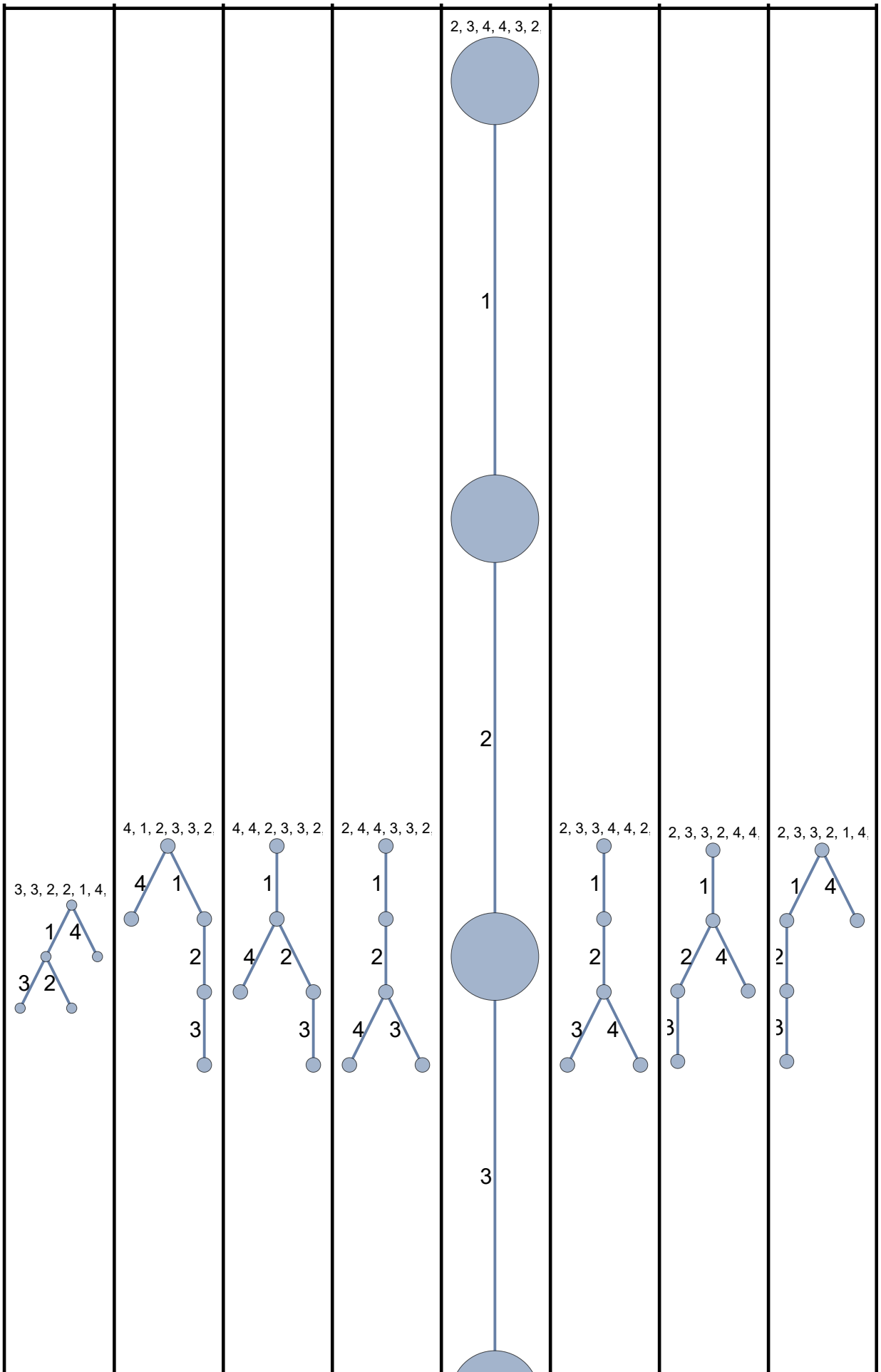
3, 3, 4, 4, 2, 2,
3, 3, 2, 4, 4, 2,
3, 3, 2, 2, 4, 4,

Out[•]=


3, 3, 2, 2, 1, 4,
4, 1, 2, 3, 3, 2,
4, 4, 2, 3, 3, 2,
2, 4, 4, 3, 3, 2,
2, 3, 4, 4, 3, 2,
2, 3, 4, 4, 2,
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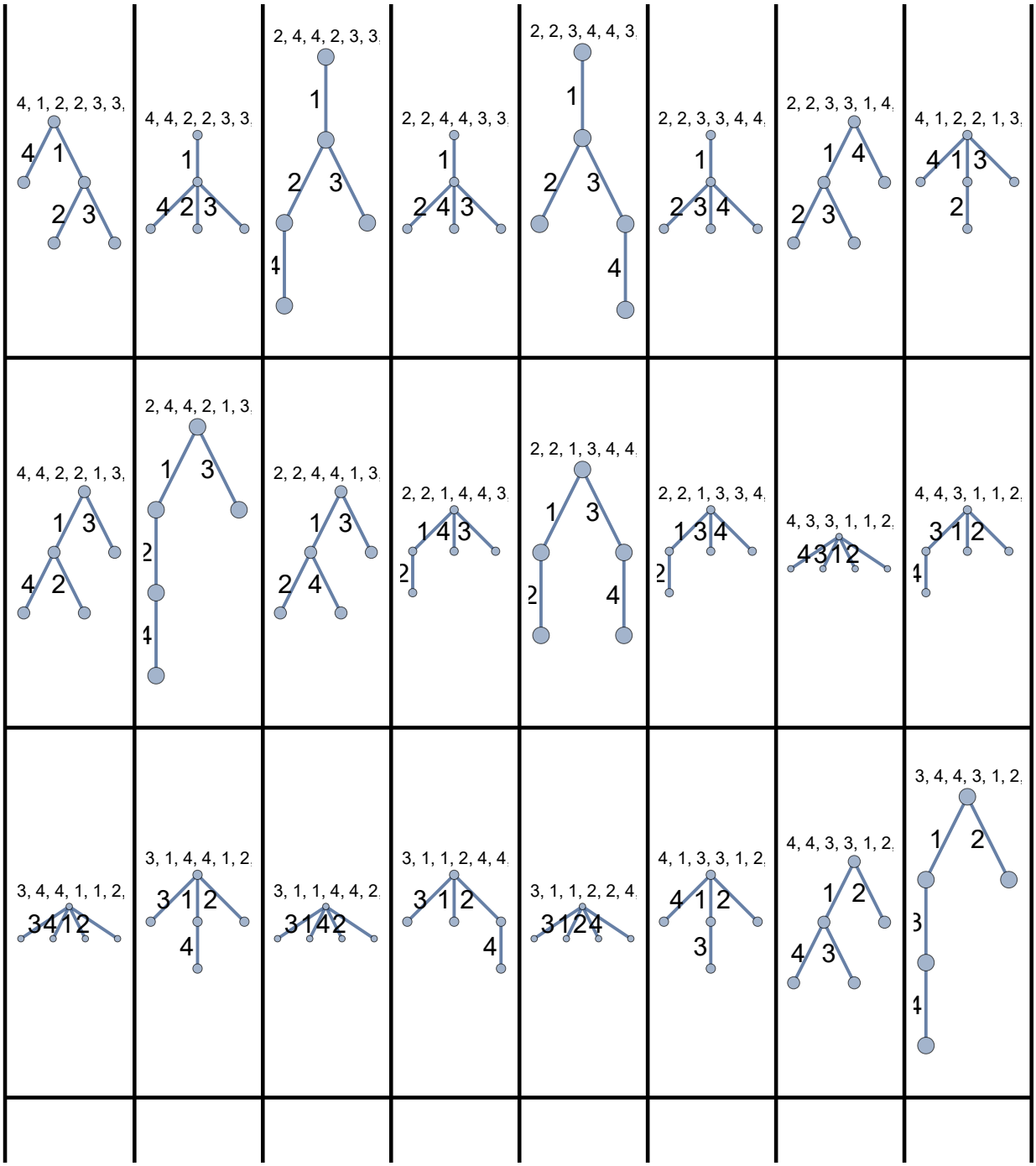
2, 3, 3, 2, 1, 4,
1
2
3


4, 1, 2, 2, 3, 3,
4, 4, 2, 2, 3, 3,
2, 4, 4, 2, 3, 3,
2, 2, 4, 4, 3, 3,
2, 2, 3, 4, 4, 3,
2, 2, 3, 3, 4, 4,
2, 2, 3, 3, 1, 4,
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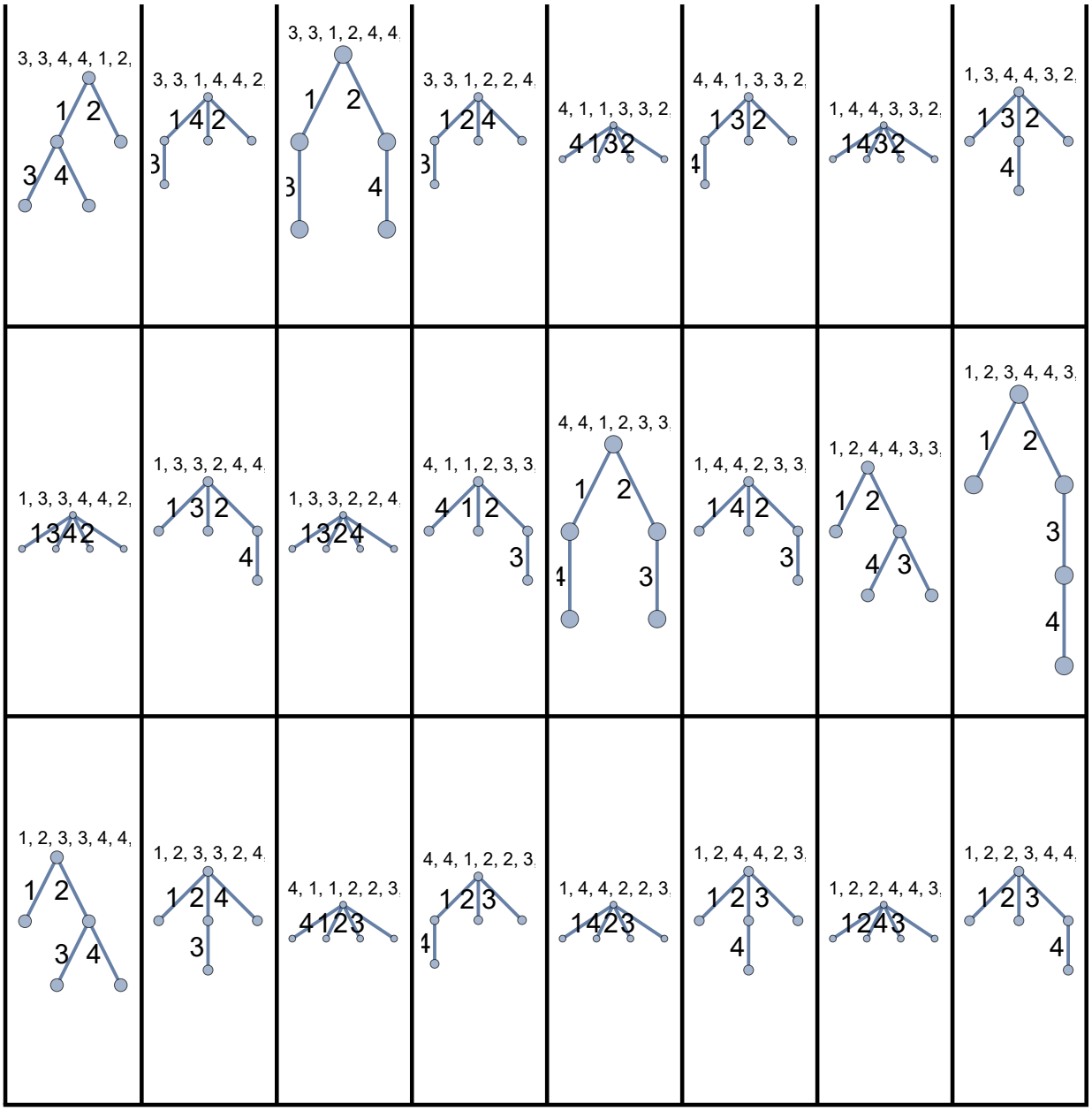
4, 4, 3, 3, 1, 2,
3, 4, 4, 3, 1, 2,



Limits for Ratio test in m(2)

In[]:=
```
case1 = (2^(1 + 2 n) n! (1 + n)!) / (2 (1 + n))!
case2 = (2^(-1 - 2 n) (2 n)!) / (n! (1 + n)!)
```

Out[]=
$$\frac{2^{1+2n}\, n!\, (1+n)!}{(2\,(1+n))!}$$

Out[]=
$$\frac{2^{-1-2n}\, (2n)!}{n!\, (1+n)!}$$

```
In[•]:= limitm2case1 = Limit[case1, n → Infinity]
        limitm2case2 = Limit[case2, n → Infinity]
Out[•]= 0
Out[•]= 0
```

Computing m(1) to m(10) from n=0 to n=2000

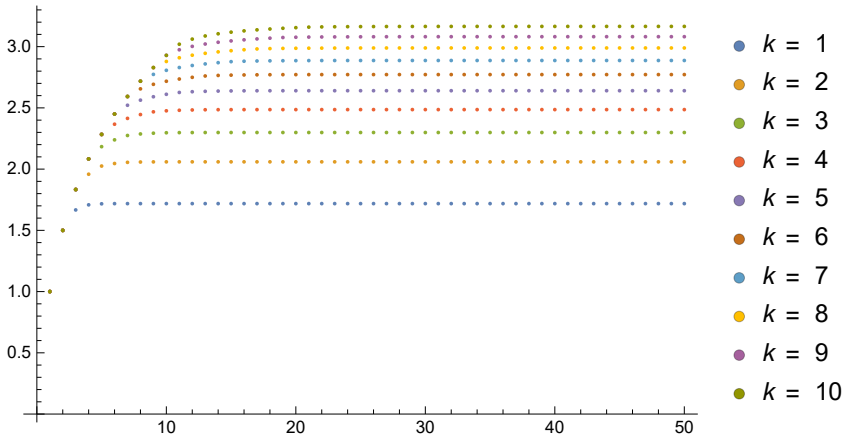
```
In[•]:= Multifactorial[n_, k_] := Abs[Apply[Times, Range[-n, -1, k]]]

In[•]:= For[i = 1, i < 11, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 150}], 20]]]
        2.7182818284590452354
        3.0594074053425761445
        3.2989135380884190034
        3.4859449774535577452
        3.6402244677338097342
        3.7719023962117584357
        3.8869596537408434954
        3.9892412126901365441
        4.0813755201688985441
        4.1652437655583845908
```

Plot of m (1) to m (1) superimposed on each other from n = 0 to n = 2000

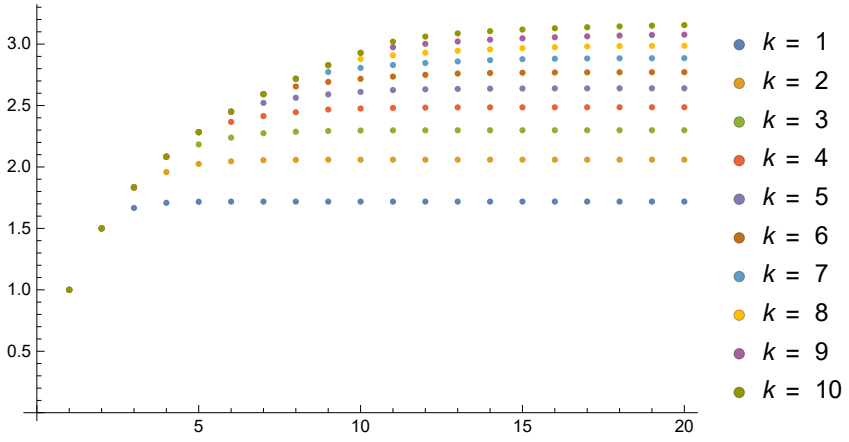
```
ListPlot[Table[Sum[1 / Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 20}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[•]=



```
ListPlot[Table[Sum[1 / Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 50}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[•]=



Computation of RMFCs using the closed form formula

In[•]:=
$$\text{ClosedFormRMFC}[n\_] := 1 + \frac{1}{n}\,\text{Exp}[1/n]\,\text{Sum}\left[n^{k/n}\,\text{Gamma}\left[\frac{k}{n}, 0, \frac{1}{n}\right], \{k, n\}\right]$$

In[•]:=
```
For[i = 1, i < 11, i++, Print[N[ClosedFormRMFC[i], 20]]]
```

```
2.7182818284590452354

3.0594074053425761445

3.2989135380884190034

3.4859449774535577452

3.6402244677338097342

3.7719023962117584357

3.8869596537408434954

3.9892412126901365441

4.0813755201688985441

4.1652437655583845908
```

In[•]:=
```
N[ClosedFormRMFC[100], 100]
```

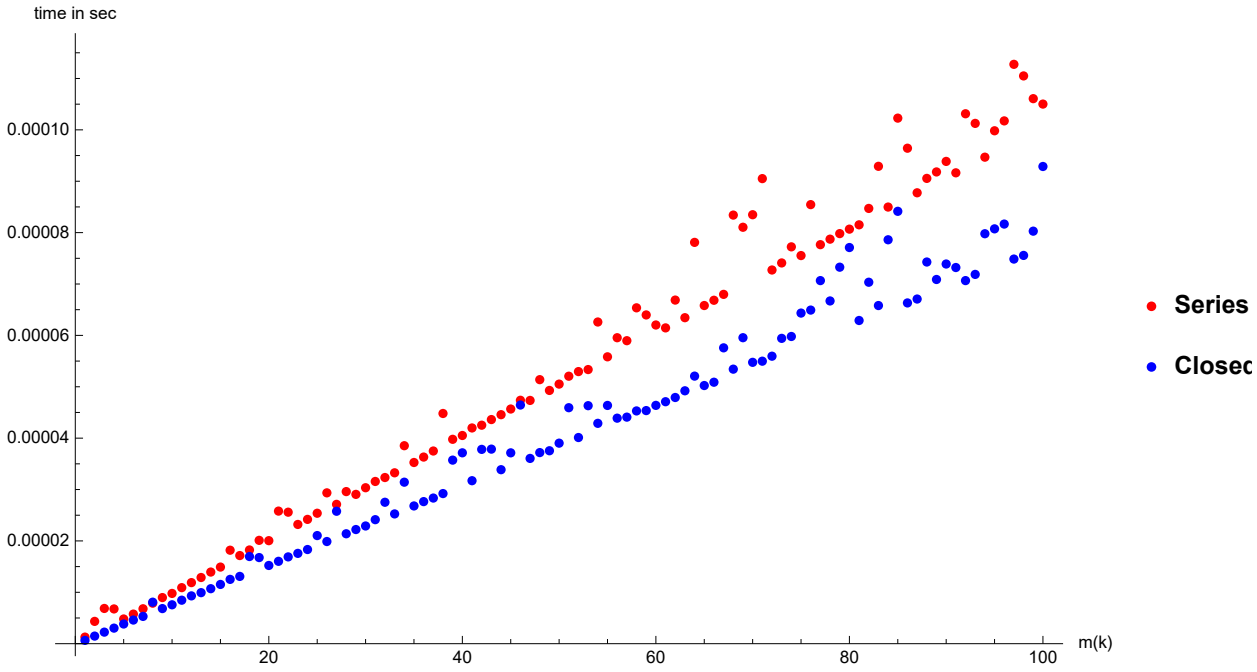
Out[•]=
```
6.2325559690048781755948333144748471914777217107326130369845865125921265925999218831125
30488409940747
```

Analysing efficiency of the two RMFC calculation methods

```
In[ ]:= test1[xx_] :=
          (For[i = 1, i < xx, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 250}], 50]] //
              Inactive] // RepeatedTiming)[[1]]
        test2[xx_] := (For[i = 1, i < xx, i++, Print[N[ClosedFormRMFC[i], 50]] // Inactive] //
            RepeatedTiming)[[1]]
```

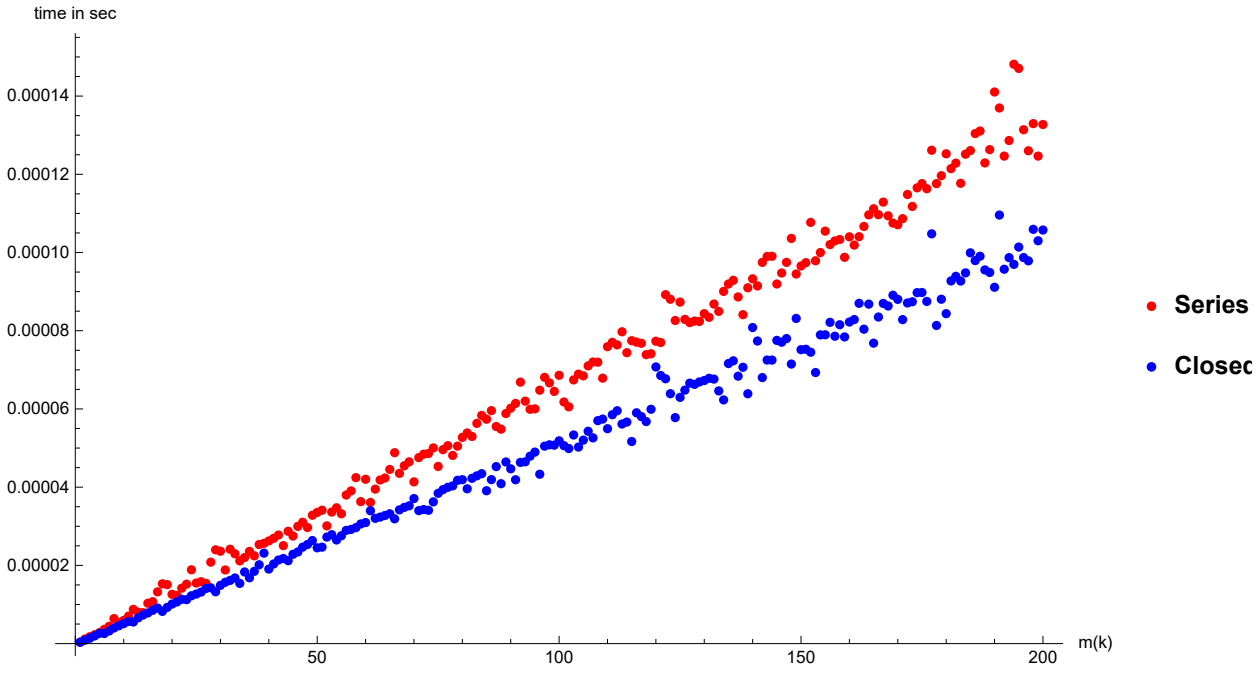
```
In[ ]:= list1 =
          ListPlot[Table[{xx, test1[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
           AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
        list2 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends →
            {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

```
In[ ]:= Show[list1, list2, ImageSize → Large]
```




```
In[ ]:= list3 =
          ListPlot[Table[{xx, test1[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
           AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
        list4 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends →
            {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

```
In[ ]:= Show[list3, list4, ImageSize → Large]
```

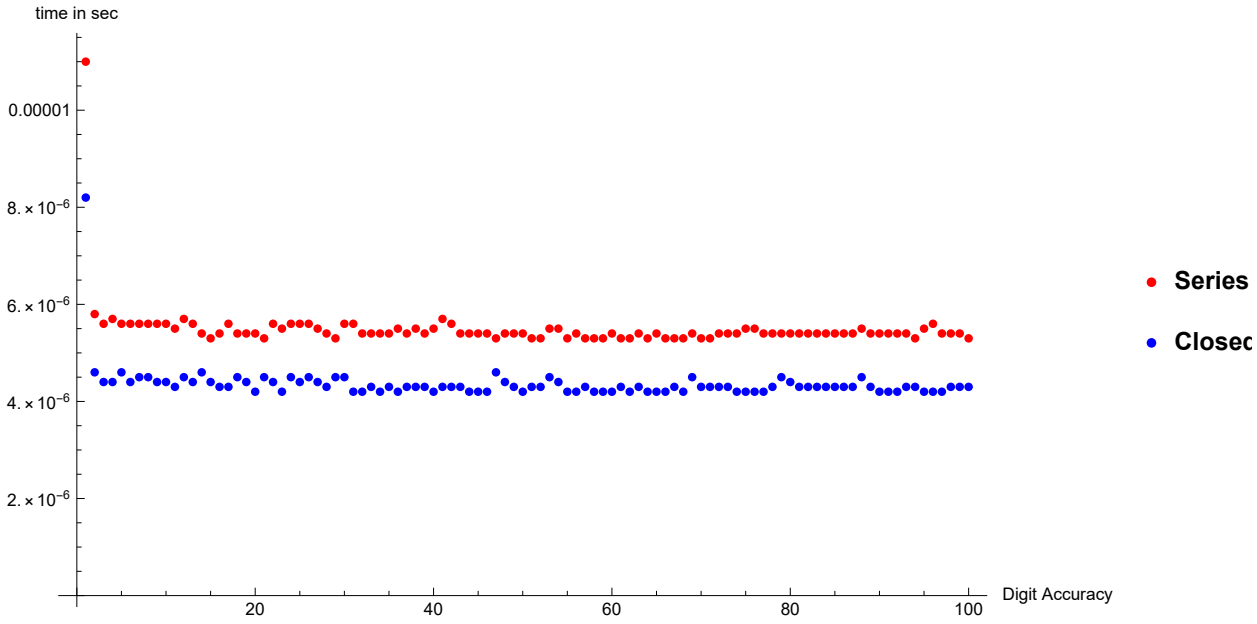



```
test3[xx_] :=
 (For[i = 1, i < 10, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 500}], xx]] //
     Inactive] // AbsoluteTiming)[[1]]
test4[xx_] := (For[i = 1, i < 10, i++, Print[N[ClosedFormRMFC[i], xx]] // Inactive] //
    AbsoluteTiming)[[1]]
```

In[ ]:=
```
list5 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

In[ ]:=
```
list6 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

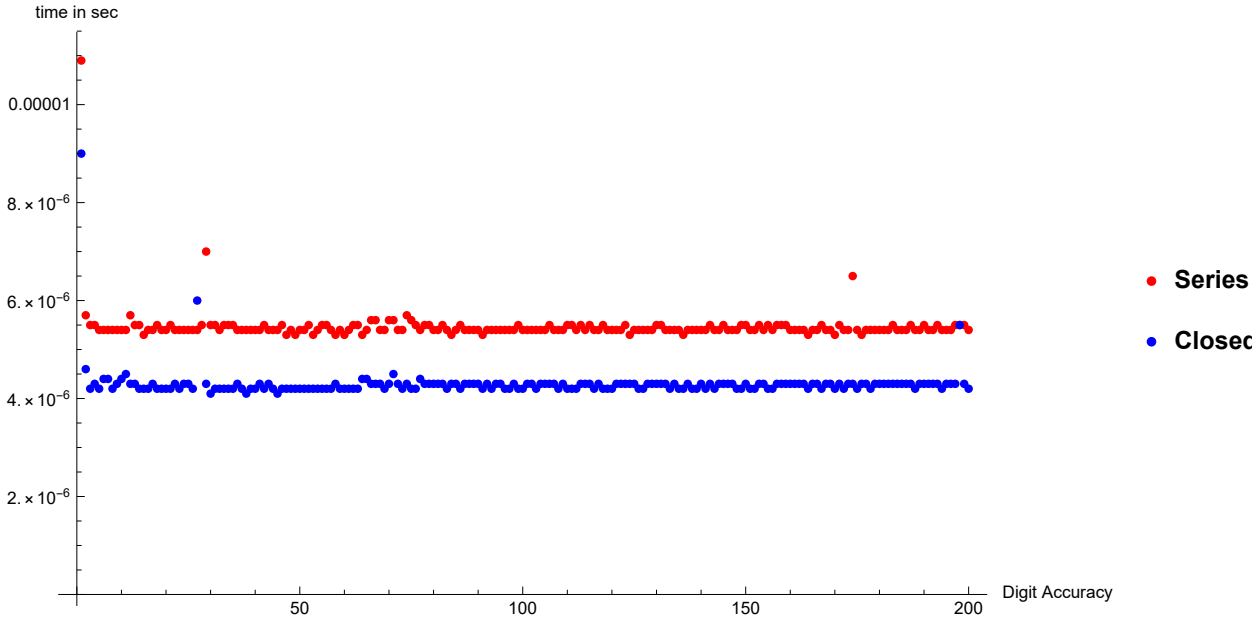
In[ ]:=
```
Show[list5, list6, ImageSize → Large]
```




```
In[ ]:= list7 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];
```

```
In[ ]:= list8 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];
```

```
In[ ]:= Show[list7, list8, ImageSize → Large]
```

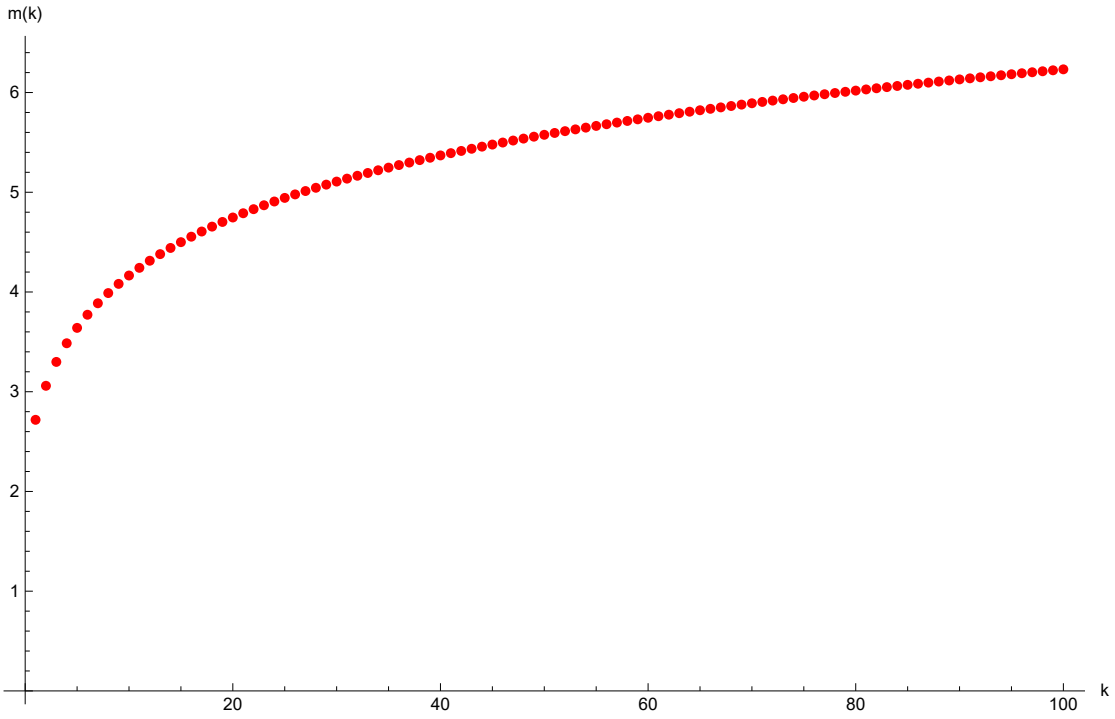


Asymptotics of Reciprocal Multifactorial Series

Simple Graph

In[]:=
```
ListPlot[Table[{x, ClosedFormRMFC[x]}, {x, 1, 100}],
 ImageSize → Large, AxesLabel → {"k", "m(k)"}, PlotStyle → {Red, Thick}]
```

Out[]=



Detailed asymptotics

```
Series[1/k Integrate[((1 - t) (1 - E^(-(t/k))))/(t (E^(-(1/k) Log[t])) - 1)), {t, 0, 1}],
 {k, Infinity, 5}]
```

Out[]=
$$\frac{\text{Log}[2]}{k}+\frac{-1-\text{Log}\left[\frac{9}{4}\right]}{4\,k^2}+\frac{5+8\,\text{Log}\left[\frac{4}{3}\right]}{48\,k^3}+\frac{-11-36\,\text{Log}\left[\frac{5}{4}\right]}{864\,k^4}+\frac{-79+108\,\text{Log}\left[\frac{6}{5}\right]}{12\,960\,k^5}+O\left[\frac{1}{k}\right]^6$$

In[]:=
```
Series[HarmonicNumber[k], {k, Infinity, 5}]
```

Out[]=
$$\left(\text{EulerGamma}+\text{Log}[k]\right)+\frac{1}{2\,k}-\frac{1}{12\,k^2}+\frac{1}{120\,k^4}+O\left[\frac{1}{k}\right]^6$$

In[]:=
```
FullSimplify[1 + (1 + 1/(2 k^2) + 1/k) (EulerGamma + 1/(6 k^2) + 1/(2 k) + Log[3]/(2 k^2) - Log[4]/(2 k) - Log[64]/(12 k^2) + Log[k])]
```

Out[]=
$$1+\frac{\left(1+2\,k\,(1+k)\right)\left(1+\text{Log}\left[\frac{27}{8}\right]+k\left(3+6\,\text{EulerGamma}\,k-\text{Log}[64]\right)+6\,k^2\,\text{Log}[k]\right)}{12\,k^4}$$

```
In[]:= FullSimplify[1 + (1 + 1/k) * (((EulerGamma + Log[k] + 1/(2 k)) - ((1/k) Log[2])))]
```

$$\text{Out[]}= 1 + \frac{(1+k)\,(1 + 2\,\text{EulerGamma}\,k - \text{Log}[4] + 2\,k\,\text{Log}[k])}{2\,k^2}$$

```
In[]:= ClosedFormRMFC[n_] := 1 + 1/n Exp[1/n] Sum[n^(k/n) Gamma[k/n, 0, 1/n], {k, n}]

In[]:= RMFCApproximation[k_] :=
         1 + ((1 + 2 k (1 + k)) (1 + Log[27/8] + k (3 + 6 EulerGamma k - Log[64]) + 6 k^2 Log[k]))/(12 k^4)
       RMFCApproximation1[k_] := 1 + ((1 + k) (1 + 2 EulerGamma k - 2 Log[2] + 2 k Log[k]))/(2 k^2)

In[]:= N[RMFCApproximation[10]]
       N[RMFCApproximation1[10]]
```

$$\text{Out[]}= 4.16492$$

$$\text{Out[]}= 4.14653$$

```
In[]:= N[ClosedFormRMFC[10]]
```

$$\text{Out[]}= 4.16524$$

```
In[]:= Table[{x, Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation[10^x]])]/
           N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

$$\text{Out[]}= \{\{0, 0.0608426\}, \{1, 0.0000779859\}, \{2, 1.14278 \times 10^{-7}\}, \{3, 1.29004 \times 10^{-10}\}, \{4, 1.37156 \times 10^{-13}\}, \{5, 0.\}\}$$

```
In[]:= Table[{x, Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation1[10^x]])]/
           N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

$$\text{Out[]}= \{\{0, 0.349539\}, \{1, 0.00449172\}, \{2, 0.0000476604\}, \{3, 4.84353 \times 10^{-7}\}, \{4, 4.87866 \times 10^{-9}\}, \{5, 4.90019 \times 10^{-11}\}\}$$

Generalized Reciprocal Multifactorial Constant

```
In[]:= GRMFC[k_, x_] := 1 + 1/k Exp[x^k/k] Sum[k^(r/k) Gamma[r/k, 0, (x^k)/k], {r, 1, k}]
       Multifactorial[n_, k_] := Times @@ Range[n, 1, -k]

In[]:= GRMFCsum[k_, x_, nn_] := Sum[x^n/Multifactorial[n, k], {n, 0, nn}] // N
```

```
In[•]:= Table[{k, 0.5 x, N[GRMFC[k, 0.5 x], 5]}, {k, 1, 5}, {x, 1, 5}]
```

```
Out[•]= {{{1, 0.5, 1.64872}, {1, 1., 2.71828}, {1, 1.5, 4.48169},
    {1, 2., 7.38906}, {1, 2.5, 12.1825}}, {{2, 0.5, 1.67697},
    {2, 1., 3.05941}, {2, 1.5, 6.42488}, {2, 2., 16.2285}, {2, 2.5, 50.9309}},
   {{3, 0.5, 1.68663}, {3, 1., 3.29891}, {3, 1.5, 9.06146}, {3, 2., 45.7755}, {3, 2.5, 589.3}},
   {{4, 0.5, 1.69039}, {4, 1., 3.48594}, {4, 1.5, 13.1533},
    {4, 2., 219.471}, {4, 2.5, 70299.3}}, {{5, 0.5, 1.69195}, {5, 1., 3.64022},
    {5, 1.5, 20.2725}, {5, 2., 2850.31}, {5, 2.5, 1.43833 × 10^9}}}
```

```
In[•]:= Table[{k, 0.5 x, N[GRMFCsum[k, 0.5 x, 250], 5]}, {k, 1, 5}, {x, 1, 5}]
```

```
Out[•]= {{{1, 0.5, 1.64872}, {1, 1., 2.71828}, {1, 1.5, 4.48169},
    {1, 2., 7.38906}, {1, 2.5, 12.1825}}, {{2, 0.5, 1.67697},
    {2, 1., 3.05941}, {2, 1.5, 6.42488}, {2, 2., 16.2285}, {2, 2.5, 50.9309}},
   {{3, 0.5, 1.68663}, {3, 1., 3.29891}, {3, 1.5, 9.06146}, {3, 2., 45.7755}, {3, 2.5, 589.3}},
   {{4, 0.5, 1.69039}, {4, 1., 3.48594}, {4, 1.5, 13.1533},
    {4, 2., 219.471}, {4, 2.5, 70299.3}}, {{5, 0.5, 1.69195}, {5, 1., 3.64022},
    {5, 1.5, 20.2725}, {5, 2., 2850.31}, {5, 2.5, 1.43833 × 10^9}}}
```