St. Xavier's College
Mumbai

FYBSc Mathematics Project

A look at Reciprocal Multifactorial Constants

Bhoris Dhanjal 462 202020

May 30, 2021

Contents

1 Introduction **2**

2 Multifactorial **3**
- 2.1 Physical Intuition 3
- 2.2 Definition of Multifactorial 7

3 Reciprocal Multifactorial Series **7**
- 3.1 Definition 7
- 3.2 Computations of Reciprocal Multifactorial Series 9

4 Closed form formula for RMFCs **11**
- 4.1 Prerequisite definitions 11
- 4.2 Proof for closed form formula for RMFCs 11
- 4.3 Computation of RMFCs using the closed form formula . . 14

5 Asymptotics of Reciprocal Multifactorial Series **17**
- 5.1 Casual observation 17
- 5.2 Proof of divergence 18
- 5.3 Alternate proof of divergence 19
- 5.4 Detailed Asymptotics 21
- 5.5 Asymptotic approximations for RMFCs 21

6 Generalized Reciprocal Multifactorial Constants **23**
- 6.1 Closed form formula 23
- 6.2 Computations of GRMFCs using the closed form 24

References **25**

A look at Reciprocal Multifactorial Constants

Bhoris Dhanjal

May 30, 2021

1 Introduction

In this project we will explore the Reciprocal Multifactorial Constants (RMFC's). The objectives of this project are as follows:

- Motivating and then defining the concept of a 'multifactorial'.
- Defining a reciprocal multifactorial series, and displaying the convergent nature for a few examples computationally.
- Simply stating some of the definitions and relationships necessary for proving the closed form formula for RMFCs.
- Providing a proof for the closed form formula for RMFCs.
- Computing examples of RMFCs using both the original definition and the closed form to showcase equality between the two.
- Exploring computational efficiency between the series definition and the closed form definition for RMFCs.
- Explaining why the Reciprocal Multifactorial Series will tend to diverge as multifactorial order approaches infinity.
- Exploring the asymptotics and providing two asymptotic approximations of the RMFCs.
- Computing absolute errors of RMFCs using the asymptotic approximations.
- Proving a closed form formula of reciprocal multifactorial power series.

The Mathematica notebook comprising of all code used for computations and plots can be viewed at my GitHub with the following link
https://github.com/BhorisDhanjal/reciprocal-multifactorial-constants

2 Multifactorial

To begin talking about Reciprocal Multifactorial Series we must first clearly define the word 'multifactorial'.

We will denote the k^{th} order multifactorial for $n \in \mathbb{N}_0$ by either $n!_{(k)}$, or using the more common notation $n \underbrace{!\ldots!}_{\text{k times}}$.

2.1 Physical Intuition

We will begin by motivating the definition of a multifactorial by giving a few examples of its occurrence.

For the trivial case of the first Multifactorial i.e. simply the factorial, we know of the most common example to be that $n!$ represents the total number of permutations for n distinct objects.

ABCD	BACD	CABD	ACBD
BCAD	CBAD	CBDA	BCDA
DCBA	CDBA	BDCA	DBCA
DACB	ADCB	CDAB	DCAB
ACDB	CADB	BADC	ABDC
DBAC	BDAC	ADBC	DABC

Figure 1: 24 permutations of 4 Objects

Let us see two non-trivial examples.

- The number of edges without common vertices which cover every vertex of a complete graph with 2n vertices is given by the double factorial $(2n-1)!_{(2)}$ or $(2n-1)!!$. Stated in brief, the number of perfect matchings for a complete graph K_{2n} is equal to $(2n-1)!!$.

- Another interesting occurrence of the double factorial is in Stirling permutations. Stirling permutations of n^{th} order are defined as the permutations of the multiset $\{1, 1, 2, 2, \ldots n, n\}$, such that for each set of two values of i, the value j in between the two i's is greater than i.

  The number of Stirling permutations for n is given by the double factorial $(2n-1)!_{(2)}$ or $(2n-1)!!$.

For further reading refer to *A combinatorial survey of identities for the double factorial* [1]. Below are a few visualizations of the above discussed examples.

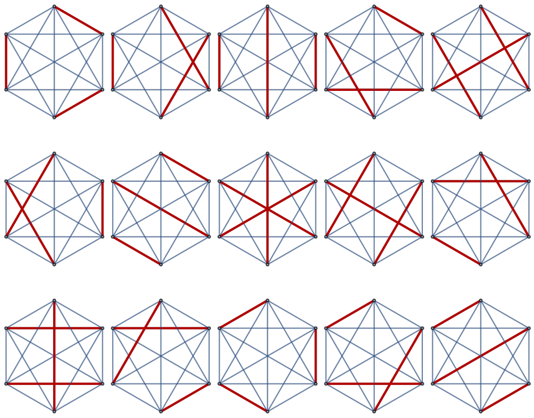

Figure 2: $(2 \cdot 3 - 1)!! = 5!! = 15$ perfect matchings for K_6

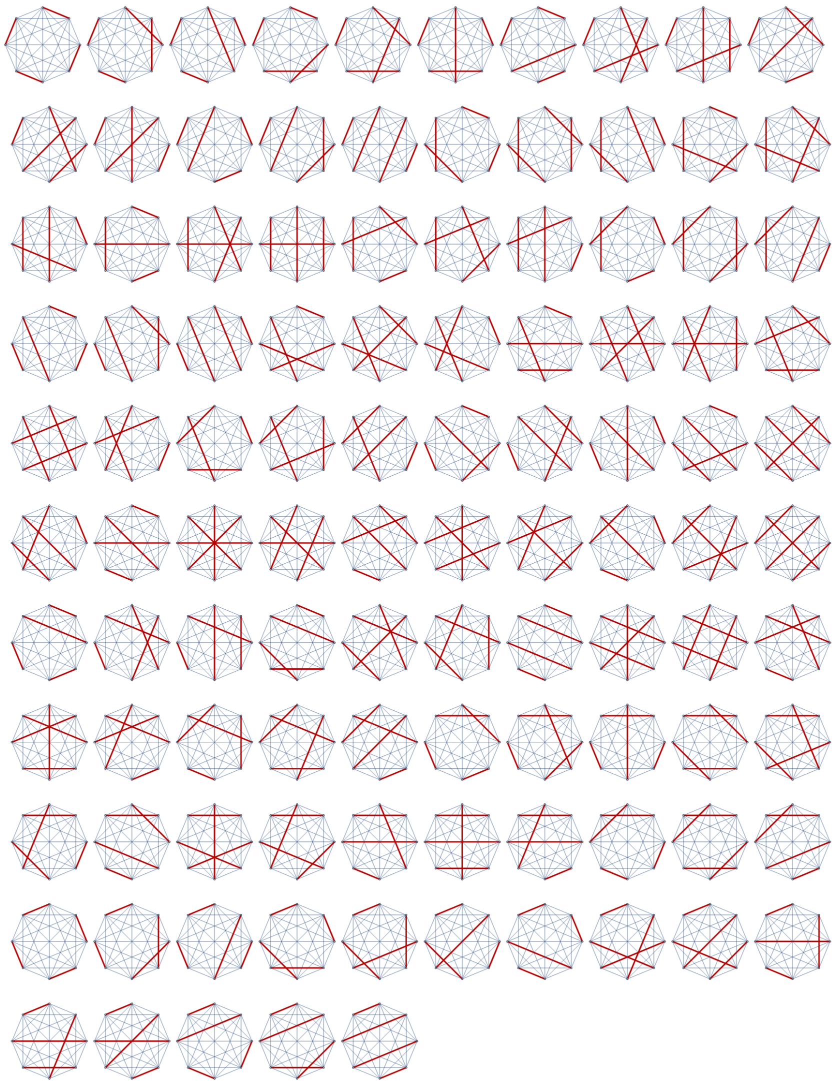

Figure 3: $(2 \cdot 4 - 1)!! = 7!! = 105$ perfect matchings of K_8

11223344	12442133	22144133	33221441	11223443	12442331
22144331	33224411	11224433	12443321	22331144	33244211
11233244	13312244	22331441	33441122	11233442	13312442
22334411	33441221	11234432	13314422	22344311	33442211
11244233	13322144	22441133	34431122	11244332	13322441
22441331	34431221	11332244	13324421	22443311	34432211
11332442	13344122	23321144	44112233	11334422	13344221
23321441	44112332	11344322	13443122	23324411	44113322
11442233	13443221	23344211	44122133	11442332	14412233
23443211	44122331	11443322	14412332	24421133	44123321
12213344	14413322	24421331	44133122	12213443	14422133
24423311	44133221	12214433	14422331	24433211	44221133
12233144	14423321	33112244	44221331	12233441	14433122
33112442	44223311	12234431	14433221	33114422	44233211
12244133	22113344	33122144	44331122	12244331	22113443
33122441	44331221	12332144	22114433	33124421	44332211
12332441	22133144	33144122	12334421	22133441	33144221
12344321	22134431	33221144			

Figure 4: $7!! = 105$ Stirling permutations of order 4

The Stirling permutations for any given n can be constructed by following an Euler path on an ordered tree with n edges. Examples of the constructions of order 3 and 4 is shown below.[2]

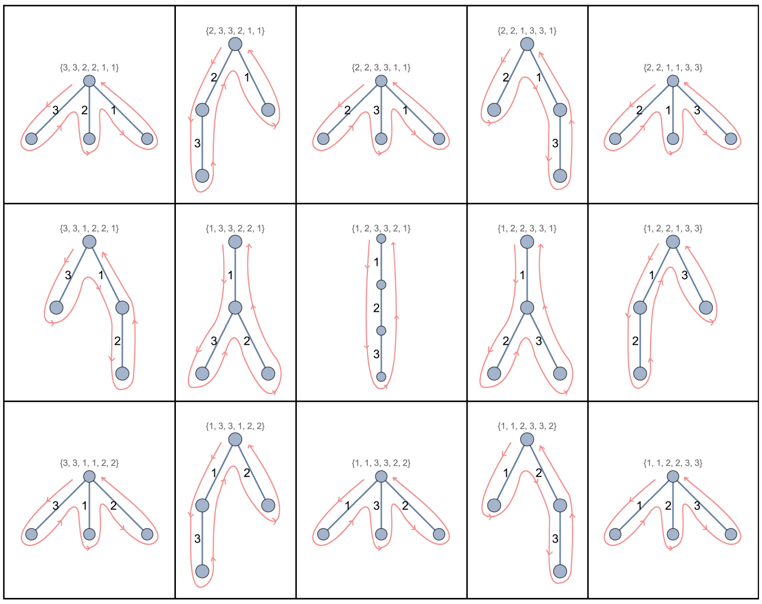

Figure 5: Construction of Stirling permutations of order 3

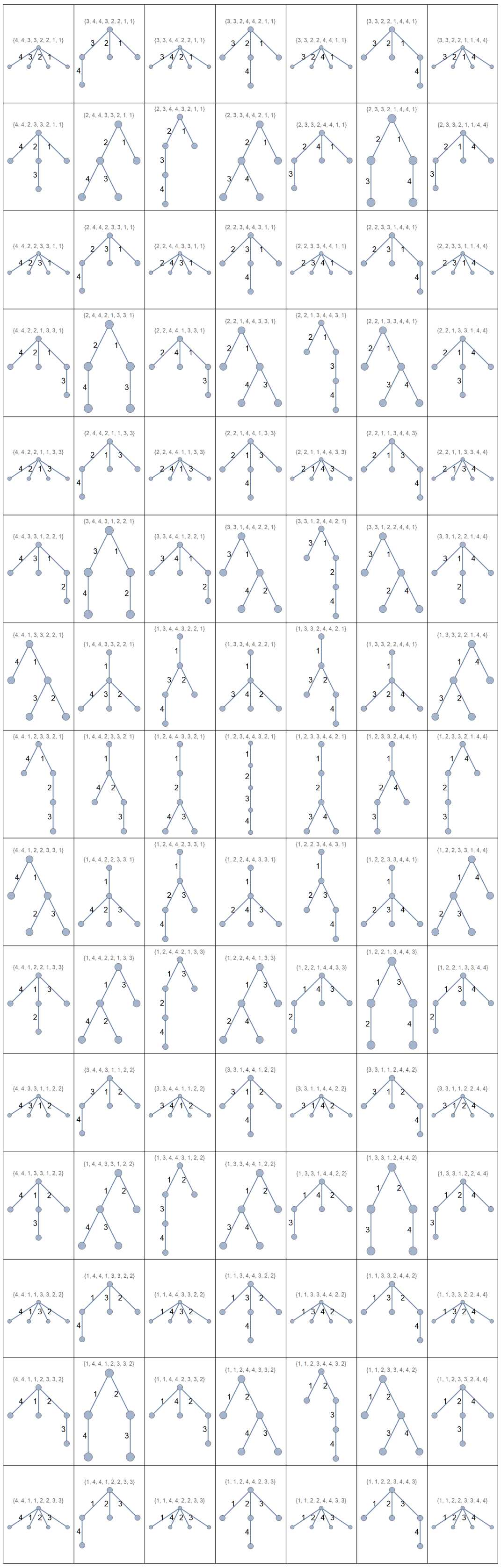

Figure 6: Construction of Stirling permutations of order 4

2.2 Definition of Multifactorial

We shall now formally define the multifactorial. The simplest way to understand the multifactorial is by comparing it with the factorial. We know that the factorial is defined as follows:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \ldots \qquad \textit{Terminates with 1}$$

Using steps of larger integer values we get,

$$\begin{aligned} n!! &= n \cdot (n-2) \cdot (n-4) \cdot (n-6) \ldots && \textit{Terminates with 2 or 1} \\ n!!! &= n \cdot (n-3) \cdot (n-6) \cdot (n-9) \ldots && \textit{Terminates with 3, 2 or 1} \\ &\vdots \end{aligned}$$

Using this we can alternatively define the multifactorial of any $n > 0$ of order $k > 0$ as the follows:

$$\begin{aligned} n!_{(k)} &= \prod_{j=0}^{q} kj + r && \text{where } n = kq + r, q \geq 0, \text{and } 1 \leq r \leq k \\ &= 1 && n = 0 \end{aligned} \tag{2.1}$$

The above definition is identical to the recursive relation:

$$n!_{(k)} = \begin{cases} 1 & \text{if } n = 0 \\ n & \text{if } 0 < n \leq k \\ n\left((n-k)!_{(k)}\right) & \text{if } n > k \end{cases} \tag{2.2}$$

3 Reciprocal Multifactorial Series

Now that we have defined multifactorials we can define the Reciprocal Multifactorial Series.

3.1 Definition

The Reciprocal Multifactorial Series for multifactorial of order k is defined as the follows:

$$m(k) = \sum_{n=0}^{\infty} \frac{1}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}} \tag{3.1}$$

(The 1 in the above representation is due to $0!_{(k)=1.)}$. We all know of the convergence for the famous case of m(1) which equates to e. However in general we can test the convergence of m(k) individually using the ratio test.

Example 1: Testing Convergence of m(1)

$$S_n = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\begin{aligned}
L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| \\
&= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{1}{n+1} = 0
\end{aligned}$$

Since $L < 1$, by Ratio Test we can say that S_n is absolutely convergent. $\square$

Example 2: Testing Convergence of m(2)

$$S_n = \sum_{n=0}^{\infty} \frac{1}{n!!}$$

Consider the relations between $n!$ and $n!!$.

$$2n!! = 2^n n! \qquad \text{where n is even}$$

$$(2n-1)!! = \frac{(2n+1)!}{2^n n!} \qquad \text{where n is odd}$$

Case I: n is even, $n+1$ is odd

$$\begin{aligned}
L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{\frac{(2(n+1))!}{2^{n+1}(n+1)!}}}{\frac{1}{2^n n!}} \right| \\
&= \lim_{n\to\infty} \frac{2^n n!}{\frac{(2(n+1))!}{2^{n+1}(n+1)!}} = 0
\end{aligned}$$

Case II: n is odd, $n+1$ is even

$$\begin{aligned}
L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{\frac{(2n)!}{2^n n!}}} \right| \\
&= \lim_{n\to\infty} \frac{\frac{(2n)!}{2^n n!}}{2^{n+1}(n+1)!} = 0
\end{aligned}$$

Since in both cases $L < 1$, by Ratio test we can say that S_n is absolutely convergent. $\square$

The above used relations between the double factorial and factorial can be derived in the following manner.

$$\begin{aligned}(2n)!! &= (2n)(2n-2)(2n-4)\cdots 2\\ &= [2(n)][2(n-1)][2(n-2)]\cdots 2\\ &= 2^n n! \qquad \square\\ (2n+1)!!2^n n! &= [(2n+1)(2n-1)\cdots 1][2n][2(n-1)][2(n-2)]\cdots 2\\ &= [(2n+1)(2n-1)\cdots 1][2n(2n-2)(2n-4)\cdots 2]\\ &= (2n+1)(2n)(2n-1)(2n-2)\cdots 2\\ &= (2n+1)! \qquad \square\end{aligned}$$

To read about some more proprieties of the double factorial in detail refer to "Double Factorial"- MathWorld.[3]

3.2 Computations of Reciprocal Multifactorial Series

We can examine the convergence of a few terms of m(k) computationally. The below table enumerates m(1) through to m(10) summed up from n=0 to n=150 and rounded to 10 significant digits. These are the first ten Reciprocal Multifactorial Constants (RMFCs).

m(k)	$\sum_{n=0}^{2000} \frac{1}{n!_{(k)}}$ Rounded to 10 significant digits
m(1)	2.718281828
m(2)	3.059407405
m(3)	3.298913538
m(4)	3.485944977
m(5)	3.640224468
m(6)	3.771902396
m(7)	3.886959654
m(8)	3.989241213
m(9)	4.081375520
m(10)	4.165243766

Table 1: Computed values of first 10 RMFCs

The values can be seen on the OEIS [4]. Below we have displayed the first 50 partial sums of first 10 Reciprocal Multifactorial Series. The upward trend of the convergence can be easily seen in these plots.

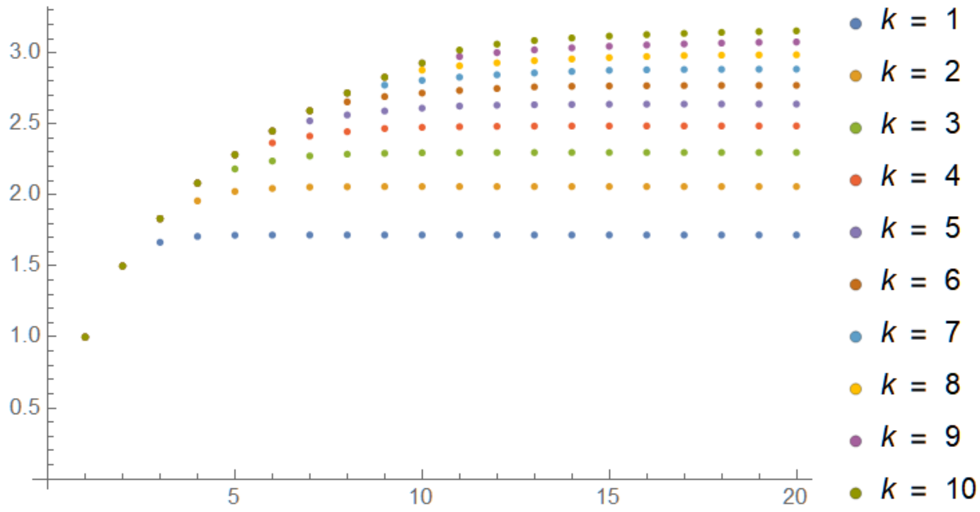

Figure 7: First 20 partial sums of first 10 Reciprocal Multifactorial Series

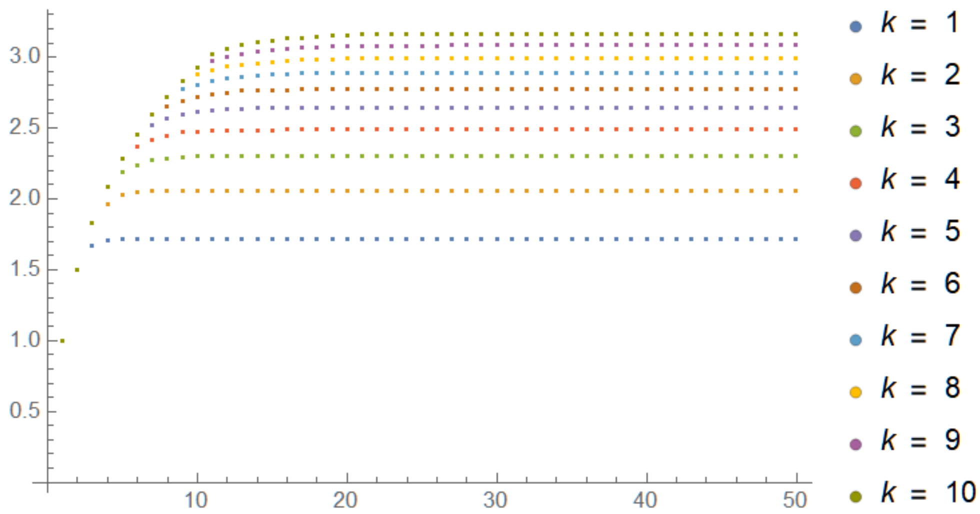

Figure 8: First 50 partial sums of first 10 Reciprocal Multifactorial Series

4 Closed form formula for RMFCs

In this section we shall prove the closed form formula for the k^{th} reciprocal multifactorial constant.

4.1 Prerequisite definitions

We shall first state a few important definitions needed for the proof. [5] [6] [7]

$$\text{Gamma Function } \Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \text{ for } \mathrm{Re}(z) > 0 \tag{4.1.1}$$

$$\text{Beta Function } \mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \text{ for } \mathrm{Re}(x), \mathrm{Re}(y) > 0 \tag{4.1.2}$$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4.1.3}$$

$$\text{Lower incomplete gamma function } \gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \tag{4.1.4}$$

4.2 Proof for closed form formula for RMFCs

Let us begin by proving a lemma that is needed for the final theorem.

Lemma 4.1. *Relation between k^{th} multifactorial and Beta function.*

Proof: Recall the definition of multifactorial as described in Eq. 2.1.

$$\begin{aligned}
n\underbrace{!\ldots!}_{\text{k times}} = n!_{(k)} = \prod_{j=0}^{q}(kj+r) &= k^{q+1}\prod_{j=0}^{q}\left(j+\frac{r}{k}\right) \\
&= k^{q+1}\frac{\Gamma(q+1+\frac{r}{k})}{\Gamma(r/k)} \\
&= \frac{k^{q+1}\Gamma(q+1)}{\mathrm{B}\left(\frac{r}{k}, q+1\right)} \qquad \text{(using 4.1.3)} \\
&= \frac{k^{q+1}q!}{\mathrm{B}\left(\frac{r}{k}, q+1\right)} \qquad \text{(q is a positive integer)}
\end{aligned}$$

We now have a relation between the multifactorial and the Beta function given as follows.

$$n!_{(k)} = \frac{k^{q+1}q!}{\mathrm{B}\left(\frac{r}{k}, q+1\right)} \tag{4.1}$$

Therefore, we have

$$\frac{1}{n!_{(k)}} = \frac{\mathrm{B}\left(\frac{r}{k}, q+1\right)}{k^{q+1}q!} \qquad \square$$

We have now established all definitions we need to prove the closed form formula for RMFCs. The proof involves using the Beta function representation of the multifactorial and then proceeding to simplify the function to arrive at a closed form formula.[8][9]

Theorem 4.2. *A closed form formula for Reciprocal Multifactorial Constants is given by the following expression*

$$m(k) = 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} r^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right)$$

Proof: Recall definition 3.1 for reciprocal multifactorial series.

$$m(k) = \sum_{n=0}^{\infty} \frac{1}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}}$$

Now, using Lemma 4.1 we get,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{k^{q+1}q!} \mathrm{B}\left(\frac{r}{k}, q+1\right)$$

Using the integral definition for the Beta function (see 4.1.2) we can say,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{k^{q+1}q!} \int_0^1 t^{r/k-1}(1-t)^q \, dt$$

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{1}{kk^q q!} \int_0^1 t^{r/k-1}(1-t)^q \, dt$$

We can now re-arrange the integral and summation signs (this is possible since the expression is strictly greater than 0)..

$$= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^1 t^{r/k-1} \sum_{q=0}^{\infty} \frac{1}{q!} \left(\frac{1-t}{k}\right)^q dt$$

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ we have,

$$= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^1 t^{r/k-1} e^{(1-t)/k} \, dt$$

Taking $t = kx$ we simplify the function as follows,

$$\begin{aligned}
&= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^{1/k} (kx)^{r/k-1} e^{(1-kx)/k} \, kdx \\
&= 1 + \frac{1}{k} \sum_{r=1}^{k} \int_0^{1/k} k^{r/k-1} x^{r/k-1} e^{1/k} e^{-x} \, kdx \\
&= 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} \int_0^{1/k} k^{r/k} x^{r/k-1} e^{-x} \, dx \\
&= 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \int_0^{1/k} x^{r/k-1} e^{-x} \, dx
\end{aligned}$$

Finally using the definition for incomplete gamma function (see 4.1.4),

$$= 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right) \qquad \square$$

4.3 Computation of RMFCs using the closed form formula

Using the closed form formula allows us to compute RMFCs with greater computational efficiency. Below we have listed the first ten RMFCs calculated up to 20 decimal place accuracy.

m(k)	Computed RMFCs
m(1)	2.71828182845904523 54
m(2)	3.0594074053425761445
m(3)	3.2989135380884190034
m(4)	3.4859449774535577452
m(5)	3.6402244677338097342
m(6)	3.7719023962117584357
m(7)	3.8869596537408434954
m(8)	3.9892412126901365441
m(9)	4.0813755201688985441
m(10)	4.1652437655583845908

Table 2: Computed values of first 10 RMFCs using closed form formula (20 digit accuracy)

We can further analyze the time taken for the computer to execute both the series definition and the closed form formula definition of a RMFCs.

We shall first test the two functions in their ability to calculate larger order RMFCs for an arbitrary decimal accuracy.

The below plots displays the time taken for the computer to compute a list of the first k RMFCs (k ranging from 1 to 200) up to a 50 digit accuracy. Using the series definition (in red) and the closed form definition (in blue).

We can see from the below plots that computing higher order RMFCs using the series definition is slower to in comparison with the closed form formula.

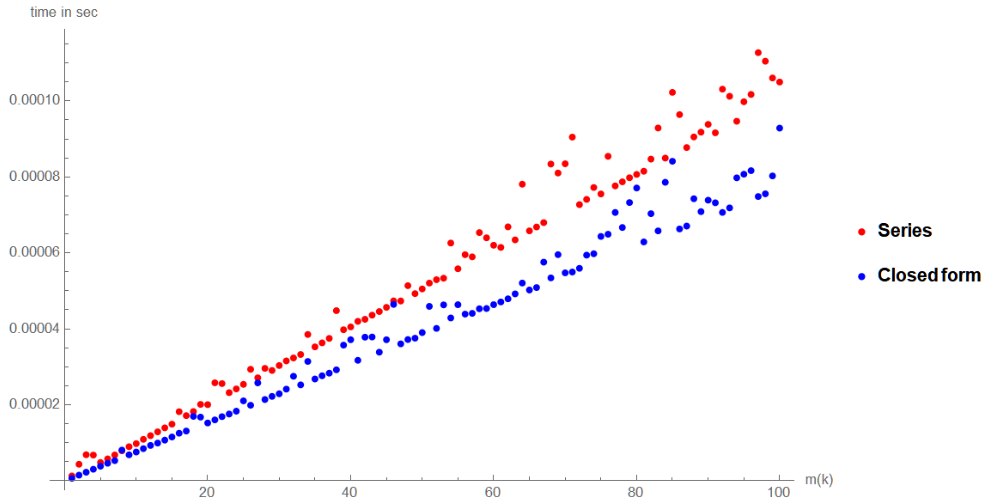

Figure 9: Time taken to compute a list of first k RMFCs (k ranging from 1 to 100) up to 50 digit accuracy.

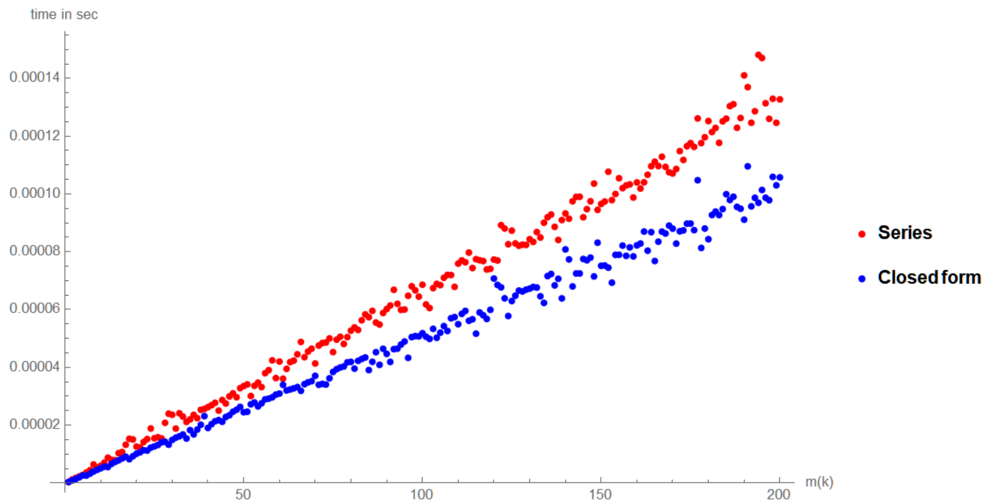

Figure 10: Time taken to compute a list of first k RMFCs (k ranging from 1 to 200) up to 50 digit accuracy.

We will now take a look at the individual computations of some arbitrary RMFCs for increasing order of digit accuracy.

The below plots display the comparison between two functions for simultaneous computation of the first 10 RMFCs with higher levels of digit accuracy.

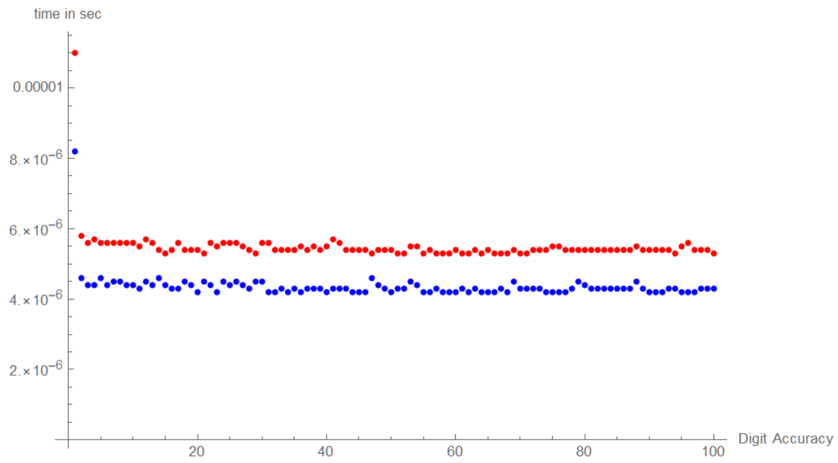

Figure 11: Time taken to compute first 10 RMFCs with increasing accuracy up to 100 digit accuracy.

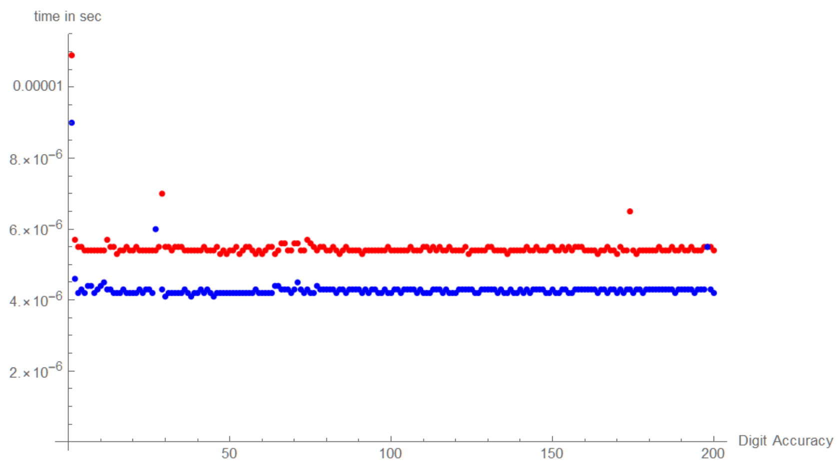

Figure 12: Time taken to compute first 10 RMFCs with increasing accuracy up to 200 digit accuracy.

From the above two plots it is seen that the closed form formula is computationally faster than the series definition for obtaining more accurate values of RMFCs. However the difference in the time taken appears to remain constant for higher levels of accuracy as opposed to the previous case (wherein the difference in time increased for higher order RMFCs of a fixed accuracy).

5 Asymptotics of Reciprocal Multifactorial Series

In this section we will examine the asymptotic behaviour of the RMFCs, i.e. we will examine what happens to the RMFC, m(k), as k approaches positive infinity.

A keen reader might have noticed that the computed values of RMFCs displayed in Tables 1 and 2 all seem to be increasing for higher orders.

This can be observed from the below graph.

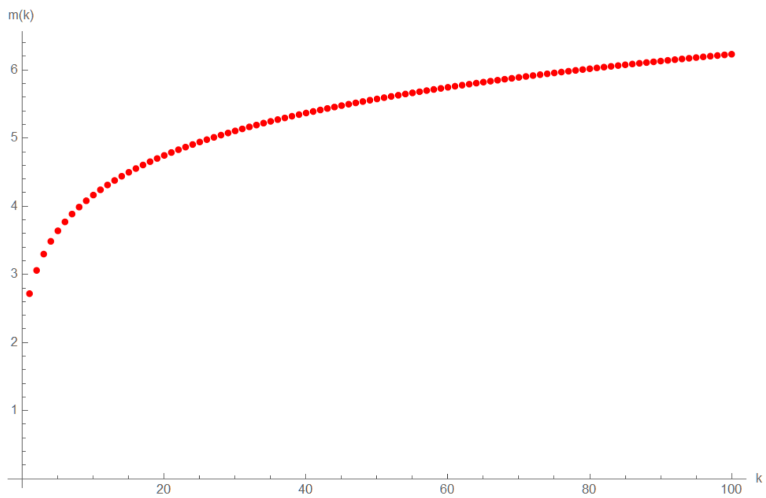

Figure 13: First 100 RMFCs

5.1 Casual observation

We shall first motivate the concept that RMFCs get arbitrarily larger by a simple observation based on definition for a multifactorial. Recall Definition 2.2,

$$n!_{(k)} = \begin{cases} 1 & \text{if } n = 0 \\ n & \text{if } 0 < n \leq k \\ n\left((n-k)!_{(k)}\right) & \text{if } n > k \end{cases}$$

Observe carefully the second line that says $n!_{(k)} = n$, if $0 < n \leq k$. One can tell that as k approaches infinity every $q!_{(k)}$ for $q < k$ will simply be equal to q. As such the Reciprocal Multifactorial Series will simply resemble

$$1 + \sum_{i=1}^{k} \frac{1}{i} = 1 + H_k, \text{ as k approaches infinity.}$$

Where H_k is the k^{th} harmonic number. We shall show this more rigorously in the following section.

5.2 Proof of divergence

We will begin with a result achieved in the proof of the closed form formula (line 6 of the proof) and proceed to prove the assertion made in the previous section.[10]

Theorem 5.1. $\lim_{k\to\infty} m(k) = 1 + H_k$

Proof: Consider the following expression (shown in Theorem 4.2)

$$\begin{aligned} m(k) &= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}e^{(1-t)/k}\,dt \\ &= 1 + \frac{e^{1/k}}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}e^{-t/k}\,dt \end{aligned}$$

Observe that $e^{-t/k} = 1 - (1 - e^{-t/k})$

$$= 1 + \frac{e^{1/k}}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}(1-(1-e^{-t/k}))\,dt$$

Since, $\int_0^1 t^{r/k-1}\,dt = \frac{k}{r}$, we can further simplify the above expression as follows

$$\begin{aligned} &= 1 + \frac{e^{1/k}}{k}\sum_{r=1}^{k}\left(\frac{k}{r} - \int_0^1 t^{r/k-1}(1-e^{-t/k})\,dt\right) \\ &= 1 + e^{1/k}\left(H_k - \frac{1}{k}\sum_{r=1}^{k}\int_0^1 t^{r/k-1}(1-e^{-t/k})\,dt\right) \end{aligned}$$

Replacing $\frac{1}{k}\sum_{r=1}^{k}\left(\int_0^1 t^{r/k-1}(1-e^{-t/k})\right)$ with Δk we get,

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$

Taking the limit as k approaches infinity we get,

$$\begin{aligned} &\lim_{k\to\infty} m(k) = \lim_{k\to\infty} 1 + e^{1/k}(H_k - \Delta k) \\ &\lim_{k\to\infty} m(k) = 1 + H_k \end{aligned}$$

□

5.3 Alternate proof of divergence

In this section we will provide an alternate proof for Theorem 5.1. To begin we will prove the a few results about the lower incomplete gamma function.[11]

In this section the notation $x^{(n)}$ will refer to rising factorials, which is defined as $x^{(n)} = x(x+1)(x+2)\dots(x+n-1)$. [12]

Lemma 5.2. *The power series for the lower incomplete gamma function is*

$$\gamma(a,x) = x^a e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{a^{(k+1)}}$$

Proof: We will derive the power series by repeated integration by parts. Recall definition (4.1.4)

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt$$

Integrating by parts, $u = e^{-t}, du = -e^{-t}dt, dv = t^{a-1}\, dt, v = \frac{t^a}{a}$

$$\begin{aligned}
&= e^{-x} \cdot \frac{x^a}{a} - \int_0^x \frac{t^a}{a} \cdot (-e^{-t}) dt \\
&= \frac{e^{-x} x^a}{a} + \frac{1}{a} \int_0^x t^a e^{-t}\, dt \\
\gamma(a,x) &= \frac{e^{-x} x^a}{a} + \frac{1}{a} \gamma(a+1, x)
\end{aligned}$$

This is the recurrence relation, further upon repeated integration by parts we get,

$$\begin{aligned}
&= e^{-x} x^a \left(\frac{1}{a} + \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} \dots \right) \\
&= x^a e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{a^{(k+1)}} \qquad \square
\end{aligned}$$

Now we will prove another useful lemma regarding the limit of the lower incomplete gamma function.

Lemma 5.3.

$$\frac{\gamma(a,x)}{x^a} \to \frac{1}{a}, \text{ as, } x \to 0$$

Proof: We will begin by using the power series for the lower incomplete gamma function.

$$\gamma(a,x) = x^a e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{a^{(k+1)}}$$
$$\frac{\gamma(a,x)}{x^a} = e^{-x} \left(\frac{1}{a} + \frac{x}{a(a+1)} + \frac{x^2}{a(a+1)(a+2)} \cdots \right)$$

As $x \to 0, e^{-x} \to 1$ and all terms with x in the numerator approach 0.

$$\frac{\gamma(a,x)}{x^a} \to \frac{1}{a} \text{ as, } x \to 0 \qquad \square$$

Theorem 5.4.

$$\lim_{k\to\infty} m(k) = 1 + H_k$$

Proof: Using the result of the closed form formula for RMFCs (Theorem 4.2),

$$m(k) = 1 + \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right)$$

Taking the limit for $k \to \infty$ (i.e . $1/k \to 0$)

$$\lim_{k\to\infty} m(k) = 1 + \lim_{k\to\infty} \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{1}{k}\right)$$

Using Lemma 5.3 on $\gamma(r/k, 1/k)$ we get

$$\lim_{k\to\infty} m(k) = 1 + \lim_{k\to\infty} \frac{e^{1/k}}{k} \sum_{r=1}^{k} k^{r/k} \frac{k}{rk^{r/k}}$$
$$\lim_{k\to\infty} m(k) = 1 + \lim_{k\to\infty} \frac{e^{1/k}}{k} \sum_{r=1}^{k} \frac{k}{r}$$
$$\lim_{k\to\infty} m(k) = 1 + \lim_{k\to\infty} \frac{e^{1/k}}{k} kH_k$$
$$\lim_{k\to\infty} m(k) = 1 + \lim_{k\to\infty} e^{1/k} H_k$$
$$\lim_{k\to\infty} m(k) = 1 + H_k \qquad \square$$

5.4 Detailed Asymptotics

In this section we will mostly refer to the following expression.

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right) \tag{5.4.1}$$

Our goal for the next section is to devise an asymptotic approximation for RMFCs. We will begin by displaying the necessary asymptotic expressions.

For $e^{1/k}$andH_k we will use known results of its asymptotic expansions are as follows.[13][14]

$$e^{1/k} = 1 + \frac{1}{k} + \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right) \tag{5.4.2}$$

$$H_k = \log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + O\left(\frac{1}{k^4}\right) \tag{5.4.3}$$

Where γ is the Euler-Mascheroni Constant. [15]

We compute the asymptotic expansion of Δk using Mathematica. Which results in the following expression.

$$\Delta k = \frac{\log(2)}{k} + \frac{1}{4k^2}\left(-1 - \log\frac{9}{4}\right) + \frac{1}{48k^3}\left(5 + 8\log\frac{4}{3}\right) + O\left(\frac{1}{k^4}\right) \tag{5.4.4}$$

5.5 Asymptotic approximations for RMFCs

We will use the asymptotic series described above, truncated up to 1^{st} and 2^{nd} order of k respectively to construct two asymptotic approximations for $m(k)$. Substituting values of (5.4.2), (5.4.3) and (5.4.4) into (5.4.1) we get the following.

Approximation 1: Using asymptotic series truncated at 1^{st} order terms

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$
$$m(k) \sim 1 + \left(1 + \frac{1}{k}\right)\left(\left(\log(k) + \gamma + \frac{1}{2k}\right) - \left(\frac{\log(2)}{k}\right)\right)$$

Which can be simplified as the following,

$$m(k) \sim 1 + \frac{(1+k)(1 + 2\gamma k - \log(4) + 2k\log(k))}{2k^2} \tag{5.5.1}$$

Approximation 2: Using asymptotic series truncated at 2^{nd} order terms

$$m(k) = 1 + e^{1/k}\left(H_k - \Delta k\right)$$
$$m(k) \sim 1 + \left(1 + \frac{1}{k} + \frac{1}{2k^2}\right)\left[\left(\log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\right) - \left(\frac{\log(2)}{k} + \frac{1}{4k^2}\left(-1 - \log\frac{9}{4}\right)\right)\right]$$

Which can be simplified as the following,

$$m(k) \sim 1 + \frac{(1 + 2k(1+k) \cdot (1 + \log \frac{27}{8} + 6k^2 \log k + k(3 + 6\gamma k - \log 64))}{12k^4} \tag{5.5.2}$$

We can compare the exact values of m(k) with the values computed using the two asymptotic definition described in (5.5.1) and (5.5.2) as follows,

m(k)	Absolute error between asymptotic approximation and exact solution
1	$3.49539 \cdot 10^{-1}$
10	$4.49172 \cdot 10^{-3}$
10^2	$4.76604 \cdot 10^{-5}$
10^3	$4.84353 \cdot 10^{-7}$
10^4	$4.87866 \cdot 10^{-9}$

Table 3: Comparison between first asymptotic equation and exact solution for a few RMFCs

m(k)	Absolute error between asymptotic approximation and exact solution
1	$6.08426 \cdot 10^{-2}$
10	$7.79859 \cdot 10^{-5}$
10^2	$1.14278 \cdot 10^{-7}$
10^3	$1.29004 \cdot 10^{-10}$
10^4	$1.37156 \cdot 10^{-13}$

Table 4: Comparison between second asymptotic equation and exact solution for a few RMFCs

We can see from the above tables that both the asymptotic approximation are good since they approximates to the exact solution with low error rates for small values of k.

6 Generalized Reciprocal Multifactorial Constants

In this section we will discuss the power series reciprocal multifactorial series. For simplicity sake we will refer to these power series as *Generalized Reciprocal Multifactorial Series*. Which we define as follows,

$$m_x(k) = \sum_{n=0}^{\infty} \frac{x^n}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}} \tag{6.0.1}$$

The radius for the above power series can be trivially seen to be infinity for all finite k.[1] Two examples of calculating α are shown on page 8.

6.1 Closed form formula

In this section we shall provide a closed form formula for Generalized Reciprocal Multifactorial Constants (GRMFCS).

Theorem 6.1. *A closed form formula for Generalized Reciprocal Multifactorial Constants is given by the following expression*

$$m_x(k) = 1 + \frac{e^{x^k/k}}{k} \sum_{r=1}^{k} k^{r/k} \gamma\left(\frac{r}{k}, \frac{x^k}{k}\right)$$

Proof: We continue similarly to Theorem 4.2.

$$m_x(k) = \sum_{n=0}^{\infty} \frac{x^n}{n!_{(k)}} = 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{(kq+r)\underbrace{!\ldots!}_{\text{k times}}}$$

Using Lemma 4.1 we get,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{k^{q+1} q!} \mathrm{B}\left(\frac{r}{k}, q+1\right)$$

Using the integral definition for the Beta function,

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{k^{q+1} q!} \int_0^1 t^{r/k-1}(1-t)^q \, dt$$

$$= 1 + \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{x^{kq+r}}{k k^q q!} \int_0^1 t^{r/k-1}(1-t)^q \, dt$$

[1]Altough the values get very large for higher x

We can now re-arrange the integral and summation signs (this is possible since the expression is strictly greater than 0).

$$
\begin{aligned}
&= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^1 x^r t^{r/k-1}\sum_{q=0}^{\infty}\frac{1}{q!}\left(x^k\frac{1-t}{k}\right)^q dt \\
&= 1 + \frac{1}{k}\sum_{r=1}^{k}\int_0^1 x^r t^{r/k-1}e^{(x^k-x^k t)/k}\,dt \\
&= 1 + \frac{e^{x^k/k}}{k}\sum_{r=1}^{k}x^r\int_0^1 t^{r/k-1}e^{(-x^k t)/k}\,dt
\end{aligned}
$$

Let, $u = x^k t/k, dt = kx^{-k}du, t = uk/x^k$

$$
\begin{aligned}
&= 1 + \frac{e^{x^k/k}}{k}\sum_{r=1}^{k}x^r\int_0^{x^k/k}\left(\frac{uk}{x^k}\right)^{r/k-1} e^{-u}kx^{-k}\,du \\
&= 1 + \frac{e^{x^k/k}}{k}\sum_{r=1}^{k}x^r(kx^{-k})(k^{r/k-1}x^{k-r})\int_0^{x^k/k} u^{r/k-1}e^{-u}\,du \\
&= 1 + \frac{e^{x^k/k}}{k}\sum_{r=1}^{k}k^{r/k}\int_0^{x^k/k} u^{r/k-1}e^{-u}\,du \\
&= 1 + \frac{e^{x^k/k}}{k}\sum_{r=1}^{k}k^{r/k}\gamma\left(\frac{r}{k},\frac{x^k}{k}\right) \qquad \square
\end{aligned}
$$

6.2 Computations of GRMFCs using the closed form

In this section we will use the closed form formula derived in the previous section to compute a few values of GRMFCs.

x\k	1	2	3	4	5
0.5	1.64872	1.67697	1.68663	1.69039	1.69195
1.0	2.71828	3.05941	3.29891	3.48594	3.64022
1.5	4.48169	6.42488	9.06146	13.1533	20.2725
2.0	7.38906	16.2285	45.7755	219.471	$2.85031 \cdot 10^3$
2.5	12.1825	50.9309	589.300	$7.02993 \cdot 10^4$	$1.43833 \cdot 10^9$

Table 5: Few computed values of GRMFCs using the closed form formula.

The values calculated are the same as the summation definition. (This can be verified by viewing the linked Mathematica notebook.)

References

[1] David Callan. A combinatorial survey of identities for the double factorial, 2009.

[2] How to visually display the stirling permutations of k^{th} order? Mathematica Stack Exchange. https://mathematica.stackexchange.com/questions/240381 (version: 2021-02-21).

[3] Eric W. Weisstein. Double factorial. From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/DoubleFactorial.html`. Last visited on May 30, 2021.

[4] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences. A001113, A143280, A288055, A288091, A288092, A288093, A288094, A288095, A288096. A342033,.

[5] Eric W. Weisstein. Gamma function. From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/GammaFunction.html`. Last visited on May 30, 2021.

[6] Eric W. Weisstein. Beta function. From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/BetaFunction.html`. Last visited on May 30, 2021.

[7] Eric W. Weisstein. Incomplete gamma function. From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/IncompleteGammaFunction.html`. Last visited on May 30, 2021.

[8] Eric W. Weisstein. Reciprocal multifactorial constant. From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/ReciprocalMultifactorialConstant.html`. Last visited on May 30, 2021.

[9] How to prove the formula for the reciprocal multifactorial constant? Mathematics Stack Exchange. https://math.stackexchange.com/q/3993774 (version: 2021-01-21).

[10] How do i examine the behavior of the reciprocal multifactorial series $\sigma\frac{1}{n!^{(k)}}$ as k approaches infinity. Mathematics Stack Exchange. https://math.stackexchange.com/q/4021528 (version: 2021-02-11).

[11] Incomplete gamma and related functions- asymptotic approximations and expansions. NIST Digital Library of Mathematical Functions. https://dlmf.nist.gov/8.11.ii, Last visited on May 30, 2021.

[12] Eric W. Weisstein. Rising factorial. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/RisingFactorial.html. Last visited on May 30, 2021.

[13] Eric W. Weisstein. Exponential function. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/ExponentialFunction.html. Last visited on May 30, 2021.

[14] Eric W. Weisstein. Harmonic number. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/HarmonicNumber.html. Last visited on May 30, 2021.

[15] Eric W. Weisstein. Euler-mascheroni constant. From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/Euler-MascheroniConstant.html. Last visited on May 30, 2021.