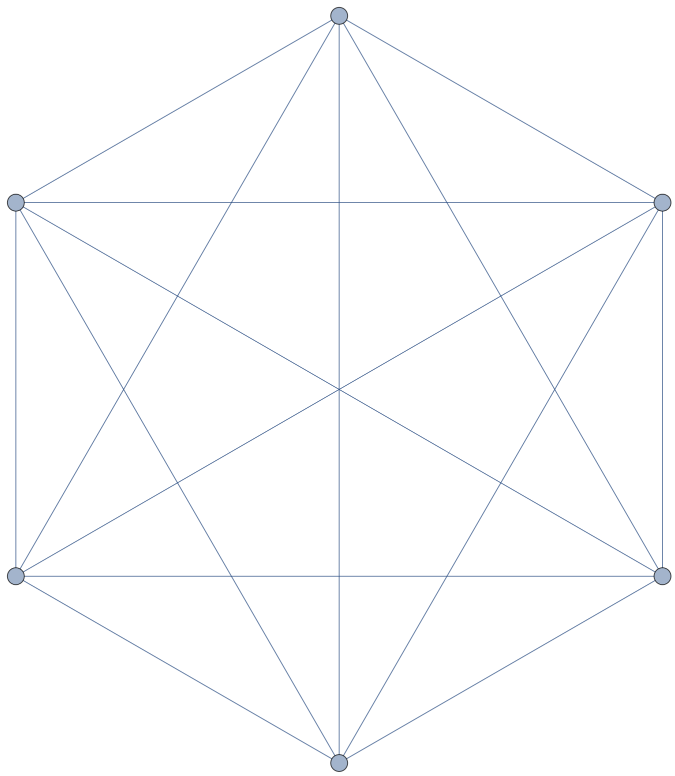
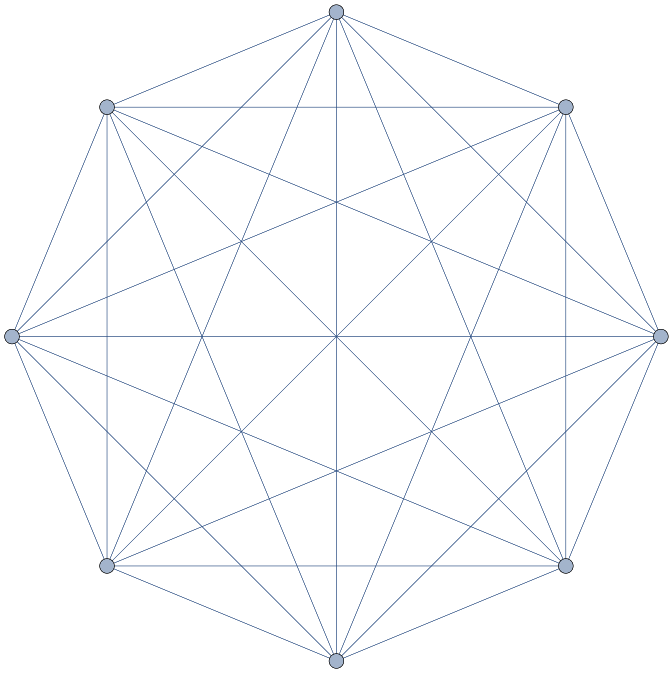
Reciprocal Multifactorial Constants

Perfect matchings for K_ 6 and K_8

In[]:=
```
k6 = CompleteGraph[6]
k8 = CompleteGraph[8]
```

Out[]=


Out[ ]=


In[ ]:=
```
l6 = Length[FindIndependentEdgeSet[k6]]
l8 = Length[FindIndependentEdgeSet[k8]]
```

Out[ ]= 3

Out[ ]= 4

```
In[•]:= esl6 = Select[Subsets[EdgeList[k6], {l6}], IndependentEdgeSetQ[k6, #] &]
        esl8 = Select[Subsets[EdgeList[k8], {l8}], IndependentEdgeSetQ[k8, #] &]

Out[•]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6}, {1 ↔ 2, 3 ↔ 5, 4 ↔ 6}, {1 ↔ 2, 3 ↔ 6, 4 ↔ 5}, {1 ↔ 3, 2 ↔ 4, 5 ↔ 6},
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Out[•]= {{1 ↔ 2, 3 ↔ 4, 5 ↔ 6, 7 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 7, 6 ↔ 8}, {1 ↔ 2, 3 ↔ 4, 5 ↔ 8, 6 ↔ 7},
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In[•]:=

In[•]:= Table[HighlightGraph[k6, h, GraphHighlightStyle → "Thick"], {h, esl6}]
```
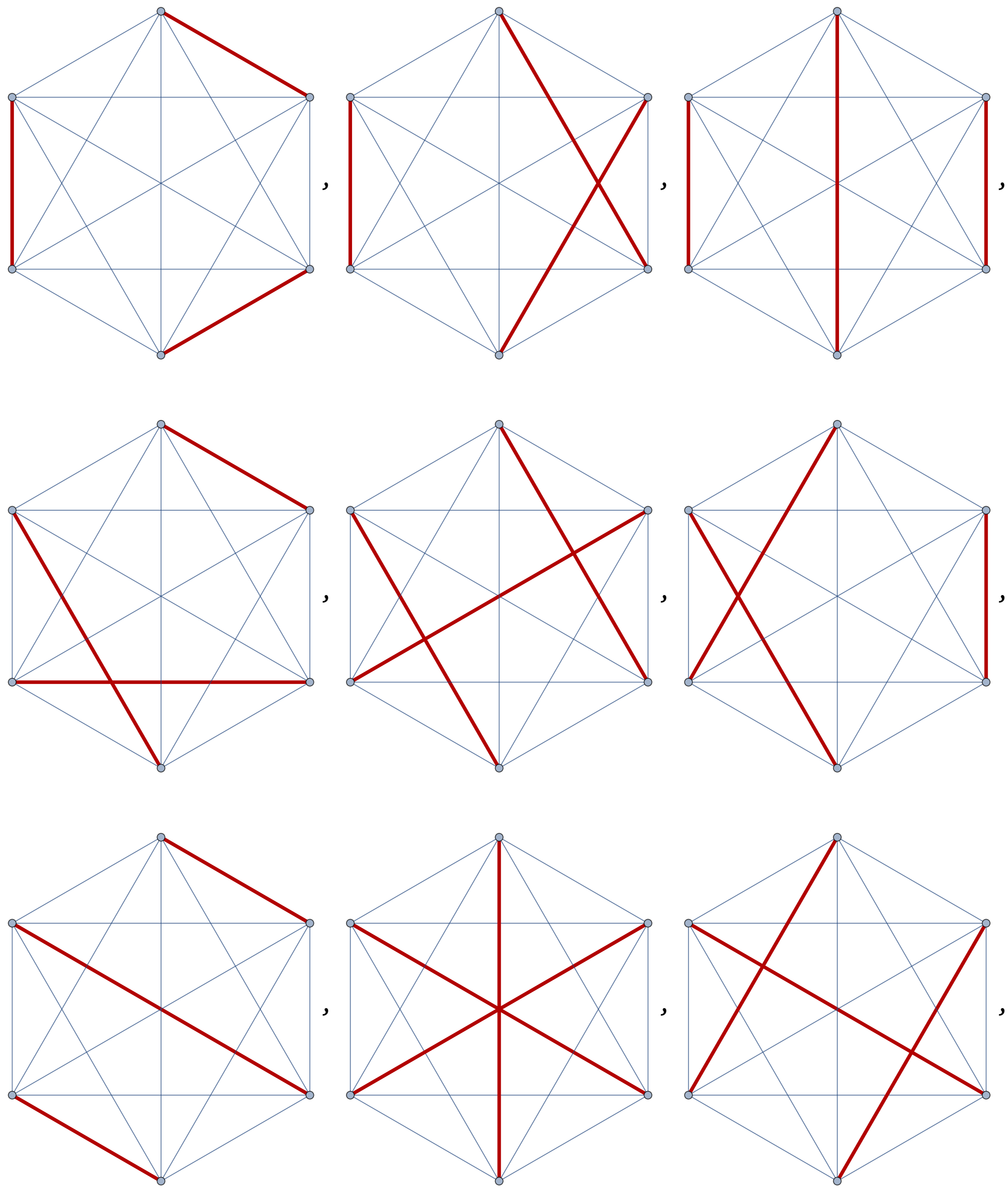
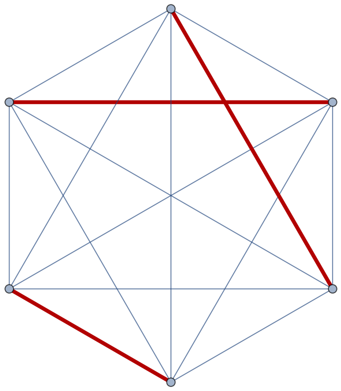
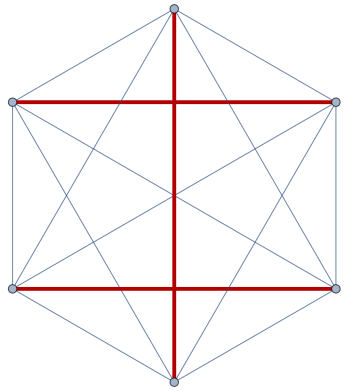
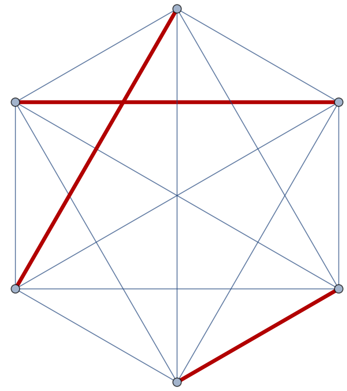
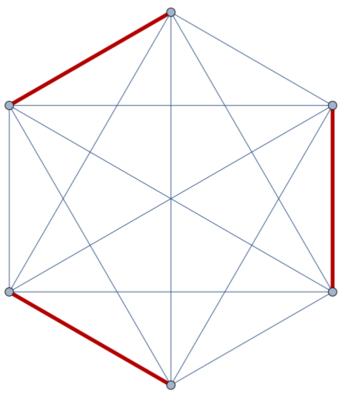
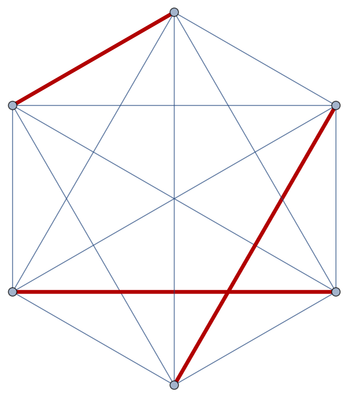
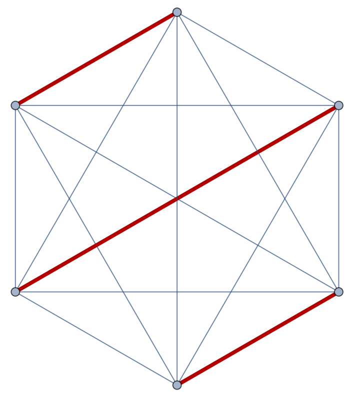
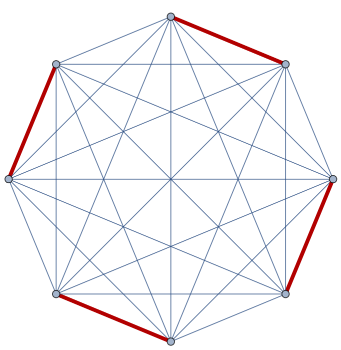
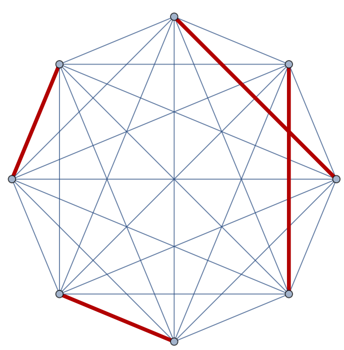
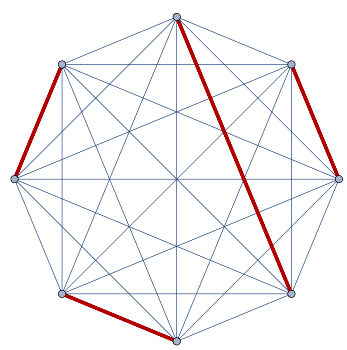

Out[•]= {

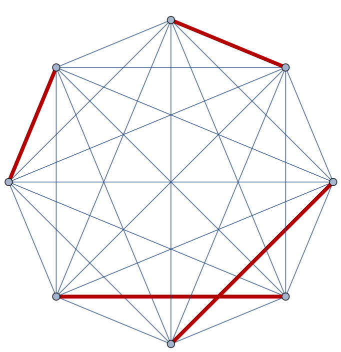
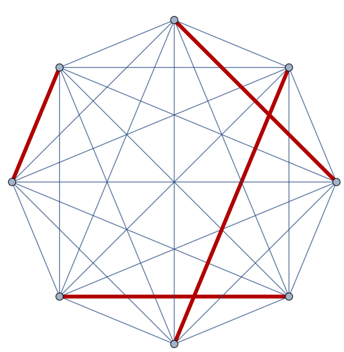
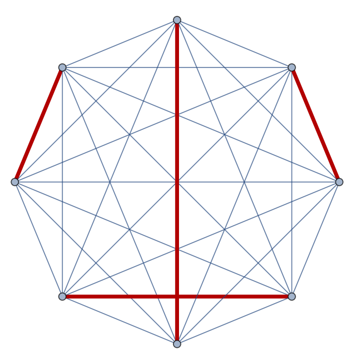


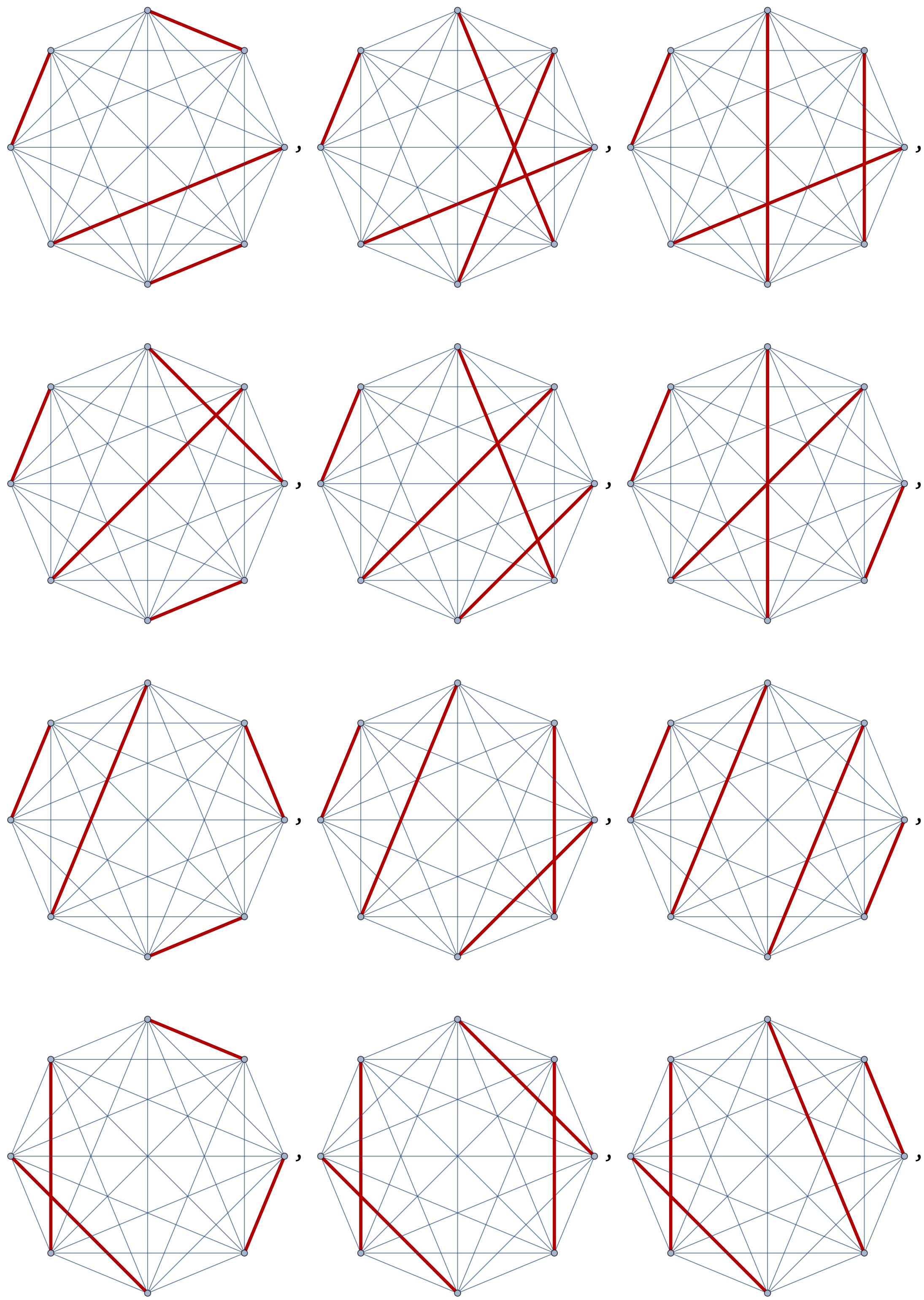
 ,
 ,
 ,

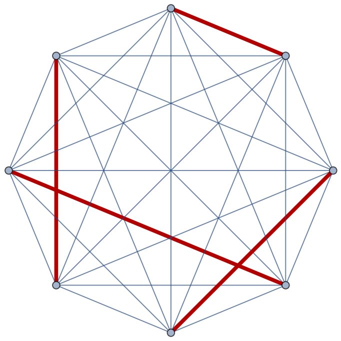
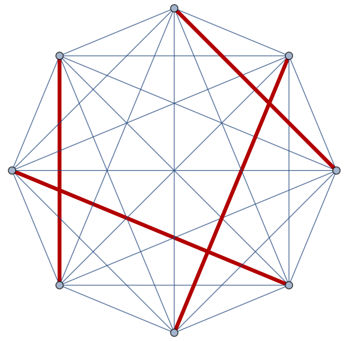
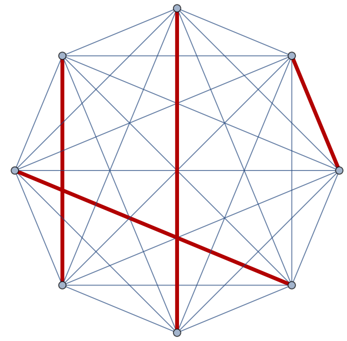
 ,
 ,
 }

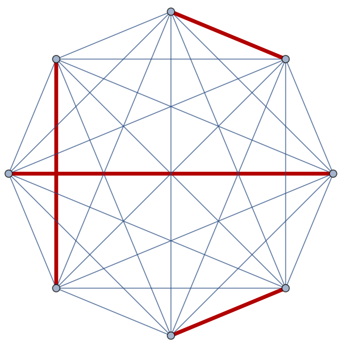
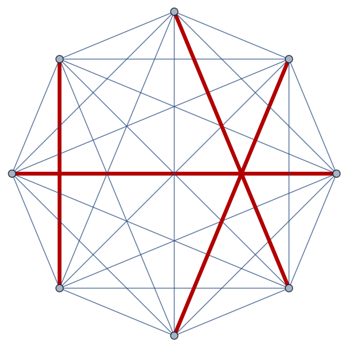
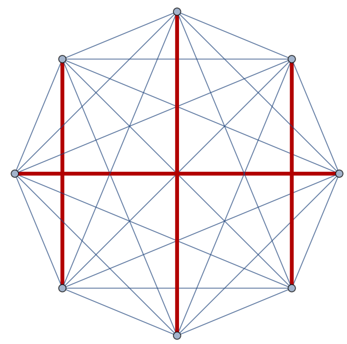
In[ ]:=
```
Table[HighlightGraph[k8, h, GraphHighlightStyle → "Thick"], {h, esl8}]
```

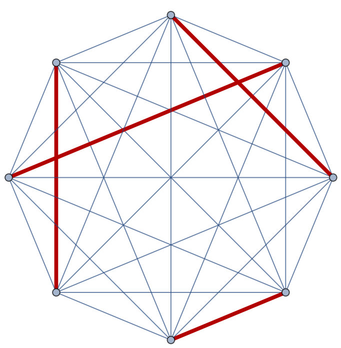
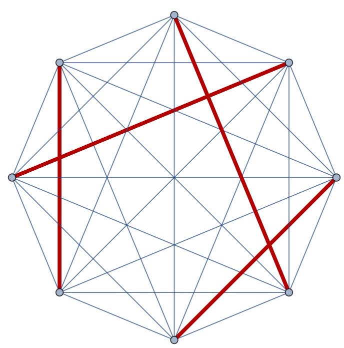
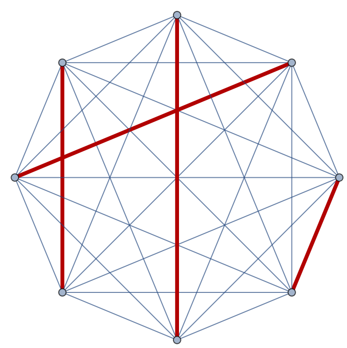
Out[ ]= {
 ,
 ,
 ,

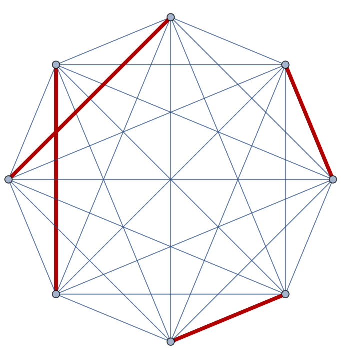
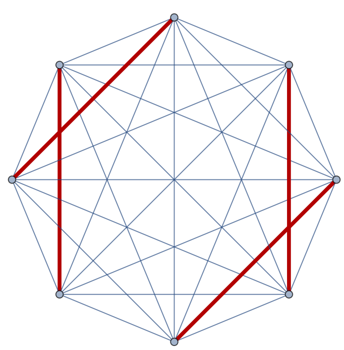
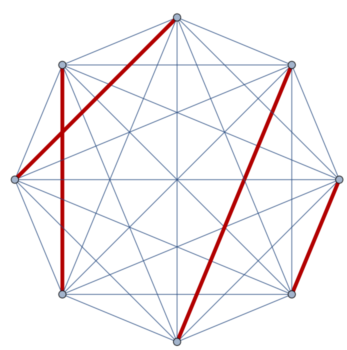
 ,
 ,
 ,

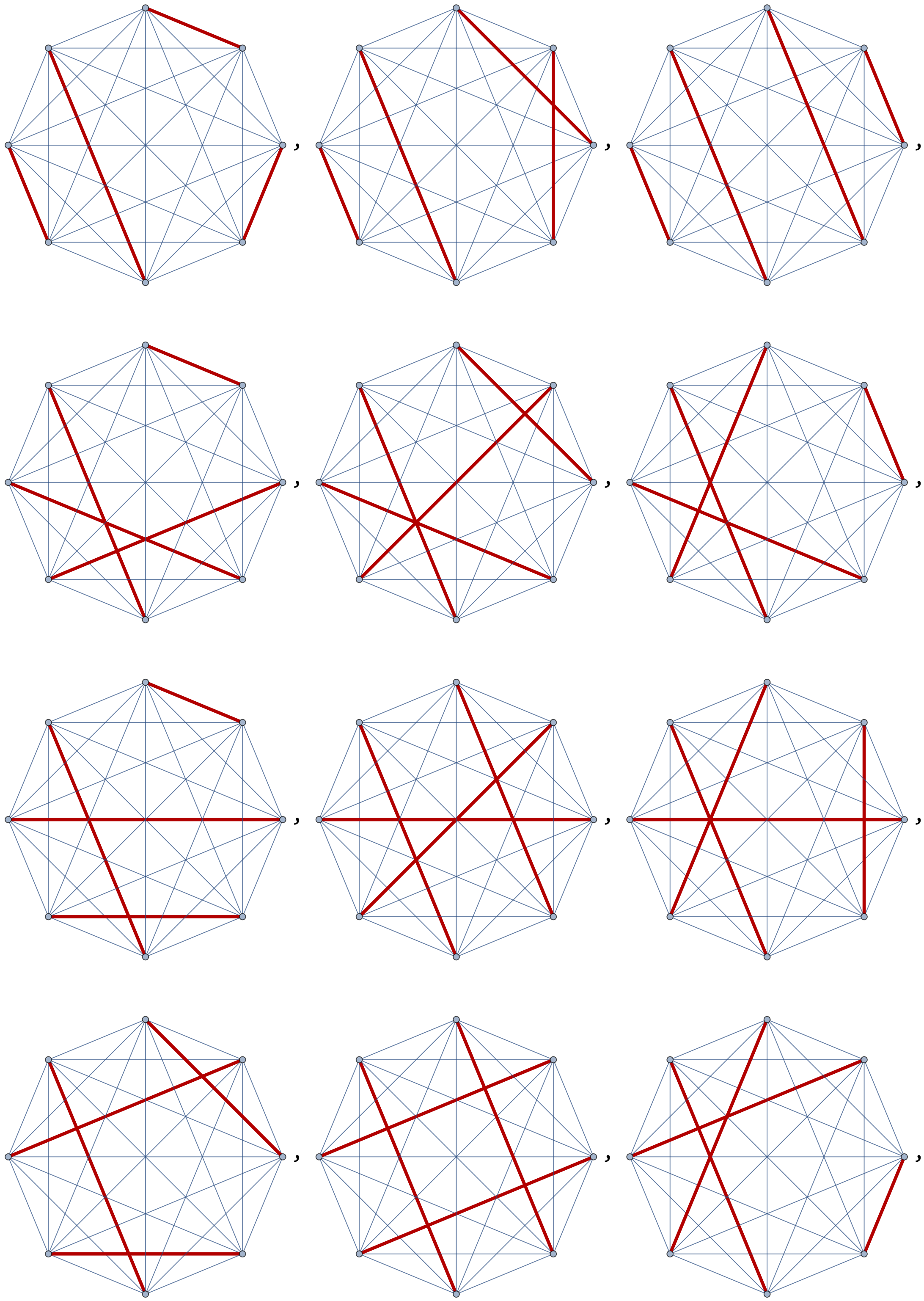


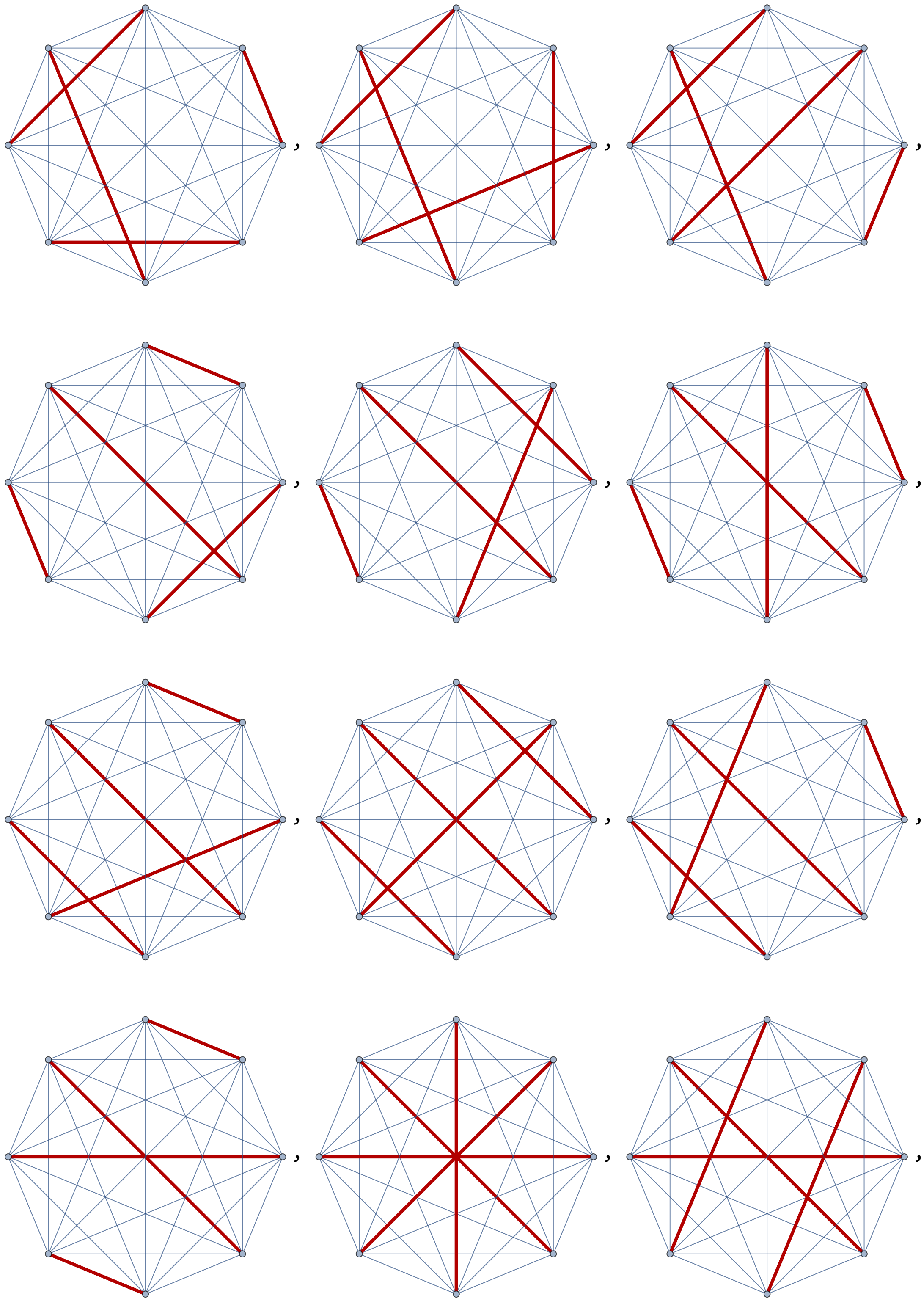
 ,  ,  ,

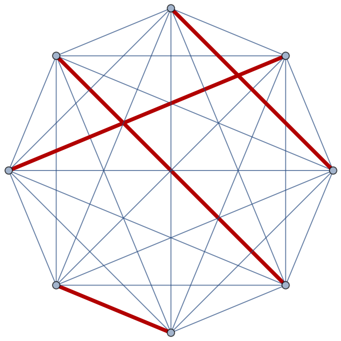
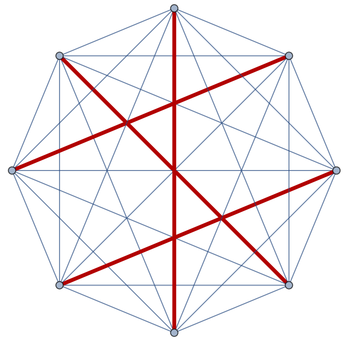
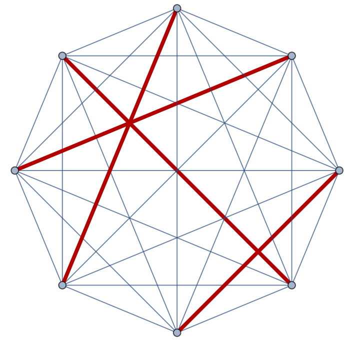
 ,  ,  ,

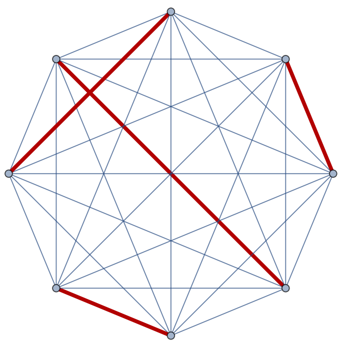
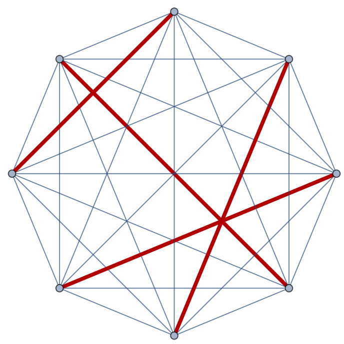
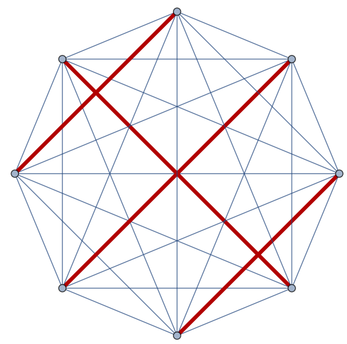
 ,  ,  ,

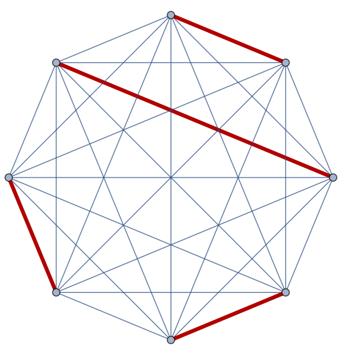
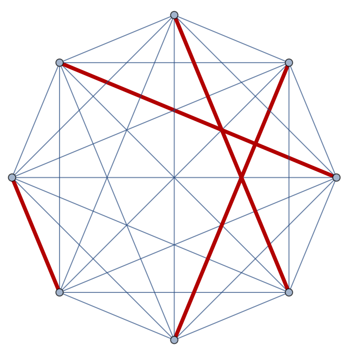
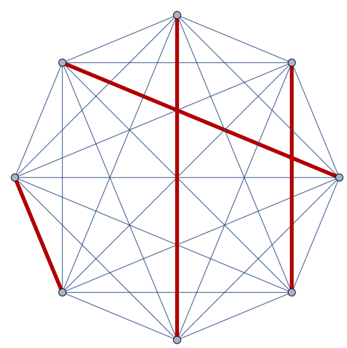
 ,  ,  ,

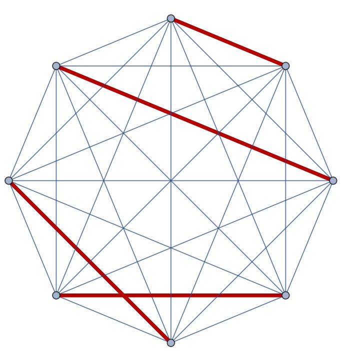
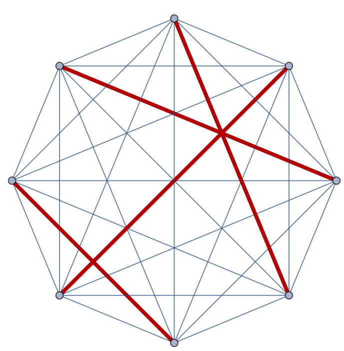
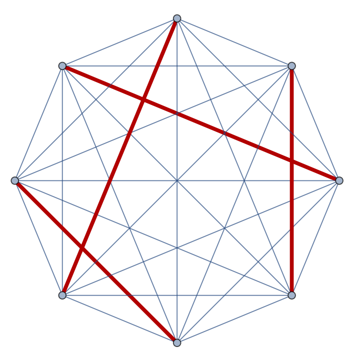


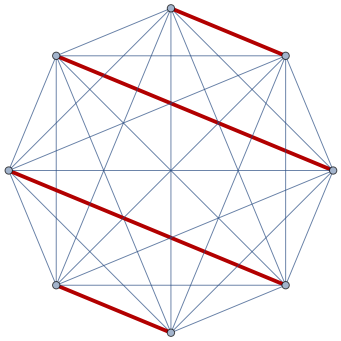
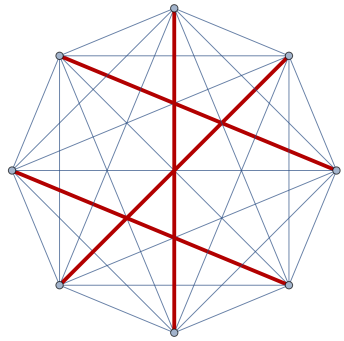
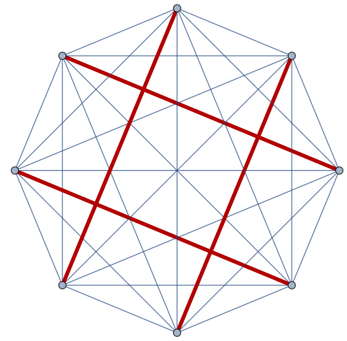


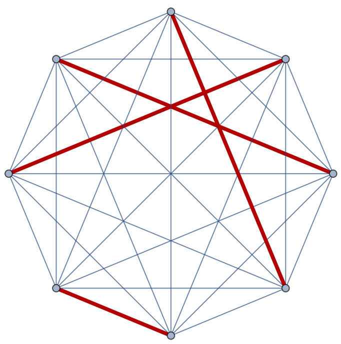
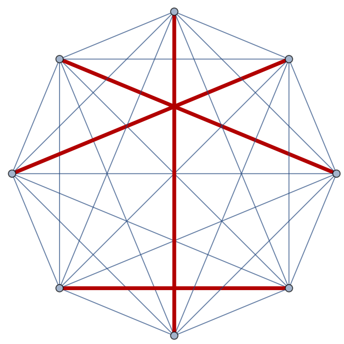
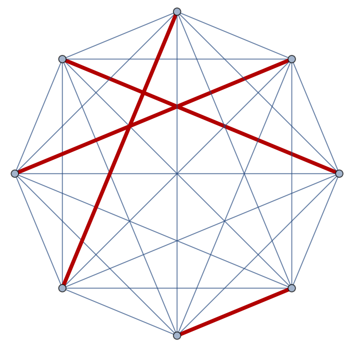
 ,  ,  ,

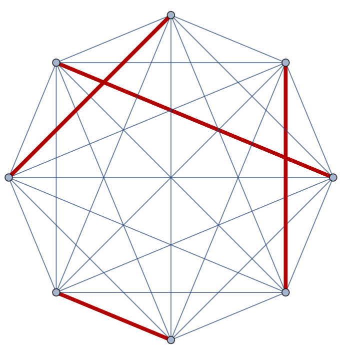
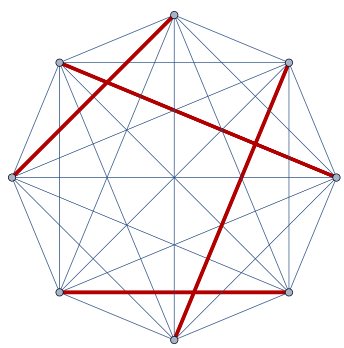
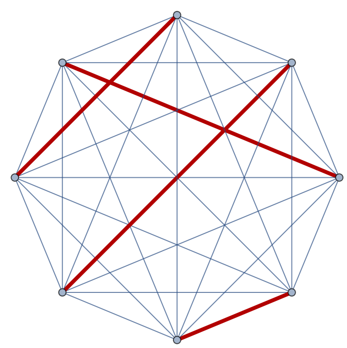
 ,  ,  ,

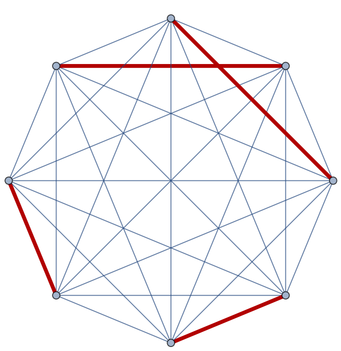
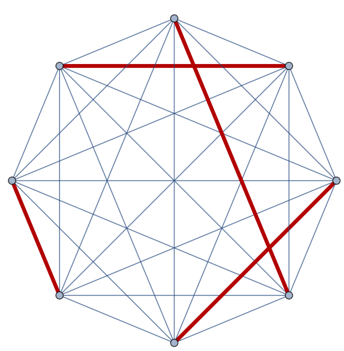
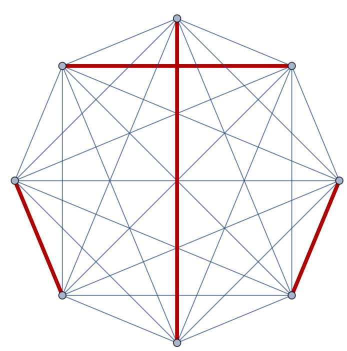
 ,  ,  ,

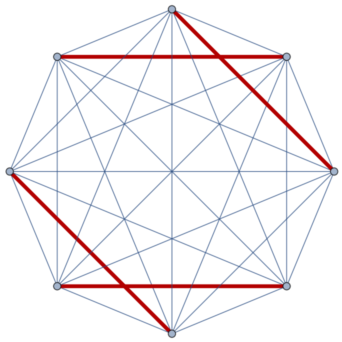
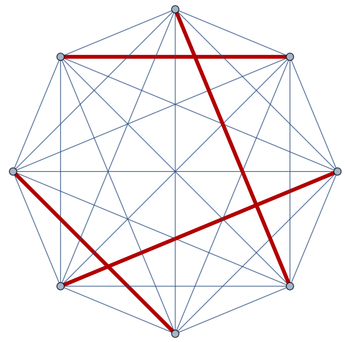
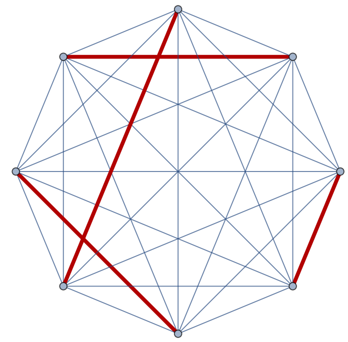
 ,  ,  ,

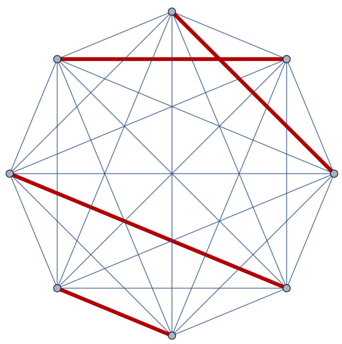
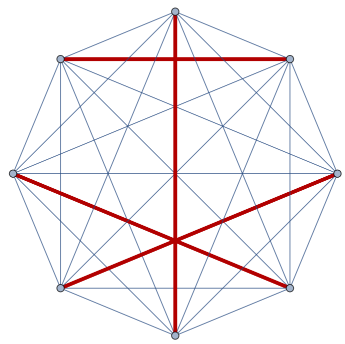
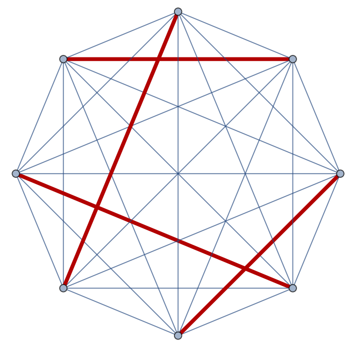
 ,
 ,
 ,

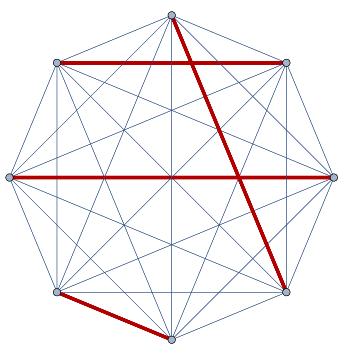
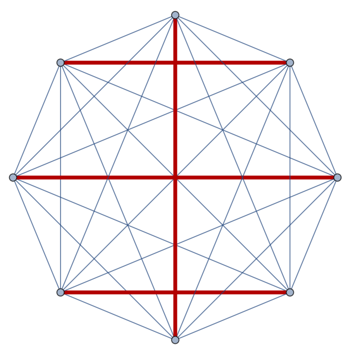
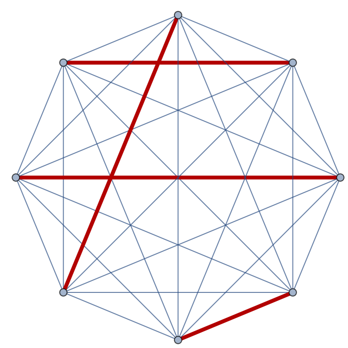
 ,
 ,
 ,

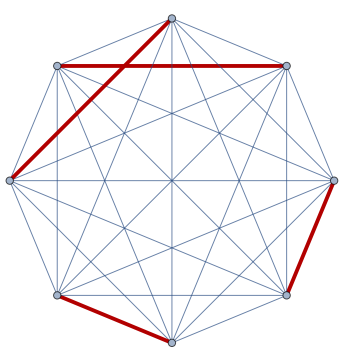
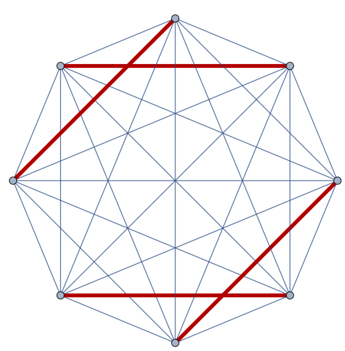
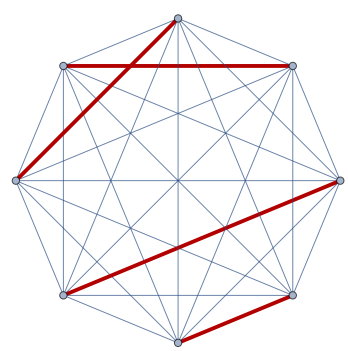
 ,
 ,
 ,

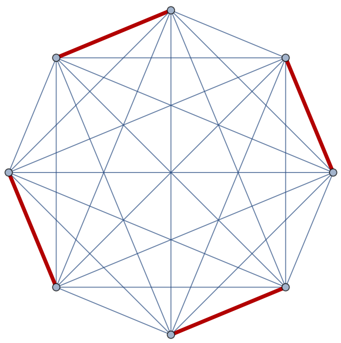
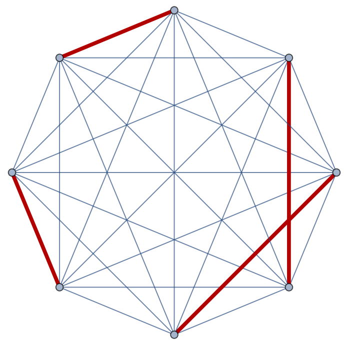
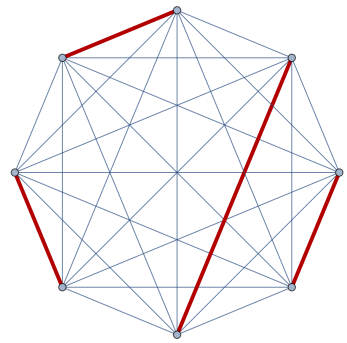
 ,
 ,
 ,

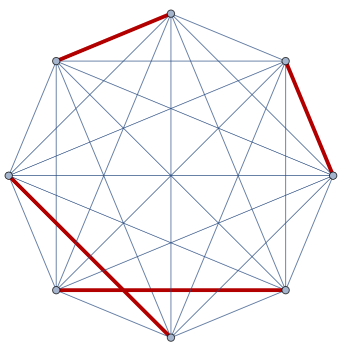
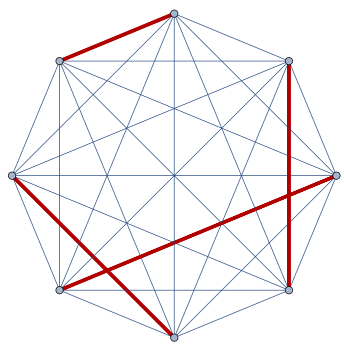
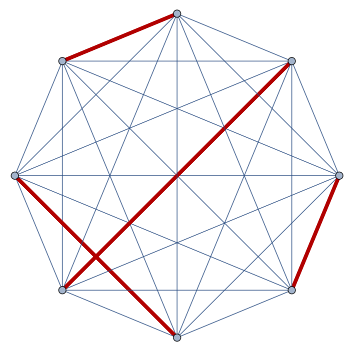
 ,  ,  ,

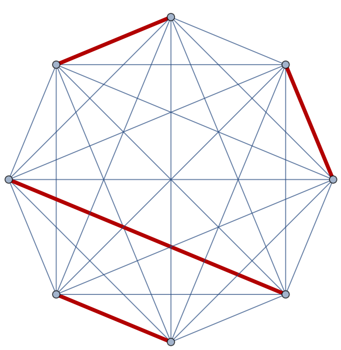
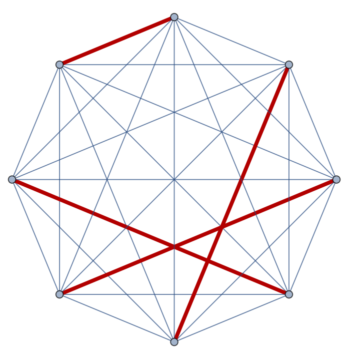
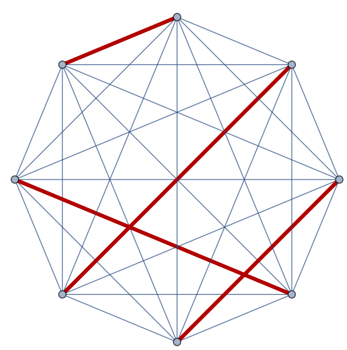
 ,  ,  ,

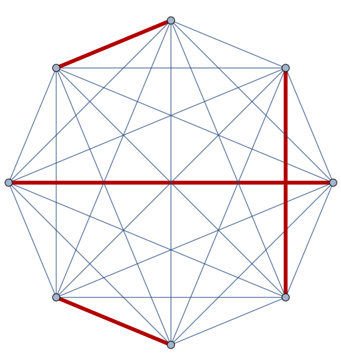
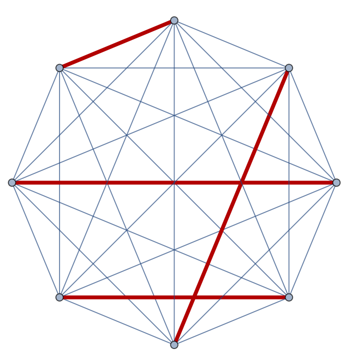
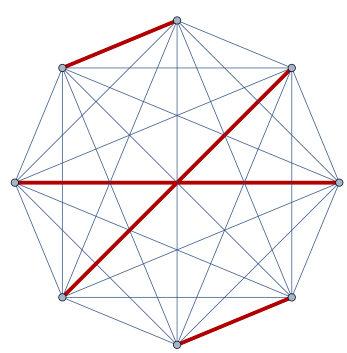
 ,  ,  ,

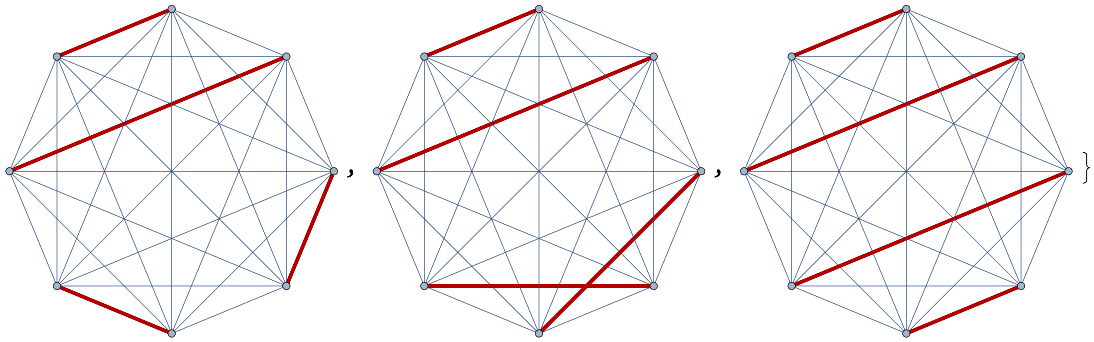
 ,  ,  ,

 ,  ,  ,

 ,  ,  ,

 ,  ,  ,

 ,  ,  ,



Stirling Permutations

In[ ]:=
```
ClearAll[strlngPermutations]
strlngPermutations[1] = {{1, 1}};
strlngPermutations[k_] := Join @@
  (Function[x, Flatten[Insert[x, {k, k}, #]] & /@ Range[2 k - 1]] /@ strlngPermutations[k - 1])
```

In[ ]:=
```
Multicolumn[Sort@strlngPermutations@4, 5, Appearance → "Horizontal"]
```

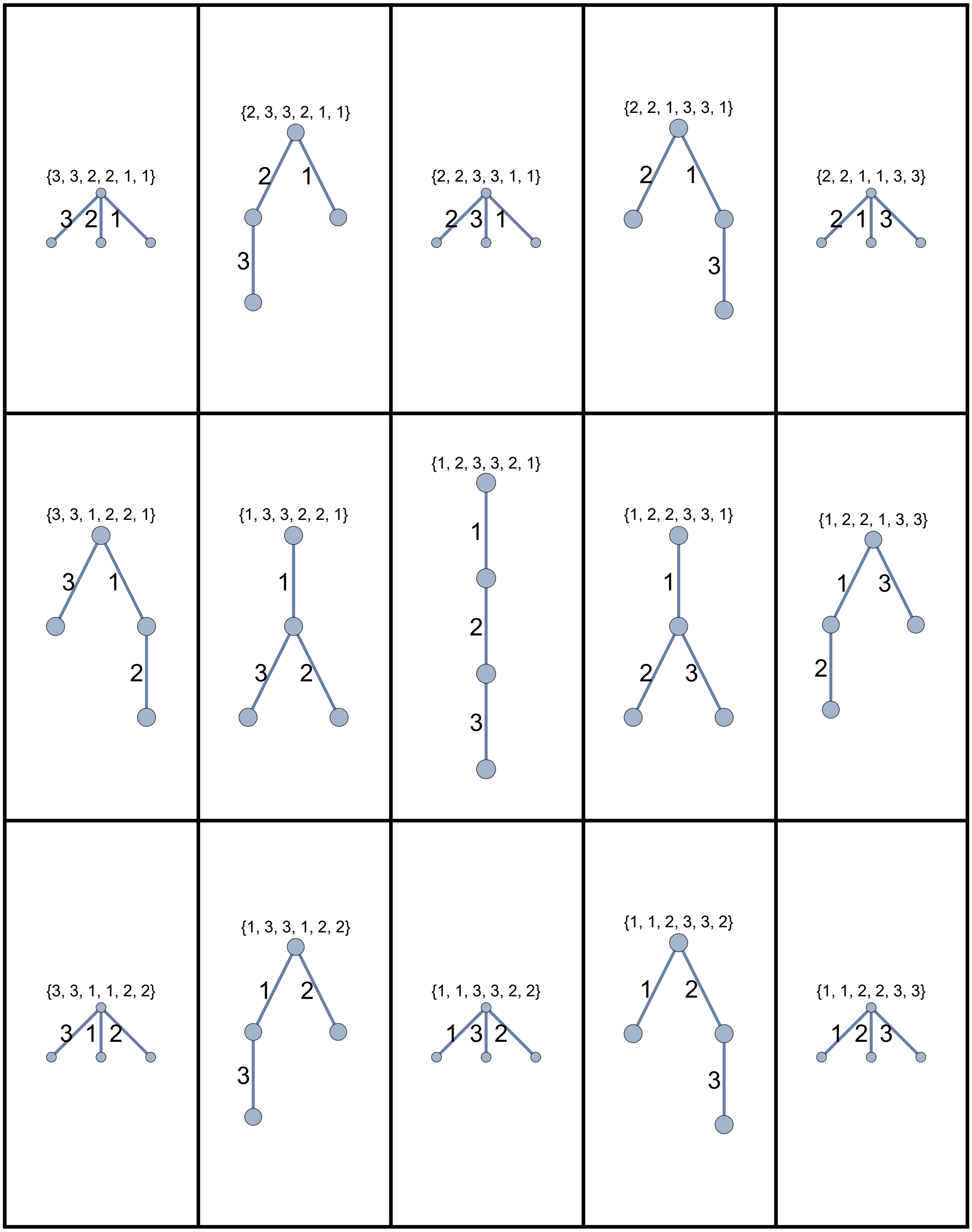
Out[ ]=

{1, 1, 2, 2, 3, 3, 4, 4}	{1, 1, 2, 2, 3, 4, 4, 3}	{1, 1, 2, 2, 4, 4, 3, 3}	{1, 1, 2, 3, 3, 2, 4, 4}	{1, 1, 2, 3, 3, 4, 4, 2}
{1, 1, 2, 3, 4, 4, 3, 2}	{1, 1, 2, 4, 4, 2, 3, 3}	{1, 1, 2, 4, 4, 3, 3, 2}	{1, 1, 3, 3, 2, 2, 4, 4}	{1, 1, 3, 3, 2, 4, 4, 2}
{1, 1, 3, 3, 4, 4, 2, 2}	{1, 1, 3, 4, 4, 3, 2, 2}	{1, 1, 4, 4, 2, 2, 3, 3}	{1, 1, 4, 4, 2, 3, 3, 2}	{1, 1, 4, 4, 3, 3, 2, 2}
{1, 2, 2, 1, 3, 3, 4, 4}	{1, 2, 2, 1, 3, 4, 4, 3}	{1, 2, 2, 1, 4, 4, 3, 3}	{1, 2, 2, 3, 3, 1, 4, 4}	{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 4, 4, 3, 1}	{1, 2, 2, 4, 4, 1, 3, 3}	{1, 2, 2, 4, 4, 3, 3, 1}	{1, 2, 3, 3, 2, 1, 4, 4}	{1, 2, 3, 3, 2, 4, 4, 1}
{1, 2, 3, 3, 4, 4, 2, 1}	{1, 2, 3, 4, 4, 3, 2, 1}	{1, 2, 4, 4, 2, 1, 3, 3}	{1, 2, 4, 4, 2, 3, 3, 1}	{1, 2, 4, 4, 3, 3, 2, 1}
{1, 3, 3, 1, 2, 2, 4, 4}	{1, 3, 3, 1, 2, 4, 4, 2}	{1, 3, 3, 1, 4, 4, 2, 2}	{1, 3, 3, 2, 2, 1, 4, 4}	{1, 3, 3, 2, 2, 4, 4, 1}
{1, 3, 3, 2, 4, 4, 2, 1}	{1, 3, 3, 4, 4, 1, 2, 2}	{1, 3, 3, 4, 4, 2, 2, 1}	{1, 3, 4, 4, 3, 1, 2, 2}	{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 1, 2, 2, 3, 3}	{1, 4, 4, 1, 2, 3, 3, 2}	{1, 4, 4, 1, 3, 3, 2, 2}	{1, 4, 4, 2, 2, 1, 3, 3}	{1, 4, 4, 2, 2, 3, 3, 1}
{1, 4, 4, 2, 3, 3, 2, 1}	{1, 4, 4, 3, 3, 1, 2, 2}	{1, 4, 4, 3, 3, 2, 2, 1}	{2, 2, 1, 1, 3, 3, 4, 4}	{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 4, 4, 3, 3}	{2, 2, 1, 3, 3, 1, 4, 4}	{2, 2, 1, 3, 3, 4, 4, 1}	{2, 2, 1, 3, 4, 4, 3, 1}	{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 4, 4, 3, 3, 1}	{2, 2, 3, 3, 1, 1, 4, 4}	{2, 2, 3, 3, 1, 4, 4, 1}	{2, 2, 3, 3, 4, 4, 1, 1}	{2, 2, 3, 4, 4, 3, 1, 1}
{2, 2, 4, 4, 1, 1, 3, 3}	{2, 2, 4, 4, 1, 3, 3, 1}	{2, 2, 4, 4, 3, 3, 1, 1}	{2, 3, 3, 2, 1, 1, 4, 4}	{2, 3, 3, 2, 1, 4, 4, 1}
{2, 3, 3, 2, 4, 4, 1, 1}	{2, 3, 3, 4, 4, 2, 1, 1}	{2, 3, 4, 4, 3, 2, 1, 1}	{2, 4, 4, 2, 1, 1, 3, 3}	{2, 4, 4, 2, 1, 3, 3, 1}
{2, 4, 4, 2, 3, 3, 1, 1}	{2, 4, 4, 3, 3, 2, 1, 1}	{3, 3, 1, 1, 2, 2, 4, 4}	{3, 3, 1, 1, 2, 4, 4, 2}	{3, 3, 1, 1, 4, 4, 2, 2}
{3, 3, 1, 2, 2, 1, 4, 4}	{3, 3, 1, 2, 2, 4, 4, 1}	{3, 3, 1, 2, 4, 4, 2, 1}	{3, 3, 1, 4, 4, 1, 2, 2}	{3, 3, 1, 4, 4, 2, 2, 1}
{3, 3, 2, 2, 1, 1, 4, 4}	{3, 3, 2, 2, 1, 4, 4, 1}	{3, 3, 2, 2, 4, 4, 1, 1}	{3, 3, 2, 4, 4, 2, 1, 1}	{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 2, 2, 1}	{3, 3, 4, 4, 2, 2, 1, 1}	{3, 4, 4, 3, 1, 1, 2, 2}	{3, 4, 4, 3, 1, 2, 2, 1}	{3, 4, 4, 3, 2, 2, 1, 1}
{4, 4, 1, 1, 2, 2, 3, 3}	{4, 4, 1, 1, 2, 3, 3, 2}	{4, 4, 1, 1, 3, 3, 2, 2}	{4, 4, 1, 2, 2, 1, 3, 3}	{4, 4, 1, 2, 2, 3, 3, 1}
{4, 4, 1, 2, 3, 3, 2, 1}	{4, 4, 1, 3, 3, 1, 2, 2}	{4, 4, 1, 3, 3, 2, 2, 1}	{4, 4, 2, 2, 1, 1, 3, 3}	{4, 4, 2, 2, 1, 3, 3, 1}
{4, 4, 2, 2, 3, 3, 1, 1}	{4, 4, 2, 3, 3, 2, 1, 1}	{4, 4, 3, 3, 1, 1, 2, 2}	{4, 4, 3, 3, 1, 2, 2, 1}	{4, 4, 3, 3, 2, 2, 1, 1}

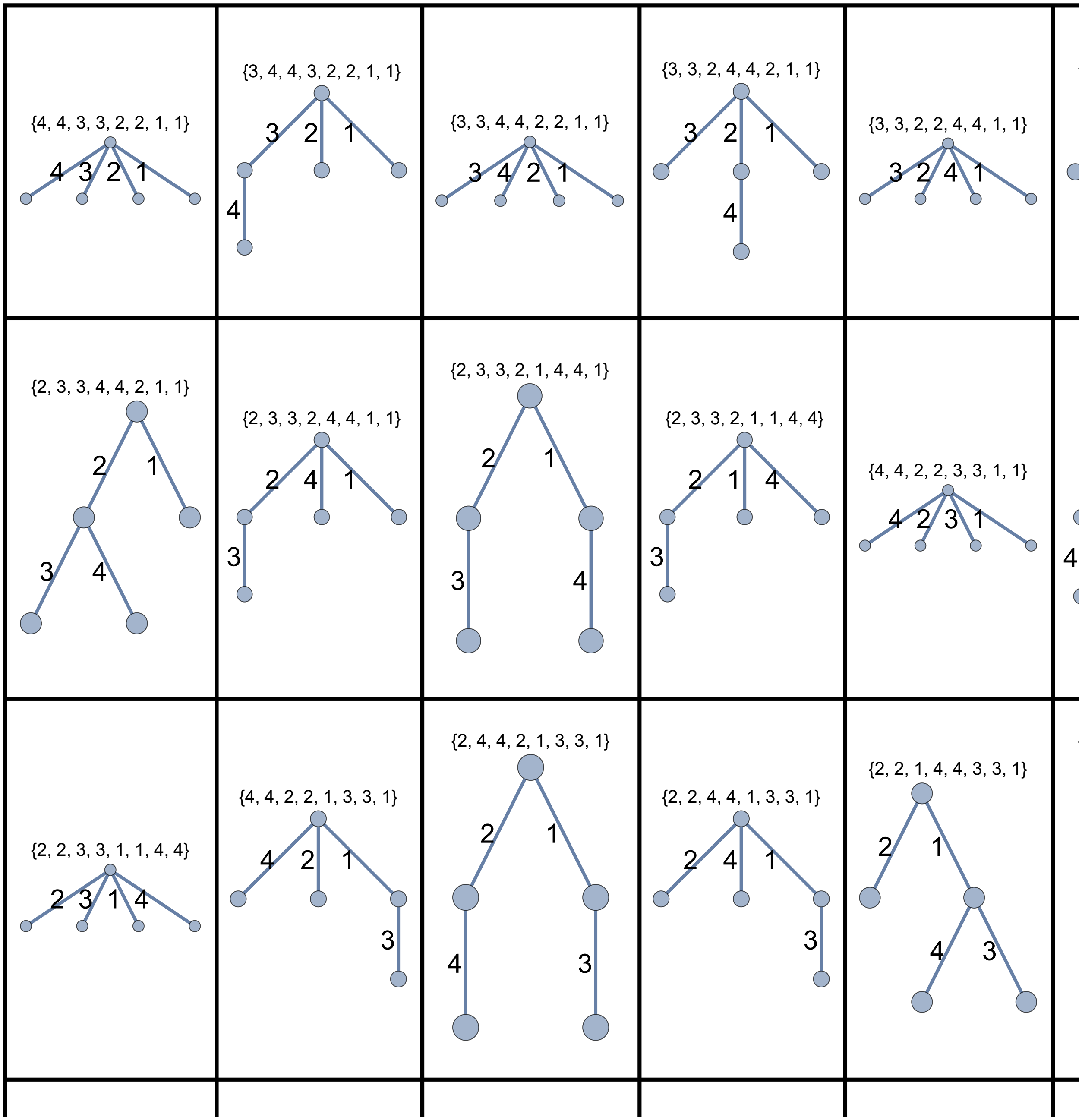
```
In[•]:= ClearAll[strlngPermGraph]
strlngPermGraph[sp_, opts : OptionsPattern[]] :=
 Module[{vl = DeleteDuplicates@sp, pos = PositionIndex@sp,
   eL = EdgeList@*TransitiveReductionGraph@*GraphUnion},
  Graph[Prepend[vl, 0], eL[Graph@Thread[0 → vl],
    SimpleGraph@RelationGraph[And @@ Between[pos@#] /@ pos[#2] &, vl]],
   GraphLayout → {"LayeredEmbedding", "RootVertex" → 0},
   EdgeLabels → {e_ :→ Placed[Last@e, {Left, "Middle"}]}, opts]]
```

In[ ]:=
```
Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
      EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
    strlngPermutations[3], 5], Dividers → All, Spacings → {4, 4}]
```

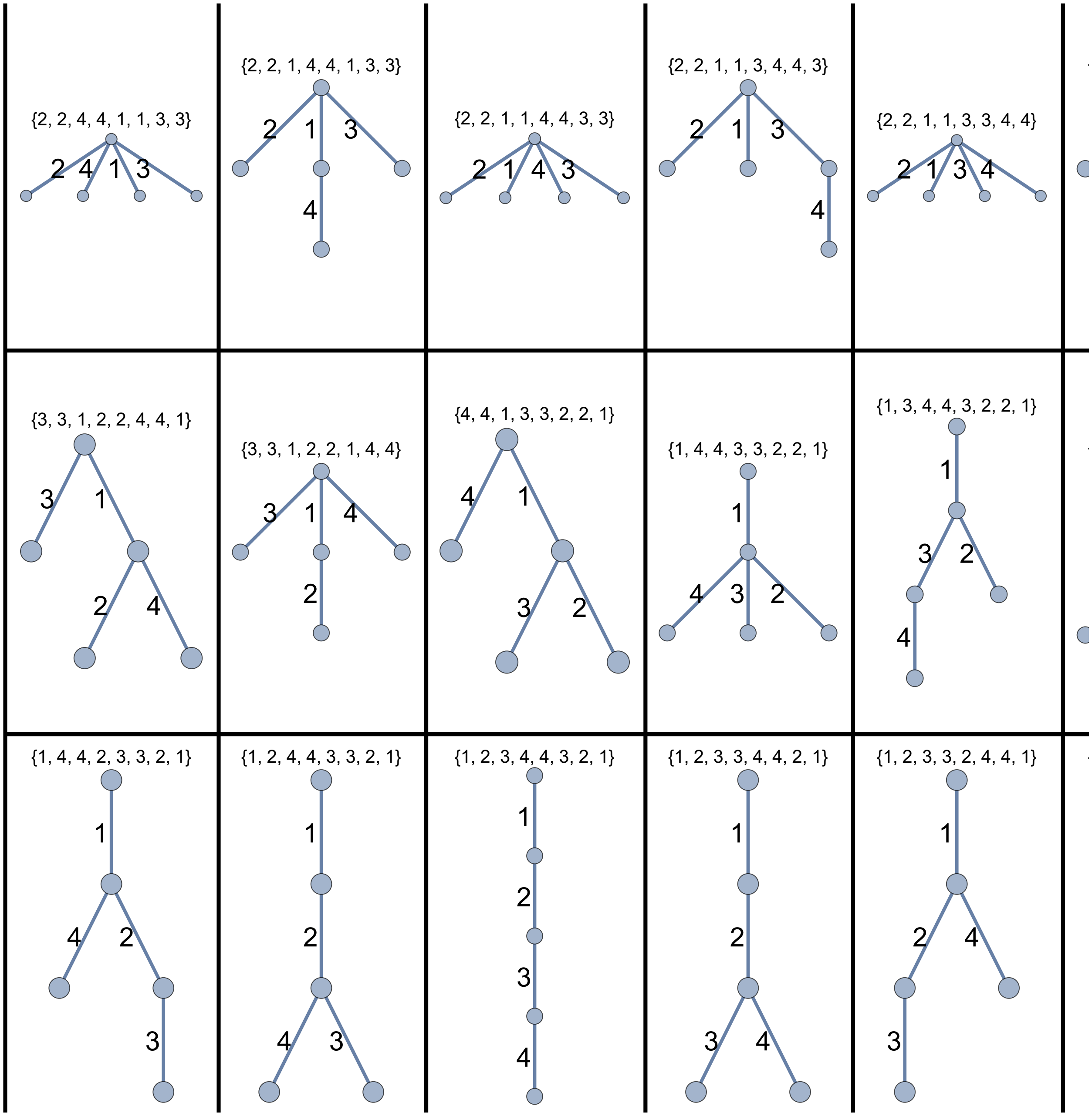
Out[ ]=

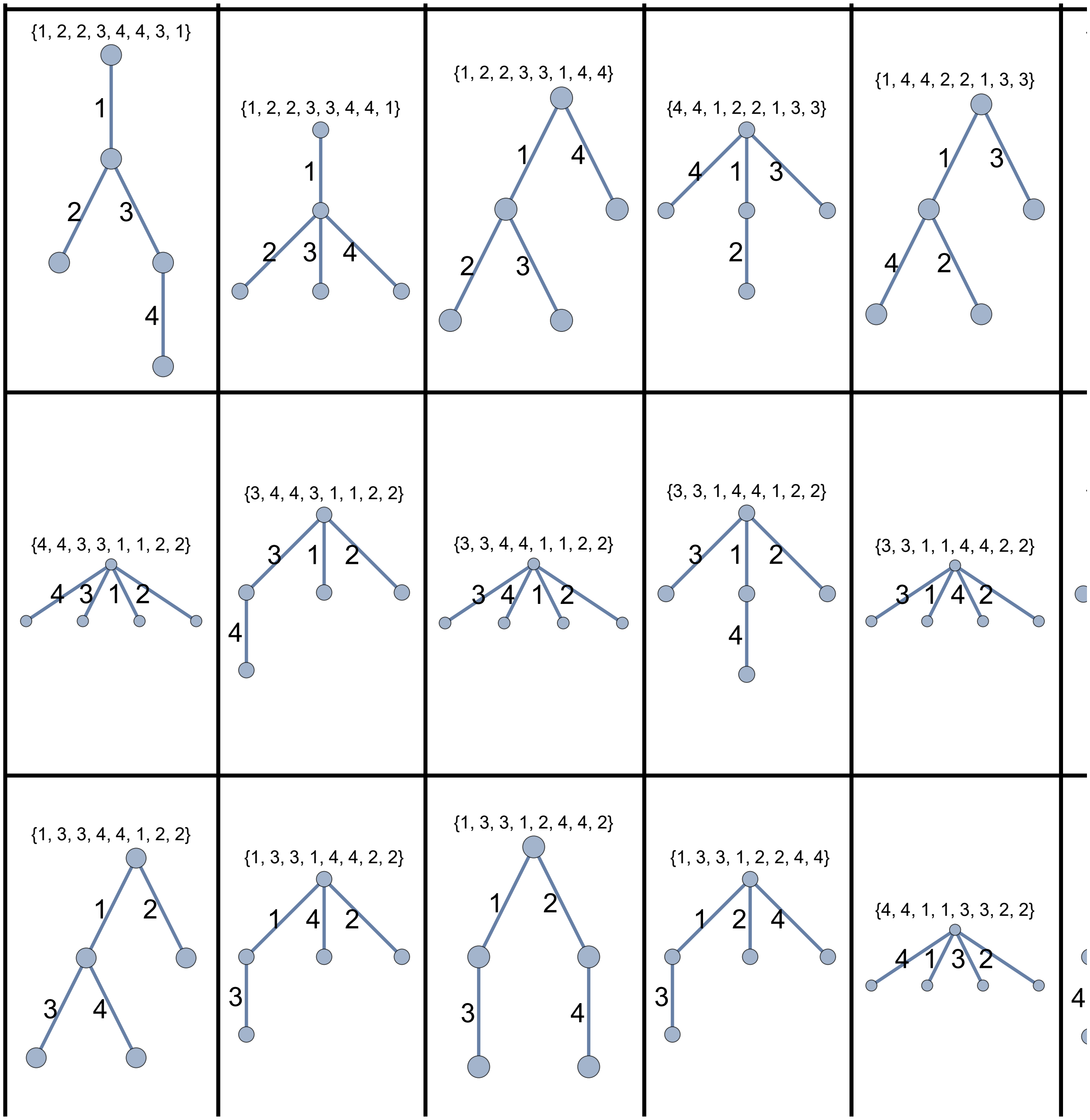


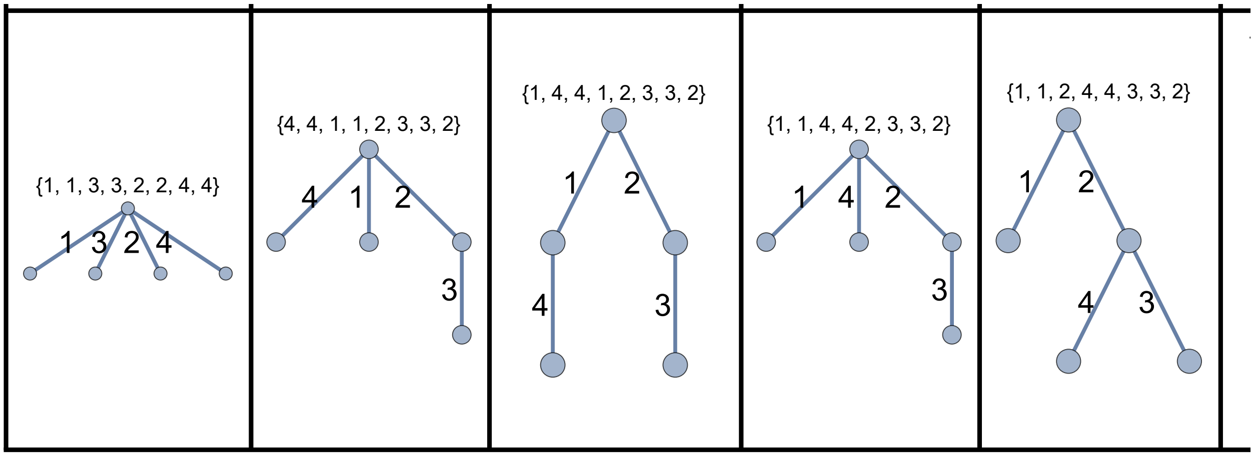
In[ ]:=
```
Grid[Partition[strlngPermGraph[#, PlotLabel → #, EdgeShapeFunction → "Line",
     EdgeStyle → Thick, EdgeLabelStyle → 16, VertexSize → Medium] & /@
   strlngPermutations[4], 10], Dividers → All, Spacings → {1, 1}]
```



Out[•]=


{2, 2, 4, 4, 1, 1, 3, 3}
{2, 2, 1, 4, 4, 1, 3, 3}
{2, 2, 1, 1, 4, 4, 3, 3}
{2, 2, 1, 1, 3, 4, 4, 3}
{2, 2, 1, 1, 3, 3, 4, 4}
{3, 3, 1, 2, 2, 4, 4, 1}
{3, 3, 1, 2, 2, 1, 4, 4}
{4, 4, 1, 3, 3, 2, 2, 1}
{1, 4, 4, 3, 3, 2, 2, 1}
{1, 3, 4, 4, 3, 2, 2, 1}
{1, 4, 4, 2, 3, 3, 2, 1}
{1, 2, 4, 4, 3, 3, 2, 1}
{1, 2, 3, 4, 4, 3, 2, 1}
{1, 2, 3, 3, 4, 4, 2, 1}
{1, 2, 3, 3, 2, 4, 4, 1}


{1, 2, 2, 3, 4, 4, 3, 1}
{1, 2, 2, 3, 3, 4, 4, 1}
{1, 2, 2, 3, 3, 1, 4, 4}
{4, 4, 1, 2, 2, 1, 3, 3}
{1, 4, 4, 2, 2, 1, 3, 3}
{4, 4, 3, 3, 1, 1, 2, 2}
{3, 4, 4, 3, 1, 1, 2, 2}
{3, 3, 4, 4, 1, 1, 2, 2}
{3, 3, 1, 4, 4, 1, 2, 2}
{3, 3, 1, 1, 4, 4, 2, 2}
{1, 3, 3, 4, 4, 1, 2, 2}
{1, 3, 3, 1, 4, 4, 2, 2}
{1, 3, 3, 1, 2, 4, 4, 2}
{1, 3, 3, 1, 2, 2, 4, 4}
{4, 4, 1, 1, 3, 3, 2, 2}



x

Limits for Ratio test in m(2)

In[•]:=
```
case1 = (2^(1 + 2 n) n! (1 + n)!) / (2 (1 + n))!
case2 = (2^(-1 - 2 n) (2 n)!) / (n! (1 + n)!)
```

Out[•]= $\frac{2^{1+2n}\, n!\, (1+n)!}{(2\,(1+n))!}$

Out[•]= $\frac{2^{-1-2n}\, (2n)!}{n!\, (1+n)!}$

In[•]:=
```
limitm2case1 = Limit[case1, n → Infinity]
limitm2case2 = Limit[case2, n → Infinity]
```

Out[•]= 0

Out[•]= 0

Computing m(1) to m(10) from n=0 to n=2000

In[•]:=
```
Multifactorial[n_, k_] := Abs[Apply[Times, Range[-n, -1, k]]]
```

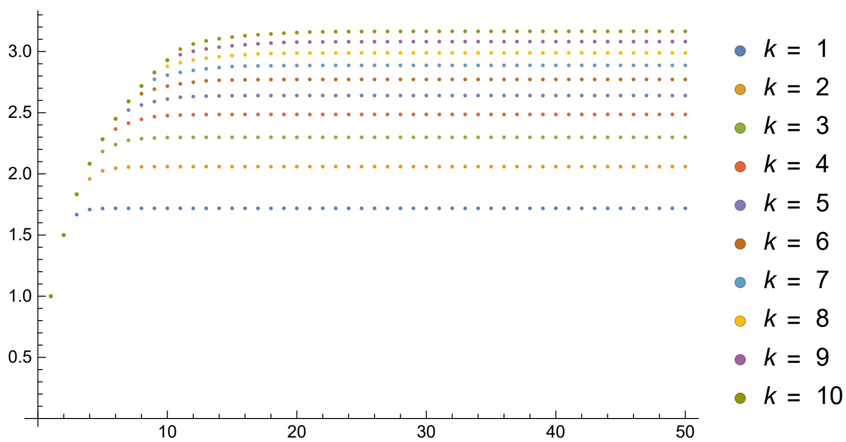
In[•]:=
```
For[i = 1, i < 11, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 150}], 20]]]
```

```
2.7182818284590452354

3.0594074053425761445

3.2989135380884190034

3.4859449774535577452

3.6402244677338097342

3.7719023962117584357

3.8869596537408434954

3.9892412126901365441

4.0813755201688985441

4.1652437655583845908
```

Plot of m (1) to m (1) superimposed on each other from n = 0 to n = 2000

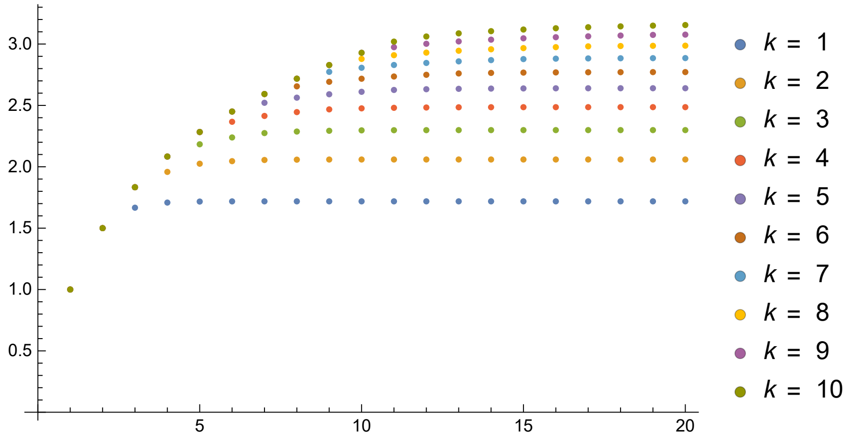
```
ListPlot[Table[Sum[1/Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 20}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[•]=



```
ListPlot[Table[Sum[1/Multifactorial[n, j], {n, 1, i}], {j, 1, 10}, {i, 1, 50}],
 PlotLegends → PointLegend[Automatic,
   PromptForm[k, #] & /@ Range[10], LegendMarkers → {Graphics[Disk[]], 6}]]
```

Out[ ]=



Computation of RMFCs using the closed form formula

In[ ]:=

$$\text{ClosedFormRMFC}[n\_] := 1 + \frac{1}{n}\,\text{Exp}[1/n]\,\text{Sum}\left[n^{k/n}\,\text{Gamma}\left[\frac{k}{n}, 0, \frac{1}{n}\right], \{k, n\}\right]$$

In[ ]:=

```
For[i = 1, i < 11, i++, Print[N[ClosedFormRMFC[i], 20]]]
```

2.7182818284590452354

3.0594074053425761445

3.2989135380884190034

3.4859449774535577452

3.6402244677338097342

3.7719023962117584357

3.8869596537408434954

3.9892412126901365441

4.0813755201688985441

4.1652437655583845908

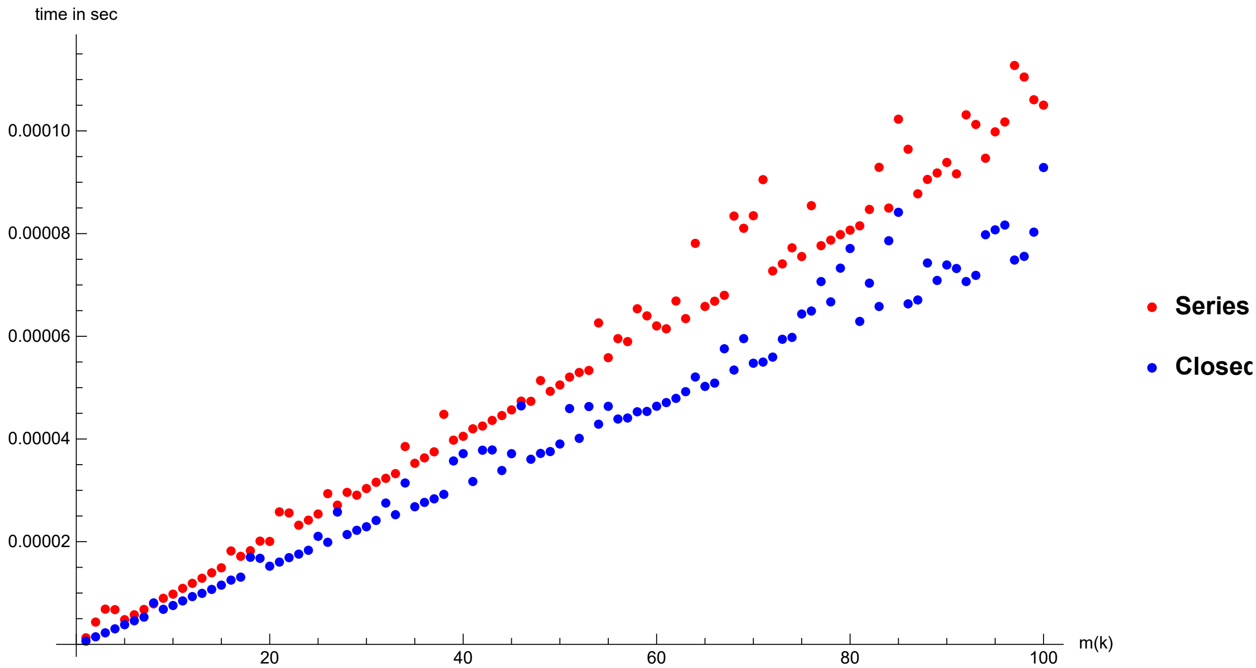
Analysing efficiency of the two RMFC calculation methods

In[ ]:=

```
test1[xx_] :=
 (For[i = 1, i < xx, i++, Print[N[Sum[1/Multifactorial[n, i], {n, 0, 250}], 50]] //
     Inactive] // RepeatedTiming)[[1]]
test2[xx_] := (For[i = 1, i < xx, i++, Print[N[ClosedFormRMFC[i], 50]] // Inactive] //
    RepeatedTiming)[[1]]
```

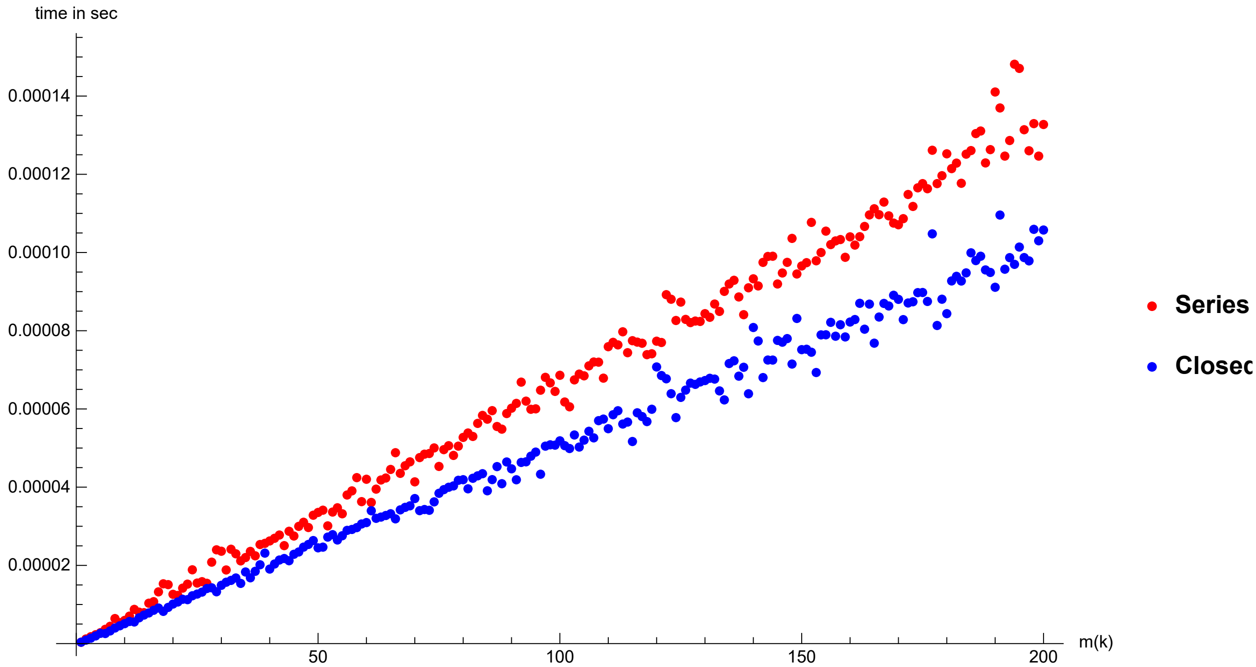
In[•]:=
```
list1 =
  ListPlot[Table[{xx, test1[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
list2 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends →
    {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

In[•]:=
```
Show[list1, list2, ImageSize → Large]
```



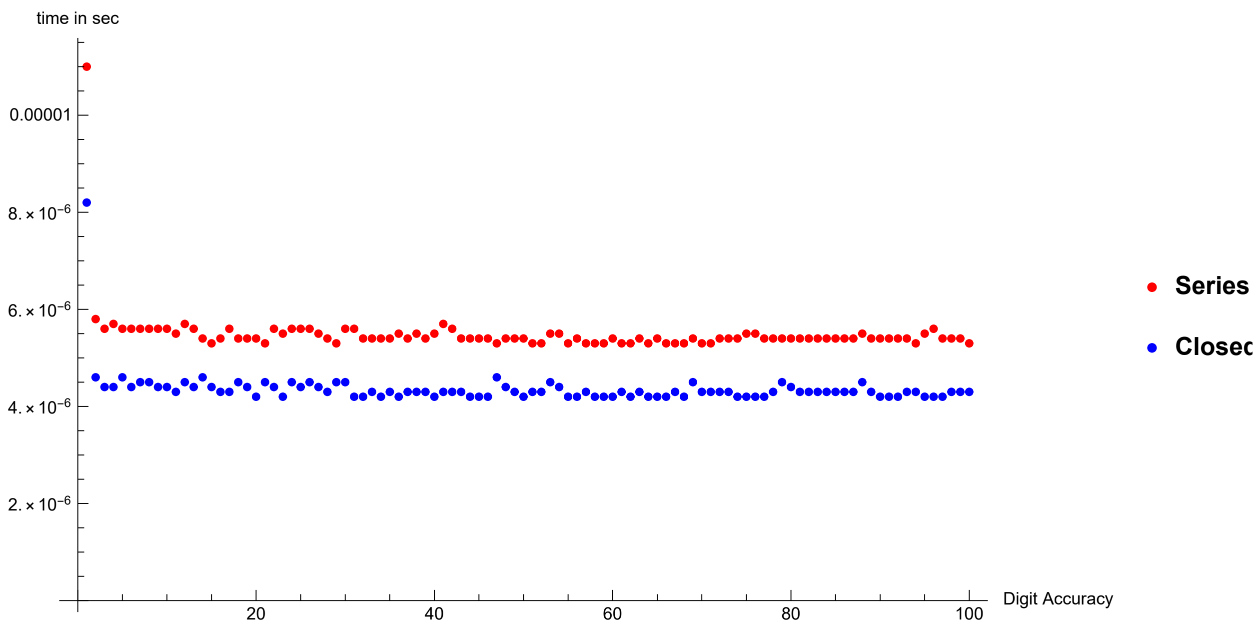
In[•]:=
```
list3 =
  ListPlot[Table[{xx, test1[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Red, Thick}];
list4 = ListPlot[Table[{xx, test2[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends →
    {Automatic}, AxesLabel → {"m(k)", "time in sec"}, PlotStyle → {Blue, Thick}];
```

In[•]:=
```
Show[list3, list4, ImageSize → Large]
```

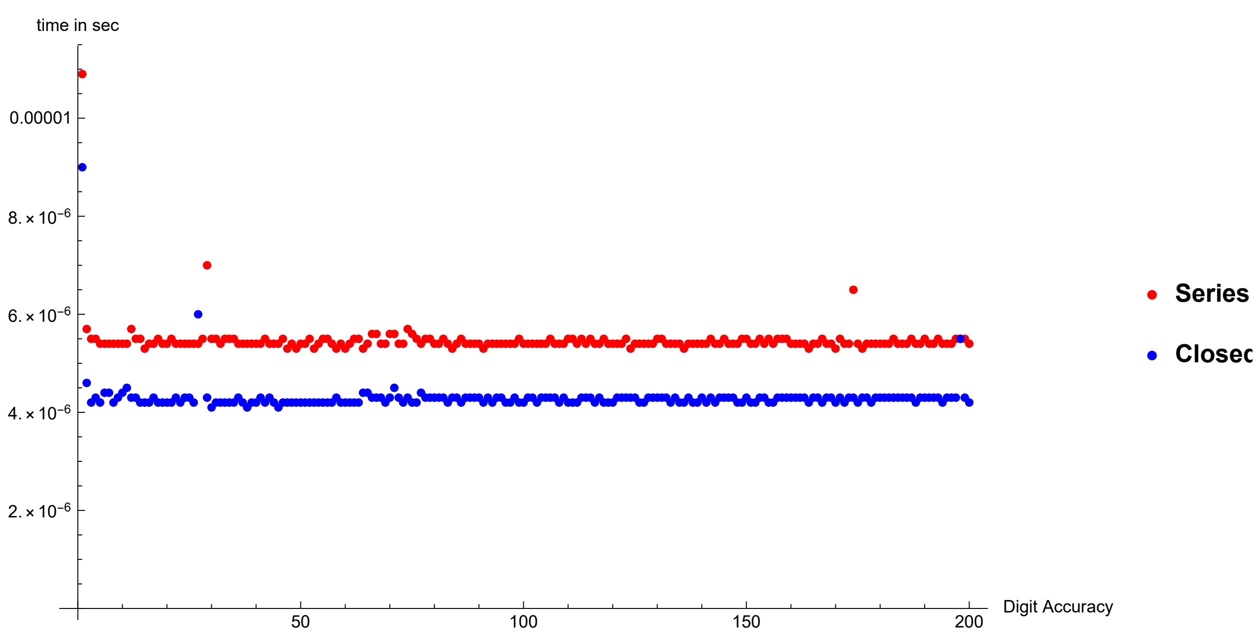



```
test3[xx_] :=
 (For[i = 1, i < 10, i++, Print[N[Sum[1 / Multifactorial[n, i], {n, 0, 500}], xx]] //
     Inactive] // AbsoluteTiming) [[1]]
test4[xx_] := (For[i = 1, i < 10, i++, Print[N[ClosedFormRMFC[i], xx]] // Inactive] //
    AbsoluteTiming) [[1]]
```

```
In[]:= list5 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];

In[]:= list6 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 100}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];

In[]:= Show[list5, list6, ImageSize → Large]
```




```
In[ ]:= list7 =
  ListPlot[Table[{xx, test3[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Red, Thick}];

In[ ]:= list8 =
  ListPlot[Table[{xx, test4[xx]}, {xx, 1, 200}], PlotRange → All, PlotLegends → {Automatic},
   AxesLabel → {"Digit Accuracy", "time in sec"}, PlotStyle → {Blue, Thick}];

In[ ]:= Show[list7, list8, ImageSize → Large]
```

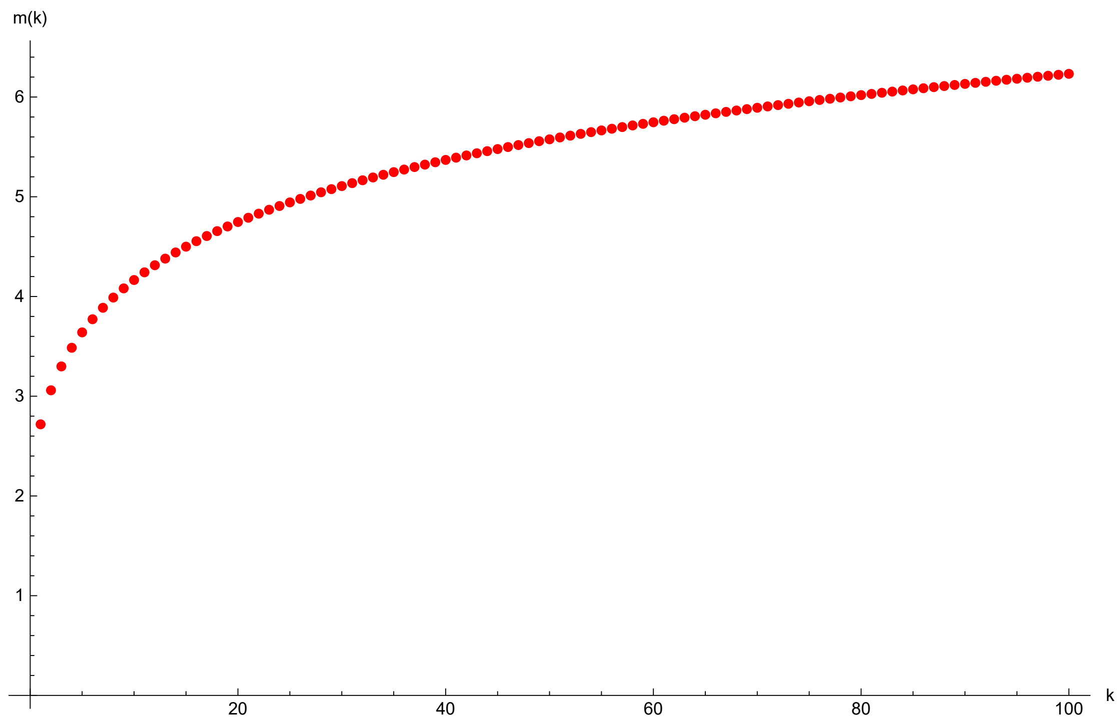


Asymptotics of Reciprocal Multifactorial Series

Simple Graph

In[ ]:=
```
ListPlot[Table[{x, ClosedFormRMFC[x]}, {x, 1, 100}],
 ImageSize → Large, AxesLabel → {"k", "m(k)"}, PlotStyle → {Red, Thick}]
```

Out[ ]=



Detailed asymptotics

In[1]:=
```
Asymptotic[λ Integrate[((1 - t) (1 - E^(-(λ t)))) / (t (E^(-(λ Log[t])) - 1)), {t, 0, 1}],
 λ → 0, SeriesTermGoal → 1]
```

Out[1]= $\lambda \operatorname{Log}[2]$

```
Asymptotic[λ Integrate[((1 - t) (1 - E^(-(λ t)))) / (t (E^(-(λ Log[t])) - 1)), {t, 0, 1}],
 λ → 0, SeriesTermGoal → 2]
```

Out[ ]= $$-\frac{\lambda^2}{4}-\frac{1}{2}\lambda^2 \operatorname{Log}[3]+\frac{1}{2}\lambda \operatorname{Log}[4]+\frac{1}{12}\lambda^2 \operatorname{Log}[64]$$

In[3]:= $$\text{FullSimplify}\left[1+\left(1+\frac{1}{2\,k^2}+\frac{1}{k}\right)\left(\text{EulerGamma}+\frac{1}{6\,k^2}+\frac{1}{2\,k}+\frac{\operatorname{Log}[3]}{2\,k^2}-\frac{\operatorname{Log}[4]}{2\,k}-\frac{\operatorname{Log}[64]}{12\,k^2}+\operatorname{Log}[k]\right)\right]$$

Out[3]= $$1+\frac{\left(1+2\,k\,(1+k)\right)\left(1+\operatorname{Log}\left[\frac{27}{8}\right]+k\,\left(3+6\,\text{EulerGamma}\,k-\operatorname{Log}[64]\right)+6\,k^2\operatorname{Log}[k]\right)}{12\,k^4}$$

In[17]:= $$\text{FullSimplify}\left[1 + \left(1 + \frac{1}{k}\right) * \left(\left(\text{EulerGamma} + \text{Log}[k] + \frac{1}{2\,k}\right) - \left(\frac{1}{k}\,\text{Log}[2]\right)\right)\right]$$

Out[17]= $$1 + \frac{(1 + k)\,(1 + 2\,\text{EulerGamma}\,k - \text{Log}[4] + 2\,k\,\text{Log}[k])}{2\,k^2}$$

In[5]:= $$\text{ClosedFormRMFC}[n\_] := 1 + \frac{1}{n}\,\text{Exp}[1/n]\,\text{Sum}\left[n^{k/n}\,\text{Gamma}\left[\frac{k}{n}, 0, \frac{1}{n}\right], \{k, n\}\right]$$

In[10]:= $$\text{RMFCApproximation}[k\_] := 1 + \frac{(1 + 2\,k\,(1 + k))\,\left(1 + \text{Log}\left[\frac{27}{8}\right] + k\,(3 + 6\,\text{EulerGamma}\,k - \text{Log}[64]) + 6\,k^2\,\text{Log}[k]\right)}{12\,k^4}$$

$$\text{RMFCApproximation1}[k\_] := 1 + \frac{(1 + k)\,(1 + 2\,\text{EulerGamma}\,k - 2\,\text{Log}[2] + 2\,k\,\text{Log}[k])}{2\,k^2}$$

In[12]:=
```
N[RMFCApproximation[15]]
N[RMFCApproximation1[15]]
```

Out[12]= 4.49958

Out[13]= 4.49055

In[●]:=
```
N[ClosedFormRMFC[15]]
```

Out[●]= 4.49969

In[●]:=
```
Table[{x, N[RMFCApproximation[10^x], 50]}, {x, 0, 5}]
```

Out[●]=
```
{{0, 2.8836692626558410221123829825576677447983026999440},
 {1, 4.1649189354117046099965674566757558205746873600579},
 {2, 6.2325552567621081237809508324268568037822079589510},
 {3, 8.4922666866575160699533708545283873810253799369197},
 {4, 10.788515528463920803804219311415785943169176345868},
 {5, 13.090260100433386932193328011615118199331396176765}}
```

In[14]:=
```
Table[{x, N[RMFCApproximation1[10^x], 50]}, {x, 0, 5}]
```

Out[14]=
```
{{0, 1.7681369686831751023785599372484517259333184031593},
 {1, 4.1465346438235424150510583658830438800192818922459},
 {2, 6.2322589228748650233038059050189145713692433849724},
 {3, 8.4922625744998130773184202912360509141546656899936},
 {4, 10.788515475831875568107750939226274215134136716216},
 {5, 13.090260099791939680800187624934920717370858843943}}
```

In[●]:=
```
Table[{x, N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

Out[●]=
```
{{0, 2.7182818284590452353602874713526624977572470937000},
 {1, 4.1652437655583845907872624104455607382280307953708},
 {2, 6.2325559690048781755948333144748471914777217107326},
 {3, 8.4922666877530555546922513543645116307527055440479},
 {4, 10.788515528465399974853138837254851897589458808237},
 {5, 13.090260100433388795106669060937341380984702346850}}
```

In[15]:=
```
Table[{x, 100 Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation[10^x]])] /
    N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

Out[15]= $\{\{0, 6.08426\}, \{1, 0.00779859\}, \{2, 0.0000114278\}, \{3, 1.29004 \times 10^{-8}\}, \{4, 1.37156 \times 10^{-11}\}, \{5, 0.\}\}$

In[16]:=
```
Table[{x, 100 Abs[(N[ClosedFormRMFC[10^x]] - N[RMFCApproximation1[10^x]])] /
    N[ClosedFormRMFC[10^x], 50]}, {x, 0, 5}]
```

Out[16]= $\{\{0, 34.9539\}, \{1, 0.449172\}, \{2, 0.00476604\}, \{3, 0.0000484353\}, \{4, 4.87866 \times 10^{-7}\}, \{5, 4.90019 \times 10^{-9}\}\}$